Supplementary Material for Online Message Passing-based Inference in the Hierarchical Gaussian Filter

Ismail Senoz
Electrical Engineering Dept.
Eindhoven University of Technology
Eindhoven, the Netherlands
i.senoz@tue.nl

Bert de Vries
Electrical Engineering Dept.
Eindhoven University of Technology & GN Hearing
Eindhoven, the Netherlands
bert.de.vries@tue.nl

I. Derivations for GCV Node

In this supplement we derive the update equations that are given in Table I in the ISIT-2020 paper entitled *Online Message Passing-based Inference in the Hierarchical Gaussian Filter*. We use a structured factorization for the parameters y, x, z, κ, ω, that is we assume

$$q(y,x,z,\kappa,\omega) = q(y,x)q(z)q(\kappa)q(\omega). \tag{1}$$

Before we proceed with the message update rules, we compute intermediate expectations $\mathbb{E}_{q_{xy}}[(x-y)^2], \mathbb{E}_{q_z q_\kappa}[\exp(\kappa z)]$ and $\mathbb{E}_{q_\omega}[\exp(\omega)]$ that are needed for computation of messages and marginals. Assuming $q(x,y) \propto \mathcal{N}(m,\Lambda)$, where $m \in \mathbb{R}^2$ is the mean and $\Lambda \in \mathbb{R}^{2\times 2}$ is the covariance of the joint distribution, we can write

$$\mathbb{E}_{q_{xy}}[(x-y)^2] = \underbrace{(m_1 - m_2)^2 + \Lambda_{11} + \Lambda_{22} - \Lambda_{21} - \Lambda_{12}}_{\gamma_4}. \tag{2}$$

Following (1) we can approximate the multiplication of two Gaussian random variables with a Gaussian random variable. Assuming independence between κ and z we write

$$z\kappa \overset{approx.}{\sim} \mathcal{N}(m_z m_\kappa, \underbrace{m_z^2 v_\kappa + m_\kappa^2 v_z + v_z v_\kappa}_{\gamma_1}). \tag{3}$$

This means $\exp(z\kappa)$ is log-normally distributed with mean and variance as in Eq. 4a. We approximate the joint expectation of this log-normally distributed product of two independent normal random variables as per Eq. 4b. To arrive at Eq. 4b we use the fact that for a log-normally distributed z, for all $n \in R$ all moments are well defined and analytically given by $\mathbb{E}_{q_z}[z^n] = \exp\left(nm_z + \frac{n^2 v_z}{2}\right)$. For $n = -1$ we obtain Eq. 4b.

$$\exp(z\kappa) \overset{approx.}{\sim} \log\mathcal{N}(m_z m_\kappa, \gamma_1) \tag{4a}$$

$$\mathbb{E}_{q_z q_\kappa}[\exp(\kappa z)] \approx \exp\left(m_z m_\kappa + \frac{\gamma_1}{2}\right) \tag{4b}$$

$$\mathbb{E}_{q_z q_\kappa}\left[\frac{1}{\exp(\kappa z)}\right] \approx \exp\left(-m_z m_\kappa + \frac{\gamma_1}{2}\right) \triangleq \gamma_2\,. \tag{4c}$$

Similarly we can write the expectation for ω as

$$\mathbb{E}_{q_\omega}[\exp(\omega)] \approx \exp\left(m_\omega + \frac{v_\omega}{2}\right) \tag{5a}$$

$$\mathbb{E}_{q_\omega}\left[\frac{1}{\exp(\omega)}\right] \approx \exp\left(-m_\omega + \frac{v_\omega}{2}\right) \triangleq \gamma_3\,. \tag{5b}$$

We are now equipped with the needed expectations to obtain expressions of Table I. We start our derivation with $\overrightarrow{\nu}(y)$. Using computation results of the paper we can write

$$\overrightarrow{\nu}(y) \propto \int \overrightarrow{\nu}(x)\tilde{f}(x,y)\mathrm{d}x \tag{6a}$$

$$\propto \int \overrightarrow{\nu}(x)\exp\left(-0.5\mathbb{E}_{\setminus q_{xy}}\left[\frac{(y-x)^2}{\exp(\kappa z + \omega)}\right]\right)\mathrm{d}x. \tag{6b}$$

Let us assume that incoming message is a Gaussian i.e. $\overrightarrow{\nu}(x) \propto \mathcal{N}(x|\overrightarrow{m}_x, \overrightarrow{v}_x)$. Using Eq. 5b,4c and recognizing that the integral in Eq. 6b is a convolution of two Gaussian forms we can write

$$\overrightarrow{\nu}(y) \propto \int \overrightarrow{\nu}(x) \exp\left(-0.5\gamma_2\gamma_3(y-x)^2\right) \mathrm{d}x \tag{7a}$$

$$\propto \mathcal{N}(\overrightarrow{m}_x, \overrightarrow{v}_x + \gamma_2\gamma_3). \tag{7b}$$

Assuming $\overleftarrow{\nu}(y) \propto \mathcal{N}(y|\overleftarrow{m}_y, \overleftarrow{v}_y)$, we can compute the message $\overleftarrow{\nu}(x)$ in a similar manner

$$\overleftarrow{\nu}(x) \propto \mathcal{N}(\overleftarrow{m}_y, \overleftarrow{v}_y + \gamma_2\gamma_3). \tag{8}$$

Using $\overleftarrow{\nu}(y)$, $\overrightarrow{\nu}(x)$ and $\tilde{f}(x,y)$ we can compute the joint by

$$q(x,y) \propto \overrightarrow{\nu}(x)\overleftarrow{\nu}(y)\tilde{f}(x,y) \tag{9a}$$

$$\propto \mathcal{N}(x|\overrightarrow{m}_x, \overrightarrow{v}_x)\mathcal{N}(y|\overleftarrow{m}_y, \overleftarrow{v}_y)\mathcal{N}(y|x, \gamma_2\gamma_3) \tag{9b}$$

$$\propto \mathcal{N}\left(\begin{bmatrix} x \\ y \end{bmatrix} \middle| \begin{bmatrix} \overrightarrow{m}_x \\ \overleftarrow{m}_y \end{bmatrix}, \begin{bmatrix} \overrightarrow{v}_x & 0 \\ 0 & \overleftarrow{v}_y \end{bmatrix}\right) \mathcal{N}\left(\begin{bmatrix} x \\ y \end{bmatrix} \middle| \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \frac{1}{\gamma_2\gamma_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) \tag{9c}$$

$$\propto \mathcal{N}\left(\begin{bmatrix} x \\ y \end{bmatrix} \middle| \Lambda^{-1}\begin{bmatrix} \overrightarrow{m}_x/\overrightarrow{v}_x \\ \overleftarrow{m}_y/\overleftarrow{v}_y \end{bmatrix}, \underbrace{\begin{bmatrix} 1/\overrightarrow{v}_x + \gamma_2\gamma_3 & -\gamma_2\gamma_3 \\ -\gamma_2\gamma_3 & \overleftarrow{v}_x + \gamma_2\gamma_3 \end{bmatrix}}_{\Lambda}\right). \tag{9d}$$

Eq.(9c) follows by grouping the first two terms of (9b) into a multivariate Gaussian distribution and the last term can be obtained by arranging the quadratic terms of the Gaussian distribution. Note however that the second term of (9c) is not a proper distribution as the determinant of covariance is 0. We can however write the (9d) by noting that summation of two quadratic terms in the exponent of (9c) results in a quadratic term which represents a multivariate Gaussian. Note that (9d) is a proper distribution.

We now compute

$$\overleftarrow{\nu}(z) \propto \exp\left(\mathbb{E}_{/q_z}[\log f(y,x,z,\kappa,\omega)]\right) \tag{10a}$$

$$\propto \exp\left(\mathbb{E}_{/q_z}\left[-\frac{\kappa z + \omega}{2}\right] + \mathbb{E}_{/q_z}\left[-\frac{(y-x)^2}{2\exp(\kappa z + \omega)}\right]\right) \tag{10b}$$

$$\propto \exp\left(-0.5\left(zm_\kappa + \underbrace{\frac{\gamma_4\gamma_3}{\exp(zm_\kappa - \frac{z^2 v_\kappa}{2})}}_{\gamma_5}\right)\right) \tag{10c}$$

$$\propto \exp\left(-0.5\left(zm_\kappa + \gamma_5\right)\right) \tag{10d}$$

where expectations of (10b) are given by (5b) and (2). Similarly for $\overleftarrow{\nu}(\kappa)$ we can write

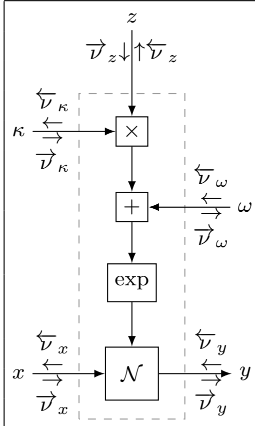
$$\overleftarrow{\nu}(\kappa) \propto \exp\left(\mathbb{E}_{/q_\kappa}[\log f(y,x,z,\kappa,\omega)]\right) \tag{11a}$$

$$\propto \exp\left(\mathbb{E}_{/q_\kappa}\left[-\frac{\kappa z + \omega}{2}\right] + \mathbb{E}_{/q_\kappa}\left[-\frac{(y-x)^2}{2\exp(\kappa z + \omega)}\right]\right) \tag{11b}$$

$$\propto \exp\left(-0.5\left(\kappa m_z + \underbrace{\frac{\gamma_4\gamma_3}{\exp(\kappa m_z - \frac{\kappa^2 v_z}{2})}}_{\gamma_5}\right)\right) \tag{11c}$$

$$\propto \exp\left(-0.5\left(\kappa m_z + \gamma_5\right)\right). \tag{11d}$$

TABLE I
MESSAGE PASSING UPDATE RULES FOR THE GCV NODE.

GCV Node	**Auxilary**	
	γ_1	$m_z^2 v_\kappa + m_\kappa^2 v_z + v_z v_\kappa$
	γ_2	$\exp\left(-m_\kappa m_z + 0.5\gamma_1\right)$
	γ_3	$\exp\left(-m_\omega + 0.5 v_\omega\right)$
	γ_4	$(m_1{-}m_2)^2{+}\Lambda_{11}{+}\Lambda_{22}{-}\Lambda_{12}{-}\Lambda_{21}$
	γ_5	$\gamma_4\gamma_3 \exp\left(-m_\kappa z{+}0.5 z^2 v_\kappa\right)$
	γ_6	$\gamma_4\gamma_2 \exp\left(-\omega\right)$
	γ_7	$\gamma_4\gamma_3 \exp\left(-m_z\kappa{+}0.5\kappa^2 v_z\right)$
	m	$\Lambda^{-1}\begin{bmatrix}\overrightarrow{m}_x/\overrightarrow{v}_x \\ \overleftarrow{m}_y/\overleftarrow{v}_y\end{bmatrix}$
	Λ	$\begin{bmatrix}1/\overrightarrow{v}_x{+}\gamma_2\gamma_3 & -\gamma_2\gamma_3 \\ -\gamma_2\gamma_3 & 1/\overleftarrow{v}_y{+}\gamma_2\gamma_3\end{bmatrix}$
	Messages	
	$\overleftarrow{\nu}(y)$	$\mathcal{N}\left(\overleftarrow{m}_y, \overleftarrow{v}_y\right)$
	$\overrightarrow{\nu}(y)$	$\mathcal{N}\left(\overrightarrow{m}_x, \overrightarrow{v}_x + \gamma_2\gamma_3\right)$
	$\overrightarrow{\nu}(x)$	$\mathcal{N}\left(\overrightarrow{m}_x, \overrightarrow{v}_x\right)$
$\mathcal{N}(y\|x, \exp(\kappa z{+}\omega))$	$\overleftarrow{\nu}(x)$	$\mathcal{N}\left(\overleftarrow{m}_y, \overleftarrow{v}_y + \gamma_2\gamma_3\right)$
	$\overrightarrow{\nu}(z)$	$\mathcal{N}(\overrightarrow{m}_z, \overrightarrow{v}_z)$
Marginals	$\overleftarrow{\nu}(z)$	$\exp\left(-0.5\left(m_\kappa z{+}\gamma_5\right)\right)$
$q(x,y) = \mathcal{N}(m, \Lambda)$	$\overrightarrow{\nu}(\kappa)$	$\mathcal{N}(\overrightarrow{m}_\kappa, \overrightarrow{v}_\kappa)$
$q(z) = \mathcal{N}(m_z, v_z)$	$\overleftarrow{\nu}(\kappa)$	$\exp\left(-0.5\left(m_z\kappa{+}\gamma_6\right)\right)$
$q(\kappa) = \mathcal{N}(m_\kappa, v_\kappa)$	$\overrightarrow{\nu}(\omega)$	$\mathcal{N}(\overrightarrow{m}_\omega, \overrightarrow{v}_\omega)$
$q(\omega) = \mathcal{N}(m_\omega, v_\omega)$	$\overleftarrow{\nu}(\omega)$	$\exp\left(-0.5\left(m_\omega + \gamma_7\right)\right)$
Entropy	**Average Energy**	
$0.5\log(2\pi e)^5\|\Lambda^{-1}\|v_z v_\kappa v_\omega$	$0.5\left(\log 2\pi + m_\kappa m_z + m_\omega + \gamma_4\gamma_3\gamma_2\right)$	

Last remaining message is towards ω. We compute

$$\begin{aligned}
\overleftarrow{\nu}(\omega) &\propto \exp\left(\mathbb{E}_{/q_\omega}\left[\log f(y,x,z,\kappa,\omega)\right]\right) && \text{(12a)}\\
&\propto \exp\left(\mathbb{E}_{/q_\omega}\left[-\frac{\kappa z+\omega}{2}\right] + \mathbb{E}_{/q_\omega}\left[-\frac{(y-x)^2}{2\exp(\kappa z+\omega)}\right]\right) && \text{(12b)}\\
&\propto \exp\left(-0.5\left(\omega + \underbrace{\frac{\gamma_4\gamma_2}{\exp(\omega)}}_{\gamma_7}\right)\right) && \text{(12c)}\\
&\propto \exp\left(-0.5\left(\omega+\gamma_7\right)\right). && \text{(12d)}
\end{aligned}$$

This completes are derivation of messages presented in Table I.

REFERENCES

[1] L. A. Aroian, "The Probability Function of the Product of Two Normally Distributed Variables," *The Annals of Mathematical Statistics*, vol. 18, no. 2, pp. 265–271, Jun. 1947. [Online]. Available: https://projecteuclid.org/euclid.aoms/1177730442