Theorem 0.0.1. *Suppose X and Y are independent random variables with*

$$f_X(x) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} \quad \textit{for } x \in \mathbb{R}$$

and $Y \sim \chi^2_\nu$, that is

$$f_Y(y) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} y^{\nu/2-1} e^{-y/2}$$

Then the random variable T defined by

$$T := \frac{X}{\sqrt{Y/\nu}}$$

follows Student's t-distribution with ν degrees of freedom.

Proof. Let T be defined as above and define $Z := \sqrt{Y/\nu}$. Then we have $x(t,z) = tz$ and $y(t,z) = \nu z^2$. We also compute the partial derivatives as

$$\frac{\partial x}{\partial t} = z \quad \frac{\partial x}{\partial z} = t \quad \frac{\partial y}{\partial t} = 0 \quad \frac{\partial y}{\partial z} = 2\nu z$$

Recalling the formula for the transformation of random variables, we can write

$$f_{T,Z}(t,z) = f_{X,Y}(x(t,z), y(t,z)) \, |J(t,z)|$$

where $J(t,z)$ is the Jacobian matrix. To start off, we compute the Jacobian as

$$\begin{vmatrix} z & t \\ 0 & 2\nu z \end{vmatrix} = 2\nu z^2$$

and, noting that $\nu \in \mathbb{N}$, we conclude $|J| = 2\nu z^2$.

Therefore,

$$\begin{aligned}
f_{T,Z} &= f_X(x(t,z)) f_Y(y(t,z)) \, |J| && \text{(by independence)} \\
&= \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2z^2} (\nu z^2)^{\nu/2-1} e^{-\frac{1}{2}\nu z^2} (2\nu z^2)}{2^{\nu/2}\Gamma(\nu/2)} \\
&= \frac{\nu^{\nu/2} z^\nu e^{-\frac{1}{2}t^2z^2} e^{-\frac{1}{2}\nu z^2}}{2^{\frac{1}{2}(1-\nu)}\Gamma(\nu/2)\sqrt{\pi}}
\end{aligned}$$

To obtain the pdf of T we compute the marginal. Note that z ranges from 0 to $+\infty$.

$$f_T(t) = \frac{\nu^{\nu/2}}{2^{\frac{1}{2}(1-\nu)}\Gamma(\nu/2)\sqrt{\pi}} \underbrace{\int_0^\infty z^\nu e^{\frac{1}{2}\left(z^2(t^2+\nu)\right)} \, \mathrm{d}z}_{I}$$

To evaluate I we change the variable to $\gamma := \frac{1}{2}(z^2(t^2+\nu))$, so that

$$z = \left(\frac{2\gamma}{t^2+\nu}\right)^{1/2} \quad \text{and} \quad \mathrm{d}z = \frac{\mathrm{d}\gamma}{z(t^2+\nu)}$$

Therefore

$$\begin{aligned}
I &= \int_0^\infty \left(\frac{2\gamma}{t^2+\nu}\right)^{\nu/2} e^{-\gamma} \frac{\mathrm{d}\gamma}{\left(\frac{2\gamma}{t^2+\nu}\right)^{1/2}(t^2+\nu)} \\
&= \int_0^\infty \left(\frac{2\gamma}{t^2+\nu}\right)^{\nu/2} e^{-\gamma} \frac{1}{\sqrt{2}\gamma^{1/2}(t^2+\nu)} \, \mathrm{d}\gamma \\
&= \sqrt{2}^{\nu-1} (t^2+\nu)^{-\frac{\nu+1}{2}} \int_0^\infty \gamma^{\frac{\nu-1}{2}} e^{-\gamma} \, \mathrm{d}\gamma \\
&= \sqrt{2}^{\nu-1} (t^2+\nu)^{-\frac{\nu+1}{2}} \int_0^\infty \gamma^{\frac{\nu+1}{2}-1} e^{-\gamma} \, \mathrm{d}\gamma = \sqrt{2}^{\nu-1} (t^2+\nu)^{-\frac{\nu+1}{2}} \Gamma\left(\frac{\nu+1}{2}\right)
\end{aligned}$$

Putting this together,

$$
\begin{aligned}
f_T(t) &= \frac{\cancel{2^{\frac{1}{2}(\nu-1)}}\nu^{\nu/2}(t^2+\nu)^{\frac{1}{2}(\nu+1)}}{\cancel{2^{\frac{1}{2}(1-\nu)}}\Gamma(\nu/2)\sqrt{\pi}}\Gamma\left(\frac{\nu+1}{2}\right) \\
&= \frac{\cancel{\nu^{\nu/2}}\left(\frac{t^2}{\nu}+1\right)^{\frac{1}{2}(\nu+1)}}{\Gamma(\nu/2)\sqrt{\pi}\cancel{\nu^{\nu/2}}\sqrt{\nu}}\Gamma\left(\frac{\nu+1}{2}\right) \\
&= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma(\nu/2)}\left(\frac{t^2}{\nu}+1\right)^{\frac{\nu+1}{2}}
\end{aligned}
$$

This is exactly the pdf of Student's t-distribution with ν degrees of freedom. $\square$