Theorem 0.0.1. *Suppose V is a vector space with $\dim(V) = n$ over a field $\mathbb{F}$. Further suppose $T : V \to V$ is a linear map whose characteristic polynomial can be written as a product of linear factors, i.e., $\chi_T(x) = \Pi_{i=1}^n (x - \lambda_i)$ for (not necessarily distinct) eigenvalues λ_i.*[1] *Then there exists a basis B of V with $[T]_B$ an upper-triangular matrix.*

Proof. We proceed by induction on n, the dimension of V.

Base case. The theorem is obvious for $n = 1$ because each matrix $[a]$ is upper-triangular.

Inductive hypothesis. We assume that, for each integer k with $1 \leq k < n$, the theorem holds.

Inductive step. Let the dimension of V be n. Now since χ_T factorises into a product of linear factors, there exists an eigenvalue $\lambda \in \mathbb{F}$. Call the corresponding eigenvector $\mathbf{w}_1$ and let $W := \mathrm{Span}(\mathbf{w}_1)$. Note that, of course, W is T-invariant. This means that we can form the quotient space ${}^V\!/\!{}_W$ and it has dimension $n - 1$.

Recall that $\chi_T(x) = \chi_{T|_W}(x)\chi_{\bar{T}}(x)$ where $T|_W$ is the map from W to W with $T|_W(\mathbf{v}) = T(\mathbf{v})$ and $\bar{T}$ is the map from ${}^V\!/\!{}_W$ to ${}^V\!/\!{}_W$ with $\bar{T}(W + \mathbf{v}) = W + T(\mathbf{v})$. Obviously, $T|_W$ and $\bar{T}$ factorise into a product of linear factors because they divide T, which does so. By the inductive hypothesis there exist bases B_W and $\bar{B}$ with $[T|_W]_{B_W}$ and $[\bar{T}]_{\bar{B}}$ upper triangular. Let B be a basis of V consisting of the counterparts of $\bar{B}$ in V and the vectors in B_W.

Let us recall that we may write this linear transformation as a block matrix

$$[T]_B = \begin{bmatrix} [T|_W]_{B_W} & * \\ 0 & [\bar{T}]_{\bar{B}} \end{bmatrix}$$

Because both the matrices on the diagonal are upper triangular, $[T]_B$ is upper triangular. This completes the proof. $\square$

The above proof also gives us a way to find the basis B that reveals the triangularised matrix. The steps are:

1. Find an eigenvector $\mathbf{w}_1$, and define $W := \mathrm{Span}(\mathbf{w}_1)$.
2. Form $[\bar{T}]_{\bar{B}}$ and find an eigenvector $W + \mathbf{w}_2$ of that, then let $W' = \mathrm{Span}(\mathbf{w}_1, \mathbf{w}_2)$.
3. Do the same thing for $\bar{\bar{T}} : {}^V\!/\!{}_{W'} \to {}^V\!/\!{}_{W'}$ and form W''.
4. Repeat the above step until $W = V$. The resulting basis $\mathbf{w}_1, \ldots, \mathbf{w}_n$ triangularises the matrix.

Warning!

Remember that the theorem only works when the matrix's characteristic polynomial can be factorised into linear factors!

[1] This will depend on the field. For instance, every matrix is triangularisable over $\mathbb{C}$, but this is certainly not so over $\mathbb{F}_p$.