Red_Tide Derived Quantities

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Here, we write equations for rotary and other derived quantities as functions of the sine and cosine coefficients of u and v. These include: clockwise amplitude $|R_{\text{cw},i}|$, counterclockwise amplitude $|R_{\text{ccw},i}|$, rotary angle θ_{rot}, rotary phase ϕ_{rot}, eccentricity ϵ, u-phase, v-phase, variance of all types (u, v, cw, ccw).

Note that I unconventionally define phase as relative to the sine of $\omega_i t$, rather than the cosine of $\omega_i t$ as most authors do; admittedly this was an initial oversight that got legacied in and by the time I realized, it was too late to conveniently change.

$$\vec{u}(t) = u(t)\hat{\imath} + v(t)\hat{\jmath}$$

$$u(t) = \sum_{i=1}^{M} u_i(t) = \sum_{i=1}^{M} \big(a_i \sin(\omega_i t) + b_i \cos(\omega_i t)\big) = \sum_{i=1}^{M} \left(\sqrt{a_i^2 + b_i^2}\sin(\omega_i t + \phi_u)\right)$$

$$= \sum_{i=1}^{M} \left(\sqrt{a_i^2 + b_i^2}\cos(\omega_i t + \phi_u - \frac{\pi}{2})\right)$$

$$v(t) = \sum_{i=1}^{M} v_i(t) = \sum_{i=1}^{M} \big(c_i \sin(\omega_i t) + d_i \cos(\omega_i t)\big) = \sum_{i=1}^{M} \left(\sqrt{c_i^2 + d_i^2}\sin(\omega_i t + \phi_v)\right)$$

$$= \sum_{i=1}^{M} \left(\sqrt{c_i^2 + d_i^2}\cos(\omega_i t + \phi_v - \frac{\pi}{2})\right)$$

$$\phi_u = \arctan(b_i/a_i)$$

$$\phi_v = \arctan(d_i/c_i)$$

$$\mathrm{var}(u) = \frac{1}{2}\sum_{i=1}^{M}\left(a_i^2 + b_i^2\right) = \frac{1}{2}\sum_{i=1}^{M} A_{u_i}^2$$

$$\mathrm{var}(v) = \frac{1}{2}\sum_{i=1}^{M}\left(c_i^2 + d_i^2\right) = \frac{1}{2}\sum_{i=1}^{M} A_{v_i}^2$$

$$u_i(t) + \mathrm{i}v_i(t) = R_{\mathrm{ccw},i}\mathrm{e}^{\omega_i t} + R_{\mathrm{cw},i}\mathrm{e}^{-\omega_i t}$$

$$R_{\mathrm{cw},i} = \frac{1}{2}(b_i - c_i + \mathrm{i}(d_i + a_i))$$

$$R_{\mathrm{ccw},i} = \frac{1}{2}(b_i + c_i + \mathrm{i}(d_i - a_i))$$

$$R_{\mathrm{cw},i}R^*_{\mathrm{cw},i} + R_{\mathrm{ccw},i}R^*_{\mathrm{ccw},i} = \frac{1}{2}(a_i^2 + b_i^2 + c_i^2 + d_i^2) = \mathrm{var}(u_i) + \mathrm{var}(v_i)$$

$$\theta_{\mathrm{rot},i} = \frac{1}{2}\left(\arctan\left(\frac{d_i - a_i}{b_i + c_i}\right) + \arctan\left(\frac{d_i + a_i}{b_i - c_i}\right)\right)$$

$$\phi_{\mathrm{rot},i} = \frac{1}{2}\left(\arctan\left(\frac{d_i - a_i}{b_i + c_i}\right) - \arctan\left(\frac{d_i + a_i}{b_i - c_i}\right)\right)$$

$$\text{Semi-major Axis}_i = \mathrm{SM}_i = |R_{\mathrm{ccw},i}| + |R_{\mathrm{cw},i}|$$

$$... = \frac{1}{2}(\sqrt{a^2 + b^2 + c^2 + d^2 + 2bc - 2ad} + \sqrt{a^2 + b^2 + c^2 + d^2 - 2bc + 2ad})$$

$$\text{Semi-minor Axis}_i = \mathrm{sm}_i = ||R_{\mathrm{ccw},i}| - |R_{\mathrm{cw},i}||$$

$$\text{Eccentricity}_i = \epsilon_i = \sqrt{1 - \frac{\mathrm{sm}_i^2}{\mathrm{SM}_i^2}}$$

$$\text{Semi-major Axis}_i = \mathrm{SM}_i = |R_{\mathrm{ccw},i}| + |R_{\mathrm{cw},i}| = \tfrac{1}{2}(a_i^2 + b_i^2 + c_i^2 + d_i^2) = \tfrac{1}{2}(A_{u_i}^2 + A_{v_i}^2)$$

$$u_3(t) = u_1(t) - u_2(t) = A_3\cos(\omega t + \phi_3) = A_1\cos(\omega t + \phi_1) - A_2\cos(\omega t + \phi_2)$$

$$= \sqrt{(A_1^2 + A_2^2 - 2A_1A_2\cos(\phi_1 - \phi_2))}\cos\left(\omega t + \tan^{-1}\left(\frac{A_1\sin(\phi_1) - A_2\sin(\phi_2)}{A_1\cos(\phi_1) - A_2\cos(\phi_2)}\right)\right)$$

Example with numbers (MATLAB language):

```
y1 = 3*sin(2*pi*0.1*tt) - 5*cos(2*pi*0.1*tt);
y2 = 7*sin(2*pi*0.1*tt) - 11*cos(2*pi*0.1*tt);
Rc = 0.5*(-5 - 7 + 1i*(-11 + 3))
Rcc = 0.5*(-5 + 7 + 1i*(-11 - 3))
y_R = Rc*exp(-2*pi*1i*0.1*tt) + Rcc*exp(2*pi*1i*0.1*tt);
figure;plot(tt,y1,'.-');hold on;plot(tt,y2,'.-')
plot(tt,real(y_R),'o--');hold on;plot(tt,imag(y_R),'o--')
```