NUFFT-Based Spectral Biot–Savart Solver

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Contents

1 Spectral Biot–Savart Formulation

We consider a current density sampled at quadrature points

$$\{\boldsymbol{x}_n = (X_n, Y_n, Z_n),\ \boldsymbol{J}_n = (J_{x,n}, J_{y,n}, J_{z,n}),\ w_n\}_{n=1}^{N},$$

representing either a volumetric current distribution or a line current.

The magnetic field is computed on a periodic Cartesian grid

$$[0, L_x) \times [0, L_y) \times [0, L_z).$$

1.1 Fourier Transform of the Current

The continuous Fourier transform

$$\hat{\boldsymbol{J}}(\boldsymbol{k}) = \int \boldsymbol{J}(\boldsymbol{x})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{d}^3x$$

is approximated by quadrature:

$$\hat{\boldsymbol{J}}(\boldsymbol{k}_{pqr}) \approx \sum_{n=1}^{N} \boldsymbol{J}_n\, w_n\, e^{-i\boldsymbol{k}_{pqr}\cdot\boldsymbol{x}_n},$$

with discrete wavevectors

$$\boldsymbol{k}_{pqr} = \left(\frac{2\pi p}{L_x}, \frac{2\pi q}{L_y}, \frac{2\pi r}{L_z}\right).$$

This operation is implemented using a type-1 NUFFT.

1.2 Spectral Biot–Savart Operator

For all nonzero modes $\boldsymbol{k} \neq 0$,

$$\hat{\boldsymbol{B}}(\boldsymbol{k}) = -\, i\, \mu_0 \frac{\boldsymbol{k} \times \hat{\boldsymbol{J}}(\boldsymbol{k})}{\|\boldsymbol{k}\|^2}, \qquad \hat{\boldsymbol{B}}(\boldsymbol{0}) = \boldsymbol{0}.$$

Explicitly,

$$\boldsymbol{k} \times \hat{\boldsymbol{J}} = \begin{pmatrix} k_y \hat{J}_z - k_z \hat{J}_y \\ k_z \hat{J}_x - k_x \hat{J}_z \\ k_x \hat{J}_y - k_y \hat{J}_x \end{pmatrix}.$$

The zero mode is removed to enforce both

$$\nabla \cdot \boldsymbol{B} = 0 \quad \text{and} \quad \langle \boldsymbol{B} \rangle = 0$$

on the periodic domain.

1.3 Inverse Transform

The magnetic field in real space is obtained by a 3D inverse FFT:

$$\boldsymbol{B}(\boldsymbol{x}_{ijk}) = \sum_{pqr} \hat{\boldsymbol{B}}(\boldsymbol{k}_{pqr})\, e^{i\boldsymbol{k}_{pqr}\cdot\boldsymbol{x}_{ijk}}.$$

2 Geometry: Circular Torus Embedding

The torus is defined by major radius R_0 and minor radius a. In cylindrical coordinates,

$$(R - R_0)^2 + Z^2 \le a^2, \qquad R = \sqrt{X^2 + Y^2}.$$

The embedding uses curvilinear coordinates $(\rho, \theta, \zeta) \in [0,1]\times[0,2\pi)\times[0,2\pi)$:

$$\begin{aligned} R &= R_0 + a\rho\cos\theta, \\ Z &= a\rho\sin\theta, \\ (X, Y) &= (R\cos\zeta,\ R\sin\zeta). \end{aligned}$$

The quadrature weight is

$$w = a^2\rho\,(R_0 + a\rho\cos\theta)\ \Delta\rho\,\Delta\theta\,\Delta\zeta.$$

3 Current Models

3.1 Volumetric Toroidal Current

The toroidal current density is defined as

$$\boldsymbol{J}(\boldsymbol{x}) = J_\phi(R, Z)\,\hat{\boldsymbol{\phi}}, \qquad \hat{\boldsymbol{\phi}} = (-\sin\zeta, \cos\zeta, 0),$$

with

$$J_\phi(R, Z) = \begin{cases} \dfrac{I}{\pi a^2}, & (R - R_0)^2 + Z^2 \le a^2, \\ 0, & \text{otherwise.} \end{cases}$$

This ensures the total current satisfies

$$I = \int_{\text{cross-section}} J_\phi \,\mathrm{d}A.$$

3.2 Toroidal Filament Current

An idealized current filament is defined along the magnetic axis:

$$\boldsymbol{x}(\zeta) = (R_0\cos\zeta,\ R_0\sin\zeta,\ 0),$$

with current density

$$\boldsymbol{J}(\zeta) = I\,(-\sin\zeta, \cos\zeta, 0),$$

and quadrature weight

$$w = \frac{2\pi R_0}{N_\zeta}.$$

This model is useful for validation and asymptotic comparisons.

4 Validation Tests

4.1 Geometry Tests

The geometry module is validated by:

- Domain inclusion: all points satisfy $(R - R_0)^2 + Z^2 \leq a^2$
- Coordinate consistency: $R = \sqrt{X^2 + Y^2}$
- Volume convergence:

$$\sum_n w_n \to 2\pi^2 R_0 a^2$$

- Vanishing first moments:

$$\sum_n X_n w_n = \sum_n Y_n w_n = \sum_n Z_n w_n = 0$$

- Jacobian consistency and symmetry tests

4.2 Current Model Tests

The current models are verified to ensure:

- Strict support inside the torus
- Purely toroidal direction
- Uniform magnitude inside the volume
- Zero current outside
- Correct total current normalization

4.3 Magnetic Field Tests

The magnetic field solver satisfies:

1. **Zero-current consistency**:

$$\boldsymbol{J} = 0 \Rightarrow \boldsymbol{B} = 0$$

2. **Midplane symmetry**:

$$B_x(x, y, z) = B_x(x, y, -z), \quad B_y(x, y, z) = B_y(x, y, -z), \quad B_z(x, y, z) = -B_z(x, y, -z)$$

3. **Linearity**:

$$\boldsymbol{B}(\alpha I) = \alpha\, \boldsymbol{B}(I)$$

4. **Solenoidality**:

$$\int_\Omega \nabla \cdot \boldsymbol{B}\, \mathrm{d}V = 0$$

5 Comparison with Direct Biot–Savart

For validation, the spectral solution is compared with direct quadrature:

$$\boldsymbol{B}_{\mathrm{BS}}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \sum_n \boldsymbol{J}_n \times \frac{\boldsymbol{r} - \boldsymbol{x}_n}{\|\boldsymbol{r} - \boldsymbol{x}_n\|^3}\, w_n.$$

The relative error is defined as

$$\varepsilon = \frac{\|\boldsymbol{B}_{\mathrm{FFT}} - \boldsymbol{B}_{\mathrm{BS}}\|}{\|\boldsymbol{B}_{\mathrm{BS}}\|},$$

and is verified to remain bounded near the box center.