Railroad Ink

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Last Move

Part A

Sets

$$\begin{aligned}
I &- \text{set of all inside squares, as (row,col) pairs}\\
O &- \text{set of all outside (start) squares, as (row, col) pairs}\\
S &- \text{set of all squares, } S = I \cup O\\
P &- \text{set of all pieces}\\
T &- \text{set of all possible tiles that can be placed, with a piece, rotation and reflection}\\
E &- \text{set of the edge types} E = \{\text{railway, highway}\}\\
P_s &- \text{set of all special pieces, } P_s \subseteq P\\
T_j &- \text{set of all junction tile, } T_j \subseteq T
\end{aligned}$$

Data

$$\begin{aligned}
numstarts &- \text{ the number of start pieces}\\
MP &- \text{ the longest possible path that can be constructed}
\end{aligned}$$

Functions

$$\begin{aligned}
\texttt{adj}(s) &- \text{ returns the squares adjacent to square } s\\
\texttt{loose-ends}(t, s) &- \text{ returns the count of loose ends on internal edges if tile } t \in T \text{ is placed at square } s \in S\\
\texttt{centre}(s) &- \text{ returns whether square } s \text{ is a centre square}\\
\texttt{clash}(e) &- \text{ returns the edge type that clashes with the given edge type}\\
\texttt{tile-at}(s) &- \text{ returns the tile currently on the board at square } s \in S \text{ if there is one}\\
\texttt{free}(s) &- \text{ returns whether a piece can be placed at square } s \in S\\
\texttt{variations}(p) &- \text{ returns all the possible tiles associated with piece } p
\end{aligned}$$

$texttthand - count(p) -$ how many of piece p are in the current hand

Variables

board variables

$x_{t,s}$ – 1 if tile $t \in T$ is placed at square $s \in S$

$y_{s,s',e}$ – whether there is a connection between $s \in S$ and $s' \in \texttt{adj}(s)$ s.t $s < s'$ of type $e \in E$

connecting flow variables

$f_{s,s',e}$ – connecting flow from $s \in S$ to $s' \in \texttt{adj}(s)$ of type $e \in E$ for joining the start squares

$f'_{s,e}$ – connecting flow from edge type $e \in E$ to the other edge type (for junctions) at square $s \in I$

g_s – connecting flow from start square $s \in O$ to the super sink

h_s – 1 if there is any connecting flow from square $s \in O$ to the super sink

j – 1 if the bonus point for connecting all start squares is earned

longest path variables

$a_{s,e}$ – whether square $s \in I$ is the start square of the longest railway of type $e \in E$

$b_{s,s',e}$ – longest path flow from $s \in I$ to $s' \in \texttt{adj}(s)$ s.t $s' \in I$ of type $e \in E$

$c_{s,s',e}$ – whether longest path flow of type $e \in E$ is permitted from $s \in I$ to $s' \in \texttt{adj}(s)$ s.t $s' \in I$

$d_{s,e}$ – whether square $s \in I$ is part of the longest path of type $e \in E$

Objective

Maximise:

$$\left(48 - 4 \cdot \sum_{s \in O} h_s + j\right) + \left(\sum_{\substack{s \in I| \\ \texttt{centre}(s)}} x_{t,s}\right) + \left(\sum_{\substack{s \in I \\ e \in E}} d_{s,e}\right) - \left(\sum_{\substack{t \in T \\ s \in I}} x_{t,s} \cdot \texttt{loose-ends}(t,s) - \sum_{\substack{(s,s',e) \in y| \\ s,s' \in I}} 2 \cdot y_{s,s',e}\right)$$

This objective function is the number of points earned, there are four components, in order:

1) Points earned from connections between start squares. Any square without a super sink connection is connected and scores points. j is the bonus point for connecting all start squares.

2) Points earned from placing pieces on the centre squares.

3) Points earned from the longest railway and highway. $d_{s,e}$ variables are only set if a square is on the longest path.

4) Points lost due to errors. Points lost to errors can be counted as the total number of railways/highways on internal edges minus the pairs that are joined together.

Constraints

Piece placement constraints

$$\sum_{t \in T} x_{t,s} <= 1 \qquad \forall s \in S \tag{1}$$

$$x_{\texttt{tile-at}(s),s} = 1 \qquad \forall s \in S \mid \sim \texttt{free}(s) \tag{2}$$

$$\sum_{\substack{s \in S \mid \texttt{free}(s) \\ t \in \texttt{variations}(p)}} x_{t,s} = \texttt{hand-count}(p) \qquad \forall p \in P \tag{3}$$

$$\sum_{\substack{s \in S \mid \texttt{free}(s), \\ t \in \texttt{variations}(p)}} x_{t,s} \leq 1 \qquad \forall p \in P_s \tag{4}$$

$$\sum_{\substack{s \in S \mid \texttt{free}(s), \\ p \in P_s,\ t \in \texttt{variations}(p)}} x_{t,s} \leq 1 \tag{5}$$

$$\sum_{\substack{s \in S,\ p \in P_s \\ t \in \texttt{variations}(p)}} \leq 3 \tag{6}$$

$$\sum_{\substack{t \in T \mid \\ t.R=e}} x_{t,s} + \sum_{\substack{t \in T \mid \\ t.L=\texttt{clash}(e)}} x_{t,s'} \leq 1 \qquad \forall s = (r,c) \in S, e \in E \mid s' = (r, c+1) \in S \tag{7}$$

$$\sum_{\substack{t \in T \mid \\ t.B=e}} x_{t,s} + \sum_{\substack{t \in T \mid \\ t.T=\texttt{clash}(e)}} x_{t,s'} \leq 1 \qquad \forall s = (r,c) \in S, e \in E \mid s' = (r+1, c) \in S \tag{8}$$

$$2 \cdot y_{s,s',e} \leq \sum_{\substack{t \in T \mid \\ t.R=e}} x_{t,s} + \sum_{\substack{t \in T \mid \\ t.L=e}} x_{t,s'} \qquad \forall s = (r,c) \in S, e \in E \mid s' = (r, c+1) \in S \tag{9}$$

$$2 \cdot y_{s,s',e} \leq \sum_{\substack{t \in T \mid \\ t.B=e}} x_{t,s} + \sum_{\substack{t \in T \mid \\ t.T=e}} x_{t,s'} \qquad \forall s = (r,c) \in S, e \in E \mid s' = (r+1, c) \in S \tag{10}$$

Joining constraints

$$\sum_{s' \in \texttt{adj}(s)} f_{s,s',e} + f'_{s,e} = \sum_{s' \in \texttt{adj}(s)} f_{s',s,e} + f'_{s,\texttt{clash}(e)} \qquad \forall s \in I, e \in E \tag{11}$$

$$1 + \sum_{\substack{s' \in \texttt{adj}(s) \\ e \in E}} f_{s',s,e} = \sum_{\substack{s' \in \texttt{adj}(s) \\ e \in E}} f_{s,s',e} + g_s \qquad \forall s \in O \tag{12}$$

$$g_s \leq numStarts \cdot h_s \qquad \forall s \in O \tag{13}$$

$$f_{s,s',e} \leq numStarts \cdot y_{s,s',e} \qquad \forall s \in S, s' \in \texttt{adj}(s), e \in E \mid s' > s \tag{14}$$

$$f_{s,s',e} \leq numStarts \cdot y_{s',s,e} \qquad \forall s \in S, s' \in \texttt{adj}(s), e \in E \mid s' < s \tag{15}$$

$$f'_{s,e} \leq numStarts \cdot \sum_{t \in T_j} x_{t,s} \qquad \forall s \in I, e \in E \tag{16}$$

$$(numStarts - 1) \cdot j \leq \sum_{s \in O} (1 - h_s) \tag{17}$$

Longest path constraints

$$\sum_{s \in I} a_{s,e} = 1 \qquad \forall e \in E \tag{18}$$

$$c_{s,s',e} \leq y_{s,s',e} \qquad \forall s \in S, e \in E, s' \in \texttt{adj}(s) \mid s' \in I \land s < s' \tag{19}$$

$$c_{s,s',e} \leq y_{s',s,e} \qquad \forall s \in S, e \in E, s' \in \texttt{adj}(s) \mid s' \in I \land s > s' \tag{20}$$

$$\sum_{\substack{s' \in \texttt{adj}(s) \mid \\ s' \in I}} b_{s',s,e} + MP \cdot a_{s,e} = \sum_{\substack{s' \in \texttt{adj}(s) \mid \\ s' \in I}} b_{s,s',e} + d_{s,e} \qquad \forall s \in I, e \in E \tag{21}$$

$$b_{s,s',e} \leq MP \cdot c_{s,s',e} \qquad \forall s \in S, e \in E, s' \in \texttt{adj}(s) \mid s' \in I \tag{22}$$

$$\sum_{\substack{s' \in \texttt{adj}(s) \mid \\ s' \in I}} c_{s,s',e} \leq 1 \qquad \forall s \in I, e \in E \tag{23}$$

These constraints are designed as follows:

(1) Only one tile can be placed at each square.

(2) All squares that are already occupied must finish with that piece there.

(3) Use pieces as required in your hand.

(4) Each special piece can only be played once.

(5) Only one special piece can be played this turn.

(6) Only three specials can be played total.

(7, 8) Cannot place two pieces next to each other that directly clash with each other (i.e have a railway run into a highway).

(9, 10) Can only set the $y_{s,s',e}$ variable (connecting two squares with edge type e) if there are matching edges between the squares.

(11) The joining inflow at internal squares (from adjacent squares or converted through a junction) must equal the outflow (to adjacent squares or passed through the junction).

(12) The joining inflow at start squares (inflow of 1 plus from adjacent square) must equal the outflow (flow to adjacent square and to the super sink).

(13) The h_s variable must be set if there is any flow to the super sink, i.e $g_s > 0$.

(14, 15) There can only be joining flow when there is a connection between the two squares of that edge type.

(16) If there is a junction at a given square, joining flow can readily flow between the two edge types. This is only not permitted for overpasses.

(17) An extra point can be awarded if all start pieces are joined together.

(18) The longest path of each type can only have one start square.

(19, 20) The longest path of a certain type can only travel on an edge that has a connection of that type.

(21) The inflow of the longest path (from adjacent squares, plus a large inflow at the start square) must be greater than the outflow (to adjacent squares, plus points allocated to this square)

(22, 23) These constraints prevent branching of the longest path. There may only be outflow in one direction from any given square.