STATISTICAL FLUID MECHANICS: Mechanics of Turbulence

Volume 1

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English edition updated, augmented and revised by the authors

Edited by John L. Lumley

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AUTHORS' PREFACE TO THE ENGLISH EDITION

The flows of fluids with which one has to deal in engineering, and which one meets in nature, are turbulent in the overwhelming majority of cases, and their description demands a statistical approach. Laminar flows, which are quite accessible to individual descriptions, occur with exotic infrequency. We are persuaded that fluid mechanics cannot be hmited to the study of these seldom encountered special cases, and that the classical description of individual laminar flows, for all its unquestioned importance and value, must be considered only as an introductory chapter to the theory of real turbulent flows, in which the objects of investigation are the properties of ensembles of flows, arising from macroscopically identical external conditions.

We would like such a view to spread through the entire fluid mechanics community, and this is one of the goals of our book. For this reason we are very happy with the appearance of the English edition, which will help spread our views on fluid mechanics in English-speaking countries.

We have tried to summarize in this book the majority of the fundamental works and ideas of modern turbulence theory, described and discussed in studies from various countries. This is why the book has turned out to be so thick. But, of course, our own scientific interests must be reflected in the selection of material, and considerable attention has been given to problems on which we ourselves have done substantial work—particularly the theory of the local structure of developed turbulence due to A. N. Kolmogorov, whose students we were, and the theory of turbulence in stratified media, which finds wide applications, primarily in geophysics.

The mechanics of turbulence is a lively, actively developing science, and after the Russian edition of our book saw the light (Volume 1 in 1965 and Volume 2 in 1967), a large number of new and interesting works appeared in this area. The present edition has been replenished with information on many new works (the bibliography included in the book has, as a result, grown by several hundred entries) and by the inclusion of a series of improvements and additions (all of this work was carried out by the second author). Particularly great changes were undergone by Section 2, dealing with hydrodynamic stability and transition to turbulence, and Section 8, in which are set forth experimental results on turbulence in a thermally stratified boundary layer.

We consider it our pleasure to express our deep gratitude to our editor, Prof. J. L. Lumley, who did a great deal to improve the English edition of our book.

A. S. Monin
A. M. Yaglom

EDITOR'S PREFACE TO THE ENGLISH EDITION

This is a translation of the first Russian edition of Статистическая гидромеханика. In editing the manuscript of the translation, extensive use was made of the translation of Volume 1 prepared by the Joint Publications Research Service at the behest of K.L. Calder of the U.S. Environmental Science Services Administration, and of the translation of Chapter 4, Volume 1, prepared at the instigation of F. Pasquill of the British Meteorology Research Division. The cooperation of both these individuals is gratefully acknowledged.

The edited English manuscript was sent to Yaglom, who made very extensive additions, corrections and revisions to the technical content, after which it was again edited. Russian technical style tends to be turgid with internal cross-references which sound redundant to an English ear; in addition, the inflected character of the language makes possible a complexity of sentence structure that is dizzying. A conscientious translator hesitates to paraphrase too freely, with the consequence that the translation retains an unmistakable flavor of the original. Yaglom has an excellent ear for English style, and as he revised he also made innumerable suggestions for freer paraphrases, which sound more natural in English. As editor, I wish to express my gratitude for his cooperation, giving him full credit for what is good in this translation, while taking the blame for remaining inadequacies.

J. L. Lumley

FOREWORD

The theory of turbulence discussed in this book is based on the usual macroscopic description of flows of liquids and gases, considered as continua, and on the classical equations of fluid mechanics. However, unlike ordinary fluid mechanics, the theory of turbulence does not study individual fluid flows but the statistical properties of an ensemble of flows having macroscopically identical external conditions. Hence, the title of this book, *Statistical Fluid Mechanics.* However, we must stress that questions dealing with the deduction of the macroscopic equations of fluid mechanics from the statistical laws of the kinetic theory of gases, which sometimes are also referred to by this title, are not discussed in the present work.

The basic concepts and ideas relating to turbulence and our approach to the presentation of the mechanics of turbulence are outlined in the Introduction, together with a brief sketch of the historical development of the theory of turbulence, and the plan of both volumes of the present work. The Introduction is intended to describe in the most general terms the main problems and methods of the recent theory of turbulence and to explain some of its practical applications. Naturally, we have had to make use here of many concepts which will be expanded in detail only in subsequent parts of the book.

The book is divided into chapters, sections and subsections. The numbering of the chapters and sections runs continuously, while the subsections are numbered within each section. The number of the section and of the subsection is indicated in the subsection number separately (e.g., the fourth subsection of the second section is denoted as 2.4). The equations are also numbered in a similar manner: e.g., Eq. (2.15) is the fifteenth equation in the second section. When referring to the works listed in the bibliographies at the end of each volume, we cite the name of the author and, in brackets, the date of publication of the work. In cases when several works of a given author are cited which all appeared in the same year, these are denoted further by letters of the alphabet. Except in the Introduction, the initials of the authors are given only in cases where one must distinguish between two authors with the same surname.

We have tried, as far as possible, to use generally accepted notation in this book. In the numerous cases where different authors use different symbols for the same quantities, we usually decided to select one of these, and not to introduce new notation. In several cases this has led to the same symbol being used for different quantities in different parts of the book. Sometimes, for various reasons, it also proved convenient to denote the same quantity by a different symbol in different sections of the book. In all these cases the notation used is specified in the text.

This book represents the combined work of both authors. We feel that we must point out the great influence on the writing of it, and on our own work in the field of turbulence, of our frequent discussions with Andrei Nikolaevich Kolmogorov, our Professor in our student days. We have tried to reflect many of Kolmogorov's ideas in this book.

We are also greatly indebted to A. M. Obukhov who was one of those who initiated the writing of this book, and with whom we discussed the selection of the material and the details of the exposition of many questions. Some parts of the manuscript were read by L. A. Dikiy, Ye. A. Novikov and V. I. Tatarskiy, who made a number of profitable comments. In discussions on the analysis of the experimental data on atmospheric turbulence, A. S. Gurvich and L. V. Tsvang were active. G. S. Golitsyn helped us in the preparation and editing of the manuscript. To all these colleagues we wish to express our sincere gratitude.

A. S. Monin
A. M. Yaglom

CONTENTS

INTRODUCTION

This book is devoted to the mechanics of turbulent flows of liquids and gases. It begins with a resumé of the basic equations of fluid mechanics and, generally speaking, demands of the reader no prior knowledge in the field of turbulence. For this reason it seemed to us that it would be worthwhile to explain here, if only briefly, what will be dealt with in the book.

It is known that all flows of liquids and gases may be divided into two sharply different types; the quiet smooth flows known as "laminar" flows, and their opposite, "turbulent" flows in which the velocity, pressure, temperature and other fluid mechanical quantities fluctuate in a disordered manner with extremely sharp and irregular space- and time-variations. As a typical example, Fig. 1 shows a trace of the time fluctuations of the wind velocity, the vertical component of velocity and the temperature in the surface layer of the atmosphere. The data of Fig. 1 were obtained by measuring the velocity and temperature with special precise, low-inertia devices. The complicated nature of these curves indicates at once that the corresponding motion of the air was turbulent. The variety of fluctuations of different periods and amplitudes observed in the traces shown in Fig. 1 illustrates the complicated structure of turbulence in contrast to simple laminar flows. This complicated structure affects many properties of flows which differ greatly in the laminar and turbulent cases. Thus, turbulent flows possess far greater ability to transfer momentum (figuratively speaking, a turbulent medium has an enormous effective viscosity) and hence in many cases they exert far greater *forces* on rigid bodies in contact with the fluid. Similarly, turbulent flows possess an increased ability to transfer heat, soluble admixtures, and suspended particles and to propagate chemical reactions (in particular, combustion). Due to the presence of internal inhomogeneities, turbulent flows are able to scatter sound waves and electromagnetic waves and to induce fluctuations of their amplitudes, phases, etc.

The indicated properties of turbulent flows may evidently be very significant for many problems in natural science and technology. Thus the question of how frequently turbulent flows are encountered is of undoubted interest. It is found that an overwhelming majority of the flows actually encountered in nature and technology are, in fact, turbulent flows, while the laminar flows, which are studied in detail in fluid mechanics, occur only as fairly rare exceptions.

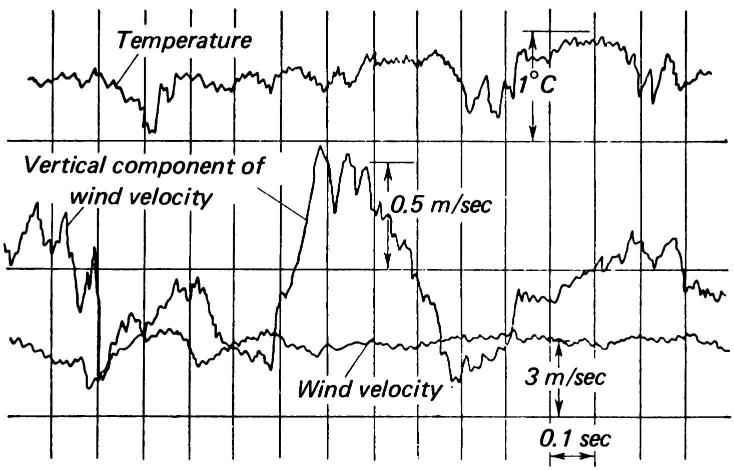


FIG. 1. Typical example of a recording of fluctuations of temperature, wind velocity and the vertical component of wind velocity.

In particular, the various motions of the air in the earth's atmosphere, from a slight breeze in the surface layer (to which the measurements of Fig. 1 correspond) up to general atmospheric circulation of planetary scale, are turbulent. Atmospheric turbulence plays a fundamental role in the transfer of heat and moisture by air masses, in evaporation from the surface of land or water, and in thermal and dynamic interaction between the atmosphere and the underlying surface which has a considerable effect on changes in the weather. Atmospheric turbulence determines also the spreading of admixtures in the air, the production of wind waves on large lakes, seas, and the ocean, the production of wind currents in the ocean, the buffeting of airplanes and other aircraft and the vibrations of many structures on the earth's surface. Finally, turbulent fluctuations of the refractive index affect significantly the propagation of light and radio waves from terrestrial and cosmic sources.

Turbulent, too, are the flows of water in rivers, lakes, seas and oceans, and also the motions of gases in interstellar nebulae having an enormous scale many orders greater than the earth. Finally, practically all flows in pipes encountered in technology and engineering are turbulent, e.g., in water-pipes, gas mains, petroleum pipelines, the nozzles of jet engines, etc; and also the motions in boundary layers over the surface of moving aircraft, in liquid or gas high-speed jets issuing from a nozzle, in the wakes behind rapidly moving rigid bodies—propeller blades, turbine blades, bullets, projectiles and rockets. Thus turbulence is literally all around us both in nature and in engineering devices using flows of liquids and gases; therefore its study is extremely important from the practical viewpoint.

Turbulent flows are also of great interest from a purely theoretical point of view as examples of nonlinear mechanical systems with a very great number of degrees of freedom. Indeed, the motion of any continuous medium, strictly speaking, is described by an infinite number of generalized coordinates (e.g., by the coefficients of the expansion of the velocity field with respect to some complete system of functions of the spatial coordinates). For laminar motion these coordinates can usually be chosen in such a way that only a few of the corresponding degrees of freedom will be excited, i.e., will actually take part in the motion. However, for turbulent motion, an enormous number of degrees of freedom are always excited, and hence the variation with time of any physical value will be described here by functions containing a vast number of Fourier components, i.e., by functions of an extremely complicated nature (cf. again Fig. 1). Therefore, in this case it is practically hopeless to attempt to describe the individual time variations of all the generalized coordinates corresponding to the excited degrees of freedom (i.e., to find a mathematical expression for the time-dependence of the fields of velocity, pressure, etc., of a single individual flow). The only possibility in the theory of turbulence is a statistical description, based on the study of specific statistical laws, inherent in phenomena *en masse*, i.e., in large ensembles of similar objects. Thus only statistical fluid mechanics, which studies the statistical properties of the ensembles of fluid flows under macroscopically identical external conditions, can provide a turbulence theory.

The theory of turbulence by its very nature cannot be other than statistical, i.e., an individual description of the fields of velocity, pressure, temperature and other characteristics of turbulent flow is in principle impossible. Moreover, such description would not be useful even if possible, since the extremely complicated and irregular nature of all the fields eliminates the possibility of using exact values of them in any practical problems. As a result, the contrasting of the "semiempirical" and statistical theories of turbulence often found in the literature is meaningless. The semiempirical theory, of course, is also statistical and differs from other theories of turbulence not by rejecting the use of statistical characteristics, but only in the means used to determine them.

In discussing the statistical nature of the theory of turbulence, comparison is often made with the kinetic theory of gases, which investigates many-particle systems of interacting molecules. This comparison is reasonable in the sense that in both these theories an exact description of the evolution of the individual mechanical system is theoretically impossible and practically useless. However, it must be remembered that between the statistical mechanics of the ensembles of molecules investigated by Gibbs, Boltzmann and others, and the statistical fluid mechanics of a viscous fluid, there is a fundamental difference. This is connected primarily with the fact that the total kinetic energy of a molecular ensemble is time-invariant (in any case, under the simplified assumptions on molecular interactions adopted in the kinetic theory of gases) while the kinetic energy of real fluid flow is always dissipated to heat due to the action of viscosity. Of less importance, but still significant, is the fact that molecular ensembles are discrete by nature and their time evolution is described by systems of ordinary differential equations, while in fluid mechanics we are dealing with the motion of a continuous medium, which is described by partial differential equations. Thus the analogy with the kinetic theory of gases is of relatively little help in the formulation of the theory of turbulence, and is useful only for a preliminary understanding of the concept of a statistical approach to physical theory.

Far more fruitful, perhaps, is the analogy between the theory of turbulence and quantum field theory, which is connected with the fact that a system of interacting fields is also a nonlinear system with a theoretically infinite number of degrees of freedom. From this follows the similarity of the mathematical techniques used in both theories. This allows us to hope that the considerable advances in the one will also have a decisive effect on the development of the other. However, at present, quantum field theory is encountering great difficulties connected with its first principles, while in the theory of turbulence there are only partial successes; consequently, the major influence of one of these theories upon the other still lies in the future.

In spite of the fact that the fluid flows encountered in nature and in technical devices are, as a rule, turbulent, in all existing courses on fluid mechanics, at best only a few sections are devoted to the theory of turbulence. These sections contain usually only some disjointed remarks on the methods of statistical description of disordered fluid flows and on some statistical characteristics of such flows. The monograph literature devoted to turbulence is also very poor and amounts in all to only a few titles (almost all of these may be found in the bibliography at the end of this book); moreover, a great many of these refer to books of relatively narrow subject matter. It is not difficult to understand why the situation is so unsatisfactory. Turbulent flows are considerably more complicated than laminar flows and require essentially new methods for their study. These methods differ from the classical methods of mathematical physics, which for almost two centuries have been considered as the only ones suitable for the quantitative study of the laws of nature. The mathematical techniques needed for the logically accurate formulation of the statistical mechanics of continuous media, i.e., the theory of random fields, was devised only in the last 25-30 years and is still almost unknown outside a small group of experts in probability theory. It was during these years also that the modern theory of turbulence was formulated (which even now is still far from being complete). However, we feel that the existing achievements in this field certainly deserve a considerably high place in the necessary bulk of knowledge of every scientist studying fluid mechanics and theoretical physics, and if this has not yet occurred it is only because the theory of turbulence is relatively new. We are convinced that in the future the place of this theory in textbooks and fluid mechanics curricula for colleges and universities, in the education plans of all specialists in fluid mechanics and theoretical physics, and in various research projects, will rapidly increase. If our book can contribute in some measure to these developments we shall indeed be happy.

* * *

Let us now turn to a brief history of turbulence. We shall enumerate the methods and results, the detailed description of which constitutes the fundamental part of this book. Simultaneously, we shall try to explain why it seems worthwhile to us to write such a book now at the present stage of development of statistical fluid mechanics.

The existence of two sharply different types of flow, now called laminar and turbulent, had already been pointed out in the first half of the nineteenth century. However, a theory of turbulence came only with the outstanding pioneering works of Osborne Reynolds (1883; 1894). In these works Reynolds turned his attention first to the conditions under which the laminar flow of fluid in pipes is transformed into turbulent flow. The study of these conditions led him to a general criterion for dynamic similarity of flows of a viscous, incompressible fluid. In the absence of external forces, and with geometric similarity, this criterion is the coincidence of the values of the so-called "Reynolds number" $\mathrm{Re} = UL/\nu$, where U and L are characteristic scales of velocity and length in the flow and ν is the kinematic viscosity of the fluid. From the dynamical viewpoint, the Reynolds number may be interpreted as the ratio of typical values of the inertial and viscous forces acting within the fluid. The inertial forces which produce mixing of the different volumes of fluid moving inertially with different velocities, also produce a transfer of energy from large- to small-scale components of motion and hence assist the formation in the flow of sharp, small-scale inhomogeneities that characterize a turbulent flow. The viscous forces, on the contrary, assist in the smoothing out of small-scale inhomogeneities. Thus we may expect that flows with sufficiently small values of Re will be laminar, while those with sufficiently large Re will be turbulent. This is a fundamental result which Reynolds formulated.

Reynolds also contributed another important step in the theory of turbulence. He proposed representing the values of all the hydrodynamical quantities in turbulent flow as sums of mean (regular) and fluctuating (irregular) components, giving up the practically hopeless attempts to describe the details of individual hydrodynamic fields, and studying only the mean values which vary comparatively smoothly with space and time. For the determination of the mean values, Reynolds proposed to use time-averaging or space-averaging, but in fact he used only certain algebraic properties of the operation of averaging. These properties permit considerable simplification in application to the equations of fluid mechanics. Consequently, although nowadays when investigating turbulence we understand averaging in a different sense from that of Reynolds, all his deductions still remain valid, since the properties of averaging which he used proved to agree completely with the present understanding of the operation.

Here we will explain how averaging is now understood in the theory of turbulence. In present-day statistical fluid mechanics, it is always implied that the fluid mechanical fields of a turbulent flow are random fields in the sense used in probability theory. In other words, every actual example of such a field is considered as a "sample" taken from the "statistical ensemble of all possible fields." Such a statistical ensemble is described by a given probability measure on the set of functions of space-time points, which satisfy the necessary kinematic and dynamic conditions (arising from the laws of fluid mechanics). The averaging of any fluid mechanical quantities may be understood then as probability averaging with respect to the corresponding statistical ensemble. Moreover, all the properties of the operation of averaging which Reynolds required follow from the well-known properties of the mean value of probability theory (i.e., the mathematical expectation) which are described in probability textbooks. This immediately eliminates many difficulties inherent in the application of time- or space-averaging. Of course, the interpretation in the real world of the results of the formal theory is (in such a formulation) found to require the use of certain assumptions of ergodicity, but this situation is common to all applications of statistical physics.

In the preceding paragraph we departed from the chronological principle of describing the fundamental stages in the formation of the theory of turbulence. However, before returning to the historical survey, let us give the present-day formulation of the general problem of the statistical description of turbulent flows (or, in short the "problem of turbulence"). For simphcity, we shall confine ourselves to the case of an incompressible fluid. In this case the flow is determined completely by a solenoidal velocity field (i.e., a velocity field without divergence) $\boldsymbol{u}(\boldsymbol{x}, t) = \{u_1(\boldsymbol{x}, t), u_2(\boldsymbol{x}, t), u_3(\boldsymbol{x}, t)\}$ where u_1, u_2, u_3 are components of velocity, $\boldsymbol{x}$ is a point of space, t is the time (the pressure field may be expressed in terms of the velocity field with the aid of the equations of motion). The problem of turbulence is reduced here to finding the probability distribution $P(d\omega)$ in the phase space of turbulent fiow $\Omega = \{\omega\}$, the points ω of which are all possible solenoidal vector fields $\boldsymbol{u}(\boldsymbol{x}, t)$ which satisfy the equations of fluid mechanics and the boundary conditions imposed at the boundaries of the flow. In many cases a narrower (i.e., a less complete) formulation of the problem of turbulence also proves useful; here, only synchronous values of the fluid mechanical quantities (i.e., those occurring at the same instant of time) are considered. With this approach, the problem of turbulence consists of finding a one-parameter family of probability distributions $P_t(d\tilde{\omega})$ in the phase spaces $\Omega_t = \{\tilde{\omega}\}$, the points $\tilde{\omega}$ of which are all possible solenoidal vector fields $\boldsymbol{u}(\boldsymbol{x})$ which satisfy the corresponding boundary conditions (these boundary conditions may be time-dependent, and as a result, we introduce subscript t into the notation Ω_t for the phase space). The time-dependence of the distributions $P_t(d\tilde{\omega})$ reflects the evolution of the velocity field in accordance with the equations of fluid mechanics. In fact, if we write the solution of these equations symbolically in the form $\boldsymbol{u}(\boldsymbol{x}, t) = T_t\boldsymbol{u}(\boldsymbol{x}, 0)$ where T_t is the corresponding nonlinear operator, then obviously for any measurable set A of the space Ω_t the equality $P_t(A) = P_0(T_{-t}A)$ holds where $T_{-t}A$ belonging to Ω_0 is the set of all initial velocity fields $\boldsymbol{u}(\boldsymbol{x}, 0)$ which under the action of the operator T_t are transformed into fields $\boldsymbol{u}(\boldsymbol{x}, t)$ belonging to the set A. Consequently, we see that the family of probability distributions $P_t(d\tilde{\omega})$ is defined in principle uniquely by the given initial distribution $P_0(d\tilde{\omega})$. As a result, in the formulation under discussion, the problem of turbulence is a problem of the evolution of the probability measure in a functional space with given initial conditions.

A complete determination of the probability distribution $P(d\omega)$ or at least $P_t(d\tilde{\omega})$ is extremely difficult, and at present is far from being solved. However, in many practical cases it is by no means necessary that the problem of turbulence be solved completely. In most practical cases it is sufficient to determine only some of the simplest numerical characteristics of the probability distributions for the fluid mechanical fields of a turbulent flow, for example, the mean values of the velocity and pressure at different space-time points, the second moments of the fluctuations of the fields at a given point (e.g., the variances of the velocity fluctuations which describe the intensity of turbulence or the second moments of velocity components and temperature which express the components of the turbulent fluxes of momentum and heat), and the correlation coefficients between the fluctuations of the fluid mechanical fields at two points of space-time (describing the space-time statistical coherence or the structure of the hydrodynamical fields of turbulent flow). Thus it is natural for the majority of investigations in the theory of turbulence to be devoted not to the problem of turbulence as a whole, but to attempts to determine (although approximately) some simplified characteristics of the probability distribution for fluid mechanical fields of turbulent flow. These investigations led to a number of valuable results on the physical nature and properties of turbulence and give satisfactory answers to many important practical questions.

Returning now to the short outline of the history of the theory of turbulence, we must mention first the interesting work of Geoffrey Taylor (1921) on turbulent transport in which the important role of the correlation functions (i.e., the mixed second moments) of the velocity field was first demonstrated (not for a simple Eulerian velocity at a given point, but for the more complicated Lagrangian velocity of a given fluid particle). However, in general, the idea that the correlation functions and other statistical moments of fluid mechanical fields must be recognized as the fundamental characteristics of turbulence was first stated by L. V. Keller and A. A. Friedmann (1924), who proposed a general method of obtaining (using the equation of motion of a real fluid) the differential equations for the moments of arbitrary order of fluid mechanical fields of turbulent flow. The determination of all such moments (given some general assumptions) is equivalent to the determination of the corresponding probability distribution in the functional space $P(d\omega)$ or $P_t(d\tilde{\omega})$, i.e., the solution of the problem of turbulence. Therefore, the total infinite Friedmann-Keller system of equations for all possible moments gives an analytical formulation of the problem of turbulence. But this system of equations is very complicated; any finite subsystem of this system is always nonclosed, i.e., it contains more unknowns than there are equations in the given subsystem (the impossibility of obtaining a closed system of equations for a finite number of moments is a direct consequence of the nonlinearity of the equations of fluid mechanics). Thus when we study the application of the Friedmann-Keller method to a finite number of moments, there arises the closure problem of the equations for the moments. This problem is in many ways analogous to the closure problem of a chain of equations for multiple distribution functions in the kinetic theory of gases.

It may be said that most of the theoretical work on the dynamics of turbulence has been devoted (and still is devoted) to ways of overcoming the difficulties associated with the closure problem. So far, this has not been completely possible. Nevertheless, many important results of great practical value have been obtained in the theory of turbulence, in two roundabout directions: the first, devoted to the description of large-scale components of turbulence (the scale is comparable to the scales of the overall flow); and the second, to the description of small-scale components. The basic difference in behavior between these two types of components lies in the fact that the large-scale characteristics of turbulence depend considerably on the geometry of the boundaries of flow and the nature of the external forces and hence will be very different for different types of flow. The small-scale characteristics, to a large extent, possess a universal character.

The large-scale components make the fundamental contribution to the transfer of momentum and heat through a turbulent medium, and thus their description is necessary for important engineering problems such as the calculation of drag and heat transfer in the fluid flows past rigid bodies. Therefore, it is natural that in the development of the theory of turbulence attention was first given to the study of the large-scale components. The urgent needs of technology demanded the carrying out of a large amount of experimental research into the properties of the large-scale components of turbulence in the case of flows in pipes, channels, boundary layers and in free turbulent flows (jets, wakes, etc.). On the basis of these experiments "semiempirical theories of turbulence" were formulated. These theories allow the experimental data to be systematized and used to predict the results of subsequent similar experiments.

The formulation of the semiempirical theories of turbulence was an important advance in the development of statistical fluid mechanics. This step started as early as the second decade of this century, but it came to full fruition in the following two decades (1920's and 1930's). However, the possibilities of the semiempirical method have not yet been fully exhausted, and valuable work is still going on in this field. Decisive advances in the development of the semiempirical approach to the theory of turbulence were made by Geoffrey Taylor (1915; 1932), Ludwig Prandtl (1925) and Theodor von Kármán (1930).

The semiempirical theories of turbulence are based on the analogy between turbulence and molecular chaos. Its fundamental concepts include the mixing length (analogous to the mean free path of molecules), the intensity of turbulence (analogous to the root-mean-square velocity of molecules) and the coefficients of turbulent viscosity, thermal conductivity and diffusion. Using the stated analogy it is postulated that a linear dependence exists between the turbulent stress tensor and the tensor of the mean rate of strain and also between the turbulent flux of heat (or passive admixture) and the mean temperature gradient (or gradient of the admixture concentration). These assumed relationships are supplemented by certain hypothetical laws which are established in general form with the aid of qualitative physical considerations, or simply by "guessing" from considerations of simplicity. These assumptions (or some simple corollaries drawn from them) are verified against empirical material, and at the same time, values of the undefined constants occurring in the semiempirical relationships are determined. If the results obtained prove satisfactory, the deductions drawn from them are extended to the whole class of turbulent flows related to the flow to which the empirical data chosen for the verification actually refer.

The semiempirical theories of the 1920's and 1930's usually considered only the simplest statistical characteristics of turbulence. As a rule, the hypotheses adopted in these theories permit the closure of the first equations of the infinite Friedmann-Keller system–the so-called Reynolds equations, containing only single-point first and second moments of the fluid mechanical field. An important role in the semiempirical theories is also played by symmetry considerations and the use of certain simplified hypotheses of similarity (which are, in particular, an essential part of all semiempirical theories of turbulent jets and wakes). However, the similarity hypotheses based on real physical ideas of the mechanism of turbulence do not form a unique basis for these theories and are always supplemented (sometimes even without any real necessity) by assumptions of a more specialized and artificial nature. Thus, for example, one of the most important deductions of the semiempirical theory is that of the universal logarithmic law for the profile of mean velocity in pipes, channels and boundary layers on a flat plate. (Universal here means that the law holds for all sufficiently large Reynolds numbers.) At present, it is known that this law may be derived from a single similarity hypothesis about the probability distribution of the various quantities in turbulent flows in a half-space or from dimensional considerations based on natural assumptions concerning the physical quantities which determine the turbulent regime in this case. Nevertheless, in semiempirical theories this result was always justified by some special hypotheses, and, unfortunately, such artificial deductions even now are predominant in textbooks on fluid mechanics.

The semiempirical theories of turbulence are valuable for solving a number of important practical problems. However, the hypotheses adopted in these theories frequently have no reliable physical foundation and contribute little to the understanding of the physical nature of turbulence. The theory of the *universal steady statistical regime of the small-scale components of turbulence for very high Reynolds numbers*, however, is a completely different matter. This theory is an immediate consequence of the new similarity hypotheses for small-scale components proposed by A. N. Kolmogorov (1941a, b) [the same conclusions were reached by A. M. Obukhov (1941) based on a special model of the physical processes affecting the evolution of these components]. The formulation of this theory was another great step in the development of statistical fluid mechanics.

Before proceeding to a description of the contributions to the theory of turbulence made by Kolmogorov and Obukhov, it is necessary to mention in this historical survey two of their predecessors. One of these was the English scientist, Lewis Richardson (1922; 1926), and the other was Geoffrey Taylor (1935a) whom we have mentioned twice before. In his outstanding paper, published in 1922, and in some other papers, Richardson put forward some penetrating ideas on the physical mechanism of turbulent mixing in the case of large Reynolds numbers (of which little notice was taken at the time). According to his assumptions, developed turbulence consists of a hierarchy of "eddies" (i.e., disturbances or nonhomogeneities) of various orders. Here, the "eddies" of a given order arise as a result of the loss of stability of larger "eddies" of the preceding order, borrowing their energy, and, in their own turn, losing their stability and generating smaller "eddies" of the following order to which they transmit their energy. Thus there arises a peculiar "cascade process," of breaking-down of eddies in which the energy of the overall flow is transmitted to motions of smaller and smaller scale, down to motion of the smallest possible scale, which is stable. To be stable, these extremely small-scale motions must be characterized by a sufficiently small Reynolds number. Thus it follows that viscosity will play an important role and, consequently, there will be considerable dissipation of the kinetic energy into heat. The corresponding physical picture of developed turbulence is expressed neatly in the following rhyme, which first appeared on page 66 of Richardson's book (1922) and has since been often quoted (usually without the exact reference and the last line):

Big whorls have little whorls,
Which feed on their velocity;
And little whorls have lesser whorls,
And so on to viscosity
(in the molecular sense).

Richardson put forward these general ideas only in qualitative form and did not make any deductions that could be formulated in the precise language of mathematics. However, his intuition was so powerful that in his paper of 1926 he was, nevertheless, able to establish by purely empirical means one of the general quantitative laws of small-scale turbulent motion, which follows in fact from the mathematical theory of the cascade process of breaking-down of eddies. This law consists of the fact that the effective diffusion coefficient for a cloud of admixtures in a developed turbulence is proportional to the four-thirds power of the characteristic scale-length of the cloud. In 1941, when Kolmogorov and Obukhov formulated the general quantitative theory of small-scale components of turbulence, Richardson's "four-thirds law" was actually the only empirical result which had indicated the existence of some simple general rules controlling the small-scale structure of turbulence.

Taylor's work also played a large part in the theory of small-scale turbulent motion. G. I. Taylor (1935a) introduced the concept of "homogeneous and isotropic turbulence," determined by the condition that all the finite-dimensional probability distributions of the fluid mechanical quantities at a finite number of space-time points are invariant under any orthogonal transformations (shifts, rotations and reflections) of a system of three-dimensional coordinates. Homogeneous and isotropic turbulence is a special case of turbulent flow in which the structure of the statistical moments of the fluid mechanical fields and the form of the corresponding Friedmann-Keller equations is extremely simple. In this simplified case, all the theoretical difficulties connected with the problem of closure of the Friedmann-Keller equations still apply. However, the corresponding equations are far more amenable to mathematical analysis than the general equations corresponding to arbitrary turbulence, and by using them it is possible to obtain many specific results which explain individual aspects of turbulent flow.

In itself, the model of homogeneous and isotropic turbulence is not suitable for describing any real turbulent flows, since the assumptions of homogeneity and isotropy are not fulfilled for real flows (three-dimensional homogeneity assumes, in particular, that there are no boundaries in the flow and that the mean velocity is strictly constant). However, the mathematical technique of homogeneous and isotropic turbulence, after certain generalizations, proves very valuable for describing the properties of small-scale components of real turbulent flows. This is because the statistical regimes of these components as, following Kolmogorov, we will explain later, may be taken quite naturally to be homogeneous and isotropic. In other words, any developed turbulence with sufficiently high Reynolds number may be considered to be locally homogeneous and locally isotropic, which immediately simplifies the mathematical investigation.

We shall now consider the fundamental ideas of Kolmogorov's "theory of locally isotropic turbulence." We shall use this comparatively short expression rather than the longer one given above. First, Kolmogorov made an important addition to the assumptions on the cascade process of transfer of energy from large-scale components (which obtain their energy immediately from the mean flow) to components of smaller and smaller scale, by noting that due to the chaotic nature of this transfer of energy, the orienting effect of the mean flow must be weakened with each breaking-down. Consequently, for sufficiently small-scale components of turbulence (i.e., for sufficiently large "order numbers") this orienting effect will produce no results at all. In other words, in spite of the fact that the mean flow and the largest-scale nonhomogeneities of any real turbulent motion are, in general, nonhomogeneous and anisotropic, the statistical regime of sufficiently small-scale fluctuations of any turbulence with very high Reynolds number may be taken to be homogeneous and isotropic. Moreover, it is natural to expect that the characteristic periods of different orders will be smaller, the greater their order. Therefore, for fluctuations of sufficiently high order (i.e., sufficiently small spatial scales), these periods will be much smaller than the time of perceptible variation of the mean flow. Therefore, the regime of such fluctuations will be quasi-steady—practically steady over an interval of time containing a large number of characteristic periods.

Thus, we see that for sufficiently small-scale fluctuations, a homogeneous, isotropic and practically steady statistical regime will prevail, characterized by the presence of a mean flux of energy $\bar{\varepsilon}$ to the largest of the fluctuations under consideration and the dissipation of energy (equal to this flux) as heat under the action of the viscosity, which is concentrated principally in the range of minimum scale fluctuations. Arguing from this, Kolmogorov formulated the hypothesis that the statistical regime of sufficiently small-scale components of velocity of any turbulence with sufficiently large Reynolds number will be universal and determined by only two dimensional parameters—the mean rate of dissipation of energy (per unit mass of fluid) $\bar{\varepsilon}$ and the coefficient of viscosity ν. Hence, with the aid of simple dimensional considerations, we may deduce that the scale of the greatest fluctuations on which the viscosity still exerts a considerable effect must be of the order $\eta = (\nu^3/\bar{\epsilon})^{1/4}$. Consequently, it is natural to assume that when the range of fluctuations subject to the universal statistical regime under discussion extends to scales much greater than η, then there must exist an "inertial subrange" of scales (many times smaller than the typical length-scale of the whole flow L, but many times greater than η), in which the viscosity will no longer play any part, i.e., the statistical regime will be determined by a single parameter $\bar{\varepsilon}$. This assumption is Kolmogorov's second main hypothesis.

These hypotheses of Kolmogorov allow a number of concrete deductions to be formulated on the statistical characteristics of the small-scale components of turbulence. The most important of them is the "two-thirds law" (deduced by Kolmogorov) according to which the mean square of the difference between the velocities at two points of a turbulent flow at a distance r apart, when r lies in the "inertial subrange," is equal to $C(\bar{\varepsilon}r)^{2/3}$, where C is a universal numerical constant. Another form of this assertion [first put forward by Obukhov (1941)] is the "five-thirds law," according to which the spectral density of the kinetic energy of turbulence over the spectrum of wave numbers k in the inertial subrange has the form $C_1\bar{\varepsilon}^{2/3}k^{-5/3}$ where C_1 is a new numerical constant (which is simply connected with C). There are also many other consequences of these hypotheses, but we shall not dwell on them here.

The work of Kolmogorov has served as a basis for all subsequent developments of the theory of the local structure of turbulence and its applications in the 1940's and 1950's. During this period investigations were made concerning the local structure of not only the velocity field, but also the fields of concentration of passive (i.e., dynamically neutral) admixtures, temperature (including the case of a thermally stratified heavy fluid in which, due to the effect of buoyancy, the temperature can no longer be considered as a "passive" admixture), pressure, and turbulent acceleration. The deductions obtained concerning the statistical properties of small-scale components of turbulence have found an application in other problems, for example, on the relative dispersion of particles and the breakdown of drops in a turbulent medium, on the generation of wind waves on the surface of the sea, on the generation of a magnetic field in a turbulent flow of conducting fluid and the structure of inhomogeneities in the electron density in the ionosphere, on fluctuations of the refractive index in the atmosphere and the scattering and fluctuations of electromagnetic waves in the turbulent atmosphere, and to many other interesting questions.

The consequences of Kolmogorov's theory, first and foremost the "two-thirds law" and "five-thirds law" given above, were tested many times during the 1940's and 1950's on results of measurements in real turbulent flows. However, it was found that in laboratory experiments (normally carried out in wind-tunnels) the Reynolds number is not large enough for a perceptible "inertial subrange" to exist in the turbulence spectrum, and, consequently, the results of wind-tunnel measurements carried out over 20 years cannot be used to verify these laws. Measurements in nature, on the other hand, where the Reynolds number has, as a rule, far higher values than those observed in laboratory conditions, have until recently given results with considerable statistical scatter. Consequently, although the general ensemble of experimental data unquestionably supported the theory, it had still not been verified in a completely direct manner, and reliable estimates could not be made for the numerical parameters occurring in the theory. Only in the 1960's has there been any real change in the situation, and now several researchers have made extremely precise measurements of the characteristics of turbulence in various natural and artificial turbulent flows with very large Reynolds numbers. The results of these measurements are in excellent agreement with each other, and furnish a definite confirmation of the correctness of the theory, finally allowing the constants C and C_1 to be determined with sufficient accuracy.

At present, strong arguments exist for considering the theoretical treatment and experimental verification of the idea of a universal statistical regime of small-scale components of velocity as being basically completed. However, for the further development of the theory of turbulence, we require essentially new ideas, the search for which we shall describe a little later. Thus it is appropriate now to sum up the results of this important period in the development of the theory of turbulence connected with the elaboration of this idea.

The attempt to carry out this task to some extent is one of the purposes of our book.

There are certain works which deal with the search for new avenues of approach in the development of statistical fluid mechanics. In the papers of A. N. Kolmogorov (1962) and A. M. Obukhov (1962) delivered a week apart at two international conferences on the theory of turbulence held in Marseilles in the early autumn 1961, a method is proposed for further refinement of the basic description of the local structure of turbulence with large Reynolds numbers. The principle of the method is as follows: as we have seen in the theoretical discussion above, the statistical characteristics of small-scale turbulence are assumed to depend only on the mean value $\bar{\varepsilon}$, of the rate of dissipation of energy. In fact, however, the field of dissipation of energy also undergoes disordered fluctuations, and there is reason to think (both theoretically and purely empirically) that its range of variation may be extremely large. The statistical characteristics of this field evidently may depend on the peculiarities of the large-scale flow: in particular, we must expect the variance of the field of ε to increase with the increase of Re. Therefore, the statistical characteristics of small-scale turbulent motion, defined by the single quantity $\bar{\varepsilon}$ must be interpreted only as conditional mean values, obtained on the condition that ε is strictly fixed (and equal to $\bar{\varepsilon}$). However, the unconditional mean values which are obtained by averaging the results of calculations with fixed ε over the fluctuations in value of this parameter will, generally speaking, differ from the conditional mean values (and may themselves be shown to be not universal, i.e., they are different for different types of large-scale flows, and depend, in particular, on Re). The works of Kolmogorov and Obukhov were devoted to methods of estimating the effect of this fact. To obtain specific quantitative results in this way, we first need to discover the statistical properties of the field of the dissipation of energy $\varepsilon(\boldsymbol{x}, t)$, i.e., in other words, we require a more detailed study of the mechanism of dissipation of energy in turbulent flow. Preliminary estimates show that the corrections to the "two-thirds" and "five-thirds" laws obtained by taking into account the fluctuations of the rate of energy dissipation will be close to the order of accuracy of the best experimental data presently available.

We must also note some special variants of closure hypotheses for the equations of turbulence, which were developed in a series of papers by R. Kraychnan (1959; 1962b and others) based on the assumption that direct interactions between triads of space Fourier components of the velocity field play a considerably greater part than do the indirect interactions (by way of all the other Fourier components). We should also mention the method of describing large-scale components of turbulence proposed by W. Malkus (1954b) [see also Townsend (1962b) and Spiegel (1962)] based on the use of the hypothetical variational principal of "maximum dissipation" and representing hydrodynamical fields in the form of a superposition of a finite number of corresponding characteristic functions. In recent years, both these approaches have evoked many controversies—they lead to some interesting and likely results, but they are based on unverified hypotheses and involve a number of difficulties. Thus, for example, the first approximation of Kraichnana's theory leads to equations for the turbulence spectrum in the small-scale range which do not agree with the predictions of Kolmogorov's theory (which have been substantiated excellently) and indicate an unspecified dependence between the statistical characteristics of the small-scale components and the mean square velocity which characterizes the large-scale, energy-containing eddies. Malkus' theory for free convection leads to results which disagree with deductions drawn from the natural similarity theory for the turbulence in a stratified medium (and indicate an unexpected dependence of the large-scale components of convective flow on the molecular thermal conductivity of the fluid). The same theory, applied to flow between parallel plates, leads to a very disputed conclusion concerning the dependence of the large-scale characteristics on the molecular viscosity, however large the Reynolds number, and to a departure from the generally accepted (and well supported by experiment) universal logarithmic law for the mean velocity profile. Therefore, at present, it is still impossible to say whether or not these approaches (or some simple modifications of them) will be fruitful for the development of the theory of turbulence.

Finally, mention must be made of the prospects of solving the general problem of turbulence which are connected with the use of the techniques of "characteristic functionals" of fluid mechanical fields. These characteristic functionals define uniquely the probability distributions $P(d\omega)$ or $P_t(d\tilde{\omega})$ in the phase space of turbulent flow, and hence finding them would give a complete solution of the problem of turbulence. In the work of Eberhard Hopf (1952) an equation is deduced for the characteristic functional of a turbulent velocity field in an incompressible field. This is an equation in functional derivatives, and its most remarkable feature is linearity. Thus, although the dynamics of a fluid is nonlinear, the fundamental problem of statistical fluid mechanics, formulated in terms of characteristic functionals, proves to be a linear problem. We should also note that Hopf's equation is formally similar to the Schwinger equations of quantum field theory, which are equations in functional derivatives for the Greens function of interacting quantum fields (we have already mentioned the strict analogy between the theory of turbulence and quantum field theory).

The solution of Hopf's equation has encountered considerable difficulties, first, because it is still not clear exactly how specific problems for this equation must be formulated, and, second, because of the absence at present of any general methods of solving equations in functional derivatives (or even of general results on the existence and uniqueness of such solutions). In recent years the new mathematical techniques of function space integrals (i.e., of integrals of functionals over some function space) has attracted a great deal of attention in connection with the theory of equations in functional derivatives. Today, it is formally possible to write down a solution of Hopf's equation in the form of a specific function space integral with respect to some "generalized measure" in a function space (which does not possess some properties of the usual measures of measure theory, and hence calls to mind the notorious "Feynman measure" arising in quantum mechanics and quantum field theory). However, writing down this solution remains a purely formal process, and is of little assistance in the effective formulation and study of the required specific solutions.

* * *

We conclude this introduction by discussing briefly the contents of this book. It is, of course, completely impossible in one or two volumes to exhaust the whole range of questions connected with the problem of turbulence. (We may note that the book by Chandrasekhar (1961), which in fact deals almost entirely with the material of our Sects. 2.6 and 2.7 and its generalizations only, contains approximately 700 pages!) We shall not even attempt to be exhaustive but will select the material which, in our opinion, assists our understanding of the physical nature of turbulence. Consequently, we shall not dwell on specific engineering applications or mathematical details connected with the calculation of the complicated statistical characteristics. We shall confine ourselves throughout to only the simplest flows and simplest problems. Thus we shall deal with flows in *straight circular* tubes only; also, we consider boundary layers only along a *flat plate* and in the *absence of a pressure gradient* in the ambient flow; diffusing particles are taken only *to be points* and to have the *same specific gravity* as the surrounding medium, etc. Further, we shall not discuss problems of turbulent flows of a conducting fluid in a magnetic field. The discussion of compressibility occupies only one section in the chapter on isotropic turbulence, so as to enable us to demonstrate some of the new physical effects connected with it. However, we include a chapter on turbulence in a medium with vertical density stratification (for the simplest case of a two-dimensional flow in a half-space $z > 0$) since the effect of stratification on turbulence has a clear physical character and may be investigated with the aid of general methods that are used widely in other parts of the book.

The inclination to make the book as "physical" as possible also dictated the choice of methods of investigation. The equations of turbulent motion are always unclosed (containing more unknowns than equations) and hence problems of the turbulence theory cannot be reduced directly to finding a unique solution of some differential equation (or equations) determined by known initial and/or boundary values. Under these conditions, it is inevitable that, in addition to the equations of motion, some additional considerations be introduced. In our opinion, among all these additional considerations, it is the considerations of similarity (based on the invariance of the problem under some group of transformations) which make the most obvious physical sense as well as dimensional considerations (based on the isolation of physical quantities which effect the turbulent flow under investigation). We have tried to consider most explicitly the deductions from the ideas of dimensionality and similarity which may be used in the theory of turbulence far more widely than is normally proposed. The semiempirical theory of turbulence which employs more specialized hypotheses occupies relatively little space in this book; our description of the "classical" applications of the semiempirical theory to flow in tubes, channels and boundary layers is especially brief (This subject is discussed in detail in the monographs of S. Goldstein (1938), L. G. Loitsyanskiy (1941) and H. Schlichting (1960), together with the semiempirical theories of "free turbulence" which are omitted from our book entirely.) However, we have included several comparatively new and less-known applications of the semiempirical theories and considered a number of applications of the semiempirical theory of turbulent transport (which normally is the only theory considered in discussions of the problem of the spreading of admixtures in turbulent flows). Moreover, we have made a detailed analysis of various hypotheses on the transfer of energy through the spectrum of isotropic turbulence, similar in character to the semiemperical hypotheses, but which throw some light on the physical mechanism of turbulent mixing and which previously have not been discussed in full.

The great attention paid in this book to similarity and dimensionality is also conditioned by the fact that Kolmogorov's theory of locally isotropic turbulence (which is based entirely on these methods) is given a great deal of space here. We have already noted that one of the principal incentives for writing this book was a desire to summarize the development of the idea of a universal local structure in any turbulent flow for sufficiently large Reynolds number. We have also discussed the improved version of this idea put forward by Kolmogorov and Obukhov in 1961, together with additional data on this question obtained later. We also consider briefly the ideas of Kraichnan on small-scale features of turbulence. But the controversial theory of Malkus which contradicts the results of some recent calculations [see Reynolds and Tiederman (1966)], except for the historical outline in the Introduction, will only be mentioned briefly.

While wishing to present a systematic description of the theoretical principles of statistical fluid mechanics, we did not want to give our book a formal mathematical character, and have tried throughout to reinforce the theoretical deductions by the analysis of experimental data. The combination of theoretical and experimental approaches, which is extremely fruitful in all investigations of natural phenomena, is especially necessary in statistical fluid mechanics where the theory is often still of a preliminary nature and is almost always based on a number of hypotheses which require experimental verification. However, we have avoided introducing experimental results which have no theoretical explanation and which do not serve as a basis for some definite theoretical deductions, even if these data are in themselves very interesting or practically important. As a source of experimental data on turbulence, in this book we have very frequently used atmospheric turbulence. To a certain extent, this is connected with the fact that both of us have worked in the Institute of Atmospheric Physics for many years and are far better acquainted with atmospheric turbulence than with other types of turbulent flow. However, in addition, there are weighty reasons which have turned our attention to this type of turbulence. The fact is, that the atmosphere, which von Kármán himself (1934) called "...a giant laboratory for 'turbulence research,'" possesses very valuable properties which make it especially suitable for the verification of the deductions of modern statistical theory. We have already observed that atmospheric motion is usually characterized by far larger Reynolds numbers than flows created in the laboratory, and therefore is far more convenient for investigating specific laws relating to the case of very large Re. Moreover, the geometrical conditions of atmospheric turbulence (namely, the conditions of a two-dimensional flow in a half-space bounded by a rigid wall, referring to the wind in the surface level of the atmosphere, where in many cases the "wall" may be considered as plane and homogeneous; or the conditions of flow in an infinite space, which give a good description of the air motion in the free atmosphere) are simpler than in most laboratory experiments. The only additional comphcation, which arises on transition from laboratory to atmosphere, is the necessity of taking into account the thermal stratification, but, as we have already observed, this complication leads to additional interesting theoretical discussions and extends the number of laws observed experimentally and permits physically based explanations.

The widespread nature of turbulent flows, their great significance for a number of diverse engineering problems, and their interest to theoreticians, has led to the fact that the literature on the subject is enormous–it numbers many tens of thousands of papers, scattered throughout a vast number of physical, mathematical, mechanical, chemical, meteorological, oceanographical, and engineering journals. Unfortunately, however, the theory of turbulence is extremely difficult, and until now it has not been advanced very far; therefore, many of the works referred to are purely empirical, or contain only the very beginnings of the theory, or are even erroneous. These facts, of course, greatly complicate any survey of the literature of statistical fluid mechanics. In this book we originally proposed to confine ourselves to the minimum number of necessary references; however, having spent a great deal of time on the study of the literature (without which it would have been impossible to decide what was necessary), we decided that it would be more expedient not to limit the references, especially to recent works. We have done this to give the reader a sufficiently complete picture of the present state of investigations in the theory of turbulence, to aid him in finding the necessary information, and, as it were, give him his bearings on the infinite sea of books and papers. In a number of places throughout the book we have even included brief surveys of the literature, in which we have tried, as far as possible to give a short description of the content of a considerable number of typical works referring to the particular section of the theory under discussion. We realize, of course, that a large number of references encumbers to some extent, the text of a book and also (since it is completely impossible to cite even the majority of the existing works) considerably increases the number of authors who could feel they had been unjustly passed over in so extensive a survey. We should like, first, to offer our apologies to all such authors and to warn the reader that we in no way claim that the works included in our bibliography are necessarily the best or the most important, nor do we pretend that it is a complete survey. We realize that as far as priority is concerned (a subject which we did not especially study) there may well be a number of slips.

Of course, in selecting material for this book our personal scientific tastes played a definite part. In particular, we have included a number of results from our own research. A large proportion of these results is given in a revised and augmented form which now seems better to us; often, also, we present the results of other authors in a revised form, taking into account later data and also the general approach, terminology and notation adopted in this book. It is natural, therefore, that in many cases the works cited contain only an equivalent (or similar) form of the equation or deduction in connection with which they are quoted.

Often, we also depart from the analysis given in the original work as a result of our desire to explain every assumption used in the course of the various deductions as clearly as possible. In particular, we always try to clarify which results follow immediately from the equations of motion (i.e., from the general laws of physics), which follow from dimensional considerations (i.e., from the specific hypothesis on the complete list of physical parameters affecting the given effect), which require the introduction of special empirical hypotheses (and what these are), and which are simply empirical facts.

Statistical fluid mechanics makes wide use of the results and methods of classical fluid mechanics and probability theory.

Therefore, a knowledge of these two subjects will greatly assist the understanding of this book. Nevertheless, we hope that our book will be accessible also to those who have only a general mathematical and physical training. With these readers in mind, we have included in the first two chapters the basic necessary concepts from classical fluid mechanics (beginning with the equations of continuity and motion) and of the theory of probablility (commencing with the concept of probability itself). In these as in all subsequent chapters we have tried to place fundamental stress on the principles involved without lingering over technical details. Similarly, we never discuss methods of solving differential equations or other standard mathematical techniques, but quote the answer (which is sometimes not easy to find) immediately. At the same time we discuss in comparative detail certain insufficiently widely known but important mathematical questions traditionally excluded from books and papers intended for those specializing in mechanics and physics (e.g., questions of the ergodic theorems or of spectral representations of random fields); this explains the fact that only two chapters of the book are devoted to the mathematical theory of random fields.

For technical reasons it has proved more convenient to issue the book in two volumes. The first volume includes questions which can be dealt with without recourse to spectral representations. (Probably many of our colleagues will be surprised that such problems have been collected into a thick book.) Here we discuss general equations of fluid mechanics and their simplest corollaries (concluding with the somewhat more specialized theory of the small oscillations of a compressible gas); the question of hydrodynamic instability and transition to turbulence is considered (including elements of the nonlinear theory of instability); some elementary notions are given from probability theory and the theory of random fields (including conditions of ergodicity); the application of dimensional and similarity considerations to turbulent flows in tubes, channels, boundary layers and in free turbulent flows (jets, wakes, etc.) are discussed in detail and there is a brief survey of the main ideas and results of the semiempirical theory of turbulence (with specific examples); the theory of similarity for turbulence in a medium with vertical density stratification is discussed in detail, and the results obtained are compared with the extensive empirical data on wind and temperature fluctuations in the surface layer of the atmosphere; the Lagrangian characteristics of turbulence and the theory of turbulent diffusion are described.

The second volume of the book will begin with a mathematical chapter devoted to the spectral theory of random fields (including fields that are inhomogeneous but locally homogeneous); there is a detailed description of the theory of isotropic turbulence (here we are concerned principally with the various methods of closure of the moment equations of incompressible isotropic turbulence, but some results relating to compressible fluids are also given); the general theory of the universal local structure of turbulence with sufficiently large Reynolds numbers is considered (including the theory of relative diffusion, i.e., the increase in size of a cloud of an admixture carried by a turbulent flow); the basic concepts relating to the propagation of electromagnetic and sound waves in a turbulent medium are given and, finally, we consider the general formulation of the problem of turbulence, based on the study of the characteristic functionals of hydrodynamic fields.

1 LAMINAR AND TURBULENT FLOWS

1. EQUATIONS OF FLUID DYNAMICS AND THEIR CONSEQUENCES

1.1 System of Dynamical Equations for an Incompressible Fluid

Turbulence is a complex physical phenomenon, and its theoretical investigation must be based on the fundamental laws of physics, expressed in the equations of hydro- and thermodynamics. Hence, we shall begin our discussion with a brief resumé of these equations and some of their more important consequences, restricting ourselves, of course, to those equations and facts which we shall need in the subsequent discussion [for a more detailed treatment of fluid mechanics see, e.g., Lamb (1932), Goldstein (1938), Landau and Lifshitz (1963), Kochin, Kibel', and Roze (1964), Longwell (1966), and Batchelor (1967)]. As usual, we shall describe the flow of fluid[1]

[1] Here and henceforth, we shall use the word "fluid" to denote any liquid or gaseous medium.

by the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)=\{u_1(x_1, x_2, x_3, t), u_2(x_1, x_2, x_3, t), u_3(x_1, x_2, x_3, t)\}$ and by the fields of any two thermodynamic quantities, e.g., the pressure $p(\boldsymbol{x}, t)$ and the density $\rho(\boldsymbol{x}, t)$ or the temperature $T(\boldsymbol{x}, t)$—in all, by five functions of four variables. In addition, we shall also need the values of the molecular transport coefficients which determine the physical properties of the fluid—the coefficient of viscosity μ, or kinematic viscosity $\nu=\frac{\mu}{\rho}$, and the second coefficient of viscosity ζ. Later, we shall need the coefficient of thermal conductivity $\varkappa$ (and thermal diffusivity or thermometric conductivity $\chi=\frac{\varkappa}{c_p\rho}$, where c_p is the specific heat at constant pressure of the fluid).

The simplest equation of fluid mechanics describing the physical law of conservation of mass is the equation of continuity

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u_\alpha)}{\partial x_\alpha}=0 \tag{1.1}$$

or

$$\frac{\partial \rho}{\partial t}+u_\alpha\frac{\partial \rho}{\partial x_\alpha}=-\rho\frac{\partial u_\alpha}{\partial x_\alpha}\ . \tag{1.2}$$

Here and henceforth, we shall always adopt Einstein's summation convention, according to which whenever an index occurs twice in a single-term expression, the summation is carried out over the three possible values of the index, so that $\frac{\partial(\rho u_\alpha)}{\partial x_\alpha}$, for example, has the same meaning as $\sum_{\alpha=1}^{3}\frac{\partial(\rho u_\alpha)}{\partial x_\alpha}$.[2] The fundamental dynamic equations expressing Newton's second law apphed to a small volume of fluid, i.e., the equation of conservation of momentum, have the form

$$\frac{\partial(\rho u_i)}{\partial t}+\frac{\partial(\rho u_i u_\alpha)}{\partial x_\alpha}=\rho X_i-\frac{\partial p}{\partial x_i}+$$
$$+\frac{\partial}{\partial x_\alpha}\left[\mu\left(\frac{\partial u_i}{\partial x_\alpha}+\frac{\partial u_\alpha}{\partial x_i}-\frac{2}{3}\frac{\partial u_\beta}{\partial x_\beta}\delta_{i\alpha}\right)\right]+\frac{\partial}{\partial x_i}\left(\zeta\frac{\partial u_\beta}{\partial x_\beta}\right),\quad i=1, 2, 3 \tag{1.3}$$

[2] In the deduction of the general equations, we shall always denote the coordinate axes by Ox_i, $i = 1, 2, 3$ in order to use widely this convenient summation convention. In specific examples, however, when the directions of the three axes are defined in some special manner, we shall often also use the letters x, y, z to denote the coordinates.

where $\rho X_i = \rho X_i(\boldsymbol{x}, t)$ is the ith component of the external force-density at the point $\boldsymbol{x}$ at the instant t. From Eq. (1.1) it follows that the density ρ may be taken outside the derivative $\frac{\partial}{\partial t}$ in the first term of the left side of Eq. (1.3), taking ρu_α outside the sign $\frac{\partial}{\partial x_\alpha}$ in the second term at the same time. For the coefficients of viscosity μ and ζ, the variation of these values in space (due to their dependence on the temperature which varies from point to point) is almost always negligible. Thus terms containing derivatives of these coefficients are usually ignored completely. However, this means that the values of μ and ζ in Eq. (1.3) may also be taken outside the derivative, so that these equations become

$$\rho\left(\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha}\right) = \rho X_i - \frac{\partial p}{\partial x_i} + \mu\,\Delta u_i + \left(\zeta + \frac{\mu}{3}\right)\frac{\partial^2 u_\alpha}{\partial x_i\,\partial x_\alpha}\,. \quad (1.4)$$

Since the four equations (1.2) and (1.4) contain five unknown functions, they do not yet form a closed system. In practice, however, the variation of the density ρ of a moving fluid particle proves to be so small in many cases that this too may be ignored. This is particularly true in the case of ordinary liquids, i.e., liquids in the normal sense of the word. In the case of gaseous media in steady motion [with velocity $\boldsymbol{u}(\boldsymbol{x})$ independent of time] the variations of density due to the variations of pressure may be ignored completely, provided the absolute velocity $u = |\boldsymbol{u}|$ at every point of the flow is small in comparison with the velocity of sound a. In general, for unsteady motion, variations of the density ρ will be negligibly small if, in addition to the condition $u \ll a$, the condition $T \gg L/a$ is also fulfilled, where T and L are the characteristic time and length scales in which the velocity of the fluid $\boldsymbol{u}(\boldsymbol{x}, t)$ undergoes a perceptible change; see Landau and Lifshitz (1963), Sect. 10. In all the above cases, the density ρ may generally be taken as constant throughout the whole space filled by the fluid. Then ρ will represent a preassigned constant value (characteristic of the properties of the medium) and the four equations (1.2) and (1.4) will now suffice for the four unknown functions $u_i(\boldsymbol{x}, t)$ $i = 1, 2, 3$, and $p(\boldsymbol{x}, t)$ to be determined (for given initial and boundary conditions).

In the present case of an incompressible fluid, these equations take a simpler form, since a number of terms containing derivatives of ρ (or expressed in terms of such derivatives) may be omitted. Thus, in this case we may omit ail the terms on the ieft side of the equation of continuity (1.2), so that

$$\frac{\partial u_\alpha}{\partial x_\alpha} = 0. \tag{1.5}$$

In the equations of motion (1.4) the last term on the right side thus becomes identically equal to zero, and these equations are then transformed into the well-known Navier-Stokes equations

$$\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha} = X_i - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \Delta u_i, \tag{1.6}$$

which are the fundamental dynamical equations of a viscous incompressible fluid. We see that they contain only a single coefficient of viscosity, since the second viscosity is important only when compressibility is taken into account.

It is easy to see that the pressure p may be eliminated from Eqs. (1.6) from the very beginning. In fact, for this purpose we need only apply to both sides of Eqs. (1.6) the operation of taking the curl, which in tensor notation is written as the operation $\varepsilon_{k\beta i} \frac{\partial}{\partial x_\beta}$, where $\varepsilon_{k\beta i}$ is a completely skew-symmetric tensor of the third rank (i.e., $\varepsilon_{k\beta i} = 0$ if the indices k, β, i are not all different, and $\varepsilon_{k\beta i} = +1$ or -1 if (k, β, i) is obtained from (1, 2, 3) by an even or odd permutation, respectively). Assuming, for simplicity, that there are no external forces, we arrive at a system of three equations

$$\frac{\partial \omega_k}{\partial t} + u_\alpha \frac{\partial \omega_k}{\partial x_\alpha} - \omega_\alpha \frac{\partial u_k}{\partial x_\alpha} = \nu \Delta \omega_k, \qquad k = 1, 2, 3, \tag{1.7}$$

where

$$\omega_k = \varepsilon_{k\beta\alpha} \frac{\partial u_\alpha}{\partial x_\beta} \tag{1.8}$$

are the components of the vorticity vector. In principle we may determine the three functions $u_i(\boldsymbol{x}, t)$ from Eqs. (1.7) and (1.8); then, in order to find the pressure field it is only necessary to solve Poisson's equation

$$\Delta p = -\rho \frac{\partial^2 (u_\alpha u_\beta)}{\partial x_\alpha \partial x_\beta}, \tag{1.9}$$

which is obtained by applying the operation $\frac{\partial}{\partial x_l}$ to Eqs. (1.6) [with $X_i=0$], i.e., by taking the divergence of the Navier-Stokes vector equation.

From Eq. (1.9) it follows that, accurate to within a harmonic function of $\boldsymbol{x}$,

$$p(\boldsymbol{x})=\frac{\rho}{4\pi}\int\frac{\partial^2[u_\alpha(\boldsymbol{x}')\,u_\beta(\boldsymbol{x}')]}{\partial x'_\alpha\,\partial x'_\beta}\frac{d\boldsymbol{x}}{|\boldsymbol{x}-\boldsymbol{x}'|}, \tag{1.9'}$$

where the integration is taken over the whole volume occupied by the fluid. In particular, if the fluid flow then takes place in the whole unbounded space, then the corresponding harmonic function may be only a constant. Since only derivatives of the pressure appear in the equation of motion, the constant term in the expression for the pressure will generally play no role at all; thus it is permissible here to assume that Eq. (1.9′) is accurate without any additions. For flows in finite regions, the harmonic addition to the right side of Eq. (1.9′) must be determined from the boundary conditions for the pressure; here, in a number of cases this may also prove to be a constant and hence may be ignored.

1.2 Simple Flows of Incompressible Fluid

To obtain a unique solution of the system of equations (1.5)–(1.6) it is necessary to fix the initial values of the fields $u_i(\boldsymbol{x},\ t)$ and to take into account that the velocity must vanish on the surfaces of all rigid bodies immersed in the flow. However, since this system is nonlinear, finding its exact solutions in explicit analytical form is, in general, very difficult. Only for a few special flows are the nonlinear terms of Eqs. (1.6) identically equal to zero; in this case the system of equations of fluid mechanics is greatly simplified, and in such cases its exact solution is usually found without much trouble. Below, we give some exact solutions of this type; examples of more complicated exact solutions of Eqs. (1.5)–(1.6) may be found, for instance, in Landau and Lifshitz (1963), Kochin, Kibel', and Roze (1964), and Batchelor (1967). In all the examples considered below we shall assume that there are no external body forces X_i[3]; moreover, the flow will always be assumed steady, i.e., time-independent.

[3]We observe that the body force of gravity $\boldsymbol{X}=(0,\ 0,\ -\rho g)$ which is always present may easily be eliminated when there is no free surface of the fluid, by subtraction from the true pressure p of the hydrostatic pressure $p_0=-\rho g x_3+\text{const}$.

We note first, that in the case of a steady flow the total momentum of the fluid in a fixed part of space does not vary with time; i.e., the momentum loss connected with the friction of rigid bodies submerged in the fluid is balanced by the gain in momentum due to the action of forces that produce the fluid motion (in the case of $X_i = 0$, due to the action of the pressure drop). In other words, in a steady regime the force of the pressure drop acting over the whole fluid to produce motion is balanced by the retarding force acting on the fluid (which differs only in sign from the total force exerted by the flow on the body submerged in it). The equation expressing this equality which allows us to establish a relationship between the characteristic velocity of the flow and the pressure drop is usually called the "*drag law.*"

The force $\boldsymbol{W}$ exerted by the flow on the body is equal to the integral over the surface of the body Σ of the flux of momentum along the normal to this surface. The flux of the ith component of momentum per unit area perpendicular to the axis Ox_k has the form $\rho u_i u_k + p\delta_{ik} - \sigma_{ik}$, where δ_{ik} is the unit tensor ($\delta_{ik} = 1$ for $i = k$, $\delta_{ik} = 0$ for $i \neq k$) and σ_{ik} is the viscous stress tensor, which in an incompressible fluid is equal to $\mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$ [see Eq. (1.61)]. Since the velocity of the fluid on the surface of a rigid body equals zero, we obtain for the ith component of the force $\boldsymbol{W}$

$$W_i = -\int_{\Sigma} (p\delta_{ik} - \sigma_{ik})\, n_k\, d\Sigma = -\int_{\Sigma} p n_i\, d\Sigma + F_i, \qquad (1.10)$$

where n_i is the component of the unit vector of the outward normal to the surface, and F_i is the integral of $\sigma_{ik} n_k$ over the surface of the body. The first term on the right side of Eq. (1.10) describes the transfer of momentum to the body by the pressure forces and is independent of the viscosity of the fluid. In several cases (e.g., in the case of flow along a flat plate or flows in straight tubes) this component is equal to zero. The second component F_i represents the skin friction drag, which in a viscous fluid is always nonzero. In the case of geometrically similar bodies, the expression $\frac{1}{2}\rho U^2 S$ may be taken as characteristic of the force $\boldsymbol{W}$, where U is the characteristic velocity and S some characteristic area, e.g., the area of the surface of the body or the area of its midsection. The dimensionless quantity

$$C_W = \frac{W}{\frac{1}{2}\rho U^2 S} \tag{1.11}$$

is called the *drag coefficient* of the body. If the drag is due only to the friction ($W_i = F_i$), the drag coefficient is denoted by C_f and called the *coefficient of friction* (or the *friction factor*, or *skin friction coefficient*).

Examples of exact solutions of the fluid-mechanical equations

1. We begin with the simplified case of a two-dimensional flow of viscous fluid between two parallel planes, one of which is stationary and the other moving with constant velocity U. We denote the distance between the planes by H and choose a coordinate system in which the equations of these planes will have the form $z = 0$ (stationary plane) and $z = H$ (moving plane), while the Ox axis is directed along the vector $\boldsymbol{U}$. In this case all the hydrodynamic quantities will be dependent only on z, while the velocity of the fluid will everywhere lie along the Ox axis. As a result, Eq. (1.5) and the second equation of Eqs. (1.6) will be satisfied identically, and the first and third equations of Eqs. (1.6) will take the forms

$$\frac{d^2u}{dz^2} = 0; \qquad \frac{dp}{dz} = 0, \tag{1.12}$$

where $u(z) = u_1(z)$ is the only nonzero component of the velocity of the flow. Consequently, $p =$ const and $u = az + b$; taking the boundary conditions $u = 0$ for $z = 0$, $u = U$ for $z = H$ into consideration, we obtain

$$u(z) = \frac{U}{H} z. \tag{1.13}$$

Thus the flow is described by a linear velocity profile, while the mean velocity $U_m = \frac{U}{2}$. The frictional force per unit area of the walls $z = 0$ and $z = H$ equals

$$\tau = \mu \left| \frac{du}{dz} \right| = \frac{\mu U}{H}. \tag{1.14}$$

On the plane $z = 0$ this force is directed along the Ox axis, while on $z = H$ it has the opposite direction. If we put

$$\tau = \frac{1}{2} \rho U_{\mathrm{m}}^2 \cdot C_f \tag{1.15}$$

where C_f is the skin friction coefficient, then

$$C_f = \frac{4\nu}{U_{\mathrm{m}} H} = \frac{4}{\mathrm{Re}}, \text{ where } \mathrm{Re} = \frac{U_{\mathrm{m}} H}{\nu}. \tag{1.16}$$

The steady plane flow of fluid between infinite parallel planes described by Eqs. (1.13) and (1.16) is clearly a mathematical idealization, which, however, may prove useful in certain cases. In handbooks on fluid mechanics, this idealization is sometimes called a *plane Couette flow*.

2. Let us now consider a steady two-dimensional flow between two fixed parallel planes $z = 0$ and $z = H$ under the action of an external force applied at infinity which produces a pressure gradient along the Ox axis. It is easy to see that in this case the velocity of the flow at all points will also be directed along the Ox axis and will depend only on z, so that $u(z) = u_1(z)$ will again be the only nonzero component of the field $\boldsymbol{u}$. Here Eq. (1.5) and the second equation of Eqs. (1.6) will also be satisfied identically, while the first and third equations of Eqs. (1.6) will take the form

$$\frac{\partial p}{\partial x} = \mu \frac{d^2 u}{dz^2}, \qquad \frac{\partial p}{\partial z} = 0. \tag{1.17}$$

It is clear therefore that in this case the pressure will be independent of z; i.e., it will be constant over the whole thickness of the layer of fluid, and consequently the first equation of Eq. (1.17) will be satisfied only for

$$\frac{\partial p}{\partial x} = \mathrm{const} = -\frac{\Delta_l p}{l}, \qquad u(z) = -\frac{1}{2\mu} \frac{\Delta_l p}{l} z^2 + az + b, \tag{1.18}$$

where $\Delta_l p$ is the drop in pressure between the planes $x = x_0$ and $x = x_0 + l$. Taking into account the boundary conditions ($u = 0$ for $z = 0$ and for $z = H$), we obtain

$$u(z) = -\frac{1}{2\mu}\frac{\Delta_l p}{l}\left[\left(z - \frac{H}{2}\right)^2 - \frac{H^2}{4}\right], \tag{1.19}$$

so that in this case the velocity profile will have a parabolic form. The sheer stress τ on both walls will equal

$$\tau = \mu\left|\frac{du}{dz}\right| = \frac{H}{2}\frac{\Delta_l p}{l}\,. \tag{1.20}$$

Hence, the dimensionless skin friction coefficient C_f of Eq. (1.15) will equal

$$C_f = \frac{12\nu}{HU_{\mathrm{m}}} = \frac{12}{\mathrm{Re}}\,, \quad \text{where} \quad \mathrm{Re} = \frac{U_{\mathrm{m}}H}{\nu} \tag{1.21}$$

(because in the present case

$$U_{\mathrm{m}} = \frac{1}{H}\int_0^H u(z)\,dz = \frac{H^2}{12\mu}\frac{\Delta_l p}{l} = \frac{2}{3}U_{\max}\bigg).$$

The idealized flow described by the exact solution (1.19) is sometimes called *plane Poiseuille flow.*

3. We now turn to the case of a steady flow of fluid in a straight, infinitely long, circular cylindrical tube of given diameter D. The axis Ox is taken along the axis of the tube; then the only nonzero component of the velocity field will be $u_1(y, z) = u(y, z)$. Thus in this case also, Eq. (1.5) will be satisfied identically; further, it follows from the second and third equations of Eqs. (1.6) that $\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$; i.e., that the pressure is constant across the section of the tube and depends only on x. We now introduce into the plane Oyz the cylindrical coordinates (r, φ). Then by symmetry of the tube, $u(y, z) = u(r)$, and hence in these coordinates the first equation of Eqs. (1.6) will take the form

$$\frac{\partial p}{\partial x} = \frac{\mu}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right). \tag{1.22}$$

Since the left side depends only on x and the right side on r, we must have $\frac{\partial p}{\partial x} = \text{const} = -\frac{\Delta_l p}{l}$, where $\Delta_l p$ is the pressure drop along a section of tube length l. Taking into account the boundary condition $u(D/2) = 0$ and the fact that $u(r)$ is bounded for all r, we obtain

$$u(r) = \frac{\Delta_l p}{4\mu l}(R^2 - r^2) \left(\text{where } R = \frac{D}{2}\right). \tag{1.23}$$

Thus also in this case, the velocity profile (along any diameter) will be parabolic. The mean velocity of flow defined by the equation

$$U_{\mathrm{m}} = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R u(r)\, r\, dr\, d\varphi, \tag{1.24}$$

is equal here to $\frac{\Delta_l p}{8\mu l} R^2 = \frac{U_{\max}}{2}$; hence, the discharge of fluid (i.e., the volume of fluid flowing through a cross section of the pipe in unit time) equals

$$Q = \pi R^2 U_{\mathrm{m}} = \frac{\pi R^4}{8\mu} \frac{\Delta_l p}{l}. \tag{1.25}$$

The shear stress on the wall is equal to $\tau = \mu \left| \frac{du}{dr} \right| = \frac{R}{2} \frac{\Delta_l p}{l}$; the same result may also be obtained by considering that the total shear stress $\pi D l \cdot \tau$ on an element of the tube of length l must be balanced by the force of the pressure drop acting on the cross section of the tube πR^2. Consequently, in this case the skin friction coefficient equals

$$C_f = \frac{8\nu}{R U_{\mathrm{m}}} = \frac{16}{\mathrm{Re}}, \quad \text{where } \mathrm{Re} = \frac{U_{\mathrm{m}} D}{\nu}. \tag{1.26}$$

The solution (1.23) of the equations of fluid mechanics also refers to an idealized flow, since the tube is taken to be infinitely long, and the flow, to be strictly constant. However, it is found that in cases of real finite tubes, the flow of fluid along them under constant pressure head $\Delta_l p$ quite often becomes practically steady extremely quickly, while at distances from the intake of the tube of the order of $\frac{R^2 U_{\mathrm{m}}}{\nu}$ [according to the theoretical calculations of Schiller (1934), cited also in Goldstein (1938), Vol. 1, Sect. 139, even for $x > 0.115 \left.\frac{R^2U_m}{\nu}\right]$ the effect of the finite length of the tube also ceases to be perceptible, and Eq. (1.23) is a good description of the flow. Thus the laws (1.23), (1.25), and (1.26) may be checked experimentally; historically speaking, they were first demonstrated experimentally (by G. Hagen in 1839 and J. Poiseuille in 1840-1841) and only later justified theoretically (by J. Stokes in 1845). Hagen also showed that these laws are correct only in the case of sufficiently high viscosity ν and sufficiently low velocity U_m—a fact which we shall discuss in detail in Sect. 2. A flow of incompressible fluid described by Eqs. (1.23) and (1.25) is often called a *circular Poiseuille flow* (or a Hagen-Poiseuille flow) or (in cases when there is no danger of confusion with the similar plane flow) simply a Poiseuille flow.

4. As our last example we shall consider the steady motion of a fluid in an annulus between two coaxial infinite cylinders of radii R_1 and $R_2 > R_1$, rotating about their axes with angular velocities Ω_1 and Ω_2; the pressure drop along the axes of the cylinders is taken equal to zero. In this case all the hydrodynamic quantities will be independent of the x coordinate, reckoned along the axis of the cylinders. Further, if in the plane Oyz perpendicular to the axis of the cylinders we introduce cylindrical coordinates (r, φ), then from considerations of symmetry it is clear that only the component of velocity $u = u_\varphi$ will be nonzero and that the velocity u and the pressure p will depend only on r. It follows that in this case the equation of continuity (1.5) and the first equation of Eqs. (1.6) will be satisfied identically; for the second and third equations of Eqs. (1.6), after transformation to cyhndrical coordinates, these will take the form

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{u^2}{r}, \qquad \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0. \tag{1.27}$$

The general solution of the second of Eqs. (1.27) has the form $u(r) = ar + \frac{b}{r}$; since at the same time $u(r)$ must equal $\Omega_1 R_1$ for $r = R_1$ and $\Omega_2 R_2$ for $r = R_2$, the unknown function $u(r)$ will be defined by the equation

$$u(r) = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} \cdot r + \frac{(\Omega_1 - \Omega_2)\, R_1^2 R_2^2}{R_2^2 - R_1^2}\,\frac{1}{r}. \tag{1.28}$$

Using the first equation of Eqs. (1.27) it is now very easy to find $p(r)$, but we shall not do so here. The flow given by Eq. (1.28), in many respects, is similar to that of Example 1, but it may be reproduced far more simply and with comparative accuracy in the laboratory. This flow is sometimes called circular Couette flow, or flow between cylinders.

1.3 The Reynolds Number and Similarity Criteria

All exact solutions of the equations of fluid mechanics given above refer to extremely simple idealized flows. In more complex cases it is generally impossible to find an exact solution, and we have to resort to a numerical solution of Eqs. (1.5)–(1.6) using some approximation method. In such cases it is extremely important to be able to estimate the orders of magnitude of the terms in our equations to determine which of the terms may be ignored in which cases. As a result, we may confine our attention to simplified equations that are much easier to integrate.

We shall assume that we are dealing with a steady flow of viscous incompressible fluid in the absence of body forces. Using Eq. (1.9) makes it easy to show that the terms of Eqs. (1.6) containing the pressure are, in general, of the same order as the nonlinear terms $u_\alpha \dfrac{\partial u_l}{\partial x_\alpha}$ so that it is only necessary to compare the orders of the terms $u_\alpha \dfrac{\partial u_l}{\partial x_\alpha}$ and $\nu \Delta u_i$ which describe the effect of the forces of inertia and the forces of viscous friction. We take U to be some typical velocity (e.g., the maximum velocity of flow, or some mean velocity) and L to be some typical length (e.g., the mean dimension of a body immersed in the flow, or the typical distance between rigid walls, or, in general, a distance along which the velocity of the flow undergoes a perceptible change ΔU of the order of U). In this case, the order of the first derivative of the velocity field will be given by the ratio U/L and the second derivative by the ratio U/L^2, so that terms describing the inertia forces will have order U^2/L and terms describing the friction forces will have order $\nu U/L^2$. The ratio of these quantities,

$$\frac{U^2}{L} : \frac{\nu U}{L^2} = \frac{UL}{\nu} = \mathrm{Re}\,, \tag{1.29}$$

is called the Reynolds number of the flow; it is a very important characteristic of flow, defining the relative role of the inertia forces and the friction forces in the dynamics [we have already introduced this number in Eqs. (1.16), (1.21), and (1.26)]. In the case of small Reynolds numbers, the viscosity has a considerable effect on the entire flow by smoothing out all existing small-scale inhomogeneities; hence space variations of the fluid mechanical quantities must take place very smoothly in the case of small Reynolds numbers. In the case of large Reynolds numbers the dominant role in the flow is played by the inertia forces, the action of which leads to the transfer of energy from the large-scale components of motion to the small-scale components and consequently to the formation of sharp local irregularities; we shall be concerned with this type of flow in all subsequent sections of this book.

The concept of the Reynolds number greatly simplifies the investigation of geometrically similar flows of fluid, such as, flows of fluid in tubes with cross sections of given form, or flow of an infinite stream of fluid past a rigid body of given form. Since in these cases we are concerned with the whole set of flows with geometrically similar boundaries, the properties of the boundaries will be determined uniquely by the same length L (in our examples, L may be taken as the mean diameter of the cross section of the tube or the mean diameter of the body washed by the flow). Furthermore, the flow itself will be described by some typical velocity U (for example, the maximum velocity in a fixed cross section of the tube, or the velocity of the flow incident on a body). Finally, the only dimensional parameter occurring in Eqs. (1.5) and (1.6) is the viscosity coefficient ν, which describes the physical properties of a fluid. Thus, for a steady flow in the absence of body forces, geometrically similar flows will depend only on the length L and the dimensional parameters U and ν, which have the dimensions

$$[U] = LT^{-1}; \qquad [\nu] = L^2T^{-1}, \tag{1.30}$$

where L and T denote, as usual, the dimensions of length and time, respectively. It is clear that from the parameters L, U, and ν, a unique dimensionless combination—Reynolds number $\mathrm{Re} = \frac{UL}{\nu}$—may be formulated. Hence, if we measure length in units of L, velocity in units of U, and pressure in units of ρU^2, then Eq. (1.5) and the boundary conditions for all geometrically similar flows will be of the same form, while Eq. (1.6) will be characterized almost entirely by the value of the dimensionless parameter Re, which will appear in this equation after transformation to dimensionless form. Consequently, in the case of geometrically similar flows, the velocity, measured in terms of U will depend only on the coordinates measured in terms of L, and on the number Re. In other words,

$$u_i(\boldsymbol{x}) = U\varphi_i\left(\frac{\boldsymbol{x}}{L},\ \mathrm{Re}\right), \tag{1.31}$$

where $\varphi_i(\boldsymbol{\xi}, \mathrm{Re})$ is some universal function describing the entire set of flows under consideration (we must point out that in Examples 1 to 4 of the previous subsection, the functions φ_i, due to various special reasons, were completely independent of Re). In the same way, we have $p(\boldsymbol{x}) = \rho U^2\varphi_0\left(\frac{\boldsymbol{x}}{L},\ \mathrm{Re}\right)$, where φ_0 is a new universal function. Further, if W is the force exerted by the flow upon some rigid body washed by the flow, then the drag coefficient $C_W = \frac{W}{\frac{1}{2}\rho U^2 L^2}$ will depend only on Re, while L^2 in the definition of C_W may also be replaced by the area S of the whole surface (washed by the flow) of the body (so that we can assume that $C_f = \frac{\tau}{\rho U^2}$, where τ is the mean friction force per unit surface). In other words, when the corresponding Reynolds numbers are equal (but, generally speaking, only when they are equal), geometrically similar flows will also exhibit mechanical similarity; i.e., they will possess geometrically similar configurations of the stream lines and will be described by the same functions of the dimensionless coordinates (this is called *Reynolds' similarity law*). This law is of great importance in the theoretical study of flows occurring under similar conditions, for coordinating the presentation of observational data on such flows, and for the modeling of flows encountered in practical problems. Furthermore, it indicates that all possible geometrically similar flows will be described by a one-parameter family of solutions of Eqs. (1.5)–(1.6); this of course greatly simplifies the problem of integrating these equations numerically.

We must stress once again that Reynolds' similarity law holds only for steady flows of incompressible fluid, on which external forces exert no appreciable effect. However, for motions that depend largely on body forces (e.g., on the force of gravity) and also for nonsteady motions described by some typical period T which differs from L/U, the similarity law is more complicated; in such a case, for mechanical similarity, it is necessary not only that the Reynolds numbers be equal, but also that certain additional dimensionless "similarity criteria" be equal. In the case of flows of a compressible fluid, the number of similarity criteria will be even greater; we shall discuss this further in Sect. 1.6.

We must also point out that in certain cases a very slight departure from geometrical similarity may lead to a total breakdown of mechanical similarity. Thus, for example, a slight change in the conditions in the intake section of a circular tube, introducing slight disturbances into the flow, may completely change the character of the flow in the tube (see Sect. 2).

1.4 Flows with Large Reynolds Numbers; The Boundary Layer

We shall now assume that the Reynolds number of the flow is very large. In this case, the nonlinear inertia terms of Eqs. (1.6) will be far greater in magnitude than the terms containing the viscosity. Thus, at first glance, the effect of the viscosity may appear to be simply ignored. However, this is not so; to ignore the terms containing ν in Eqs. (1.6) would reduce the order of these differential equations. Moreover, the solutions of the simplified equations of an ideal fluid thus obtained would no longer satisfy the "no slip" boundary conditions which require that the velocity vanish upon all rigid surfaces bounded by the fluid. At the same time, we know that for a viscous fluid (however small the viscosity) "adhesion" is bound to take place. Therefore, in the motion of a viscous fluid with a large Reynolds number, only at a distance from the rigid walls does the flow become close to that which would occur in the case of an ideal fluid (with zero viscosity); close to the walls a thin layer exists in which the velocity of the flow changes very rapidly from zero on the wall to the value on the outer boundary of the layer. This value is very close to that which would be obtained in the case of an ideal fluid. The rapid variation of velocity within this layer (known as the boundary layer) means that within it, the effect of the friction forces is by no means small, but of the same order as the effect of the forces of inertia.

Thus, for large Reynolds numbers, the viscosity of the fluid plays an effective role only in the region close to the boundary walls. In this region the terms containing ν cannot be ignored; however, here the equations of fluid mechanics can be simplified considerably due to specific features of the motion of a thin film of fluid in contact with a rigid body. Therefore, in free space, the flow will be determined by the equations of an ideal fluid, while for the boundary conditions we need to use the conditions on the upper surface of the corresponding boundary layer (or layers).

Let us now consider the simple case of a boundary layer on a flat plate of length L and infinite width lying in the plane Oxy (so that it occupies the region $0 \leqslant x \leqslant L$ of this plane) and washed by a flow of fluid, moving with given velocity U in the Ox direction. First, before considering the mathematical solution of this problem, let us make a few qualitative preliminary remarks. Since in this case the characteristic length will be L and the velocity U, the nonlinear terms in Eqs. (1.6) [or, more precisely, in the first of these equations] will be of the order U^2/L in the region of the boundary layer. On the other hand, the longitudinal velocity $u = u_1$ will vary in this region from $u = 0$ on the surface of the plate to a value of the order of U on the outer edge of the boundary layer.

Let us take the thickness of the boundary layer to be δ.[4]

We now obtain that $\frac{\partial^2 u}{\partial z^2}$ is of the order of U/δ^2, and the main term $\nu \frac{\partial^2 u}{\partial z^2}$ describing the friction force will be of the order of $\nu U/\delta^2$. However, within the boundary layer the friction force will have the same order as the inertia force. Hence $U^2/L \sim \nu U/\delta^2$, i.e.,

$$\delta \sim \sqrt{\frac{\nu L}{U}} = \frac{L}{\sqrt{\mathrm{Re}}}, \quad \mathrm{Re} = \frac{UL}{\nu}. \tag{1.32}$$

Thus the thickness of the boundary layer depends on the Reynolds number of the flow. The greater this number (i.e., the smaller the viscosity of the fluid and the greater the free stream velocity), the thinner will be the boundary layer.

To determine the thickness of the boundary layer above a given point of the plate at a distance x from its leading edge, we must replace the total length of the plate L by the length x. Thus

[4]This is, of course, not an absolute concept since the boundary layer does not have a clearly defined edge, and the velocity u only tends asymptotically to U with increasing distance from the plate in the Oz direction. Hence δ is understood to denote the distance from the plate (in the Oz direction) at which the velocity u attains a given, sufficiently great fraction of U, e.g., $0.99U$ (this definition of δ is often used in fluid mechanics). There exist other somewhat less formal definitions of the thickness of the boundary layer which, however, in practice give equivalent results—see the closing remarks of this subsection.

$$\delta \sim \sqrt{\frac{\nu x}{U}} \quad \text{at the point } x, \tag{1.33}$$

i.e., as the distance x from the leading edge increases, the thickness of the boundary layer increases in proportion to $\sqrt{x}$. As far as the shear stress τ on the wall is concerned (τ is equal to the friction force per unit area of the plate), we have $\tau = \mu \frac{\partial u}{\partial z}$ (where $\frac{\partial u}{\partial z}$ must be taken for $z = 0$). Since $\frac{\partial u}{\partial z} \sim \frac{U}{\delta}$, we have $\tau \sim \frac{\mu U}{\delta}$ or, in view of Eq. (1.32)

$$\tau \sim \rho \sqrt{\frac{\nu U^3}{L}}. \tag{1.34}$$

Thus the shear stress here is proportional to 3/2 power of the velocity; the corresponding skin friction coefficient

$$C_f = \frac{\tau}{\rho U^2} \sim \frac{1}{\sqrt{\mathrm{Re}}} \tag{1.35}$$

naturally depends only on the Reynolds number and decreases as this number increases.

We now turn to the deduction of the equations defining the velocity field within the boundary layer. Since in the case under discussion the motion will be strictly two-dimensional (in the plane Oxz) and independent of y, the general equations (1.5) and (1.6) take the form

$$\begin{gathered}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right), \\
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right), \\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,
\end{gathered} \tag{1.36}$$

where $u = u_1$, $w = u_3$. We shall now attempt to determine the order of magnitude of the individual terms of these equations. We recall that within the boundary layer, u is equal to some finite fraction of the velocity U, i.e., it is of the same order as U, while the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial z}$ are respectively of the order U/L, and U/δ. Integrating the third equation of Eqs. (1.36) with respect to z from 0 to some $z = c\delta$, $0<c<1$, constituting a finite part of the total thickness δ of the boundary layer, we obtain

$$w = \int_0^{c\delta} \frac{\partial u}{\partial x} dz \sim \delta \frac{U}{L} = \frac{\delta}{L} U. \tag{1.37}$$

Therefore, it is clear that w is approximately $\frac{L}{\delta} = \sqrt{\mathrm{Re}}$ times less than u. Taking into account Eq. (1.32), we may assert that in the first equation of Eqs. (1.36) the terms $u\frac{\partial u}{\partial x}$, $w\frac{\partial u}{\partial z}$ and $\nu\frac{\partial^2 u}{\partial z^2}$ are all of the order of U^2/L, while the term $\nu\frac{\partial^2 u}{\partial x^2}$ is of much smaller order $\frac{U^2}{L}\frac{1}{\mathrm{Re}}$; hence this latter term may be ignored. Further, it is clear from the second equation of Eqs. (1.36) that the term $\frac{1}{\rho}\frac{\partial p}{\partial z}$ will be of an order not greater than $\frac{U^2\delta}{L^2} = \frac{U^2}{L}\cdot\frac{\delta}{L}$ (since no term of this second equation is of a higher order); thus with accuracy to terms of relative order $\frac{\delta}{L}$ we may ignore transverse variations of pressure and assume that $\frac{\partial p}{\partial z} = 0$, i.e., $p = p(x)$. However, in this case the pressure $p(x)$ within the boundary layer may be replaced by the value on the upper edge. Hence the term $-\frac{1}{\rho}\frac{\partial p}{\partial x}$ in the first equation of Eqs. (1.36) may be determined independently from the equations of an ideal fluid which describe the flow outside the boundary layer; putting $\nu = 0$ in the first of Eqs. (1.6) we find that $-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}$ or (in the steady case) $-\frac{1}{\rho}\frac{\partial p}{\partial x} = U\frac{\partial U}{\partial x} = \frac{1}{2}\frac{\partial U^2}{\partial x}$. Thus, to find u and w we obtain a system of two equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}, \tag{1.38}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{1.39}$$

This system, together with the boundary conditions

$$u = w = 0, \text{ for } z = 0,\ 0 \leqslant x \leqslant L;\ u \to U(x,\ t) \text{ as } z \to \infty \tag{1.40}$$

permits the functions $u(x, z, t)$ and $w(x, z, t)$ to be determined uniquely from the initial values of one of these functions.

Equations (1.38) and (1.39) make up the well-known *boundary layer* equations which were first obtained by Prandtl in 1904. Later, both Prandtl and other authors proposed several different derivations of equations. In particular, it was established that Prandtl's equations are also correct for the two-dimensional flow past a curved surface (if the curvature is not too great) and that they may be obtained formally from the general equations of fluid mechanics as a first approximation by the expansion of all the terms in series in powers of $\frac{1}{\mathrm{Re}}$ [see Kochin, Kibel', Roze (1964), part 2, Chapt. II, Sect. 29; Goldstein (1938), Vol. 1, Chapt. IV; see also the special monographs by Loitsyanskiy (1941; 1962b) and Schlichting (1960)]. In general, z must be taken as the coordinate reckoned along the normal to the surface and x as the longitudinal coordinate in the tangent plane.

Let us now consider in greater detail the simplest case of a flat plate washed by a steady flow with constant velocity U in the Ox direction. In this case, the terms $\frac{\partial u}{\partial t}$ and $\frac{1}{\rho}\frac{\partial p}{\partial x}$ in Eq. (1.38) disappear, so that this equation takes the particularly simple form

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2}. \tag{1.41}$$

We assume that the length L is infinite. Since it is natural to think that in the flow considered the velocities in every section $x = \text{const}$ of the boundary layer will not depend greatly on the velocities in the next sections downstream, the assumption of infinite length will have no great effect on the solution anywhere except in the narrow neighborhood of the trailing edge and in the region $x \geqslant L$ lying behind the plate, which we shall not deal with here. With $L = \infty$ the boundary conditions (1.40) become

$$u = w = 0 \text{ for } z = 0,\quad x \geqslant 0;\quad u \to U \text{ as } z \to \infty. \tag{1.42}$$

Thus we only have to find a solution of Eqs. (1.39) and (1.41) which satisfies the conditions (1.42).

To eliminate ν and U from the equations, we turn to the dimensionless values

$$x_1 = \frac{x}{L}, \quad z_1 = \frac{z\sqrt{\mathrm{Re}}}{L} = z\sqrt{\frac{U}{\nu L}}, \qquad u_1 = \frac{u}{U}, \quad w_1 = \frac{w\sqrt{\mathrm{Re}}}{U} = w\sqrt{\frac{L}{\nu U}}, \tag{1.43}$$

defined in accordance with Eqs. (1.32) and (1.37); the length L in Eq. (1.43) can now be chosen as desired (since the conditions of the problem do not contain any characteristic lengths). Substituting Eqs. (1.43) into Eqs. (1.41), (1.39), and (1.42), we obtain

$$u_1\frac{\partial u_1}{\partial x_1} + w_1\frac{\partial u_1}{\partial z_1} = \frac{\partial^2 u_1}{\partial z_1^2}, \quad \frac{\partial u_1}{\partial x_1} + \frac{\partial w_1}{\partial z_1} = 0, \tag{1.44}$$

$$u_1 = w_1 = 0 \text{ for } z_1 = 0,\ x_1 \geqslant 0;\ u_1 \to 1 \text{ as } z_1 \to \infty. \tag{1.45}$$

However, since the length L may be chosen arbitrarily, only a solution independent of L has any meaning. Thus it is clear that $u_1 = \frac{u}{U}$ can depend only on the combination $\eta = \frac{z_1}{\sqrt{x_1}} = z\sqrt{\frac{U}{\nu x}}$ of the variables z_1 and x_1 which does not contain L. The function $\sqrt{x_1}\,w_1 = w\sqrt{\frac{x}{\nu U}}$, which is also expressed in terms of w, x, ν and U only, must depend on the same combination. Thus we see that

$$u = Uf\left(\sqrt{\frac{U}{\nu}}\frac{z}{\sqrt{x}}\right), \quad w = \sqrt{\frac{U\nu}{x}}f_1\left(\sqrt{\frac{U}{\nu}}\frac{z}{\sqrt{x}}\right) \tag{1.46}$$

where f and f_1 are two universal functions of a single variable (independent of U and ν). From Eq. (1.46), in particular, it follows that the velocity profiles for u and w above all points of any plate washed by a flow at constant velocity must be similar to each other. If we put

$$f(\eta) = \varphi'(\eta), \tag{1.47}$$

Eq. (1.39) is transformed into the form $f_1(\eta) = \frac{1}{2}(\eta\varphi' - \varphi)$; then Eqs. (1.41) and (1.42) are easily reduced to the form:

$$\varphi\varphi'' + 2\varphi''' = 0; \quad \varphi(0) = \varphi'(0) = 0, \quad \varphi'(\infty) = 1. \tag{1.48}$$

Thus our system of equations reduces to a single nonlinear, third-order, ordinary differential equation with three boundary conditions. This equation may be integrated numerically, for example, by resolving the function $\varphi(\eta)$ as a power series close to $\eta = 0$, and as an asymptotic series as $\eta \to \infty$ (this method of solution was used in 1908 by Blasius, who was the first to investigate the problem); later, other numerical methods were applied to Eq. (1.48) by Töpfer, Barstow, Goldstein, Howarth and others [see the references in Goldstein (1938), Schlichting (1960), Longwell (1966) and Loytsyanskiy (1941, 1962b)]. The profiles obtained in these works for the longitudinal and vertical velocities in a boundary layer above a plate are given in Figs. 2 and 3. The thickness δ of the boundary layer (defined as the value of z for which $u(z) = 0.99U$) is given by

$$\delta \approx 5\sqrt{\frac{\nu x}{U}} \tag{1.49}$$

[cf. Eq. (1.33)]. The shear stress at a point of the plate distant x from the leading edge is given by the equation

$$\tau(x) = \mu\left(\frac{\partial u}{\partial z}\right)_{z=0} = \mu\sqrt{\frac{U^3}{\nu x}}\,\varphi''(0);$$

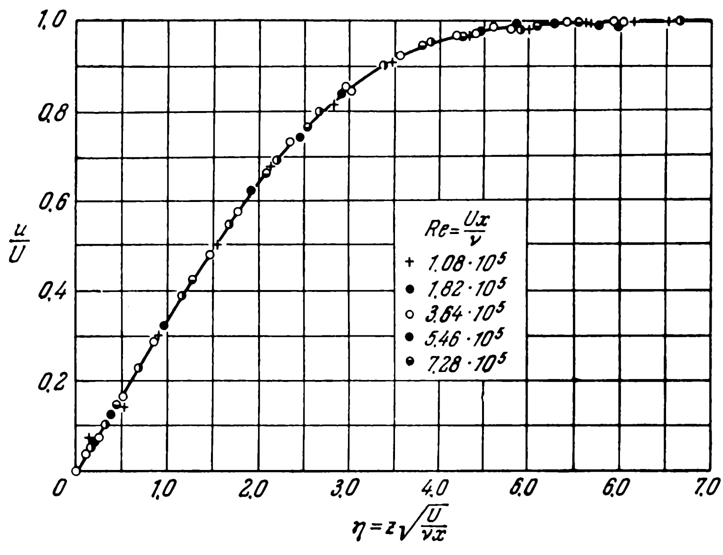


FIG. 2. Longitudinal velocity profile in the boundary layer on a flat plate. Experimental data according to Nikuradse's measurements [see Schlichting (1960)].

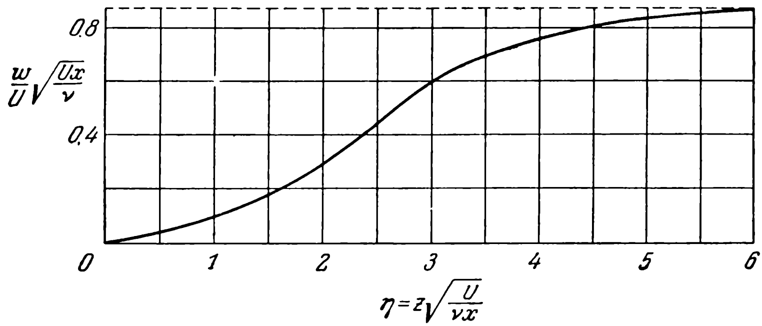


FIG. 3. Vertical velocity profile in the boundary layer on a flat plate.

since the numerical integration shows that $\varphi''(0) = 0.332$, then consequently

$$\tau(x) = 0.332\rho\sqrt{\frac{\nu U^3}{x}}, \quad c_f(x) = \frac{\tau(x)}{\rho U^2} = \frac{0.332}{\sqrt{\mathrm{Re}_x}}, \quad \mathrm{Re}_x = \frac{Ux}{\nu}. \tag{1.50}$$

For a plate of length L and wetted on both sides, the total friction force (per unit breadth of the plate) will equal

$$F = 2\int_0^L \tau(x)\,dx = 1.328\,\rho\sqrt{\nu U^3 L}, \tag{1.51}$$

i.e., it will be proportional to the 3/2 power of the velocity and the square root of the length of the plate (the comparatively slow increase of F with increase of L is explained quite naturally by the fact that the stress $\tau(x)$ decreases with increase of the distance from the leading edge due to the increase of thickness of the boundary layer). Instead of F, we normally use the dimensionless skin friction coefficient C_f, which for a flat plate of length L and breadth B is determined from the equation

$$C_f = \frac{FB}{\rho U^2 LB} = \frac{1.328}{\sqrt{\mathrm{Re}}}, \quad \mathrm{Re} = \frac{UL}{\nu}. \tag{1.52}$$

The experimental verification of the theory of flow past a flat plate was carried out, in particular, by I. M. Burgers and B. G. Van der Hegge Zijnen in 1924, M. Hansen in 1928 and J. Nikuradse in 1942 [for references to the original works see, e.g., Schlichting (1960)]. It was established that if the Reynolds number $\mathrm{Re} = \frac{UL}{\nu}$ is sufficiently large, the observed values of the skin friction coefficient are described with sufficient accuracy by Eq. (1.52). In exactly the same way, the measured values of the thickness of the boundary layer δ and the form of the longitudinal velocity profile $u(z)$ within this layer for values of $\mathrm{Re}_x = \frac{Ux}{\nu}$ that are neither too large nor too small (i.e., everywhere except in a narrow strip near the leading edge of the plate where the condition $\left|\frac{\partial^2 u}{\partial x^2}\right| \ll \left|\frac{\partial^2 u}{\partial z^2}\right|$ is not fulfilled, and also, perhaps the traihng edge of the plate in the case when $\frac{UL}{\nu}$ is very great) prove to be in extremely close agreement with the theoretical predictions. The slight divergence of theory and experiment observed in early measurements may be explained entirely by the effect of the finite thickness of the plate and the form of the sharpening on the leading edge, and also by the presence of fairly

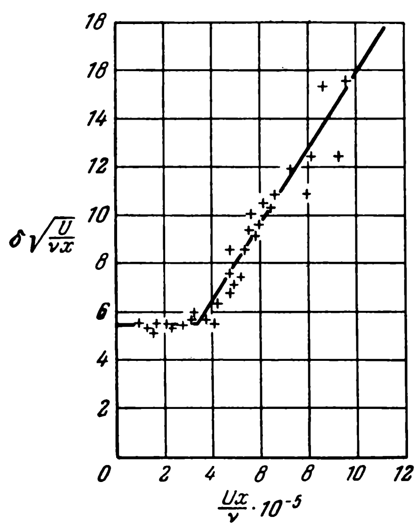


FIG. 4. Dependence of the boundary-layer thickness δ on a flat plate on the distance x from the leading edge of the plate [according to the data of Hansen (1928)].

small pressure gradients in the flow; these phenomena were specially considered in Nikuradse's experiments, and he was able to obtain an almost ideal agreement with the calculated profile of $u(z)$ [see Fig. 2 in which Nikuradse's data are plotted]. However, for large values of the velocity U and length L, beginning from some value z for which Re_x is of the order 3×10^5–3×10^6, all the laws deduced above are greatly distorted [see, for example, Fig. 4, where results of Hansen's measurements of the thickness of a boundary layer are given. These results show that beginning at about $\mathrm{Re}_x = 3.2 \times 10^5$, this thickness begins to increase far more rapidly than would be expected from Eq. (1.49)]. The causes producing these distortions will be discussed in detail in Sect. 2.

So far, we have defined the thickness of the boundary layer on a purely conditional basis as the value of z for which the longitudinal velocity $u(z)$ attains a value equal to some predetermined fraction of U sufficiently close to unity (for example, equal to 99% of U). We were forced to adopt this somewhat vague definition because $u(z)$ approaches U only asymptotically as $z \to \infty$, and nowhere attains a terminal value; strictly speaking, we should regard the boundary layer as not having a sharp boundary. Nevertheless, the transverse dimension of the boundary layer may be described by quantities which have definite physical meaning. One such quantity is the *displacement thickness* δ^*, defined by the equation

$$\delta^* = \int_0^\infty \left[1 - \frac{u(z)}{U}\right] dz. \tag{1.53}$$

The meaning of this quantity is that δ^* is equal to the distance through which the flow is displaced outwards due to the frictional reduction of the longitudinal velocity in the boundary layer. Let us in fact consider a streamline at a distance z_0 from the plate at its leading edge (see Fig. 5). Since the flux of fluid through every section of the stream tube contained between this streamline and the surface of the plate must be the same, while the longitudinal velocity u decreases downstream due to friction on the plate, the section of the stream tube must increase with increase of x. At a distance x from the leading edge of the plate, the streamline is a distance z_1 from the plate, i.e., it has been displaced through a distance $z_1 - z_0$, which may be determined from the condition of constant flux of the fluid through the initial section of the stream tube and through the section with longitudinal

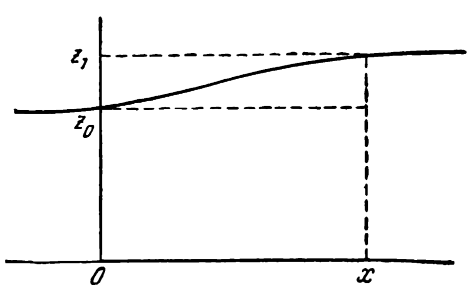


FIG. 5. Schematic form of a streamline in the boundary layer on a flat plate.

coordinate x:

$$Uz_0 = \int_0^{z_1} u(x, z)\, dz,$$

where

$$z_1 - z_0 = \int_0^{z_1} \left(1 - \frac{u}{U}\right) dz. \tag{1.54}$$

As $z_1 \to \infty$ the difference $z_1 - z_0$ tends precisely to the displacement thickness δ^*. For the case of steady flow past a flat plate

$$\delta^* = \sqrt{\frac{\nu x}{U}} \int_0^{\infty} [1 - \varphi'(\eta)]\, d\eta = \lim_{\eta \to \infty} [\eta - \varphi(\eta)] \cdot \sqrt{\frac{\nu x}{U}}, \tag{1.55}$$

where we may obtain the equation

$$\delta^* \approx 1.73 \sqrt{\frac{\nu x}{U}}. \tag{1.56}$$

We note in this case that our previous calculation of the thickness of the boundary layer δ (defined according to the relationship $u(\delta) = 0.99\, U$) gives a value almost three times the displacement thickness δ^*. Due to the "displacement" of the streamlines, the vertical component of velocity w does not tend to zero as $z \to \infty$ (see Fig. 3) but approaches some asymptotic value, which proves to be equal to

$$w(\infty) = 0.895\, U \sqrt{\frac{\nu}{Ux}}. \tag{1.57}$$

Thus the displacement shows that the flow is distorted even outside the boundary layer (however the thickness of the layer is defined), i.e., the boundary layer has an effect on the outer flow which sometimes must be taken into account in calculations.

Another quantity that characterizes the transverse dimension of a boundary layer is the *momentum thickness*, defined as

$$\delta^{**} = \int_0^{\infty} \frac{u}{U}\left(1 - \frac{u}{U}\right) dz. \tag{1.58}$$

The term momentum thickness derives from the fact that the momentum transmitted by the friction forces per unit time to an element of the plate of unit breadth and extending from the leading edge $x=0$ to a section of the plate at a given x (i.e., the momentum lost per unit time by the column of fluid lying above this part of the plate) may be shown to be equal to $\rho U^2 \delta^{**}$. That is, it is characterized by the length δ^{**}. In the case of a steady flow of constant velocity U past a plate, this length will be [from Eqs. (1.46), (1.47), and (1.48)]:

$$\delta^{**}=\sqrt{\frac{\nu x}{U}}\int_0^\infty \varphi'(1-\varphi')\,d\eta=2\varphi''(0)\sqrt{\frac{\nu x}{U}}=0.664\sqrt{\frac{\nu x}{U}}; \qquad (1.59)$$

therefore, δ^{**} here is almost three times smaller than δ^{*}.

We may note further that boundary layer theory also permits us to explain why Poiseuille flow which represents an exact solution of the equations of fluid mechanics occurs only at distances of the order of $\frac{R^2U}{\nu}$ from the intake of a tube (see subsection 1.3, Example 4). Close to the mouth of the tube the retarding action of the walls is exerted only within the thin boundary layer, while in the central part of the tube, the fluid will move with constant velocity, experiencing no viscous effects. Proceeding further into the tube, the thickness of the boundary layer on the walls of the tube will increase in proportion to $\sqrt{\frac{\nu x}{U}}$, where x is the distance from the mouth. Finally, at a distance $x\sim\frac{R^2U}{\nu}$ from the mouth, the boundary layer fills the whole cross section of the tube, and viscous effects become evident through the whole flow. Only then, of course, can the theoretical parabolic velocity profile given by Eq. (1.23) be established [cf. Schiller (1932; 1934) or Goldstein (1938), Vol. 1, Chap. 7, Sect. 139].

1.5 General Equation of the Heat Budget and the Thermal Conduction Equation; Forced and Free Convection

In what follows, we shall be concerned mainly with flows of an incompressible fluid which are described by Navier-Stokes equations (1.6) and the continuity equation (1.5). However, since Chapt. 4 will be devoted to the more general case of a thermally inhomogeneous fluid, and the second part of the book briefly considers turbulence in a compressible medium, it will be beneficial at this point to discuss the general system of equations of mechanics of a compressible fluid. The results of this discussion allow us to write down the equation for the temperature of an incompressible thermally inhomogeneous fluid, which will be important in the subsequent analysis.

In a compressible medium the continuity equation (i.e., the balance of mass) and the dynamical equations (i.e., the balance of the three components of momentum) take the forms (1.2) and (1.4), respectively. Since these four equations contain five unknowns, to obtain a closed system we must add to them a fifth equation—the equation of the budget of heat, which expresses the physical law of conservation of energy. In the most general form, this equation is

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2}+\rho e\right)=-\frac{\partial}{\partial x_\alpha}\left[\rho u_\alpha\left(\frac{u^2}{2}+w\right)-u_\beta\sigma_{\beta\alpha}-\varkappa\frac{\partial T}{\partial x_\alpha}\right]+u_\alpha\rho X_\alpha, \qquad (1.60)$$

where e is the internal energy of unit mass of fluid [so that the sum $\rho(u^2/2+e)$ gives the total energy per unit volume of moving fluid], $w=e+\frac{p}{\rho}$ is known as the enthalpy,[5] $\varkappa$ is the coefficient of thermal conductivity and T is the temperature. Further, in Eq. (1.60),

$$\sigma_{ij}=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\frac{\partial u_\alpha}{\partial x_\alpha}\delta_{ij}\right)+\zeta\frac{\partial u_\alpha}{\partial x_\alpha}\delta_{ij} \tag{1.61}$$

is the viscous stress tensor which appears under the derivative $\frac{\partial}{\partial x_\alpha}$ on the right side of the momentum equation (1.3) [see, for example, Landau and Lifshitz (1963), Sect. 49]. From the energy equation (1.60) we may also deduce the equation for the entropy budget; in fact, using the thermodynamic equations expressing de and dw in terms of dp, $d\rho$, and ds, where s is the entropy of unit mass of fluid, and also taking into account Eqs. (1.2) and (1.4), we may transform Eq. (1.60) into the form

$$\rho T\left(\frac{\partial s}{\partial t}+u_\alpha\frac{\partial s}{\partial x_\alpha}\right)=\sigma_{\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta}+\frac{\partial}{\partial x_\alpha}\left(\varkappa\frac{\partial T}{\partial x_\alpha}\right) \tag{1.62}$$

[see Landau and Lifshitz (1963)]. Thus, when Eqs. (1.2) and (1.4) hold, Eqs. (1.60) and (1.62) are equivalent to each other, and either may be used equally well for deriving a closed system of equations for flows of a compressible fluid.

To obtain a closed system, it is necessary only to express the thermodynamic quantities e and w or s in terms of the pressure p and the density ρ (or temperature T) with the aid of the general equations of thermodynamics and the equations of state of the medium under consideration (connecting p, ρ, and T). We shall confine ourselves here to the simplified case when the equation of

[5] We should note at this point that the use here and henceforth of thermodynamic quantities and relationships requires special reservations, since a moving fluid with nonzero velocity and temperature gradients does not constitute a system in thermodynamic equilibrium. It may be shown, however, that for the moderate gradients encountered in real fluid flows, the fundamental thermodynamic quantities may be defined in such a way that they satisfy the ordinary equations of thermodynamics of media in equilibrium. [See, e.g., Landau and Lifshitz (1963), Sect. 49, and also Tolman and Fine (1948), and special manuals on the kinetic theory of gases—Chapman and Cowling (1952), Hirschfelder, Curtiss and Bird (1954)].

state is that for an ideal gas, i.e., of the form

$$p = R\rho T \tag{1.63}$$

where T is the temperature in degrees Kelvin, and the constant R is equal to the difference between the specific heats of the medium at constant pressure c_p and constant volume c_v:

$$R = c_p - c_v. \tag{1.64}$$

Moreover, in accordance with the deductions of the kinetic theory of gases and with the experimental data for real fluids, we may assume that the specific heats c_p and c_v are individually constant (i.e., independent of temperature). In this case it is not difficult to show that $e = c_v T + e_0$, $w = e + \frac{p}{\rho} = c_p T + e_0$ and $s = -R \ln \rho + c_v \ln T + \text{const} = -R \ln p + c_p \ln T + \text{const}$ [see, for example, Landau and Lifshitz (1958), Sect. 43]. Substituting these expressions for e and w into Eq. (1.60) and applying Eqs. (1.2) and (1.4), without difficulty we may transform the general equation of the heat budget into the form:

$$c_v\rho\left(\frac{\partial T}{\partial t} + u_\alpha \frac{\partial T}{\partial x_\alpha}\right) = -p\frac{\partial u_\alpha}{\partial x_\alpha} + \frac{\partial}{\partial x_\alpha}\left(\varkappa \frac{\partial T}{\partial x_\alpha}\right) + \rho\varepsilon, \tag{1.65}$$

or, which is equivalent, into the form

$$c_p\rho\left(\frac{\partial T}{\partial t} + u_\alpha \frac{\partial T}{\partial x_\alpha}\right) = \frac{\partial p}{\partial t} + u_\alpha \frac{\partial p}{\partial x_\alpha} + \frac{\partial}{\partial x_\alpha}\left(\varkappa \frac{\partial T}{\partial x_\alpha}\right) + \rho\varepsilon, \tag{1.65'}$$

where

$$\rho\varepsilon = \sigma'_{\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} = \mu \frac{\partial u_\alpha}{\partial x_\beta}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} - \frac{2}{3}\frac{\partial u_\gamma}{\partial x_\gamma}\delta_{\alpha\beta}\right) + \zeta\delta_{\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta}\frac{\partial u_\gamma}{\partial x_\gamma} =$$
$$= \frac{\mu}{2}\sum_{\alpha,\beta}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} - \frac{2}{3}\frac{\partial u_\gamma}{\partial x_\gamma}\delta_{\alpha\beta}\right)^2 + \zeta\left(\frac{\partial u_\gamma}{\partial x_\gamma}\right)^2. \tag{1.66}$$

The quantity ε is ρT multiplied by the increase in the entropy s per unit time, caused by the transition of a part of the kinetic energy into heat as a result of internal friction; in other words, ε is the quantity of heat liberated per unit time per unit mass of fluid. With the additional presence of influx of heat due to radiation, chemical reactions, phase transitions or any other causes, a further term ρQ must be added to the right sides of Eqs. (1.60), (1.62), (1.65), and (1.65′), where Q is the additional influx of heat transmitted to unit mass per unit time. Equations (1.2), (1.4), and (1.65) or (1.65′), together with Eqs. (1.63) and (1.66) give a general system of five equations for determining the five unknown functions $u_i(\boldsymbol{x}, t)$, $i = 1, 2, 3$; $p(\boldsymbol{x}, t)$ and $\rho(\boldsymbol{x}, t)$ or $T(\boldsymbol{x}, t)$.

An equation of the type (1.65′) may also be obtained in the case of thermally inhomogeneous liquids (in the usual sense of the word). Here the role of the state equation will be played by the law of thermal expansion of the fluid

$$\rho - \rho_0 = -\beta\rho_0(T - T_0) \tag{1.67}$$

where ρ and ρ_0 are the densities at temperatures T and T_0, respectively, and β is the coefficient of thermal expansion. It may be shown that under these conditions, Eq. (1.62) takes the form

$$\rho c_p\left(\frac{\partial T}{\partial t} + u_\alpha\frac{\partial T}{\partial x_\alpha}\right) = \beta T\left(\frac{\partial p}{\partial t} + u_\alpha\frac{\partial p}{\partial x_\alpha}\right) + \frac{\partial}{\partial x_\alpha}\left(\varkappa\frac{\partial T}{\partial x_\alpha}\right) + \rho\varepsilon \tag{1.68}$$

while in the expression for ε in the case of a liquid we may assume that $\frac{\partial u_\gamma}{\partial x_\gamma} = 0$, so that

$$\varepsilon = \frac{1}{2}\nu\sum_{\alpha,\beta}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)^2 \tag{1.69}$$

[see, for example, Howarth (1953), Vol. II, Chapt. 14]. The coefficient β for ordinary liquids is very small (for example, for water $\beta \approx 1.5 \times 10^{-4}$ per degree at 15°C); thus in Eq. (1.69) we may ignore the term containing this coefficient. We then obtain

$$\frac{\partial T}{\partial t} + u_\alpha\frac{\partial T}{\partial x_\alpha} = \chi\,\Delta T + \frac{\varepsilon}{c_p} \tag{1.70}$$

where $\chi = \frac{\varkappa}{\rho c_p}$ is the coefficient of thermal diffusivity (or thermometric conductivity) of the medium, which here and henceforth will be assumed to have a constant value [cf. the transition from Eq.

(1.3) to Eq. (1.4) in Sect. 1.1]. The term $\frac{\varepsilon}{c_p}$ in Eq. (1.70) describes the general heating of the medium caused by the internal friction of the fluid; this heating under real-life conditions generally plays a completely insignificant role and may be entirely ignored. Then Eq. (1.70) simplifies still further and transforms into the ordinary equation of heat conduction in a moving medium

$$\frac{\partial T}{\partial t} + u_\alpha \frac{\partial T}{\partial x_\alpha} = \chi \Delta T. \tag{1.71}$$

We shall make wide use of this equation in subsequent work.

It is most important that Eq. (1.71) be applicable not only to liquids, but also to gases, provided that the velocity of these gases is much less than the corresponding sound velocity $a = \sqrt{\gamma \frac{p}{\rho}}$, $\gamma = \frac{c_p}{c_v}$. It may be shown that at such velocities, variations in pressure will play a considerably smaller role in Eq. (1.65′) than variations in temperature. Therefore terms containing p on the right side may be ignored; then Eq. (1.70) may be obtained quickly or, ignoring also the heating of the medium due to internal friction, Eq. (1.71). We should observe that comparison of Eqs. (1.70) and (1.65) shows that in Eq. (1.65), unlike Eq. (1.65′) it is impossible to ignore the term $-p \frac{\partial u_\alpha}{\partial x_\alpha}$; the local compressions and expansion caused by variations of density during heating and cooling, must be taken into account in the heat equation, even in the case of small velocities. However, in the dynamic equations, for motion at velocities much less than the sound velocity, the velocity field may be assumed to be completely "incompressible," provided the variations of density produced by the temperature inhomogeneities are small in absolute value, i.e., provided the absolute differences of temperature in the flow are small in comparison with the mean absolute temperature T_0. Under these conditions, it may also be assumed in the expression for ε that $\partial u_\alpha / \partial x_\alpha = 0$, i.e., ε may once again be taken in the form of Eq. (1.69). In the case of an incompressible medium $T ds = de$; thus, in an incompressible fluid ε will be exactly equal to the increase of internal energy per unit mass per unit time, i.e., the amount of kinetic energy dissipated (transformed into heat) per unit time per unit mass of fluid [cf. Landau and Lifshitz (1963), Sect. 16]. Although in the calculation of the temperature field this quantity may generally be ignored, it is nevertheless a very important physical characteristic of the motion; for brevity, we shall refer to it simply as the "rate of energy dissipation."

We must stress once again that Eq. (1.72) is identical in form to the diffusion equation

$$\frac{\partial \vartheta}{\partial t}+u_\alpha \frac{\partial \vartheta}{\partial x_\alpha}=\chi \Delta \vartheta \tag{1.72}$$

which describes the variation in concentration $\vartheta(\boldsymbol{x}, t)$ of some passive admixture in the medium (i.e., an admixture which has no effect on the dynamics of the flow). The only difference is that in the latter case the coefficient χ must be interpreted not as the coefficient of thermal diffusivity, but as the molecular coefficient of diffusion. Since Eqs. (1.71) and (1.72) are identical in form, henceforth, when considering Eq. (1.72), we shall always bear in mind that the quantity ϑ may denote here either the concentration of some passive admixture or the temperature. Taking into account the importance of the temperature variations for many applications, for brevity, we shall always refer to ϑ simply as the temperature; however, it must be remembered that the symbol ϑ may in fact always denote the concentration of some passive admixture, and in those cases when we do not consider the temperature as a passive admixture, we shall denote it by T instead of ϑ.

The passive admixture assumption implies that the velocity field $u_i(\boldsymbol{x}, t)$ may be determined independently of the field ϑ from the ordinary system of equations of fluid mechanics of an incompressible fluid and then substituted into Eq. (1.72). If we take ϑ to be the temperature, then the temperature differences between different points of the fluid must be sufficiently small for the changes which they produce in the physical properties of the fluid to have no effect on the motion. However, at the same time, these temperature differences must be large enough for the heating resulting from internal friction to be negligible in comparison with them. In such cases the temperature inhomogeneities will simply move with the fluid, at the same time being smoothed out under the influence of molecular thermal conductivity; the motion of masses of non-uniformly heated fluid which arises under such conditions is usually called *forced convection*.

An important case of flows, in which the temperature cannot be considered as a passive admixture, is that of a nonuniformly heated fluid in a gravitational field. Flows of this class arise due to Archimedean forces, which cause an upward buoyancy of the warmer volumes of fluid and a downward sinking of the cooler volumes. Such a motion of a temperature-inhomogeneous fluid is called *free convection*. Let us determine the equations of motion in this case. We assume that the velocity of our motion is sufficiently small for the variations in the density produced by variations of pressure (but not of temperature) to be ignored. Therefore, it follows that we may use the ordinary equation of incompressibility (1.5) and the Navier-Stokes equations (1.6), taking into account in Eqs. (1.6), however, the body force $\boldsymbol{X} = -g\boldsymbol{e}_3$ (where $\boldsymbol{e}_3$ is the unit vector of the axis $Ox_3 = Oz$) and remembering that the density ρ depends on the temperature. We now assume that the (absolute) temperature $T(x_1\ x_2,\ x_3,\ t) = T(x,\ y,\ z,\ t)$ may be put in the form $T = T_0 + T_1$, where T_0 is some constant mean value, and T_1 is a small deviation from T_0. It is clear in this case that $\rho = \rho_0 + \rho_1$, where ρ_0 is the constant density corresponding to T_0, and $\rho_1 = \rho - \rho_0$ is defined as

$$\rho_1 = -\beta\rho_0 T_1 \tag{1.73}$$

(the coefficient of thermal expansion $\beta = -\frac{1}{\rho_0}\left(\frac{\partial\rho_0}{\partial T}\right)_p$ for a gas which satisfies Eq. (1.63) will clearly be equal to $1/T_0$). We note that for $T = T_0 =$ const and $\rho = \rho_0 =$ const, the pressure p_0 will not, however, be constant but will decrease with height as the weight of the column of liquid above the point under discussion decreases:

$$p_0 = -\rho_0 g x_3 + \text{const.} \tag{1.74}$$

Putting $\rho = \rho_0 + \rho_1$, with accuracy to small first-order quantities, we have

$$\frac{1}{\rho}\frac{\partial p}{\partial x_3} = \frac{1}{\rho_0}\frac{\partial p_0}{\partial x_3} + \frac{1}{\rho_0}\frac{\partial p_1}{\partial x_3} - \frac{\rho_1}{\rho_0^2}\frac{\partial p_0}{\partial x_3} = -g + \frac{1}{\rho_0}\frac{\partial p_1}{\partial x_3} + g\beta T_1.$$

Thus it is clear that the third Navier-Stokes equation will be written in this case in the form

$$\frac{\partial u_3}{\partial t} + u_\alpha\frac{\partial u_3}{\partial x_\alpha} = -\frac{1}{\rho_0}\frac{\partial p_1}{\partial x_3} + \nu\Delta u_3 - g\beta T_1 \tag{1.75}$$

(the presence in this equation of a term containing T_1 shows immediately that the temperature cannot be considered a passive admixture in this case). The first and second Navier-Stokes equations, however, will have the usual form (1.6) in this case; the term ρ may be replaced from the very beginning by a constant ρ_0, while p is taken to mean the deviation p_1 of the pressure from the mean value of the pressure p_0 and is dependent on $z=x_3$. Finally, the equation for the temperature, as always when the medium may be assumed incompressible, will take the form of the ordinary equations of heat conduction of a moving fluid

$$\frac{\partial T_1}{\partial t}+u_\alpha\frac{\partial T_1}{\partial x_\alpha}=\chi\Delta T_1 \tag{1.76}$$

(as usual, we ignore the term with ε in the equation for the temperature). We thus obtain an approximate system of five equations (1.5), (1.6) with $i = 1, 2$; (1.75) and (1.76) with respect to five unknown functions $u_i(\boldsymbol{x}, t)$, $i = 1, 2, 3$, $p_1(\boldsymbol{x}, t)$ and $T_1(\boldsymbol{x}, t)$. These equations describe the free convection of the fluid; they were first considered by J. Boussinesq as far back as the nineteenth century. Therefore they are called the equations of free convection or Boussinesq equations (or Boussinesq approximation for fluid mechanics equations). A detailed discussion of the approximations which lead to these equations can be found, e.g., in Chandrasekhar (1961) or in Mihaljan (1962).

1.6 Similarity Criteria for a Thermally Inhomogeneous Fluid; The Thermal Boundary Layer

We have already seen that two geometrically similar flows of incompressible fluid will also be mechanically similar provided the Reynolds numbers of the two flows are identical. In the case of a thermally inhomogeneous or a compressible fluid, however, this assertion is not true. For mechanical and thermal similarity to exist between two geometrically similar flows we require that several dimensionless characteristics, "similarity criteria"—which we shall now consider—should be equal simultaneously.

Let us take first the simple case of flows of a nonuniformly heated fluid in which the temperature may be considered as a passive admixture. In this case the flow will be described by the usual equations (1.5)–(1.6) of incompressible fluid mechanics (with constant ρ) to which, however, must be added the equation of heat conduction (1.72) for the temperature. For simplicity, we shall consider only cases of steady motion, i.e., we assume that u_i, p and ϑ are all time-independent. There are two constant coefficients ν and χ in our equations (both having the dimension L^2T^{-1}, where L and T symbohze the dimensions of length and time). Furthermore, the boundary conditions of the problem in the case of geometrical similarity will be described by some length scale L, velocity scale U, and also a temperature difference scale $\vartheta_1 - \vartheta_0$ (for example, a typical temperature difference between the rigid walls and the flow). However, since the temperature is passive in this case, and has no effect on the dynamics, the unit of temperature may be chosen arbitrarily. Thus we must assume that

$$[\vartheta_1 - \vartheta_0] = \Theta,$$

where Θ is a new dimension symbol independent of L and T (for example, a "centigrade degree"). However, in this case, it is possible to formulate only two independent dimensionless combinations of ν, χ, L, U, and $\vartheta_1 - \vartheta_0$; we may take for example, the Reynolds number $\mathrm{Re} = \frac{UL}{\nu}$ and the Prandtl number

$$\mathrm{Pr} = \frac{\nu}{\chi}. \tag{1.77}$$

When ϑ is the concentration of a passive admixture and not the temperature, so that χ is the diffusion coefficient, the ratio ν/χ is sometimes called the *Schmidt number* and is denoted by Sc.

Sometimes instead of Pr it is convenient to use the Peclét number

$$\mathrm{Pe} = \frac{UL}{\chi} = \mathrm{Re} \cdot \mathrm{Pr} \tag{1.78}$$

which determines the order of the ratio of the advective terms $u_\alpha \frac{\partial \vartheta}{\partial x_\alpha}$ and the term $\chi \nabla^2 \vartheta$ in Eq. (1.73), and which plays a role in the investigation of the temperature field analogous to that of the Reynolds number in dynamic problems. However, the Prandtl number has an advantage over the Peclét number in that it depends only on the nature of the moving fluid (i.e., it is a characteristic of the medium) and not on the special features of the motion. For air (and other diatomic gases), $\mathrm{Pr} \approx 0.7$ and is almost independent of the temperature; in general, for most gases, Pr is of the order of unity. For liquids (except liquid metals), the Prandtl number is usually greater than unity and decreases sharply with rise in temperature; for example, for water (at atmospheric pressure), Pr ≈ 7 at 20°C, and Pr ≈ 2 at 80°C. For very viscous technical oils the Pr number is much greater than for water, and may reach values of several hundreds or even thousands. On the other hand, for liquid metals, which are distinguished by their very great thermal diffusivity, the Prandtl number takes values very much less than unity. For example, for mercury, Pr ≈ 0.023 at 20°C; for other molten metals this number often proves to be even smaller (of the order of 10^{-3}). When χ is the coefficient of diffusion of a passive admixture, the Prandtl number, or (what is exactly the same thing) the Schmidt number, also possesses a very different order for gases and for liquids. In gaseous media the coefficients of diffusion and viscosity are of the same order, so that $\nu/\chi \sim 1$. For liquids of the type of water, $\nu/\chi \sim 10^3$, and for very viscous liquids this ratio becomes still larger and may reach values of the order of 10^6 and more.

To obtain similarity of the temperature fields in two geometrically similar flows, both the Reynolds numbers Re and the Prandtl numbers Pr (or Re and Pe) must be equal. In general, for flows that are only geometrically similar, the temperature fields will be determined by

$$\vartheta(\boldsymbol{x}) - \vartheta_0 = (\vartheta_1 - \vartheta_0)\,\varphi\left(\frac{\boldsymbol{x}}{L},\ \mathrm{Re},\ \mathrm{Pr}\right) \tag{1.79}$$

which differs from the similar equation (1.31) for the velocity field in that here the function φ now depends on two dimensionless parameters.

In exactly the same way, if we let q denote the density of the heat flux through the surface of a body immersed in the fluid (i.e., we put $q = -\chi \frac{\partial \vartheta}{\partial n}$, where n is the outward normal to the surface of the body), then

$$q = \frac{\chi(\vartheta_1 - \vartheta_0)}{L}\,\psi\left(\frac{\boldsymbol{x}}{L},\ \mathrm{Re},\ \mathrm{Pr}\right).$$

In other words, at every point on the boundary of the body the dimensionless ratio $\mathrm{Nu} = \frac{qL}{\chi(\vartheta_1 - \vartheta_0)}$ will depend only on the Reynolds and Prandtl numbers. However, if we replace the heat flux at the point by the mean heat flux q_m per unit area of the surface of the body, which characterizes the total heat-exchange between the body and the fluid, then in the equation

$$\mathrm{Nu} = \frac{q_m L}{\chi(\vartheta_1 - \vartheta_0)} = \psi(\mathrm{Re}, \mathrm{Pr}) \tag{1.80}$$

the function ψ will now be a universal function of the two variables, which determine the dependence of the heat-exchange on the physical properties of the fluid and the scales of length, velocity, and temperature difference for the whole set of geometrically similar flows. In heat engineering, the quantity Nu is called the *Nusselt number*. Instead of the Nusselt number we may also use as the characteristic of heat-exchange the *heat-transfer coefficient* (otherwise called the *Stanton number*)

$$c_h = \frac{q}{c_p \rho U(\vartheta_1 - \vartheta_0)} = \frac{\mathrm{Nu}}{\mathrm{Re}\,\mathrm{Pr}} = \frac{\mathrm{Nu}}{\mathrm{Pe}}. \tag{1.80'}$$

Let us now turn to the case of free convection. Here the differential equations of the problem contain three dimensional coefficients ν, χ and $g\beta$ (where for an ideal gas $\beta = \frac{1}{T_0}$); the boundary conditions will now be described by the typical length scale L and the typical temperature difference scale $T_m - T_0$ (note that there will be no typical velocity in this case). It is clear from these quantities that it is possible to form two dimensionless combinations. For example, we may take the Prandtl number and the *Grashof number*

$$\mathrm{Gr} = \frac{g\beta L^3 (T_m - T_0)}{\nu^2}. \tag{1.81}$$

Instead of the Grashof number, we may also use the *Rayleigh number* derived from it:

$$\mathrm{Ra} = \frac{g\beta L^3 (T_m - T_0)}{\nu\chi} = \mathrm{Gr} \cdot \mathrm{Pr}. \tag{1.81'}$$

Thus, in the case of free convection in geometrically similar flows, the velocity and temperature fields will be similar only when the similarity criteria Pr and Gr (or Pr and Ra) are equal; generally speaking, however, in the case of geometric similarity we shall obtain only relationships of the form

$$u_i = \frac{\nu}{L} f_i\left(\frac{x}{L},\ \mathrm{Pr},\ \mathrm{Gr}\right),\quad T_1 = (T_m - T_0)\,\varphi\left(\frac{x}{L},\ \mathrm{Pr},\ \mathrm{Gr}\right),$$
$$q_m = \frac{\chi(T_m - T_0)}{L}\,\psi(\mathrm{Pr},\ \mathrm{Gr}), \tag{1.82}$$

where f_i, $i = 1, 2, 3$, φ and ψ are new universal functions. In general, for flows of an incompressible fluid, described by the complete system of equations (1.2), (1.4), (1.63), (1.65), and (1.66), the number of similarity criteria is even greater; however, we shall not be concerned with this.

For flows with very large Peclét numbers $\mathrm{Pe} = \frac{UL}{\chi}$ (e.g., when the Reynolds number is great and the Prandtl number is less than or of the order of unity), the term $\chi\nabla^2\vartheta$ in the equation of heat conduction will be far smaller than the nonlinear terms $u_\alpha \frac{\partial\vartheta}{\partial x_\alpha}$; thus, to a first approximation, the effect of the molecular thermal conductivity may be ignored. However, just as in the case of the viscous effect in dynamical problems, this effect may be ignored only at a distance from the rigid bodies immersed in the flow. A thin thermal boundary layer may be formed close to the surfaces of such bodies. In this layer the temperature varies rapidly from the temperature of the body ϑ_1, to a temperature close to the temperature ϑ_0 which would be observed in the free flow if no rigid body were present. Moreover, the effect of the thermal conductivity in this layer will be of the same order as the effect of the temperature advection, described by the terms $u_\alpha \frac{\partial\vartheta}{\partial x_\alpha}$. Therefore, we may deduce [in a manner similar to the deduction of Eq. (1.32)] that the thickness δ_1 of the thermal boundary layer will be of the order $\frac{L}{\sqrt{\mathrm{Pe}}} = \frac{L}{\sqrt{\mathrm{Re}\cdot\mathrm{Pr}}}$. In other words, it will also be inversely proportional to $\sqrt{\mathrm{Re}}$ and for Prandtl numbers of the order of unity it will be of the same order as δ. Taking into account that $q = -\chi\frac{\partial\vartheta}{\partial n} \sim \chi\frac{\vartheta_1 - \vartheta_0}{\delta_1}$, we easily find that for large Reynolds numbers

$$\mathrm{Nu} = \frac{qL}{\chi(\vartheta_1 - \vartheta_0)} \sim \sqrt{\mathrm{Re}}.$$

Within the thermal boundary layer the heat-conduction equation may be simplified in a manner analogous to the simplification of the equation of motion within the ordinary boundary layer. In particular, in the case of a steady flow of fluid at a given temperature ϑ_0 past a flat plate maintained at some other temperature ϑ_1, the heat-conduction equation within the thermal boundary layer will be written in the form

$$u\frac{\partial\vartheta}{\partial x} + w\frac{\partial\vartheta}{\partial z} = \chi\frac{\partial^2\vartheta}{\partial z^2} \tag{1.83}$$

which is analogous to Eq. (1.41). For the boundary conditions, it is clear that in this case they will take the form $\vartheta = \vartheta_1$ for $z = 0$ and $\vartheta \to \vartheta_0$ as $z \to \infty$. Putting $\vartheta(x, z) = \vartheta_1 - (\vartheta_1 - \vartheta_0)\,\Theta\left(\sqrt{\frac{U}{\nu}}\frac{z}{\sqrt{x}}\right)$ and substituting this equation together with Eqs. (1.46) and (1.47) into Eq. (1.83) we may easily obtain

$$\Theta'' + \frac{\mathrm{Pr}}{2}\varphi\Theta' = 0;\quad \Theta(0) = 0,\quad \Theta(\infty) = 1 \tag{1.84}$$

where φ is a solution of Eq. (1.48). Thus, a simple linear equation is obtained for $\Theta(\eta)$. The explicit solution of this was first given by Pohlhausen in 1921 [see Goldstein (1938), Vol. 2, Sect. 268; Howarth (1953), Vol. 2, Chapt. 14, Sect. 13; Schlichting (1960), Chapt. 14, Sect. 7, Longwell (1966), Chapt. 2, Sect. 7-15]. . In particular, when Pr = 1 comparison of Eqs. (1.84) and (1.48) shows that the solution of $\Theta(\eta)$ has the form $\Theta(\eta) = \varphi'(\eta)$. Thus in this case the temperature profile proves to be exactly similar to the velocity profile. However, for other values of Pr, the function $\Theta(\eta)$ may easily be found numerically from the function $\varphi'(\eta)$ known from Fig. 2 (see Fig. 6).

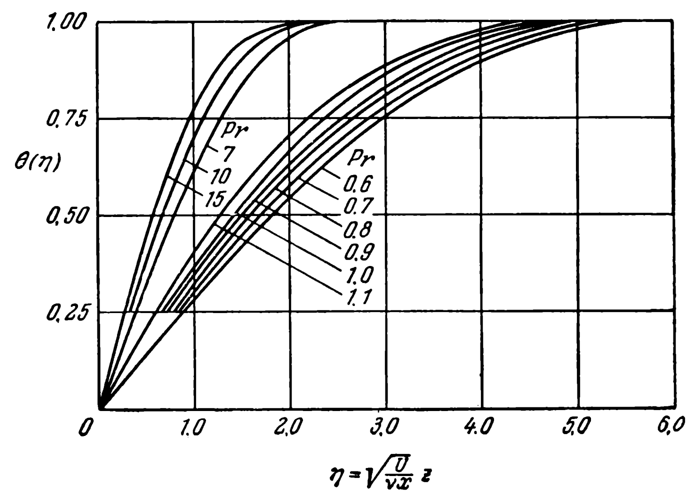


FIG. 6. Temperature profiles in the boundary layer on a flat plate for different Prandtl numbers.

1.7 Small Oscillations of a Compressible Fluid

Flows of compressible fluid described by the general system of equations (1.2), (1.4), (1.63), (1.65), and (1.66) generally are very complex, and their theoretical study involves great difficulties. We shall restrict ourselves at this stage to the simple case of small oscillations about a position of rest (or motion with constant velocity) for which the method of linearization of the equations may be used. Carrier and Carlson (1946), Yaglom (1949) and Kovásznay (1953) have shown that in this case all possible motions may be divided into oscillations of three types, with considerable differences in character. This division will play an important role in the discussion of isotropic turbulence in a compressible gas and the processes of wave propagation in a turbulent medium in Volume 2 of this book.

Let us consider a stationary mass of ideal gas of constant density ρ_0 and constant pressure p_0 (for a gas moving at constant velocity, u may, of course, be reduced to the case of a stationary gas by transforming to a new inertial system of coordinates). We now assume that at the initial instant small disturbances arise in the gas, characterized by small variations of the hydrodynamic quantities $u_l(x, t)$, $\rho'(x, t) = \rho(x, t) - \rho_0$ and $p'(x, t) = p(x, t) - p_0$, where $\frac{|\rho'|}{\rho_0} \ll 1$, $\frac{|p'|}{p_0} \ll 1$ and $\frac{|u|}{a_0} \ll 1$ (here $a_0 = \sqrt{\gamma \frac{p_0}{\rho_0}}$ is the sound velocity in the undisturbed medium, $\gamma = \frac{c_p}{c_v}$). In this case, the time-variation of the fields $u_l(x, t)$, $\rho'(x, t)$ and $p'(x, t)$ in the first approximation may be determined from the system of equations (1.2), (1.4), (1.63), (1.65), and (1.66), linearized with respect to the velocity components u_i, and with respect to the deviations of pressure, density and temperature from the corresponding undisturbed values p_0, ρ_0 and $T_0 = \frac{p_0}{R\rho_0}$. Then the equation of state (1.63) permits us to eliminate immediately temperature from the unknowns and to obtain a closed system of five equations with five unknowns. As unknowns, we may take, for example, the three components of velocity and some two single-valued functions of pressure and density. According to Kovásznay (1953) and Chu and Kovásznay (1958), we may take as these two functions the dimensionless pressure $P = \frac{p}{\gamma p_0}$ and the entropy divided by c_p: $S = \frac{s}{c_p} = \log \frac{p^{1/\gamma}}{\rho} + \text{const}$. Then to within small quantities of higher order:

$$\frac{\rho - \rho_0}{\rho_0} = (P - P_0) - (S - S_0), \quad \frac{T - T_0}{T_0} = (\gamma - 1)(P - P_0) + (S - S_0).$$

The linearized system of hydrodynamic equations in the variables u_i, P and S will take the form:

$$\frac{\partial P}{\partial t} - \frac{\partial S}{\partial t} + \frac{\partial u_\alpha}{\partial x_\alpha} = 0, \tag{1.85}$$

$$\frac{\partial u_l}{\partial t} + a_0^2 \frac{\partial P}{\partial x_l} - \nu \nabla^2{}_l - \left(\frac{\nu}{3} + \eta\right) \frac{\partial^2 u_\alpha}{\partial x_l \partial x_\alpha} = 0, \quad \eta = \frac{\zeta}{\rho_0}, \quad i = 1, 2, 3, \tag{1.86}$$

$$\frac{\partial S}{\partial t} - \chi \nabla^2 S - (\gamma - 1) \chi \nabla^2 P = 0. \tag{1.87}$$

Transforming Eqs. (1.85) and (1.86) by introducing as unknowns instead of the components of the velocity field u_i, the components of the vorticity $\omega_k = \varepsilon_{k\alpha\beta} \frac{\partial u_\beta}{\partial x_\alpha}$ and of the divergence $D = \frac{\partial u_\alpha}{\partial x_\alpha}$, from Eq. (1.86) we have

$$\frac{\partial \omega_k}{\partial t} - \nu \nabla^2 \omega_k = 0, \quad k = 1, 2, 3, \tag{1.88}$$

$$\frac{\partial D}{\partial t} - \nu_1 \nabla^2 D + a_0^2 \nabla^2 P = 0, \quad \nu_1 = \frac{4\nu}{3} + \eta, \tag{1.89}$$

while Eq. (1.85), taking Eq. (1.87) into account, becomes

$$\frac{\partial P}{\partial t} - (\gamma - 1) \chi \nabla^2 P - \chi \nabla^2 S + D = 0. \tag{1.90}$$

First, we observe that the vorticity components ω_k occur only in Eqs. (1.88) which agree with the linearized equations for a vorticity field in an incompressible fluid. In this connection we recall that in an incompressible fluid we may formulate the velocity field u_i uniquely in terms of the vorticity field ω_k and the corresponding boundary conditions; in a compressible medium, on the other hand, the velocity field may be presented in the form of a sum of incompressible (solenoidal) and irrotational (potential) components, the latter of which is independent of the vorticity field. Thus for motion which represents only a weak perturbation of the state of rest, the system of hydrodynamic equations in the first approximation divides into a closed system of equations in the components of the vorticity ω_k, describing the incompressible flow, and a system of equations in D, P and S, describing the irrotational potential flow. In this approximation, the fluctuations of pressure and entropy are connected only with the compressible irrotational flow, i.e., they are absent in the incompressible (vorticity) component of flow. In higher approximations in the theory of small disturbances, these two components will interact with each other, thus causing additional changes in pressure and entropy (we shall discuss this briefly at the end of this subsection).

In order to investigate in detail the types of motion described by the linearized system (1.87)–(1.90) and, in particular, to show that the equations in D, P and S actually describe two different types of oscillations, we use the ordinary Fourier methods. For this purpose we assume that all the fields $\omega_j(\boldsymbol{x}, t)$, $D(\boldsymbol{x}, t)$, $P(\boldsymbol{x}, t)$, and $S(\boldsymbol{x}, t)$ are periodic functions of the space coordinates with a given wave-number vector $\boldsymbol{k}$. Moreover, we replace the ordinary time t by the dimensionless time $\tau = a_0 k t$, where $k = |\boldsymbol{k}|$, and replace the divergence by the dimensionless divergence $\dfrac{D}{a_0 k}$:

$$\omega_j(\boldsymbol{x}, t) = \omega_j(\tau)\, e^{i\boldsymbol{k}\boldsymbol{x}}, \qquad \frac{D(\boldsymbol{x}, t)}{a_0 k} = D(\tau)\, e^{i\boldsymbol{k}\boldsymbol{x}},$$

$$P(\boldsymbol{x}, t) = P(\tau)\, e^{i\boldsymbol{k}\boldsymbol{x}}, \qquad S(\boldsymbol{x}, t) = S(\tau)\, e^{i\boldsymbol{k}\boldsymbol{x}}.$$

Then Eqs. (1.88) take the form

$$\frac{d\omega_j(\tau)}{d\tau} = -\frac{\nu k}{a_0}\omega_j, \quad j = 1, 2, 3, \tag{1.91}$$

and the remaining equations are transformed into the following system of ordinary linear equations:

$$\begin{aligned} \frac{dD(\tau)}{d\tau} &= -\frac{\nu_1 k}{a_0} D + P; \\ \frac{dP(\tau)}{d\tau} &= -D - (\gamma - 1)\frac{\chi k}{a_0} P - \frac{\chi k}{a_0} S; \\ \frac{dS(\tau)}{d\tau} &= -(\gamma - 1)\frac{\chi k}{a_0} P - \frac{\chi k}{a_0} S. \end{aligned} \tag{1.92}$$

Therefore it follows that the amplitudes $\omega_j(t)$ of the components of the vorticity field will be damped with time, according to the law

$$\omega_j(t) = \omega_j(0)\, e^{-\nu k^2 t}, \tag{1.93}$$

while the amplitudes $D(t)$, $P(t)$, and $S(t)$ of the velocity divergence, pressure and entropy fields will be of the form

$$A_1 e^{\lambda_1 a_0 kt} + A_2 e^{\lambda_2 a_0 kt} + A_3 e^{\lambda_3 a_0 kt},$$

where λ_1, λ_2 and λ_3 are three roots of the characteristic equation of the system (1.92):

$$\lambda^3 + \frac{(\nu_1 + \gamma\chi)\, k}{a_0}\lambda^2 + \left(1 + \frac{\gamma\nu_1\chi k^2}{a_0^2}\right)\lambda + \frac{\chi k}{a_0} = 0. \tag{1.94}$$

A further simplification of the results may be obtained if we observe that Eqs. (1.91)–(1.94) contain dimensionless parameters of very different orders of magnitude. In fact, all the coefficients of the system (1.92) and Eq. (1.94) may be expressed in terms of the following three dimensionless parameters:

$$\delta_1 = \frac{\nu_1 k}{a_0}, \qquad \pi_1 = \frac{\nu_1}{\chi}, \qquad \gamma = \frac{c_p}{c_v}; \tag{1.95}$$

however, in Eqs. (1.91), the dimensionless parameter $\delta = \frac{\nu k}{a_0}$ appears, which differs from δ_1 only in the change of the viscosity coefficient $\nu_1 = \frac{4}{3}\nu + \eta = \frac{4}{3}\frac{\mu}{\rho_0} + \frac{\zeta}{\rho_0}$ which defines the internal friction in a potential flow, from the ordinary kinematic viscosity ν. But π_1 is analogous to the ordinary Prandtl number $\mathrm{Pr} = \frac{\nu}{\chi}$ and is of the same order of magnitude; thus in a gaseous medium $\pi_1 \sim 1$; just as $\gamma \sim 1$ (for air $\gamma \approx 1.4$). At the same time $\delta = \frac{\nu k}{a_0}$ is of the same order as the ratio $\frac{l}{\Lambda}$ of the length of the mean free path of the gas l to the wavelength of the disturbance $\Lambda = \frac{2\pi}{k}$ (since $\nu \sim vl$ where v is the root mean square velocity of the thermal motion of the molecules, $a \sim v$ and $k \sim 1/\Lambda$). Thus, in all cases when the motion of the gas may be described by ordinary fluid mechanical equations $\delta \ll 1$ (in particular, for air under normal conditions $\frac{\nu}{a_0} \approx 0.5 \times 10^{-5}$ cm, i.e., even if $\Lambda \approx 1$ mm, $\delta < 10^{-4}$). Moreover, $\delta_1 = \frac{\nu_1}{\nu}\delta$ is of the same order of magnitude, and hence $\delta_1 \ll 1$.

Now, ignoring in Eqs. (1.91)–(1.92) the terms of order δ (or δ_1), we obtain the simplified system

$$\frac{d\omega_j(t)}{dt} = 0, \qquad \frac{dS(t)}{dt} = 0, \qquad \frac{dD(t)}{dt} = a_0^2 k^2 P, \qquad \frac{dP(t)}{dt} = -D. \tag{1.96}$$

In accordance with the first two equations of Eq. (1.96), the vorticity field $\omega_j(\boldsymbol{x}, t)$ and the entropy field $S(\boldsymbol{x}, t)$ in the approximation under discussion will be fixed in space. The last two equations of Eq. (1.96) show that

$$\frac{d^2 D(t)}{dt^2} + a_0^2 k^2 D(t) = 0, \qquad \frac{d^2 P(t)}{dt^2} + a_0^2 k^2 P(t) = 0,$$

or, proceeding at once from the single Fourier component to the total fields $D(\boldsymbol{x}, t)$ and $P(\boldsymbol{x}, t)$,

$$\frac{\partial^2 D}{\partial t^2} = a_0^2 \nabla^2 D, \quad \frac{\partial^2 P}{\partial t^2} = a_0^2 \nabla^2 P. \tag{1.97}$$

Thus for the pressure field $P(\boldsymbol{x}, i)$ and the field of the velocity divergence $D(\boldsymbol{x}, t)$ we have obtained identical wave equations which describe waves propagated at the sound velocity a_0.

We see that in the zeroth-order approximation (with respect to a small parameter δ) the disturbances of the hydrodynamic quantities break down into three noninteracting components. These are called the *incompressible vorticity mode* which is described by the vorticity field $\omega_j(\boldsymbol{x})$ and is time-invariant (or displaced without change at the unperturbed velocity $\boldsymbol{u}$), the *entropy mode* which is described by the also stationary (or displaced with velocity $\boldsymbol{u}$) entropy field $S(\boldsymbol{x})$ produced by initial temperature inhomogeneities, and the *potential* or *acoustic mode* which is connected with fluctuations of the potential part of the velocity field and with fluctuations of pressure, and which is represented by a set of waves propagated at the unperturbed sound velocity a_0.

If we also take into account effects of order δ, then we find at once that the velocity field of the vorticity mode will be slowly damped under the influence of viscosity in accordance with Eq. (1.93). If we wish to write down the entropy and acoustic modes also with accuracy of δ, it is convenient first, to rewrite Eq. (1.94) in the form

$$\lambda^3 + \left(1 + \frac{\gamma}{\pi_1}\right)\delta_1\lambda^2 + \left(1 + \frac{\gamma}{\pi_1}\delta_1^2\right)\lambda + \frac{\delta_1}{\pi_1} = 0, \tag{1.94$'$}$$

which contains only the parameters (1.95). It is then easy to determine the three roots of Eq. (1.94$'$) to terms of order δ_1 by the usual power series method. As a result, we obtain

$$\begin{aligned} \lambda_1 &= i - \frac{\pi_1 + \gamma - 1}{2\pi_1}\delta_1, \\ \lambda_2 &= -i - \frac{\pi_1 + \gamma - 1}{2\pi_1}\delta_1, \\ \lambda_3 &= -\frac{1}{\pi_1}\delta_1. \end{aligned} \tag{1.98}$$

These equations show that if we restrict ourselves to terms of order not exceeding δ_1, and eliminate the vorticity mode from the discussion, then in a compressible medium there will exist three different waves with the same wave-number vector $\boldsymbol{k}$. Two of these waves will be damped with the same decrement $\frac{\pi_1 + \gamma - 1}{2\pi_1}\delta_1 a_0 k = \frac{\nu_1 + (\gamma - 1)\chi}{2}k^2$ and will be propagated with the undisturbed sound velocity a_0 in the direction of the vectors $\boldsymbol{k}$ and $-\boldsymbol{k}$; the third wave will be stationary with respect to the undisturbed flow, and will have decrement $\frac{\chi k}{a_0}a_0 k = \chi k^2$. (It is not difficult to show that even if terms of order δ_1^2 are taken into account, this changes only the imaginary parts of the roots λ_1 and λ_2, i.e., it leads to the appearance of a weak dependence of the velocity of propagation of the acoustic waves on the coefficients of viscosity and thermal diffusivity [cf. Yaglom (1948)].) By assuming further that the dependence of $D(\tau)$, $P(\tau)$ and $S(\tau)$ on τ will be determined by the factor $e^{\lambda_j\tau}$, $j = 1, 2, 3$, and substituting the corresponding particular solutions in the system (1.92), it is easy to show that the general solution of this system, to terms of order δ_1 inclusive, may be presented in the form of a sum of the following two solutions:

$$S(\tau) = S_0 e^{\lambda_3\tau}, \qquad P(\tau) = 0, \qquad D(\tau) = -\frac{\delta_1}{\pi_1}S_0 e^{\lambda_3\tau} \tag{1.99}$$

and

$$D(\tau) = D_0^{(1)} e^{\lambda_1 \tau} + D_0^{(2)} e^{\lambda_2 \tau}, \qquad S(\tau) = -\frac{\gamma - 1}{\pi_1} \delta_1 \left[D_0^{(1)} e^{\lambda_1 \tau} + D_0^{(2)} e^{\lambda_2 \tau} \right],$$

$$P(\tau) = D_0^{(1)} \left[i + \frac{\pi_1 - \gamma + 1}{2\pi_1} \delta_1 \right] e^{\lambda_1 \tau} + D_0^{(2)} \left[-i + \frac{\pi_1 - \gamma + 1}{2\pi_1} \delta_1 \right] e^{\lambda_2 \tau}. \tag{1.100}$$

Thus it is clear that in the approximation under consideration, the entropy mode of our disturbance will correspond to an arbitrary initial entropy (or temperature) distribution which does not move in space but which is gradually smoothed out under the action of thermal conductivity. Connected also with this mode is the very weak (of order δ_1) volume distribution of the potential mode of velocity and the velocity divergence $D = \frac{\partial u_\alpha}{\partial x_\alpha}$. This distribution is set up by thermal expansion and compression of elementary volumes of the thermally inhomogeneous medium. The acoustic mode, however, will be composed of pressure waves and velocity divergence waves which are of the form $D_0 e^{i(kx \pm a_0 kt) - \frac{\nu_1 + (\gamma - 1)\chi}{2} k^2 t}$, propagated with velocity a_0, and slightly damped by molecular viscosity and thermal conductivity; these waves generate also an extremely weak transfer of entropy (of the order of δ_1).

This resolution of arbitrary disturbances of hydrodynamic fields into vorticity, entropy and acoustic modes may easily be extended to all orders of magnitude with respect to δ and δ_1; however, in practice, in all cases it is sufficient to restrict ourselves to the terms given above. Of somewhat greater significance is the question of the changes induced in individual modes of the disturbances by the nonlinear terms in the hydrodynamic equations. In order to take this effect into account, it is necessary to proceed to the next approximation in small-disturbance theory which consists of preserving in Eqs. (1.87)–(1.90) terms that are bilinear with respect to the hydrodynamic quantities. In these terms the basic quantities may be defined according to the equations of the first approximation, i.e., they are assumed to agree with the solutions of linearized equations given above. Thus Eqs. (1.87)–(1.90) in the subsequent approximation will contain some fairly small additional components of known functional form, which are naturally transferred to the right side of the equation and considered as three-dimensional "sources" of the corresponding hydrodynamic fields. The solution of these equations will differ from that given above for the homogeneous equations (1.87)–(1.90) by the additional terms produced by the corresponding sources, i.e., by bilinear combinations or "interactions" of individual components of the linear theory with each other. Again writing the solution of the linearized equations in the form of the sum of vorticity, entropy and acoustic modes, we obtain six different paired bilinear (and quadratic) combinations of these three modes, i.e., six different "interactions," each of which, in its turn, can make definite additions to solutions describing any of the three modes, that is, they "generate" this mode.

However, it is natural that many of the 18 second-order effects thus produced will be expressed by bilinear combinations that contain the factors δ or δ_1, and will therefore play a very small role. Thus, for example, from the fact that (to terms of order δ_1) the entropy mode contains only fluctuations of entropy, we can deduce that in the zeroth approximation with respect to δ, this mode will not interact with itself at all, while its interaction with the vorticity mode will introduce a definite contribution [connected, evidently, with the advective terms $u_\alpha \frac{\partial s}{\partial x_\alpha}$ of Eq. (1.62)] only in the entropy mode of the disturbance field. A complete analysis from this viewpoint of all the possible "interactions" may be found in the paper by Chu and Kovásznay (1958). The results of this analysis are illustrated by the table below, which indicates the order with respect to δ_1 of all 18 second-order effects, with a brief explanation of the physical meaning of the zero-order effects. In addition to this, for all zero-order effects with respect to δ_1, Chu and Kovásznay

Interaction	Generation		
	Acoustic Mode	Vorticity Mode	Entropy Mode
acoustic mode with acoustic mode	$O(1)$ "sound scattering by sound" and "nonlinear distortion of soundwaves	$O(\delta_1)$	$O(\delta_1)$
vorticity mode with vorticity mode	$O(1)$ "generation of sound by vorticity"	$O(1)$ "stretching of vortex lines" or " inertial transfer of the vorticity by a rotational velocity field	$O(\delta_1)$
entropy mode with entropy mode	$O(\delta_1)$	$O(\delta^2)$	$O(\delta_1)$
acoustic mode with vorticity mode	$O(1)$ "scattering of sound by rotational inhomogeneities of the velocity field"	$O(1)$ "transfer of vorticity by sound-waves"	$O(\delta_1)$
acoustic mode with entropy mode	$O(1)$ "scattering of sound by temperature inhomogeneities"	$O(1)$ "generation of vorticity by the interaction of entropy and pressure" (Bjerknes effect).	$O(1)$ "transfer of heat by sound-waves"
vorticity mode with entropy mode	$O(\delta_1)$	$O(\delta_1)$	$O(1)$ "transfer of heat by vorticity"

give explicit expressions for the corresponding three-dimensional sources which must be added to the right side of the linearized equations. However, we shall not study in detail all the zero-order effects, a consiserable proportion of which, as may be seen from the table, refer to a simple shift of hydrodynamical inhomogeneities by the velocity field from one fluid element to another. We shall consider a more detailed discussion of only the most important interaction of the vorticity mode with itself (recalling that in an ordinary, weakly compressible medium, the vorticity mode as a rule, considerably exceeds all the others in magnitude).

Since the vorticity mode is connected only with the disturbances of the velocity field, its interaction with itself will arise only from bilinear terms with respect to velocity in the hydrodynamic equations. In the entropy equation (i.e., heat budget) there are such bilinear terms (in the term $\rho\varepsilon$) but with the coefficient of viscosity, from which it is clear that the entropy mode generation by this interaction will be of order not exceeding δ. A much more important role will be played by the terms of the equations of motion $u_\alpha \dfrac{\partial s}{\partial x_\alpha}$ which are bilinear with respect to the velocity; these will determine all the "significant" second-order effects, connected with the interaction of the vorticity mode with itself.

Keeping the terms $u_\alpha \frac{\partial u_l}{\partial x_\alpha}$ in the equations of motion, and then proceeding to the vorticity equation, we obtain Eq. (1.7)

$$\frac{\partial \omega_k}{\partial t} + \nu \nabla^2 \omega_k = - u_\alpha \frac{\partial \omega_k}{\partial x_\alpha} + \omega_\alpha \frac{\partial u_k}{\partial x_\alpha},$$

which differs from Eq. (1.88) in the right side which describes the process of generation of vorticity due to the stretching of the vortex lines. Assuming in this right side that the field u_α is incompressible and satisfies Eq. (1.88), we obtain the first significant "second-order effect"–the generation of the vorticity mode of the velocity field by the interaction of this mode with itself. The effect of this interaction on the acoustic mode may be determined in the zero approximation with respect to δ, omitting in the hydrodynamic equations all terms containing the coefficients of viscosity and thermal diffusivity, and all nonlinear terms except the terms $u_\alpha \frac{\partial u_l}{\partial x_\alpha}$ in the equations of motion. But in this case it is easy to see that instead of the last equation of Eq. (1.97) we obtain

$$\frac{\partial^2 P}{\partial t^2} - a_0^2 \nabla^2 P = - \frac{\partial}{\partial x_\beta} u_\alpha \frac{\partial u_\beta}{\partial x_\alpha}. \tag{1.101}$$

Here the right side evidently describes the additional fluctuations of pressure which arise from the fluctuations of the velocity field in accordance with Eq. (1.9) which have the form

$$\nabla^2 p = - \rho \frac{\partial}{\partial x_\beta} \left(u_\alpha \frac{\partial u_\beta}{\partial x_\alpha} \right).$$

Substituting into the right side of Eq. (1.101) for $u_i(\boldsymbol{x}, t)$ an incompressible velocity field satisfying the "vorticity equation" (1.88) or (if we do not assume that the velocity fluctuations are small) Eq. (1.7), we may find the sound field generated by the interaction of an incompressible velocity field with itself; the right side of Eq. (1.101), evidently in this case may also be written as $-\frac{\partial u_\alpha}{\partial x_\beta} \frac{\partial u_\beta}{\partial x_\alpha}$ or as $-\frac{\partial^2 (u_\alpha u_\beta)}{\partial x_\alpha \partial x_\beta}$. The generation of sound by a rotational flow described by Eq. (1.101) was studied by Lighthill (1952; 1954); this is the second significant second-order effect connected with the interaction of the vortex mode of the velocity field with itself.

2. HYDRODYNAMIC INSTABILITY AND TRANSITION TO TURBULENCE

2.1 Concept of Turbulence; Empirical Data on Transition to Turbulence in Tubes and Boundary Layers

In the preceding section, equations were given describing fluid motion, and some of the simpler solutions were indicated. We pointed out also that these solutions do not by any means always correspond to any flow that may actually be observed. Thus, in Sect. 1.2, Example 3, we noted that the flow in a tube is described by Eqs. (1.23)–(1.26) only in the case of sufficiently high viscosity and sufficiently low mean velocity, while in Sect. 1.4 we stressed that the Blasius solution of the boundary-layer equation on a flat plate gives good agreement with the data only in the case of fairly small values of Ux/ν. The same is true for all other examples of fluid flows. As a rule, the theoretical solutions of the equations of fluid mechanics, whether exact or approximate, gives a satisfactory description of real flows only under certain special conditions. If these conditions are not met, the nature of the flow undergoes a sharp change, and instead of a regular smooth variation of the fluid mechanical quantities in space and time, we observe irregular fluctuations of all these quantities which have the complex nature indicated in Fig. 1. Thus, flows of fluid may be divided into two very different classes: smooth, quiet flows which vary in time only when the forces acting or the external conditions are changing, known as *laminar* flows; and flows accompanied by irregular fluctuations of all the fluid mechanical quantities in time and space, which are called *turbulent* flows.

The difference between laminar and turbulent regimes of flow is revealed in a number of phenomena which are of great significance for many engineering problems. For example, the action of a flow on rigid walls (i.e., friction on walls) in the case of a turbulent regime is considerably greater than in the case of a laminar regime (since the transfer of momentum in a turbulent medium is much more intensive). The presence of irregular fluctuations of the velocity leads also to a sharp increase in mixing of the fluid; extremely intensive mixing is often considered as the most characteristic feature of turbulent motion. With increase of the mixing, there is also a sharp increase in the thermal conductivity of the fluid, etc. For all these reasons, the determination of the conditions of transition from a laminar to a turbulent regime is a very pressing problem. Moreover, finding the mechanism of the initiation of turbulence must aid our understanding of its nature and facilitate the study of the laws of turbulent flow, which are of great importance in practical work.

A detailed survey of the experimental data relating to the transition from laminar to turbulent flow is to be found in the articles of Schlichting (1959), Dryden (1959), and Tani (1967), containing many additional references. Many hundreds of scientific papers and several dozens of survey articles are devoted to the theory of this question [see, e.g., the surveys by Stuart (1963; 1965), Shen (1964), Reid (1965), Segel (1966), Drazin and Howard (1966), and Görtler and Velte (1967)]; see also the special monographs of Lin (1955), Chandrasekhar (1961), and Betchov and Criminale (1967). In this book we shall confine ourselves to a consideration of the most important (and simplest) flows only, and shall devote most of our attention to the principles involved. For the details of the mathematical calculations and a detailed description of the experiments, and also for matters relating to more complex flows, the reader is referred to the aforementioned sources and special articles.

The first results on the conditions for transition to turbulence were obtained by Hagen (1839). Hagen studied flows of water in straight circular tubes of fairly small radius, and established that with gradual decrease in the viscosity of water (brought about by increasing its temperature), the velocity of flow for the same pressure-head first increases to some limit and then begins to decrease again. The jet of water issuing from the tube before this limit is reached has a smooth form, but after passing through this limit it suffers sharp fluctuations. Hagen explained these phenomena by saying that for sufficiently low viscosity values, internal motions and vortices are produced, leading to increased resistance, and, consequently, to a decrease in the velocity of flow. Hagen showed also that variation in the nature of the flow may be effected by changing the pressure-head (i.e., the mean velocity) or the radius of the tube; he could not, however, obtain any general criterion for the transition from laminar to turbulent flow.

A general criterion for transition to turbulence was established by O. Reynolds (1883) using the concept of mechanical similarity of flows of viscous fluid. This criterion consists of the fact that the flow will be laminar so long as the Reynolds number $\mathrm{Re} = UL/\nu$ does not exceed some critical value Re_{cr}, while for $\mathrm{Re} > \mathrm{Re}_{cr}$ it will be turbulent. As explained in Sect. 1.3, the Reynolds number has the significance of a ratio of characteristic values of the forces of inertia and viscosity. The inertia forces lead to the approach of initially remote fluid volumes and thus contribute to the formation of sharp inhomogeneities in the flow. The viscous forces, on the other hand, lead to the equalizing of velocities at neighboring points, i.e., to the smoothing of small-scale inhomogeneities. Thus for small Re, when the viscous forces predominate over the inertia forces, there can be no sharp inhomogeneities in the flow, i.e., the fluid mechanical quantities will vary smoothly and the flow will be laminar. For large Re, on the other hand, the smoothing action of the viscous forces will be weak, and irregular fluctuations—sharp, small-scale inhomogeneities—will arise in the flow, i.e., the flow will become turbulent. The Reynolds criterion is explained by these considerations.

To verify this criterion experimentally and to measure Re_{cr}, Reynolds carried out a series of experiments with water flows in circular glass tubes connected with a reservoir. In these experiments a source of colored fluid was placed on the axis of the tube at the intake. For small Re, the colored water took the form of a thin, clearly defined jet, which indicated a laminar regime of flow. As Re increased, at the instant of passing through the critical value, the form of the colored jet sharply changed; at quite a small distance from the intake into the tube, the jet spread out and waves appeared in it; farther on, separate eddies were formed and towards the end of the tube the whole of the liquid was colored. If in such an experiment the flow is illuminated by an electric spark, it may be seen that the colored mass consists of more or less distinct swirls which indicate the presence of vorticity. Transient phenomena are observed for subcritical Re numbers close to Re_{cr} in a laminar flow. These phenomena consist of the appearance of short-term flashes of high-frequency fluctuations, which were first observed by Reynolds, in the form of peculiar "turbulent slugs" which fill the entire cross section of the tube but only for fairly short sections of its length. (According to Schiller (1934), the appearance of such "bursts" is connected with the alternate formation and breakdown of large-scale vortices on the inner wall of the tube, at its upstream end). In the initial part of the tube, with $\mathrm{Re} > \mathrm{Re}_{cr}$, the flow has a similar character. But as Re increases, the length of this initial part where the flow is not entirely turbulent, decreases rapidly. According to Rotta's measurements (1956), in an intermediate regime of this kind, the mean fraction of the time in which a turbulent regime is observed at a given point (the "coefficient of intermittency") increases monotonically with increase of the distance x from the intake into the tube. The existing experimental data (the most detailed being Lindgren's (1957; 1959; 1961) allow this increase to be explained by the fact that the local velocity of the leading edge of a "turbulent slug" exceeds the local velocity of its trailing edge and hence the "slug" increases in length as it moves along the tube (occasionally some slugs will overtake others, coalescing to form a single, larger slug).

Reynolds' experiments were carried out in tubes of various diameters with a smooth intake connected to the reservoir. Variation of the numbers $\mathrm{Re} = U_m D/\nu$ (where U_m is the mean velocity of flow) was accomplished both by changing to a new tube (with another value of the diameter D), and by varying the rate of flow and the viscosity of the water (by altering the temperature). The value of $\mathrm{Re}_{\mathrm{cr}}$ in these experiments was on the average close to 12,830. However, such results were obtained only by exercising the greatest care for the absence of disturbances in the water before entering the tube. Reynolds' investigations showed that the value of $\mathrm{Re}_{\mathrm{cr}}$, corresponding to the transition from laminar to turbulent flow, depends considerably on the degree of disturbance in the water entering the tube (or, as we say, the "initial turbulence," which is determined principally by the conditions at the intake of the tube) and therefore in different experiments may prove to differ very much. It is impossible to produce an absolutely steady flow in the tube. Therefore, disturbances will always be in the flow, characterized by relatively rare fluctuations of velocity, the origin of which is partly connected with the vortices separating from the leading edge of the tube. The intensity of these disturbances (which may be characterized by the parameter U'/U_m where U' is the typical magnitude of the fluctuations of the x-component of velocity) may be fairly large, but due to their infrequent recurrence these disturbances by themselves do not alter the laminar nature of the flow and have no considerable effect on the velocity profile, the pressure drop or other mean characteristics of the flow. For sufficiently small Reynolds numbers, the disturbances which arise are attenuated when displaced downstream. However, as the Reynolds number increases, at the instant that it attains its critical value (which depends on the intensity of the disturbances and possibly on their scale and frequency), the disturbances suddenly generate turbulence. It is found that the value of $\mathrm{Re}_{\mathrm{cr}}$ corresponding to the transition to turbulence is the smaller, the greater the intensity of the disturbances. Thus, for example, in the case of a tube with a sharp entrance, pushed through the plane wall of the reservoir, the end of the tube will create considerable disturbances, and $\mathrm{Re}_{\mathrm{cr}}$ will equal approximately 2800. Conversely, if the degree of disturbance at the intake into the tube is decreased strongly by some means or other, we can delay the transition from laminar to turbulent flow until very high Reynolds numbers (this is called *persistence of the laminar regime*). Thus Barnes and Coker (1905) were able to delay the transition to turbulence in tubes to values $\mathrm{Re_{cr}} \approx 20{,}000$, and Ekman (1911) up to $\mathrm{Re_{cr}} \approx 50{,}000$. Later, Comolet (1950) obtained laminar flow in a tube at Reynolds numbers in the range 70,000–75,000, and Pfenninger (1961) [who arranged twelve special screens for damping the flow disturbances between the inlet of the tube and the atmosphere] found that a tube flow can be fully laminar even at $\mathrm{Re} = 100{,}000$.

These results show that the Reynolds number in itself is not a unique criterion for transition to turbulence; for a flow in a tube it is apparently impossible to find a universal critical value $\mathrm{Re_{cr}}$ such that for $\mathrm{Re} > \mathrm{Re_{cr}}$, the flow regime is bound to be turbulent. To establish the upper value of Re for laminar flows in tubes it is necessary to have some knowledge of the level of inlet turbulence of the laminar flows considered. Thus the upper bound of the Re numbers of laminar flows will generally be a function of the parameter U'/U_m (which apparently increases monotonically as the parameter decreases).

Without knowing the degree of disturbance of the laminar flow, we can find only a rather weak criterion which indicates the conditions under which only a laminar regime of flow is possible. For this we must determine the critical Reynolds number $\mathrm{Re_{cr\,min}}$ corresponding to the transition from laminar to turbulent flow for the largest possible level of disturbance of the laminar flow at the intake into the tube. With $\mathrm{Re} < \mathrm{Re_{cr\,min}}$ the flow will always remain laminar, i.e., any disturbance, regardless of its intensity, will be damped.

Experiments to measure $\mathrm{Re_{cr\,min}}$ were performed by Reynolds himself. Since in these experiments it is necessary to introduce into the tube fluid that is as disturbed as possible, the method of dye injection is clearly unsuitable here. As a result, we must determine the transition to a turbulent regime in some other way (for example, by the variation of the skin friction law which determines the dependence of the mean velocity on the pressure drop). In Reynolds' experiments the minimum critical value of $\mathrm{Re} = U_m D/\nu$ was found to be $\mathrm{Re_{cr\,min}} \approx 2030$. Values close to this (lying between 1800 and 2320) were also obtained in all subsequent investigations [including the recent works by Lindgren (1959; 1961) and Sibulkin (1962)]. Similar results were also valid for flows in channels of rectangular cross section [see, e.g., Narayanan and Narayana (1967)].

Results on the criteria for transition to turbulence in many ways similar to the above were obtained in the study of boundary-layer flows around a solid body. Let us consider, for example, the boundary layer formed on a flat plate by a flow of constant velocity U, flowing parallel to the plate. The Reynolds number of the boundary layer may be defined, e.g., by $\mathrm{Re}_\delta = U\delta/\nu$ where δ is the thickness of the boundary layer. Alternatively we may use the more easily measured number $\mathrm{Re}_x = Ux/\nu$ where x is the distance from the leading edge of the plate measured along the flow. The numbers Re_δ and Re_x are connected by a functional dependence; for example, for a laminar flow, by the results of Sect. 1.4, $\mathrm{Re}_\delta \approx 5\sqrt{\mathrm{Re}_x}$ [cf. Eq. (1.49)]. Proceeding downstream, both Re_δ and Re_x increase, and at some point x_{cr}, they attain the "critical value" when the flow sharply changes its nature and becomes turbulent. Thus, for $x < x_{cr}$ (more precisely, for $\mathrm{Re}_\delta < \mathrm{Re}_{\delta\,cr}$ and $\mathrm{Re}_x < \mathrm{Re}_{x\,cr}$) the flow in the boundary layer will be laminar while for $x > x_{cr}$ (i.e., $\mathrm{Re}_\delta > \mathrm{Re}_{\delta\,cr}$ and $\mathrm{Re}_x > \mathrm{Re}_{x\,cr}$) it will be turbulent. In the immediate neighborhood of $x = x_{cr}$ a "mixed regime" is formed in which only discrete "bursts" of turbulence are observed; these arise at a point in the form of "turbulent spots" which grow in size and coalesce with each other as they move downstream [see, e.g., the survey papers by Schlichting (1959), Sect. 14, Coles (1962), Stuart (1965) and also Elder (1960)]. The appearance of "turbulent spots" leads to the presence, at points near x_{cr} of alternating laminar and turbulent regimes; at the beginning of the transition region both the frequency of occurrence and the duration of the turbulent regime is negligible, while at the end of it, the laminar regime occurs only very rarely and briefly.

The first measurements of the critical Reynolds number of a boundary layer were carried out in 1924 by Burgers and Van der Hegge Zijnen who studied a flow of air past a flat glass plate in a wind-tunnel. Shortly thereafter, similar measurements were also made by Hansen (1928) [see, in particular, Fig. 4]. According to the data of these authors

$$\mathrm{Re}_{x\,cr} = \left(\frac{Ux}{\nu}\right)_{cr} \sim 3 \cdot 10^5 \div 5 \cdot 10^5,$$

which corresponds to

$$\mathrm{Re}_{\delta\,cr} = \left(\frac{U\delta}{\nu}\right)_{cr} \sim 2750 \div 3500,$$

which is of the same order of magnitude as the value of Re_{cr} for flow in a tube. Later, it was shown that as in the case of flows in tubes, the critical Reynolds number of a boundary layer depends considerably on the disturbance level of the ambient flow, depending on which Re_x may vary from 1×10^6 to almost 3×10^6 (see Fig. 7, which illustrates the dependence of $Re_{x\,cr}$ on U'/U, where $3U'^2$ is taken equal to the mean square velocity fluctuation of the ambient flow). However, we observe in Fig. 7 that the value of $Re_{x\,cr}$ will not increase without bound, but will tend to a definite limit (of the order 3×10^6) above which the boundary layer will be turbulent no matter how small the disturbance level of the initial flow. On the other hand, in the case of flow in a circular tube, experiment does not indicate a sharp reduction in the rate of increase of Re_{cr} as U'/U tends to zero; thus it seems likely that here $Re_{cr} \to \infty$ as $U'/U \to 0$ (i.e., for any value of Re, the flow in a tube will remain laminar provided that a sufficiently low disturbance level of the intake flow can be guaranteed). We shall discuss this important difference between flows in boundary layers and in tubes in greater detail in Sect. 2.8.

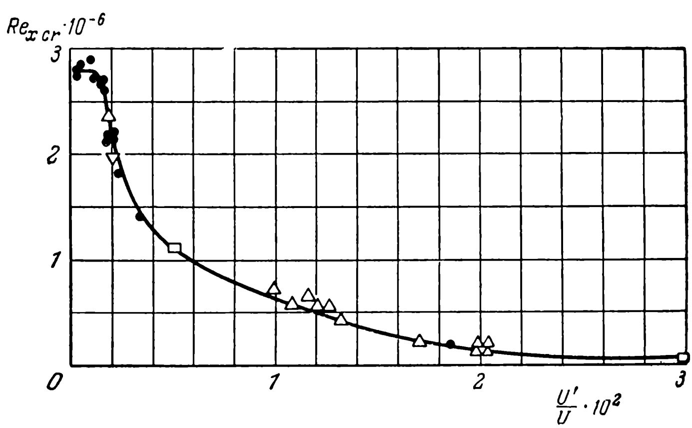


FIG. 7. Dependence of the critical Reynolds number for the boundary layer on a flat plate on the free-stream disturbance level [after Dryden (1949)]. (The different symbols on the figure denote the data of different investigators.)

The fact that the critical Reynolds number is considerably smaller for flow along a rough plate with natural or artificial irregularities is connected with the turbulizing effect of disturbances in the ambient flow. According to the data of numerous measurements, even one isolated irregularity can bring about the transition of a laminar boundary layer into a turbulent one, provided that the height h_0 of this irregularity is of the order of the "displacement thickness" $\delta^* \approx 0.3\delta$ of the boundary layer in its neighborhood. Even more important is the effect of a number of irregularities scattered over the whole plate [see, e.g., Schlichting (1959), Chapt. X or Dryden (1959), Sect. 5]. The value of $\mathrm{Re}_{x\,\mathrm{cr}}$ is also changed considerably by even quite small longitudinal pressure gradients in the incident flow. However, we shall not dwell on this point, but confine ourselves to considering only one special effect connected with the presence of a negative pressure gradient.

2.2 Turbulent Flow Past Solid Bodies; Boundary-layer Separation, the Drag Crisis and the Mechanism of Boundary-layer Transition

Transition to turbulence in a flow of viscous fluid past a solid body may occur not only in the boundary layer, but also by the formation of a turbulent wake behind the body due to the separation of macroscopic vortices from its surface.

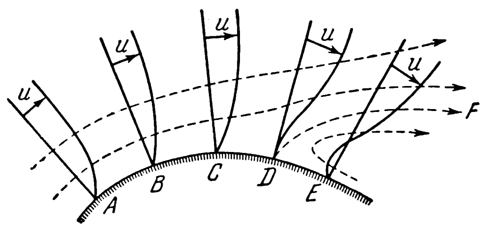


FIG. 8. Schematic form of the streamlines (dotted) and the velocity profiles (solid) at different points in the flow past a right cylinder.

The formation of a turbulent wake is generally connected with the retarding action of a negative longitudinal pressure gradient in the flow. Let us consider, for example, a right circular cylinder washed by an irrotational flow perpendicular to its axis (see Fig. 8 which shows the flow past the upper part of the cylinder). Outside the boundary layer the fluid may be assumed ideal and its motion—irrotational. The streamlines of this potential motion are closest together on the upper part of the cylinder (point C) where, consequently, the tangential velocity u will attain a maximum. By the Bernoulli equation

$$\frac{u^2}{2} + p/\rho = \text{const}$$

[which follows from the equation $u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x}$, given in Sect. 1.4, immediately before Eq. (1.38)] the pressure at C in the outer flow will attain a minimum, so that it will decrease along AC and increase along CE. Such changes in the pressure along the surface of the body will also take place in the boundary layer (since across the boundary layer the pressure hardly varies). Consequently, on CE the fluid in the boundary layer must move in the direction of increasing pressure, which leads to its retardation. This retardation will have the strongest effect, of course, on the fluid particles moving on the very surface of the cylinder, i.e., possessing the least velocity. At some point D downstream from C, these particles will come to a standstill; beyond D they will be moving backwards with respect to the fluid particles further from the cylinder surface which have not yet been retarded. The formation of counterflows on the surface of the body beyond D forces the outer flow away from the surface of the cylinder; and there occurs what we call "separation of the boundary layer" from the surface, with formation of a dividing stream-surface DF in the fluid. It is clear that if the velocity U decreases sufficiently rapidly beyond C, separation of the boundary layer is bound to occur. Even if the boundary layer is laminar before separation, after separation it will behave as a free submerged jet and will quickly become turbulent (for considerably smaller Re than for an unseparated boundary layer since the presence of the wall has a stabilizing effect on the flow). The dividing stream-surface DF, which is a surface of tangential discontinuity of the velocity, is very unstable (see below) and is quickly transformed into one or more vortices. In the region FDE beyond the dividing stream-surface a large-scale vortex is formed close to the cylinder, with a second such vortex being formed on the lower part of the cylinder. These vortices separate in turn from the cylinder, are carried downstream and gradually are dispersed; in their place new vortices are formed (see, e.g., the figures in Hinze (1959), p. 8).

As a result, a turbulent wake is formed behind the body in which the motion is rotational, while outside the layer the motion is irrotational (i.e., potential). In fact, the fluid outside the boundary layer may be assumed to be ideal; it follows, therefore, that during its motion, the circulation of the velocity along any closed contour is conserved, and hence in the case of steady motion the curl of the velocity is constant along streamlines. Therefore, it is evident that a region of turbulent rotational flow at a distance from a body can only arise when streamlines leave the boundary layer (in which the motion will be rotational due to the viscosity), i.e., when there is a direct mixing of the fluid from the boundary layer with that of the space outside it.

It is also clear that the streamlines cannot leave the region of flow in which the curl of the velocity is nonzero, i.e., the region of the turbulent wake (although they can enter the wake from the region of potential flow). In other words, fluid can flow into the turbulent wake from the potential region but cannot flow out of the turbulent wake. At the same time, turbulent fluctuations of velocity can penetrate from the wake into the region of potential flow, although with considerable attenuation. In fact, for potential motion of an incompressible fluid, the equations of motion (1.7) will be satisfied identically. Therefore, in this case the flow will be described by the single condition of incompressibility (1.5). This condition is equivalent to Laplace's equation $\nabla^2 \varphi = 0$ for the velocity potential φ, which defines the velocity: $u_i = \partial\varphi/\partial x_i$. Let z be the coordinate across the wake. Then the field $\varphi(x, y, z)$ which describes the velocity fluctuations may conveniently be decomposed into periodic components of the form $\varphi = \varphi_0(z)e^{i(k_1x+k_2y)}$. From $\nabla^2 \varphi = 0$, it follows that $\frac{d^2\varphi_0}{dz^2} = k^2\varphi_0$, where $k = \sqrt{k_1^2 + k_2^2}$ is the wave number, inversely proportional to the horizontal (Oxy-plane) scale of the periodic fluctuations under consideration. Discarding the physically meaningless solution for φ_0, which increases with increasing z, we find that the attenuation of the fluctuation amplitude in the region $z > 0$ is given by the factor e^{-kz}. Thus the fluctuations are attenuated the quicker, the smaller the scale. Consequently, at a sufficient distance inside the potential motion, only comparatively smooth large-scale fluctuations arise. For such fluctuations the energy dissipation does not play a large role; thus almost all the dissipation in the flow will take place in the rotational turbulent wake.[7]

The considerable energy dissipation in the whole region of the turbulent wake, together with the formation of a dividing streamsurface leads to a considerable increase in drag of bodies having

[7] The analysis described belongs to Landau and Lifshitz (1963); it also plays the central role in detailed investigations of the velocity field outside a turbulent region of flow made by Phillips (1955) and Stewart (1956).

separated boundary layers. This drag, as a rule, will be the smaller, the narrower the turbulent wake, i.e., the further along the surface of the body is the point of separation. For sufficiently large Reynolds numbers for which, however, the boundary layer remains laminar up to the separation point, the drag coefficient $C_W = \frac{W}{\frac{1}{2}\rho U^2 S}$ (where W is the total drag and S is the area of the body or its cross section) is independent of Re because the position of the separation point is independent of Re (it is found from the equation $\left(\frac{\partial u}{\partial z}\right)_{z=0} = 0$ in which it may be shown that the Reynolds number does not occur). However, as we approach Reynolds numbers for which the boundary layer becomes turbulent before the separation point for a laminar boundary layer, the separation point will move downstream, and in this case the turbulent wake becomes considerably narrower and the drag of the body decreases sharply (by several times). This phenomenon is called the *drag crisis*. It is explained by the fact that the momentum transfer within a boundary layer increases sharply when it becomes turbulent. Therefore, the entrainment of fluid from the outer flow by the boundary layer is considerably increased so that the fluid particles in the boundary layer move in the direction of increasing pressure much longer than in the case of a laminar boundary layer.

The drag crisis in the case of flow past a sphere was first observed by Eiffel (1912). The transition from large to small drag occurs in this case for Reynolds number Re $= UD/\nu$ (where D is the diameter of the sphere) close to 5.0×10^5; the drag coefficient C_W decreases, approximately from 0.5 at Re $= 10^5$ to 0.15 at Re $= 10^6$. Later, it was also found that the minimum value of C_W in very careful experiments is less than 0.1. The coefficient C_W for a circular cylinder behaves in the same way. The dependence of the drag coefficient for a sphere and a circular cylinder on Re is shown in Fig. 9. It is clear from the above discussion that the drag crisis will arise the earlier, the greater the disturbance level of the ambient flow, i.e., the smaller the critical Reynolds number for transition to a turbulent regime in the boundary layer. This is confirmed clearly by the experiments of Prandtl (1914) who tried to achieve passage through the drag crisis in the flow past a sphere by fitting a wire ring around the sphere, i.e., by introducing additional disturbances into the flow to cause transitions in the boundary layer.

The necessary condition for separation of the boundary layer is an increase of pressure in the direction of flow along some part of the surface of the body. This condition is satisfied not only for flow past convex surfaces, but also in other cases, e.g., flow in an expanding conical tube (diffuser) or in a sharply bent tube. In these cases, also, separation of the boundary layer may occur.

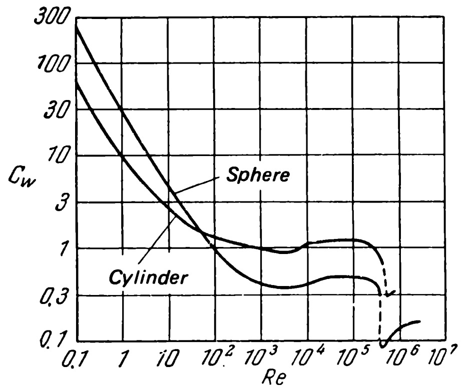


FIG. 9. Dependence of the drag coefficients of a sphere and a circular cylinder on the Reynolds number.

The separation of the boundary layer under the action of a negative longitudinal pressure gradient may also explain to a certain extent the effect of disturbances in the ambient flow on the value of $Rc_{x\,cr}$. We may assume that this effect is connected with the generation of fluctuations of the longitudinal pressure gradient by these disturbances, leading to the random formation of individual spots of unstable S-shaped velocity profiles (as at point E, Fig. 8) and hence to separation and transition of the boundary layer. On the basis of this hypothesis, G. I. Taylor (1936a) tried to estimate theoretically the dependence of the critical Reynolds number $Re_{x\,cr}$ on the initial turbulence level of the ambient flow (later, Wieghardt (1940) gave a shorter deduction of Taylor's results).

In his work, Taylor proceeded from the approximate von Kármán–Pohlhausen theory of a laminar boundary layer in the presence of a longitudinal pressure gradient $\partial p/\partial x$. According to the theory the form of the velocity profile in different sections of the boundary layer depends only on a single dimensionless parameter $\Lambda = -\dfrac{\delta^2}{\nu U \rho}\dfrac{\partial p}{\partial x}$. In a boundary layer on a flat plate $\partial p/\partial x = 0$, but there may exist fluctuations of pressure. Thus Taylor proposed that the character of the motion in a fixed section is determined here by the parameter $\Lambda = -\dfrac{\delta^2}{\nu U \rho}\dfrac{\delta p'}{\delta x}$ (where p' is the pressure fluctuation and $\delta/\delta x$ signifies a typical value of the derivative $\partial/\partial x$). In other words, according to Taylor, the point of transition from laminar to turbulent flow is determined by the parameter Λ attaining some critical value. But the equations of motion show that $-\dfrac{1}{\rho}\dfrac{\delta p'}{\delta x}$ must be of the same order of magnitude as $u' \dfrac{\delta u'}{\delta x} = \dfrac{1}{2}\dfrac{\delta u'^2}{\delta x}$ where u' is the fluctuation of the longitudinal velocity in the ambient flow. Further, we may put $\dfrac{\delta u'^2}{\delta x} \sim \dfrac{U'^2}{\lambda}$, where U' is a typical value of the velocity fluctuations, and λ is the so-called Taylor microscale of turbulence which is determined from the condition $\left(\dfrac{\delta u'}{\delta x}\right)^2 = \dfrac{U'^2}{\lambda^2}$ (this scale will be used several times in later sections of the book). The scale λ may be expressed in terms of the external (integral) scale of turbulence L (which defines the order of magnitude of the greatest distance at which some relation between the instantaneous values of the velocity fluctuations is apparent) as follows: the mean rate of energy dissipation $\bar{\varepsilon} \sim \nu\left(\dfrac{\delta u'}{\delta x}\right)^2$ is, on the one hand, proportional to $\dfrac{\nu U'^2}{\lambda^2}$ and, on the other, for large values of $U'L/\nu$, it is also proportional to U'^3/L (we shall discuss this fact in greater detail in Volume 2 of this book). Consequently,

$$\lambda \sim L\left(\frac{U'L}{\nu}\right)^{-1/2}, \quad \frac{1}{\rho}\frac{\delta p'}{\delta x} \sim \frac{U'^2}{\lambda} \sim \left(\frac{U'^5}{\nu L}\right)^{1/2}.$$

Taking into account that for a laminar boundary layer on a flat plate $\delta \sim (\nu x/U)^{1/2}$, we obtain

$$\Lambda = -\frac{\delta^2}{\nu U\rho}\frac{\delta p'}{\delta x} \sim \left(\frac{U'}{U}\right)^{5/2}\left(\frac{x}{L}\right)^{1/2}\left(\frac{Ux}{\nu}\right)^{1/2}.$$

Thus, taking $\mathrm{Re}_{x\,\mathrm{cr}} = \varphi(\Lambda_{\mathrm{cr}})$, we obtain

$$\mathrm{Re}_{x\,\mathrm{cr}} = \left(\frac{Ux}{\nu}\right)_{\mathrm{cr}} = F\left[\frac{U'}{U}\left(\frac{x}{L}\right)^{1/5}\right]. \tag{2.1}$$

We may take the length L in this equation to be some characteristic dimension of the device generating the turbulence (for example, if the turbulence is set up by a grid in a wind-tunnel, then L will be approximately equal to the distance between the rods of the grid).

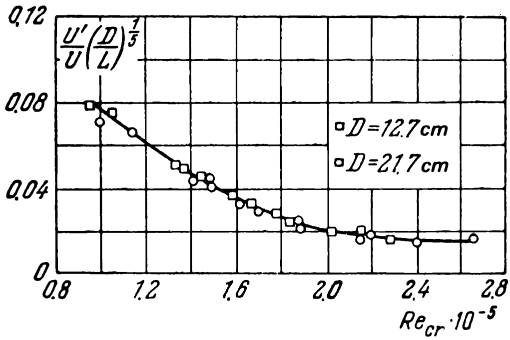


FIG. 10. Dependence of $\mathrm{Re}_{\mathrm{cr}}$ on $\dfrac{U'}{U}\left(\dfrac{D}{L}\right)^{1/5}$ for flow past a sphere.

The result (2.1), according to which Re_{cr} depends only on $\frac{U'}{U}\left(\frac{x}{L}\right)^{1/5}$ has been deduced here for the case of flow past a flat plate. However, it may be generalized to flows past other bodies by replacing the coordinate x with s, the distance from the point where the flow impinges on the body to the point of transition of the boundary layer from a laminar to a turbulent regime, reckoned along the contour of the body, or by some characteristic dimension of the body D, and by using, instead of $\mathrm{Re}_x = Ux/\nu$, $\mathrm{Re}_s = Us/\nu$ or $\mathrm{Re} = UD/\nu$. In spite of the lack of rigor in its deduction, this result is in very good agreement with experimental data (see, e.g., Fig. 10 which gives data of Dryden, Schubauer, Mock and Skramstad (1937), who measured the value of Re_{cr} for boundary layers on spheres of different radii with different values of the intensity and scale of the turbulence in the free stream).

2.3 Hydrodynamic Instability

In the theoretical analysis of the transition problem, we must start from the fact that the velocity and pressure fields in any fluid flow, whether laminar or turbulent, are solutions of the equations of fluid mechanics for given initial and boundary conditions. Steady laminar flow, in particular, is described by steady solutions of these equations; however, in the case of turbulent flow, each individual example of the flow corresponds to a very complex nonsteady solution of the dynamical equations. The impossibility of the existence of laminar flow for sufficiently high Reynolds numbers (in spite of the fact that the equations of fluid mechanics have a steady solution for any Re) clearly shows that not every solution corresponds to a real fluid motion. It is natural to associate this fact with the familiar postulate that real motion must not only satisfy the dynamical equations but must also be stable to small disturbances. In other words, small disturbances of the motion which are always present, must be damped in time so as not to change the general nature of the flow. On the other hand, if these disturbances increase with time, then considerable distortion of the initial motion occurs and, consequently, this motion cannot continue to exist for any considerable time.

Therefore, we may expect that the value of Re_{cr} corresponds to the point at which stability is lost; for $\mathrm{Re} < \mathrm{Re}_{cr}$ the laminar flow is stable, and with $\mathrm{Re} > \mathrm{Re}_{cr}$ it is unstable and becomes turbulent under the influence of the existing small disturbances. But if this is the case, then by the mathematical study of laminar solutions of the equations of motion, we may (at least in principle) theoretically determine the corresponding critical Reynolds number.

Turbulence is characterized by very complicated irregular fluctuations of the velocity and other characteristics of the flow. The mechanics of the initiation of these fluctuations may be explained from the viewpoint of general mechanics. It is only necessary to consider the flow of fluid as a dynamical system with a very large number of degrees of freedom in which self-excited oscillations arise as a result of the influx of energy from outside.

The application of the concept of degrees of freedom necessitates immediate introduction of some generalized coordinates which describe the configuration of the flow uniquely. To define these generalized coordinates we may begin from a decomposition of the motion into elementary components. These components must be such that the sum of all their energies is equal to the total energy of the flow while the state of each of them is characterized by a fairly small number of parameters. The parameters of all the elementary components of motion will be generalized coordinates of the flow, and the number of these coordinates which can vary under given external conditions will be the total number of degrees of freedom of the flow. From a mathematical viewpoint, decomposition of the motion into elementary components is equivalent to the expansion of the velocity field in terms of an orthogonal system of functions; each of these functions will describe the velocity field of the corresponding elementary component of the motion, while the coefficients of the expansion will be generalized coordinates of the flow. The choice of an arbitrary orthogonal system of functions is dictated by the form of the boundaries of the flow. For flows in a finite volume, the orthogonal system of coordinates will always be countable; thus, these flows will have no more than a countable set of generalized coordinates.[8]

For steady laminar flow the values of the generalized coordinates will be defined uniquely by given external and boundary conditions so that the number of degrees of freedom of a laminar flow is zero. The number of degrees of freedom of a turbulent flow in a finite volume will be very great, but it is also finite. In fact when the velocity field is expanded in a series in terms of orthogonal functions, the various components will describe elementary motions

[8] The concept of a flow in an infinite space will always be an idealization. Using this idealization, instead of expanding the velocity field as a series in terms of an orthogonal system of functions, we must use an expansion similar to that of a Fourier-Stieltjes integral (which will be used extensively in Volume 2) and to allow the existence of a continuous spectrum. Therefore, an unbounded flow of fluid may possess even a continuum of generalized coordinates. However, in bounded space-time regions, fluctuations with a continuous set of wave numbers, of course, may also be approximated by a countable number of harmonic oscillations.

of different scales. As the order of the component increases indefinitely, the corresponding scale tends to zero. However, due to the viscosity, fluctuations of too small a scale cannot exist. Thus with steady external conditions the coefficients of the expansion of the velocity field in terms of orthogonal functions of sufficiently high order will be independent of time. This means that the number of degrees of freedom of the flow will be finite. Also, the number of degrees of freedom must increase with decrease of the coefficient of viscosity, in other words, with increase of Re. According to the estimate of Landau and Lifshitz (1963), the number of degrees of freedom of a turbulent flow in a finite volume will be proportional to $\mathrm{Re}^{9/4}$ for large enough Re, where Re is the Reynolds number of the overall flow (this result will be discussed in Volume 2). Consequently, the number of degrees of freedom will increase rapidly with increase of Re, and for developed turbulence with large Reynolds numbers it will reach enormous values.

If we are to describe not only the configuration of a system but also its change in time, we require in addition to the values of its generalized coordinates the values of the corresponding generalized velocities. The choice of values of all the generalized coordinates and generalized velocities defines some point in the phase space of a system which characterizes completely its instantaneous state. The change of the state of the system is described in the phase space by a definite line—the phase trajectory of the system. The plotting of the phase trajectory is a convenient way of describing the evolution of a system.

Let us consider the evolution of a fluid flow under fixed steady external conditions (in particular, with constant influx of energy) but with various initial conditions. Each set of initial conditions has a corresponding phase trajectory, coming from the corresponding phase point. The question of the behavior of these trajectories over long intervals of time is an interesting one. From statistical mechanics it is known that dynamical systems with a large number of degrees of freedom and steady external conditions are inclined to converge to some limiting equilibrium regime in which the mean influx of energy equals the mean dissipation of energy of the system, and the total energy has a fixed value and fixed distribution among the different degrees of freedom. We may put forward the hypothesis that for a broad class of flows two possible limiting regimes exist—laminar and turbulent. Thus every phase trajectory of a flow will in the course of time either tend asymptotically to the point corresponding to laminar flow or else approach some "limit cycle" corresponding to a steady turbulent regime. The criterion for transition to turbulence must permit us to predict from the starting point of the phase trajectory which of the two limiting regimes will occur.

The possibility of decomposing the motion of the fluid into elementary components means that a flow may be considered as a set of interacting elementary nonlinear oscillators. In each of these oscillators, due to the influx of energy, self-excited oscillations may arise. The possibility of such oscillations arising is determined by the relationship between the energy E^+ acquired by the oscillator and the energy E^- lost by it for various amplitudes of the oscillations a (see Fig. 11). If $E^- > E^+$ for all amplitudes (Fig. 11a), then the oscillations will evidently be damped for any initial amplitude, and the system will be stable to any disturbances. If $E^- < E^+$ for $a_1 < a < a_0$, but $E^- > E^+$ for $a < a_1$ or $a > a_0$ (Fig. 11b), then the oscillations with initial amphtude $a<a_1$ will be damped, but those with initial amplitude $a>a_1$ will increase, until their amplitude attains the equilibrium value a_0. In this case, the system will be stable to small disturbances but unstable to disturbances of sufficiently large amplitude (such a system is called a system with *hard self-excitation*). Finally, if $E^+ > E^-$ for any amplitude, however small (Fig. 11c), the system will be unstable to infinitely small disturbances (i.e., absolutely unstable) and will practically always be in a regime of self-excited oscillation with amplitude a_0 (system with *soft self-excitation*).

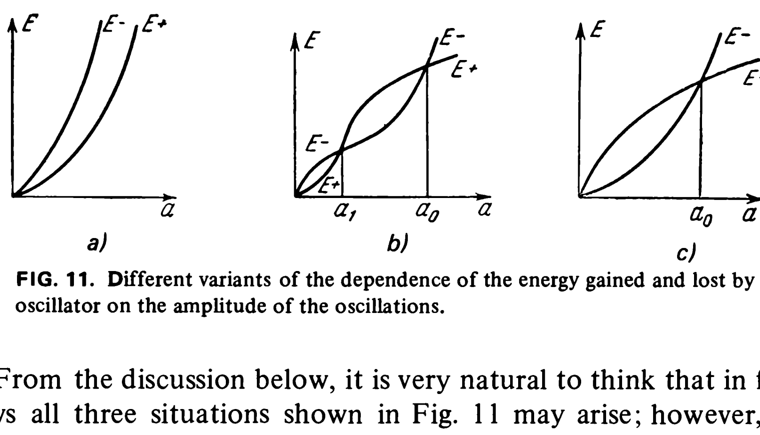


FIG. 11. Different variants of the dependence of the energy gained and lost by an oscillator on the amplitude of the oscillations.

From the discussion below, it is very natural to think that in fluid flows all three situations shown in Fig. 11 may arise; however, the exact conditions which would allow us to determine in every case which situation actually exists in a given flow are still unknown.

2.4 Simple Examples of Absolutely Unstable Fluid Flows

We cite above the experimental data relative to the dependence of Re_{cr} for flows in tubes and boundary layers on the intensity of the initial disturbances and the persistence of the laminar regime. These data show that for Re slightly greater than $Re_{cr\,min}$ the flows will be self-excited systems with hard excitation (one possible mechanism of the excitation of oscillations in such systems is given by Taylor's theory outlined in Sect. 2.2).

We shall now show some simple examples of fluid motions that are unstable even with respect to infinitely small disturbances; i.e., from the viewpoint of the theory of oscillations they are systems with soft excitation.

One of the simplest examples of an absolutely unstable flow is the flow near a surface of tangential velocity discontinuity which we have mentioned above. In this case, the absolute instabihty may be explained qualitatively with the aid of simple physical considerations. Let us consider an ideal fluid of zero viscosity, two layers of which slide over each other with equal and opposite velocities U and $-U$, forming a surface of discontinuity of velocity. Let us assume that as a result of some disturbance on the surface of discontinuity, a small-amplitude wave is formed (see Fig. 12). For simplicity, we assume that this wave is nonprogressive. Under these circumstances, the streamlines above the wavecrest will draw closer together, i.e., the velocity will increase, while in the troughs the streamlines will become further apart and the velocity will decrease. According to Bernoulli's equation, $u^2/2 + p/\rho = \text{const}$, the pressure will fall above the crest and rise in the troughs (in Fig. 12 this is denoted by the plus and minus signs). Thus a transverse pressure gradient arises in the fluid, tending to increase the amplitude of the wave. Later, this increase in amplitude leads to the wave disintegrating into individual vortices, forming the beginning of the turbulent zone.

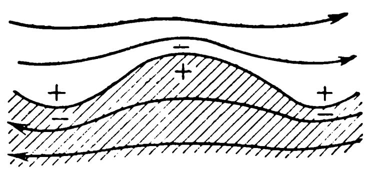


FIG. 12. Schematic form of the streamlines and the pressure distribution close to a disturbed surface of tangential velocity discontinuity.

In a real fluid, of course, the waves which arise are progressive, but the processes of their evolution are similar. These processes may be observed for example, in experiments with a jet issuing from an orifice and then expanding in a space filled with the same (but motionless) fluid (the boundary of such a jet may be considered as a surface of tangential velocity discontinuity). An accurate quantitative analysis of the instability of a surface of tangential velocity discontinuity was obtained first by Helmholtz (1868) [cf. Lamb (1932), Sect. 232, or Landau and Lifshitz (1963), Sect. 30]. For viscous fluid, the sliding of the two layers over each other is, of course, impossible, and instead of the surface of discontinuity, there exists between the two flows a narrow transition layer in which the velocity profile will be S-shaped. The investigation of the stability of such a layer will be more complicated; however, here we may show also (both theoretically and experimentally) that it is very unstable (see below, Sect. 2.8). We note further that the absolute instability of a surface of tangential velocity discontinuity is the simplest case of so-called *Helmholtz instability*—the absolute instability of a special type of surface of discontinuity, separating two regions of flow, fllled with the same or different fluids, moving with different velocities. A survey of the main results in this field and references to later literature may be found in Birkhoff (1962).

Another simple example of absolute instability is the equilibrium in a gravitational field of stationary stratified fluid with variable density $\rho = \rho(z)$ increasing with height. Consequently, with any function $\rho(z)$ the equations of motion of incompressible fluid will allow a solution $\boldsymbol{u}(x, y, z, t) = 0$, corresponding to a state of rest; the gravitational field will only produce a vertical pressure variation according to the law

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g, \quad \text{i.e.,} \quad p(z) = g\int_z^\infty \rho(z')\,dz' + \text{const.} \tag{2.2}$$

Now suppose that as a result of some disturbance, some element of the fluid is displaced from level z to a new level $z' = z + h$. If the density ρ decreases with height, then for $h > 0$ the element will tend to move downwards under the force of gravity, and for $h < 0$ it will tend to rise under the action of the buoyancy, so that the equilibrium will be stable. However, if the density increases with height, then for any value of h the displaced element will tend to move even further from its original position, and the state of equilibrium will be absolutely unstable. Moreover, for an ideal fluid (without friction), the equations of motion will also have a steady solution for any $\rho = \rho(z)$ and arbitrary vertical profile of the x-component of velocity $u = u(z)$ [and zero components of velocity along the other axes]. Using the same argument, this flow will be absolutely unstable for $d\rho/dz > 0$. For $d\rho/dz < 0$, the question of stability of the flow is considerably more complex; at this early stage in the discussion we can only say, by similarity, the criterion of stability here must be expressed in terms of the so-called *Richardson number*, i.e., the dimensionless parameter

$$\mathrm{Ri} = \frac{-\dfrac{g}{\rho}\dfrac{d\rho}{dz}}{\left(\dfrac{\partial u}{\partial z}\right)^2}. \tag{2.3}$$

The case of a fluid that is stratified with respect to the z axis is of great interest for meteorological problems, where such stratification arises from the temperature profile $T = T(z)$. However, in this case, we cannot assume that the fluid is incompressible, but must use the equation of state and the elementary thermodynamic identities [see, e.g., Landau and Lifshitz (1963), Sect. 4]. Then we find that a fluid element displaced from level z to level $z + h$, will be lighter for $h > 0$ than the surrounding air, but heavier for $h < 0$ when and only when

$$\frac{dT}{dz} < -\frac{gT}{c_p V}\left(\frac{\partial V}{\partial T}\right)_p, \tag{2.4}$$

where T is now the absolute temperature and V is the specific volume. Condition (2.4) will be a condition of absolute instability of the state of rest in the presence of a temperature profile $T = T(z)$. When the medium may be considered as an ideal gas, $\left(\dfrac{\partial V}{\partial T}\right)_p = \dfrac{R}{p} = \dfrac{V}{T}$, so that the criterion of instability will take the form

$$\frac{dT}{dz} < -\frac{g}{c_p} = -\frac{\gamma - 1}{\gamma}\frac{g}{R}, \qquad \gamma = \frac{c_p}{c_v} \tag{2.5}$$

(the criterion will be found in this form in all textbooks of dynamical meteorology). In meteorology, $G_a = \dfrac{\gamma - 1}{\gamma}\dfrac{g}{R}$ is called the *adiabatic temperature gradient* (for air, this gradient is approximately 1°C/100 m). The thermal stratification of the air for which $-(dT/dz)$ is greater, equal or less than G_a, is called, respectively, stable, neutral, or unstable stratification.

Another representation of the conditions of instability (2.4) or (2.5) is often used in meteorology; this is connected with the introduction of the so-called *potential temperature*, defined by

$$\theta = T\left(\frac{p_0}{p}\right)^{\frac{\gamma - 1}{\gamma}}, \tag{2.6}$$

where p_0 is some standard pressure (usually taken as the normal sea-level pressure), instead of the ordinary temperature T. By the entropy equation for an ideal gas (Sect. 1.7), $c_p \ln \theta = s +$ const. Therefore, the potential temperature does not vary in adiabatic processes, so that θ is equal to the temperature which the air would attain if brought adiabatically to standard pressure p_0. It is easy to see that $\frac{d\theta}{dz} \sim \frac{dT}{dz} - G_a$. Thus using the concept of the potential temperature, the criterion of instability (2.4) may be formulated as follows: the state of rest will be unstable if $d\theta/dz < 0$ (i.e., if the potential temperature decreases with height) and stable if $d\theta/dz > 0$ (i.e., if the potential temperature increases with height).

If an arbitrary wind velocity profile exists, the motion in the case of unstable stratification will hkewise be unstable; for stable stratification, however, the stability or instability of the motion must be determined in some way by the value of the Richardson number

$$\mathrm{Ri} = \frac{\frac{g}{T}\left(\frac{dT}{dz} - G_a\right)}{\left(\frac{du}{dz}\right)^2} = \frac{\frac{g}{\theta}\frac{d\theta}{dz}}{\left(\frac{du}{dz}\right)^2}. \tag{2.3'}$$

2.5 Mathematical Formulation of the Stability Problem for Infinitesimal Disturbances

The conditions for transition to turbulence are by no means as easy to find in every case as in the above examples. In general, the most effective means of investigating the stability is the general method of small disturbances. We shall now discuss the basic idea of this method with reference to the flow of an incompressible fluid of constant density ρ.

The method of small disturbances is based on writing the velocity field $u_i(\boldsymbol{x}, t)$ and the pressure $p(\boldsymbol{x}, t)$ which satisfy the dynamic equations in the form $u_i = U_i + u_i'$, $p = P + p'$, where $U_i(\boldsymbol{x}, t)$ and $P(\boldsymbol{x}, t)$ are particular solutions of the equations under investigation, and u_i', p' are small disturbances. Taking into account that U_i and P themselves satisfy the equations of motion, and ignoring quadratic terms in the disturbances, we obtain linear equations for u_i' and p' in the form

$$\begin{gathered} \frac{\partial u_i'}{\partial t} + U_\alpha \frac{\partial u_i'}{\partial x_\alpha} + u_\alpha' \frac{\partial U_i}{\partial x_\alpha} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \nabla^2 u_i', \\ \frac{\partial u_\alpha'}{\partial x_\alpha} = 0. \end{gathered} \tag{2.7}$$

Differentiating Eq. (2.7) containing $\partial u_i'/\partial t$ with respect to i, summing with respect to x_i, and using the last of Eq. (2.7), we may express p' in terms of u_i' with the aid of an equation analogous to Eq. (1.9′). Then the general solution of Eq. (2.7) will be determined by fixing only the initial values $u_i'(\boldsymbol{x}, 0)$ of the function $u_i'(\boldsymbol{x}, t)$. We may thus (at least in principle) attempt to establish conditions under which not all solutions of the corresponding initial value problem will be damped in time; these conditions will also be conditions of absolute instability[9] of the solution U_i, P. Of course, if the corresponding conditions are not satisfied, so that the solution U_i, P will be absolutely stable (with respect to infinitesimal disturbances), there still remains the possibility that with respect to finite disturbances u_i', p' (described by essentially nonlinear equations), this solution may yet be unstable, i.e., that the flow described by this solution will be a system with hard self-excitation. However, to verify this possibility we shall require quite different methods of investigation (see below, Sect. 2.9).

When the solution $U_i = U_i(\boldsymbol{x})$, $P = P(\boldsymbol{x})$ describes a steady laminar flow of fluid, the coefficients of the system of equations (2.7) will evidently be time-independent. In this case, the system will have particular solutions of the form

$$\boldsymbol{u}'(\boldsymbol{x}, t) = e^{-i\omega t}\boldsymbol{f}_\omega(\boldsymbol{x}), \qquad p'(\boldsymbol{x}, t) = e^{-i\omega t}g_\omega(\boldsymbol{x}), \tag{2.8}$$

the time-dependence of which is given by the exponential factor $e^{i\omega t}$ with, generally speaking, a complex "frequency" ω. The permissible values of the characteristic frequency ω and the corresponding amplitudes $\boldsymbol{f}_\omega(\boldsymbol{x})$, $g_\omega(\boldsymbol{x})$ will then be determined from the eigenvalue problem for a linear system of partial differential equations. When the coefficients of this system are independent of some space coordinates, the number of independent variables in the system may be reduced by assuming that the dependence of the amplitudes $\boldsymbol{f}_\omega$ and g_ω on the corresponding coordinates will also be exponential, with a given "wave number" (i.e., the spatial scale of the disturbances is prescribed in the directions of the coordinate axes along

[9]We must point out that this terminology has not been completely settled. Often the condition of absolute instability is taken to mean the condition that Eq. (2.7) will have an increasing solution, i.e., cases where only damped and neutrally stable solutions are present are included in the stable class. Here, however, it will be more convenient to consider as unstable solutions those for which there exists at least one undamped disturbance.

which the undisturbed flow is homogeneous. Thus, for example, if the undisturbed flow depends only on the coordinate x_3 then we may put

$$f_\omega(x) = e^{i(k_1x_1+k_2x_2)} f_{\omega;\, k_1,\, k_2}(x_3), \quad g_\omega(x) = e^{i(k_1x_1+k_2x_2)} g_{\omega;\, k_1,\, k_2}(x_3); \quad (2.9)$$

where the characteristic frequency $\omega = \omega(k_1\ k_2)$ and the amplitudes $f_{\omega;\, k_1,\, k_2}$ and $g_{\omega;\, k_1,\, k_2}$ will be determined from the eigenvalue problem for a system of ordinary differential equations containing the parameters k_1 and k_2. Similar equations, with $\exp\,[i(k_1x_1+k_2x_2)]$ replaced by $\exp\,(ik_1x_1)$, and $f_{\omega;\, k_1,\, k_2}(x_3)$ and $g_{\omega;\, k_1,\, k_2}(x_3)$, will be obtained for flows which depend essentially on x_2 and x_3 only.

The system of eigenfunctions $f_\omega(x)$ will frequently be complete in the space of all vector functions $f = (f_1, f_2, f_3)$ satisfying the continuity equation $\partial f_i/\partial x_i = 0$ and the necessary boundary conditions (if $f_\omega(x) = e^{ik_1x_1} f_{\omega;\, k_1}(x_2, x_3)$ or $f_\omega(x) = e^{i(k_1x_1+k_2x_2)} f_{\omega;\, k_1,\, k_2}(x_3)$, then it is sufficient that the functions $f_{\omega;\, k_1}(x_2, x_3)$ or $f_{\omega;\, k_1,\, k_2}(x_3)$ for any fixed k_1 or fixed k_1, k_2 will form a complete system in the corresponding space of vector functions of one or two variables).[10]

When the system of eigenfunctions is complete, any initial values $u(x, 0)$ can be expanded in a series (or integral) in terms of these eigenfunctions. Thus the general solution of the initial value problem for Eqs. (2.7) may be expressed as a superposition of elementary exponentially time-dependent solutions. Therefore the general stability problem may be reduced here to the corresponding eigenvalue problem. For the stability of such a laminar flow with respect to infinitesimal disturbances it is necessary and sufficient that all characteristic frequencies ω have a negative imaginary part $\mathrm{Im}\,\omega < 0$. When there is a space homogeneity with respect to one or more coordinates, the various characteristic frequencies ω will, generally speaking, depend on the space "scales" of the disturbances (i.e., on the wave numbers k_1 or k_1 and k_2) and on Re. As Re $\to 0$, the imaginary parts of all frequencies ω will tend to negative values

[10]However, the problem of completeness in the theory of hydrodynamic stability is not a simple one. From a mathematical viewpoint, the eigenvalue problems which arise here are those of a linear nonself-adjoint operator in a function space. To establish the completeness of the system of eigenfunctions (or at least of the eigenfunctions and related associated functions) of this operator, we may often use the theorem of Keldysh (1951) [see also Keldysh and Lidskiy (1963)]. However, in much of the literature, the assumption of completeness is used without any justification, and in some applications the assumption is clearly incorrect (see the end of this section).

(because for ρ = const, the state of rest is always stable). However, as Re increases, the imaginary parts of certain frequencies may increase, and, finally, become positive. Let us assume that the eigenvalue problem corresponding to a fixed disturbance scale (i.e., fixed k_1 or k_1 and k_2) has a discrete spectrum of characteristic frequencies ω_j (this assumption is fulfilled for many important flows). Here, the critical value of Re corresponding to the transition to instability of a laminar flow with respect to infinitesimal disturbances of a given scale will be determined from the equation $\max \operatorname{Im} \omega_j = 0$. The smallest of these critical values of Re for disturbances of various scales will be the critical number Re_{cr} of the flow, i.e., for $\mathrm{Re} > \mathrm{Re}_{cr}$, the laminar flow will be absolutely unstable and for $\mathrm{Re} < \mathrm{Re}_{cr}$ it will be stable. Of course this Re_{cr} characterizing the instability with respect to infinitesimal disturbances, must not be less than the critical Reynolds number characterizing the stability of the flow with respect to finite disturbances. Thus, with the notation of Sect. 2.1, it might be denoted by $\mathrm{Re}_{cr\,max}$. On the other hand, the instability of a flow for $\mathrm{Re} > \mathrm{Re}_{cr\,max}$ means only that for such Reynolds numbers the corresponding laminar flow cannot exist; however, it does not signify that the flow is bound to become turbulent. In fact, it is possible that after loss of stability, a given laminar flow may be transformed into a new laminar flow which is now stable; also, transition to a turbulent regime occurs only after loss of stability of this new laminar flow, at Reynolds numbers considerably greater than $\mathrm{Re}_{cr\,max}$ (see the examples in subsections 2.6–2.7).

Before ending our discussion, we must stress once again that the possibility of expanding an arbitrary solution of the system (2.7) in a series in terms of particular solutions of the form of Eq. (2.8) will occur often, but not always–this fact is frequently forgotten in presentations of the theory of hydrodynamic stability. In particular, the situation is more complicated if the system (2.7) is singular (i.e., for example, if a coefficient of a leading derivative in this system becomes zero at some point). In this case, the completeness of the system of eigenfunctions cannot be proved simply, and even the very concept of eigenfunctions must be defined with care. Often, even with a fixed disturbance scale a continuous part of the spectrum of eigenvalues arises with corresponding eigenfunctions satisfying unusual boundary conditions or possessing a more complicated structure (e.g., not vanishing at infinity or having discontinuities of the derivatives at a singular point). In applications, these "improper" eigenfunctions sometimes simply go unnoticed and, hence, the system of "elementary solutions" of form (2.8) is obviously incomplete [see Case (1962); Lin (1961); Lin and Benney (1962); Drazin and Howard (1966)]. If for some reason, the system of eigenfunctions is incomplete, the investigation of the corresponding eigenvalue problems is clearly insufficient for the solution of the stability problem. For a complete study of such cases, we must investigate the behavior of the general solution of the corresponding initial value problem. This investigation is very complicated; however, in the special case of an ideal fluid with $\nu = 0$, it nevertheless has been possible to obtain a number of conclusive results (see the works of Dikiy (1960a,b) and Case (1960a,b) which we shall discuss in detail later).

2.6 Stability of Flow between Two Rotating Cylinders

One important example of absolute instability, which is amenable to complete mathematical analysis, is the instability of steady circular Couette flow between rotating cylinders. Let R_1 and Ω_1 be the radius and angular velocity of the inner cylinder, and R_2 and Ω_2 those of the outer cylinder. In cylindrical coordinates r, φ, z with the Oz axis along the axis of the cylinders, the velocity field of such a Couette flow will be defined by the familiar equations

$$U_r = U_z = 0, \quad U_\varphi = Ar + \frac{B}{r},$$
$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}, \quad B = -\frac{R_1^2 R_2^2 (\Omega_2 - \Omega_1)}{R_2^2 - R_1^2}, \tag{2.10}$$

(see Eq. (1.28) subsection 1.2).

First, let us ignore the effect of viscosity. Then we may define the criterion of instability from the following elementary physical considerations. In a steady laminar flow, the centrifugal force acting on an element of the fluid will be balanced by the radial pressure gradient. Now let an element of mass m move under the action of the disturbance from a position with coordinate r_0 to a position with coordinate $r > r_0$. Then by the law of conservation of angular momentum $mrU_\varphi(r)$, in the new position its velocity will equal $r_0 U_\varphi(r_0)/r$ and therefore a centrifugal force $m\frac{r_0^2}{r^3}U_\varphi^2(r_0)$ will act on it. The equilibrium will be unstable if this force is greater than the radial pressure gradient at a distance r from the axis, which is equal in magnitude to the undisturbed value of the centrifugal force at the distance r. Hence, the condition of instability [which, for such an inviscid Couette flow was established by Rayleigh (1916a)], will have the form

$$[r_0 U_\varphi(r_0)]^2 - [r U_\varphi(r)]^2 > 0 \quad \text{for} \quad r > r_0,$$

or, in other words,

$$\frac{\partial}{\partial r}(rU_\varphi)^2 < 0. \tag{2.11}$$

Following Coles (1965), and taking into account that U_φ/r is the angular velocity of the flow, and that $d(rU_\varphi)/rdr$ is the axial vorticity, we may easily rephrase the criterion (2.11) as follows: *a flow is unstable if the vorticity (local rotation) is opposite in sense to the angular velocity (overall rotation).* In such a form, this criterion is apparently valid for many inviscid circulatory flows.

Using Eq. (2.10), Rayleigh's criterion of instability may be reduced to the form $(\Omega_2 R_2^2 - \Omega_1 R_1^2) U_\varphi < 0$. If the cylinders are rotating in opposite directions, then U_φ will change sign somewhere between the cylinders, and in this case the flow will certainly be unstable. When both rotate in the same direction, we may put $\Omega_1 > 0$, $\Omega_2 > 0$, and then $U_\varphi > 0$ everywhere; in this case the Rayleigh criterion of instability takes the form

$$\frac{\Omega_2}{\Omega_1} < \left(\frac{R_1}{R_2}\right)^2. \tag{2.12}$$

A rigorous mathematical derivation of this result, obtained by applying disturbance theory to the inviscid fluid was given by Synge (1933) for the case of axisymmetric (i.e., independent of φ) disturbances of the velocity [see also Shen (1964)], and by Chandrasekhar (1960) for the case of arbitrary disturbances of the flow.

A more complete analysis (taking into account also the effect of the viscosity) may be carried out only by the method of small disturbances, first applied to this problem in a classical paper by G. I. Taylor (1923). Since the undisturbed velocity field (2.10) here depends only on the r-coordinate, then, by Eqs. (2.8) and (2.9), the disturbances of the velocity and pressure may be sought in the form

$$\begin{aligned} u_r'(\boldsymbol{x}, t) &= e^{i(kz+n\varphi-\omega t)} f^{(r)}(r), & u_\varphi'(\boldsymbol{x}, t) &= e^{i(kz+n\varphi-\omega t)} f^{(\varphi)}(r), \\ u_z'(\boldsymbol{x}, t) &= e^{i(kz+n\varphi-\omega t)} f^{(z)}(r), & p'(\boldsymbol{x}, t) &= e^{i(kz+n\varphi-\omega t)} g(r). \end{aligned} \tag{2.13}$$

Here $2\pi/k$ is the wavelength of the disturbance in the Oz direction, n is a nonnegative integer, determining the dependence of the disturbance on the angle φ, and $\boldsymbol{f}(r)=\boldsymbol{f}_{\omega;\,k,\,n}(r)=[f^{(r)}(r), f^{(\varphi)}(r), f^{(z)}(r)]$ and $g(r)=g_{\omega;\,k,\,n}(r)$ are the r-dependent "amplitudes" of the disturbance with given axial wavenumber k, azimuthal wavenumber n and characteristic frequency ω. Substituting from Eqs. (2.10) and (2.13) into the general system of equations (2.7) and taking into account the boundary conditions $\boldsymbol{u}\,(r, \varphi, z, t)=0$ for $r=R_1$ and $r=R_2$, we arrive at the eigenvalue problem determining the spectrum of permissible frequencies for given k and n. It may be shown [see, for example, Di Prima (1961)], that this problem, after certain transformations (including, in particular, the elimination of $f^{(z)}$ and g), may be reduced to the following system of two differential equations in two unknown functions $f^{(r)}$ and $f^{(\varphi)}$:

$$\begin{aligned} &\frac{d}{dr}\left[N\left(\frac{df^{(r)}}{dr}+\frac{f^{(r)}}{r}\right)\right]-k^2\left(N+\frac{\nu}{r^2}\right)f^{(r)}= \\ &\qquad\qquad=-2k^2\left(A+\frac{B}{r^2}-i\,\frac{n\nu}{r^2}\right)f^{(\varphi)}-in\,\frac{d}{dr}\,N\left(\frac{f^{(\varphi)}}{r}\right), \\ &-k^2\left(N+\frac{\nu}{r^2}\right)f^{(\varphi)}-\frac{n^2}{r}\,N\left(\frac{f^{(\varphi)}}{r}\right)= \\ &\qquad\qquad=2k^2\left(A-i\,\frac{n\nu}{r^2}\right)f^{(r)}-\frac{in}{r}\,N\left(\frac{df^{(r)}}{dr}+\frac{f^{(r)}}{r}\right), \end{aligned} \tag{2.14}$$

where A and B are determined from Eq. (2.10) and N is the differential operator

$$N=-\nu\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}-k^2\right)-i\left(\omega-\frac{nU_\varphi(r)}{r}\right). \tag{2.15}$$

The boundary conditions for which the sixth-order system (2.14) must be solved, take the form

$$f^{(r)}(r)=f^{(\varphi)}(r)=\frac{df^{(r)}}{dr}=0 \quad \text{for } r=R_1, \qquad r=R_2. \tag{2.16}$$

The eigenvalue problem (2.14)–(2.16) has no singularities. Therefore it is apparently not difficult to show that this problem will always have a discrete set of eigenvalues $\omega_j=\omega_j(k, n, \Omega_1, \Omega_2, R_1, R_2)$, while the corresponding eigenfunctions $f^{(r)}_{\omega;k,n}(r)$, $f^{(\varphi)}_{\omega;k,n}(r)$ form a complete system in the space of function pairs $(f^{(r)}, f^{(\varphi)})$, satisfying the boundary conditions (2.16). Consequently, it follows that the initial velocities $\boldsymbol{u}'(\boldsymbol{x}, 0)$, defined with the aid of Eq. (2.13) and corresponding to the eigenfunctions with all possible k and integral nonnegative n, will form a complete system in the space of vector functions of $\boldsymbol{x} = (r, \varphi, z)$ satisfying the solenoidal condition and becoming zero for $r = R_1$ and $r = R_2$. Thus, investigation of the eigenvalue problem (2.14)–(2.16) completely exhausts the stability investigation of Couette flow between rotating cylinders. However, this eigenvalue problem is very complicated, and only recently have there appeared several works devoted to it. We shall discuss these works at the end of this section; first, however, let us consider the simplified approach proposed by G. I. Taylor (1923) and used by almost all subsequent investigators.

Instead of solving the complete eigenvalue problem for the general system of equations (2.14), almost all investigators studying the stability problem for circular Couette flow have assumed that $n = 0$, i.e., they limited their attention to axisymmetric disturbances of the velocity of the basic flow. Under this assumption, the system is considerably simplified, and may be rewritten as

$$\begin{gathered}\left(L - k^2 + \frac{i\omega}{\nu}\right)(L - k^2) f^{(r)}(r) = \frac{2k^2}{\nu}\left(A + \frac{B}{r^2}\right) f^{(\varphi)}(r),\\ \left(L - k^2 + \frac{i\omega}{\nu}\right) f^{(\varphi)}(r) = \frac{2}{\nu} A f^{(r)}(r),\end{gathered} \tag{2.17}$$

where

$$L = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} = \frac{d}{dr}\left(\frac{d}{dr} + \frac{1}{r}\right). \tag{2.17'}$$

The boundary-value problem (2.17)–(2.16) for fixed k has a denumerable number of eigenvalues $\omega_j = \omega_j(k, \Omega_1, \Omega_2, R_1, R_2)$ and eigenfunctions $\boldsymbol{f}_j(r) = \boldsymbol{f}^{(\varphi)}_{\omega_j;k}(r)$. However, the corresponding functions $e^{ikz}\boldsymbol{f}_{\omega_j;k}(r)$ now will not form a complete system in the space of permissible initial fields $\boldsymbol{u}'(\boldsymbol{x}, 0) = \boldsymbol{u}'(r, \varphi, z)$ [if only because all these functions are independent of φ]. Nevertheless, the region of the (Ω_1, Ω_2)-plane to which there correspond unstable disturbances of the form $\exp[i(kz - \omega t)]\boldsymbol{f}(r)$ [i.e., disturbances of this form with $\operatorname{Im}\omega \geqslant 0$] is usually identified simply with the whole region of instability [see, e.g., Lin (1955); Chandrasekhar (1961); Stuart (1963)]. In other words, it generally is assumed, even if this is not specifically stated, that for all values R_1, R_2, Ω_1 and Ω_2, for which there is at least one eigenvalue $\omega_j(k, \Omega_1, \Omega_2, R_1, R_2)$ with $n \neq 0$ and $\mathrm{Im}\,\omega_j \geqslant 0$, there will also be an eigenvalue $\omega_j(k, \Omega_1, \Omega_2, R_1, R_2)$ [corresponding to $n = 0$] with $\mathrm{Im}\,\omega_j \geqslant 0$ (i.e., as the Reynolds number increases, a nonnegative imaginary part will first appear in the case $n = 0$). Until recently there was no doubt that this assumption was correct and in fact most of the existing experimental data (although not all) give excellent confirmation of the identity of the region of stability determined by considering only axisymmetric disturbances with the entire region of stability of a flow between rotating cylinders. However, in the last few years, some new results (both experimental and theoretical) have been obtained. These results show that the study of the eigenvalue problem (2.17)–(2.16) [which corresponds to $n = 0$] is not sufficient for a complete solution of the Couette flow stability problem. We shall discuss these new results later; first, we will consider the older results, based on the assumption that it is permissable to take $n = 0$.

Let us now assume that the radii R_1 and R_2 are fixed. It is easy to see that for sufficiently small Ω_1 and Ω_2, all the eigenvalues $\omega_j(k, \Omega_1, \Omega_2)$ and $\omega_j(k, n, \Omega_1, \Omega_2)$ will have negative imaginary part (because the state of rest is always stable). If we now increase the angular velocities Ω_1 and Ω_2 without changing the ratio Ω_2/Ω_1 (that is, we increase the Reynolds number without loss of geometric similarity), then for certain Ω_2/Ω_1 there will be no zero or negative imaginary part for any $\omega_j(k, \Omega_1, \Omega_2)$ at any Re (i.e., the motion will always remain stable). For other values of Ω_2/Ω_1 at some $\mathrm{Re}_{cr} = \Omega_2 R_2^2/\nu$, there will first appear a value of k_{cr} such that some $\omega_j(k_{cr}, \Omega_1, \Omega_2)$ will have a zero imaginary part (i.e., the motion will become unstable).

It is interesting that in this problem for all Ω_2/Ω_1, the transition from stability to instability will occur by the appearance of an eigenvalue ω_j such that not only does $\mathrm{Im}\,\omega_j = 0$, but in fact $\omega_j = 0$ (a rigorous mathematical proof of this fact, in general, is difficult, and apparently, has been given only for the special case when $R_2 - R_1 \ll (R_1 + R_2)/2$, and under some additional restirictions, by Pellew and Southwell (1940) and Meksyn (1946). However, both the experimental data and the results of direct numerical procedures show that it has a more general character). This means that the loss of stability in circular Couette flow leads to the development of a new (secondary) steady motion with velocity field $\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{u}'$, where $\boldsymbol{u}' = \exp(ikz)\,\boldsymbol{f}(r)$. This motion will have the form of so-called *Taylor* *vortices*—cellular toroidal vortices spaced regularly along the axis of the cylinders (the schematic form of the streamline pattern of these vortices, computed from the corresponding eigenfunctions $f(r)$, is shown in the figure on p. 101).

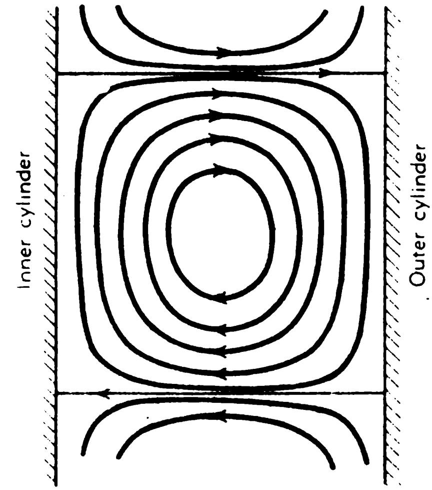


FIG. 12a. Streamline pattern of vortex disturbance for flow between rotating cylinders. (*After Shen, 1964*).

The fact that the transition from stability to instability proceeds through a steady state, corresponding to the purely zero eigenvalue $\omega = 0$, is sometimes called the "principle of exchange of stabilities" in the theory of hydrodynamic stability. This principle was often assumed and given great significance in the past, but it is now known that it holds only in certain special cases, and breaks down for many simple flows (e.g., for plane Poiseuille flow).

With further increase of the Reynolds number in Couette flow, there will be a whole range of wavenumbers k for which there are eigenvalues ω_j with $\mathrm{Im}\,\omega_j \geqslant 0$. However, experiment shows that in this case the disturbed motion preserves the form of a regular sequence of toroidal vortices for a fairly long time, but with an axial wavenumber which cannot be deduced from the linear theory (see below, Sect. 2.9). With further significant increase of Re, this periodic motion becomes unstable and is transformed into another regular motion (this time nonaxisymmetric); with still further increase of the velocity other (more comphcated) forms of flow appear and disappear in an orderly sequence, until finally the flow takes the form of disordered turbulent motion [see Coles (1965) and also subsection 2.9].

The boundary-value problem (2.17)–(2.16) is considerably simpler than (2.14)–(2.16).[11] Nevertheless, solving Eqs. (2.17)–(2.16)

[11] We may note, in particular, that since the principle of exchange of stabilities applies for Eqs. (2.17)–(2.16), when determining the boundaries of the region of stability without pretention to rigor, we may assume in this case that $\omega = 0$. For problem (2.14)–(2.16) with $n \neq 0$, however, this principle does not apply, and thus the additional parameter Re ω occurs in the calculations.

still causes considerable difficulty. Consequently, the numerical calculation of eigenvalues was first performed by Taylor (1923) only for the case of a very small gap between the cylinders (i.e., for $R_2 - R_1 \ll (R_1 + R_2)/2$). In this case both the equations and the numerical work may be simplified considerably. Taylor's method is based on the expansion of the solution of Eqs. (2.17) in terms of a special set of orthonormal functions; the differential equations then reduce to an infinite system of linear algebraic equations. Later, this method was improved somewhat by Synge (1938); other numerical methods (in particular, a method based on the reduction of the boundary-value problem (2-17)–(2.16) to a variational problem, Galerkin's method and some direct numerical methods) also applicable only in the small-gap approximation were worked out by Pellew and Southwell (1940), Meksyn (1946), Chandrasekhar (1954; 1961), Di Prima (1955), Steinman (1956), Duty and Reid (1964), Walowit, Tsao and Di Prima (1964), Reed and Harris (1964) and others. Only rather recently has a more general approach been proposed [Witting, (1958)] which uses an expansion of the solution in powers of the parameter d/R_0 (where $d = R_2 - R_1, R_0 = (R_1 + R_2)/2$). In the works of Chandrasekhar (1958), Chandrasekhar and Elbert (1962), Kirchgässner (1961), Sparrow, Munro and Jonsson (1964), Yu Yun-Sheng and Sun Dah-Dhen (1964), Roberts (1965), and Capriz, Ghelardoni and Lombardi (1966), still more general methods have been developed for the numerical calculation of the eigenvalues and eigenfunctions of (2.17)–(2.16) for arbitrary R_1 and R_2; see also surveys by Chandrasekhar (1961) and Di Prima (1963). We should also point out that the existence of the instability of the flow of a viscous fluid between rotating cylinders for certain values of Ω_1 and Ω_2 was rigorously proved by Krylov (1963) and by Yudovich (1966a) without using any numerical methods, but this proof provides no means of determining the boundary of the region of instability. However, the numerical methods allow us to find this boundary with high accuracy and to show that it has roughly the same form in all cases.

For an example, Fig. 13a shows the region of the $(\Omega_1, \ \Omega_2)$ plane in which unstable disturbances independent of φ are possible for $R_2/R_1 = 1.13$ (according to Taylor's calculations and in excellent agreement with all later calculations). Figure 13b gives the corresponding region for $R_2/R_1 = 2$ (according to Chandrasekhar's calculations). In both cases (and also for all other R_2/R_1) the numerical procedures lead to the finding of unstable disturbances only if $\Omega_2/\Omega_1 < (R_1/R_2)^2$. Therefore it is natural to think that in the region $\Omega_2/\Omega_1 > (R_1/R_2)^2$ instability will not occur for any Re. Thus here the viscosity has only a stabilizing effect [though in fact for viscous flow with $\Omega_2/\Omega_1 > (R_1/R_2)^2$, rigorous proof of stability has been given only for axisymmetric disturbances and for some very special nonaxisymmetric disturbances: see, e.g., Chandrasekhar (1961), Sect. 70, and Schultz-Grunow (1967)]. When Ω_2/Ω_1 = const $< (R_1/R_2)^2$, however, according to the data of Fig. 13, instability of infinitesimal disturbances must arise for sufficiently great Re.

The flows corresponding to the unshaded region of Fig. 13 are certainly stable with respect to infinitesimal disturbances, but as experiment shows, for sufficiently large Re they are unstable with respect to finite disturbances, in this case loss of stability leading to the spontaneous formation of regions of developed turbulence (this is an example of "catastrophic transition" in the terminology of Coles (1965), who performed a brilliant experimental investigation of different types of transitions in circular Couette flow). The boundary of the region of stabihty to infinitesimal disturbances, calculated by Taylor, is in excellent agreement with the experimental data found by Taylor himself and by other investigators, using various visualization techniques. This was the first great (even epoch-making) success of the hydrodynamic disturbance theory (see Fig. 13a, where the dots denote the measured values of a number of pairs (Ω_1, Ω_2) for which loss of stability first occurred). Calculations for the case $R_2/R_1 = 2$ are also in excellent agreement with the experimental data

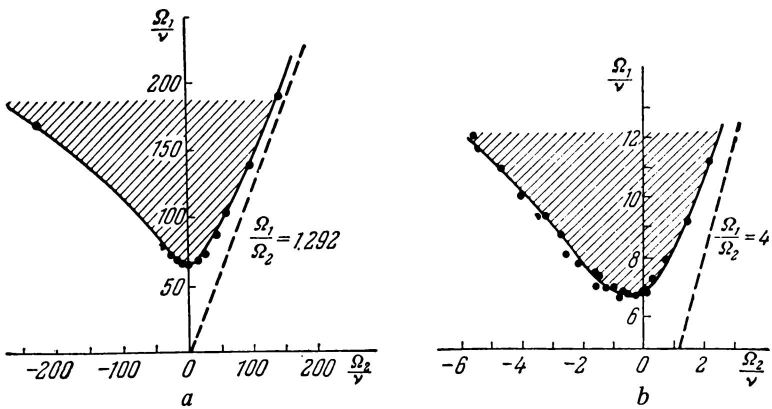


FIG. 13. Position of the region of instability in the plane (Ω_1, Ω_2) for Couette flow between cylinders, for values of the ratio R_2/R_1. The region of instability is shaded in the figure. The dotted lines indicate the boundary of the region of instability for the corresponding flow of an inviscid fluid (after Rayleigh).

of Donnelly and Fultz (1960) [dots in Fig. 13b, see also Donnelly (1962)]. Snyder (1968b) made a detailed comparison of all the existing stability calculations (for axisymmetric disturbances of circular Couette flow) with the experimental results of various investigators (including his own). With a few exceptions, excellent agreement was found. The exceptional cases can usually be explained by end effects; another explanation related to the results considered below is discussed by the author.

The experimental findings of G. I. Taylor (1923) and Donnelly and Fultz (1960) [shown in Fig. 13], appear to confirm the agreement between the regions of instability associated with axisymmetric disturbances and the regions of general instability; in other words, they are in excellent agreement with the assumption that for monotonically increasing Reynolds number, circular Couette flow always becomes unstable first to disturbances with $n = 0$. At the same time, however, as early as the work of Lewis (1928) it had been shown that for $\mu = \Omega_2/\Omega_1$ negative and rather large (i.e., for comparatively high rotational speeds of the cylinders in opposite directions), upon loss of stability motions appear which are sometimes said to be of "pulsating form," i.e., which do not appear to be axisymmetric. For many years Lewis' observation did not attract the attention of anyone. However, recent experiments by Nissan, Nardacci and Ho (1963) and Snyder [see Snyder and Karlsson (1965); Snyder (1968a); Krueger, Gross and Di Prima (1966)] completely confirmed its correctness. Furthermore, from a purely mathematical viewpoint, the investigation of the stability of circular Couette flow cannot be considered complete until the behavior of nonaxisymmetric disturbances has been analyzed. Finally, extensive experiments (which will be discussed further in Sect. 2.9) show that the loss of stability of axisymmetric Taylor vortices in the gap between the cylinders, arising upon further increase of the Reynolds number, very often leads to the formation of a new nonaxisymmetric flow. Consequently, the examination of the behavior of nonaxisymmetric disturbances of the flow between cylinders must be considered an important and interesting problem.

Apparently, Di Prima (1961) was the first whose work was devoted to the mathematical analysis of the behavior of nonaxisymmetric disturbances of circular Couette flow. He examined the general eigenvalue problem (2.14)–(2.16), and under a series of simplifying assumptions (including, in particular, the assumption of smallness of the gap $d = R_2 - R_1$ by comparison with $R_0 = (R_1 + R_2)/2$, and that of $\mu = \Omega_2/\Omega_1 \geq 0$) a method was displayed whereby successive (rather crude) numerical estimates are obtained for the smallest value of $\mathrm{Re} = \Omega_1 d^2/\nu$ at which, for n = 0, 1, 2 and 3, one of the eigenvalues of the system (2.14)–(2.16) appeared for the first time with a nonnegative imaginary part (specific calculations were carried out for the cases d/R_0 = 1/8, $\mu = 0$ and d/R_0 = 1/8, μ = 0.5). The results obtained show (as anticipated) that $\mathrm{Re}_{\mathrm{cr}}$ corresponds to axisymmetric disturbances. However, a somewhat surprising result was that the critical Reynolds numbers for nonaxisymmetric disturbances with small values of n turned out to be only slightly larger (by a few percent in all cases) than $\mathrm{Re}_{\mathrm{cr}}$ for axisymmetric disturbances with $n = 0$.

Later, Roberts (1965) calculated more precise values of $\mathrm{Re}_{\mathrm{cr}}$ for several fixed values of $\eta = R_1/R_2$ (to be exact, for η = 0.95, 0.90, 0.85 and 0.75), several values of n (in all cases not exceeding 4) and the value μ = 0 (i.e., $\Omega_2 = 0$). He also found that $\mathrm{Re}_{\mathrm{cr}}$ for μ = 0 always increases (although not strongly) with increasing n. The first work in which an asymptotic analysis of the eigenvalue problem (2.14)–(2.16) was carried out also for $\mu < 0$ (again in the limit of small gap), was the dissertation of Krueger (1962). According to his estimates for the case of small gap between the cylinders and $\mu \lesssim -0.8$, nonaxisymmetric disturbances with $n = 1$ become unstable for lower angular velocities of the cylinders than those at which axisymmetric disturbances first become unstable. Krueger, Gross and Di Prima (1966) made a more complete investigation of this question. They used a straightforward numerical procedure, and calculated (with the help of a digital computer) critical values of the so-called Taylor number Ta (proportional to $\Omega_1^2 R_1 d^3/\nu^2$, so that Ta $\propto$ $(\mathrm{Re})^2 = (\Omega_1 R_1/\nu)^2$ for fixed $\delta = d/R_1$) for a series of values of n and $\mu = \Omega_2/\Omega_1$ in the interval $0 \geq \mu \geq -1.25$ (in the hmit of small gap between the cylinders). Thus it turned out that if $\mu > -0.78$, then $\mathrm{Ta}_{\mathrm{cr}}$ increases with increasing n (i.e., axisymmetric disturbances become unstable first), but for $\mu \leqslant -0.78$ the minimum value of $\mathrm{Ta}_{\mathrm{cr}}$ is attained for $n \neq 0$, and with decreasing μ the critical value of n (giving the absolute minimum of $\mathrm{Ta}_{\mathrm{cr}}$ or $\mathrm{Re}_{\mathrm{cr}}$) increases (of course, not continuously; n cannot take on nonintegral values). For example, for $\delta = d/R_1$ = 1/20 and μ = –0.70, –0.80, –0.90, –1.00, and –1.25, with increasing rotational speed instability first appears for disturbances with n = 0, 1, 3, 4, and 5, respectively. On the other hand, for fixed μ, the values n_{cr} do not increase with increasing dimensionless gap width $\delta = d/R_1$, i.e., with decreasing $\eta = R_1 R_2$

(calculations with the approximations of small gap not made, show that for $\mu = -1$ and $\eta = 0.95, 0.90, 0.80, 0.70$, and 0.60, the greatest instability is attained for disturbances with $n = 4, 3, 2, 2$, and 2, respectively). However, it is important to note that the difference between the minimum value Ta_{cr} or Re_{cr} and the value Ta_{cr} or Re_{cr} determined from the outcome of an examination of axisymmetric disturbances only, is usually found to be rather small; for example, with $\delta = 1/20$ and $\mu = -1$, the critical value Re_{cr} (corresponding to $n = 4$) is only roughly 4% less than that Re for which axisymmetric disturbances first become unstable. This last circumstance induced Krueger, Gross and Di Prima to carry out additional cumbersome calculations using the complete eigenvalue problem (2.14)–(2.16) [without the simplification resulting from the assumption of small gap δ], to verify that the conclusions reached previously would not change. It was found that for values of μ of order -1, even for $\delta = 1/20$, using the approximation of small gap noticeably changes the numerical values Ta_{cr}; however, the change is almost identical for all n and k, so that all qualitative conclusions bearing on the comparison of the stability of disturbances for various values of n, are exactly the same in the small-gap approximation and in the exact formulation of the problem.

The results of Krueger, Gross and Di Prima are in good agreement with the experimental results of Nissan, Nardacci and Ho (1963) and Snyder and Karlsson (1965), according to which, in particular, for μ less than roughly -0.75, the secondary motion which appears on the loss of stability of circular Couette flow always turns out to be nonstationary and nonaxisymmetric (having a wavy or spiral structure). The conclusions of Coles (1965; and other works) also agree with these results. Coles carried out a detailed investigation of transition in Couette flow between two relatively short cylinders (appreciably shorter than those of most of the other investigators). Finally, the work of Snyder (1968a) was particularly concerned with verification of the conclusions of Krueger, Gross and Di Prima. Snyder investigated experimentally the stability of circular Couette flow (for four different values of $\eta = R_1/R_2$: 0.96, 0.80, 0.50, and 0.20) over a wide range of $\mu = \Omega_2/\Omega_1$, including large negative values. Here, for $\eta = 0.96$ (corresponding obviously to small gap) excellent agreement with the above-mentioned theory is found throughout; in particular, the values of μ corresponding to the first appearance of the most unstable disturbances with $n = 1, 2, 3$, and 4 were almost exactly as if they had been based on the calculations of Krueger, Gross and Di

Prima. Snyder obtained analogous results also for $\eta = 0.80$, 0.50, and 0.20 (where the assumption of small gap cannot be considered well satisfied). According to all these results, the form of the stability boundary shown in Fig. 13a must be changed slightly in the upper left corner of the figure to take into account the nonaxisymmetric disturbances (in Fig. 13b the range of values η, in which it is necessary to take into account disturbances with $n \neq 0$, is not shown).

In closing, we should note that the problem of the stability of flow between cylinders is one which has been subjected to the most detailed study in the theory of hydrodynamic stability. In addition to the results mentioned above, a great many problems also exist which have been subjected to general analysis, arising from the addition to the original problem of axial or circumferential pressure gradients, and/or the addition of axial, circumferential or radial magnetic fields, the replacement of the usual fluid by a non-Newtonian fluid, etc. [see, for example, Chandrasekhar (1961), Di Prima (1963), Krueger and Di Prima (1964), and a series of other works].

2.7 Stability of a Layer of Fluid Heated from Below

When a vertical, downwards temperature gradient exists in a fluid, there is additional destabilization of the flow due to buoyancy, which is similar to the effect of the centrifugal forces in curved flow when the rotational velocity of the fluid decreases with distance from the center of curvature. On the other hand, an upwards temperature gradient has a stabilizing effect on the flow, e.g., the case of curved flow with velocity increasing with distance from the center of curvature. Thus it is not surprising that the stability problem of a thin layer of fluid between two infinite planes at different temperatures is very similar mathematically to the stability problem of an incompressible fluid between two rotating cylinders.[12]

To solve this new problem we must take as our starting point the Boussinesq equations of free convection, discussed at the end of Sect. 1.5. For a layer of fluid bounded by the rigid planes $x_3 = 0$ and $x_3 = H$, which are maintained at constant temperatures T_0 and T_1, the

[12] In fact Jeffreys (1928), showed that the first of these problems is strictly equivalent to a particular case of the second; cf. Lin (1955) Sect. 7.3. The analogy between the centrifugal and buoyancy effects also plays a central role in Görtler's (1959) stability investigation; the same idea was used by Debler (1966) to obtain numerical results on the stability of a horizontal layer of fluid with heat sources directly from data on stability of circular Couette flow.

boundary conditions will have the form: $u_i = 0$, $T = T_0$ for $x_3 = 0$; $u_i = 0$, $T = T_1$ for $x_3 = H$. In the case of a free constant-temperature surface of the layer, the boundary condition of the velocity is changed so as to ensure constant pressure on this surface. There are also other types of mathematical boundary conditions, corresponding to different physical conditions on the boundaries of the layer [see, e.g., Sparrow, Goldstein and Jonsson (1964), or Hurle, Jakeman and Pike (1967)]. The steady state (the stability of which we are discussing), independent of the boundary conditions, will be the state of rest, for which[13]

$$\begin{aligned} &u_i = 0, \qquad T = T(x_3) = T_0 + \frac{T_1 - T_0}{H} x_3, \\ &p = p(x_3) = p_0 - g\rho x_3. \end{aligned} \tag{2.18}$$

Putting $T = T(x_3) + T'$, $p = p(x_3) + p'$ and linearizing the Boussinesq equations with respect to the disturbances u_i, T' and p', we obtain a system of five equations with five unknowns, from which it is easy to eliminate all the variables except T'. Now, if we transform to dimensionless variables, and then, following Eqs. (2.8) and (2.9) we seek $T'(\boldsymbol{x}, t)$ in the form of a product

$$T'(\boldsymbol{x}, t) = \exp\left[i\left(\frac{k_1 x_1}{H} + \frac{k_2 x_2}{H} - \frac{\omega \nu t}{H^2}\right]\theta(\zeta), \quad \zeta = \frac{x_3}{H}, \tag{2.19}$$

we obtain the following eigenvalue problem:

$$\left(\frac{d^2}{d\zeta^2} - k^2\right)\left(\frac{d^2}{d\zeta^2} - k^2 + i\omega\right)\left(\frac{d^2}{d\zeta^2} - k^2 + i\omega \cdot \mathrm{Pr}\right)\theta + k^2 \mathrm{Ra} \cdot \theta = 0, \tag{2.20}$$

$$k^2 = k_1^2 + k_2^2, \quad \mathrm{Pr} = \nu/\chi, \quad \mathrm{Ra} = \frac{g\beta(T_0 - T_1)H^3}{\nu\chi},$$

with boundary conditions on rigid surfaces with fixed temperature

$$\theta = \frac{d^2\theta}{d\zeta^2} = \frac{d}{d\zeta}\left(\frac{d^2}{d\zeta^2} - k^2 + i\omega\,\mathrm{Pr}\right)\theta = 0 \tag{2.20'}$$

$$\theta = \frac{d^2\theta}{d\zeta^2} = \frac{d^4\theta}{d\zeta^4} = 0 \tag{2.20''}$$

[13]Of course, in this analysis, we have assumed following the usual assumptions of the Boussinesq approximation, that the variation of density ρ with height is negligibly small, as far as the disturbance equations are concerned. Therefore, our results are applicable only to the case of relatively thin layers, with small values of H. However, in the case of very thin layers of fluid these results once again become inapplicable due to the breakdown of the conditions of linearity of the temperature profile and certain other conditions assumed in the theory under discussion [see Sutton (1950), Segel and Stuart (1962)].

[see, for example, Chandrasekhar (1961), Stuart (1963), and Lin (1955)]. Similar boundary conditions are also obtained for rigid boundaries of finite conductivity or for boundaries with variable temperature, but fixed heat flux or fixed linear relation between the heat flux and the temperature. In every case, we end up with an eigenvalue problem which, for given Prandtl number Pr, contains only two parameters: the wave number k and the Rayleigh number Ra. Consequently, to any given values of k and Ra will correspond an associated set of eigenvalues $\omega_j(k, \mathrm{Ra})$. It is found that for negative or small positive values of Ra (i.e., when the lower boundary is at the lower temperature, or when the lower boundary is only slightly warmer than the upper), all the eigenvalues $\omega_j(k, \mathrm{Ra})$for all values of k will have a negative imaginary part. However, after some "critical value" $\mathrm{Ra}_{\mathrm{cr}}$ (i.e., after some critical temperature difference $T_0 - T_1$, which also depends on the distance H between the planes), there will appear a value $k = k_{\mathrm{cr}}$ for which one of the eigenvalues $\omega_j(k_{\mathrm{cr}}, \mathrm{Ra}_{\mathrm{cr}})$ has a zero imaginary part. In this case, just as in the problem of the stability of flow between rotating cylinders, the principle of exchange of stabilities will apply, i.e., the first eigenvalue $\omega_j(k, \mathrm{Ra})$ with $\mathrm{Im}\,\omega_j = 0$ proves to be zero [a rigorous mathematical proof of this fact was given by Pellew and Southwell (1940)]. Thus the loss of stability of the state of rest on attaining the critical temperature difference $T_0 - T_1$ leads to the initiation of a steady convection periodic with respect to x_1 and x_2.

From a qualitative viewpoint, all the basic features of the transition from stability to instability in a layer of fluid heated from below were described very early by Rayleigh (1916b), who analyzed the mathematically much simpler (but physically unreal) problem of convection in a layer of fluid between two free boundaries at constant temperature. Mathematically, this problem reduces to an eigenvalue problem for the differential equation (2.20) with the boundary conditions (2.20″) for $\zeta = 0$ and $\zeta = 1$. To find the values of $\mathrm{Ra}_{\mathrm{cr}}$ and k_{cr} it is sufficient to consider Eq. (2.20) with $\omega = 0$, i.e.,

$$\left(\frac{d^2}{d\zeta^2} - k^2\right)^3 \theta + k^2\,\mathrm{Ra}\,\theta = 0. \tag{2.21}$$

Solutions of this equation, which satisfy Eq. (2.20″) for $\zeta = 0$ and $\zeta = 1$, are all of the form $\theta = \sin \pi n\zeta$, $n = 1, 2, \ldots$. Consequently. for sufficiently large values of Ra, we shall have a sequence of different neutral disturbances (with $\omega = 0$). the wave numbers of which will satisfy

$$(\pi^2 n^2 + k^2)^3 = k^2 \,\mathrm{Ra}, \tag{2.22}$$

where n is a positive integer. The minimum Rayleigh number for every given k will correspond to a disturbance with $n = 1$; Ra_{cr} will accordingly be defined from the condition $\mathrm{Ra}_{cr} = \min\limits_{k} \frac{(\pi^2 + k^2)^3}{k^2}$. Thus we obtain

$$\mathrm{Ra}_{cr} = 27\pi^4/4 \approx 657.5, \quad k_{cr} = \pi\sqrt{2}/2 \approx 2.2.$$

For the real problem of convection in a layer between rigid fixed-temperature boundaries, the analogous calculation demands the use of numerical methods because of the more complicated boundary conditions. Such calculations were carried out by Jeffreys (1962), Low (1929), and Pellew and Southwell (1940) [see also Lin (1955), Reid and Harris (1958), and Chandrasekhar (1961)]; it is found that in this case $\mathrm{Ra}_{cr} \approx 1708$ and $k_{cr} \approx 3.12$. The values of k and Ra for the first 10 eigenfunctions of this eigenvalue problem can be found in Catton (1966). Further, we note that the value of k determines only the periodicity of the flow arising in the (x_1, x_2) plane but not its form; in fact, it is not difficult to see that the function $\exp\left[i(k_1x_1/H + k_2x_2/H)\right]$ in Eq. (2.19), without changing the conclusions which follow, may be replaced by an arbitrary function $\varphi(x_1, x_2)$ satisfying the following equation:

$$\frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} + \frac{k^2}{H^2}\varphi = 0. \tag{2.23}$$

The explicit form of the function φ will determine the form of the cells into which the convection breaks down, but it cannot be determined uniquely on the basis of linear disturbance theory. However, the data of numerous experiments (described, for example, in Chandrasekhar (1961) [see also Stuart (1963) and the references cited there] show that the flow which arises most often breaks down into a set of cells (known as *Bénard cells*) in the form of hexagonal prisms, where in the middle the fluid moves upward and on the edges it moves downward, or vice versa.[14] To such cells there corresponds

[14]It is worth noting that hexagonal cells are not the only form of cellular convection observed in the experiments. More detailed discussion of this question will be given in Sect. 2.9 in connection with the nonlinear theory of thermal convection.

a completely defined form of the function φ, which was found by Christopherson (1940). Comparison of the velocity field corresponding to such an eigenfunction of our problem with the data of observations of Bénard cells may be found in Chandrasekhar (1961) and in the article by Stuart (1964) [who also used the results of nonlinear convection theory, which we shall discuss in Sect. 2.9]; it gives good agreement with experiment.

The situation is analogous when the upper boundary of the fluid is a free boundary with fixed temperature; in this case we need only replace the boundary condition (2.20′) on $\zeta = 1$ by (2.20″). As a result, we arrive at a new eigenvalue problem, for which numerical calculations show that the loss of stability occurs at $\mathrm{Ra}_{cr} \approx 1100$ [Pellew and Southwell (1940); Chandrasekhar (1961)]. This value of Ra_{cr}, similar to the value of Ra_{cr} for convection between rigid boundaries is confirmed excellently by experiment [cf., for example, Chandrasekhar (1961), Sutton (1950), Thompson and Sogin (1966)]. Values of Ra_{cr} corresponding to some other boundary conditions for temperature, may be found, for example, in the articles of Sparrow, Goldstein and Jonsson (1964), Hurle, Jakeman and Pike (1967) and Nield (1967). There are also many papers on other effects in the convection problem (first of all, on surface tension effects which apparently play an important role in Bénard's original experiments); the reference to these papers may be found in Stuart (1965) and Nield (1967; 1968).

With further increase of $T_0 - T_1$ (i.e., with increase of Ra) above the value of Ra_{cr} the steady "cellular" convection first preserves its character, but then becomes unstable, and, for $\mathrm{Ra} \sim 500{,}000$, disordered turbulent motion arises. It is important to note that the transition to turbulent flow takes place gradually in this case–in the form of a sequence of discrete "jumps," characterizing transitions to a sequence of different regimes of convection each of which is more complicated and less ordered than the previous one [see Malkus (1954a), Willis and Deardorff (1967)].

2.8 Stability of Parallel Flows

Above, we considered two examples of the application of the method of small disturbances to the investigation of hydrodynamic stability. However, from the viewpoint of the experimenter or engineer, both these examples are rather specialized. Far more suitable for experimental verification and for practical application are the cases of flows in a circular tube and along a flat plate (which is why they were given particular attention at the beginning of this section). Nevertheless, to illustrate the small disturbance method, first, we considered the flow between rotating cylinders and free convection in a layer between two planes of constant temperature. This was done because in these two examples (due to the presence of additional effects connected with a centrifugal force in the first case and with buoyancy in the second), the method of small disturbances leads to comparatively simple eigenvalue problems without any singularities even in the limiting case $\nu = 0$); therefore quite definite results can be obtained in these examples. For flows in tubes and boundary layers, however, use of the small disturbance method encounters quite considerable mathematical difficulties, which even now cannot be considered to have been completely solved.

Because of the complexity of the mathematical analysis of stability of flows in tubes and boundary layers, practically all of the relevant investigations have considered only the simplest plane-parallel, two-dimensional flows, which to some degree may be regarded as models of the real flows in question. Therefore, we shall also begin our discussion with the stability problem for plane-parallel flows and only then shall we point out what is known concerning the stability of real flows in tubes and boundary layers. Further, we note that the equations of fluid dynamics (1.6) show that a steady plane-parallel flow of viscous fluid with only one nonzero component of velocity $U = u_1$, dependent only on the coordinate $x_3 = z$, is possible only on the condition that the dependence of the profile of $U(z)$ on z is quadratic. In other words, such a flow is always a combination of a Couette flow with linear velocity profile and a Poiseuille flow with a parabolic profile (see Examples 1 and 2, at the end of Sect. 1.2). However, keeping in mind that we are interested in plane-parallel flows as possible models of more complicated real flows, we shall also consider more complicated profiles of $U(z)$ in the hope that the results obtained may be applicable to flows that are not strictly plane-parallel.

We shall assume the axis $Ox_1 = Ox$ to be directed along the flow, and that the velocity of the undisturbed flow is independent of $x_2 = y$ and is given by an arbitrary function $U(x_3) = U(z)$. It is important to note that *in finding the criterion of instability of such a plane-parallel flow, we may limit our consideration to only two-dimensional disturbances of the form* $\boldsymbol{u}' = \{u'(x, z, t), w'(x, z, t)\}$, since the more general three-dimensional disturbances lose their stability later (for higher Reynolds numbers) than two-dimensional disturbances. This assertion, proved by Squire (1933) [see also Lin (1955), Sect. 3.1, Betchov and Criminale (1967), Sect. 18] may be explained by writing the disturbance as the superposition of elementary plane-wave components. Two-dimensional disturbances will correspond to waves of the form $\exp i(kx - \omega t)$ propagated along the undisturbed flow, and general three-dimensional disturbances to waves of the form $\exp i(k_1 x + k_2 y - \omega t)$ propagated in a direction which does not coincide with the direction of the undisturbed flow.

If the coordinate frame is rotated in the (x, y) plane so that the new x-axis is in the direction of the wave, then the basic flow has two components $U(z)$, $V(z)$, and the wave propagates in the x-direction, and is independent of y. Further, the equations for u', ω', p' are independent of V and v' in such a case. These considerations show that a wave component of a disturbance with a given wave-number vector is in fact affected only by the component of velocity of the undisturbed flow in the direction of this wave-number vector, which equals Uk_1/k, where k_1 is the x-component of $\boldsymbol{k}$, and k is its modulus. Mathematically, this is seen by the fact that for plane-parallel flow with velocity $U = U(z)$, substitution of Eqs. (2.8) and (2.9) with $x_1 = x$, $x_2 = y$, $x_3 = z$ in the general system (2.7) after certain transformations [and elimination of all unknown functions except the function $f(z) = f^{(w)}(z)$, which gives the component $u_3'(\boldsymbol{x}, t) = w'(\boldsymbol{x}, t)$, of the disturbance $\boldsymbol{u}'(\boldsymbol{x}, t)$] leads to the equation

$$(k_1 U - \omega)\left(\frac{d^2}{dz^2} - k^2\right) - k_1 \frac{d^2 U}{dz^2}\Big] f = - i\nu \left(\frac{d^2}{dz^2} - k^2\right)^2 f, \quad k^2 = k_1^2 + k_2^2, \tag{2.24}$$

This equation together with the corresponding boundary conditions determines the eigenfrequencies of the problem, $\omega = \omega_j$, $j = 1, 2, \ldots$. Now, if we consider a two-dimensional disturbance with the same modulus of the wave-number vector $\boldsymbol{k}$ (i.e., a disturbance of the form (2.8)–(2.9) with $k_1 = k$ and $k_2 = 0$), then the new equation (2.24) will differ from the old one only in the replacement of $k_1 U$ by kU, i.e., it will be identical to the above equation, but with the velocity $U(z)$ increased by a factor of k/k_1. Thus, for wave disturbances propagated at some angle to the undisturbed flow, the effective Reynolds number is less than for disturbances with the same wavelength propagated in the direction of the undisturbed flow, and therefore the critical Reynolds number for two-dimensional disturbances will be subcritical for three-dimensional disturbances. Consequently, in finding the criterion of stability for plane-parallel flows, it is always permissible to assume that $k_2 = 0$, i.e., that the disturbance is independent of y. We can also assume then that $u_2'(\boldsymbol{x}, t) = v'(\boldsymbol{x}, t) = 0$, because the fluctuations v' have no effect on the system of equations in u' and w' (and will be damped in time).[15]

For two-dimensional disturbances of a plane-parallel flow, the general linearized equations (2.7) take the form

$$\begin{gathered}
\frac{\partial u'}{dt} + U\frac{\partial u'}{\partial x} + w'\frac{dU}{dz} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} + \nu\nabla^2 u', \\
\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + \nu\nabla^2 w', \\
\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0,
\end{gathered} \tag{2.25}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

is the two-dimensional Laplace operator. The boundary conditions for these equations are that both components of velocity u' and w' become zero on the solid boundaries of the flow (or, in the case of an unbounded flow, at infinity, i.e., as $z \to \infty$ or as $z \to -\infty$ or as $z \to \pm\infty$, depending on the nature of the unbounded flow). Using the equation of continuity, in the first two equations of Eq. (2.25), we transform from the components of velocity u' and w' to the stream function ψ, putting

$$u' = -\frac{\partial\psi}{\partial z}, \qquad w' = \frac{\partial\psi}{\partial x}.$$

After eliminating the pressure, we obtain the following partial

[15]This does not mean, of course, also that in the case $\mathrm{Re} > \mathrm{Re}_{cr}$ only two-dimensional disturbances independent of y are important. Since the existing data definitely show that in transition to turbulence of a two-dimensional boundary layer, three-dimensional disturbances play a major role [see, for example, Klebanoff, Tidstrom and Sargent (1962)], the investigation of the behavior of such disturbances at supercritical Reynolds numbers is also of interest. Such an investigation was carried out, in particular, by Watson (1960b) and Michael (1961). They showed that within the framework of the linear disturbance theory, there always exists a range of values of Re, $\mathrm{Re}_{cr} < \mathrm{Re} < \mathrm{Re}_l$ for any plane-parallel flow within which, of all the unstable wave disturbances, the most rapidly increasing (i.e., that possessing the greatest value of $\mathrm{Im}\,\omega$) will be some two-dimensional disturbance (although, in the case of disturbances with fixed wave number k, for certain values of k the most unstable will be three-dimensional disturbances). Apparently, however, three-dimensional disturbances will often begin to play a fundamental role for values of Re only slightly greater than Re_{cr}, as a result of essentially nonlinear effects (see below, Sect. 2.9).

differential equation for the stream function:

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2\psi-\frac{d^2U}{dz^2}\frac{\partial\psi}{\partial x}=\nu\nabla^2(\nabla^2\psi) \tag{2.26}$$

To find the instability criterion, it is necessary to find the solution $\psi(x, z, t)$ of this equation for the given boundary conditions and given initial value $\psi(x, z, 0) = \psi_0(x, z)$. Here, $\psi_0(x, z)$ is an arbitrary function, nonzero within a finite region of space (and satisfying the boundary conditions and small disturbance condition $|\nabla\psi_0| \ll U$). The nature of the time-variation of $\max\limits_{(x, z)} |\nabla\psi(x, z, t)|$ will depend on the Reynolds number of the undisturbed flow; for sufficiently small Re, this value will always be damped, and for sufficiently large Re, it may sometimes prove to increase with time. The boundary between these regions of values of Re (if they exist) will define the critical Reynolds number.

In general, the solution of this initial value problem is very complicated. However, since the coefficients of Eq. (2.26) are independent of t and x, this equation may have particular solutions of the form

$$\psi(x, z, t)=e^{i(kx-\omega t)}\varphi(z)=e^{ik(x-ct)}\varphi(z), \tag{2.27}$$

where $c = \omega/k$. If there are a sufficient number of such "wave-like solutions" that it is always possible to resolve the arbitrary solution $\psi(x, z, t)$ of the initial value problem into solutions of the form (2.27) with all possible (real) wave numbers k, then the general problem of finding the value of Re_{cr} will be reduced to determining only the permissible values of c. As a result, we must substitute Eq. (2.27) into (2.26); we then obtain the following ordinary differential fourth-order equation for $\varphi(z)$:

$$(U-c)(\varphi''-k^2\varphi)-U''\varphi=-\frac{i\nu}{k}(\varphi^{IV}-2k^2\varphi''+k^4\varphi) \tag{2.28}$$

(where the primes denote differentiation with respect to z). Equation (2.28) is called the *Orr-Sommerfeld equation*. The boundary conditions for this equation are that φ and φ' become zero on the boundaries of the flow (which may be at infinity). The homogeneous equation (2.28) with the given homogeneous boundary conditions will have a nontrivial solution only for certain special values of the parameter c. These *eigenvalues of the problem* will be, generally speaking, complex numbers $c = c_1 + ic_2$, dependent on the wave number k and on the viscosity ν (i.e., on the Reynolds number Re). The real part c_1 will have the meaning of the phase velocity of propagation of the wave, and the imaginary part c_2 will define the time variation of the amplitude of the wave, which is described by the factor e^{kc_2t}. Therefore, for $c_2 < 0$ the amplitude of the wave will be damped, for $c_2 = 0$ it will remain constant (in this case the wave is called *neutral*), while for $c_2 > 0$ it will increase with time. Thus $\mathrm{Re}_{\mathrm{cr}}$ can be defined as the value of Re for which $c_2(k, \mathrm{Re})$ is first equal to zero, at least for one value of k.

To find the instability criterion, it is only necessary to define c_2 as a function of k and Re. Further, we note that at the value $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$ where $c_2(k, \mathrm{Re})$ first becomes equal to zero, the real part c_1 of the eigenvalue $c(k, \mathrm{Re})$ with $\operatorname{Im} c = c_2 = 0$ will not, generally speaking, be equal to zero (breakdown of the "principle of exchange of stabilities"). This means that the eigenvalue c usually corresponds to propagation of the wave (2.27) along the Ox axis. For values of Re somewhat larger than $\mathrm{Re}_{\mathrm{cr}}$, a fairly small range of values of k will be found, for which c_2 will be positive (and c_1 nonzero). Waves with values of k from this range will form a wave packet, which will grow with time, moving simultaneously downstream. In fact, the value of kc_2 will be greatest somewhere near the center of this range of values of k; thus, close to this center $\frac{\partial}{\partial k}(kc_2) = 0$, and the group velocity $\frac{\partial}{\partial k}(kc) \approx \frac{\partial}{\partial k}(kc_1)$ of our wave packet will be real, i.e., it will represent the true velocity of displacement of the packet [cf. Landau and Lifshitz (1963), Sect. 29]. In this respect, the instability under discussion will differ from that of the flows considered in Sects. 2.6 and 2.7, where the unstable disturbances did not move, and, at a given point, increased to a finite value.

Stability of Plane-Parallel Flows of an Ideal Fluid

Substituting some actual velocity profile $U(z)$ into Eq. (2.28) we arrive at a very complicated eigenvalue problem, the solution of which requires the use of sophisticated and cumbersome mathematical procedures. To simphfy these procedures, we begin with an attempt to use experimental data, according to which the critical Reynolds number for many plane-parallel flows is extremely large. Consequently, we may expect that for Reynolds numbers close to the critical value, the terms on the right side of the Orr-Sommerfeid equation (2.28), which describe the action of the viscous forces on a small disturbance, will be small compared with the terms on the left side. Therefore, first, we may try considering the fluid as ideal, i.e., we ignore the right side of Eq. (2.28) and consider the abridged equation:

$$(U-c)(\varphi''-k^2\varphi)-U''\varphi=0, \tag{2.29}$$

which was studied in detail by Rayleigh (1880; 1887; 1895; 1913). Since this equation (called the Rayleigh equation) is not of fourth order but only of the second, we can no longer demand that four boundary conditions be satisfied. However, we must confine ourselves to the ordinary requirements for a flow of ideal fluid, e.g., that on the boundary walls only the normal component of velocity is equal to zero. Bearing in mind the definition of the stream function ψ and Eq. (2.27), we conclude that the condition $\varphi = 0$ must be satisfied on the walls.

However, by completely ignoring the viscosity and using the abridged equation (2.29), we encounter a number of difficulties. These are connected, first, with the fact that for real eigenvalues c (corresponding to neutral waves, which are of considerable interest for finding the instability criterion) there may be a value of z (say, $z = z_0$) for which the velocity of the undisturbed flow $U(z_0) = U_0$ will equal c, so that z_0 is a singular point of Eq. (2.29). Moreover, Rayleigh [and later, by another method, Tollmien and Pretsch (1946)] showed that in the case of plane-parallel flow of an ideal fluid, *the phase velocity c of the neutral waves will always be between the minimum and the maximum velocity of the undisturbed flow* (i.e., $U_{min} \leqslant c \leqslant U_{max}$), so that there must be a singular point z_0 within the flow.[16] If $U'(z_0) \neq 0$, then, in the neighborhood of this

[16]The simplest proof is the following [cf. Drazin and Howard 1966)]. Let us rewrite the Rayleigh equation (2.29) as

$$((U-c)^2F')'-k^2(U-c)^2F = 0\ ,\quad F = \varphi/(U-c)\,.$$

Multiplying this equation by the complex conjugate function F^* and integrating from z_1 to z_2 (where z_1 and z_2 are boundaries of the flow, so that $F = 0$ at $z = z_1$ and $z = z_2$) one obtains for nonsingular F

$$\int_{z_1}^{z_2}(U-c)^2\{|F'|^2+k^2|F|^2\}dz = 0\,.$$

The last equation implies that F cannot be nonsingular when c is real and therefore that c

(cont'd p. 118)

point we may put $U(z) - c \approx U'(z_0)\,[z - z_0]$, and, as $z \to z_0$, the function φ'' will tend to infinity as $\frac{U''(z_0)\,\varphi(z_0)}{U'(z_0)\,(z - z_0)}$. Thus the x-component of velocity of a neutral disturbance in the neighborhood of a singular point will take the form

$$u' \sim \varphi' \sim \frac{U''(z_0)\,\varphi(z_0)}{U'(z_0)} \ln\,(z - z_0).$$

Thus, for a neutral wave, one of the two linearly independent solutions of the abridged equation (2.29) will be discontinuous and multivalued, which gives rise to the question: which branch of the multivalued function are we to take?

There is the further difficulty that when c in Eq. (2.29) [corresponding to the eigenfunction $\varphi(z)$] is a complex number, its complex conjugate c^* will also be an eigenvalue of this equation [corresponding to the eigenfunction $\varphi^*(z)$]. Consequently, in addition to the damped wave, the equation for the stream function will always have a solution in the form of a growing wave. Therefore, for an inviscid fluid it is meaningless to define the stable case as the case when only damped oscillations occur. Thus the very definition of stability based on the consideration of elementary wave-like solutions must be changed. Stability now must be defined as the absence of growing wave-like disturbances (so that the criterion for stability is that the Rayleigh equation has only real eigenvalues c for any k).

The usual method of overcoming all these difficulties consists of going back to the complete Orr-Sommerfeld equation (2.28). Here, we select the proper branch of the solution of the abridged equation (2.29), which describes neutral oscillations, as well as the wave solutions of the equation with $\operatorname{Im} c \neq 0$ which have a physical meaning, by carefully investigating the asymptotic behavior of solutions of the fourth-order equation (2.28) as $\nu \to 0$ (i.e., as $\mathrm{Re} \to \infty$).[17]

cannot lie beyond the range of U.

Note further that for complex $c = c_1 + ic_2$ with $c_2 \neq 0$ the imaginary part of the integral relation above implies that $c_1 = \operatorname{Re} c$ *cannot lie beyond the range of* U (this result was also first demonstrated by Rayleigh). Finally, simple manipulations with both the real and imaginary parts of the same integral give *the semicircle theorem* by Howard (1961). This theorem states that all eigenvalues c with $c_2 = \operatorname{Im} c > 0$ (corresponding to unstable waves) lie in a semicircle of the complex c-plane with center in the real point $\frac{1}{2}(U_{max} + U_{min})$ and radius $\frac{1}{2}(U_{max} - U_{min})$, so that *any eigenvalue c, real or complex, must lie in or on the circle with center* $\frac{1}{2}(U_{max} + U_{min})$ *and radius* $\frac{1}{2}(U_{max} - U_{min})$.

[17] We note that in the case of neutral waves—with real c—we need to take into account the viscosity only in the neighborhood of the singular point $z = z_0$, where an additional "internal boundary layer" is introduced.

In particular, by using this approach it was found that the amplified and damped wave solutions of Eq. (2.29) which have physical meaning, are not conjugate to each other, and that in certain cases, such solutions must be constructed by "sticking together" different expressions in different domains of z [see, for example, Lin (1955), Chapt. 8; Stuart (1963); Shen (1964); and Reid (1965)]. Of course, this approach to inviscid stability theory is fairly cumbersome, and the advantages given by the relative simplicity of Eq. (2.29) are in fact lost. However, there is another approach to the same problem which allows the use of the relative simplicity of the inviscid equations; this will be discussed later.

Let us now give the main results of the stability theory of plane-parallel inviscid flows. The important works of Rayleigh (mentioned above) served as a starting point of the theory. In particular, as early as 1880 he showed that *if* $U''(z)$ *is nonzero everywhere within the flow, then the abridged equation (2.29) cannot possess complex eigenvalues with* $\operatorname{Im} c \neq 0$. The proof of this theorem is very simple. Rewriting Eq. (2.29) in the form

$$\varphi'' - k^2\varphi - \frac{U''\varphi}{U - c_1 - ic_2} = 0,$$

multiplying throughout by the complex-conjugate function φ^* and integrating from $z = z_1$ to $z = z_2$ (where z_1 and z_2 are the boundaries of the flow on which $\varphi(z)$ is equal to zero) we obtain

$$\int_{z_1}^{z_2} (|\varphi'|^2 + k^2|\varphi|^2)\,dz + \int_{z_1}^{z_2} \frac{(U - c_1 + ic_2)\,U''\,|\varphi|^2}{(U - c_1)^2 + c_2^2}\,dz = 0. \qquad (2.30)$$

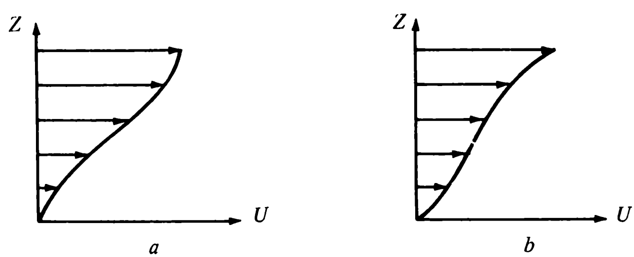
With $c_2 \neq 0$, the imaginary part of this equation can become zero only if $U''(z)$ changes sign somewhere between z_1 and z_2. Thus we obtain the result cited. It was widely accepted for many years that this Rayleigh theorem gave complete proof of the stability (in the sense of the absence of growing disturbances) of plane-parallel flows of ideal fluid, the velocity profile of which does not possess an inflection point. (We note that for an ideal fluid with $\nu = 0$, any velocity profile $U(z)$ will be permissible from the viewpoint of the equations of motion, unlike the case of a viscous fluid.) However, this, in fact, is misleading. The singularity of Eq. (2.29) connected with the coefficient of a higher derivative becoming zero, leads to the appearance of a continuous spectrum of eigenvalues c (in addition to the usual discrete spectrum) while the eigenfunctions for eigenvalues of the continuous spectrum are "generahzed functions," i.e., they do not satisfy the usual conditions imposed on eigenfunctions (for example, they may contain Dirac's δ-functions) and very often simply go unnoticed (in particular, the proof of Rayleigh's theorem refers only to simple discrete eigenvalues). It is interesting that the existence of the continuous spectrum was known to Rayleigh [(1894), Vol. 2, pp. 391–400], but its importance was completely disregarded until recently. If we confine ourselves to a discrete spectrum, the boundary-value problem for the Rayleigh equation (2.29) will usually have no more than a finite number of eigenfunctions for every fixed k. Consequently, the set of these functions will evidently be incomplete (e.g., for a Couette flow, where $U'' \equiv 0$, there will be no eigenfunctions which vanish on the two boundaries; later, we shall see that this situation occurs rather often). However, if we also consider the continuous part of the c-spectrum, we must first define precisely what we mean by corresponding eigenfunctions and investigate their behavior as $t \rightarrow \infty$; then we must prove that the system of functions obtained is complete, which is no easy task. Thus, in fact, the proof of stability of an inviscid flow by considering only elementary wave-like disturbances does not follow from Rayleigh's theorem alone, but requires additional sophisticated procedures.

In this connection Case (1960a,b) and Dikiy (1960a,b) independently of each other, stated that for inviscid stability analyses it is advantageous to discard the consideration of elementary wave-like solutions of the form (2.27). Instead, one should solve the general initial value problem for the partial differential equation (2.26) with zero right side, i.e., with $\nu = 0$, and initial conditions $\psi(x, z, 0) = \psi_0(x, z)$ [this is the second approach to the inviscid stability problem which we mentioned earlier]. Both authors solved the initial value problem by using the Laplace transform with respect to time, and obtained the solution in the form of a Laplace integral, the asymptotic behavior of which as $t \rightarrow \infty$ may be studied by ordinary methods of the theory of functions of a complex variable. The integrand in the corresponding Laplace integral may also be used for accurate determination of the set of eigenfunctions (for both the continuous and the discrete spectrum) of Eq. (2.29); the completeness of this set of functions will then be obtained automatically (since the solution of an arbitrary initial value problem is expanded in it). However, as already mentioned, the consideration of elementary wave-like solutions satisfying Eq. (2.29) is quite unnecessary for the study of stability, since the behavior as $t \to \infty$ of the general solution of the initial value problem will immediately determine whether or not the flow will be stable. Case (1960a) and Dikiy (1960b) independently of each other, outlined the proof of the fact that in all cases, increasing solutions of the initial value problem for Eq. (2.26) can exist only when the corresponding equation (2.29) has discrete complex eigenvalues c, for which $\operatorname{Im} c > 0$ (or multiple discrete real eigenvalues c). More precisely, they also outlined the proof that the discrete spectrum alone is associated with instability (while the part of the Laplace integral respresentation of $\psi(x, z, t)$ over the continuous spectrum always decays like $1/t$ as $t \to \infty$). According to Rayleigh's theorem, complex eigenvalues of Eq. (2.29) cannot exist if the profile $U(z)$ has no inflection points. A more detailed investigation of Eq. (2.29), described in Lin (1955), Sect. 8.2, shows that a change of sign of $U''(z)$ within the flow is also necessary for the existence of purely real eigenvalues c; in other words, in the absence of an inflection point of the velocity profile Eq. (2.29) has no discrete eigenvalues at all. Therefore, the results of Case and Dikiy give the first rigorous justification for the deduction which is generally made from Rayleigh's theorem, concerning the stability of inviscid plane-parallel flows without inflection points of the velocity profile. It is interesting that for the special case of a plane Couette flow in an ideal fluid, the approach described here was used as early as Orr (1906–1907), who solved the general initial value problem for a small disturbance in such a flow and showed that the solution always decays [cf. Drazin and Howard (1966), p. 28]; later, however, this result of Orr's was set aside. Dikiy and Case showed also [cf. especially Case (1961)] that the solution of the problem with initial condition $\psi(x, z, 0) = \psi_0(x, z)$ for the complete equation (2.26) always tends to the solution of the same problem for the abridged equation as $\nu \to 0$. Thus, in this respect also the initial value problem has a considerable advantage over the boundary-value problem for Eq. (2.28), in which there is no simple connection between the eigenvalues and eigenfunctions of the complete and abridged equations [cf. Lin (1961), and Lin and Benney (1962)]. An extensive review of the initial-value-problem approach to the study of inviscid hydrodynamic stability can be found in Drazin and Howard (1966); rigorous proofs of some results in this field, only outlined by Case and Dikiy, were given (under additional restrictions) by Rosencrans and Sattinger (1966).

Let us now assume that the velocity profile has an inflection point, i.e., that $U''(z)$ becomes zero at least at one point. According to the theorem of Rayleigh, this condition is necessary but not sufficient for instability of inviscid plane-parallel flow. One very simple and stronger condition necessary for instabihty was noted by Fjørtoft (1950) and Hølland (1953). To obtain this condition we must only add the real part of Eq. (2.30) to the imaginary part of the same equation multiplied by the factor $(c_1 - U_s)/c_2$ where $U_s = U(z_0)$ is the value of the velocity at the inflection point. Then we get

$$\int_{z_1}^{z_2} \frac{U''(U - U_s)|\varphi|^2}{(U - c_1)^2 + c_2^2} dz = -\int_{z_1}^{z_2} (|\varphi'|^2 + k^2|\varphi|^2)\, dz < 0 .$$

Therefore, not only must $U''(z)$ vanish in some point $z = z_0$, but also $U''(U - U_s)$ must be negative somewhere in the field of flow. In particular, if $U(z)$ is a monotonic function and $U''(z)$ vanishes in one point z_0 only, then for instability the inequality $U''(U - U_s) < 0$ must be fulfilled at every $z \neq z_0$ (so that, for example, the flow with the velocity profile shown in the figure below *a* may be unstable, but the



Two types of velocity profiles with one inflection point.

flow with the velocity profile of the figure *b* is always stable). This new necessary condition of instability is also insufficient, in general. For example, Tollmien (1935) showed that a flow with the velocity profile $U(z) = \sin z$ in the layer $z_1 \leq z \leq z_2$, $z_1 < 0 < z_2$, is stable if $z_2 - z_1 < \pi$, although the Fjørtoft–Hølland condition is vahd here [cf. Drazin and Howard (1966), p. 35]. However, in the same paper, Tollmien proved that the presence of an inflection point of the velocity profile is sufficient for instability in very important cases of channel flows with symmetric velocity profiles and flows of the boundary-layer type with velocity profiles of the form shown in Fig. *a* [see also, Lin (1955), Sect. 8.2; Shen (1964), Sect. G, 7; and Drazin and Howard (1966)]. For all such flows the vanishing of $U''(z)$ at some point implies that the Fjørtoft-Høiland condition is also fulfilled and that the Rayleigh equation (2.29) will have eigenvalues c such that $\operatorname{Im} c > 0$. One other necessary condition of instability which is also sufficient in the case of a monotonic profile $U(z)$ with only one inflection point was given by Rosenbluth and Simon (1964).

The results of Case, Dikiy, and Rosencrans and Sattinger indicate that the continuous spectrum of the Rayleigh equation is generally (or even always) stable. Thus the investigation of the instability characteristics of various plane-parallel flows of nonviscous fluid is equivalent to the finding of all discrete eigenvalues and eigenfunctions of the corresponding Rayleigh equation. This eigenvalue problem is difficult to solve explicitly in the case of smoothly varying functions $U(z)$. However, when $U(z)$ is a piecewise-linear polygonal contour, one can easily find exponential solutions of the stability equation for different z-regions and then obtain the explicit eigenvalue relation by using the natural "joining conditions" at the corners of $U(z)$. This method was first used by Helmholtz (for the schematic case of a tangential velocity discontinuity), Kelvin and Rayleigh [see especially the classical book by Rayleigh (1894), Vol. 2, Chapt. XXI]; it was also recently used by Drazin and Howard (1962; 1966) and Michalke and Schade (1963).

The introduction of electronic computers changed the situation; now it is easy to solve the eigenvalue problem for the Rayleigh equation numerically even in the case of smooth profiles $U(z)$. Many examples of such solutions can be found in surveys by Drazin and Howard (1966), Sect II.4, and Betchov and Criminale (1967), containing references to original papers.

The results for the cases of a Bickley jet with $U(z) = \operatorname{sech}^2 z$, $-\infty < z < \infty$, and a hyperbolic-tangent free boundary layer with $U(z) = 0.5(1 + \tanh z)$, $-\infty < z < \infty$, were compared with experimental data obtained by Sato (1960), Freymuth (1966) and others who measured the growth of artificial disturbances introduced in jets and jet boundary layers by sound from a loudspeaker [see the review by Michalke and Freymuth (1966)].

The first results of the comparison were satisfactory but more detailed investigations show that the experimental data deviate in some respects from the calculated ones. One possible explanation is that the theory considers strictly wave-like disturbances growing in time, while in the experiments we have time-periodic disturbances growing spatially in the direction of the mean flow. To use the theory for the explanation of the experimental data, one transforms the temporal growth rate linearly into a spatial growth rate by means of the disturbance phase velocity (as was done first by Schubauer and Skramstad for the case of a viscous boundary layer; we shall discuss this work later). However, the transformation indicated is not a rigorous one, as was explained by Gaster (1962; 1965). Therefore the theory of time-periodic spatially growing disturbances based on the investigation of the eigenvalue problem for the Rayleigh equation (2.29) with fixed (real) frequency $\omega = kc$ and unknown eigenvalues (generally complex) k is more suitable for the description of this experimental data. Such a theory was developed by Michalke (1965b) for the hyperbolic-tangent shear layer and by Betchov and Criminale (1966; 1967) for several inviscid jet and wake profiles. The results of Michalke's calculations were in excellent agreement with the experimental findings of Freymuth (1966) for all moderate Strouhal numbers $\mathrm{St} = \omega\delta/U_0$ (where δ and U_0 are characteristic scales of boundary-layer thickness and jet velocity). For greater values of the Strouhal number it is necessary to take into account the deviation of the real velocity profile from the hyperbolic tangent (see Michalke (1968a) on spatially growing disturbances in free shear layers of different velocity profiles).

In the case of a fluid having density varying with height (i.e., with the z coordinate), we will retain even with nonzero viscosity the same difficulties which arose in the instability investigation of inviscid plane-parallel flows; i.e., even when $\nu \neq 0$, the corresponding Orr-Sommerfeld equation will have a singularity at the point where $U(z) = c$ [see, for example, Dikiy (1960a)].[18]

Consequently, we are not justified in transferring to this case the constant density rules for selecting the proper branch of the multivalued solutions, as was done by Schlichting (1935b). The instability of a horizontally stationary layer of a heavy fluid of variable density was studied in the so-called *Rayleigh-Taylor instability theory* [see, e.g., Chandrasekhar (1961), Chapt. X; Selig

[18] In fact, if in addition to nonzero viscosity, we also take into account nonzero thermal conductivity, we shall once again obtain an ordinary eigenvalue problem (this time of the sixth order) without any singularity in the coefficient of the leading term [see Koppel (1964)].

(1964)]. At present, the results for viscous heterogeneous fluid in horizontal motion are very incomplete [cf. Drazin (1962)]. We must note that for the case of variable density, the reduction of the determination of the instability criterion to an eigenvalue problem, without considering a continuous spectrum, is unjustified even if $\nu \neq 0$, due to the incompleteness of the corresponding system of eigenfunctions. Therefore, for flow of fluid with density varying with height, both for $\nu = 0$ and $\nu \neq 0$, a strict stability analysis demands the study of the asymptotic behavior as $t \to \infty$ of the solution of the corresponding initial value problem. This analytical problem is very difficult and what progress has been made in this field was achieved fairly recently and only on the assumption that $\nu = 0$ (i.e., for an ideal fluid).

Thus, Eliassen, Høiland and Riis (1953) and, later, more accurately, Dikiy (1960a) and Case (1960b), investigated the asymptotic behavior of the solution of the initial value problem for the stream function of the disturbance for a two-dimensional flow of a nonhomogeneous heavy fluid with exponentially decreasing density, filling an unbounded half-space and possessing a linear velocity profile. They showed that just as in the case of plane-parallel flows of an ideal fluid, instability (in the sense of the presence of increasing disturbances) can arise in this case only when there exist wave-like disturbances of the form (2.27) with increasing amplitudes (i.e., with $\operatorname{Im} c > 0$).

By virtue of the results of the investigation of the eigenvalue problem carried out by G. I. Taylor in 1915 [but published much later; see Taylor (1931)], supplemented recently by a more strict mathematical analysis of the behavior of the zeros of the confluent hypergeometric functions, carried out independently by Dikiy (1960c) and Dyson (1960), it follows that such a flow of inhomogeneous fluid will be stable (in the sense indicated) for all positive values of Richardson number Ri (and not only for $\mathrm{Ri} > 1/4$ as was asserted in the well-known books and survey paper of Prandtl (1949) and Schlichting (1960; 1959). In fact, in this case the corresponding eigenvalue problem has no discrete eigensolutions when $0 < \mathrm{Ri} < 1/4$ and only stable ones when $\mathrm{Ri} > 1/4$. Later, a similar means of investigation was used by Miles (1961) for the general plane-parallel flow of inviscid heterogeneous heavy fluid with a practically arbitrary density profile $\rho(z)$ [such that $\rho'(z) < 0$ for all z, and that the density decreases monotonically with height], and any monotonic velocity profile $U(z)$. Miles showed that in this case also, it follows from the existence of unstable disturbances that unstable waves exist with stream function of the form (2.27). In addition, he gave a detailed investigation of possible wave-like disturbances in this flow, and as a result, was able to show that the flow under discussion will certainly be stable if at all of its points $\mathrm{Ri} > 1/4$ (special cases of the last result are contained in the works of Goldstein (1931) and Drazin (1958) on wave-like disturbances in some special flows of ideal fluid of variable density). Miles' general result was also proved much more simply (with the aid of some integral inequalities similar to those used by Rayleigh) and for more general conditions (without restrictions on the velocity and density profiles) by Howard (1961). In the same work the semicircle theorem was also proved for general variable-density flows (in fact the theorem was first demonstrated for this general case). In addition, Case (1960c) also analyzed the behavior of the solution of the initial value problem for disturbances in a motionless ideal fluid in the half-plane $z > 0$ with exponentially increasing density; he naturally found a solution which increases in time without limit (see above, Sect. 2.4), and estimated its maximum rate of increase. Some other examples of the analysis of initial value problems for disturbances in fluids with density increasing with z (but with nonzero velocity gradients) may be found in the work of Eliassen, Hølland and Riis (1953). A detailed investigation of the possible wave-like disturbances in some special plane-parallel flows with density stratification was made by Miles (1963) and Howard (1963). They showed that the principle of "exchange of stabilities" (see Sect. 2.6) holds only under certain rather special conditions, while the region of instability in the (k, Ri)-plane may consist here of several disconnected parts.

A detailed discussion of the inviscid stability theory of plane-parallel flows (both for homogeneous and heterogeneous fluids and also under the action of various force fields) can be found in the excellent survey by Drazin and Howard (1966).

The General Theory of Instability for Plane-Parallel Flows of Viscous Fluid

Let us now return to the more complex case of flows of viscous fluid; the density ρ, as usual will now be assumed constant. In this case, the Orr-Sommerfeld equation (2.28) plays a central role. It is a nonsingular equation which has a purely discrete spectrum; therefore there is no reason to doubt that any solution of Eq. (2.26) may be represented by superposition of "plane waves" of the form (2.27).[19] Here we may limit our attention entirely to the study of the usual eigenvalue problem for a discrete spectrum of the Orr-Sommerfeld equation (2.28). The first attempts at such a study were made (using mathematical methods that were not very strict) circa 1910 by a number of authors (W. Orr, A. Sommerfeld, R. von Mises, L. Hopf, et al.) for the special case of plane Couette flow with a linear velocity profile. These attempts led to the result that such a flow is stable for all Reynolds numbers. This result seemed, on the one hand, fairly natural (since Orr (1906–1907) had shown rigorously that in the absence of viscosity, a Couette flow is stable, while the action of viscosity is naturally assumed to be stabilizing) but, on the other hand, it clearly contradicts the empirical facts on the transition to turbulence of all known flows for sufficiently great Reynolds numbers. At the beginning of the 1920's, Prandtl (1921) and Tietjens (1925) considered the question of the stability of flows with a velocity profile consisting of segments of straight lines. They came to the most unexpected conclusion that in the presence of viscosity, such flows are unstable for any Reynolds numbers (however small). During the same few years, Heisenberg's important work (1924) on the stability of a plane Poiseuille flow, using the method of small disturbances, appeared. In this work, a result was obtained which, at the time, seemed paradoxical (but which was, nevertheless, correct). By investigating in detail the asymptotic behavior of the solution of the corresponding Orr-Sommerfeld equation with large Re (i.e., small ν), a Poiseuille flow which in the absence of viscosity will be stable to small disturbances, will become unstable in the case of a viscous fluid, for sufficiently large Reynolds numbers. Heisenberg's result was obtained by sophisticated mathematical procedures, and on a physical (rather than mathematical) level of rigor; therefore it is not surprising that it aroused serious doubts for a long time, and proofs of the stability of Poiseuille flows continued to be published for many more years (apparently the last such "proof" was that of C. L. Pekeris in 1948). The physical mechanism of the very interesting phenomenon of instability caused by viscosity was outhned by Prandtl in the 1920's, but did not become clearly understandable

[19]Nevertheless, we must note that the question of rigorous proof of the possibility of such a representation was apparently studied only in a very old paper by Haupt (1912) and probably also in the unpublished dissertation of Schensted (1960). Hence there is no reason at present to consider the question as being completely solved.

until much later [Lin (1955), Chapt. 4; see also, Betchov and Criminale (1967)].

In the work of Tollmien (1929), the method of small disturbances was used to investigate the stability of boundary-layer flow, this being considered as plane-parallel, with a velocity profile composed of straight-line and parabolic segments. Tollmien was the first to obtain the form of the curve of neutral stability in the (k, Re) plane, dividing the regions of stable and unstable disturbances. Later, Tollmien (1930; 1947) and Schlichting (1933a, b; 1935a) extended these results to arbitrary velocity profiles. In 1944–1945, the whole theory of stability of plane-parallel flows was reconsidered critically by Lin (1945) who reworked the basic examples and improved the numerical results of Tollmien and Schlichting. Nevertheless, the complexity of the methods used for the asymptotic analysis of the solution of the Orr-Sommerfeld equation (2.28) leads to the fact that even now the results obtained cannot be taken as definitive in certain respects. The difficulty is that the asymptotic series used normally have a singularity at the point z where $U(z) - c = 0$, while the initial equation is regular at that point. Therefore, there is great interest in the problem of finding uniformly convergent asymptotic expansions, but the formulation of such expansions encounters great difficulties [see, for example, Lin and Benney (1962)].

Many recent results in the extensive field of refined mathematical analysis of various asymptotic approximations for solutions of the Orr-Sommerfeld equation can be found in Stuart (1963), Shen (1964) and Reid (1965; 1966); see also, Goering (1959); Chen, Joseph and Sparrow (1966); Graebel (1966); Betchov and Criminale (1967), Appendix II; Tam (1968); and Huppert (1968).

We have already pointed out that, strictly speaking, a plane-parallel flow of viscous fluid can only be a combination of Couette and Poiseuille flows. Therefore it is natural to begin the study of special plane-parallel flows of viscous fluid with consideration of these two examples. Let us first consider a *plane Couette flow*. At present, there is little doubt that this flow is stable with respect to any infinitely small disturbance; nevertheless, so far, a strict proof of this fact has not been given. Most investigations of this problem use certain asymptotic expansions for the study of limiting cases (usually cases of very large Re) and direct numerical procedures for the calculation of eigenvalues and eigenfunctions at moderate parameter values. This approach was developed first by Southwell and Chitty (1930) [however, they confined their attention to purely imaginary eigenvalues, taking on faith the "principle of exchange of stabilities"). Recently, this method has been developed widely in connection with considerable improvements to the technique of asymptotic approximation and the use of computers for numerical calculations. One of the most important contributions was made by Grohne (1954), who discovered the existence of a sequence of modes of wave-like disturbances in a plane Couette flow at given k and Re and computed the complex eigenvalues $c = c(k, \mathrm{Re})$ for the twelve lowest modes. All these eigensolutions correspond to damped oscillations with $c_2 = \mathrm{Im}\, c < 0$ for all k and Re. The form of the twelve functions $c_2 = c_2(\mathrm{Re})$ for one fixed k given by Grohne can be found also in Shen (1964) and Betchov and Criminale (1967) [although it must be borne in mind that numerically Grohne's results do not agree very well with those of subsequent computations by Gallagher and Mercer (1964)]. In subsequent numerical works using computers, Riis (1962) [who also investigated wave-like disturbances in a flow of heterogeneous viscous fluid with a linear velocity profile], Gallagher and Mercer (1962; 1964), Deardorff (1963), Birikh (1965) [he gave special consideration to the case of small values of Re], Hains (1967), Ponomarenko (1968c), and others, only damped oscillations were found. Therefore, it is very likely that no unstable disturbances exist in plane Couette flow, although the problem cannot be considered definitely settled at present since the entire domain of the parameters, which cannot be handled by asymptotic estimates, is not exhausted. Apparently, the only exact mathematical result relating to the eigenvalues of Eq. (2.28) with $U(z) = Az$, for arbitrary Reynolds number, belongs to Dikiy (1964), who proved (without use of numerical methods) that all the purely imaginary eigenvalues c of this equation will satisfy the inequality $\mathrm{Im}\, c < -k/\mathrm{Re}$ (where Re is the Reynolds number, defined for the value $H_1 = H/2$ of the half-width of the channel, and the maximum velocity of the flow $U = 2AH_1$). This result, in particular, shows that in this case the transition to instability cannot occur while satisfying the "principle of exchange of stabilities," but it does not guarantee that instability, in general, is impossible.

The situation for *plane Poiseuille flow* is far more satisfactory. Here, Lin's careful calculations based on asymptotic expansions of a special type, and applicable for large Re, confirmed the fundamental conclusion of Heisenberg on the instability of this flow for large enough (but finite) Re and permitted the form of the curve of neutral stability $\mathrm{Im}\, c(k, Re) = 0$ to be found. Afterwards the problem was reexamined by Thomas (1953) with the aid of the numerical solution of the corresponding eigenvalue problem, and then also by Shen (1954), Lock (1955), Nachtsheim (1964), Grosch and Salwen (1968), and others, who used improved forms of the asymptotic expansion and/or better computers. The results of all the calculations are in satisfactory agreement with each other (see, for example, Fig. 14). The instability of the plane Poiseuille flow at sufficiently large Re was also rigorously proved by Krilov (1964) without the use of numerical methods. Shen (1954) used Lin's methods to calculate the form of a family of stability curves $\operatorname{Im} c(k, \mathrm{Re}) =$ const, which define the set of disturbances of plane Poiseuille flow with a given growth rate [cf. Lin (1955); Shen (1964); Betchov and Criminale (1967)]. The same calculations were later repeated by Grosch and Salwen (1968), whose results agree well with those of Shen.

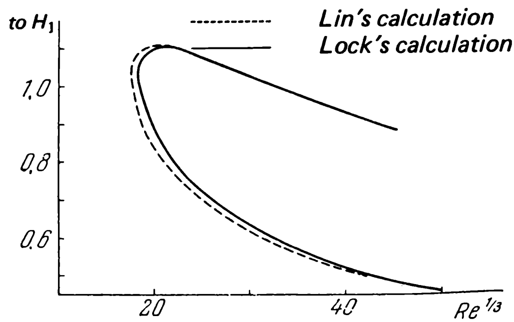


FIG. 14. The form of the neutral curve in the (k,Re) plane for a plane Poiseuille flow according to the calculations of Lin (1945) and Lock (1955).

The form of the "neutral curve" $\operatorname{Im} c(k, Re), = 0$, where $\mathrm{Re} = UH_1/\nu$ (H_1 is the half-width of the channel and U is the maximum velocity of the undisturbed flow), found by Lin and Lock is shown in Fig. 14. The critical Reynolds number $\mathrm{Re}_{\mathrm{cr}}$ (corresponding to the point of this curve farthest to the left) is equal to about 6000 according to Lock, and approximately 5300 according to Lin; in both calculations, it corresponds to the value $k_{\mathrm{cr}} \approx 1/H_1$ (close values of $\mathrm{Re}_{\mathrm{cr}} \approx 5780$ and $k \approx 1.02/H_1$ were obtained by Thomas whereas according to Nachtsheim Re_{cr} 5767, $k_{cr} \approx 1.02/\mathrm{H}_1$ and according to Grosch and Salwen $\mathrm{Re}_{cr} \approx 5750$, $k_{cr} \approx 1.025/\mathrm{H}$. As $\mathrm{Re} \to \infty$, both branches of the neutral curve (both upper and lower) tend to zero (the upper branch is asymptotically proportional to

$\mathrm{Re}^{-1/11}$, and the lower is proportional to $\mathrm{Re}^{-1/7}$). Thus, as Re increases, the disturbances with fixed (but not too great) k lie first in the region of stability (i.e., they are damped), then fail in the region of instability, and, finally, once again lie in the region of stability. From this we see why in the limit as $\mathrm{Re} \to \infty$ (i.e., as $\nu \to 0$) the flow becomes stable with respect to any disturbance.

Grohne (1954) proved that the Orr-Sommerfeld equation has higher modes for the eigenvalues also in the case of the basic Poiseuille velocity profile. He calculated the twelve lowest complex eigenvalues $c = c(k,\mathrm{Re})$ and found that all the modes, except for the first (which is the only mode studied by all the authors mentioned above), are strongly damped. Some of Grohne's results for plane Poiseuille flow are reprinted in Shen (1964) and Betchov and Criminale (1967). Later, the eigenvalues and eigenfunctions of the higher modes were recalculated by Grosch and Salwen (1968). They obtained results in full qualitative (but not too close quantitative) agreement with Grohne's results.

The most general, stricfly steady, and plane-parallel flow of viscous fluid is the combined *plane Couette-Poiseuille flow* arising in a plane channel with a moving upper wall in the presence of a constant longitudinal pressure gradient. The velocity profile of such a flow may be written in nondimensional form as $U_1(z_1) = (4-A)z_1-(4-2A)z_1^2, 0 \leqslant z_1 \leqslant 1$, where $z_1 = z/H, U_1(z_1) = U(z_1H)/U(0.5H)$, and $A = U_1(1)$ is the nondimensional parameter proportional to the velocity of the upper wall and changing from $A = 0$ for pure Poiseuille flow to $A = 2$ for pure Couette flow. The stability of such Couette-Poiseuille flow was studied by Potter (1966) and Hains (1967) whose results were in agreement with each other. According to their results the superposing of a Couette flow on a Poiseuille flow always has a stabilizing effect and increases Re_{cr} considerably; for example, the change of the upper wall velocity from zero (for Poiseuille flow) to 10% of the maximum Poiseuille velocity leads to an increase of Re_{cr} of 236%. The plane Couette-Poiseuille flow apparently becomes stable to all infinitesimal disturbances (i.e., Re_{cr} tends to infinity) when A reaches a value of about 0.55—long before the linear velocity profile is reached when $A = 2$.

Similar results were obtained by Potter (1967) also for the case of model flow with a symmetric parabolic nondimensional velocity profile of the form $U_1(z_1) = (A-1)z_1^2 + Az_1 + 1, \ 0 \geqslant z_1 \geqslant -1$; $U_1(z_1) = U_1(-z_1), \ 0 \leqslant z_1 \leqslant 1$. Here, also, $A = 0$ corresponds to Poiseuille flow and increasing A corresponds to increasing Re_{cr}; complete stability to all infinitesimal disturbances at all finite Reynolds numbers is reached apparently when $A \approx 0.437$ (i.e., long before the symmetrical linear velocity profile is reached when $A = 1$).

Further Results on the Stability of Plane-parallel Flows of Viscous Fluid. The Case of Boundary-layer Flows

Let us now consider the most important case of flow in a boundary layer over a flat plate. According to the results of Sect. 1.4, since the thickness of the boundary layer increases comparatively slowly and the vertical velocity is considerably less than the horizontal velocity, it is reasonable to assume that in this case also we may use, with fairly good precision, the theory of stability for plane-parallel flows.[20] Hence in their calculations Tollmien and Schlichting paid considerable attention to the stability of plane-parallel flows with velocity profiles identical or close to the Blasius profile (Fig. 2, Sect. 1.4). As a result, they were able to plot the neutral curve $\operatorname{Im} c(k,\mathrm{Re}) = 0$, i.e., to determine the boundary of the region of instability in the (k,Re)-plane, which, according to their data, has approximately the same form as the curve $\operatorname{Im} c(k, \mathrm{Re}) = 0$ in Fig. 14. The critical Reynolds number $\mathrm{Re}_{\delta^* \mathrm{cr}} = \left(\frac{U\delta^*}{\nu}\right)_{\mathrm{cr}}$, where δ^* is the displacement thickness of the boundary layer defined by Eq. (1.53), which, according to Tollmien's calculations equals 420, by Schlichting's equals 575.

According to Eq. (1.56), this means that $\mathrm{Re}_{x\,\mathrm{cr}} = (Ux/\nu)_{\mathrm{cr}} = 0.6 \times 10^5$, according to Tollmien and $\mathrm{Re}_{x\,\mathrm{cr}} = 1.1 \times 10^5$, according to Schlichting. These values are somewhat lower than those given in Sect. 2.1 because the loss of stability does not have to be accompanied by an instantaneous transition to the turbulent regime. At the point of loss of stability only some growing oscillations occur which are amplified as they move downstream. For some larger value of x transition to developed turbulence occurs.

On the other hand, however, the values obtained for $\mathrm{Re}_{\mathrm{cr}}$ cannot in themselves be taken as a confirmation of Tollmien-Schlichting

[20]Some preliminary estimates of the effect of additional terms in the Orr-Sommerfeld equations for almost (but not exactly) plane-parallel flows, which permit this assumption to be partly justified, were obtained as early as by Pretsch (1941a). Later, this problem was also studied by other authors, and was discussed by Reid (1965) and Betchov and Criminale (1967), Sect. 54, where references to other works can also be found. Nevertheless, the question of the exactness of the plane-parallel approximation in the study of stability of quasi-parallel flows is at present far from being definitely settled and the main argument in favor of the permissibility of such approximation is the rather good agreement between the theoretical results and the experimental data.

theory, and thus it is not surprising that as early as G.I. Taylor (1938), doubt was expressed as to the legitimacy of applying this theory to real flows in a boundary layer that is not, in fact, plane-parallel. However, the direct experimental verification of the Tollmien-Schlichting theory, carried out by Schubauer and Skramstad (1947), using the example of a flow in a boundary layer, confirmed in basic outline all the deductions of the theory, showing that doubts as to its correctness were unfounded. After this, Lin (1945) and Shen (1954) repeated the calculations of Tollmien and Schlichting and obtained once again the value $\mathrm{Re}_{\delta^* cr} = 420$. However, the form they obtained for the neutral curve differed slightly from that found previously (their result is shown in Fig. 15). More recently, the same problem was reexamined several times by different investigators with the aid of machine computations [see the comprehensive review by Betchov and Criminale (1967); cf. also Reid (1965)] without finding any significant disagreement with the results of Lin and Shen.

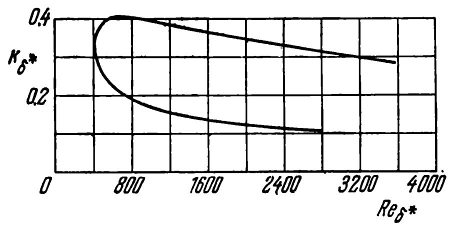


FIG. 15. The form of the neutral curve in the $(k, \mathrm{Re}_{\delta^*})$ plane for a boundary layer on a flat plate.

The experiments of Schubauer and Skramstad were carried out at the National Bureau of Standards in Washington, D.C., in a wind-tunnel having extremely low initial turbulence. The parameter U'/U in this tunnel, when certain precautionary measures are taken, may attain values of the order 0.0003–0.0002. This proves to be very important, since there are some data showing that when the values of U'/U exceed 0.002 (i.e., in particular, for the values obtained in all the older tests), the transition to turbulence in the boundary layer is apparently affected strongly by the influence of the free-stream disturbances [in accordance with Taylor's scheme described in Sect. 2.2, or as a result of another mechanism of interaction of the free-stream with the boundary layer flow studied by Criminale (1967)]. However, when $U'/U < 0.002$, the main role in transition is played by random two-dimensional disturbances of sinusoidal form, the amplitude of which under certain conditions increases downstream in accordance with the deductions of the linear disturbance theory. The presence of such regular oscillations was demonstrated as early as 1940 by Schubauer and Skramstad using careful hot-wire anemometer observations. Later, for more accurate verification of the deductions of the theory, they used a thin metal ribbon placed in the boundary layer, set in oscillation by an electromagnet and producing artificial disturbances of fixed frequency ω. They succeeded in demonstrating neutral (neither growing nor damped) almost purely sinusoidal fluctuations of velocity, corresponding to points of the neutral curve in the stability diagram. This classical experiment of Schubauer and Skramstad was repeated in subsequent years by other members of the National Bureau of Standards group, confirming the previous findings (cf. Betchov and Criminale (1967), Chapt. VI). Similar experiments have also been performed by Wortmann (1955) using water as the fluid and special flow visualization technique, and by Burns, Childs, Nicol and Ross (1959), who used a silvered lamina placed in the flow instead of a hot-wire anemometer to observe the oscillations of the reflection of a light beam directed on to the lamina caused by the z-component of the velocity disturbances. The observations by Wortmann and by Burns et al., lead to results which agree closely with those of Schubauer and Skramstad.

To compare the experimental data with the deductions of the theory, it is convenient to consider, instead of the more usual space-periodic disturbances of the form (2.27) with fixed (real) wave number k, time-periodic disturbances of the form $\psi = e^{i(kx-\omega t)}\varphi(z)$ with fixed (real) frequency ω and variable (generally speaking, complex) wave number k. In other words, it is expedient to investigate the space variation along the plate (i.e., with increase of x) of the disturbances of given frequency ω, set up at a fixed point of the flow. At the time of the Schubauer-Skramstad experiment, the theory was developed for space-periodic disturbances only; therefore, these authors simply transformed the temporal growth rate linearly into a spatial growth rate by means of the disturbance phase velocity. However, this procedure is not a rigorous one and if there is any dispersion it seems more reasonable to transform x from temporal to spatial growth using the group rather than the phase velocity. This question was studied later by Gaster (1962; 1965), who showed that for disturbances in the boundary layer with comparatively small amplification rates, for which the group and phase velocities are roughly equal, the Schubauer-Skramstad transformation has rather high precision, i.e., it can be justified (see also Betchov and Criminale (1967), Sect. 53). On the other hand, when modern computers are utilized, the transition from the eigenvalue problem for unknown complex c (with fixed real k) to the eigenvalue problem for unknown complex k (with fixed real ω) does not lead to any serious additional difficulties. Therefore, it is not surprising that numerical stability calculations for spatially growing disturbances in the Blasius boundary layer have been carried out independently during 1964–1966 by several investigators (all the references and some of the results obtained can be found in Sect. 53 in Betchov and Criminale).

Figure 16 gives the comparison of the experimental data by Schubauer and Skramstad (1947) [black dots] and by Burns et al. (1959) [crosses] on the frequency of the neutral disturbances with the theoretical neutral curve $\operatorname{Im} k(\omega, \mathrm{Re}) = 0$ on the (ω, Re)-plane, obtained with the aid of the phase-velocity transformation from the data of Shen (1954). The agreement between theory and experiment proves to be completely satisfactory. Let us note in this respect that the neutral curve for the space-periodic and for the time-periodic disturbances is always exactly the same, because if $\operatorname{Im} c = 0$, we have $\operatorname{Im} \omega = \operatorname{Im} kc = 0$. Figure 17 gives the data of Schubauer and Skramstad relating this time to the wave numbers of the neutral disturbances; these data also lie fairly well on the theoretical curve. The agreement proves to be even slightly better when the theoretical curve obtained by digital computation in 1959–1964 is used instead of the older curve shown in Fig. 15 (this new neutral curve can be found in Betchov and Criminale's book). The rate of spatial growth $\operatorname{Im} k = k_2(\omega, \mathrm{Re})$ measured by Schubauer and Skramstad also is in quite good agreement with the results of the theoretical computations. Further material on the comparison of the data with the deductions of the theory may be found in surveys of Lin (1955), Schlichting (1959;

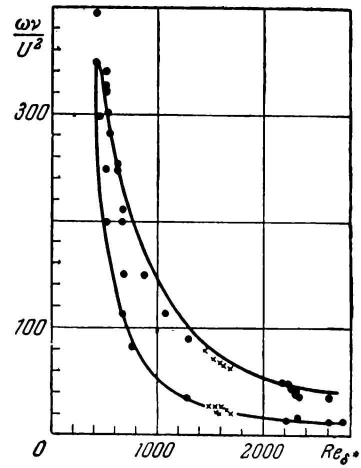


FIG. 16. Calculated form of the neutral curve in the $(\omega, \mathrm{Re}_{\delta^*})$ plane for the boundary layer on a flat plate, and experimental data on the frequency of the neutral oscillation.

1960), Dryden (1959), Stuart (1963), Shen (1964), and Betchov and Criminale (1967).

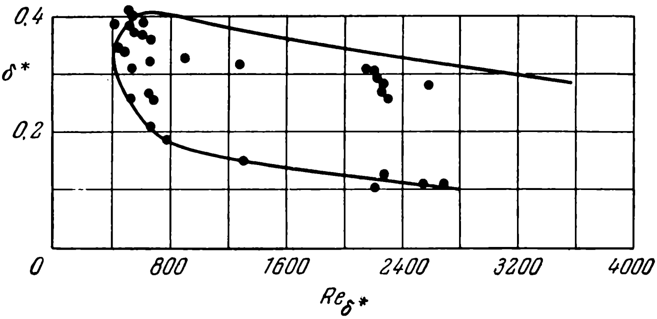


FIG. 17. Values of the wave numbers of neutral disturbances in the boundary layer on a flat plate according to the data of Schubauer and Skramstad.

Analogous calculations of the neutral curve and the stability diagram were also carried out for many other boundary-layer profiles, corresponding, for example, to flows past curved surfaces or in the presence of a favorable or adverse pressure gradient, to flow over a porous plate when there is suction of the fluid through the plate, to wall-jet flow, and so on. The results of these calculations may be found in Lin (1955), Shen (1964), Reid (1965), Hughes and Reid (1965), Betchov and Criminale (1967), and many other sources. Results of the same type for plane-parallel channel flow between solid walls with nonquadratic velocity profiles may be found, for example, in Birikh (1966) and Chen and Sparrow (1967).

It is interesting that the form of the neutral curve obtained depends considerably upon whether or not the velocity profile has a point of inflection (i.e., whether $\frac{d^2U}{dz^2}$ becomes zero for some z). In the latter case, the neutral curve on the stability diagram will have the same character as in cases of plane Poiseuille flow or the Blasius boundary layer (see Figs. 14 and 15), i.e., as $\mathrm{Re} \to \infty$ both its branches are drawn towards the abscissa. Thus, when the velocity profile has no inflection point, any disturbances of fixed wavelength (or fixed frequency) with increasing Reynolds number will finally become stable. However, when the velocity profile does possess an inflection point, the upper branch of the neutral curve will have a finite asymptote as $\mathrm{Re} \to \infty$ (see Fig. 18) and the ordinate of this asymptote depends upon the distance of the inflection point from the wall. In other words, there exists here a range of wavelengths (or frequencies) such that the corresponding disturbances are unstable however large the value of the Reynolds number. Obviously, this is connected directly with the definitive role played by the presence of an inflection point in the velocity profile, for the inviscid instability problem.

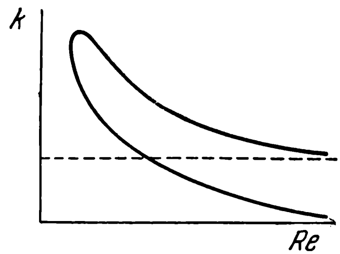


FIG. 18. Schematic form of the neutral curve in the (k, Re) plane for a flow with velocity profile possessing an inflection point (e.g., for a boundary layer in the presence of an adverse pressure gradient).

So far, we have spoken only of neutral and unstable disturbances in boundary-layer flows. It is natural to think that all the results mentioned above relate to the lowest eigenmode and that there is also an infinite sequence of strongly damped higher modes similar to that found by Grohne (1954) for the cases of plane Couette and Poiseuille flow. However, nothing is presently known about these higher eigenmodes. Information about all the modes is of some importance in investigating the fate of a spatially localized disturbance introduced into the flow at an initial time. In fact, here we must consider the initial value problem with rather general initial conditions decomposed into an integral over all the space-periodic eigenmodes of the form $\psi = e^{i(k_1 x + k_2 y - k c_n t)} \varphi_n(k_1, k_2, z)$ where $k = (k_1 + k_2^2)^{1/2}$ and the index n refers to the number of the mode. Because all the evidence at hand indicates that only the first mode can be unstable and that all the others are damped, it is reasonable to hope that in the asymptotic estimates for sufficiently large values of t, only the terms of the expansion with $n = 1$ will play an important role. However, the corresponding approximation is difficult to handle because of our rather poor knowledge of the eigenfunctions of the Orr-Sommerfeld equation, even for the case of the first mode. Therefore in the concrete example of the initial value problem for a three-dimensional localized disturbance in the Blasius boundary layer considered by Criminale and Kovásznay (1962) [see also, Betchov and Criminale (1967)] some rather crude simplifying assumptions were made, and the results obtained are not too close to the experimental data of Vasudeva (1967), who tried to check the theory. However, the qualitative features of the solution of Criminale and Kovásznay agree well with the experimental results. The same is true for the calculations of Tam (1967) who used the method of Case (1960a) and Dikiy (1960b) and showed that strongly localized "pointlike" disturbances in a slightly supercritical plane flow develop into a three-dimensional turbulent spot when moving downstream; such spots were observed by many experimentalists. On the other hand, Iordanskiy and Kulikovskiy (1965) have sketched a very general asymptotic analysis of the fate of a two-dimensional localized [in the (x,z)-plane] disturbance in a two-dimensional flow at high enough Reynolds numbers. They assumed that the wave numbers tend to zero on the branches of the neutral curve $\operatorname{Im} c(k, \mathrm{Re}) = 0$ and concluded that in this case the localized disturbance apparently must decay as $t \to \infty$ in any finite region of the flow (see also related paper by Kulikovskiy (1966), analyzing the stability of plane Poiseuille flow in a channel of finite length).

Stability of Certain Plane-Parallel Flows in an Unbounded Space

As a concrete example of a flow with a profile having an inflection point, let us consider a plane-parallel flow in the infinite space $-\infty < z < \infty$, which has a velocity profile of the form

$$U(z) = U_0 \operatorname{sech}^2 \frac{z}{H} = \frac{4U_0}{(e^{z/H} + e^{-z/H})^2} \tag{2.31}$$

(see Fig. 19a). According to the calculations of Schlichting and Bickley (see, for example, Schlichting (1960), or Goldstein (1938),

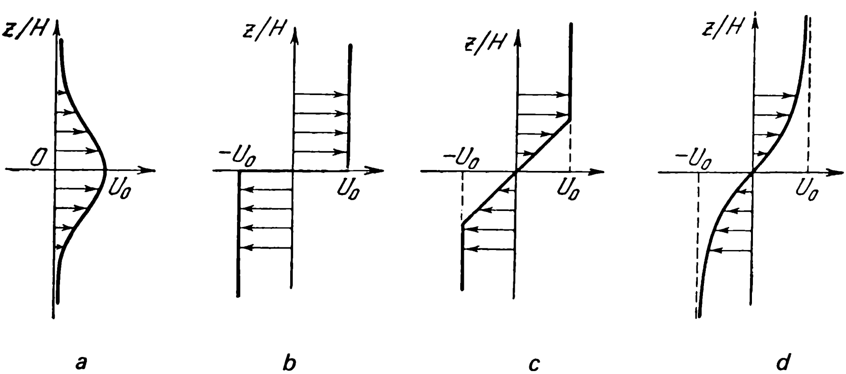


FIG. 19. Schematic form of the velocity profiles for some plane-parallel flows in an unbounded space: a–plane jet; b–flow with a tangential velocity discontinuity; c– and d– mixing layers of two plane flows of different velocity.

Vol. 1, Sect. 57) an equation of this type represents the similarity solution for the longitudinal velocity profile $u_1 = U$ in a laminar plane jet (issuing from an infinitely thin linear aperture along the line $x = 0$, $z = 0$, into a space filled with the same fluid). Here the parameters U_0 and H will depend on x only comparatively weakly, while the transverse velocity $u_3 = w$, will be small in comparison with the longitudinal component. Thus a plane-parallel flow with profile (2.31) may be considered as a model of a plane jet at a great distance from the aperture; in this connection its stability has been studied by a number of authors [in particular, by Curle (1956); Tatsumi and Kakutani (1958); Howard (1959); Klenshow and Elliott (1960); and Soprunenko (1965)].

As always, determination of the neutral curve in the plane (k, Re) for a flow with profile (2.31) leads to the investigation of an eigenvalue problem for the corresponding Orr-Sommerfeld equation (2.28). The unboundedness of this flow permits some simphfication of the calculation in comparison with the case of channel flow between rigid walls (since in this case it is only necessary to consider those solutions of Eq. (2.28) which are damped at infinity). Moreover, because of the symmetry of the profile (2.31), all the eigenfunctions $\varphi(z)$ in this case may be divided into even and odd with respect to z corresponding to antisymmetric and to symmetric disturbances of velocity. Since both experiment and crude theoretical estimates [cf., for example, Petrov (1937)] show that disturbances with even $\varphi(z)$ are always unstable, in the detailed calculations we may confine ourselves only to the region $0 \leqslant z < \infty$ and the boundary conditions $\varphi(\infty) = \varphi'(\infty) = 0$ and $\varphi'(0) = \varphi'''(0) = 0$. The results of these calculations, carried out by the two different methods of Curle (1956) and of Tatsumi and Kakutani (1958) are shown in Fig. 20. We observe that they are generally in fairly good agreement (except in the region of small values of k, where Curle's method is not sufficiently accurate). These results were also confirmed by the machine computations of Kaplan, who found simultaneously the stability curves $\operatorname{Im} c = c_2\ (k,\ Re) = \text{const}$ [see Betchov and Criminale (1967)]. The numerical results of Soprunenko showed some small deviations from the data of Fig. 20 (rather minor ones), but they contain much additional information (for example, Soprunenko also investigated symmetric velocity disturbances and found that $\mathrm{Re}_{cr} \approx 90$, i.e., it is much greater than Re_{cr} for antisymmetric disturbances, as was expected). According to Fig. 20, as $\mathrm{Re} \to \infty$, all disturbances with $k < 2/H$ are unstable; moreover, as Re increases, the region of unstable wave numbers expands monotonically. Thus, increasing the viscosity can only have a stabilizing effect on the disturbances (unlike the situation for flows with a velocity profile without an inflection point). The Re_{cr} here prove to be very small (which is also typical for flows with velocity profile having an inflection point); according to the careful numerical calculations of Klenshow and Elliot, $\mathrm{Re}_{cr} = \left(\frac{U_0 H}{\nu}\right)_{cr} \approx 3.7$ and $k_{cr} \approx 0.25/H$ (we note that for real jets the value $\mathrm{Re} \approx 4$ is generally attained in a region of the flow in which the jet cannot be considered plane-parallel). The theoretical results for the flow considered were compared with the experimental data of Sato and Sakao (1964) on the development of artificial disturbances induced in a plane jet by a loudspeaker both by the experimenters themselves and by Soprunenko. The experimental data of Sato and Sakao are rather scattered, but in the mean they do not contradict the deductions from theoretical stability calculations.

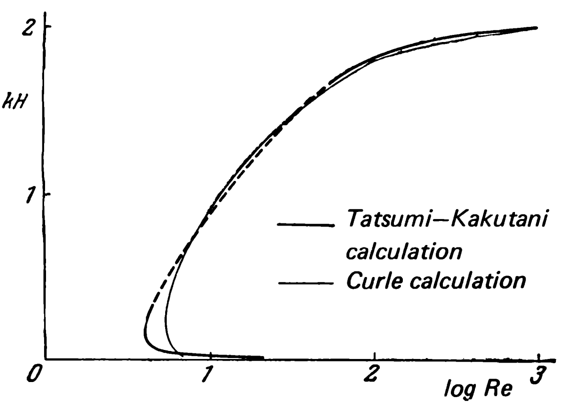


FIG. 20. Form of the neutral curve on the (k, Re) plane for a plane jet, according to the calculations of Curie (1956) and Tatsumi and Kakutani (1958).

The stability of the idealized model of a plane jet with polygonal velocity profile of the form $U(z) = U_0$ when $|z| \leqslant H$ and $U(z) = 0$ when $|z| > H$, was studied by Yamada (1964), who used the corresponding Orr-Sommerfeld equation with natural "matching conditions" at the corners of the velocity profile. He found that in this case, $\mathrm{Re}_{cr} \approx 4$ and $k_{cr} \approx 0.25/H$ for antisymmetric velocity disturbances, whereas $\mathrm{Re}_{cr} \approx 20$ and k_{cr} $1/H$ for symmetric disturbances in reasonable agreement with the results for the jet with a smooth velocity profile. More interesting is that Yamada disclosed the existence of a sequence of eigenmodes of both antisymmetric and symmetric types; to every mode there corresponds a special neutral curve, only the first mode leading to the values of Re_{cr} and k_{cr} mentioned above.

Another important type of flow in an unbounded space with a velocity profile $U(z)$ having an inflection point is that for which $U(z) \to U_0$ as $z \to +\infty$ and $U(z) \to -U_0$ as $z \to -\infty$. The simplest flow of this type is the idealized flow with a broken velocity profile shown in Fig. 19b; this flow describes a plane surface of tangential velocity discontinuity. More real profiles of the same type are shown in Figs. 19c and 19d; these correspond quantitatively to the laminar mixing zone of two plane-parallel flows, flowing one above the other with different velocities. As already mentioned when considering the stability of ideal fluid flows, neglecting viscosity the instabihty of flows with the velocity profiles shown in Figs. 19b and 19c was strictly proved as early as Helmholz, Kelvin, and Rayleigh; the instability of the flow of a nonviscous fluid with a profile of the form shown in Fig. 19d has now also been investigated in every detail [see, for example, Michalke (1964)]. Taking the viscosity into account, the stability of a flow with a profile of the type in Fig. 19d was studied by Lessen (1950); however, Lessen's results refer only to the case of sufficiently large Reynolds numbers and actually show only that here $\mathrm{Re}_{cr} < 20$ [in Lin (1955), it was assumed erroneously from this that in this case Re_{cr} is close to 20]. Later, Esch (1957) investigated the eigenvalue problem for the Orr-Sommerfeld equation (2.28) numerically, with the function $U(z)$ shown in Fig. 19c, and for all values of Re. He was the first to find that the flow with such a velocity profile is, in fact, unstable for all Re. Still later, Tatsumi and Gotoh (1960) showed that this result is not connected with the special form of the profile studied by Esch, but is related to a wide range of profiles, corresponding to a plane mixing zone of two parallel flows [although Lessen and Ko (1966) studied the stability of a half-jet flow similar to the plane mixing zone, they obtained numerically a neutral curve of the same form as shown in Fig. 20 and a nonzero value of Re_{cr} of the order of a few units]. The general form of the neutral curve in the (k, Re)-plane for a plane mixing zone, which is in accordance with the deductions of Tatsumi and Gotoh, is shown in Fig. 21. This figure presents the results obtained by

Betchov and Szewczyk (1963) by numerical integration of the Orr-Sommerfeld equation for the hyperbolic-tangent velocity profile $U(z) = U_0 \tanh(z/H)$ (of the type shown in Fig. 19d). The details of this integration procedure and many additional results of it may be found in Betchov and Criminale (1967), Sect. 13.

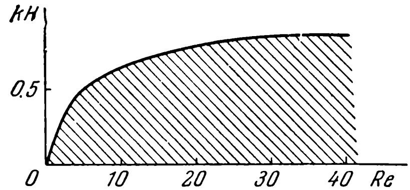


FIG. 21. Position of the neutral curve and the region of instabihty (shaded) in the (k,Re) plane for the mixing layer between two plane-parallel flows according to the data of Betchov and Szewczyk (1963).

The Stability of Circular Poiseuille Flow and other Axisymmetric Flows

The problem of the instability and transition to turbulence of Poiseuille flow in a circular tube is probably the most intriguing and interesting stability problem closely related to the classical experiments of Osborne Reynolds described at the beginning of this section. However, this problem is also very difficult and our knowledge in this field until now has been rather poor. Therefore, we shall begin our discussion with the general analysis of stability for axisymmetric flows (with some applications to axisymmetric jets and wakes) and only at the end shall we pass to the special case of circular Poiseuille flow.

First, let us consider a general steady axisymmetric flow, i.e., parallel flow with the velocity $U(y,z) = U(r)$, $r = (y^2 + z^2)^{1/2}$, which is everywhere parallel to the x-axis and depends only on the distance r from the line $y = z = 0$. Applying the small disturbance theory, we have to use cylindrical coordinates (x,r,φ). Then the coefficients of the system of linearized equations will not depend on x and φ; hence disturbances of the velocity and pressure may be sought here in the form $\boldsymbol{u}'(\boldsymbol{x},t) = \mathrm{f}(r)\exp[i(kx + n\varphi - \omega t)]$, $p(\mathrm{x},t) = g(r)\exp[i(kx + n\varphi - \omega t)]$ (cf. Eq. (2.13), Sect. 2.6). Upon substitution of these expressions into the equations we obtain, as usual, a complicated eigenvalue problem, related (but of course not identical) to the problem (2.14)–(2.16).

When considering the stability of plane-parallel flows we assume first that the Reynolds number is so high that it is possible to disregard the viscosity, i.e., to consider the fluid as ideal. Let us do the same for the axisymmetric flow. Then our eigenvalue problem is simplified considerably and after eliminating the functions f^x, f^φ and g (related to the axial and circumferential components of velocity and pressure) from the system of linearized equations for the disturbance amplitudes, we obtain the following single equation for one unknown function $f^{(r)}(r) = F(r)$:

$$(U - c)\frac{d}{dr}\left[\frac{r}{n^2 + k^2r^2}\frac{d(rF)}{dr}\right] - (U - c)F - rF\frac{d}{dr}\left(\frac{rU'}{n^2 + k^2r^2}\right) = 0$$

where as usual $c = \omega/k$ is the eigenvalue of the problem. This equation was given by Rayleigh (1892), who founded the theory of hydrodynamic stability of ideal fiuid flows in the axisymmetric case too [see also the important papers by Batchelor and Gill (1962), and Schade (1962), on the general analysis of the stability of axisymmetric flows]. The corresponding boundary conditions state that $F(r) = 0$ on the solid (cylindrical) boundaries of the flow and $F(r) \to 0$ as $r \to \infty$ for unbounded flow, while $F(0) = 0$ if $n \neq 1$ and $F(0)$ is bounded if $n = 1$ in the case of a flow enveloping the axis $r = 0$.

The Rayleigh equation for $F(r)$, after dividing by $U - c$, multiplying by the complex conjugate of rF, and integrating the imaginary part of the new equation, gives

$$c_2\int_{r_1}^{r_2} |q(r)|^2 Q'(r)\,dr = 0\,,$$

where $c_2 = \operatorname{Im} c$, $q(r) = rF(r)/(U - c)$, $Q(r) = rU'/(n^2 + k^2r^2)$, and r_1 and r_2 are the boundaries of the flow (the cases $r_1 = 0$ and/or $r_2 = \infty$ are not excluded). Therefore, the necessary condition for the existence of complex eigenvalues c (i.e., of growing wave-like disturbances) is that $Q'(r)$ should change sign at some point of the flow (this condition also was demonstrated first by Rayleigh). It follows, for example, that in circular Poiseuille flow (with $U(r) \sim R^2 - r^2$) nonaxisymmetric growing inviscid disturbances are not possible. The case of axisymmetric disturbances (with $n = 0$) in Poiseuille flow is exceptional in that Q' is then identically zero (similar to the case of plane-parallel Couette flow where U'' is identically zero). The Rayleigh equation for $F(r)$ in this case has no eigensolutions at all.

This necessary condition for instability is evidently similar to the necessity of the inflection point in the velocity profile for instability of plane-parallel inviscid flow. The stronger necessary condition of Fjørtoft-Høiland for the instability of plane-parallel flow also has a simple counterpart in the stability theory of axisymmetric flows: if U_s is the velocity at the point where Q' changes sign, then a complex eigenvalue c may exist only if the inequality $Q'(U - U_s) < 0$ holds somewhere in the field of the flow [the corresponding proof is similar to that for the Fjørtoft-Høiland condition; see Batchelor and Gill (1962)]. Rayleigh showed also that $c_1 = \operatorname{Re} c$ cannot lie beyond the range of $U(r)$; moreover, Howard's semicircle theorem which restricts the range of the possible values c in the complex c-plane for plane-parallel primary flow, is also vahd (and has exactly the same formulation) for axisymmetric primary flow [Batchelor and Gill (1962)]. Finally, the results by Tollmien (1935) showing that the simplest necessary conditions for inviscid instability of plane-parallel flows are in many important cases also sufficient, can be transferred to the axisymmetric flows [see Schade (1962)]. There is little doubt that the results of Case and Dikiy on the initial value problem for disturbances in plane-parallel inviscid flow can also be proved for axisymmetric flow, although this problem has apparently not been studied by anyone.

If we disregard the possible continuous spectrum of the c-values, which, in general, is apparently stable (on grounds of similarity with the plane-parallel case), the investigation of the instability characteristics of specific axisymmetric inviscid flow is equivalent to finding the discrete eigenvalues and eigenfunctions of the Rayleigh equation for $F(r)$. This problem was studied by Batchelor and Gill (1962) for the idealized jet with top-hat profile (i.e., $U(r) = U_0$ if $r \leqslant R$ and $U(r) = 0$ if $r > R$) and for a more realistic jet profile of the form $U(r) \sim (r^2 + R^2)^{-1}$; in the latter case, only disturbances with $n = 1$ appear to be unstable. Similar results were obtained by Sato and Okada (1966) for the axisymmetric wake with profile of the form $U(r) = U_0 - \exp(-br^2)$; here also unstable disturbances may exist only for $n = 1$. Sato and Okada computed the growth rate of the unstable disturbances and compared the results with experimental data on the evolution of artificial disturbances induced in the wake of a slender and sharp-tailed axisymmetric body; the agreement between theory and experiment was rather good. Stability calculations for axisymmetric jet flows were also performed by Michalke and Schade (1963).

Now let us return to the considerably more difficult problem of the stability of axisymmetric flows of viscous fluid. The effect of the viscosity on the stability of axisymmetric disturbances in the axisymmetric jet was considered briefly by Gill (1962), but the overwhelming majority of work in this field is connected with the investigation of circular Poiseuille flow. The corresponding eigenvalue problem is extremely complicated; therefore all the investigators have considered only special cases. Particular attention was paid to the case of φ-independent axisymmetric disturbances ($n = 0$). This special problem was studied by Sexl (1927a, b), Pretsch (1941b), Pekeris (1948), Belyakova (1950), Corcos and Sellars (1959), Sexl and Spielberg (1959), Gill (1965a) and some others. In all cases only damped eigenmodes were found. A comprehensive review of all previous results was given by Gill (1965a) who also computed the spatial damping rate of some disturbances with the aid of a numerical solution of the eigenvalue problem with fixed (real) frequency ω and unknown (complex) k, and compared them with the experimental data of Leite (1959) on the damping of artificial periodic nearly axisymmetric disturbances in pipe flow. Since no growing axisymmetric solution of the disturbance equations has yet been found, the majority of the workers in the field are sure that a Poiseuille flow in a tube (like a plane Couette flow) is stable for all Reynolds numbers with respect to infinitesimal axisymmetric disturbances. However, we must note that the situation with regard to a rigorous proof of this fact is even slightly less satisfactory than for a plane Couette flow.

Until now only a first step has been made in the direction of solving the general problem of the behavior of nonaxisymmetric disturbances ($n \neq 0$) in circular Poiseuille flow. Until 1968 only very special nonaxisymmetric disturbances were studied at all [for example, the x-independent ones with $k = 0$, or such that $u_r(\mathbf{x}, t) \equiv 0$; see Kropik (1964), where additional references may be found]. All these disturbances appear to be stable also. Therefore, it had been customary to think that Poiseuille flow in a circular tube is stable to any infinitesimal disturbance and that instability here can take place only when disturbances of finite amplitude are introduced (in other words, that circular Poiseuille flow is a self-excited system with hard excitation). This idea agrees well with the experimental data on the persistence of the laminar regime in tubes and we have already mentioned it in this connection in Sects. 2.1 and 2.4. However, the experimental findings of Lessen, Fox, Bhat and Liu (1964) and especially of Fox, Lessen and Baht (1968), raised some doubts as to the correctness of this view of the problem. These investigators produced artificially (by means of a small vibrating vane, whose plane is parallel to the tube axis, with the motion in a direction perpendicular to that of the main flow) in laminar circular Poiseuille flow, the first mode of the azimuthally periodic disturbance (with $n = 1$). Observing the development of the disturbance as it propagates downstream, they found both damped and growing disturbances (and also almost neutral ones). By varying both the Reynolds number of the flow and the frequency of vane vibration, they succeeded in obtaining the experimental neutral curve on the (ω, Re)-plane (which corresponds to the eigenvalue problem with fixed real ω and unknown complex k for a disturbance proportional to $\exp\,[i\,(\pm\varphi + kx - \omega t)]$). The neutral curve is similar in form to the curve shown in Fig. 16 (but it is smoother); the corresponding value of Re_{cr} is about 2130. Therefore, Fox et al. assumed that circular Poiseuille flow is unstable to infinitesimal nonaxisymmetric disturbances with $n = 1$ when Re is greater than about 2130. Taken at face value, this assumption did not seem to be completely improbable. This was because nothing was known at that time about the solutions of the complex eigenvalue problem for the system of coupled ordinary differential equations which describes the behavior of the disturbances with $n \neq 0$ in circular Poiseuille flow. However, the careful numerical investigation of the problem undertaken by Lessen, Sadler and Liu (1968) for the case $n = 1$ showed that several eigenmodes exist for the problem, but all of them are damped in the range of wave and Reynolds numbers which Fox, Lessen and Bhat (1968) found to be unstable. Hence, no linear instability exists in this range for disturbances with $n = 1$. In addition, Lessen et al. investigated limited portions of the (k, Re)-plane corresponding to much higher Re values (up to Re = 30,000) and also observed no instabilities, nor any indication of the likelihood of them for higher Reynolds number. Disturbances with $n > 1$ were not considered by Lessen et al.; however, they present some qualitative arguments indicating that such disturbances would be more stable than those with $n = 1$. Their expectations are in good agreement with the finding of Salwen and Grosch (1968), who gave results of a numerical investigation of the behavior of disturbances with $n \leqslant 5$; they found no instability for $\mathrm{Re} \leqslant 10{,}000$. Therefore, it is now clear that the instabilities observed by Fox, Lessen and Bhat were not caused by the linear instability of infinitesimal disturbances, and the hypothesis that circular Poiseuille flow is stable to any infinitesimal disturbance, now seems again to be highly probable.

The stability problem for rotating axisymmetric flows (with a circumferential component of velocity $V(r)$ different from zero) is similar in many respects to the problem for parallel axisymmetric flows. One such problem was considered in detail in Sect. 2.6; two other special cases of this problem were studied by Pedley (1967) and Michalke and Timme (1967). The stability theory of general parallel flows [with velocity $U(y, z) = (r, \varphi)$] parallel to the x-axis but dependent on two variables was begun by Hocking (1968).

2.9 Stability to Finite Disturbances; Growth of Disturbances and Transition to Turbulence

The small disturbance theory discussed above in many cases permits the conditions of loss of stability of laminar flows to be determined theoretically. These results allow some important facts connected with the transition to turbulence to be explained. However, it is quite clear that the loss of stability does not in itself constitute such a transition, and that the linear theory discussed in Sects. 2.5 through 2.8 can, at best, describe only the very beginning of the process of initiation of turbulence, but cannot give a complete picture of this process. Moreover, for certain important flows, for example, Couette flow between two planes or Poiseuille flow in a tube, the small disturbance theory cannot, in principle, aid our understanding of the transition to turbulence observed experimentally, since these flows are apparently stable with respect to all infinitely small disturbances. On the other hand, for example, in the case of a Poiseuille flow between two plates (close to a real flow in a plane channel) stability theory, although it implies a possible loss of stability, indicates a considerably higher value of Re_{cr} than that at which transition actually occurs. The data of Davies and White (1928) show that transition of a plane Poiseuille flow actually takes place for $\mathrm{Re} = UH_1/\nu \approx 1000$, while, according to the linear theory, $\mathrm{Re}_{cr} \approx 6000$. Finally, the cardinal difference between the motion which arises after loss of stability in the flow between rotating cylinders, or in the state of rest of a fluid heated from below, on the one hand, and plane Poiseuille flow of boundary-layer flow, on the other hand, cannot be explained from the viewpoint of linear disturbance theory. All these facts demand that great stress be placed on working out a more complete nonlinear theory of the initiation of turbulence; at present, this is only in the initial stage of development.

Energy Balance of a Finite Disturbance

The simplest approach to the investigation of the stability of a flow with respect to disturbances of finite amplitude is connected with the use of the "energy method," originated by O. Reynolds (1894) [hence it is older than the small disturbance theory]. The essence of this method consists in determining the energy balance of a disturbance in a given flow, to ascertain the conditions under which the energy of such a disturbance will increase (or decrease) with time.

Let $U_i(\boldsymbol{x}, t)$ and $P(\boldsymbol{x}, t)$ be the velocity and pressure of the undisturbed flow, and $u_i'(\boldsymbol{x}, t)$ and $p'(\boldsymbol{x}, t)$ the velocity and pressure of the disturbance. In this case, both fields $u_i = U_i + u_i'$ and U_i must satisfy the Navier-Stokes equations (1.6) and the continuity equation (1.5) [as usual, the fluid is assumed incompressible]. Subtracting the equations for u_i and U_i from each other, we obtain the following equations for the disturbance velocity $\boldsymbol{u}' = (u_1', u_2', u_3')$:

$$\begin{gathered}\frac{\partial u_i'}{\partial t} + U_\alpha \frac{\partial u_i'}{\partial x_\alpha} + u_\alpha' \frac{\partial U_i}{\partial x_\alpha} + u_\alpha' \frac{\partial u_i'}{\partial x_\alpha} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu\nabla^2 u_i', \\ \frac{\partial u_\alpha'}{\partial x_\alpha} = 0\end{gathered} \tag{2.32}$$

(these differ from Eqs. (2.7) only by the presence in the first of them of a nonlinear term in $\boldsymbol{u}'$, which is omitted in the linear disturbance theory). Then, multiplying the first of Eqs. (2.32) by $u_i'/2$, summing over i and integrating the result over the whole region of flow, we obtain the energy balance equation, which determines the variation in time of the total energy of the disturbance. In particular, if the flow takes place in a finite volume, bounded by rigid walls (moving or nonmoving), on which $u_i' \equiv 0$, the disturbance energy balance equation may be written in the form

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}\int \sum_i u_i'^2\, dV\right) = -\int u_i' u_\alpha' \frac{\partial U_i}{\partial x_\alpha}\, dV - \nu \int \sum_{i,j} \left(\frac{\partial u_i'}{\partial x_j}\right)^2 dV, \tag{2.33}$$

where dV is an element of volume [it was actually used in this form by O. Reynolds (1894) and Orr (1906–1907)]. The same result will also be true for many unbounded flows; for example, it will be correct if the disturbance may be considered periodic for all coordinates with respect to which the flow is unbounded [in this case, the integration in Eq. (2.33) along such axes must be carried out over one period of the disturbance only].

The first term on the right side of Eq. (2.33) describes the exchange of energy between the undisturbed flow and the disturbance and, as a rule, is positive (the transfer of energy usually takes place from the undisturbed flow to the disturbance). However, the second term on the right describes the dissipation of energy due to the viscosity, and is always negative. The relative value of the two terms on the right side of Eq. (2.33) will determine whether the energy of the disturbance decreases or increases.

Now if we transform Eq. (2.33) to dimensionless quantities, measuring distance in units of the characteristic length L, velocity in units of the characteristic velocity U, and time in units of L/U, the dimensional coefficient ν in the second term on the right side will be transformed into the dimensionless coefflcient $1/\mathrm{Re}$. Hence, it is clear that if the Reynolds number is sufficiently small, the negative second term on the right side will dominate the positive first term, and the energy of any disturbance will be damped, i.e., the flow will be stable to disturbances of any magnitude. Proceeding from Eq. (2.33), we may obtain certain estimates from below of $\mathrm{Re}_{cr\ \min}$, which bounds the range of "sufficiently small" Reynolds numbers, within which the energy of any disturbance can only decrease. Thus, for example, Serrin (1959) showed that for any pair of solenoidal vector fields $\boldsymbol{u}'(\boldsymbol{x})$ and $\boldsymbol{U}(\boldsymbol{x})$ in an arbitrary bounded region with diameter D, such that $\boldsymbol{u}'$ becomes zero on the boundary of the region, the inequality

$$-\int u_i'u_\alpha' \frac{\partial U_i}{\partial x_\alpha}\,dV \leqslant \frac{\nu}{2}\int \sum_{i,j}\left(\frac{\partial u_i'}{\partial x_j}\right)^2 dV + \frac{U^2_{\max}}{2\nu}\int \boldsymbol{u}'^2\,dV,$$
$$\int \sum_{i,j}\left(\frac{\partial u_i'}{\partial x_j}\right)^2 dV \geqslant \frac{3\pi^2}{D^2}\int \boldsymbol{u}'^2\,dV, \qquad (2.34)$$

will hold, where $U_{\max}$ is the maximum of the modulus of the field $\boldsymbol{U}(\boldsymbol{x})$. Substituting these equations into Eq. (2.33), we find that in the case of a flow in a region of diameter D, the energy $E(t)$ of the disturbance obviously satisfies the inequality

$$E(t) \leqslant E(0) \exp\left[\left(\frac{U_{\max}^2}{\nu} - \frac{3\pi^2\nu}{D^2}\right) t\right],$$

such that

$$\mathrm{Re}_{\mathrm{cr\,min}} = \left(\frac{U_{\max} D}{\nu}\right)_{\mathrm{cr\,min}} \geqslant \sqrt{3}\,\pi \approx 5.44.$$

Serrin obtained analogous inequalities for the fields $\boldsymbol{u}'(\boldsymbol{x})$ and $\boldsymbol{U}(\boldsymbol{x})$, corresponding to flows in a straight tube of arbitrary cross section with maximum diameter D or in a channel of maximum width D. All that is necessary is to replace the constant $3\pi^2$ in the second equation of Eq. (2.34) in the first case by $2\pi^2$ and in the second case by π^2; Serrin likewise showed that $\mathrm{Re}_{\mathrm{cr\,min}} \geqslant \sqrt{2}\,\pi \approx 4.43$ for flows in a straight tube (of arbitrary section) and $\mathrm{Re}_{\mathrm{cr\,min}} \geqslant \pi \approx 3.14$ for flows in a plane channel. Later, Velte (1962) improved the value of the constant coefficients in equations of the type of the second equation (2.34) and showed that $\mathrm{Re}_{\mathrm{cr\,min}} \geqslant \sqrt{6}\,\pi \approx 7.7$ for flows in a bounded region of diameter D, $\mathrm{Re}_{\mathrm{cr\,min}} \geqslant \sqrt{4.7}\,\pi \approx 6.8$ for a flow in a straight tube of diameter D and $\mathrm{Re}_{\mathrm{cr\,min}} \geqslant \sqrt{3.7}\,\pi \approx 6.0$ for a flow in a channel of width D. At the same time, Velte showed that his numerical values are very close to the largest values which can be obtained with the aid of general inequalities of the type (2.34) for fairly general domains of the type considered. Still later, Sorger (1966) considered more special flow regions (such as the interior of a sphere or a circle, or the domain between two concentric spheres or circles), and for these regions found the exact values of the coefficients in the inequalities, thus further improving the stability bounds obtained previously for wider classes of stability problems. Sorger's results have some points in common with the previous finding by Serrin (1959) that inequalities of the same type can be found also for fields $\boldsymbol{u}'(\boldsymbol{x})$ and $\boldsymbol{U}(\boldsymbol{x})$ in the annulus between concentric cylinders of radii R_1 and R_2. Using them, Serrin obtained the following sufficient condition for stability to arbitrary disturbances of Couette flow between cylinders:

$$\frac{|\Omega_2 - \Omega_1|}{\nu} < (R_2^2 - R_1^2)\left[\frac{\pi}{R_1 R_2 \ln(R_2/R_1)}\right]^2 \qquad (2.35)$$

Figure 22 shows the region (2.35) of the (Ω_1, Ω_2)-plane for the case studied experimentally by Taylor, with $R_1 = 3.55$, $R_2 = 4.03$ together with the region of instability (transferred from Fig. 13a) of the corresponding flow with respect to infinitesimal disturbances.

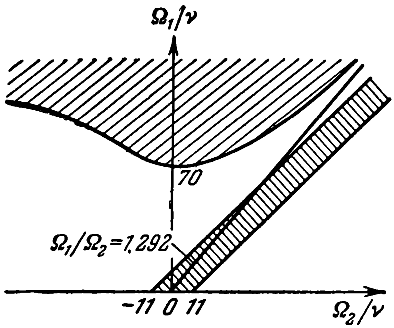


FIG. 22. Position of the instability region to infinitesimal disturbances and the stability region to any finite disturbance for Couette flow between cylinders studied by Taylor. The upper shaded region corresponds to instability to infinitesimal disturbances, while flows corresponding to points of the shaded strip are definitely stable to any finite disturbance. The continuous line in the figure is the boundary of the region of instability for the case of an inviscid fluid (cf. Fig. 13a).

The above-mentioned results of Serrin, Velte and others, are deduced in a fairly simple manner, and are extremely general, but they give very rough estimates of $\mathrm{Re}_{\mathrm{cr\,min}}$ lower than the experimental data by several orders of magnitude (we recall, for example, that the experimental value of $\mathrm{Re}_{\mathrm{cr\,min}}$ for a straight tube is close to 2000). Sufficient conditions for the stability of circular Couette flow, found by Serrin, later were somewhat strengthened by Sorger (1967) utilizing more exact inequalities. Somewhat better results are obtained if, instead of using general (but rough) inequalities of type (2.34), we find the disturbances $\boldsymbol{u}'(\boldsymbol{x})$ for which the ratio of the influx of energy from the primary flow to the dissipation of energy due to the action of viscosity, will be the maximum possible. Let us consider, for example, a plane-parallel flow between the planes $z = 0$ and $z = H$, with velocity $\mathrm{U} = U(z)$, parallel to the x-axis and for simplicity, take the disturbance $\boldsymbol{u}'$ also to be two-dimensional: $\boldsymbol{u}' = [(u'(x, z, t),\ w'(x, z, t)]$. Measuring distance in units of H, velocity in units of $U_{\max} = \max U(z)$ and time in units of $H/U_{\max}$, we may rewrite the general energy balance equation (2.33) in the form

$$\frac{dE}{dt} = \frac{d}{dt}\int\int \frac{u'^2 + w'^2}{2}\,dx\,dz =$$

$$= -\int\int u'w'\,\frac{dU}{dz}\,dx\,dz - \frac{1}{\mathrm{Re}}\int\int\left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}\right)^2 dx\,dz, \qquad (2.33')$$

$$\mathrm{Re} = U_{\max}H/\nu.$$

By the equation of continuity [the second equation of Eq. (2.32)] we may now introduce the stream function ψ, putting $u' = -\partial\psi/\partial z$,

$w' = \partial\psi/\partial x$. Then the condition of instability $dE/dt \geqslant 0$ will be written in the form

$$\frac{1}{\mathrm{Re}} \leqslant \frac{\int\int \frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial z}\frac{dU}{dz}\,dx\,dz}{\int\int (\Delta\psi)^2\,dx\,dz} = F[\psi]. \tag{2.36}$$

The maximum of the functional $F[\psi]$ over the set of all possible functions ψ satisfying the given boundary conditions, in this case will also have the meaning of $1/\mathrm{R}_{\mathrm{cr\,min}}$ (since for Reynolds numbers less than the reciprocal of this maximum, the energy of the disturbance will certainly decrease; of course, for two-dimensional disturbances only). It may be expected that the values of $\mathrm{Re}_{\mathrm{cr\,min}}$ obtained in this way will be larger than those which follow from the general inequalities (2.34), which do not take the specific form of the primary flow into account. It is clear, however, that they may certainly be considerably lower than the true values of $\mathrm{Re}_{\mathrm{cr\,min}}$, since using this method we consider the velocity field of the disturbance (and the derivative dE/dt) only at one moment of time, and do not take into account that the time evolution of the sum of this field and the velocity field of the undisturbed flow must satisfy the system of fluid dynamic equations. In fact, attempts to determine the instability critierion by the energy method, working out values of $F[\psi]$ for certain special functions ψ (leading to a relatively high value of this functional), or by solving the corresponding variational problem (usually only for some special set of disturbances), which were undertaken by Lorentz (1907), Orr (1906–1907), Hamel (1911), von Kármán (1924) and others, all led to low values of $\mathrm{Re}_{\mathrm{cr\,min}}$ (approximately an order of magnitude less than the values of $\mathrm{Re}_{\mathrm{cr\,min}}$ actually observed). Analyzing this fact, Petrov (1938) came to the conclusion that the values of ψ at which $F[\psi]$ takes a maximum, taking into account the time variations of all the functions, will apparently in no case generate a dynamically possible motion. Thus the energy method can never give an exact value of $\mathrm{Re}_{\mathrm{cr\,min}}$, but is only suitable for making preliminary, very rough estimates of this value.

It is interesting to note that a similar approach can also be developed for thermal convection problems and here it proves to be considerably more successful. This fact was demonstrated by Sorokin (1953; 1954) and Joseph (1965; 1966).

In convection problems the Navier-Stokes equations must evidently be replaced by the Boussinesq equations (see Sect. 1.5). The additional term $-g\beta T'$ on the right side of the Boussinesq equation for the component u_3' of the velocity, produces an extra term on the right side of the equation of the energy balance (2.33) which now takes the form

$$\frac{dE}{dt} = -\int u_i' u_\alpha' \frac{\partial U_i}{\partial x} dV - g\beta \int u_3' T' dV - \nu \int \sum_{i,j} \left(\frac{\partial u_i'}{\partial x_j}\right)^2 dV. \qquad (2.33'')$$

The extra term in this equation can easily be estimated by the inequality

$$g\beta \int u_3' T' dV \leq g\beta \left[\int u_3'^2 dV \cdot \int T'^2 dV\right]^{1/2} \leq 2g\beta (EH)^{1/2}$$

where $H = \frac{1}{2}\int T'^2 dV$ is the intensity of the temperature disturbances. The heat-conduction equation for T and the condition $T =$ const on the boundaries of the flow (together with the periodicity conditions for unbounded flows) imply the equation of the balance of the intensity H:

$$\frac{dH}{dt} = -\int T' u_i' \frac{\partial T}{\partial x_i} dV - \chi \int \sum_i \left(\frac{\partial T'}{\partial x_i}\right)^2 dV. \qquad (2.33''')$$

Clearly, $-\int T' u_i' \frac{\partial T}{\partial x_i} dV \leq 2\gamma (EH)^{1/2}$ where $\gamma = (\nabla T)_{max}$ is the maximal value of the gradient of undisturbed temperature. Moreover, if $T' = 0$ on the boundaries of the flow, then the following analog of the second inequality (2.34) can be proved:

$$\int \sum_i \left(\frac{\partial T'}{\partial x_i}\right)^2 dV \geq \frac{\alpha\pi^2}{D^2} \int T'^2 dV = \frac{2\alpha\pi^2}{D^2} H. \qquad (2.34')$$

Here $\alpha = 3$ for a bounded flow region of diameter D and $\alpha = 1$ for flow in a channel of width D (in the latter case the disturbances must be considered as spatially periodic in planes parallel to the channel walls and all the integrals must be taken over one period only). Using

Eq. (2.34) and all the other above-mentioned inequalities, it is not difficult to derive the following general deduction from the two balance equations (2.33″) and (2.33‴):

$$\sqrt{E(t)} + \lambda\sqrt{\frac{g\beta\nu}{\gamma\chi}H(t)} \le \left[E(0) + \lambda\sqrt{\frac{g\beta\nu}{\gamma\chi}H(0)}\right]\exp[-\xi(\lambda\alpha\pi^2 - \sqrt{\mathrm{Ra}})t]$$

where $\mathrm{Re} = U_{\max}D/\nu$, $\mathrm{Ra} = g\beta\gamma D^4/\nu\chi$, $\lambda^2 = (\delta - \mathrm{Re}^2)/2\alpha\pi^2$ is assumed to be positive, δ/D^2 is the best value of the coefficient before the integral in the right side of the second inequality (2.34), and $\xi = \lambda\nu/D^2$ when $\lambda^2\nu/\chi \le 1$ whereas $\xi = \chi/\lambda D^2$ when $\lambda^2\nu/\chi \ge 1$ [cf. Joseph (1965)]. It follows from this result that the convective motion will be universally stable to any disturbance of the velocity and/or temperature if

$$0 \le \mathrm{Ra} < \frac{\pi^2(\delta - \mathrm{Re}^2)}{2}$$

For the special case of a motionless horizontal fluid layer (for which $\alpha = 1$, $\delta \approx 3.7\pi^2$, and $\mathrm{Re} = 0$) we obtain the result: $\mathrm{Ra}_{\mathrm{cr\ min}} > 1.85\pi^4 \approx 180$. The last result can easily be improved; in fact the first inequality (2.34) is clearly unsatisfactory in the case of motionless fluid when its left side is equal to zero. If we simply omit the first term in the right side of Eq. (2.33′) and then repeat all the arguments, we obtain twice as good an estimate: $\mathrm{Ra}_{\mathrm{cr\ min}} > 360$ (this compares with the value $\mathrm{Ra}_{\mathrm{cr}} \approx 1708$ obtained from linear theory). A similar improvement can also be made in the estimate of the boundary of the universal stability region on the (Ra,Re)-plane if one uses a different estimate of the first term in the right side of Eq. (2.33′) [giving zero for the fluid at rest] and another definition of Re-number [see Joseph (1965; 1966)].

The preceding results on the limits of stability of convective flows are similar to the Serrin-Velte-Sorger results for nonconvective flows: they do not depend on any specific details of the flow geometry or the distributions of the primary velocity and temperature field. Considerably stronger results can be achieved by using all the available information on the basic flow. With that end in view, Joseph proposed first to fix the positive parameter $\mu = \mathrm{Re}/(\mathrm{Ra})^{1/2}$ and another positive parameter λ and then to seek the greatest value of Re compatible with the condition $\frac{d}{dt}[\gamma E(t) + \lambda g\beta H(t)] < 0$ as a solution of a specific variational problem for a given primary flow. If

$\mathrm{Ra} = \mathrm{Ra}(\lambda, \mu)$ is the solution of the problem, we can easily find the largest of the values of $\mathrm{Ra}(\lambda, \mu)$ over λ for a fixed point μ [i.e., the value $\mathrm{Ra}_\mu = \max \mathrm{Ra}(\lambda, \mu)$]; then the point $(\mathrm{Ra}_\mu, \mu\sqrt{\mathrm{Ra}_\mu})$ on the (Ra,Re)-plane will belong to the optimum boundary of the stability region.

For the special case of pure convection in a motionless fluid, Sorokin (1953;1954) developed the same approach considerably earlier and produced surprisingly strong results. For simplicity, let us consider only the case of a horizontal fluid layer of width D bounded by rigid planes. It is easy to see that the derivative $\frac{d}{dt}[\gamma E(t) + \lambda g\beta H(t)]$ takes the smallest value when $\lambda = 1$; hence we must consider this value of λ only. After introducing the new variables $v_i = u_i'(\gamma\nu/g\beta\chi)^{1/2} = u_i'[(T_1 - T_0)\nu/Dg\beta\chi]^{1/2}$ of the dimension of temperature and the nondimensional coordinates $y_i = x_i/D$, the condition

$$\frac{d}{dt}[\gamma E(t) + g\beta H(t)] \leq 0$$

takes the form

$$(\mathrm{Ra})^{1/2}\int v_3 T' dV + \frac{1}{2}\int\left[\sum_{i,j}\left(\frac{\partial v_i}{\partial y_j}\right)^2 + \sum_i\left(\frac{\partial T'}{\partial y_i}\right)^2\right] dV \geq 0.$$

Therefore the optimal value of Ra compatible with this inequality (i.e., the value of $\mathrm{Ra}_{\mathrm{cr\,min}}$ can be found as the solution of the following variational problem:

$$-\int v_3 T' dV = \frac{1}{2(\mathrm{Ra}_{\mathrm{cr\,min}})^{1/2}} = \mathrm{Max}$$

where the maximum is considered over the class of functions $\boldsymbol{v}(\boldsymbol{y}, t)$, $T'(\boldsymbol{y}, t)$ satisfying the conditions $\boldsymbol{v} = T' = 0$ at $y_3 = 0$ or 1, $\operatorname{div} \boldsymbol{v} = 0$,

$$\int\left[\sum_{i,j}\left(\frac{\partial v_i}{\partial y_j}\right)^2 + \sum_i\left(\frac{\partial T'}{\partial y_i}\right)^2\right] dV = 1.$$

Using ordinary techniques of variational calculus we can reduce this problem to determining a minimum eigenvalue of a specific eigenvalue problem for the system of partial differential equations. It is not difficult to verify that this eigenvalue problem agrees with the eigenvalue problem of linear stability theory corresponding to the case of neutrally stable disturbances (i.e., to disturbances meeting the principle of exchange of stabilities) in a motionless layer of fluid heated from below. This fact was demonstrated by Sorokin (1953), who gave simultaneously a very simple proof of the validity of the principle of exchange of stabilities in thermal convection problems (and also of the sharper result that all the characteristic frequencies ω_i corresponding to the eigenvalue problem of small disturbance theory in a motionless fluid heated from below are purely imaginary); later, the same fact was discovered by Joseph (1965). The unknown minimum eigenvalue $\mathrm{Ra}_{\mathrm{cr\,min}}$ replaces the parameter Ra of linear disturbance theory in the boundary-value problem obtained. Hence the best estimate of $\mathrm{Ra}_{\mathrm{cr\,min}}$ which can be achieved with the aid of the energy method agrees with the minimal value of Ra compatible with the existence of infinitesimal neutrally stable disturbances, i.e., $\mathrm{Ra}_{\mathrm{cr\,min}} = \mathrm{Ra}_{\mathrm{cr}} \approx 1708$ (because $\mathrm{Ra}_{\mathrm{cr\,min}}$ cannot be greater than $\mathrm{Ra}_{\mathrm{cr}}$). We see that the energy method in the case of pure convection, gives the exact value of $\mathrm{Ra}_{\mathrm{cr\,min}}$; simultaneously it shows that in the framework of Boussinesq's approximate theory of convection, a layer of fluid heated from below is stable to arbitrary periodic disturbances (and not only to infinitesimal ones) at all Rayleigh numbers below that given by the linear disturbance theory.

Sorokin and Joseph give more general results for pure convection problems relating to more general boundary conditions and other flow regions; in all cases the $\mathrm{Re}_{\mathrm{cr\,min}}$ turn out to coincide with a value given by linear disturbance theory. In particular, steady disturbances of any magnitude do not exist at Ra below $\mathrm{Ra}_{\mathrm{cr}}$ and, therefore, the state of rest is the only steady solution of the Boussinesq equations at subcritical values of Rayleigh number; this special result was proved by various methods and by Ukhovskiy and Yudovich (1963), Howard (1963), Sani (1964), and Platzman (1965). Joseph (1966) also studied in detail the case of a plane Couette flow heated from below. He found that in this case the variational technique of the energy method gives the stability region in the form:$(\mathrm{Re})^2 + \mathrm{Ra} < 1708$. It is worth noting that for pure convection (when $\mathrm{Re} = 0$), the corresponding estimate ($\mathrm{Ra}_{\mathrm{cr\,min}} \approx 1708$) is exact, whereas for the pure dynamical problem (when $\mathrm{Ra} = 0$) this estimate ($\mathrm{Re}_{\mathrm{cr\,min}} > 41$) is poor. Other examples of the application of the energy method to thermal convection problems (corresponding to flows in the presence of internal heat sources) can be found in Joseph and Shir (1966), Joseph and Carmi (1966), and Joseph, Goldstein and Graham (1968).

The Admissibility of Linearization in the Investigation of Hydrodynamic Stability

In dynamical problems we have seen that the energy method leads to very rough estimates of Re_{cr}. This may be explained by the fact that it makes only very incomplete use of the dynamical equations, since it is based exclusively upon equalities (and inequahties) relating to a single instant. Substantially more promising (but also much more difficult) is the approach based upon tracing the temporal development of a finite disturbance after a long time with the aid of the full system of equations of fluid dynamics.

The first question which arises in connection with this approach concerns the interrelation between the linear theory, described in the previous sections, and the strict nonlinear theory of finite disturbances. Restating it differently, the question concerns the extent to which the stability (or instability) of solutions of the linearized equations of fluid mechanics entails also the stability (or instability) of the corresponding solutions of the complete nonlinear system of equations.

Before answering this question it is necessary to define exactly when the solution of the nonlinear system is called stable. Such a definition had already been given at the end of the last century by Lyapunov. To state it, first, it is necessary to choose some sort of "norm" in function space, which allows one to measure the magnitude of a function (such a norm, for example, may be the maximum of the function, or the integral of its square, or, in the case of vector functions $\mathbf{u}(\mathbf{x})$, the spatial average of the value of the integral of $u^2(x) + [\nabla u(x)]^2$, etc.). Having done this, a solution of the nonlinear equations is called *stable according to Lyapunov,* if, from the smallness of a disturbance of this solution at the initial instant $t = 0$, it follows that this disturbance will be small for all $t > 0$. Stated exactly, the solution is called *stable* if, for arbitrarily small $\epsilon > 0$, one can find a $\delta = \delta(\epsilon)$, such that for an "initial disturbance" (i.e., of a difference in initial conditions at $t = 0$ from the intial conditions of the solution under investigation) which is less than δ according to the norm, the norm of the difference between the two corresponding solutions ("disturbance") for all $t > 0$ will not exceed ϵ.

This definition differs from that which we used in the linear theory, where basically we had a spectrum of eigenvalues ω_j, and for stability we required that $\mathrm{Im}\,\omega_j \leq 0$ for all j. However, usually in cases where the linear theory leads to instability, Lyapunov's stability condition is also not satisfied. Actually, if the initial disturbance is chosen very small then it will apparently be well described by the linearized equations. For this reason if $\mathrm{Im}\,\omega_j > 0$ for at least one j, the initial disturbance can be chosen such that for small t it grows proportional to $\exp[\omega_j' t]$, where $\omega_j' = \mathrm{Im}\,\omega_j > 0$. Then, as the disturbance becomes relatively large, the linear approximation ceases to apply; the influence of the nonlinear terms usually leads first, to the disturbance beginning to grow slower than would follow from the linear theory, and then even ceasing to grow, i.e., it remains bounded (we will encounter examples of such behavior below). However, it is important to note that if we decrease further the magnitude of the initial disturbance (keeping the same form), we obtain only a longer time interval during which the linear theory is a suitable description of the flow, the subsequent fate of the disturbance being the same; thus the maximum value achieved is not less than in the first case. This means that the solution of the nonlinear equation (or system) is not stable according to Lyapunov.

Apparently, under very broad conditions the reverse implication holds–from the stability of the linearized equations it follows that the solution of the complete nonlinear system of equations is stable in the sense of Lyapunov. The assumption that linearization of the equations of motion is possible for stability investigations has just this sense (namely, that a one-to-one relationship exists between the absence of eigenvalues of the linearized equations having positive imaginary parts, and the stability in the sense of Lyapunov of the solution of the complete nonlinear system of equations). In stability theory this assumption is always taken on faith [see, for example, Lin (1955), Sect. 1.1], but, in general, it is not at all easy to prove rigorously. Further, let us note that, if this assumption is true, then the linear approximation gives a complete answer to the question of whether or not the given fluid flow allows self-exciting oscillations with soft excitation; obviously the study of oscillations with hard excitation always requires the use of the complete system of nonlinear equations.

However, in several specific cases, the presence of stability (in the sense of Lyapunov) of the solutions of the nonlinear equations of fluid mechanics, for which the corresponding linearized systems are stable, may be proven comparatively simply and elegantly. This was substantially clarified by Arnol'd (1965a; 1966a,b). Let us examine that simplest of the examples he studied—plane parallel flow of an inviscid and incompressible fluid in a two-dimensional channel between rigid walls—and let us, following Arnol'd, limit our consideration to two-dimensional disturbances of the flow. For simplicity, we will measure all lengths with a characteristic length L_0, and all velocities with a characteristic velocity U_0, so that henceforth all quantities may be considered nondimensional (which means that we may take arbitrary numerical functions of them, and add together any two quantities). Let $U(z) = -\frac{\partial \Psi(z)}{\partial z}$ be the nondimensional profile of velocity of the primary flow, and

$$\mathbf{u}' = \left\{ u'(x,z,t),\, w'(x,z,t) \right\} = \left\{ -\frac{\partial \psi'}{\partial z},\, \frac{\partial \psi'}{\partial x} \right\}$$

the nondimensional disturbance velocity (so that Ψ is the stream function of the basic flow, and ψ' is the stream function of the disturbance; $\Delta \Psi = \frac{d^2 \Psi}{dz^2}$ and $\Delta \psi' = \frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}$ the vorticity of the primary flow and the vorticity of the disturbance). The equations of motion here reduce to a single equation for the conservation of vorticity $\Delta \psi = \Delta(\Psi + \psi')$:

$$\frac{\partial}{\partial t}\Delta\psi - \frac{\partial \psi}{\partial z}\frac{\partial \Delta\psi}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \Delta\psi}{\partial z} = 0. \tag{2.37}$$

We consider disturbances which are periodic in the coordinate x and in doing so, we do not lose generality because the result will not depend on the length of the period, which may be taken to be arbitrarily large. From Eq. (2.37) it follows immediately that the total energy of the flow $E = \frac{1}{2}\iint (\nabla\psi)^2\, dxdz$ (where the integral in dx may conveniently be taken over a single period) is conserved in time (let us recall that we are considering the case of zero viscosity). It is known also that in the flow of an ideal fluid the vorticity $\Delta\psi$ is conserved, and hence an arbitrary function of the vorticity $\Phi(\Delta\psi)$; consequently, the quantity $J_\Phi = \iint \Phi(\Delta\psi)\, dxdz$ also does not change in time [the equality $\frac{dJ_\Phi}{dt} = 0$ may easily be obtained from Eq. (2.37)].

Now let us assume that the velocity profile $U(z) = -\frac{\partial \Psi}{\partial z}$, has no inflection points. Then $U''(z) = -\frac{d^3\Psi}{dz^3} \neq 0$ for all z, so that $\Delta\Psi = \frac{d^2\Psi}{dz^2}$ is a monotonic function of z and $\Delta\Psi$ may be taken instead of z as a new transverse coordinate. In particular, it follows from this that the stream function $\Psi = \Psi(z)$ may also be considered a function of $\Delta\Psi$;

$$\Psi = \Lambda(\Delta\Psi). \tag{2.38}$$

In discussing now the invariant integral $G = E + J\Phi$, we shall show that the function of a single variable $\Phi(s)$ may be chosen in such a form that the functionals $G[\psi]$ of the stream function $\psi(x, z)$ [satisfying the boundary condition ψ = const on the rigid walls of the channel] takes on its minimum value for $\psi(x, z) = \Psi(z)$ [i.e., for the plane-parallel steady flow under investigation). This minimum value is understood here to be of a local character–minimum relative to nearby flows differing from the basic flow by small disturbances. However, from this it follows immediately that if the initial disturbance $\psi'(x, z, 0)$ is small, then in the function space of the fields $\psi(x, z)$ the "level lines" of $G[\psi] = G[\Psi(z) + \psi'(x, z, 0)]$ topologically will present the appearance of small ellipsoids surrounding the extremal point $\psi = \Psi(z)$. Since $G[\psi]$ is an invariant functional, $\psi(x, z, t) = \Psi(z) + \psi'(x, z, t)$ for all t will belong to the given "level line," i.e., the disturbance $\psi'(x, z, t)$ will remain small for all time. To make the foregoing assertion rigorous, we must still define how the magnitude of the disturbance is to be measured, i.e., we must introduce a norm into the function space of fields $\psi'(x, z)$. For such a norm it is convenient to take the value of the functional

$$H[\psi'] = \iint [(\nabla\psi')^2 + (\Delta\psi')^2]\,dxdz =$$

$$\iint [\mathbf{u}'^2 + (\nabla \times \mathbf{u}')^2]\,dxdz$$

(which is well defined, since we have nondimensionalized both $\nabla\psi'$ and $\Delta x'$). In this case it is easy to show, with the help of equations for the first and second variations of $G[\psi]$ which will be introduced below, that there exist a function Φ a system of coordinates x, z, and constants $K_1 > 0$, $K_2 > K_1$ and $\epsilon_1 > 0$ such that

$$K_1 H[\psi'] < G[\Psi + \psi'] - G[\Psi] < K_2 H[\psi']$$

if $H[\psi'] < \epsilon_1$ for chosen Φ and coordinate system. Moreover, $G[\psi']$ does not change with time, and $H[\psi']$ changes continuously; therefore, it follows immediately that if $H[\psi'] < K_1\epsilon/K_2$ for $t = 0$ and $\epsilon < \epsilon_1$, then also $H[\psi'] < \epsilon$ for all $t > 0$, i.e., that the solution $\psi(x, z, t)$ of the nonlinear equation (2.37) is stable in the sense of Lyapunov.

Let us proceed now to the construction of the functional

$$G[\psi] = \iint [\tfrac{1}{2}(\nabla\psi)^2 + \Phi(\Delta\psi)]\,dxdz ,$$

which possesses the properties referred to above. First, let us find the variation of this functional at the point $\psi = \Psi(z)$, i.e., the leading term of the difference $G[\Psi + \psi'] - G[\Psi]$. With the help of the usual rules of variational calculus, we obtain

$$\delta G[\psi]|_{\psi=\Psi} = \iint [\nabla\Psi\nabla\psi' + \Phi'(\Delta\Psi)\Delta\psi']\,dxdz = \iint [\Phi'(\Delta\Psi) - \Psi]\Delta\psi'\,dxdz$$

where Φ' is the derivative of the function Φ (while ψ' designates the disturbance stream function). By virtue of Eq. (2.38) $\delta G = 0$, if $\Phi'(s) = \mathrm{A}(s)$, i.e., $\psi = \Psi(z)$ will then be the extremal point of the functional $G[\psi]$. Analogously the second variation of the functional $G[\psi]$ with $\Phi' = \mathrm{A}$ may be calculated:

$$\delta^2 G[\psi]|_{\psi=\Psi} = \iint [\Lambda'(\Delta\Psi)\Delta\psi' - \psi']\Delta\psi' dxdz =$$

$$\iint \left[\frac{d\Psi/dz}{d^3\Psi/dz^3}(\Delta\psi')^2 + (\nabla\psi')^2\right] dxdz = \iint \left[\frac{U(z)}{U''(z)}(\Delta\psi')^2 + (\nabla\psi')^2\right] dxdz$$

where $\Lambda'(s) = d\Lambda/ds$ and Eq. (2.38) is used for the calculation of $\Lambda'(\Delta\Psi)$. However, the assumption of the absence of inflection points of the velocity profile implies that $U''(z)$ has the same sign for all z. Since the equations of fluid mechanics are Galilean invariant, we may always transform to a new system of coordinates, moving with a constant velocity relative to the old, so that $U(z)$ has only one sign for all z, namely, the same as $U''(z)$. In such a case, $U/U'' > 0$, i.e., $\delta^2 G[\psi] > 0$ for $\psi = \Psi(z)$.. This means that the point $\psi = \Psi$ really corresponds to a local minimum of the functional $G[\psi]$, which was to be proven. The inequalities relating $G[\Psi + \psi'] - G[\Psi]$ and $H[\psi']$ follow at once from the fact that U/U'' is bounded from above and from below.

We already know that Rayleigh had shown that plane-parallel flow of an inviscid fluid in the absence of inflection points in the velocity profile, is stable in the sense that no eigenvalues ω of the linearized system exist having positive imaginary parts. Now we see that in this case it is possible also to prove rigorously that if we choose the initial disturbance sufficiently small, then within the full nonlinear theory it will be small for all $t > 0$, i.e., that here soft excitation of finite oscillations is not possible (at any event, within our limitation to two-dimensional disturbances). In Arnold's works it was also shown that the assumption of the absence of inflection points of $U(z)$ may be replaced by several other weaker conditions; with these, analogous arguments permit the proof of stability in the sense of Lyapunov of many plane-parallel flows with profiles $U(z)$, having inflection points, for which stability of the linearized system of equations was shown by Tollmien (1935) [of course, also only within the limitation to two-dimensional disturbances]. Further, the same methods are shown to be applicable for the proof of stability (without the use of the linearized equations) of many two-dimensional flows of inviscid incompressible fluids, having curvilinear stream-lines. An analogous proof of stability in the sense of Lyapunov is shown to be applicable to zonal flows of an ideal (or even viscous) fluid on a sphere [cf. Dikiy (1965b)]. However, for three-dimensional disturbances of the flow (and three-dimensional flows), the reasoning presented here is shown to be insufficient. It permits only a few partial results to be obtained and does not resolve finally the question of the stability in the sense of Lyapunov of the solutions of the nonlinear equations of fluid mechanics [cf. Arnold (1965b); Dikiy (1965a)].

Yudovich (1965a) has outlined quite a different approach to the proof of the general theorem of admissibility of linearization for the investigation of stability of stationary solutions of the equations of motion of a viscous incompressible fluid (i.e., to the proof of the assertion of a one-to-one relation between the existence of eigenvalues of the Orr-Sommerfeld equation lying in the upper half-plane of the complex ω-plane, and the stability in the sense of Lyapunov of the solutions of the nonlinear system of Navier–Stokes equations). However, in this work, considerably more complicated norms in velocity-field space are introduced and sophisticated mathematical techniques are used; consequently, both the statement of the results and their proof seem to be rather awkward and comparatively unenlightening.

Landau's Theory

The investigation of the admissibility of linearization in the studies of hydrodynamic stability is significant for rigorous justification of the highly developed linear stability theory; however, it is not the primary function of the nonlinear stability theory. Considerably more attractive of course is the hope of obtaining new fundamental physical results on the behavior of finite disturbances in various fluid flows, with the aid of the complete system of nonlinear dynamical equations. It is of interest both in the investigation of disturbances in a flow with Reynolds numbers[21] less than the value $\mathrm{Re}_{cr} = \mathrm{Re}_{cr\ max}$ defined from the linear stability theory (to determine the critical Reynolds number for finite disturbances of a given amplitude) and in the investigation of disturbances with $\mathrm{Re} > \mathrm{Re}_{cr}$ (to study the further evolution of weak disturbances, which increase exponentially according to the linear theory). At present, however, there exist only a few isolated results for both these cases and almost all of them relate only to Reynolds numbers only slightly different from Re_{cr} [for a more exact estimate of the range of permissible values of Re, see, e.g., Stuart (1960)].

The most general results on the behavior of finite disturbances with Re close to Re_{cr} which are independent even of the actual form of the equations of fluid dynamics, were demonstrated by Landau (1944) [see also, Landau and Lifshitz (1963), Sect. 27]. Let us assume that $\mathrm{Re} > \mathrm{Re}_{cr}$, but that $\mathrm{Re} - \mathrm{Re}_{cr}$ is small. Since with $\mathrm{Re} = \mathrm{Re}_{cr}$ there first appears a disturbance with "frequency" ω which has a zero imaginary part, with small positive $\mathrm{Re} - \mathrm{Re}_{cr}$ there will exist an infinitesimal disturbance with velocity field of the form

$$\boldsymbol{u}(\boldsymbol{x}, t) = A(t)\boldsymbol{f}(\boldsymbol{x}), \tag{2.39}$$

where $A(t) = e^{-i\omega t} = e^{\gamma t - i\omega_1 t}$, $\gamma = \operatorname{Im}\omega > 0$ and $\gamma \to 0$ as $\mathrm{Re} \to \mathrm{Re}_{cr}$ (so that $\gamma \ll |\omega_1|$ with sufficiently small $\mathrm{Re} - \mathrm{Re}_{cr}$) and $f(\boldsymbol{x})$ is the eigenfunction of the corresponding eigenvalue problem. Therefore, it is clear that $A(t)$ satisfies the equation

$$\frac{d|A|^2}{dt} = 2\gamma |A|^2. \tag{2.40}$$

However, Eq. (2.40) is correct only within the framework of the linear disturbance theory. As $A(t)$ increases, there will inevitably come an instant when the theory is no longer valid and must be replaced by a more complete one which takes into account terms in the dynamical equations that are nonlinear in the disturbances. Then the

[21] For simplicity, we shall speak about Reynolds number only, although in some cases the initiation of instability will be determined by transition through a critical value of some other nondimensional parameter of the same type.

right side of Eq. (2.40) may be considered as the first term of the expansion of $\frac{d|A|^2}{dt}$ in series in powers of A and A^* (where the asterisk denotes the complex conjugate). In the subsequent approximation (which applies for larger t) it is also necessary to take into account terms of the next order of the series—the third-order terms; however, it must be considered that the motion (2.39) is accompanied by rapid (in comparison with the characteristic time $1/\gamma$ of increase of the amplitude) periodic oscillations, described by the factor $e^{-i\omega_1 t}$ in the expression for $A(t)$. These periodic oscillations do not interest us; hence to exclude them, it is convenient to average the expression $d|A|^2/dt$ over a period of time that is large in comparison with $2\pi/\omega_1$ (but small in comparison with $1/\gamma$). Since third-order terms in A and A^* will evidently all contain a periodic factor, they will disappear during the averaging. For the fourth-order terms, after averaging there will remain only a term proportional to $|A|^4$. Thus, retaining terms of not higher than the fourth-order, we will have an equation of the form

$$\frac{d|A|^2}{dt} = 2\gamma|A|^2 - \delta|A|^4 \tag{2.41}$$

(since the period of averaging is much less than $1/\gamma$ the terms $|A|^2$ and $|A|^4$ practically will not change with averaging so that Eq. (2.41) may be considered an exact equation for the amplitude of the averaged disturbance). The sign of the coefficient δ cannot be ignored; generally speaking, it must be expected that it can be either positive or negative (and can also be zero, but only in exceptional cases).

The general solution of Eq. (2.41) may be written in the form

$$|A(t)|^2 = \frac{Ce^{2\gamma t}}{1 + \frac{\delta}{2\gamma} Ce^{2\gamma t}}, \tag{2.42}$$

where C is an undetermined constant of integration. From Eq. (2.42) it follows that if $\delta > 0$ and $|A(0)|^2 = \frac{C}{1 + C\delta/2\gamma}$ is sufficiently small, the amplitude $A(t)$ will first increase exponentially (in accordance with the linear theory), but then the rate of the increase slows, and as $t \to \infty$ the amplitude will tend to a finite value $A(\infty) = (2\gamma/\delta)^{1/2}$ independent of $A(0)$. We now note that γ is a function of the Reynolds number which becomes zero at $\mathrm{Re} = \mathrm{Re}_{cr}$ and may be expanded as a series in powers of $\mathrm{Re} - \mathrm{Re}_{cr}$ (the latter fact may be deduced from the small disturbance theory) while $\delta \neq 0$ for $\mathrm{Re} = \mathrm{Re}_{cr}$. Thus $\gamma \sim (\mathrm{Re} - \mathrm{Re}_{cr})$ and, consequently, $A(\infty) = |A|_{max} \sim (\mathrm{Re} - \mathrm{Re}_{cr})^{1/2}$ for small $\mathrm{Re} - \mathrm{Re}_{cr}$.

However, if $\delta < 0$, the solution (2.42) of Eq. (2.41) will formally become infinite for $t = [\ln(2\gamma/C|\delta|)]/2\gamma$. Consequently, considerably earlier it will have attained such large values that henceforth one cannot use Eq. (2.41), which is obtained by keeping only the first two terms in the expansion of $d|A|^2/dt$ in powers of $|A|^2$. The ultimate amplitude $A(\infty)$ will be determined in this case by the coefficients of higher powers of $|A|^2$ which are different from zero at $\mathrm{Re} = \mathrm{Re}_{cr}$. Therefore $A(\infty)$ will be finite and roughly constant however small $\mathrm{Re} - \mathrm{Re}_{cr}$. In this case Eq. (2.41) is evidently inapplicable for analysis of the processes at $\mathrm{Re} > \mathrm{Re}_{cr}$, but it may be applied to the investigation of the behavior of finite disturbances for $\mathrm{Re} < \mathrm{Re}_{cr}$. In fact, for $\mathrm{Re} < \mathrm{Re}_{cr}$, the coefficient γ will be negative, i.e., small disturbances of the form (2.39) will be damped. Since the second term on the right side of Eq. (2.41) [equal to $-\delta|A|^4$] will be positive for $\delta < 0$, then, for sufficiently large $|A|^2$, the derivative $d|A|^2/dt$ (averaged over the specifically chosen period of time) may become positive, i.e., the motion will become unstable even for $\mathrm{Re} < \mathrm{Re}_{cr}$ with respect to finite disturbances. For amplitudes that are not too large, for which Eq. (2.41) may be used, the amplitude $|A|$ will increase if $|A| > (2|\gamma|/|\delta|)^{1/2}$. Considering that $|\gamma| \sim (\mathrm{Re}_{cr} - \mathrm{Re})$, we find with respect to disturbances of given amplitude $|A|$ that the flow will be unstable for $\mathrm{Re} > \mathrm{Re}_{A\ cr} = \mathrm{Re}_{cr} - \alpha|A|^2$, where $\alpha > 0$. Therefore, in this case $\mathrm{Re}_{cr\ min}$ will be less than the value of Re_{cr} given by the linear theory. An exact evaluation of $\mathrm{Re}_{cr\ min}$, however, cannot be carried out on the basis of the approximate equation (2.41), which is applicable only for small $|A|$, since such an evaluation requires $\mathrm{Re}_{A\ cr}$ to be defined for arbitrarily large values of $|A|$.

Now let us return to the case of $\delta > 0$, $\mathrm{Re} > \mathrm{Re}_{cr}$. Here the increase of the disturbances (2.39) for Re slightly greater than Re_{cr} may be described as a soft self-excitation of an elementary oscillator, leading, finally, to the establishment of steady periodic oscillations with a small (but finite) amplitude proportional to $(\mathrm{Re}-\mathrm{Re}_{cr})^{1/2}$. An essential feature here is that Eq. (2.41) defines only the amplitude of these oscillations; the phase, however, is not defined uniquely by the external conditions, but depends on the random initial phase of the disturbances, i.e., it may in fact be arbitrary. Thus the ultimate regime of steady oscillations of such an oscillator is characterized by the presence of one degree of freedom (unlike the case of steady laminar flow which is defined uniquely by the boundary conditions, and hence does not possess any degrees of freedom at all).

With further increase of Re, this ultimate periodic motion may itself become unstable to small disturbances $\boldsymbol{u}_2(\boldsymbol{x}, t)$. The instability of a flow with velocity field $\boldsymbol{U}(\boldsymbol{x})+\boldsymbol{u}_1(\boldsymbol{x}, t)$ [where $\boldsymbol{u}_1$ is the ultimate value of the disturbance (2.39) which depends also on Re $-\mathrm{Re}_{cr}$] may in principle be investigated with the aid of the ordinary method of disturbances. For this purpose, it is only necessary to investigate the particular solutions of a linear equation in the disturbance $\boldsymbol{u}_2(\boldsymbol{x},t)$ of the form $\boldsymbol{u}_2=\exp(-i\omega t)\boldsymbol{f}(\boldsymbol{x}, t)$ [where $\boldsymbol{f}$ is a periodic function of t with period $2\pi/\omega_1$] and to determine the frequency $\omega=\omega_2$, with which, as Re increases, there will first appear (for Re $= \mathrm{Re}_{2\,cr}$) a positive imaginary part. Then we may expect that with Re somewhat greater than $\mathrm{Re}_{2\,cr}$, oscillations with this frequency will increase to some finite limit. Therefore, as $t\to\infty$ quasi-periodic oscillations occur with two periods $2\pi/\omega_1$ and $2\pi/\omega_2$ now possessing two degrees of freedom (the phases of the oscillations). With further increase of Re, a series of new oscillators will subsequently be excited (i.e., lead to an oscillatory regime). It is natural to think that the intervals between corresponding "critical" Reynolds numbers will decrease continuously, and that oscillations which arise will be of higher and higher frequency and smaller scale. Consequently, for sufficiently large Re, the motion will have very many degrees of freedom and be very complex and disordered. Such motion corresponds to the "limit cycle" of phase trajectories in which certain generalized coordinates of the flow assume fixed values, and only the coordinates corresponding to the phases of the corresponding oscillators will vary with time (according to equations of the form $\varphi(t)=\omega t+\alpha$).

A trajectory describing a "limit cycle" occupies a region in the phase space which corresponds to all possible sets of initial phases of the oscillations of the oscillators, and in the course of time, passes through practically all points of this region. In fact, at instants of time $t_n = 2\pi n/\omega_1, n = 0, 1, 2, \ldots$, at which the phase $\varphi_1(t)=\omega_1 t+\alpha_1$ takes the value α_1, the phase $\varphi_2(t)$ of any other oscillation will take values $2\pi n\omega_2/\omega_1+\alpha_2$, $n = 0, 1, 2, \ldots$. Since individual frequencies ω_1 and ω_2 will, generally speaking, be incommensurable (excluding very special cases which are most unhkely), the latter set will contain values, which after reduction to the interval $[0, 2\pi]$, will be as close as desired to any preassigned number in this interval. Thus it follows that the developed turbulent motion which arises in this case will possess a definite "ergodicity," revealed in the fact that in time, the fluid will pass through states as close as desired to any possible state of motion.

These general considerations are the essence of Landau's theory of the initiation of turbulence. They are very obvious and physically convincing, but not rigorous and cannot be considered as complete. In fact, Eq. (2.41) is based on the assumption that only one unstable disturbance will be excited at small positive values of $\mathrm{Re} - \mathrm{Re}_{cr}$ although many such disturbances will often exist for $\mathrm{Re} > \mathrm{Re}_{cr}$ and the interactions between them may be of considerable importance (see, e.g., Eckhaus (1965), where a rather full discussion of Landau's approach to nonlinear instability can be found). There are also data showing that the exceptional cases cannot be removed from the analysis of hydrodynamic instability when at some critical value of the parameters of the flow, not only $\gamma = 0$ but also $\delta = 0$ (thus the term of the equation for $|A|^2$ of the order $|A|^6$ plays an important part; see, e.g., Ponomarenko (1965). However, the most important defect of Landau's theory is that, so far, it has not been verified by direct calculations in a single concrete case, and that the mechanism of transition to turbulence which it describes is certainly not universal. Thus, for example, in the cases of plane Couette flow or circular Poiseuille flow in the absence of nonaxisymmetric disturbances the Reynolds number $\mathrm{Re}_{cr} = \mathrm{Re}_{cr\,max}$ defined by linear stability theory will apparently equal infinity (i.e., Eq. (2.40) will be meaningless). Consequently, turbulent motion must occur as a result of instability with respect to finite disturbances, while probably from the very beginning, it possesses a very large number of degrees of freedom.[22]

In the case of boundary-layer transition to turbulence, it is possible that an important part is played by the fact that the

[22]However, we note that Tatsumi (1952) made an attempt to explain the initiation of turbulent flow in a circular tube on the basis of the linear theory of axisymmetric disturbances, applied to the intake region of flow where a parabolic Hagen-Poiseuille profile has not yet been formed successfully. On the other hand, Gill (1965b) found that only a small change in the linear velocity profile of a plane Couette flow or in the parabolic profile of circular Poiseuille flow is required to change them from stable to unstable with respect to infinitesimal disturbances. Therefore he proposed the "partially nonlinear" mechanism for instability of these flows, based on the assumption that small (but not infinitesimal) disturbances (axisymmetric in the case of circular Poiseuille flow) produce a small distortion of the primary velocity profile which makes possible the subsequent development of growing disturbances in the framework of the ordinary hnear stability theory.

unstable disturbances which arise at some value of the Reynolds number will be carried downstream into a region with higher Reynolds number. In any case, the existing data on transition in a boundary layer (which we shall discuss below also will not fit into the framework of the above considerations. The described mechanism of initiation of turbulence recalls far more the generation of a turbulent wake in the flow past finite rigid bodies (see above, Sect. 2.2). Apparently, the data referring to this process may be explained on the basis of Landau's theory if we assume that for flow past a body, $\delta < 0$ and the critical Reynolds numbers for instabilities of different orders are close to each other. Unfortunately, the calculation of the critical Reynolds number for flow past a finite body, even within the framework of the linear theory, is an extremely complex problem which does not yield an exact mathematical solution. Therefore, quantitative comparison of Landau's theory with the empirical data on transition in a wake is still not possible.

The cases of circular Couette flow, free convection between parallel plates, and plane Poiseuille flow have been studied far more completely. In these cases, the linear stability theory can yield quite definitive results (see above, Sects. 2.6–2.8). Below, we shall discuss in greater detail the question of the application of Landau's theory (or, more precisely, Landau's equation (2.41) and some of its generalizations) to these three types of flow.

Nonlinear Instability Effects in a Circular Couette Flow and a Layer of Fluid Heated from Below

In either of the above-mentioned two cases the motion arising for $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$ (or $\mathrm{Ra} = \mathrm{Ra}_{\mathrm{cr}}$) is nonperiodic but steady (i.e., $\omega_1 = 0$). Nevertheless, the considerations developed above are also completely applicable to these cases since the unstable motion proves here to be periodic with respect to certain space coordinates (the z-coordinate in the first case, and the x- and y-coordinates in the second); therefore, instead of time-averaging, we may average over these coordinates. The second, more important complication arises because the unstable disturbances refer here to a continuous spectrum (they depend on the continuously varying wave number k); strictly speaking, for $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$ (or $\mathrm{Ra} > \mathrm{Ra}_{\mathrm{cr}}$) a continuous set of distinct unstable disturbances will always exist. Nevertheless, this also finally proves not to be a matter of principle, since the data clearly show that for small $\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}$ (or $\mathrm{Ra} - \mathrm{Ra}_{\mathrm{cr}}$) there always "survives" (and attains a finite value) only a single disturbance with sharply defined wave number. We shall discuss the possible cause of this later; meanwhile we shall take it as an axiom that in both cases it is permissible to confine ourselves to investigation of individual disturbances of the form (2.41) with fixed wave number k and fixed value $\omega = i\gamma$, defined by the linear theory.

All the data on flows between concentric cylinders, and on layers of fluid heated from below, show that at subcritical values of a characteristic nondimensional parameter (i.e., for $\mathrm{Re} < \mathrm{Re}_{cr}$ or, respectively, $\mathrm{Ra} < \mathrm{Ra}_{cr}$) there exists no steady motion different from the laminar Couette flow or, respectively, the state of rest. In discussing the applications of the energy method to thermal convection problems, it was mentioned that for a layer of fluid heated from below, this result was strictly proved by several investigators [namely, Sorokin (1954); Ukhovskiy and Yudovich (1963); Howard (1963); Sani (1964); Platzman (1965); and Joseph (1965)] with the aid of the existence conditions for the solutions of the complete system of nonlinear dynamical equations. The corresponding result for fluid in the annulus between rotating cylinders apparently has not been proved rigorously until now, even for the simplest axisymmetric flows (although some related theoretical deductions can be found in the works cited below); nevertheless, there is no doubt that this result is also valid for the flow in the annulus. At the same time, for $\mathrm{Re} = \mathrm{Re}_{cr}$ (or $\mathrm{Ra} = \mathrm{Ra}_{cr}$) in both cases "branching" of the steady solutions of the corresponding nonlinear equations will apparently occur. At this point, additional steady solutions will arise, differing from the ordinary solution by the presence of periodic terms in z (or in the x- and y-coordinates), the amplitude of which for small values of $\mathrm{Re} - \mathrm{Re}_{cr}$ (or $\mathrm{Ra} - \mathrm{Ra}_{cr}$) is proportional to $(\mathrm{Re} - \mathrm{Re}_{cr})^{1/2}$ [or $(\mathrm{Ra} - \mathrm{Ra}_{cr})^{1/2}$]. Therefore, for slightly supercritical Reynolds numbers (or Rayleigh numbers) two different branches of steady solutions exist which coincide for the critical Reynolds (Rayleigh) number. By slowly increasing the nondimensional parameter above the critical value, we can reach a second critical value, at which a new branching of the steady solutions will occur. The laminar-turbulent transition in both cases may be understood as a process of repeated branching of steady solutions of nonlinear equations in full accordance with the general scheme proposed by Landau.

These statements are very plausible from the viewpoint of existing experimental evidence concerning a circular Couette flow and a layer of fluid heated from below (see, e.g., the data of Malkus (1954a), and of Willis and Deardorff (1967), on the discrete transitions in a layer of fluid, and the data of Coles (1965), and of Schwarz, Springett and Donnelly (1964), on discrete transitions in circular Couette flow), although in Couette flow transitions to nonsteady periodic regimes also appear to be important. Their strict proof must be the first step in the rigorous mathematical theory of transition phenomena; later, it would be necessary to explain also why one particular solution is realized physically and another possible solution is not, and why only one particular value of wave number k is observed in the experiments for slightly supercritical non-dimensional parameter values although a continuum of k-values is mathematically possible. However, even this first step is not an easy one and requires the very heavy machinery of the advanced topological and functional-analytical methods.

Sorokin (1954) was the first to formulate the general result regarding the branching of steady solutions of the nonlinear Boussinesq equations at $\mathrm{Ra} = \mathrm{Ra}_{\mathrm{cr}}$; he demonstrated its plausibility by the use of a formal expansion technique. Later, the same formal technique was used by Sorokin (1961) to analyze the branching phenomena in a bounded flow region in which the fluid is driven by moving walls [the mathematically rigorous analysis of these phenomena was given considerally later; see Yudovich (1967)]. Related phenomena were also demonstrated by Brushlinskaya (1965) for some specific simplified "model" fluid dynamical problems allowing formulation in terms of a finite system of ordinary differential equations; in her proof she used substantially finite-dimensional methods. The general topological theory of the bifurcations of the solutions of operator equations in Banach spaces was developed by Krasnosel'skiy (see, for example, Krasnosel'skiy, 1964) and was first applied to fluid dynamical problems by Velte (1964) and Yudovich (1965b). Later, the same techniques was used by Velte (1966) to show that in circular Couette flow with $\Omega_2 = 0$ (i.e., in the case of a stationary outer cylinder) an additional branch of steady solutions forming Taylor vortices occurs at the critical Reynolds number $\mathrm{Re} = \mathrm{Re}(k)$ for any choice of the wave number k; simultaneously even more general results (relating to an arbitrary value of $\mu = \Omega_2/\Omega_1 \geqslant 0$ and also to higher-order critical Reynolds numbers) were demonstrated similarly by Yudovich (1966a) and Ivanilov and Yalovlev (1966). A rigorous proof that for circular Couette flow the branching solution has an expansion in powers of $(\mathrm{Re} - \mathrm{Re}_{cr})^{1/2}$ was first given by Kirchgässner [see Görfler and Velte (1967)] who used the general analytical method of solving nonlinear integral equations developed by Lyapunov and Schmidt at the beginning of this century.

Convection in a fluid layer heated from below (and of some other convection problems) was investigated in detail by Yudovich (1966b; 1967a,b) with the aid of a combination of the topological method of Krasnosel'skiy and the analytical method of Lyapunov-Schmidt. He showed that when Ra slowly increases and goes through the critical value Ra_{cr} two new steady solutions of a given periodicity in the (x,y)-plane occur, both having an expansion in powers of $(Ra - Ra_{cr})^{1/2}$. Moreover, the equilibrium solution turns out to be unstable for supercritical Ra numbers, and the other two solutions are stable with respect to small disturbances of the same periodicity (so that consideration of disturbances of different periodicities is necessary to explain the special role of hexagonal cells in convection phenomena demonstrated by experiment).

Later, Ovchinnikova and Yudovich (1968) investigated analytically the case of circular Couette flow between two cylinders rotating in the same direction; in particular, a rigorous proof of many of Kirchgässner's results can be found in this paper. Specifically, using the small gap approximation, Ovchinnikova and Yudovich proved that at a given wave number k Couette flow becomes unstable when the Re number reaches (from below) the critical value $Re_{cr}(k)$; with $Re > Re_{cr}(k)$ a second steady flow of the same periodicity arises which is stable to any axisymmetric disturbance, and has an expansion in powers of $(Re - Re_{cr})^{1/2}$. These results were not proved rigorously by the authors without the use of a small gap approximation, since they depend on the positivity of two constants formed from complicated integrals; however, at any fixed $Re > Re_{cr}(k)$ and k, these constants may be evaluated numerically, and Ovchinnikova and Yudovich considered some examples of this type (in all of them the constants turn out to be positive). The numerical solutions for the new steady flow with $k = k_{cr}$ were used by Ovchinnikova and Yudovich also for the evaluation of the torque for $Re > Re_{cr}$; their results agree very well with existing experimental data, and with the theoretical results of Davey (1962) discussed later. Meyer (1967) evaluated the torque in a similar way by direct numerical solution of the time-dependent equations for a two-dimensional disturbance in a circular Couette flow (with the outer cylinder fixed, i.e., for $\Omega_2 = 0$) at $Re > Re_{cr}$. The numerical solutions of Meyer all approached a steady state as $t \rightarrow \infty$, which corresponds to a torque close to that calculated by Davey. Convection in a fluid layer heated from below and of a flow between rotating cylinders (with $\Omega_1\Omega_2 > 0$) at $\mathrm{Ra} \gg \mathrm{Ra}_{cr}$ or $\mathrm{Re} \gg \mathrm{Re}_{cr}$, respectively, were considered briefly by Ivanilov (1966; 1968) who outlined a proof of the existence of a great number of steady flow regimes (apparently unstable) in these cases with relatively close values of velocity distributions and energy.

Let us now discuss a more elementary approach which does not use advanced mathematical techniques (and does not pretend to be mathematically rigorous) but applies the dynamical equations for nonrigorous quantitative explanation of the existing experimental data. The main method in such an approach is to obtain equations of the type of Landau's equation (2.34) which describe the evolution of disturbances that are unstable according to the linear theory directly from the dynamic equations of the problem (combined with specific plausible approximations). Stuart (1958) and Davey (1962) consider this problem in connection with Couette flows between cylinders. In the first of these works it is assumed that $d = R_2 - R_1 \ll (R_1 + R_2)/2 = R_0$ and that only the inner cylinder rotates (with angular velocity Ω_1). Here, instead of the Reynolds number it is convenient to use the Taylor number $\mathrm{Ta} = \Omega_1^2 R_1 d^3/\nu^2 \sim (\mathrm{Re})^2$; the Couette flow becomes unstable for $\mathrm{Ta} > \mathrm{Ta}_{cr} \approx 1708$. Stuart further assumed that the difference $\mathrm{Ta} - \mathrm{Ta}_{cr}$ is small (but positive) and that at the instant $t = 0$ there arises a disturbance periodic in z that is unstable according to the linear theory. He assumed also that the form of this disturbance (which can be found from linear stability theory) varies only slightly with time (so that approximate equation (2.39) for the amplitude $A(t)$ is applicable). Consequently, the dependence of the amplitude $A(t)$ on time can easily be determined from the equation of balance of disturbance energy (i.e., Eq. (2.33), which is a corollary of the dynamical equations). Hence, one obtains for wws an equation of the form (2.41) with coefficients γ and δ expressed explicity in terms of Taylor's number Ta, the Reynolds number $\mathrm{Re} = \Omega_1 R_1 d/\nu$, the wave number $k = k_{cr}$ and the eigenfunction corresponding to the eigenvalue problem (2.16)–(2.17), evaluated by Chandrasekhar (1953). The most important result of the calculations is in fact $\delta < 0$. This result agrees with the nonexistence of unstable disturbances at $\mathrm{Ta} < \mathrm{Ta}_{cr}$ and means that there exists a finite value $|A|_{max} = 2(\gamma/\delta)^{1/2}$, while $|A|_{max} \sim (1 - \mathrm{Ta}_{cr}/\mathrm{Ta})^{1/2} \sim (\mathrm{Ta} - \mathrm{Ta}_{cr})^{1/2}$ for small $\mathrm{Ta} - \mathrm{Ta}_{cr} > 0$. Knowing $|A|_{max}$ and the spatial form of the unstable disturbance (which is given by the linear disturbance theory), Stuart was also able to calculate the torque, i.e., the moment of the frictional forces acting on the surface of the cylinders. The values which he obtained for this torque proved to be in very close agreement with G. I. Taylor's data obtained by direct measurement (1936b) up to values of Ta approximately ten times greater than $\mathrm{Ta}_{\mathrm{cr}}$.

A more precise equation for the amplitude of a disturbance in circular Couette flow was obtained by Davey (1962) who used a method similar to that of Stuart (1960) and Watson (1960) [their works will be discussed later]. Davey took into account that an initial axisymmetric disturbance of the form $\boldsymbol{u}'(\boldsymbol{x}) = A\boldsymbol{f}(r)e^{ikz}$ due to the nonlinearity of the dynamical equations, will also generate higher harmonics (proportional to e^{inkz}, $n = 2, 3, \ldots$) and that the dependence on r of the form of this disturbance will also vary slightly with time. Hence the velocity field of the disturbance will be written here in the form

$$\boldsymbol{u}'(\boldsymbol{x}, t) = \boldsymbol{u}_0(r, t) + \boldsymbol{u}_1(r, t)e^{ikz} + \boldsymbol{u}_2(r, t)e^{2ikz} + \ldots \quad (2.43)$$

(where the term $\boldsymbol{u}_0(r, t)$ describes the distortion of the form of the laminar Couette flow produced by the disturbance). Further, it is assumed that as $t \to 0$, on the right side of Eq. (2.43) only the term $\boldsymbol{u}_1(r, t)$ is conserved, while, for very small $t > 0$, this term becomes the solution $\boldsymbol{u}' = \boldsymbol{f}(r)e^{\gamma t + ikz}$, defined from the linear disturbance theory. Then, for slightly greater positive t, this term will become the leading term, while $\boldsymbol{u}_1(r, t)$ for such values of t may be written as

$$\boldsymbol{u}_1(r, t) = A(t)\boldsymbol{f}(r) + \text{higher-order terms.} \quad (2.44)$$

Substituting Eqs. (2.43) and (2.44) into the nonlinear equations (2.32) [instead of using a sole equation (2.33) as was done by Stuart (1958)], we may once again obtain for $A(t)$ an equation of the form (2.41), where γ is determined from the linear disturbance theory and

$$\delta = \delta_1 + \delta_2 + \delta_3. \quad (2.45)$$

Here $\delta_1 > 0$ defines the influx of energy from the basic disturbance $\boldsymbol{u}_1(r, t)e^{ikz}$ to the laminar Couette flow which produces the distortion $\boldsymbol{u}_0(r, t)$ [only this term, in fact, is taken into account in Stuart's work where the changes in disturbance form were neglected], δ_2 describes the generation of higher harmonics by the basic disturbance, and δ_3 is the distortion of its radial form. For all three terms on the right side of Eq. (2.45), Davey obtained cumbersome equations (containing solutions of the corresponding eigenvalue problem of the linear theory). Then for the cases a) $d \ll R_0$, $\Omega_2 = 0$; b) $d \ll R_0$, $\Omega_2/\Omega_1 \approx 1$; and c) $R_2 = 2R_1$, $\Omega_2 = 0$, he calculated the values of these terms numerically. In all cases the coefficient δ turns out to be positive; in case a) its values agree approximately with the results of Stuart's less precise calculations, and in cases b) and c) they also lead to values of the torque under supercritical conditions which agree very well with existing experimental data.

The agreement between the calculated and observed values of the frictional torque is convincing evidence that Landau's equation (2.41) obtained by Stuart and Davey with $\delta > 0$ gives a rather accurate description of the real growth of an axisymmetric disturbance that is unstable according to the linear theory. However, this is still indirect evidence since it is not the value of the amplitude itself which is compared with experiment, but the integral characteristic of the flow computed according to that characteristic—the total torque acting on the cylinder. A more direct experimental verification of the applicability of Landau's theory (and of Davey's calculations) to flow between rotating cylinders was carried out by Donnelly and Schwarz (1965) [see also Donnelly (1963), and Donnelly and Schwarz (1963)] and by Snyder and Lambert (1966). Donnelly and Schwarz used a special ion technique for measuring the flow disturbances. They filled the gap between the cylinders with an electrolyte CCl_4 (where the nonrotating outer cylinder was of radius $R_2 = 2$ cm and seven interchangeable inner cylinders were used of various radii R_1, most of which corresponded to small gap $d = R_2 - R_2 \ll R_2$), and measured the current passing through it to a collector—a small plate on the outer cylinder, which could move with constant velocity along the Oz axis. For $\mathrm{Ta} \gg \mathrm{Ta}_{cr}$, in the electrolyte between the cylinders a regular set of steady toroidal vortices occurs, the velocity field of which takes the form $\boldsymbol{u}'(\boldsymbol{x}) = A\boldsymbol{f}(r)e^{ikz}$ where the coefficient A is $A(\infty) = A_{\max}$ of Landau's theory. These vortices disrupt electrically charged fluid layers around the electrodes, and hence affect the current passing through the electrolyte. In calculating this phenomenon, it is seen that the appearance of vortices must correspond to the appearance in the expression for the current j of an additional term of the form $\Delta j = CA \cos kz$, where C is a specific constant. The measurement results confirm that for $\Omega_1 \gg \Omega_{cr} = (\nu^2 \mathrm{Ta}_{cr}/R_1 d^3)^{1/2}$ such a component periodic in z will in fact exist. The square of its amplitude $(CA)^2$ turns out to be proportional to $\Omega^2 - \Omega_{cr}^2 \sim \mathrm{Ta} - \mathrm{Ta}_{cr}$ (which is in complete accordance with Landau's theoretical prediction) right up to the considerably greater value $\Omega_1 = \Omega_{2cr}$, after which the sharp decrease in ion signal begins (see Fig. 23 which corresponds to measurements at $R_1 =$ 1.9 cm; other examples of such curves can be found in Donnelly and Schwarz (1965). Based on the general propositions of Landau's theory, we may assume that the sharp break in the law $A_{max} \sim (Ta - Ta_{cr})^{1/2}$ when $\Omega_1 = \Omega_{2cr}$, is connected with Taylor's number attaining its "second critical value" Ta_{2cr}, where the toroidal vortices become unstable and decompose into more complex disturbances.

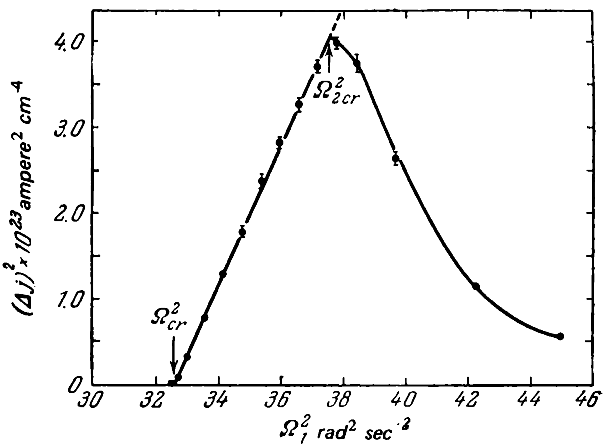


FIG. 23. Dependence of the square of the amplitude of the additional current $(\Delta j)^2$ in an electrolyte between rotating cylinders on the angular velocity of the internal cylinder [According to Donnelly (1963)].

The data in Fig. 23 concern only the ultimate amplitude $A(\infty) = A_{max}$ of Landau's theory. In addition, Donnelly and Schwarz (1963; 1965) produced another verification of Eq. (2.41), based on the use of Eq. (2.42) for finite values of t. With this goal they rewrote Eq. (2.42) in the form

$$\frac{A^2(t)}{A^2_{max}} = \frac{e^{2\gamma t}}{3 + e^{2\gamma t}}, \tag{2.42'}$$

where the undetermined constant C was successfully eliminated through a selection of the origin of time from the condition

$A(0) = \frac{1}{2}A_{\max}$. Donnelly and Schwarz then produced a series of experiments, in each of which the speed $\Omega_1 = 3$ rad/sec (which appears to be subcritical from the conditions of the experiment) was suddenly increased to the value shown on the right of each of the curves in Fig. 23a. As a result, fluctuations in the ionic current immediately arose with a variable amplitude proportional to $A(t)$. The change in this current with time, measured from the moment corresponding to the amplitude $A = \frac{1}{2}A_{\max}$, is plotted in Fig. 23a together with the points calculated from Eq. (2.42′) with values of γ also indicated to the right of each of the curves. The agreement between theory and experiment is excellent. Values of γ corresponding to each of the curves in Fig. 23a may also be calculated from the solution of the linearized equations [i.e., of the eigenvalue problem (2.16)–(2.17)] with given values of R_1, R_2, Ω_1 and $\Omega_2 = 0$. The associated wave number k in Eq. (2.17) may be chosen equal to k_{cr}, since experiments show that for moderate supercritical values of Re

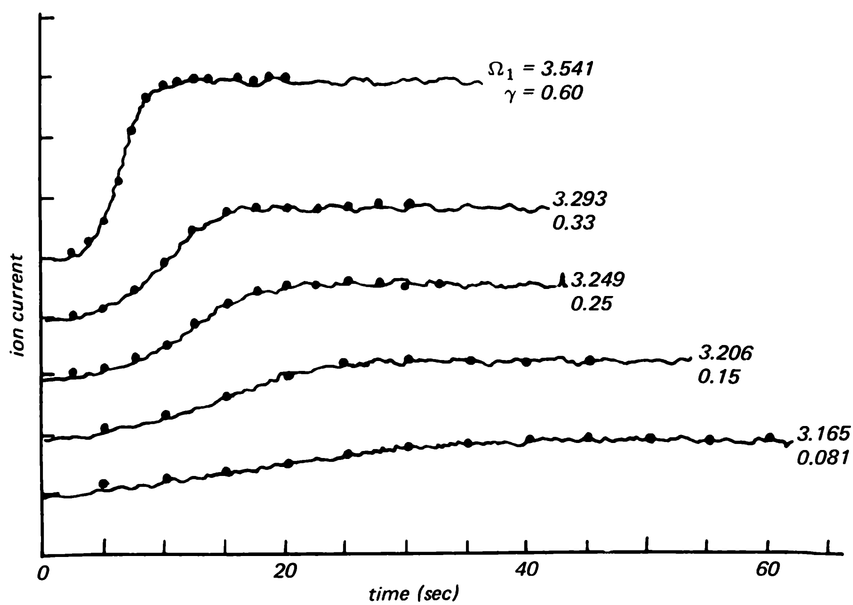


FIG. 23a. Ion current amplitude as a function of time upon suddenly increasing the speed Ω_1 from below critical to various supercritical values. The dots represent corresponding theoretical curves for values γ shown at the right of each trace (After Donnelly and Schwarz 1965).

or Ta the axial wavelength of the Taylor vortices remains approximately constant, i.e., almost indistinguishable from $2\pi/k_{cr}$; instead of this one may choose the value of k from the maximum instability criterion (i.e., from the condition $\mathrm{Im}\,\omega = \gamma = \max$), which, for moderate Ta-Ta_{cr}, leads to practically the same result. Such calculations of the value of γ were carried out by Roberts (1965); they led to results which agreed well with the values of γ determined from the curves in Fig. 23a (which again confirms that $\gamma \sim (\mathrm{Ta}-\mathrm{Ta}_{cr})$, as also follows from the data in Fig. 23).

The results of the brilliant experiments of Donnelly and Schwarz substantially confirm the theoretical concepts developed by Landau; however, quantitatively, they do not appear to be sufficiently complete. Equation (2.42′) contains a single parameter γ; for this reason the data of Fig. 23a permit verification only of the results of the calculation of this parameter, following from the linear theory (together with the qualitative conclusion regarding the validity of equations like Eq. (2.41) with a positive coefficient δ). No conclusions regarding the values of δ may be elicited from the data of Fig. 23. From this viewpoint, the results of Snyder and Lambert (1966) appear to be more informative. They used a special hot thermistor anemometer, measuring the total velocity gradient at the gap wall. Since Davey in his calculations considered only axisymmetric disturbances, while circular Couette flow for small gap becomes unstable with respect to nonaxisymmetric disturbances for Ta only slightly in excess of the critical value Ta_{cr}, Snyder and Lambert chose for verification the third case studied by Davey ($R_2/R_1 = 2$, $\Omega_2 = 0$), where instability with respect to nonaxisymmetric disturbances ensues for substantial values of Ta-Ta_{cr}. In their apparatus a small axial flow (the corresponding axial Reynolds number was about 2), was created by which they were able to fix the dependence of the velocity gradient on the z-coordinate, and thereupon single out (with the help of a spectral analyzer) the basic disturbance, proportional to $\cos kz$, and its first, second and third harmonics. They found that the values obtained for the amplitudes of the basic disturbance and its first harmonic (proportional to $\cos 2kz$) are in excellent agreement with the results of the calculations based on Davey's values of γ, δ and δ_2, over a very wide range of Taylor numbers Ta (ranging right up to $4\mathrm{Ta}_{cr}$).

The theoretical conclusions of Stuart (1958) and Davey (1962) are in good agreement with the experimental results of Donnelly and

Schwarz (1963; 1965) and Snyder and Lambert (1966), and taken together, they describe satisfactorily the nonlinear development of axisymmetric Taylor vortices. However, this does not mean that all the principal nonlinear effects in circular Couette flow may be regarded as understood. In fact, the experiments of many investigators [and first of all the work of Coles (1965)] show, that upon an increase of the Reynolds number (or Taylor number) above the critical value, transitions to new nonaxisymmetric flow regimes appear rather quickly. Coles identified a whole series of successive discrete transitions from one regime to another; if one characterizes the state of flow with m vortices between the two ends of the cylinders, each of which has an azimuthal wave number (number of azimuthal waves) n, by the symbol m/n, then for $\Omega_2 = 0$ and increasing speed of rotation Ω_1, the following states appeared successively at quite definite (and repeatable) speeds: 28/0 (Taylor vortices); 28/4; 24/5; 22/5; 22/6; etc. (in all 25 different states.[23] Among these, in all cases with $n \neq 0$ the boundaries between successive vortices had wave-like form. The results of Coles differ somewhat from the results of other investigators; thus, for example, Schwarz, Springett and Donnelly (1964), having undertaken an investigation of Couette flow in an apparatus of much greater length (containing up to 260 Taylor vortices) and with a much smaller gap between the cylinders, found the first transition to a nonaxisymmetric state with $n = 1$ (for Ta only 3–8% in excess of Ta_{cr}). However, it is important to note that these experiments also confirm the appearance of nonaxisymmetric states of flow.

The experiments described above create the impression that with increasing Ta (or Re) for comparatively moderate supercritical values axisymmetric Taylor vortices become unstable to nonaxisymmetric disturbances, having roughly the same axial wave number k, but a differing axial phase (otherwise the boundary between successive vortices would remain plane, and not become wavy). The same conclusion follows also from the theoretical calculations of Davey, Di Prima and Stuart (1968) in which a nonstandard method was used to investigate the stability of Taylor vortices (in the case of small gap). These investigators assumed that at the moment $t = 0$ in

[23] As already noted in Sect. 2.6 in the case of circular Couette flow, which remains stable to infinitesimal disturbances for all Reynolds numbers, an increase in the Reynolds number leads to a quite different type of transition to turbulence, characterized by the appearance of separate turbulent regions in the flow [cf. Coles (1965); Van Atta (1966); Coles and Van Atta (1967)].

circular Couette flow there arise four independent disturbances of the following form:

$$A_c f_0(r) \cos kz;\ A_s f_1(r) \sin kz;\ B_c f_2(r) \cos kze^{in\varphi};\ B_s f_3(r) \sin kze^{in\varphi}. \tag{2.46}$$

With the evolution of time, all four amplitudes $A_c = A_c(t)$, $A_s = A_s(t)$, $B_c = B_c(t)$ and $B_s = B_s(t)$ will vary, interacting among one another; in addition, they will create a distortion of the velocity profile of the basic Couette flow, which will change slightly their own form, and will produce higher axial and circumferential harmonics. All these effects may be taken into account if one postulates a velocity field $\boldsymbol{u}' = \boldsymbol{u}'(r, \varphi, z, t)$ in the form of a Fourier series in z and φ, substitutes this series into the Navier-Stokes equations (taking advantage in the boundary conditions as well, of the simplification arising from the assumption of small gap), and sets equal to each other the coefficients of similar terms of the Fourier series in the right and left sides of the resulting equality. Carrying out the necessary calculations and retaining in the resulting equations only terms of order no higher than fourth in the amplitudes A and B [which corresponds to retaining only terms of order $|A|^2$ and $|A|^4$ on the right side of Eq. (2.41)] Davey et al. obtained a system of four nonlinear differential equations for the amplitudes A_c, A_s, B_c, B_s. The system obtained is similar in form to the systems (2.49) and (2.52), written below (but much more complex—eight lines are required to write it); its right side contains nine numerical coefficients [of the type of coefficients γ, a, δ_1, and δ_2 of the system (2.52)], defined through cumbersome manipulation of the solution of the corresponding eigenvalue problems (2.14)–(2.16), describing the separate disturbances (2.46). Davey et al. investigated all the stationary solutions of this system for supercritical values of Ta; one of these solutions turned out to be, naturally, the Taylor-vortex flow, for which $A_c = (2\gamma/\delta)^{\frac{1}{2}}$, $A_s = B_c = B_s = 0$ [here γ and δ are the same as in Eqs. (2.41) and (2.45)]. Considering further the case of small disturbances to Taylor vortices, i.e., postulating $A_c(t) = (2\gamma/\delta)^{\frac{1}{2}} + a_c(t)$, $A_s = A_s(t)$, $B_c = B_c(t)$, $B_s = B_s(t)$ where a_c, A_s, B_c, B_s are of small magnitude, after linearization of the general amplitude equations, one obtains new equations of the form

$$\frac{da_c}{dt} = -2\gamma a_c,$$

$$\frac{dA_s}{dt} = 0,\quad \frac{dB_c}{dt} = (\gamma_B + \gamma\delta_{1B}/\delta)B_c,\quad \frac{dB_s}{dt} = (\gamma_B + \gamma\delta_{2B}/\delta)B_s.$$

These new equations determine (within the framework of linearized stabihty theory) the temporal evolution of disturbances of the form (2.46) to a flow of Taylor-vortex form. Here γ_B is a coefficient describing within the hnearized theory the temporal evolution of a nonaxisymmetric disturbance, proportional to $\exp[i(kz + n\varphi)]$, and δ_{1B} and δ_{2B} are two more of the nine coefficients mentioned above, occurring in the amplitude equations.

Since $\gamma > 0$ for $\mathrm{Ta} > \mathrm{Ta}_{\mathrm{cr}}$, $a_c(t)$ is damped. Consequently, stability or instability of Taylor vortices is determined by the sign of the real part of the combinations $\gamma_B + \gamma\delta_{1B}/\delta$ and $\gamma_B + \gamma\delta_{2B}/\delta$.

Exact calculations of the coefficients γ_B, δ_{1B} and δ_{2B} for various κ, n, Ω_1, Ω_2, and R_1, R_2 appear to be an extremely complicated and cumbersome problem. For this reason Davey et al. considered only the case $\Omega_2 = 0, k = k_{\mathrm{cr}}$, and introduced several simplifying approximations. Here for the simplification adopted, $\mathrm{Re}[\gamma_B + \gamma\delta_{1B}/\delta] < 0$ for all $\mathrm{Ta}\text{-}\mathrm{Ta}_{\mathrm{cr}} > 0$ (i.e., the Taylor vortices are stable to relatively in-phase nonaxisymmetric disturbances). At the same time $\mathrm{Re}[\gamma_B + \gamma\delta_{2B}/\delta]$ becomes positive for $\mathrm{Ta}/\mathrm{Ta}_{\mathrm{cr}} \approx 1.08$; consequently, the Taylor vortices become unstable to out-of-phase nonaxisymmetric disturbances for Ta only 8% in excess of $\mathrm{Ta}_{\mathrm{cr}}$ (as a result of the simplifications made in the numerical computations this result can be considered only as a rough estimate of the magnitude $\mathrm{Ta}_{2\mathrm{cr}}$). Moreover, the dependence of $\mathrm{Re}[\gamma_B + \gamma\delta_{2B}/\delta]$ on the azimuthal wave number n turns out to be rather weak, but, in general, one may say that $\mathrm{Re}[\gamma_B + \gamma\delta_{2B}/\delta]$ decreases with increasing n, so that the initial instability must occur for $n = 1$. All these conclusions agree rather well with the experimental findings of Schwarz, Springett and Donnelly, obtained under conditions more similar to those assumed in the theory of Davey et al. (where the length of the cylinders was considered infinite, and the gap small), than those characterizing the experiments of Coles.

The theory of Davey et al., of course, is not quite exact—they considered only the interaction of four disturbances of the special form (2.46), while for every $\mathrm{Ta} > \mathrm{Ta}_{\mathrm{cr}}$ in circular Couette flow there will exist infinitely many unstable disturbances (corresponding to the interval of unstable wave numbers k, containing k_{cr} and expanding with increasing $\mathrm{Ta}\text{-}\mathrm{Ta}_{\mathrm{cr}}$). For this reason, even the question of why, for moderate positive $\mathrm{Ta} - \mathrm{Ta}_{\mathrm{cr}}$ (or $\mathrm{Re}\text{-}\mathrm{Re}_{\mathrm{cr}}$), strongly z-periodic states of flow between cylinders arise, characterized by a single value of the wave number k, cannot be regarded at present as completely resolved. Some general considerations concerning this question (and not even using in a specific form the equations of fluid mechanics, i.e., having to do with a wide class of physical processes, described by nonlinear partial differential equations) may be found in Ponomarenko (1968a).

Let us turn now to the mathematically much simpler problem of convection in a layer of fluid heated from below. We have already pointed out that for $\mathrm{Ra} > \mathrm{Ra}_{\mathrm{cr}}$, stationary solutions describing states of rest turn out to be nonunique; in addition to these additional stationary cellular solutions arise, the amplitude of which is proportional to $(\mathrm{Ra}-\mathrm{Ra}_{\mathrm{cr}})^{\frac{1}{2}}$ [for small $(\mathrm{Ra}-\mathrm{Ra}_{\mathrm{cr}})/\mathrm{Ra}$]. This fact agrees well with the idea that for $\mathrm{Ra} > \mathrm{Ra}_{\mathrm{cr}}$ in a layer of fluid heated from below, there occur softly excited oscillations (spatially, rather than temporally) corresponding to Landau's schema with $\delta > 0$. Additional cellular solutions of the nonlinear Boussinesq equations were studied by Sorokin (1954), Gor'kov (1957), Malkus and Veronis (1958), Kuo (1961) [in this paper a numerical method of analysis was used for the first time]; Bisshopp (1962), and others. It is important that stationary solutions of the nonhnear convection equations be strongly nonunique.

Let us begin with the simplest approach of Stuart (1958), based on an approximate assumption (2.39) [the so-called "shape assumption"] that the form of the disturbances does not change with time, and coincides with the form of the unstable disturbances which arise for $\mathrm{Ra} = \mathrm{Ra}_{\mathrm{cr}}$. Then, for every choice of solution $\varphi(x_1, x_2)$ of Eq. (2.23) with $k = k_{\mathrm{cr}}$, describing dependence of the unstable disturbance on the horizontal coordinate, one may easily obtain an equation of the form (2.41) for the amplitude $A = A(t)$ through the balance equations (2.33″) and (2.33‴), where $\gamma \sim (\mathrm{Ra}-\mathrm{Ra}_{\mathrm{cr}})$ is obtained from linear disturbance theory, while δ is simply related to the eigenfunctions (of the eigenvalue problem) which describes the neutrally stable infinitesimal disturbance [cf., for example, Roberts (1966)]. The assumption of preservation ("shape assumption") of the Landau-Stuart equation (2.39) does not appear to be exact, and is suitable only as a first approximation for Ra slightly in excess of $\mathrm{Ra}_{\mathrm{cr}}$; however, the more exact methods of Gor'kov, Malkus and Veronis, Kuo, Bisshopp and others, also permit a large number of distinct stationary solutions to be obtained in the form of two-dimensional waves, proportional to $\cos(k_1x_1 + k_2x_2)$, $k_1^2 + k_2^2 = k_{\mathrm{cr}}^2$ (so-called rolls), square cells, hexagonal cells, etc. An extremely general method for the construction of stationary solutions to the nonlinear convection equations for moderate values of $\mathrm{Ra} > \mathrm{Ra}_{\mathrm{cr}}$

[suitable, however, up to Ra several times $\mathrm{Ra}_{\mathrm{cr}}$; cf. Busse (1967a)] was developed by Schlüter, Lortz and Busse (1965). This method utilized an expansion in powers of a small parameter ϵ (related to the expansions used by Gor'kov and by Malkus and Veronis for the construction of some specific solutions) of the form

$$\mathbf{u}(\mathbf{x}) = \epsilon \mathbf{u}^{(1)}(\mathbf{x}) + \epsilon^2 \mathbf{u}^{(2)}(\mathbf{x}) + \cdots, T(\mathbf{x}) = \epsilon T^{(1)}(\mathbf{x}) + \epsilon^2 T^{(2)}(\mathbf{x}) + \cdots,$$
$$\mathrm{Ra} = \mathrm{Ra}_{\mathrm{cr}} + \epsilon \mathrm{Ra}^{(1)} + \epsilon^2 \mathrm{Ra}^{(2)} + \cdots .$$

If we set equal the coefficients of equal powers of ϵ on both sides of the system of Boussinesq equations, and each time use the boundary conditions of the problem, we obtain for $u^{(1)}(x)$, $T^{(1)}(x)$ the usual linearized convection equations; for terms of higher order in the series, a sequence of systems of inhomogeneous partial differential equations is obtained. It is well known that such an inhomogeneous system will have solutions only if appropriate existence conditions are satisfied (namely, if a scalar product is defined in the space of the pairs $\mathbf{u}(x)$, $T'(x)$, when the right side of the system must be orthogonal to all solutions of the adjoint homogeneous system). These existence conditions allow one to successively define all the $\mathrm{Ra}^{(m)}$. Subsequently, $u^{(m)}(x)$ and $T^{(m)}(x)$, $m = 2, 3, \ldots$ in many cases may be found uniquely with the help of solutions of the corresponding inhomogeneous system (if for $u^{(1)}(x)$, $T^{(1)}(x)$ we choose some suitably defined solution of the linearized problem), and finally we may define even the parameter ϵ from the equality $\mathrm{Ra} - \mathrm{Ra}_{\mathrm{cr}} = \epsilon \mathrm{Ra}^{(1)} + \epsilon^2 \mathrm{Ra}^{(2)} + \cdots$ [cf. specifically, an example of the application of this approach to the solution of a simple model nonlinear partial differential equation analyzed in the survey paper by Segel (1966)].

Schlüter et al. studied the stationary solutions obtained if the zero-order approximation $u^{(1)}(x)$, $T^{(1)}(x)$ is chosen proportional to the function

$$\varphi(x_1, x_2) = \sum_{n=-N}^{N} C_n \exp[i(\kappa_1^{(m)} x_1 + \kappa_2^{(m)} x_2)], \tag{2.47}$$

where $\kappa_i^{(-n)} = -\kappa_i^{(n)}$, $i = 1, 2$: $C_{-n} = C_n^*$; $(\kappa_1^{(n)})^2 + (\kappa_2^{(n)})^2 = \kappa^2 = \text{const}$. In this case the existence conditions for the solution of the second-order system (for $u^{(2)}$, $T^{(2)}$) reduced to the condition $\mathrm{Ra}^{(1)} = 0$ (thus if we limit ourselves to terms of order no higher than ϵ^2, then $\epsilon \sim (\mathrm{Ra} - \mathrm{Ra}_{cr})^{\frac{1}{2}}$). If we take $\mathrm{Ra}^{(1)} = 0$, then the system of equations $u^{(2)}$, $T'^{(2)}$ has unique solutions for arbitrary choice of the functions (2.47) in the zero-order approximation. However, the existence conditions for the solution of the third-order system not only define $\mathrm{Ra}^{(2)}$ uniquely, but also impose a restriction on the function $\varphi(x_1, x_2)$: if we are given a set of 2-vectors $\kappa^{(1)} = (\kappa_1^{(1)}, \kappa_2^{(1)}), \kappa^{(2)}, \cdots, \kappa^{(n)}$ in expression (2.47), then a stationary solution of the nonlinear third-order system will exist only for a rather narrow choice of the coefficients $|C_1|, \ldots, |C_N|$. Thus, not every solution of the linearized problem can be a zero-order approximation to the stationary solution of the nonlinear system of convection equations. In addition, Schlüter et al. showed that the number of functions $\varphi(x_1, x_2)$ from which one may construct stationary solutions $u(x)$, $T'(x)$ up to terms of all orders in ϵ, nevertheless, appears to be infinite. Thus, for example, all existence conditions can easily be satisfied in the "regular case" (in which all angles between neighboring k-vectors are equal, and $|C_1|^2 = \cdots = |C_N|^2 = 1/2N$; the regular case includes rolls, square cells and hexagons). This is also true for even more general "semiregular" cases [cf. Segel (1965); Busse (1967c)].

For Ra much greater than Ra_{cr} the expansion procedures of Gor'kov, Malkus and Veronis, and Schlüter et al. are difficult to apply. In these cases the direct numerical procedures seem to be preferable. Such numerical procedures (based on various simplifying approximations) were used for finding specific steady solutions of the Boussinesq equations (mostly the simplest two-dimensional rolls) for different boundary conditions and different Pr values in the works of Kuo (1961), Herring (1963; 1964), Deardorff (1964), Fromm (1965), Veronis (1966), Busse (1966b), Roberts (1966), Schneck and Veronis (1967), Plows (1968) and some other authors. The results obtained in these works agree well in many respects with the existing data on convective heat transfer at large enough Rayleigh numbers, and on the mean characterisitcs of temperature and velocity fields under such conditions. In particular, it is worth noting that the computations of all these investigators show that the vertical profile of the mean (i.e., averaged over horizontal coordinates) temperature at large Ra differs strongly from the linear profile which is observed at $\mathrm{Ra} < \mathrm{Ra}_{cr}$. Namely, as Ra is increased, a thick region in the center of the fluid layer achieves a nearly isothermal state in the mean (the thickness of this region increases with increasing Ra), and almost all change in the mean temperature is concentrated in two thin thermal boundary layers near the boundaries of the flow. The temperature field for large values of Ra is characterized by a large mass of nearly isothermal fluid in the center and a specific mushroom-shaped form of the isotherms. It is an interesting detail that in the almost isothermal central region a small positive vertical temperature gradient (i.e., a reversal of temperature gradient) occurs when $\mathrm{Ra}/\mathrm{Ra}_{cr}$ is greater than several units. These results are most distinct on the figures presented by Veronis (1966); however, they can be observed on the data of all other investigators beginning with Kuo (1961). The reversal of the temperature gradient in the thick central region of the layer of fluid (at $\mathrm{Ra}/\mathrm{Ra}_{cr} = 16$) was observed also experimentally by Gille (1967) with the aid of precise interferometric measurements.

Let us now discuss a very important question concerning the existence of a preferred disturbance mode which is the only one occurring in real physical layers of fluid. This question is closely connected with the stability theory for cellular convective motions which is at present far from being complete. As we know from Sect. 2.7, the linear stability theory leads to the conclusion that for $\mathrm{Ra} > \mathrm{Ra}_{cr}$ there must exist an infinite set of unstable infinitesimal disturbances (with exponential growth rates) corresponding to some range of values of the wave number k surrounding the value $k = k_{cr}$ at which instability first appears. The most unstable (i.e., the most rapidly increasing) disturbances will correspond to one definite k-value, but there will be an infinite set of such disturbances also [since for given k the horizontal form of a disturbance may be described by an arbitrary function $\varphi(x_1, x_2)$ satisfying Eq. (2.23)]. Experiments show, however, that under each specific set of conditions there will always arise only a disturbance having a strictly defined form (corresponding largely to a division of the horizontal plane into a set of regular hexagonal cells, although there are other possibilities), and strictly defined finite amplitude. Landau's theory permits only this steady amplitude to be found [with the aid of Eq. (2.41) obtained from the nonlinear Boussinesq equations on the assumption that the function (x_1, x_2) is known; cf. Gor'kov (1957)]. However, this theory says nothing of why disturbances with several different values of k never arise in the fluid, and why among all the disturbances with given k, only those with one definite form $\varphi(x_1, x_2)$ are actually observed. The fact that in a number of cases, nonlinear interactions of disturbances differing in wave number may lead to the vigorous growth of distrubances of one given wave number, due to the suppression of all the rest, is explained partially by the calculations of Segel (1962). In this work a simple "pair interaction" of two rolls independent of the x_2 coordinate in a layer bounded both above and below by plane-free boundary conditions) was considered. In other words, Segel studied the evolution of a disturbance for which the velocity component $u_3(x,t) = u_3(\xi,\eta,\zeta,t)$ where $\xi = x_1/H$, $\eta = x_2/H$, $\zeta = x_3/H$, and H is the depth of the layer, takes the form:

$$u_3(\boldsymbol{x}, t) = A_1(t)\cos k\xi f_1(\zeta) + A_2(t)\cos l\xi f_2(\zeta) + \text{small complement.} \tag{2.48}$$

Then, applying the methods of Stuart (1960) and Watson (1960a), which we shall discuss later, Segel obtained in the first nonlinear approximation the "amplitude equations" for the functions A_1 and A_2 of the following form:

$$\begin{aligned} \frac{dA_1}{dt} &= \gamma_1 A_1 - (\delta_1 A_1^2 + \beta_1 A_2^2) A_1, \\ \frac{dA_2}{dt} &= \gamma_2 A_2 - (\beta_2 A_1^2 + \delta_2 A_2^2) A_2, \end{aligned} \tag{2.49}$$

which for $A_2 = 0$ or $A_1 = 0$, clearly yield an equation which is equivalent to Landau's equation (2.41) for the amplitude of a single disturbance. The system (2.49), evidently, has the following steady solutions:

$$\begin{aligned} &\text{(I)}\ A_1 = A_2 = 0, \\ &\text{(II)}\ A_1 = 0,\ A_2 = (\gamma_2/\delta_2)^{1/2}, \\ &\text{(III)}\ A_1 = (\gamma_1/\delta_1)^{1/2},\ A_2 = 0, \\ &\text{(IV)}\ A_1 = (\gamma_1\delta_2 - \gamma_2\beta_1)^{1/2}(\delta_1\delta_2 - \beta_1\beta_2)^{-1/2}, \\ &\qquad A_2 = (\gamma_2\delta_1 - \gamma_1\beta_2)^{1/2}(\delta_1\delta_2 - \beta_1\beta_2)^{-1/2}. \end{aligned}$$

The stability of these solutions may be verified with the ordinary methods of stability theory of differential equations (or the nonlinear theory of oscillations). It is found that in the most important case, when $\gamma_1 > 0, \gamma_2 > 0, \delta_2 > 0, \delta_2 > 0$, it follows from the stability of even one of the solutions (II) and (III) that the solution (IV) cannot be stable. Thus it is clear that a wide class of situations exists in which the ultimate state will always contain only one roll (with a definite wave number) but not a mixture of both. A more detailed analysis of the equations when Ra is just above $\mathrm{Ra}_{\mathrm{cr}}$ shows that if the linear theory growth rate γ_1 of the first roll is more than twice the growth rate γ_2 of the second roll, then in the final equilibrium state only the first roll appears (i.e., (III) is the only stable steady solution). Similarly if $\gamma_2 > 2\gamma_1$, only solution (II) is stable. That is, if one of the two competing primary disturbances has a sufficiently great advantage according to the linear theory, only the advantaged disturbance will appear in the final state while the disadvantaged one ultimately decays. On the other hand, if $\gamma_2 < \gamma_1 < 2\gamma_2$, then both solutions (II) and (III) are locally stable and the final state is either (II) or (III), depending on the initial conditions. However, in this case the initial conditions for very small disturbances are much more likely to be such that the ultimate state corresponds to solution (III) [cf. Segel (1966)]; in this sense, the disturbance with the greater linear growth rate will always be more advantaged in the nonlinear theory also. In principle, it is also possible [for certain values of the coefficients in Eq. (1.49)] that the interaction of one stable and one unstable disturbance (i.e., the case with $\gamma_1 > 0$ and $\gamma_2 < 0$) may lead to increase of the unstable disturbance and the final establishment of a "mixed state" corresponding to a solution of the type (IV). It is probable that such a situation may also sometimes be encountered in mechanically driven flow (for example, in the case of plane-parallel flow). However, for thermal convection problems with Ra number only slightly greater than $\mathrm{Ra_{cr}}$ and instability of both primary rolls (i.e., with $\gamma_1 > 0$, $\gamma_2 > 0$) one of the two rolls will necessarily decay. This explains to some extent how the interaction between disturbances lead to the fact that for small $\mathrm{Ra}-\mathrm{Ra_{cr}}$, out of the whole range of unstable disturbances, only that with one definite value of wave number k is actually observed. The same results were obtained by Segel for the more general case of N parallel "nonreplicated" rolls (such that if k and l are the wave numbers of two of the rolls, then $(k + l)/2$ is not among the wave numbers considered; here also a single disturbance ultimately appears and if one of the rolls has much greater linear growth rate than all the others, it necessarily survives [cf. Segel (1966)]. A more general approach to the problem of the mechanism of selection of a single preferred wave number from the whole band of unstable wave numbers for given $\mathrm{Ra} > \mathrm{Ra_{cr}}$ is outlined by Ponomarenko (1968a).

Now let us consider disturbances with a given wave number k only. Here, also, a question arises on the selection of the preferred mode because there are an infinite set of different disturbance forms with the horizontal wave number k. However, only one of these forms is observed in real experiments performed under fixed external conditions. Therefore the investigation of the stability of the different steady solutions of the nonlinear Boussinesq equations for $\mathrm{Ra} > \mathrm{Ra_{cr}}$ is of great importance. One of the most complete investigations of this type was carried out by Schlüter, Lortz and Busse (1965). These authors considered all the steady solutions evolved from infinitesimal disturbances of the horizontal form (2.47) at a slightly supercritical value of the Rayleigh number and studied the stability of the finite amplitude cellular motions so obtained with respect to infinitesimal disturbances of similar form. As a result, they came to the rather unexpected conclusion that all the cellular solutions with the exception of the simplest two-dimensional rolls [which correspond to $N = 1$ in Eq. (2.47)] are certainly unstable. For the exceptional case of the rolls with given horizontal wave number k, Schlüter et al. showed that they are stable to all infinitesimal disturbances with the same wave number k if only this wave number belongs to the band of unstable (according to the linear stability theory) wave numbers. Finally, Schluter et al. investigated the stability of two-dimensional rolls of finite amplitude to infinitesimal disturbances of horizontal wave number $k_1 \neq k$. They found that when $\mathrm{Ra} - \mathrm{Ra}_{\mathrm{cr}}$ is small enough the rolls with wave number $k < k_{\mathrm{cr}}$ (where k_{cr} is the wave number of the infinitesimal disturbance

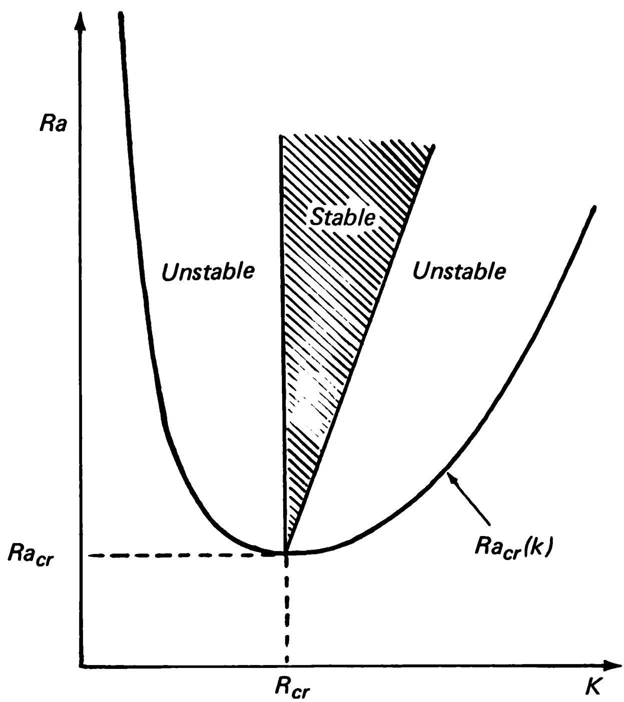


FIG. 23b. Stability range of rolls at Rayleigh numbers close to critical [After Schlüter, Lortz, and Busse (1965)].

which is neutrally stable at Ra_{cr}) cannot be stable to disturbances with arbitrary wave numbers. However, if k is greater than k_{cr} and $k - k_{cr}$ is small enough (of the order of $Ra - Ra_{cr}$) the rolls with wave number k are stable with respect to all possible infinitesimal disturbances. The full range of the stable two-dimensional rolls for small enough values of the difference $Ra - Ra_{cr} > 0$ found by Schlüter et al. is shown in Fig. 23b.

The results of Schlüter, Lortz and Busse are obtained by an expansion procedure in powers of a small parameter ϵ and are valid for Rayleigh numbers close to the critical value only. The general stability analysis of the solutions of the Boussinesq equations at higher Rayleigh numbers is difficult to perform. However, in the particular limiting case of infinite Prandtl number, the Boussinesq equations undergo considerable simplification and the stability problem becomes accessible to analysis. Using numerical methods, Busse (1967b) computed steady solutions of the Boussinesq equations with $Pr = \infty$ in the form of rolls for a wide range of Ra numbers and investigated the stability of the solutions obtained with the aid of the usual linear stability theory. He found that stable rolls correspond to a narrow elongated region on the (Ra, k)-plane [see Fig. 23c]. The range of stable wave numbers for all Rayleigh

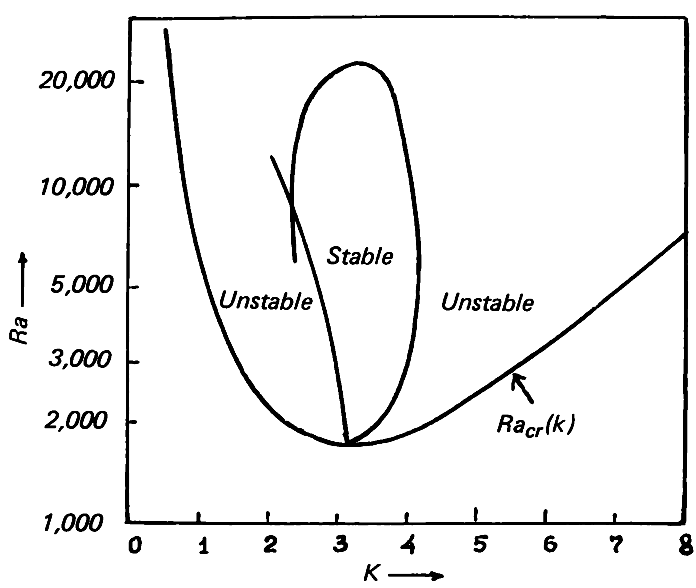


FIG. 23c. Stability region of rolls for a wide range of Rayleigh numbers in the case $Pr = \infty$ [*After Busse*(1967)].

numbers below 22,600 is restricted to a small band (almost independent of the value of Ra) surrounding k_{cr}. At Ra = 22,600 all two-dimensional solutions of the Boussinesq equations (at least when Pr = ∞) become unstable. It is important to note that the value 22,600 is of the same order of magnitude as the value of Ra at which the second discrete transition (in cellular convection) was observed experimentally.

The results shown in Figs. 23b and 23c are in qualitative agreement with the fact that the characteristic scale of the horizontal cells does not change considerably when the Rayleigh number increases. However, there is another contradiction between these theoretical results and the experimental findings. The theory asserts that the only stable form of finite-amplitude cellular convection is two-dimensional rolls. Such two-dimensional convective cells were in fact observed by several investigators including Silveston (1958; 1963), Koschmieder (1966), Chen and Whitehead (1968), Rossby (unpublished Ph.D. thesis; MIT, 1966), and some others. However, even more usual are observations in which cellular convection takes the form of beautiful regular hexagonal Bénard cells; this noteworthy fact was twice stated by us in a previous discussion of the convection problem. Now we must explain this fact from the viewpoint of stability theory.

The explanation is based on the finding that additional effects neglected in the Boussinesq approximation often play a dominating part in the real laboratory experiments. At the end of Sect. 2.7 we mentioned that surface tension effects played an important role in the original Bénard experiments; the same is apparently true for many other experiments with liquid layers bounded above by an air surface [cf. Koschmieder (1967); see also the corresponding theory of Scanlon and Segel]. The experimental data by Koschmieder (1966; 1967) and Chen and Whitehead (1968) show also that the size and the form of the tank in which the convection takes place is of great importance for the form of the convection patterns [cf. also the theory by Davis (1969)]. However, Palm (1960) was probably the first to point out that the usual Boussinesq equations of free convection cannot in principle provide a satisfactory explanation of the basic peculiarities of hexagonal cellular convections. He justified his categorical statement with reference to the experiments of Tippelskirch (1956), which definitely show that the character of the circulation in the cells is determined by the form of dependence of the coefficient of viscosity ν on the temperature T (with $d\nu/dT < 0$, the fluid rises in the center of the cells and sinks at the edges, and with $d\nu/dT > 0$ it rises on the edges and sinks at the center). Hence Palm took as his basis more complicated equations which also take into account the possible dependence of ν on T, (and estimated the effect of this dependence on the value of Ra_{cr}. He assumed further that at the initial instant of time there arises in the fluid some "basic disturbance" in the form of a roll (for example, proportional to cos $(kx_2/H) = \cos k\eta$, where $\eta = x_2/H$, H is the depth of the fluid layer, and independent of $\xi = x_1/H$) on which is then imposed a weak "background" of various other disturbances of small amplitude, with the same (most unstable) value of the wave-number vector k. In this case it is natural to assume that the fundamental role will be played by the "pair interactions" of the basic disturbance with the others. In accordance with this, Palm confined himself to the study of the evolution of disturbances with vertical velocity $u_3(\boldsymbol{x}, t) = u(\xi, \eta, \zeta, t)$ where $\zeta = x_3/H$ of the form

$$u_3(\xi, \eta, \zeta, t) = [A_1(t)\cos k\eta + A_2(t)\cos k_1\xi \cos k_2\eta] f(\zeta),$$
$$k_1^2 + k_2^2 = k^2. \tag{2.50}$$

The disturbances with $k_1^2 + k_2^2 = k^2$ will apparently be especially closely connected with the basic disturbance because quadratic combinations of these disturbances (which enter into the equations of fluid mechanics) may once again generate terms of the same form as the basic disturbance. One may further assume that the interaction of such disturbances with the basic one may, under certain conditions, lead to the mutual amphfication of both, as a result of which it is only these disturbances that will finally play an important role. On the basis of such heuristic considerations, Palm proposed, first, to consider the special case when

$$u_3(\xi, \eta, \zeta, t) = [A_1(t)\cos k\eta + A_2(t)\cos(\sqrt{3}\,k\xi/2)\cos(k\eta/2)] f(\zeta) \tag{2.51}$$

(since here the two terms *are* strongly connected with each other). Restricting the boundary conditions, for simplicity, to the physically unreal "free-free" case of convection between two free surfaces of constant temperature [this simplification was later adopted also by Segel and Stuart (1962), Palm and Øiann (1964), and Segel (1965)] and taking for definiteness, $f(\zeta) = \sin \lambda\zeta$, Palm deduced a system of differential equations for the amphtudes $A_1(t)$ and $A_2(t)$. After dropping terms of order higher than the third in the amplitudes, this system has the form

$$\begin{aligned}\frac{dA_1}{dt} &= \gamma A_1 - \frac{1}{4}\sigma A_2^2 - \delta_1 A_1^3 - (2\delta_2 - \delta_1/2) A_1 A_2^2,\\ \frac{dA_2}{dt} &= \gamma A_2 - \sigma A_1 A_2 - \delta_2 A_2^3 - (4\delta_2 - \delta_1) A_1^2 A_2,\end{aligned} \tag{2.52}$$

where γ, σ, δ_1, δ_2 are constant coefficients, $\gamma \sim \mathrm{Ra} - \mathrm{Ra}_{\mathrm{cr}}$ and $\sigma \sim |d\nu/dT|$ [cf. Segel and Stuart (1962)]. We see that the variation of viscosity with temperature generates second-order terms on the right side of the amplitude equations, whereas only first- and third-order terms are present in these equations for a fluid with temperature-independent properties [cf. Eqs. (2.49)]. It is important that the system (2.52) has simple steady solutions of the form

$$A_2 = \pm 2A_1, \tag{2.53}$$

corresponding exactly to hexagonal prismatic cells. Moreover, Palm showed that when $\sigma \neq 0$ (i.e., when $d\nu/dT \neq 0$, but not when $d\nu/dt = 0$) only solutions of this type will be stable with respect to small disturbances of amplitudes A_1 and A_2, so that precisely these will be realized in the hmit as $t \to \infty$. Since, as one can show, it is to these stable solutions that the simple link between direction of circulation in the cells and the sign of $d\nu/dT$, observed experimentally corresponds, Palm concluded that these results fully explain the fundamental experimental facts.

Later, Palm's theory was critically reconsidered and generalized by Segel and Stuart (1962), Palm and Øiann (1964), Segel (1965), Busse (1967), Palm, Ellingsen and Gjevik (1967), and Davis and Segel (1968). All these authors also took as starting point the third approximation of disturbance theory. It was shown that certain conclusions of Palm (1960) were not completely correct. First, it was found that the solutions (2.53) of Eqs. (2.52) are in fact stable only for values of $\mathrm{Ra} - \mathrm{Ra}_{\mathrm{cr}}$ that are not too large (i.e., Ra must be smaller than some value $\mathrm{Ra}_1 > \mathrm{Ra}_{\mathrm{cr}}$ which depends on a typical "scale" σ of the variation of viscosity and tends to Ra_{cr} as $\sigma \to 0$). Moreover, when $\sigma \neq 0$, then even for a small range of subcritical values of Ra (namely, for $\mathrm{Ra}_0 < \mathrm{Ra} < \mathrm{Ra}_{\mathrm{cr}}$ where $\mathrm{Ra}_{\mathrm{cr}} - \mathrm{Ra}_0$ is of order σ^2), hexagonal steady motions of quite definite finite amplitude will exist, which are stable to infinitesimal disturbances. Therefore the hexagonal convection cells were stable with respect to all infinitesimal disturbances with the same horizontal wave numbers (and apparently also to all other infinitesimal disturbances with no exception; cf. Busse (1967c), and Ponomarenko (1968b) in the whole range $Ra_0 < Ra < Ra_1$ of Ra-values. For $Ra > Ra_1$, the only stable solution of the amplitude equations is that which corresponds to a convection pattern in the form of two-dimensional rolls. Moreover, the rolls are stable to all infinitesimal disturbances not only for $Ra > Ra_1$, but for a wider range $Ra > Ra_2$ where $Ra_{cr} < Ra_2 > Ra_1$. All other forms of convection pattern are certainly unstable; hence for the range $Ra_0 < Ra < Ra_2$, only hexagons are stable for $Ra_2 < Ra < Ra_1$ both hexagons and rolls are stable, and for $Ra > Ra_1$ only rolls are stable. When the Rayleigh number Ra is slowly increased the convection pattern starts growing at Ra_{cr} and takes the form of the stable steady hexagonal cells. At $Ra = Ra_1$ the hexagonal convection pattern becomes unstable and transforms into rolls, which are the only stable form of convection at such high Ra.

With decreasing Rayleigh number the transition from rolls to hexagons occurs at $Ra = Ra_2$ (when rolls become unstable) and the convection decays after $Ra = Ra_0$ has been passed. We see that when the Rayleigh number at first increases slowly and then slowly decreases, a hysteresis effect must be observed. As $\sigma \to 0$ (i.e., the viscosity variations disappear), all the values Ra_0, Ra_1, and Ra_2 tend to Ra_{cr} and the results become identical with those of Schlüter, Lortz, and Busse.

The results presented above on the influence of viscosity variation were obtained by Segel (1965) for a model case of "free-free" boundary conditions and one special form of dependence of viscosity or temperature. Later, Palm, Ellingsen, and Gjevik (1967) considered all possible types of boundary conditions with combinations of rigid and free planes, and calculated all the "critical values" Ra_0, Ra_1, and Ra_2 for these cases. However, the first and most general results were obtained by Busse in his dissertation in 1962 [which was published considerably later; see Busse (1967c)]. Busse used the parametric expansion approach (described above in connection with the work by Schlüter et al.) and took into account the slight variation with temperature not only of the viscosity but of the thermal conductivity, specific heat at constant pressure, and thermal expansion coefficient (for all the combinations of "free" and "rigid" boundary conditions). He found that all the effects considered imply the same stability situation, which was described above for the case when only viscosity depends on temperature. Later, Davis and Segel (1968) showed that even in a fluid with constant properties, the hexagonal cells will appear for Ra sufficiently near to $\mathrm{Ra}_{\mathrm{cr}}$ if the boundary condition at the free top surface allows for its deformation. A very general approach to the problem of establishing hexagonal convection cells was outlined by Ponomarenko (1968b), who did not use a specific form of dynamic equation but emphasized the leading role of second-order terms on the right side of the amplitude equations [one can note that these terms disappear for constant fluid properties and that they make the equation for the amplitude of one isolated disturbance different from the usual Landau equation (2.41)].

It is worth noting that the transition from hexagonal convection cells to rolls when the Rayleigh number is slowly increasing above the critical value was really observed in the experiments of Silveston (1958). However, much more experimental work is clearly needed for complete verification of all the theoretical predictions presently known.

Nonlinear Instability Effects in Plane-parallel Flows and Boundary Layers. Transition to Turbulence

Let us now discuss the nonlinear development of disturbances in plane-parallel flows. We shall first consider plane Poiseuille flow which has the advantage that the linear stability theory was completely successful in this case. However, the nonlinear analysis proves to be much more complicated. The first attempt in this direction was made by Meksyn and Stuart (1951). They found, with a number of simplifying assumptions, that subcritical finite-amplitude instabilities exist in a plane Poiseuille flow, i.e., that the critical Reynolds number for two-dimensional disturbances of finite amplitude $|A|$ is less than $\mathrm{Re}_{\mathrm{cr}}$ of the linear stability theory and decreases as $|A|$ increases. This result compels us to assume that here $\delta < 0$. On the other hand, in the later work of Stuart (1958), the integral equation of energy balance (2.33′) was used to obtain a Landau equation (2.41) for the amplitude of finite unstable disturbance; here, with other simplifying assumptions (the most important being that the spatial form of the disturbance does not change with time and agrees strictly with the form of the eigenfunction of the linear stability equations), it was found that $\delta > 0$. A desire to resolve this controversy stimulated Stuart (1960) and Watson (1960a) to carry out a more complete analysis of the behavior of two-dimensional, wave-like finite disturbances in a plane

Poiseuille flow with Re close to Re_{cr}. The Stuart-Watson analysis is based on the use of the complete system of dynamic equations and expansions of the type (2.43) and (2.44) [this analysis was later taken as a model in Davey's work (1962) which we have discussed earlier]. Here also some approximations are made, but these seem to be more natural and reasonable than those of Meksyn and Stuart (1951) and Stuart (1958). The results of Stuart and Watson were somewhat extended and obtained additional support in the work of Eckhaus (1965) based on expansion of all the functions in terms of the eigenfunctions of the linear Orr-Sommerfeld equation. Another elegant formulation of an expansion method closely related to the initial Stuart-Watson approach (but adapted to the treatment of both two- and three-dimensional, wave-like disturbances) was proposed by Reynolds and Potter (1967).

All the above-mentioned works lead to the justification of an equation of the form (2.41) for the amplitude of the disturbance, and show that the coefficient δ is made up of three terms δ_1, δ_2, and δ_3 [which have the same meaning as the terms in Eq. (2.45)]. Not all these terms were correctly taken into account by Meksyn and Stuart (1951) and by Stuart (1958). In the latter work only the term δ_1 was considered, whereas in the earlier paper the terms δ_1 and δ_3 were estimated approximately and δ_2 was fully disregarded. It was also found that for all three terms, explicit expressions may be given which contain the eigenvalues and eigenfunctions of the corresponding linear Orr-Sommerfeld equation (and also of the adjoint equation) in a complex manner. The numerical calculation of these terms (and of their sum δ) is a very complex problem in numerical analysis which, however, is accessible for modern high-speed computers. The corresponding computations were performed independently by Reynolds and Potter (1967) [who used their own modification of the Stuart-Watson approach] and by Pekeris and Shkoller (1967) [based on the Eckhaus eigenfunction expansion method]. The results of these two papers do not coincide numerically (one reason being that they use different normalizations and somewhat different definitions of the amplitude $|A|$), but both results have the same general behavior and imply close values of the ratios of $\delta = \delta(k,\mathrm{Re})$ at different points of the (k,Re)-plane.

Reynolds and Potter computed the values δ_1, δ_2, δ_3, and $\delta = \delta_1 + \delta_2 + \delta_3$ at the critical point (k_{cr}, Re_{cr}), at four other points of the neutral stability curve on the (k, Re)-plane, and at two points in the neighborhood of the neutral curve. They found that δ_1 is positive at all points [in complete agreement with the previous results of Stuart (1958)] and that at the critical point and the points on the upper branch of the neutral curve δ_3 is negative and much greater in absolute value than δ_1 [so that $\delta_1 + \delta_3$ is negative; this result agrees with the approximate deductions of Meksyn and Stuart (1951)]. The value δ_2 is also positive and of the same order as δ_1; therefore the sum $\delta = \delta_1 + \delta_2 + \delta_3$ is negative at the critical point and points of the upper branch of the neutral curve (and is determined here primarily by the process of distortion of the vertical profile of the basic disturbance). Of course, the most important is the result obtained at the critical point: it indicates that finite disturbances in plane Poiseuille flow lose stability earlier than infinitesimal ones (i.e., subcritical finite-amplitude instabilities exist) and that supercritical finite-amplitude equilibrium states of the type known for a circular Couette flow and a layer of fluid heated from below are rather unlikely to be observed in plane Poiseuille flow (in full agreement with the experimental evidence). However, the Reynolds and Potter computations at the points of the lower branch of the neutral stability curve reveal a surprising result (confirmed also by the data of Pekeris and Shkoller) that δ is positive (and comparatively small in absolute value) at these points. Hence, periodic motions of finite amplitude could theoretically exist in the case of plane Poiseuille flow if the disturbance could be kept very "pure," i.e., higher wave-number contributions could be suppressed (apparently this would be very difficult to achieve experimentally).

Pekeris and Shkoller evaluated the coefficient $\delta = \delta(k,\mathrm{Re})$ for an extensive region of the (k,Re)-plane (using equations which are reasonable in the vicinity of the neutral curve) and obtained results that, in general, agreed with Reynolds and Potter's conclusions; their main result is presented in Fig. 23d.

We have already pointed out that Landau's equation (2.41), for $\delta > 0$, leads to physically interesting results only for disturbances in flows with $\mathrm{Re} > \mathrm{Re}_{cr}$, and in the case $\delta > 0$ only for finite disturbances in flows with $\mathrm{Re} < \mathrm{Re}_{cr}$. For $\delta > 0$ and $\mathrm{Re} < \mathrm{Re}_{cr}$ this equation is less interesting and for $\sigma < 0$, and $\mathrm{Re} > \mathrm{Re}_{cr}$ it quickly becomes inapplicable; therefore in both cases it is reasonable to supplement this equation by subsequent terms of the expansion in powers of $|A|^2$. The general equation for $d|A|^2/dt$, taking into account all terms of such an expansion, will evidently be of the form

$$\frac{d|A|^2}{dt} = |A|^2 \sum_{m=0}^{\infty} a_m |A|^{2m} \tag{2.54}$$

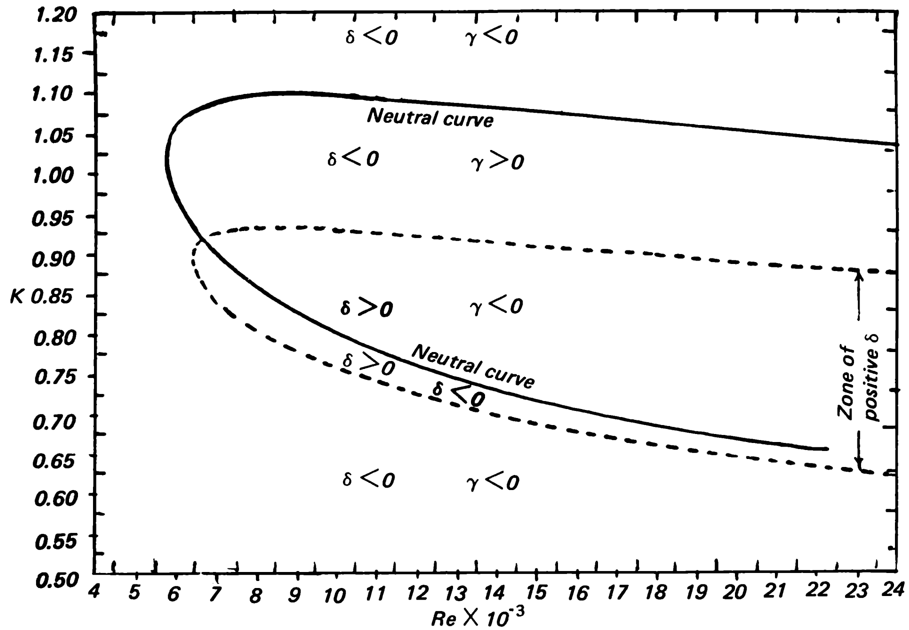


FIG. 23d. The regions of positive γ and of positive δ on the (k, Re)-plane for the case of plane Poiseuille flow [After Pekeris and Shkoller (1967)].

(from which Landau's equation is obtained, if we take into account only the first two terms of the right side). An equation of the form (2.54) for the amplitude of a two-dimensional disturbance in a plane-parallel flow was obtained by Watson (1960a) with the aid of a special expansion technique, proceeding from the equations of motion. Here, however, the expressions for the coefficients a_m with $m > 1$ are far more complex than for the coefficient $a_1 = -\delta$, so that their evaluation still seems to be almost impossible. In the subsequent work of Watson (1962) a similar analysis was carried out for plane Poiseuille flow, based on the representation of the two-dimensional disturbances $\boldsymbol{u}'(\boldsymbol{x}, t)$ in the form $A(x)\boldsymbol{f}(z, t)$ where the amplitude $A(x)$ in the linear approximation is equal to $e^{(k_1+ik_2)x}$. Here, equations are obtained for $A(x)$ which differ from Eqs. (2.41) and (2.54) only in the replacement of t by x. The evaluation of the coefficients of these equations naturally rests on the same difficulties as the evaluation of the coefficients in the equations for the time-dependent amplitude $A(t)$.

Similar results may be obtained also for many other types of plane-parallel flows. As an example, Reynolds and Potter (1967) investigated the case of the combined plane Couette-Poiseuille flow [i.e., the flow studied within the framework of linear stability theory by Potter (1966), and Hains (1967)] for different relative intensities of the Couette component (in the range of intensities for which unstable infinitesimal disturbances exist). For all such flows, the critical point on the (k,Re)-plane was found through application of linear stability theory and nonlinear analysis was carried out at the critical point only. Calculations of Reynolds and Potter show that δ is negative at the critical point for all flows considered and is determined primarily by the summand δ_3. The dependence of $|\delta|$ on intensity of the Couette component of the flow is rather complex (with two local minima); however, $\delta \to -\infty$ as $\mathrm{Re}_{cr} \to \infty$ and this result makes it probable that finite disturbances are unstable even if all infinitesimal disturbances are stable (i.e., if $\mathrm{Rc}_{cr} = \infty$).

Unfortunately, the Stuart-Watson approach based on a specific expansion in the vicinity of the neutral stability curve is inapplicable to fluid flows with $\mathrm{Re}_{cr} = \infty$ (for example, to plane Couette flow which is apparently unstable to disturbances of finite amplitude according to experimental evidence). Therefore in the paper of Kuwabara (1967) on nonlinear instability of plane Couette flow, the approximate method of Meksyn and Stuart (1951) was used in combination with the mathematical method of Galerkin. As a result the "neutral stability surface" in three-dimensional (k,Re,A)-space was obtained, where A is the amplitude of the disturbance (measured in terms of its mean energy). According to Kuwabara's result plane Couette flow is stable for any (infinitesimal or finite) disturbance if $\mathrm{Re} < \mathrm{Re}_{cr\ min} \approx 45{,}000$. Above this critical value at every fixed Re, instability appears only in a rather small region of (k, A)-plane. The crude method used is evidently insufficient for obtaining quantitatively exact results; nevertheless, one can hope that Kuwabara's results are qualitatively correct.

For free plane-parallel flows without rigid boundaries and with a velocity profile having an inflection point (some examples of such profiles are shown in Fig. 19) linear instability occurs even with $\nu = 0$. Therefore it seemed natural to expect that the nonlinear development of disturbances in such flows could be explained within the framework of the mechanics of an ideal (nonviscous) fluid. The first attempt in this direction was made by Schade (1964), who considered the problem of the determination of the value of

Landau's coefficient δ for a flow of ideal fluid in the whole space with a velocity profile of the form $U(z) = U_0 \tanh(z/H)$. Schade introduced some simplifying assumptions and using them determined analytically the value of δ for this flow [see also the discussion of Schade's work by Michalke (1965a), and Stuart (1967)]. Since the value of δ so obtained turns out to be positive, the unstable disturbances in a free shear layer with a hyperbolic-tangent velocity profile must tend to a finite periodic equilibrium state as $t \to \infty$ according to Schade's theory. This conclusion and some related results on the qualitative features of the equilibrium state obtained similarly by Michalke (1965a) and Stuart (1967) agree satisfactorily with the findings of Sato (1956; 1960), Freymuth (1966), Browand (1966) and other experimenters for a region relatively close to the origin of the shear layer (which corresponds to not too great values of t). The analytical result of Schade is nevertheless quite doubtful since some of his assumptions are apparently incorrect [see the discussion of this question in the survey lecture of Michalke (1968)]. It is also worth noting that Schade's assumption that the basic flow will not be modified by the disturbance contradicts the exact solution of the nonlinear equations of the neutral stability problem for a free shear layer with hyperbolic-tangent velocity profile found by Stuart (1967). Schade's result disagrees also with that of Gotoh (1968) who considered the same problem for a viscous fluid flow and found an expression for the constant δ as a function of the Reynolds number which does not tend to a finite limit as $\mathrm{Re} \to \infty$ (i.e., in the ideal fluid approximation). However, Michalke has noted (in a letter to one of the authors of this book) that some of the objections to Schade's work are valid relative to Gotoh's work too; therefore the question of the exact value of the Landau constant for a shear layer with a hyperbolic-tangent velocity profile (either in the inviscid case, or in the viscid one) cannot be considered as solved.

Even more important, however, is the fact that all the data on the evolution of disturbances in a free shear layer at greater downstream distances show peculiar behavior which it is impossible to explain with the aid of Landau's equation for the amplitude of a disturbance for any value of the coefficient δ. In fact, although near the origin of the layer only oscillations with the most unstable frequency are significant, further downstream some other harmonic components also appear, and still further, the frequency spectrum becomes continuous and the free shear layer becomes turbulent. Moreover, when additional harmonic components appear in the spectrum a remarkable effect is observed. At first, only the higher harmonics of the most unstable oscillation (henceforth called the fundamental oscillation), with two and three times the frequency of the fundamental, are observed. However, somewhat further downstream, a subharmonic component with half the frequency of the fundamental also appears, and shortly after its appearance, it becomes the most significant harmonic component of the flow (see, for example, the excellent data of Sato (1959), and Browand (1966), obtained in the presence of artificially produced fundamental oscillation). The simplest nonlinear mechanism of self-interaction of the fundamental with itself (through quadratic terms of the dynamic equations) would explain the existence of higher harmonics (taken into account in the Stuart-Watson expansion procedure); however, the origin of subharmonic oscillation could not be explained in that way.

Theoretically it is possible that the subharmonic is related to the slow growth of shear-layer thickness with distance downstream; however, such an explanation does not seem probable. It is more natural to think that the great difference between the behavior of the real oscillations in a free shear layer and the predictions of Landau's theory could be explained without drawing on the small deviations of the shear-layer flow from strict plane-parallelism. The transition to turbulence of the shear layer at great downstream distances leads one to suppose that the finite-amplitude equilibrium state is unstable here to small disturbances. Michalke and Timme (1967) attempted to verify this supposition using a special vortex model of the disturbed shear-layer flow. They studied the inviscid instability of an isolated cylindrical (i.e., two-dimensional) vortex and found that a single vortex of the type which occurs in a shear layer can be unstable with respect to cylindrical disturbances. This fact explains the breakdown of vortices forming the equilibrium state of a disturbed shear layer and the transition to turbulence. However, the Michalke-Timme model is a qualitative one and their approach is rather crude. A more realistic theory must deal with exact solutions of the nonlinear equations of fluid dynamics representing disturbed shear-layer flow, and study the stability of these solutions. An equivalent formulation consists of a consideration of the evolution of several different wave-like disturbances in a free shear layer, taking into account their mutual interactions, i.e., the effect of each of them on the development of the rest (the existence of such interactions is obvious from the nonlinearity of the dynamic equations).

As already mentioned in connection with other types of fluid flows, the question of the interaction of finite disturbances is one of the least explored in fluid mechanics. Kelly (1967) obtained the first results which touch upon this interaction for a free shear layer. He investigated the stability of inviscid plane-parallel shear flow, consisting of a nonzero mean component, together with a component periodic in the direction of flow and with time (such a flow describes the equilibrium state of a single disturbance in a shear layer). It was found that the interaction of the periodic component of the flow with a disturbance with twice the wavelength and frequency of the basic periodic component can produce a wave which is of the same wave number and frequency as the disturbance and which can therefore reinforce it. As a result, the subharmonic component with half the frequency of the basic periodic component begins to grow in full agreement (which is even quantitative in some details) with the experimental data of Sato (1959) and Browand (1966). Later, Kelly (1968) also considered some model examples of interaction of neutrally stable disturbances in two specific shear flows. He showed that resonant interaction is possible and increases the rate at which energy can be transferred from the mean flow to each disturbance, leading to simultaneous amplification of both disturbances.

The question of interaction of disturbances seems to be important also for the analysis of nonlinear processes in a boundary layer on a flat plate. The hnear analysis for this case was considered in detail in Sect. 2.8. In particular, we have already pointed out that the experiments of Schubauer and Skramstad show the initial development of unstable disturbances in such a flow to be in complete agreement with the deductions of the linear Tollmien-Schiichting-Lin theory. However, subsequent very interesting experimental observations of several research groups [see, for example, Schubauer and Klebanoff (1956); Hama, Long, and Hegarty (1957); Klebanoff and Tidstrom (1959); Klebanoff, Tidstrom, and Sargent (1962); Kovásznay, Komoda, and Vasudeva (1962); Tani and Komoda (1962); Hama and Nutant (1963); and the review articles by Kovásznay (1965); Stuart (1965); and Tani (1967)] showed conclusively that this is the case only in the first stage of development of the disturbances. After this first stage, the situation changes considerably and a sequence of events occurs in a definite order finishing with transition to developed turbulence. The most important in this sequence of events are the following: 1) the appearance of distinctly three-dimensional disturbances; 2) the nonlinear amplification of three-dimensional waves and the appearance of a streamwise vortex system; 3) the development of a high-shear layer and the generation of high-frequency fluctuations; and 4) the development of random "turbulent spots," the growth and repeated coalescing of which lead to transition to turbulence of the whole boundary layer.

The generation of weak three-dimensional disturbances which vary with lateral ("spanwise") position can be explained by the influence of small spanwise irregularities in the free-stream or in the upstream boundary layer. For supercritical Reynolds numbers, such three-dimensional waves may be unstable, and for sufficiently large $\mathrm{Re}-\mathrm{Re}_{cr}$, three-dimensional waves may even become the most rapidly increasing (i.e., the most unstable). The growth of unstable three-dimensional waves will clearly lead an initially two-dimensional disturbance into a three-dimensional form. The development of three-dimensionality in the boundary-layer flow is observed best in "controlled" experiments with a special vibrating-ribbon technique producing waves that are made spanwise-periodic artificially at the ribbon position [see, e.g., Klebanoff and Tidstrom (1959); Klebanoff, Tidstrom and Sargent (1962); Kovásznay, Komoda and Vasudeva (1962); Tani and Komoda (1962); Komoda (1967)]. These experiments show that a three-dimensional wave leads to development of a whole system of longitudinal vortices with axes directed along the mean stream (these streamwise vortices are of course very different from the two-dimensional spanwise vortices observed when a free shear layer begins to roll up without losing its two-dimensional structure). The development of a streamwise-vortex system leads to a flow with a very distinct three-dimensional structure and a sharp redistribution of the intensity of fluctuation in the spanwise direction $Ox_2 = Oy$ [see Fig. 24 taken from the paper by Klebanoff, Tidstrom, and Sargent (1962)]. The time evolution of disturbances at this stage is nonlinear and its theoretical analysis is impossible without taking account of the interactions between the different waves present.

For this purpose Stuart (1962) considered the behavior in a plane-parallel flow of a disturbance which is composed of a two-dimensional and a three-dimensional wave of the same streamwise wave number:

$$u(x, y, z, t) = A_1(t)\cos kx\, f_1(z) + A_2(t)\cos kx \cos ly\, f_2(z) + \\ + \text{small complement.} \tag{2.55}$$

Using the expansion technique given in Stuart (1960) and Watson (1960a), Stuart obtained for the amplitudes A_1 and A_2 a system of

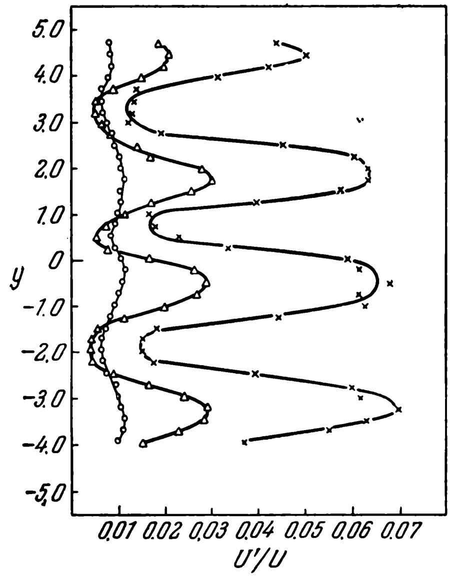


FIG. 24. Dependence of the relative magnitude of a typical fluctuation of longitudinal velocity U' on the "lateral" coordinate y in a boundary layer on a flat plate at different distances x from the oscillating metal strip which creates the disturbance: O–for $x = 7.6$ cm; Δ–for $x = 15.2$ cm; X–for $x = 19$ cm.

equations of the form (2.49). Thus in this case there will also exist steady solutions of the four types (I)–(IV) and the most important question will be the stability of these steady solutions to some disturbance. Unfortunately, the quantitative investigation of stability, even with respect to only a specific disturbance of the form (2.55), requires the determination of the values of the coefficients of the corresponding system of equations (2.49), which is extremely difficult (we have pointed out earlier in this section, that the evaluation of the coefficient δ_1, which is equal to the coefficient 2δ in the corresponding Landau equation (2.41) for a single amplitude A_1, was only recently performed for one example of plane-parallel flow, namely, for a plane Poiseuille flow). However, the data of Fig. 24 create the impression that for certain k and l a mixed steady solution of type (IV) may prove to be stable, the stability being not only with respect to disturbances of the form (2.55), but also with respect to many other ordinary types of disturbances.

A similar conclusion may also be reached on the basis of the results of Benney (1961; 1964) and Lin and Benney (1962) who approached the same problem along different lines [this approach is also presented in Betchov and Criminale (1967) Chapt VIII]. These authors proposed a solution of the equations of fluid mechanics in the form of a series of disturbances of increasing order

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}^{(0)}(\boldsymbol{x}) + a\boldsymbol{u}^{(1)}(\boldsymbol{x}, t) + a^2\boldsymbol{u}^{(2)}(\boldsymbol{x}, t) + \ldots, \tag{2.56}$$

where $a \ll 1$ is a dimensionless coefficient, defining the ratio of the amplitude of the disturbance to the amplitude of the undisturbed flow $\boldsymbol{u}^{(0)}(\boldsymbol{x})$. In actual calculations, in Benney (1961) and Lin and Benney (1962), the example chosen as $\boldsymbol{u}^{(0)}(\boldsymbol{x})$ is that of plane-parallel flow in an unbounded space, with velocity profile $U_0(z) = U_0 \tanh z$, shown in Fig. 19d, while in Benney (1964) it is a flow in the half-space $z \geqslant 0$ with profile $U_0(z)$ linearly increasing to some value $z = H$, and then assuming a constant value $U_0(H)$. The primary disturbance $\boldsymbol{u}^{(1)}(\boldsymbol{x}, t)$ in both cases is taken to be of the form

$$\boldsymbol{u}^{(1)}(\boldsymbol{x}, t) = [\mu \boldsymbol{U}_1(z) + \lambda \boldsymbol{u}_1(z) e^{ily}] e^{ik(x-ct)}. \tag{2.57}$$

Here $\boldsymbol{U}_1(z) = [U_1(z), V_1(z)]$, k and c are found from the ordinary ("two-dimensional") linear stability theory (and correspond to a slightly unstable disturbance at Re somewhat greater than Re_{cr}), $\boldsymbol{u}_1(z) = [u_1(z), v_1(z), w_1(z)]$ is determined from the linear theory for small three-dimensional disturbances, and the ratio μ/λ describes the relative role of two- and three-dimensional disturbances. The calculations are carried out accurately to second-order disturbances (of the order of a^2); they show that even with $\mu/\lambda \gg 1$ the interaction of two- and three-dimensional disturbances leads to the formation of secondary longitudinal (streamwise) vortices and to considerable redistribution of the energy of the disturbances in the Oy direction. As a result, the overall motion [described by the three terms on the right side of Eq. (2.56)] is very close to that which is actually observed in a boundary layer [which has a completely different velocity profile $U_0(z)$].

The elucidation of the important role of three-dimensional disturbances in the process of transition to turbulence in a boundary layer, prompted Meksyn (1964) to return once again to the problem of the stability of a plane Poiseuille flow with respect to finite disturbances. Adopting assumptions close to those used earlier in the work of Meksyn and Stuart (1951), Meksyn carried out a similar analysis with reference now to finite three-dimensional disturbances (with velocity field of the form $\boldsymbol{u}(\boldsymbol{x}, t) = \sum_{n_1, n_2} e^{i[n_1(kx-\omega t)+n_2 ly]}\boldsymbol{u}_{n_1, n_2}(z)$, where n_1 and n_2 are arbitrary integers). After some very cumbersome approximate calculations, he found that for certain finite three-dimensional disturbances, the critical Reynolds number $\mathrm{Re}_{\mathrm{cr}}$ may certainly be fairly close to 1000, the value given experimentally for $\mathrm{Re}_{\mathrm{cr\,min}}$ for this flow.

The theoretical investigation of the next two stages of transition to turbulence in a boundary layer, leading to the appearance of "random" turbulent spots and subsequent transition to developed turbulence, is very difficult and has not, at present, proceeded very far. It is possible that the appearance of such a "spot" is connected with many elementary disturbances becoming unstable at once, and as a result, a complex "mixed" regime is set up, possessing a large number of degrees of freedom. It is clear, however, that in this case the development of an instantaneous velocity profile with a region of large shear (and associated inflection point) plays an important part. Such high-shear layers with an inflection point were predicted by Betchov (1960) as a result of a superposition of a secondary streamwise vortex flow on the primary two-dimensional flow; they were observed by Kovásznay, Komoda, and Vasudeva (1962) and by many subsequent investigators [see, e.g., Tani (1967), and Komoda (1967)]. A high-shear layer in a flow field will usually be unsteady; its thickness and the corresponding drop in the mean velocity may vary rapidly with time, leading to the development, immediately in front of the spots, of a region of flow characterized by sharply increased instability. The fairly simple calculations carried out by Greenspan and Benney (1963) [within the framework of linear disturbance theory] for a model time-dependent shear layer with velocity profile of the type given in Fig. 19c, but with changing velocity-difference $2U_0 = 2U_0(t)$ and thickness $2H = 2H(t)$, showed that the development of such a layer may lead to violent generation of small-scale (i.e., high-frequency) fluctuations, the energy of which attains high values in a very short time. These results are in good qualitative agreement with the existing data relating to breakdown (i.e., the instantaneous generation of intense high-frequency fluctuations) and transition to turbulence of a boundary layer flow [see, in particular, Miller and Fejer (1964)].

The vortices disintegrated from the high-shear layer [the "hairpin eddies" in the terminology of Klebanoff et al. (1962)] are the embryo turbulent spots. When traveling downstream (at a speed greater than the wave speed of the primary unstable disturbance) they break down into smaller vortices, which again break down into even smaller vortices, etc. (the observation of this cascade process of wave breakdown was made by Hama, Long and Hegarty (1957), with the aid of water-flow visualization by hydrogen bubbles). During this stage the distribution loses its regular "wavelike" form and transfroms into a complex spot-like structure. The final stage is the growth of localized spots and their joining up in a developed turbulent boundary layer, characterized by the wholly irregular ("random") nature of the fluctuations of all fluid mechanical quantities in all points and all moments of time. The investigation of such developed turbulence will be the subject of all remaining chapters of this book. However, we shall require a number of results from the mathematical theory of probability and the theory of random functions, and we now turn to a short exposition of these ideas.

2 MATHEMATICAL DESCRIPTION OF TURBULENCE. MEAN VALUES AND CORRELATION FUNCTIONS

3. METHODS OF TAKING AVERAGES. FIELDS OF FLUID DYNAMIC VARIABLES AS RANDOM FIELDS

3.1 Practical Methods of Averaging and Reynolds Conditions

We have already mentioned that the characteristic feature of turbulent motion of a liquid or gas is the presence of disordered fluctuations of the fluid dynamic variables of the flow. As a result, both the spatial and temporal dependence of the instantaneous values of the fluid dynamic fields have a very complex and confused nature. Moreover, if turbulent flow is set up repeatedly under the same conditions, the exact values of these fields will be different each time. Let us return once again to Fig. 1 which shows the dependence of certain fluid dynamic quantities in a turbulent flow on time. We see that all these curves consist of a set of fluctuations of diverse periods and amplitudes, superimposed upon each other without any obvious regularity. The distributions of instantaneous values of the fluid dynamic variables in space have a similar nature; they constitute a disordered set of three-dimensional fluctuations of diverse amplitude, wavelength and orientation. Due to this extreme disorder and the sharp variation in time and space of the fields of all the fluid dynamic quantities, in the study of turbulence it is necessary to use some method of averaging which will enable us to pass from the initial fluid dynamic fields to smoother, more regular *mean values* of the flow variables. These variables may then be investigated by means of the usual methods of mathematical analysis.

The question of the definition of mean values is a delicate one in the theory of turbulence, and has a long history. In practice, to determine the mean values we most generally use time- and space-averaging over some interval of time or region of space. We may also consider a more general space-time averaging of the functions $f(x_1, x_2, x_3, t) = f(\boldsymbol{x}, t)$, given by the equation

$$\overline{f(x_1, x_2, x_3, t)} = \\ = \int\int\int\int_{-\infty}^{\infty} f(x_1-\xi_1, x_2-\xi_2, x_3-\xi_3, t-\tau)\,\omega(\xi_1, \xi_2, \xi_3, \tau)\,d\xi_1\,d\xi_2\,d\xi_3\,d\tau. \tag{3.1}$$

Here the overbar indicates averaging and $\omega(\boldsymbol{\xi}, \tau)$ is some weighting function (usually nonnegative) which satisfies the normalization condition

$$\int\int\int\int_{-\infty}^{\infty} \omega(\xi_1, \xi_2, \xi_3, \tau)\,d\xi_1\,d\xi_2\,d\xi_3\,d\tau = 1. \tag{3.2}$$

If the function ω is equal to zero outside some four-dimensional region and takes a constant value within it, then Eq. (3.1) is simple averaging over a given region of space-time. Putting $\omega(\boldsymbol{\xi}, \tau) = \omega(\boldsymbol{\xi})\,\delta(\tau)$ or $\omega(\boldsymbol{\xi}, \tau) = \omega(\tau)\,\delta(\boldsymbol{\xi})$, where δ is Dirac's delta-function, and $\omega(\boldsymbol{\xi})$ and $\omega(\tau)$ are functions which possess a constant value on some parallelepiped or segment and are equal to zero outside it, we obtain space- or time-averaging, respectively. However, it is clear that the mean value defined by Eq. (3.1), generally speaking, will depend on the form of the weighting function ω (in particular, when averaging over some time interval or region of space, it will depend on the length of the interval or the form and volume of the region). Thus Eq. (3.1) gives rise to many different "mean values," and it is necessary to discover which of these is the "best."

In choosing some particular "averaging rule," one must first formulate the general requirements which such a rule should have. From the viewpoint of the theory of turbulence, the most important of these general requirements is, of course, that the application of this rule to the differential equations of fluid dynamics will allow us to obtain sufficiently simple equations for the mean values of the fluid dynamic variables. Reynolds, who founded the theory of turbulence, was well aware of this fact; he used only the simplest form of averaging over some time interval, but at the same time he indicated the natural general conditions which any averaging applicable in fluid dynamics must satisfy. However, although not all the necessary general conditions were formulated correctly by Reynolds (1894), by slightly refining his postulates, it is easy to arrive at the conclusion that the following five relationships must be satisfied:

$$\overline{f+g}=\bar{f}+\bar{g}, \tag{3.3}$$

$$\overline{af}=a\bar{f}, \quad \text{if} \quad a=\text{const}, \tag{3.4}$$

$$\bar{a}=a, \quad \text{if} \quad a=\text{const}, \tag{3.5}$$

$$\overline{\frac{\partial f}{\partial s}}=\frac{\partial \bar{f}}{\partial s}, \quad \text{where } s \text{ is } x_1,\ x_2,\ x_3 \text{ or } t, \tag{3.6}$$

$$\overline{\bar{f}g}=\bar{f}\bar{g}. \tag{3.7}$$

At present, conditions (3.3)–(3.7) are generally known as *the Reynolds conditions.*

Condition (3.6) may also be replaced by a more general condition of commutivity of the operations of averaging and going to the limit:

$$\overline{\lim_{n\to\infty} f_n}=\lim_{n\to\infty}\bar{f}_n. \tag{3.6$'$}$$

Substituting into Eq. (3.7) successively $g=1$, $g=\bar{h}$, and $g=h'=h-\bar{h}$ (we shall always use the prime to indicate the fluctuation of the corresponding quantity, i.e., deviation from its average value) and using also Eqs. (3.5) and (3.3) we obtain the following important consequences from the Reynolds conditions:

$$\bar{\bar{f}}=\bar{f}, \quad \bar{f'}=\overline{f-\bar{f}}=0, \quad \overline{\bar{f}\bar{h}}=\bar{f}\bar{h}, \quad \overline{\bar{f}h'}=\bar{f}\bar{h'}=0. \tag{3.7$'$}$$

It is clear that conditions (3.3), (3.4), (3.5), and (3.6) [or (3.6′)] will be fulfilled for any averaging (3.1) with arbitrary weighting function ω, satisfying Eq. (3.2). The situation is different with the more complex condition (3.7). Thus, for example, if we use time- or space-averaging over some interval, then, strictly speaking, we can show that this condition will not be satisfied exactly for any choice of interval. However, it is not difficult to argue in favor of the fact that the averaging interval may be chosen in such a way that this condition will be satisfied approximately with comparatively high accuracy. For this it is necessary only that the averaging interval be large in comparison with the characteristic periods of the fluctuating quantity $f' = f - \bar{f}$, but small in comparison with the periods of the averaged quantity $\bar{f}$ [see, for example, Kochin, Kibel', Roze (1964), Vol. 2, Chapt. III, Part C]. Reynolds confined himself to this type of argument; however, at present these qualitative considerations can hardly be called convincing.

The possibility of choosing the averaging interval to be intermediate between the periods of the fluctuating and mean fields assumes that turbulent motion may be resolved into a comparatively smooth and slowly varying "mean motion" with a very irregular "fluctuating motion" superimposed on it, while between the frequency range characteristic of the one motion and that of the other, there is a considerable gap. In other words, it is assumed here that the Fourier transform (with respect to time or the coordinates) of the function f is different from zero in some region close to zero and in some high-frequency region (or region of high wave numbers), and is equal to zero in the intervening gap between these regions. This picture corresponds more or less to reality in the case of a number of artificial turbulent flows set up in the laboratory. However, in the case, for example, of natural turbulent motion in the earth's atmosphere and in the sea, it is by no means always apphcable, since atmospheric and marine turbulence often possess a wide continuous spectrum.

The most logical deduction from the idea that the mean value $\bar{f}$ and the fluctuation f' of the function f differ principally in their characteristic periods (or wavelengths) consists of defining the mean value $\bar{f}$ as that part of the representation of the function f (as a Fourier integral) which corresponds to integration over the range of values of the corresponding variable (frequency or wave number) which are less in absolute value than some fixed number p_0. It is easy to see that here the conditions (3.3), (3.4), (3.5), and (3.6) will be satisfied, since this averaging is a special case of the averaging defined by Eq. (3.1). Similarly, the first two conditions of Eq. (3.7′) will be satisfied. However, the condition (3.7) will not, generally speaking, hold; for this to be satisfied, it is necessary to impose on the functions f and g some very special conditions that are incompatible with the assumption that their Fourier transform is everywhere different from zero [on this point, see the detailed investigation of Izakson (1929), and the note of Kampé de Fériet (1951)].

We shall mention further that Birkhoff, Kampé de Fériet, Rotta and several others have published a number of papers [references to which may be found, for example, in Kampé de Fériet's survey article (1956) and in Rotta's article (1960)], which are devoted to the investigation of general "averaging operations." It was assumed in these works that the averaging operation under discussion satisfies the Reynolds conditions (3.3)–(3.7) [or some related conditions of the same type] exactly, and is defined over a specific subset of functional space (i.e., over sets of functions which satisfy some special conditions). In certain cases the results obtained may give a complete description (in abstract algebraical terms) of all such operations. However, all these investigations are of a formal mathematical nature and their results do not find direct application in the theory of turbulence. Furthermore, they are not even necessary, since in present-day turbulence theory the question of the meaning of averaging is resolved in a completely different manner, and, moreover, in such a way that all the Reynolds conditions are evidently satisfied (this, however, raises the new question of satisfying the ergodicity conditions which will be discussed below).

3.2 Random Fields of Fluid Dynamic Variables and Probability Averaging

The use of time-, space- or space-time averaging, defined by some equations of the form (3.1), is very convenient from the practical viewpoint, but leads to a great many unavoidable analytical difficulties in theoretical calculations. Moreover, this type of averaging has the great disadvantage that the question of the form of the function $\omega(\xi, \tau)$ most suitable for the given problem must be resolved each time before use. For all these reasons it is desirable in the theory of turbulence to avoid the use of this type of averaging altogether, and to adopt instead some other method of defining the mean value, a method that has simpler properties and is more universal. A convenient definition of this type, which we shall use throughout this book, is found in the probability-theory treatment of the fields of fluid dynamic variables in a turbulent flow as random fields.

The basic feature of the probability-theory approach (or, more commonly, the statistical approach) to the theory of turbulence is the transition from the consideration of a single turbulent flow to the consideration of *the statistical ensemble of all similar flows,* created by some set of fixed external conditions. To understand the implication of this, let us consider as an example a particular class of flows past a right circular cylinder arising in a wind-tunnel. The fundamental difference between laminar and turbulent flow past such a cylinder is as follows: for laminar flow, if we place two identical cylinders in two identical wind-tunnels in similar positions (or, what is essentially the same thing, repeat the experiment twice with the same cylinder and the same wind-tunnel) then, at a given time t after switching on the motor, and at a given point $\boldsymbol{x}$ of the working section of the tunnel we shall obtain the same value of $u_1(\boldsymbol{x}, t)$ [the component of velocity along the axis Ox_1] and likewise identical values of the other fluid dynamic variables of the flow (which, in principle, may in every case be found with the aid of the solution of some problem with boundary and initial conditions for the Navier-Stokes equations). However, for turbulent flow, the effect of small uncontrollable disturbances in the flow and in the initial conditions leads to a situation in which, when an experiment is performed a second time under practically the same conditions we shall obtain two different values of $u_1(\boldsymbol{x}, t)$ and the other fluid dynamic variables. In this case one may therefore introduce the concept of the "ensemble of all values of $u_1(\boldsymbol{x}, t)$ obtained in all possible experiments on turbulent flow past a cylinder under given external conditions" and the value of $u_1(\boldsymbol{x}, i)$ obtained in any actual experiment is then considered as one "realization" chosen at random from this ensemble.[1]

If we now fix the external conditions and repeat the experiment many times under these conditions, recording each time the value obtained for $u_1(\boldsymbol{x}, t)$, then the arithmetic mean of all these values will, in practice, be fairly stable. In other words, if initially we have a sufficiently large number of experiments, then when this number is increased still further, the mean value will usually vary very little, oscillating about some constant value (this stability of the mean value indicates that our collection of similar experiments does, in fact, constitute a statistical ensemble). In this case the value about which the arithmetical mean of $u_1(\boldsymbol{x}, t)$ oscillates is called *the probability mean* of the velocity $u_1(\boldsymbol{x}, t)$ and is denoted by the symbol $\overline{u_1(\boldsymbol{x}, t)}$. (Henceforth, we shall always use an overbar to denote only the probability mean.)

Similarly, the mean values of all the other fluid dynamic variables taken over the whole ensemble of similar experiments, are found to be stable and for a sufficiently large number of experiments, generally deviate only slightly from some constant value. Of especial interest for our purposes is the indicator function $\chi_{u_1(\boldsymbol{x}, t)}(u', u'')$, $u'' > u'$, which is equal to zero if the value of $u_1(\boldsymbol{x}, t)$ is greater than u'' or less than u', and equal to unity otherwise. The number $p(u', u'')$ about which the arithmetic mean of this indicator function oscillates

[1] We must note here that in the case of natural turbulence (e.g., atmospheric turbulence), the choice of the set of similar experiments will present greater difficulty, because the "external conditions" (in this case, primarily, the meteorological conditions) cannot be repeated whenever we wish. However, in this case also, we are usually able to select a number of situations in which all the conditions which are essential for determining a given measured value (e.g., for the measurement of the wind velocity at a height of 2 m, there exist the mean wind velocity, the wind and the temperature gradients and wind direction) will be practically identical. Moreover, the ensemble of these situations will also form a statistical ensemble of "similar experiments" which will be similar to the ensemble of all possible flows past equivalent cylinders in identical wind-tunnels.

is evidently equal to the frequency of occurrence of experiments in which the value of $u_1(\boldsymbol{x}, t)$ satisfies the inequality $u' \leqslant u_1(x, t) \leqslant u''$. This number $p(u', u'')$ is called *the probability* that $u_1(\boldsymbol{x}, t)$ will take a value in the range between u' and u''. Usually this number $p(u', u'')$ may be represented as an integral from u' to u'' of some nonnegative function $p(u)$ called *the probability density function* (or, briefly, the *probability density*) of $u_1(\boldsymbol{x}, t)$. Therefore, the set of all u for which $p(u) \neq 0$, will give the "set of possible values of $u_1(\boldsymbol{x}, t)$" of which we have already spoken; we shall call the actual value $u_1(\boldsymbol{x}, t)$ observed in one of the experiments *a sample value* (or *a realization*) of the x_1 component of velocity. The fact of the existence of the density $p(u)$ is sometimes expressed in the following form:

$$P\{u < u_1(\boldsymbol{x}, t) < u + du\} = p(u)\,du,$$

where the symbol $P\{\ldots\}$ denotes the probability of the condition specified in the braces being satisfied. The probability mean $\overline{u_1(\boldsymbol{x}, t)}$ of $u_1(\boldsymbol{x}, t)$, therefore, may obviously be expressed in terms of $p(u)$ with the aid of the equation

$$\overline{u_1(\boldsymbol{x}, t)} = \int_{-\infty}^{\infty} up(u)\,du. \tag{3.8}$$

At the same time, knowledge of the probability density $p(u)$ enables us to determine also the probability mean of arbitrary functions of $u_1(\boldsymbol{x}, t)$:

$$\overline{F[u_1(\boldsymbol{x}, t)]} = \int_{-\infty}^{\infty} F(u)\,p(u)\,du. \tag{3.8$'$}$$

In probability theory, variables u having a definite probability density are called *random* variables; the set of all possible probabilities $p(u', u'') = P\{u' < u < u''\}$ corresponding to u is called its *probability distribution.*

Thus we may conclude that from the viewpoint of probability theory, the value of the velocity at a point of a turbulent flow is a random variable described by a definite probability distribution.

So far, we have discussed only the values $u_1(\boldsymbol{x}, t)$ of the velocity component at a fixed point $\boldsymbol{x}$ and at a fixed instant t. However, we may apply a similar approach to the whole field of values of $u_1(\boldsymbol{x}, t)$, i.e., to the function $u_1(\boldsymbol{x}, t) = u_1(x_1, x_2, x_3, t)$ of the four variables. Repeating the same experiment (to establish some turbulent flow) several times and under the same external conditions, we shall obtain a new field $u_1(\boldsymbol{x}, t)$ every time. Here, also, we may speak of the "ensemble of possible fields $u_1(\boldsymbol{x}, t)$." Moreover, each individual field observed in some actual turbulent flow is considered as a "representative" chosen at random from this ensemble [in other words, as a *sample value* or *a realization* of the random field $u_1(\boldsymbol{x}, t)$]. It now remains only to consider here how to rephrase the proposition of the existence of the probability density which we used for a single value $u_1(\boldsymbol{x}, i)$.

For the field $u_1(\boldsymbol{x}, t)$ to be random, it is necessary, first, that the value $u_1(M) = u_1(\boldsymbol{x}, t)$ of this field at a fixed space-time point $M = (\boldsymbol{x}, t)$ be a random variable. Hence to every combination of values of $\boldsymbol{x}$ and t there must correspond its own probability density $p_M(u)$ dependent on $M = (\boldsymbol{x}, t)$. However, this is not all: if we choose two values $u_1(M_1) = u_1(\boldsymbol{x}_1, t_1)$ and $u_1(M_2) = u_1(\boldsymbol{x}_2, t_2)$ of our velocity component, the arithmetic mean of any function of these two values should also be statistically stable. This means that for values $u_1(M_1)$ and $u_1(M_2)$ there must exist a *two-dimensional probability density* $p_{M_1M_2}(u_1, u_2)$ defined by the relationship

$$P\{u_1 < u_1(M_1) < u_1 + du_1, \quad u_2 < u_1(M_2) < u_2 + du_2\} = \\ = p_{M_1M_2}(u_1, u_2)\, du_1\, du_2.$$

In other words, for the variation of the values of $u_1(M_1)$ and $u_1(M_2)$ in a large number of turbulent flows with the same external conditions, the proportion of cases in which the value of $u_1(M_1)$ will lie in the range between u_1' and u_1'', while at the same time the value of $u_1(M_2)$ lies in the range between u_2' and u_2'' must oscillate about some fixed value [equal to the double integral from u_1' to u_1'' and from u_2' to u_2'' of some nonnegative function $p_{M_1M_2}(u_1, u_2)$]. Moreover, if $M_1 = (\boldsymbol{x}_1, t_1)$, $M_2 = (\boldsymbol{x}_2, t_2), \ldots, M_N = (\boldsymbol{x}_N, t_N)$ are N arbitrary space-time points, then there must exist a corresponding function of the N variables,

$$p_{M_1M_2\ldots M_N}(u_1, u_2, \ldots, u_N), \tag{3.9}$$

defined by the relationship

$$P\{u_1 < u_1(M_1) < u_1 + du_1,\ u_2 < u_1(M_2) < u_2 + du_2, \ldots, \\ u_N < u_1(M_N) < u_N + du_N\} = \\ = p_{M_1M_2\ldots M_N}(u_1, u_2, \ldots, u_N)\, du_1\, du_2 \ldots du_N.$$

This function is the *N-dimensional probability density* of the values of the N random variables $u_1(M_1)$, $u_1(M_2)$, ..., $u_1(M_N)$. The existence of all possible probability densities justifies considering the field $u_1(\boldsymbol{x}, t)$ as random; to determine it completely (i.e., to determine the probability distribution in the functional space of all possible field values) it is necessary to determine the whole family of functions (3.9) corresponding to all possible positive integers N and all possible choices of N points of space-time. Thus we may consider the turbulent flows to be identical if they have identical (one-dimensional and multidimensional) probability densities. Consequently, if some set of densities is close to that which describes a given turbulent flow, then this set will define some approximate statistical model of the flow.

The functions (3.9) clearly must all be nonnegative and such that the integral of each over all variables is equal to unity. Moreover, they must also satisfy some conditions of symmetry and consistency. Thus by the very definition, the density (3.9) for any M_1, M_2, ..., M_N must satisfy the equation

$$p_{M_1M_2\ldots M_N}(u_1, u_2, \ldots, u_N) = p_{M_{i_1}M_{i_2}\ldots M_{i_N}}(u_{i_1}, u_{i_2}, \ldots, u_{i_N}), \tag{3.10}$$

where $i_1, i_2, \ldots, i_N$ is any permutation of the integers $1, 2, \ldots, N$. Further, if $n < N$, then for any N points $M_1, M_2, \ldots, M_n, M_{n+1}, \ldots, M_N$ the equation

$$p_{M_1\ldots M_n}(u_1, \ldots, u_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{M_1\ldots M_nM_{n+1}\ldots M_N}(u_1, \ldots, u_n, u_{n+1}, \ldots, u_N)\, du_{n+1}\ldots du_N \tag{3.11}$$

must be satisfied. Any family of nonnegative functions (3.9) possessing the properties (3.10)–(3.11) and the property that $\int_{-\infty}^{\infty} p_M(u)du = 1$ for all $M = (\boldsymbol{x}, t)$ defines some *probability distribution in the space of functions* $u_1(M) = u_1(\boldsymbol{x}, t)$ *of four variables* [i.e., it defines a *random field* $u_1(M) = u_1(\boldsymbol{x}, t)$]. The probability mean $\bar{F}$ of an arbitrary function $F(u_1, u_2, \ldots, u_N)$ of the values

$$u_1 = u_1(M_1), \quad u_2 = u_1(M_2), \quad \ldots, \quad u_N = u_1(M_N)$$

will then be defined as the integral

$$\bar{F}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}F(u_1, u_2, \ldots, u_N)\times$$
$$\times\, p_{M_1M_2\ldots M_N}(u_1, u_2, \ldots, u_N)\,du_1\,du_2\ldots du_N, \qquad (3.12)$$

where $p_{M_1M_2\ldots M_N}(u_1, u_2, \ldots, u_N)$ is the corresponding probability density.

It is natural to assume that in a turbulent flow the field $u_1(\boldsymbol{x}, t)$, the fields of the remaining velocity components, and also the fields of pressure $p(\boldsymbol{x}, t)$, density $\rho(\boldsymbol{x}, t)$ [in the case of a compressible fluid], temperature $T(\boldsymbol{x}, t)$ [when the temperature of the fluid is not homogeneous] and the other fluid dynamic variables, will also be random fields. In this case each of these fields will have a corresponding system of multidimensional probability densities (3.9). Moreover, the different fluid dynamic fields in a turbulent flow are statistically interconnected, and account must be taken that for these fields there also exist *joint probability densities* of the values of one of the fields at some given N_1 points of space-time, values of a second field at given N_2 points, values of a third field at given N_3 points, etc. Thus it follows that if we have any function of the fluid dynamic variables of a turbulent flow, we may determine its mean value as the integral of the product of this function with the joint probability densities of all its arguments, extended over the whole range of variation of these arguments [cf. Eq. (3.12)]. Then the conditions (3.3)–(3.7) will be transformed into well-known properties of probability means, the proof of which is given in textbooks on probabihty theory; therefore, they are satisfied exactly and require no special justification.

3.3 Concept of Ergodicity. Statistical Formulation of the Fundamental Turbulence Problem

The approach discussed in Sect. 3.2, which treats the fields of hydrodynamic variables of a turbulent flow as random fields, was initiated by the works of Kolmogorov and his school [see, e.g., Millionshchikov (1939)] and the work of Kampé de Fériet (1939). At present, this approach is generally accepted in all investigations on the theory of turbulence [see, e.g., the special survey articles of Kampé de Fériet (1953), and Obukhov (1954) and the monographs by Hinze (1959) and Lumley and Panofsky (1964)]. Adopting the assumption of the existence of probability distributions for all fluid dynamic fields, we may further make wide use of the mathematical techniques of modern probability theory; the operation of averaging is then defined uniquely and has all the properties naturally required of it. However, it is essential to note that with this approach, an important additional question arises concerning the comparison of theoretical deductions with the data of direct measurements.

According to our new definition, the mean value is understood as the mean taken over all possible values of the quantity under discussion. Thus to determine empirically mean values with comparatively high accuracy we should need results of a large number of measurements carried out in a long series of repeated similar experiments. In practice, however, we generally do not have such a series of experiments, and thus are obliged to determine the mean values from data taken in the course of a single experiment.[2] In all such cases based on a single experiment, we normally use simplified averaging of the data over some time or space interval. Thus we see that the assumption of the existence of probability distributions does not by itself eliminate the problem of the validity of using ordinary time or space mean values in the theory of turbulence, but only alters the formulation of the problem. Instead of investigating the special properties of particular methods of averaging, we must now discover how close the empirical mean values obtained by these methods lie to the probability mean value (the theory is concerned only with this). The position is completely analogous to that in ordinary statistical mechanics for systems with a finite number of degrees of freedom, where the theoretical "mean over all possible states of the system" (more often called the "ensemble mean") may also be replaced by the directly observed time-mean. In statistical mechanics it is well known that such a change is generally made on the basis of the assumption that as the averaging interval becomes infinitely great, the time-means converge to the corresponding ensemble means. In certain special cases, the vahdity of this assumption may be proved strictly (e.g., with the aid of G. D. Birkhoff's ergodic theorem) and in all other cases it is adopted as an additional, highly likely, hypothesis (the "ergodic hypothesis"). In the theory of turbu-

[2] In this respect, the only exception is that of turbulent diffusion experiments, in which a whole cloud of identical particles is generally released (e.g., a puff of smoke), and then "averaging over the cloud" is carried out. This averaging over the cloud is in a definite sense, equivalent to averaging over a set of similar experiments.

lence when the averaging interval is made infinitely large, the concept of the convergence of time or space means to the corresponding probability mean sometimes is introduced also as a special "ergodic hypothesis." With regard to the time mean, the correctness of this hypothesis in a number of cases is supported, in particular, by Landau's general ideas on the nature of developed turbulence, described in Sect. 2.9. We note, however, that in several cases, the legitimacy of replacing the probabihty means of the fluid dynamic fields by space or time means may also be proved strictly with the aid of the "ergodic theorems" of the modern theory of random processes and fields. Due to the great importance of this question, we shall deal with it in greater detail in Sect. 4.7; at this point, however, we shall attempt to formulate the general turbulence problem as a problem of the probability distributions for the corresponding fluid dynamic fields.

We have already seen that for laminar motion the fluid dynamic equations permit a single-valued determination of all the fluid dynamic variables at any future instant according to the initial values of the fluid dynamic fields (and the corresponding boundary conditions). For an incompressible fluid, it is sufficient to know only the initial values of the velocity field (or the field of the vorticity); in the case of a compressible fluid, however, the initial values of five independent fluid dynamic fields must be given (e.g., three components of velocity, pressure and temperature). In turbulent flows, the initial values of the corresponding fluid dynamic fields will also determine, with the aid of the fluid dynamic equations all of their future values.[3] Here, however, these future values will depend considerably on extremely small, uncontrollable disturbances of the initial and boundary conditions. Moreover, they will be so complex and confused in form that their exact determination is useless and the integration of the corresponding differential equations practically impossible. Here, only the probability distributions for the corresponding fluid dynamic fields are of interest and not the exact

[3]We must note here, however, that in the literature on the theory of turbulence, it is sometimes stated that in a turbulent flow the fluid dynamic equations, in general, are inapplicable. If one ignores completely unjustified assertions, then the only important question here is whether the molecular fluctuations can cause random "splashes" capable of transmitting energy to smaller-scale fluid dynamic disturbances, and, thus, for example, stimulating transition to turbulence. At present, it is almost widely agreed that even if such processes are possible, their role is, in every case, extremely small, so that it may be completely ignored to first approximation (see below, the beginning of Sect. 5.1).

values. Consequently, for turbulent flows, the fluid dynamic equations will be used only for investigating the corresponding probability distributions or of values defined by these distributions.

Further, we note that for it to be possible to apply the equations of fluid dynamics to random fields defined by their probability distributions, these distributions must satisfy some regularity conditions, which ensure that the realization of the corresponding fields may be assumed continuous and sufficiently smooth—having all the space and time derivatives which enter into the dynamic equations. Let us now assume that the probability distributions referring to values of the fields at a fixed initial instant of time $t = t_0$ satisfy these regularity conditions. In this case, every actual realization of the fluid dynamic fields will vary regularly in time in accordance with the time-variation of the solution corresponding to the given initial (and boundary) conditions. Consequently, the whole set of possible initial fluid dynamic fields will have changed after time $\tau > 0$ into a strictly defined set of functions of the space coordinates corresponding to the instant $t = t_0 + \tau$. Therefore, it follows that the probability density for any fluid dynamic field at the instant $t > t_0$ may be determined (in principle, in every case) from the initial probability densities. To do this, we need only evaluate with the aid of the fluid dynamic equations what set of initial conditions will correspond to one or another range of values of the field at the instant t, and then find the probability of this set of initial conditions. Thus, in a turbulent flow, the equations of fluid dynamics will determine uniquely the evolution in time of the probability distribution of all the fluid dynamic fields. This means that a more or less arbitrary choice (taking into account only certain "regularity conditions") may be made of the probability distribution at only one fixed instant of time; then, all remaining probability distributions, corresponding to the values of the fluid dynamic fields at all possible points of space time, will be determined uniquely by the equations of motion. Consequently, the fundamental problem of the theory of turbulence (e.g., for the case of an incompressible fluid) may be formulated as follows. Given the probability distribution of the values of the three velocity components at different points of space at the instant $t = t_0$, concentrated on a set of doubly differentiable solenoidal vector fields, it is required to determine the probability distribution of the values of the velocity and pressure fields at all subsequent times (including distributions for values at several different times). For a compressible fluid, instead of the probability distributions of the three components of velocity, it is necessary only to proceed from the probability distributions of the values of five independent fluid dynamic quantities. Unfortunately, this general problem is too difficult, and at present no approach to its complete solution is yet envisaged. Therefore, we shall postpone further consideration of this problem until the concluding chapter of the second volume of our book; in the remaining chapters we shall deal only with more particular problems where instead of probability distributions, less complete statistical characteristics of random fields are considered.

3.4 Characteristic Functions and the Characteristic Functional

In many cases instead of the probability densities (3.9) it is convenient to consider their Fourier transforms

$$\varphi_{M_1M_2\ldots M_N}(\theta_1, \theta_2, \ldots, \theta_N) = \\ = \int\int_{-\infty}^{\infty}\cdots\int e^{i\sum_{k=1}^{N}\theta_k u_k} p_{M_1M_2\ldots M_N}(u_1, u_2, \ldots, u_N)\, du_1\, du_2 \ldots du_N. \tag{3.13}$$

These Fourier transforms are called the *characteristic functions* of the corresponding probability distributions; according to Eq. (3.12), they may also be written in the form

$$\varphi_{M_1M_2\ldots M_N}(\theta_1, \theta_2, \ldots, \theta_N) = \overline{\exp\left(i\sum_{k=1}^{N}\theta_k u_k\right)}. \tag{3.14}$$

It is clear that the characteristic function defines uniquely its corresponding probability distribution; in fact

$$p_{M_1M_2\ldots M_N}(u_1, u_2, \ldots, u_N) = \\ = \frac{1}{(2\pi)^N}\int\int_{-\infty}^{\infty}\cdots\int e^{-i\sum_{k=1}^{N}\theta_k u_k} \varphi_{M_1M_2\ldots M_N}(\theta_1, \theta_2, \ldots, \theta_N)\, d\theta_1\, d\theta_2 \ldots d\theta_N \tag{3.15}$$

by the well-known inversion formula of Fourier integrals. Thus, giving the characteristic function is equivalent to giving the corresponding probability density.

By definition, the characteristic functions are complex-valued continuous functions of the arguments $\theta_1, \theta_2, \ldots, \theta_N$, possessing the following properties:

$$\varphi_{M_1M_2\ldots M_N}(0, 0, \ldots, 0) = 1 \tag{3.16}$$

and

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\varphi_{M_1M_2\ldots M_N}\left(\theta_1^{(k)} - \theta_1^{(l)}, \theta_2^{(k)} - \theta_2^{(l)}, \ldots, \theta_N^{(k)} - \theta_N^{(l)}\right) c_k c_l^* \geqslant 0 \tag{3.17}$$

(where the asterisk denotes complex conjugate) for any integer n, any real $\theta_1^{(1)}, \ldots, \theta_1^{(n)}$, $\theta_2^{(1)}, \ldots, \theta_2^{(n)}; \ldots; \theta_N^{(1)}, \ldots, \theta_N^{(n)}$ and any complex $c_1, \ldots, c_n$. In fact, the left side of Eq. (3.17) is equal to the mean value of the nonnegative quantity

$$\left| \sum_{k=1}^{n} c_k \exp\left(i \sum_{l=1}^{N} \theta_l^{(k)} u_l \right) \right|^2$$

whence follows the given inequality. It may also be shown that any continuous function of N variables possessing the properties (3.16) and (3.17) will be the characteristic function of some N-dimensional probability distribution [which, of course, may have no probability density and correspond, for example, to a discrete type; see Bochner (1933; 1959)]. We shall refer to this fact in another connection and in the second volume of the book.

It is not difficult to see that the conditions of symmetry and compatibility (3.10) and (3.11), applied to the characteristic functions, will become

$$\varphi_{M_1 M_2 \ldots M_N}(\theta_1, \theta_2, \ldots, \theta_N) = \varphi_{M_{i_1} M_{i_2} \ldots M_{i_N}}(\theta_{i_1}, \theta_{i_2}, \ldots, \theta_{i_N}), \tag{3.18}$$

$$\varphi_{M_1 \ldots M_n}(\theta_1, \ldots, \theta_n) = \varphi_{M_1 \ldots M_n M_{n+1} \ldots M_N}(\theta_1, \ldots, \theta_n, 0, \ldots, 0). \tag{3.19}$$

Thus the random field of any fluid dynamic quantity may also be defined by a family of characteristic functions (3.14) satisfying Eqs. (3.18) and (3.19).

From Eq. (3.19) it is seen that the characteristic function of the probability distribution of the values of a field on a given set of N points determines, in an extremely simple manner, the characteristic functions of the values of the field on any subset of these points. Thus it is natural to make an immediate attempt to determine all the probability distributions characterizing the field with the aid of a single quantity—"the characteristic function of the probability distribution for values of the field at all possible points." It is found that such a definition of a random field with the aid of a single quantity—"the characteristic functional"—is actually possible (this is one of the most important advantages of using the approach based on characteristic functions instead of on the probability density). The possibility of such a definition of random functions was first pointed out by Kolmogorov (1935); since then, a number of works have been devoted to this question, both from the purely mathematical as well as the applied viewpoints (among the latter, special mention must be made of the important paper of Hopf (1952), which we shall discuss in greater detail in the second volume of the book). Here we shall give only a brief description of the essence of the matter, without discussing the mathematical details.

For simplicity, we shall consider first, instead of a random field of four variables, a random function $u(x)$ of a single variable defined in the finite segment $a \leqslant x \leqslant b$ of the x axis. The function $u(x)$ is defined by all probability distributions for the values $u(x_1)$, $u(x_2), \ldots, u(x_N)$ of this function over an arbitrary system of N points $x_1, x_2, \ldots, x_N$, such that $a \leqslant x_k \leqslant b$. We now let the number N become infinitely great, choosing the points x_k in such a way that all the distances between two neighboring points tend to zero, and choosing for the parameters θ_k the products of $x_{k+1} - x_k$ and the values at the points x_k of some function $\theta(x)$ defined on $[a, b]$. If the function $\theta(x)$ is such that the integral

$$u[\theta(x)] = \int_a^b \theta(x) u(x)\, dx \tag{3.20}$$

exists for almost all realizations of the function $u(x)$,[4] then $\sum_{k=1}^{N} \theta_k u_k$ will tend to the

[4]That is, all realizations excluding, perhaps, some set of exceptional realizations, with total probability equal to zero.

integral (3.20) as $N \to \infty$. Taking the limit as $N \to \infty$ in Eq. (3.14) we obtain

$$\Phi[\theta(x)] = \overline{\exp\{iu[\theta(x)]\}} = \overline{\exp\left\{i \int_a^b \theta(x)\, u(x)\, dx\right\}}. \tag{3.21}$$

$\Phi[\theta(x)]$ is the value of the characteristic function of the random variable $u[\theta(x)]$ when the argument of this function is equal to unity. Thus, for a given $\theta(x)$, this will be some complex number. Equation (3.21) assigns to each function $\theta(x)$ some complex number, i.e., $\Phi[\theta(x)]$ is a function of a function or, as it is usually called, a *functional*. We shall call this functional the *characteristic functional* of the random function $u(x)$.

If we know the characteristic functional for some random function $u(x)$, then we can determine all finite-dimensional probability densities $p_{x_1, x_2, \ldots, x_N}(u_1, u_2, \ldots, u_N)$. For this, it is sufficient to substitute as the functional argument $\theta(x)$ of the functional $\Phi[\theta(x)]$ the special function

$$\theta(x) = \theta_1 \delta(x - x_1) + \theta_2 \delta(x - x_2) + \ldots + \theta_N \delta(x - x_N), \tag{3.22}$$

where $\theta_1, \ldots, \theta_N$ are arbitrary numbers, and $\delta(x)$ is Dirac's delta-function, so that Eq. (3.22) is equal to zero everywhere except at the points $x_1, x_2, \ldots, x_N$.[5]

Substituting Eq. (3.22) into (3.21), we obtain

$$\Phi[\theta(x)] = \overline{\exp\left\{i \sum_{k=1}^{N} \theta_k x_k\right\}} = \varphi_{x_1, \ldots, x_N}(\theta_1, \ldots, \theta_N). \tag{3.23}$$

[5] We note at this point that in the discussion of the characteristic functional (3.21) it is normally assumed that $\theta(x)$ is a sufficiently smooth function (e.g., a continuous or continuous and n-times differentiable function, n being a given number). Consequently, strictly speaking, instead of the "improper function" (3.22) we must consider a sequence of smooth functions $\theta^{(n)}(x)$, $n = 1, 2, \ldots$ such that for any interval $[\alpha, \beta]$ containing none of the points x_i, $i = 1, 2, \ldots N$

$$\lim_{n \to \infty} \int_\alpha^\beta \theta^{(n)}(x)\, dx = 0$$

while for any sufficiently small $\varepsilon > 0$ and $i = 1, 2, \ldots, N$

$$\lim_{n \to \infty} \int_{x_i - \varepsilon}^{x_i + \varepsilon} \theta^{(n)}(x)\, dx = \theta_i.$$

In this case

$$\lim_{n \to \infty} \Phi[\theta^{(n)}(x)] = \Phi[\theta(x)],$$

where $\theta(x)$ is the function (3.22). Therefore, the value of $\Phi[\theta(x)]$ for such $\theta(x)$ will always be defined by the characteristic functional of smooth functions $\theta^{(n)}(x)$.

In this case the characteristic functional will be transformed into the characteristic function of a multidimensional probability distribution for $u(x_1)$, $u(x_2)$, ..., $u(x_N)$, and the corresponding probability density may be found with the aid of the inversion formula for Fourier integrals.

The characteristic functional $\Phi[\theta(x)]$ possesses the following properties, which are analogous to the properties (3.16) and (3.17) of the characteristic functions

$$\Phi[\theta(x)]\,|_{\theta(x)\equiv 0} = 1, \tag{3.14}$$

$$\sum_{k=1}^{n}\sum_{l=1}^{n} \Phi[\theta_k(x) - \theta_l(x)]\, c_k c_l^* \geqslant \theta \tag{3.25}$$

for any function $\theta_1(x), \ldots, \theta_n(x)$ and complex numbers $c_1, \ldots, c_n$ (the latter property is called the property of *positive definiteness* of the functional $\Phi[\theta(x)]$). However, the converse assertion, i.e., a continuous (in some natural sense) positive-definite functional $\Phi[\theta(x)]$ which possesses property (3.24) will always be a characteristic functional of some random function, will be correct only if a considerably more general definition of a random function is adopted than that used here [see Gel'fand and Vilenkin (1964), Prokhorov (1961)].

When we consider, instead of a random function of one variable $u(x)$, a random field $u_1(\boldsymbol{x}, t)$ dependent on four variables, the procedure is completely analogous. Here the characteristic functional is given by

$$\Phi[\theta(\boldsymbol{x}, t)] = \overline{\exp\left\{ i \int\int_{-\infty}^{\infty}\int\int \theta(\boldsymbol{x}, t)\, u_1(\boldsymbol{x}, t)\, dx_1\, dx_2\, dx_3\, dt \right\}}, \tag{3.26}$$

which contains as argument the function $\theta(\boldsymbol{x}, t)$ of four variables.[6]

In this case the characteristic functional will define uniquely all the probability distributions for the field $u_1(\boldsymbol{x}, t)$[and will possess properties analogous to Eqs. (3.24) and (3.25)].

When considering several statistically interrelated random functions of random fields, we must consider a characteristic functional which is dependent on several functions. Thus, for example, the velocity field of a turbulent flow $\boldsymbol{u}(\boldsymbol{x}, t) = \{u_1(\boldsymbol{x}, t), u_2(\boldsymbol{x}, t), u_3(\boldsymbol{x}, t)\}$ is defined uniquely by the characteristic functional

$$\Phi[\boldsymbol{\theta}(\boldsymbol{x}, t)] = \Phi[\theta_1(\boldsymbol{x}, t), \theta_2(\boldsymbol{x}, t), \theta_3(\boldsymbol{x}, t)] =$$

$$= \overline{\exp\left\{ i \int\int_{-\infty}^{\infty}\int\int \sum_{k=1}^{3} \theta_k(\boldsymbol{x}, t)\, u_k(\boldsymbol{x}, t)\, d\boldsymbol{x}\, dt \right\}}, \tag{3.27}$$

which is dependent on three functions of four variables. In general, for an N-dimensional random field $u(\boldsymbol{x}) = \{u_1(\boldsymbol{x}), \ldots, u_N(\boldsymbol{x})\}$ in the space of points $\boldsymbol{x}$, we will have

$$\Phi[\boldsymbol{\theta}(\boldsymbol{x})] = \Phi[\theta_1(\boldsymbol{x}), \ldots, \theta_N(\boldsymbol{x})] = \overline{\exp\left\{ i \int \sum_{l=1}^{N} \theta_k(\boldsymbol{x})\, u_k(\boldsymbol{x})\, d\boldsymbol{x} \right\}}. \tag{3.27'}$$

[6] To avoid difficulties of convergence of the integrals at infinity, from the outset we may restrict ourselves to only those functions $\theta(\boldsymbol{x}, t)$ which are all identically equal to zero outside some bounded region of four-variable space.

The functional $\Phi[\boldsymbol{\theta}(\boldsymbol{x})]$ will also possess properties (3.24) and (3.35) [replacing the scalar argument $\theta(x)$ by the vector argument $\boldsymbol{\theta}(\boldsymbol{x})$]. The probability distribution for the velocity field at a given time t will be defined by the functional

$$\Phi[\boldsymbol{\theta}(\boldsymbol{x}), t] = \overline{\exp\left\{ i \int_{-\infty}^{\infty}\int\int \sum_{k=1}^{3} \theta_k(\boldsymbol{x})\, u_k(\boldsymbol{x}, t)\, d\boldsymbol{x} \right\}}, \tag{3.28}$$

which is dependent on the triad of functions $\boldsymbol{\theta}(\boldsymbol{x}) = \{\theta_1(\boldsymbol{x}), \theta_2(\boldsymbol{x}), \theta_3(\boldsymbol{x})\}$ of three variables and one scalar parameter t. In accordance with the remarks made at the beginning of this subsection on single-valued dependence of the probability distribution for the velocity field of an incompressible fluid at every instant of time on the probability distribution at the initial instant $t=0$, the characteristic functional $\Phi[\boldsymbol{\theta}(\boldsymbol{x}), t]$ must, in the case of an incompressible fluid, be defined uniquely by its initial value $\Phi[\boldsymbol{\theta}(\boldsymbol{x}), 0]$ and, moreover, the functional $\Phi[\boldsymbol{\theta}(\boldsymbol{x}, t)]$ of Eqs. (3.27) must also be defined uniquely with respect to $\Phi[\boldsymbol{\theta}(\boldsymbol{x}), 0]$). For a compressible flow, the position is more complicated. Here we must consider a functional of the type

$$\Phi[\boldsymbol{\theta}(\boldsymbol{x}), \theta_4(\boldsymbol{x}), \theta_5(\boldsymbol{x}), t] =$$

$$= \overline{\exp\left\{ i \int_{-\infty}^{\infty}\int\int \left[\sum_{k=1}^{3} \theta_k(\boldsymbol{x})\, u_k(\boldsymbol{x}, t) + \theta_4(\boldsymbol{x})\, \rho(\boldsymbol{x}, t) + \theta_5(\boldsymbol{x})\, T(\boldsymbol{x}, t) \right] d\boldsymbol{x} \right\}} \tag{3.29}$$

($\rho(\boldsymbol{x}, t)$ is the density field, $T(\boldsymbol{x}, t)$ is the temperature field), which is dependent on five functions of three variables and one scalar argument. For this functional there must be, in the case of a compressible fluid, a single-valued dependence at any instant $t > 0$ on the corresponding initial value $\Phi[\boldsymbol{\theta}(\boldsymbol{x}), \theta_4(\boldsymbol{x}), \theta_5(\boldsymbol{x}), 0]$.

4. MOMENTS OF FLUID DYNAMIC FIELDS

4.1 Moments and Cumulants of Random Variables

In the previous section we saw that for the complete statistical specification of the fields of the fluid dynamic variables of a turbulent flow we must define all multidimensional probability distributions for the values of these variables over all possible sets of space-time points. However, the determination of these multidimensional distributions is very complex, and can rarely be carried out with sufficient accuracy; moreover, the distributions themselves are often not at all suitable for practical application because of their awkwardness. Therefore, in practice, one almost always restricts oneself to a consideration of only the simpler statistical parameters, which describe some particular statistical property of the flow.

The most important of these parameters are the *moments* of the probability distributions. If we have a system of N random variables

$u_1, u_2, \ldots, u_N$ with an N-dimensional probability density $p(u_1, u_2, \ldots, u_N)$, then the moments of these variables are defined by the expressions

$$B_{k_1 k_2 \ldots k_N} = \overline{u_1^{k_1} u_2^{k_2} \ldots u_N^{k_N}} =$$
$$= \int\int_{-\infty}^{\infty} \ldots \int u_1^{k_1} u_2^{k_2} \ldots u_N^{k_N} p(u_1, u_2, \ldots, u_N)\, du_1\, du_2 \ldots du_N, \tag{4.1}$$

where $k_1, k_2, \ldots, k_N$ are nonnegative integers, the sum of which gives the order of the moment. In particular, the moments of first order are the mean values of the quantities $u_1, u_2, \ldots, u_N$.

In addition to the ordinary moments $B_{k_1 k_2 \ldots k_N}$, it is sometimes convenient to consider special combinations of them. For example, we often use the *central moment*, i.e., the moment of the deviations of $u_1, u_2, \ldots, u_N$ from their corresponding mean values:

$$b_{k_1 k_2 \ldots k_N} = \overline{(u_1 - \bar{u}_1)^{k_1} (u_2 - \bar{u}_2)^{k_2} \ldots (u_N - \bar{u}_N)^{k_N}}. \tag{4.2}$$

Opening up the brackets in the right side of Eq. (4.2), it is easy to express the central moment $b_{k_1 k_2 \ldots k_N}$ in terms of $B_{k_1 k_2 \ldots k_N}$ and ordinary lower-order moments. In particular, for $N = 1$, we have

$$\begin{gathered} b_1 = 0, \quad b_2 = B_2 - B_1^2, \quad b_3 = B_3 - 3B_1B_2 + 2B_1^3, \\ b_4 = B_4 - 4B_1B_3 + 6B_1^2B_2 - 3B_1^4, \ldots \end{gathered} \tag{4.3}$$

(the moment $b_2 = \sigma_u^2$ is called the *variance* of u, and $\sigma_u = \sqrt{B_2}$ is called the standard deviation of u. Similarly, the general second central moment $b_{11} = \overline{(u_1 - \bar{u}_1)(u_2 - \bar{u}_2)}$ is called the covariance of the variables u_1 and u_2. If $\bar{u}_i = 0$, $i = 1, \ldots, N$, then the central moments will coincide with the ordinary moments, so that $b_{k_1 k_2 \ldots k_N}$ is a special case of the moment $B_{k_1 k_2 \ldots k_N}$. If u_i has some definite dimensionality, then the corresponding moments or central moments will also be dimensional; however, the ratios

$$\frac{b_3}{b_2^{3/2}} = S, \quad \frac{b_4}{b_2^2} = \delta, \tag{4.4}$$

for example, are always dimensionless. The quantity s is called the *skewness* or the *skewness factor* of the random variable u (or of the corresponding probability distribution) and δ is its *flatness factor* while the difference $(\delta - 3)$ is called the *excess*.

Other combinations of the moments $B_{k_1k_2 \ldots k_N}$ of special interest are the *cumulants* (or *semiinvariants* $S_{k_1k_2 \ldots k_N}$. The general definition of these quantities will be given a little later (see Sect. 4.2); for now we shall merely note that the cumulant $S_{k_1k_2 \ldots k_N}$ (like the central moment $b_{k_1k_2 \ldots k_N}$) is obtained by subtracting from the moment $B_{k_1k_2 \ldots k_N}$ a special polynomial in the lower-order moments. In particular, for $N = 1$, the cumulants of the first five orders are given by the following equations:

$$
\begin{aligned}
&S_1 = B_1,\ S_2 = B_2 - B_1^2 = b_2,\ S_3 = B_3 - 3B_1B_2 + 2B_1^3 = b_3, \\
&S_4 = B_4 - 4B_1B_3 - 3B_2^2 + 12B_1^2B_2 - 6B_1^4 = b_4 - 3b_2^2, \\
&S_5 = b_5 - 10b_2b_3.
\end{aligned} \tag{4.5}
$$

In the multidimensional case, the cumulants of the second and third orders also coincide with the corresponding central moments, while the general fourth-order cumulant is given by

$$S_{1111} = b_{1111} - b_{1100}b_{0011} - b_{1010}b_{0101} - b_{1001}b_{0110}. \tag{4.6}$$

It will be clear from what follows that in certain cases the cumulants will be especially suitable to characterize the probability distributions; however, at present, we shall restrict our discussion as already indicated.

The various moments of $u_1, u_2, \ldots, u_N$ cannot assume arbitrary values, but must satisfy certain conditions which take the form of inequalities. Thus, for example, if all the exponents $k_1, k_2, \ldots, k_N$ are even, then the moment $B_{k_1k_2 \ldots k_N}$ obviously cannot be negative. Further, if $N = 1$, then

$$\sum_{k=0}^{n} \sum_{l=0}^{n} B_{k+l}c_kc_l \geqslant 0, \tag{4.7}$$

where $B_0 = 1$, and $c_0, c_1, \ldots, c_n$ are arbitrary real numbers, since the left side of Eq. (4.7) is equal to the mean value of the nonnegative quantity $\left[\sum_{k=0}^{n} c_ku^k\right]^2$. Taking $n = 2$ and $c_0 = 0$, we obtain, in particular

$$|B_3| \leqslant (B_2 B_4)^{\frac{1}{2}} \quad \text{and} \quad |s| \leqslant \delta^{\frac{1}{2}} \tag{4.8}$$

(since b_k is a special case of the moment B_k). Similar inequalities may also be deduced for higher-order moments and moments of multi-dimensional distributions. Nevertheless, even within these limits, the arbitrariness in the choice of possible values of the various moments is still very great; hence the determination of all the moments without exception provides considerable information about the corresponding probability distributions.

In many cases, defining all the moments is simply equivalent to defining the distribution itself (see below, the end of this subsection). Thus, using all the corresponding moments instead of the probability density does not lead to any loss in completeness of the statistical description. In practice, however, all of the moments are never known, and therefore one usually considers only certain lower-order moments. These, of course, do not provide a single-valued definition of the distribution, but describe only some particular properties of it. Nevertheless, the approach to the study of random variables, based on the consideration of only a few of their lower-order moments, often proves to be very valuable; later, we shall see that in the theory of turbulence, this approach allows us to obtain a whole series of results of considerable interest.

It is easy to see that the moments of the random variables $u_1, \ldots, u_N$ may be expressed simply in terms of the corresponding characteristic function $\varphi(\theta_1, \ldots, \theta_N)$. In fact, it follows from the comparison of Eqs. (4.1) and (3.13) that

$$B_{k_1 k_2 \ldots k_N} = (-i)^K \left. \frac{\partial^K \varphi(\theta_1, \theta_2, \ldots, \theta_N)}{\partial \theta_1^{k_1} \partial \theta_2^{k_2} \ldots \partial \theta_N^{k_N}} \right|_{\theta_1 = \theta_2 = \ldots = \theta_N = 0}, \tag{4.9}$$
$$K = k_1 + k_2 + \ldots + k_N.$$

In particular, it follows that if the characteristic function can be represented by its Taylor series, then

$$\varphi(\theta_1, \theta_2, \ldots, \theta_N) = \sum_{k_1, k_2, \ldots, k_N} i^K \frac{B_{k_1 k_2 \ldots k_N}}{k_1! \, k_2! \ldots k_N!} \theta_1^{k_1} \theta_2^{k_2} \ldots \theta_N^{k_N}. \tag{4.10}$$

Thus, in this case, if we know all the moments of the distribution, we can determine the characteristic function (and hence the probability density) uniquely. It is not difficult to show that the uniqueness of the determination of the density from the set of moments will also hold when the series (4.10) is convergent only in some region of values of $\theta_1, \theta_2, \ldots, \theta_N$ [for the one-dimensional case, the general condition for this uniqueness is given, for example, in the book of Akhiezer (1965)].

Also using the characteristic function, it is easy to formulate the general definition of cumulants of random variables. For this purpose we must consider the logarithm of the characteristic function $\psi(\theta_1, \theta_2, \ldots, \theta_N) = \ln \varphi(\theta_1, \theta_2, \ldots, \theta_N)$; the cumulants $S_{k_1 k_2 \ldots k_N}$ are then defined as follows:

$$S_{k_1 k_2 \ldots k_N} = (-i)^K \left. \frac{\partial^K \psi(\theta_1, \theta_2, \ldots, \theta_N)}{\partial \theta_1^{k_1} \partial \theta_2^{k_2} \ldots \partial \theta_N^{k_N}} \right|_{\theta_1 = \theta_2 = \ldots = \theta_N = 0} \tag{4.11}$$

$$K = k_1 + k_2 + \ldots + k_N.$$

Taking into account that $\varphi(0, 0, \ldots, 0) = 1$, it is now easy to obtain Eqs. (4.5)–(4.6) and, in general, to express any cumulant in terms of moments or central moments.

4.2 Moments and Cumulants of Random Fields

In the theory of turbulence we are concerned with random fields—random functions $u(M)$ of a point M of four-dimensional space-time. The Kth-order moments of such a field are the mean values of products of K values of the field

$$B_{uu \ldots u}(M_1, M_2, \ldots, M_K) = \overline{u(M_1) u(M_2) \ldots u(M_K)}. \tag{4.12}$$

These moments depend on the coordinates of the points at which the values are taken. Thus a Kth-order moment, generally speaking, is a function of $4K$ variables. However, we must keep in mind that some of the points $M_1, M_2, \ldots, M_K$ may coincide with each other; the number of different points among them defines the "type" of the moment. In this connection, we shall distinguish moments of the one-point, two-point, three-point, etc., types (more briefly, one-point, two-point, three-point, etc., moments). If the type of the moment is less than its order, then the corresponding moment

$$\overline{[u(M_1)]^{k_1} [u(M_2)]^{k_2} \ldots [u(M_N)]^{k_N}}$$

will be denoted by the symbol $B_{u \ldots u,\, u \ldots u,\, \ldots,\, u \ldots u}(M_1, M_2, \ldots, M_N)$, where the subscript groups corresponding to different points of space time are separated by commas.

The mean values of the products of values of several different, statistically related random fields are called *joint moments* of these fields. Thus, for example, the field of the velocity vector $\boldsymbol{u}(M) = \{u_1(M), u_2(M), u_3(M)\}$ will have 3^K different (ordinary and joint) moments of order K, which together constitute a single three-dimensional *moment tensor* of rank K. In particular, we have the most important two-point, second- and third-order moment tensors of the velocity field:

$$B_{ij}(M_1, M_2) = \overline{u_i(M_1)\, u_j(M_2)},$$
$$B_{ij,\,k}(M_1, M_2) = \overline{u_i(M_1)\, u_j(M_1)\, u_k(M_2)} \qquad (4.13)$$

(where the tensor $B_{ij,\,k}$ is clearly symmetric with respect to the indices i and j). Similar notation will be used for the joint moments of other fluid dynamic fields; for example, the two-point joint moments of the pressure and velocity or the pressure and temperature will be denoted by the symbols $B_{pj}(M_1, M_2)$, $j = 1, 2, 3$ or $B_{pT}(M_1, M_2)$, respectively. For moments of order greater than their type, the groups of indices relating to different points will be separated by commas; for example, $B_{ij,\,p,\,Tkl}(M_1, M_2, M_3)$ will denote a three-point, sixth-order moment of velocity, pressure and temperature, containing four components of velocity and forming a fourth-rank tensor that is symmetric in the pairs of indices i, j and k, l. The central moments (i.e., the moments of the fluctuations of the fluid dynamic fields—their deviations from their mean values) for fields with nonzero mean values, will be denoted in the same way as the ordinary moments, but with B replaced by b or else with the addition of "primes" to the corresponding indices. One-point moments may also be denoted conveniently by placing a bar above the relevant symbol (for example, $\overline{u^k}$ or $\overline{uv}$); also, for the variances $b_{uu}(M) = \overline{[u(M) - \overline{u(M)}]^2}$ of the field of $u(M)$ we shall also use sometimes one of the special symbols σ_u^2 or U'^2 (we have already used the latter in Sect. 2; see, for example, Figs. 7 and 24).

When the arguments $M_1, M_2, \ldots, M_K$ are arbitrary points of four-dimensional space time, we shall call the corresponding moments *space-time moments*. Very frequently, however, in the theory of turbulence one considers only moments in which the values of all the fields refer to the same instant; these are normally called *space moments*. Sometimes, also, we deal with *time moments*—mean values of products of values of the fluid dynamic fields at the same point (but at different instants). Henceforth, when we speak simply of "moments" we shall always mean space moments; on the other hand, if we are discussing time or space-time moments, we shall always make special mention of the fact.

In this book, we shall often be dealing with *correlation functions*, i.e., two-point, second-order moments.[7] The correlation function $B_{uu}(M_1, M_2) = \overline{u(M_1)u(M_2)}$ of the field $u(M)$ is symmetrically dependent on the arguments M_1 and M_2

$$B_{uu}(M_1, M_2) = B_{uu}(M_2, M_1). \tag{4.14}$$

Moreover, it possesses the property that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} B_{uu}(M_i, M_j)\, c_i c_j \geqslant 0 \tag{4.15}$$

for any nonnegative integer n and any choice of n points $M_1, \ldots, M_n$ and n real numbers $c_1, \ldots, c_n$, since the left side of Eq. (4.15) is identical to the mean value of the nonnegative quantity $\left[\sum_{i=1}^{n} u(M_i)\, c_i\right]^2$. In particular, with $n = 2$, we have the inequality

$$|B_{uu}(M_1, M_2)| \leqslant [B_{uu}(M_1, M_1)]^{\frac{1}{2}} [B_{uu}(M_2, M_2)]^{\frac{1}{2}}, \tag{4.16}$$

which follows from Eq. (4.15). Later, we shall see that any function $B_{uu}(M_1, M_2)$ which satisfies Eqs. (4.14) and (4.15) may be the correlation function of some random field (see Sect. 4.3). The joint two-point moment $B_{uv}(M_1, M_2) = \overline{u(M_1)v(M_2)}$ is often called the *cross-correlation function* of the fields u and v. This fimction satisfies an inequality, analogous to Eq. (4.16)

$$|B_{uv}(M_1, M_2)| \leqslant [B_{uu}(M_1, M_1)]^{\frac{1}{2}} [B_{vv}(M_2, M_2)]^{\frac{1}{2}}; \tag{4.17}$$

moreover, it is obvious that

$$B_{uv}(M_1, M_2) = B_{vu}(M_2, M_1). \tag{4.18}$$

If we define $B_{ij}(M_1, M_2) = \overline{u_i(M_1)u_j(M_2)}$, where $u_1(M), \ldots, u_N(M)$ are N arbitrary fields, then

[7] Other terminology is often encountered in the literature. For example, in mathematical works the function $B_{uu}(M_1, M_2)$, or else the *centered* function $b_{uu}(M_1, M_2)$, is often called the *covariance* (or *covariance function*) and the term correlation function (or *autocorrelation function*) is often reserved for a correlation coefficient $b_{uu}(M_1, M_2)/\sigma_u(M_1)\sigma_u(M_2)$.

$$\sum_{i=1}^{n} \sum_{j=1}^{n} B_{k_i k_j}(M_i, M_j) c_i c_j \geqslant 0 \tag{4.19}$$

for any choice of points $M_1, \ldots, M_n$, real numbers $c_1, \ldots, c_n$ and integers $k_1, \ldots, k_n$ (which take values from 1 to N). Two-point moments of order greater than 2 will represent correlation functions of some new fields which are products of the original fields; such two-point moments are sometimes called *higher-order correlation functions.*

The central two-point, second-order moments

$$b_{uu}(M_1, M_2) = \overline{[u(M_1) - \overline{u(M_1)}][u(M_2) - \overline{u(M_2)}]} = \\ = B_{uu}(M_1, M_2) - \overline{u(M_1)}\,\overline{u(M_2)} \tag{4.20}$$

and

$$b_{uv}(M_1, M_2) = \overline{[u(M_1) - \overline{u(M_1)}][v(M_2) - \overline{v(M_2)}]} = \\ = B_{uv}(M_1, M_2) - \overline{u(M_1)}\,\overline{v(M_2)} \tag{4.20'}$$

give the correlation functions of the fluctuations of the corresponding fields. Sometimes, when there is no risk of confusion, we shall call them simply correlation functions (in accordance with some authors). Of course, the correlation functions of fluctuations have all the properties of ordinary correlation functions. Another extremely important result is that when we divide the function $b_{uu}(M_1, M_2)$ by $\sigma_u(M_1)\sigma_u(M_2)$ or divide $b_{uv}(M_1, M_2)$ by $\sigma_u(M_1)\sigma_v(M_2)$ we obtain the correlation coefficient between $u(M_1)$ and $u(M_2)$ or between $u(M_1)$ and $v(M_2)$; hence the correlation functions of the fluctuations will become zero whenever the corresponding correlation coefficient is zero. It is natural to assume that for any fluid dynamic variable u or pair of such variables u and v the statistical connection between the values $u(M_1)$ and $u(M_2)$ or $u(M_1)$ and $v(M_2)$ [which is characterized by the value of the corresponding correlation coefficient] will become infinitely attenuated as the points M_1 and M_2 become infinitely far apart (in space and/or in time). Consequently, the correlation functions of the fluctuations of the fluid dynamic fields will always tend to zero as M_1 and M_2 become infinitely far apart. This fact defines an important property of the correlation functions of fluctuations, which ordinary correlation functions, generally speaking, do not possess.

It is well known that for independent random variables the mean value of a product is equal to the product of the mean values of individual factors. Since the values of the fluid dynamic variables at extremely remote points are almost independent, it follows that any central moments of the fluid dynamic fields (taken, for example, at points $M_1, \ldots, M_N$) with order equal to their type, approach zero when one of the points is infinitely far from the others. However, when the order of the central moments is greater than their type, this assertion does not hold. In exactly the same way, central moments of order $K \geqslant 4$ will not, generally speaking, tend to zero if it is not one point, but a group of points that is infinitely distant from the others. At the same time, when, for example, the general central moment is of the fourth order, it is not difficult to verify that the difference

$$\begin{gathered} b_{puvw}(M_1, M_2, M_3, M_4) - b_{pu}(M_1, M_2)\, b_{vw}(M_3, M_4) - \\ - b_{pv}(M_1, M_3)\, b_{uw}(M_2, M_4) - b_{pw}(M_1, M_4)\, b_{uv}(M_2, M_3) = \\ = S_{puvw}(M_1, M_2, M_3, M_4) \end{gathered} \tag{4.21}$$

will tend to zero whenever the position of the points M_1, M_2, M_3, M_4 changes in such a way that the distance between at least two of them becomes infinitely great. It may be shown that for any other moment (central or ordinary, it makes no difference) we may always choose a combination of lower-order moments such that the difference between the original moment and this combination will tend to zero, when any two of the points on which the moment depends become infinitely far apart (see below; the text printed in small type). The corresponding differences between a moment of order K and specially chosen combinations of lower-order moments will coincide exactly with the cumulants of the random variables discussed in Sect. 4.1. Therefore they are called the cumulants (or semiinvariants) of order K of these fields.

Using the definition of a cumulant given at the end of Sect. 4.1, it is not difficult to show that the cumulants of the fluid dynamic variables of a turbulent flow actually do possess this property. In fact, let us consider an arbitrary cumulant

$$\begin{gathered} S_{k_1 k_2 \ldots k_N}(M_1, M_2, \ldots, M_N) = \\ = (-i)^K \left. \frac{\partial^K \ln \varphi(\theta_1, \theta_2, \ldots, \theta_N)}{\partial \theta_1^{k_1} \partial \theta_2^{k_2} \ldots \partial \theta_N^{k_N}} \right|_{\theta_1 = \theta_2 = \ldots = \theta_N = 0}, \\ K = k_1 + \ldots + k_N, \end{gathered} \tag{4.22}$$

where $\varphi(\theta_1, \theta_2, \ldots, \theta_N)$ is the characteristic function of the values $u_1(M_1), u_2(M_2), \ldots, u_N(M_N)$ of the variables $u_1, u_2, \ldots, u_N$ (some or all of these may be identical) at the points $M_1, M_2, \ldots, M_N$ (some of which may also be identical). Now let the system of points $M_1, M_2, \ldots, M_N$ vary in such a way that the difference between at least two of them (for example, between M_i and M_j) becomes infinitely great. In this case, the system is bound to divide into at least two subsystems, such that every point of the first is infinitely far from every point of the second subsystem (it is sufficient, for example, to take as the first subsystem the set of all points infinitely far from the point M_j). But since the statistical connection between any two fluid dynamic variables becomes infinitely attenuated as the distance increases between the points at which the variables are taken, it follows that the N random variables $u_1(M_1), \ldots, u_N(M_N)$ divide into at least two groups such that the variables of the first group are finally practically independent of the variables of the second group. We now use the fact that the characteristic function of a set of two groups of statistically independent random variables is equal to the product of the characteristic functions of each group separately. [This follows from the fact that the multidimensional probability density of two groups of independent random variables is equal to the product of the probability densities for these two groups because of Eq. (3.13).] Therefore, when the set of points $M_1, M_2, \ldots, M_N$ varies as described, the function $\varphi(\theta_1, \theta_2, \ldots, \theta_N)$ tends to decompose into the product $\varphi(\theta_1, \ldots, \theta_n) \cdot \varphi(\theta_{n+1}, \ldots, \theta_N)$, where $n<N$. Substituting this expression for $\varphi(\theta_1, \theta_2, \ldots, \theta_N)$ into Eq. (4.22) and taking into account that $k_i \geqslant 1$ for all i we demonstrate that the cumulant $S_{k_1 k_2 \ldots k_N}(M_1, M_2, \ldots, M_N)$, when the system of points $M_1, M_2, \ldots, M_N$ varies in this manner, will tend to zero.

4.3 Random Fields with a Normal Probability Distribution (Gaussian Fields)

Similar to random variables, the complete determination of the probability distribution for random fields assumes, generally speaking, the determination of all moments of all possible orders. The only exceptions in this respect will be cases where some additional conditions exist on the probability distributions, which enable us to use certain given moments to determine the rest. We shall consider in this subsection, a particular but very important case of this kind, where we may confine ourselves to finding only the first- and second-order moments. Specifically, we consider the case of a Gaussian field, i.e., when all the probability distributions of its values are multidimensional normal (or Gaussian) distributions.

We recall that an N-dimensional probability distribution is said to be *normal* (or *Gaussian*) if the corresponding probability density takes the form

$$p(u_1, u_2, \ldots, u_N) = C \exp\left\{-\frac{1}{2}\sum_{j,k=1}^{N} g_{ik}(u_j - a_j)(u_k - a_k)\right\}. \quad (4.23)$$

Here a_j, $j = 1, \ldots, N$ are arbitrary real constants; g_{jk}, $j, k = 1, \ldots, N$ are real constants such that $\| g_{jk} \|$ is a positive-definite matrix (i.e.,

$\sum_{j,k} g_{jk}c_jc_k > 0$ for any real $c_1, \ldots, c_N$ not all equal to zero), and C is a constant, determined from the condition

$$C \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum_{j,k=1}^{N} g_{jk}(u_j-a_j)(u_k-a_k)} du_1 \ldots du_N = 1 \qquad (4.24)$$

(it is not difficult to verify that $C = G^{\frac{1}{2}}/(2\pi)^{\frac{N}{2}}$, where $G = |g_{jk}| = \det \|g_{jk}\|$).[8]

The constants a_j and g_{jk} in Eq. (4.23) are simply connected with the first and second moments of the distribution. In fact, substituting Eq. (4.23) into Eqs. (4.1) and (4.2) it is easy to show that

$$\overline{u}_j = a_j, \quad b_{jk} = \overline{(u_j - \overline{u}_j)(u_k - \overline{u}_k)} = \frac{G_{jk}}{G}, \qquad (4.25)$$

where, as before, $G = \|g_{jk}\|$ and $G_{jk} = \frac{\partial G}{\partial g_{jk}}$ is the co-factor of the element g_{jk} in the determinant G (so that the matrices $\| g_{jk} \|$ and $\| b_{jk} \|$ are mutually inverse). From Eq. (4.25) we also obtain an expression for the ordinary (noncentral) second moments of the distribution (4.23):

$$B_{jk} = \overline{u_j u_k} = \frac{G_{jk}}{G} + a_j a_k. \qquad (4.26)$$

We see that for a normal probability distribution, the first and second moments completely define the probability density. Consequently, they define, in general, all the statistical characteristics of the corresponding random variables, and, in particular, all the higher-order moments. Since the ordinary (noncentral) moments of any order are expressed simply in terms of the central moments and the mean values, it is sufficient to consider here only the problem of the evaluation of the central higher-order moments. It is not difficult

[8] Generally speaking, we may also consider cases when $\|g_{jk}\|$ is only a nonnegative matrix, i.e., when $\sum g_{jk}c_jc_k$ may become zero for certain nonnegative values of $c_1, \ldots, c_N$. In this case, it is only necessary to consider that the random variables $u_1, \ldots, u_N$ are linearly dependent, and that the probability distribution with density (4.23) is concentrated entirely in some linear subspace of the N-dimensional space with dimensionality less than N. Such probability distributions are called degenerate (or improper) normal distributions; however, they are not of interest for our purposes.

to see that all the central moments of odd order of a normal distribution will equal zero; for the central moments of even order, these may be evaluated with the aid of a general rule deduced by Isserlis (1918). According to this rule, if $w_1, w_2, \ldots, w_{2K}$ are $2K$ arbitrary random variables (some of which may be identical), which have a jointly normal probability distribution with zero mean, then

$$\overline{w_1 w_2 \ldots w_{2K}} = \sum \overline{w_{i_1} w_{i_2}} \cdot \overline{w_{i_3} w_{i_4}} \ldots \overline{w_{i_{2K-1}} w_{i_{2K}}}, \tag{4.27}$$

where the sum on the right side is extended over all possible divisions of the $2K$ indices $1, 2, \ldots, 2K$ into K pairs $(i_1, i_2), (i_3, i_4), \ldots, (i_{2K-1}, i_{2K})$ [it is not difficult to calculate that the number of terms on the right side of Eq. (4.27) is equal to $\frac{(2K)!}{2^K K!} = 1 \cdot 3 \cdot 5 \ldots (2K-1)$]. Thus, it follows that with $k_1 + k_2 + \ldots + k_N = 2K$

$$b_{k_1 k_2 \ldots k_N} = \overline{(u_1 - \bar{u}_1)^{k_1} (u_2 - \bar{u}_2)^{k_2} \ldots (u_N - \bar{u}_N)^{k_N}} = \\ = \sum b_{i_1 i_2} b_{i_3 i_4} \ldots b_{i_{2K-1} i_{2K}}, \tag{4.28}$$

where the factors b_{ij} have the same meaning as in Eq. (4.25) and the sum on the right side is taken over all $(2K)!/2^K K!$ partitions of the $2K$ indices $1, 1, \ldots, 1; 2, 2, \ldots, 2; \ldots; N, N, \ldots, N$ (where 1 is repeated k_1 times, ..., N is repeated k_N times) into K pairs (i_1, i_2), $(i_3, i_4), \ldots, (i_{2K-1}, i_{2K})$. In particular,

$$b_{1111} = \overline{(u_1 - \bar{u}_1)(u_2 - \bar{u}_2)(u_3 - \bar{u}_3)(u_4 - \bar{u}_4)} = \\ = b_{12} b_{34} + b_{13} b_{24} + b_{14} b_{23}, \tag{4.29}$$

while the central moments of the sixth order will be composed of 15 terms, etc. The method of providing this general rule is shown below.

From a comparison of Eqs. (4.6) and (4.29) it follows that for a normal distribution $S_{1111} = 0$. It may be shown that this result has a very general character; all cumulatants of order $K \geqslant 3$ of any multidimensional Gaussian distribution are identically equal to zero (see below).

We note that the normality of the probability distribution of the random vector $\boldsymbol{u} = (u_1, u_2, \ldots, u_N)$ is an independent property of the special choice of the coordinate system; this follows directly from the fact that any linear combination of normally distributed random variables will also have a normal probability distribution.

Let us now turn to the Gaussian random fields $u(M)$ or $\boldsymbol{u}(M) = \{u_1(M), \ldots, u_N(M)\}$, where the probability distributions of any finite number of its values are normal. As we have shown above, the complete statistical determination of these fields reduces to the determination of their mean values and correlation functions. All other moments may then be found by Eq. (4.28) and the fact that the central moments of odd order must be identically equal to zero. It is essential to note that for any (one-dimensional or multidimensional) random field with finite moments of the first two orders, it is always possible to choose a Gaussian field which will have the same mean and the same correlation function (or functions). For example, for a one-dimensional field $u(M)$, it follows from Eq. (4.15) applied to the correlation function of fluctuations of the field $b_{uu}(M_1, M_2)$, that for any choice of points $M_1, M_2, \ldots, M_N$ an N-dimensional normal distribution may be found, with density $p_{M_1M_2\ldots, M_N}(u_1, u_2, \ldots, u_N)$, having mean values $\overline{u(M_i)}$, $i = 1, \ldots, N$, and second moments $B_{uu}(M_i, M_j) = b_{uu}(M_i, M_j) + \overline{u(M_i)}\,\overline{u(M_j)}$. Corresponding probability distributions for all possible choices of points will, of course, satisfy the conditions of symmetry (3.1) and compatibility (3.11), i.e., they will define some random field which has the same moments of the first two orders as the initial random field $u(M)$. This will also apply in the case of a multidimensional random field $\boldsymbol{u}(M) = \{u_1(M), \ldots, u_N(M)\}$ with the only difference being that here, instead of Eq. (4.15), Eq. (4.19) must be used.[9] Thus using only data on the moments of the first and second orders in the approximate study of random fields, we may always assume that these fields have normal probability distributions. Later, we shall see that the random fields of the fiuid dynamic variables of a turbulent flow often in fact do prove to be close to Gaussian in many respects; we shall make frequent use of this fact.

In the study of normal velocity distributions it is very convenient to use characteristic functions. It is not difficult to show that in the case of the probability density (4.23)

[9]Since in this case we use only the properties (4.15) and (4.19) of the correlation functions, it follows, in particular, that any function $B_{uu}(M_1, M_2)$, or system of functions $B_{jk}(M_1, M_2)$, possessing these properties will be a correlation function (or functions) of some (one-dimensional or multidimensional) random field.

$$\varphi(\theta_1, \theta_2, \ldots, \theta_N) = \int_{-\infty}^{\infty} \cdots \int e^{i(\theta_1 u_1 + \cdots + \theta_N u_N)} p(u_1, \ldots, u_N)\, du_1 \ldots du_N =$$

$$= \exp\left\{ i \sum_{k=1}^{N} a_k \theta_k - \frac{1}{2} \sum_{j,\,k=1}^{N} b_{jk} \theta_j \theta_k \right\}, \tag{4.30}$$

where the constants a_j and b_{jk} are the same as in Eq. (4.25) [in the deduction of this equation it is convenient to use the reduction to principal axes of the quadratic form in the exponent of Eq. (4.23)]. Thus the characteristic function of a normal probability distribution has the form of an exponential function of a second-order polynomial in the variables $\theta_1, \theta_2, \ldots, \theta_N$ with the constant term equal to zero. The converse assertion is also true, i.e., to such a characteristic function there will always correspond a probability density of the form of Eq. (4.23).[10]

From Eq. (4.30) it follows that the first- and second-order cumulants of the Gaussian distribution are equal to constants a_j, $j = 1, \ldots, N$ and b_{jk}, j, $k = 1, \ldots, N$, respectively, while all higher-order cumulants are identically equal to zero. In addition, it is easy to obtain from this equation the general rule for calculating the central moments of even order; for this purpose, we must use the general equation (4.9) in which, instead of $\varphi(\theta_1, \ldots, \theta_N)$ we must substitute the function

$$\exp\left\{ -\frac{1}{2} \sum_{j,\,k=1}^{N} b_{jk} \theta_j \theta_k \right\},$$

having first expanded it as a power series in $\theta_1, \ldots, \theta_N$.

The method of characteristic functions enables us to give a simple proof of the fact that any linear combination of random variables having a normal probability distribution will also have a normal distribution. In fact, if $v_j = \sum_{k=1}^{N} c_{jk} u_k$, $j = 1, \ldots, M$, the characteristic function of the variables $v_1, \ldots, v_M$ will obviously equal

$$\varphi_{(v)}(\theta_1, \ldots, \theta_M) = \overline{\exp\left\{ i \sum_{j=1}^{M} \left(\sum_{k=1}^{N} c_{jk} u_k \right) \theta_j \right\}} =$$

$$= \overline{\exp\left\{ i \sum_{k=1}^{N} \left(\sum_{j=1}^{M} c_{jk} \theta_j \right) u_k \right\}} = \varphi\left(\sum_{j=1}^{M} c_{j1} \theta_j, \ldots, \sum_{j=1}^{M} c_{jN} \theta_j \right), \tag{4.31}$$

where $\varphi(\theta_1, \ldots, \theta_N)$ is the characteristic function of $u_1, \ldots, u_N$. Therefore, if

[10] In fact, Eq. (4.30) also includes characteristic functions of degenerate normal distributions; in this case $\|b_{jk}\|$ will not be positive-definite but only a nonnegative-definite matrix. It is easy to see that a function of the form of Eq. (4.30) may be the characteristic function of some probability distribution only if $\|b_{jk}\|$ is a nonnegative-definite matrix, and thus it follows that the characteristic functions will take the form (4.30) only for (nondegenerate or degenerate) normal distributions.

$\varphi(\theta_1, \ldots, \theta_N)$ is given by Eq. (4.30), then $\varphi_{(v)}(\theta_1, \ldots, \theta_M)$, will also be an exponential function of a second-degree polynomial in the variables $\theta_1, \ldots, \theta_M$. In particular, it follows that for complete specification of the probability distribution of $v_1, \ldots, v_M$, it is only necessary to determine their first- and second-order moments, which may easily be found from the first- and second-order moments of $u_1, \ldots, u_N$.

This allows us to write in explicit form the characteristic functional of an arbitrary Gaussian random function. First, let us consider the random function $u(x)$ of a single variable x, defined in the interval $a \leqslant x \leqslant b$. According to Eq. (3.21), the characteristic functional of this function is identical to the value of the characteristic function of the random variable

$$u\{\theta(x)\} = \int_a^b u(x)\,\theta(x)\,dx$$

with the argument of this function equal to unity. But if the probability distributions of any collection of values of the function $u(x)$ are normal distributions, then all approximating sums of the integral $u[\theta(x)]$, which are linear combinations of the random variables $u(x_k)$, $a \leqslant x_1 < x_2 < \ldots < x_n \leqslant b$, will have normal probability distributions. Thus the integral $u[\theta(x)]$ will also be a random variable with a normal distribution law. Therefore, to find an explicit equation for $\Phi[\theta(x)]$ it is only necessary to find the first two moments of the random variable $u[\theta(x)]$. These two moments are given by the equations

$$\overline{u[\theta(x)]} = \overline{\int_a^b \theta(x)\,u(x)\,dx} = \int_a^b \theta(x)\,\overline{u(x)}\,dx, \tag{4.32}$$

$$\overline{\{u[\theta(x)]\}^2} = \overline{\int_a^b \int_a^b \theta(x_1)\,\theta(x_2)\,u(x_1)\,u(x_2)\,dx_1\,dx_2} =$$

$$= \int_a^b \int_a^b \theta(x_1)\,\theta(x_2)\,B(x_1, x_2)\,dx_1\,dx_2. \tag{4.33}$$

Thus it follows that

$$\overline{\{u[\theta(x)]\}^2} - \{\overline{u[\theta(x)]}\}^2 = \int_a^b \int_a^b \theta(x_1)\,\theta(x_2)\,b(x_1, x_2)\,dx_1\,dx_2, \tag{4.34}$$

where $b(x_1, x_2)$ is the correlation function of the fluctuations of the random function $u(x)$. Substituting the moments (4.32) and (4.34) of the random variable $u[\theta(x)]$ into Eq. (4.30) with $N = 1$, we obtain

$$\Phi[\theta(x)] = \exp\left\{ i\int_a^b \theta(x)\,\overline{u(x)}\,dx - \frac{1}{2}\int_a^b \int_a^b \theta(x_1)\,\theta(x_2)\,b(x_1, x_2)\,dx_1\,dx_2 \right\}. \tag{4.35}$$

This is the general expression for the characteristic functional of a Gaussian random function given in Kolmogorov's first note (1935) devoted to the characteristic functionals. Similarly, the characteristic functional $\Phi[\theta(M)]$ (or $\Phi[\theta_1(M), \ldots, \theta_N(M)]$) of the Gaussian random field $u(M)$ [or $\boldsymbol{u}(M)=\{u_1(M), \ldots, u_N(M)\}$] is equal to

$$\Phi[\theta(M)] = \exp\left\{ i\int \theta(M)\,\overline{u(M)}\,dM - \right.$$
$$\left. -\frac{1}{2}\int\int \theta(M_1)\,\theta(M_2)\,b(M_1, M_2)\,dM_1\,dM_2 \right\} \quad (4.36)$$

or

$$\Phi[\theta_1(M), \ldots, \theta_N(M)] = \exp\left\{ i\int \sum_{j=1}^{N} \theta_j(M)\,\overline{u_j(M)}\,dM - \right.$$
$$\left. -\frac{1}{2}\int\int \sum_{j,k=1}^{N} \theta_j(M_1)\,\theta_k(M_2)\,b_{jk}(M_1, M_2)\,dM_1\,dM_2 \right\}; \quad (4.37)$$

naturally, in all cases it will be defined uniquely by the corresponding mean values and correlation functions.

4.4 Determination of the Moments and Cumuiants of a Random Field According to Its Characteristic Functional

Since the characteristic functional contains a full statistical specification of a random field, it will also clearly define all moments and cumuiants of this field. We shall now deduce explicit equations connecting the moments and cumulants with the functional $\Phi[\theta]$; we shall thus obtain a natural extension of Eqs. (4.9) and (4.11) which relate to the finite-dimensional case.

Once again, we begin by considering the random function $u(x)$, $a<x<b$ of a single variable. Since, as we know, moments and cumulants of the random vector $u=\{u_1, \ldots, u_N\}$ are expressed in terms of the partial derivatives of the corresponding characteristic function $\varphi(\theta_1, \ldots, \theta_N)$, first, we have to generalize the concept of a derivative to the case of functions of an infinite number of variables, i.e., of the functional $\Phi[\theta(x)]$ with respect to the function $\theta(x)$. We recall that in the finite case the function $\varphi(\theta_1, \ldots, \theta_N) = \varphi(0)$ is said to be differentiable at the point $\theta^0 = \{\theta_1^0, \ldots, \theta_N^0\}$ if its increment $d\varphi = \varphi(\theta^0 + d\theta) - \varphi(\theta^0)$ for a small variation $d\theta = \{d\theta_1, \ldots, d\theta_N\}$ of the values of the argument may be written as

$$d\varphi = \sum_{k=1}^{N} A_k\,d\theta_k + o(|d\theta_1| + \ldots + |d\theta_N|)$$

(i.e., with accuracy to higher-order corrections it is linearly dependent on $d\boldsymbol{\theta}$). The partial derivative $\frac{\partial\varphi(\theta^0)}{\partial\theta_k}$, moreover, may be defined as the coefficient A_k of $d\theta_k$ in the linear part of the increment $d\varphi$, so that

$$d\varphi \approx \sum \frac{\partial\varphi}{\partial\theta_k}\,d\theta_k\,.$$

Similarly, we say that the functional $\Phi[\theta(x)]$ is differentiable for $\theta=\theta_0(x)$, if, when a small increment $\delta\theta(x)$ is added to $\theta_0(x)$ the principal part of the increment $\delta\Phi[\theta_0(x)]$ of this functional is linearly dependent on $\delta\theta(x)$

$$\delta\Phi[\theta_0(x)]=\Phi[\theta_0(x)+\delta\theta(x)]-\Phi[\theta_0(x)]=$$
$$=\int_a^b A(x)\,\delta\theta(x)\,dx+o\left[\int_a^b|\delta\theta(x)|\,dx\right] \tag{4.38}$$

(in other words, if there exists a derivative $\delta_0\Phi[\theta_0(x)]=\frac{\partial}{\partial h}\Phi[\theta_0(x)+h\theta_1(x)]|_{h=0}$, which may be put in the form $\delta_0\Phi[\theta_0(x)]=\int_a^b A(x)\,\theta_1(x)\,dx$). The value of the function $A(x)$ at the point $x=x_1$ will, in this case, be called *functional derivative* of $\Phi[\theta_0(x)]$ with respect to $\theta(x)$ at $x=x_1$. Taking into account that $A(x)$ is the coefficient of $\delta\theta(x)dx$ in the linear part of the increment $\delta\Phi[\theta_0(x)]$, it is convenient to adopt the notation

$$\frac{\delta\Phi[\theta_0(x)]}{\delta\theta(x)dx}=A(x) \tag{4.39}$$

for the functional derivative. This notation stresses that the functional derivative is a double limit

$$\frac{\delta\Phi[\theta_0(x)]}{\delta\theta(x)\,dx}=\lim_{\substack{|\delta\theta(x)|\to0\\ \Delta x\to0}}\frac{\Phi[\theta_0(x)+\delta\theta(x)]-\Phi[\theta_0(x)]}{\int_{\Delta x}\delta\theta(x)\,dx},$$

where $\delta\theta(x)$ now indicates a function that is nonzero only in a small interval of length Δx surrounding the point x. The requirement that the functional $\Phi[\theta(x)]$ be differentiable imposes certain limitations upon this functional which, for example, are not satisfied by so simple a functional as the value $\theta(x_0)$ of the function $\theta(x)$ at the fixed point $x=x_0$. The simplest example of a differentiable functional is the functional

$$\Phi[\theta(x)]=\int_a^b A(x)\,\theta(x)\,dx,$$

for which

$$\frac{\delta\left[\int_a^b A(x)\,\theta(x)\,dx\right]}{\delta\theta(x)\,dx}=A(x). \tag{4.40}$$

In Eq. (4.40) the functional derivative $\frac{\delta\Phi[\theta(x)]}{\delta\theta(x)\,dx}$ is independent of the value of the function $\theta(x)$ at which this derivative is taken. However, in general, this will certainly not be so, and hence it is more correct to write Eqs. (4.38) and (4.39) in the form

$$\delta\Phi[\theta_0(x)]\approx\int_a^b A[\theta_0(x);\,x_1]\,\delta\theta(x_1)\,dx_1,\qquad \frac{\delta\Phi[\theta_0(x)]}{\delta\theta(x_1)\,dx_1}=A[\theta_0(x);\,x_1].$$

Thus the functional derivative of the functional $\Phi[\theta(x)]$ is once again a functional of $\theta(x)$, which is still dependent on the point x_1 as a parameter. Consequently, this functional derivative will have derivatives of double type; it may be differentiated with respect to x_1 in the usual manner and may also have its own functional derivative with respect to $\theta(x)$ at the point $x=x_2$ which is the second functional derivative of the original functional $\Phi[\theta(x)]$:

$$\frac{\delta}{\delta\theta(x_2)\,dx_2}\left[\frac{\delta\Phi[\theta(x)]}{\delta\theta(x_1)\,dx_1}\right]=\frac{\delta^2\Phi[\theta(x)]}{\delta\theta(x_1)\,dx_1\,\delta\theta(x_2)\,dx_2}\,. \tag{4.41}$$

The second functional derivative will be a functional $A[\theta(x);\ x_1,\ x_2]$, dependent on the pair of points x_1 and x_2; however, this functional will not be arbitrary, but will satisfy the condition of symmetry $A[\theta(x);\ x_1, x_2]= A[\theta(x); x_2, x_1]$ which follows from the easily provable identity

$$\frac{\delta^2\Phi[\theta(x)]}{\delta\theta(x_1)\,dx_1\,\delta\theta(x_2)\,dx_2}=\frac{\delta^2\Phi[\theta(x)]}{\delta\theta(x_2)\,dx_2\,\delta\theta(x_1)\,dx_1}\,. \tag{4.42}$$

One of the simplest examples of a twice differentiable functional is

$$\Phi[\theta(x)]=\int_a^b\int_a^b A(x,\ x')\,\theta(x)\,\theta(x')\,dx\,dx';$$

in this case, it may easily be verified that

$$\frac{\delta^2\left[\int_a^b\int_a^b A(x,\ x')\,\theta(x)\,\theta(x')\,dx\,dx'\right]}{\delta\theta(x_1)\,dx_1\,\delta\theta(x_2)\,dx_2}=A(x_1,\ x_2)+A(x_2,\ x_1). \tag{4.43}$$

Functional derivatives of higher orders are defined in a similar manner; the *n*th functional derivative

$$\frac{\delta^n\Phi[\theta(x)]}{\delta\theta(x_1)\,dx_1\ \ldots\ \delta\theta(x_n)\,dx_n}$$

(if it exists), will be a functional with respect to $\theta(x)$, dependent on n points $x_1,\ \ldots,\ x_n$.

Now let $\Phi[\theta(x)]$ be the characteristic functional of the random function $u(x)$, given by Eq. (3.21). In this case

$$\delta_0\Phi[\theta(x)]=\frac{\partial}{\partial h}\overline{\left\{\exp\left[i\int_a^b u(x)[\theta(x)+h\theta_1(x)]\,dx\right]\right\}}_{h=0}$$

$$=\overline{i\int_a^b u(x)\,\theta_1\,dx\,\exp\left[i\int_a^b u(x)\,\theta(x)\,dx\right]}$$

(sincc averaging and differentiation are commutative operations). Thus it follows that

$$\frac{\delta\Phi[\theta(x)]}{\delta\theta(x)\,dx}=\overline{i\,u(x)\,\exp\left[i\int_a^b u(x)\,\theta(x)\,dx\right]}\,. \tag{4.44}$$

Analogously, we may obtain the more general equation

$$\frac{\delta^n \Phi [\theta (x)]}{\delta\theta (x_1)\, dx_1 \ldots \delta\theta (x_n)\, dx_n} = \overline{i^n u(x_1)\ldots u(x_n) \exp \left[i \int_a^b u(x)\theta(x)\,dx \right]}, \quad (4.45)$$

which indicates that

$$B_{u\ldots u}(x_1, \ldots, x_n) = \overline{u(x_1) \ldots u(x_n)} = (-i)^n \left. \frac{\delta^n \Phi [\theta (x)]}{\delta\theta (x_1)\, dx_1 \ldots \delta\theta (x_n)\, dx_n} \right|_{\theta(x)=0}. \quad (4.46)$$

Equation (4.46) is a generalization of Eq. (4.9) for the infinite dimensional case.

The functional derivatives of the functionals $\Phi[\theta(x)]$ and $\Phi[\theta(\boldsymbol{x})] = \Phi[\theta_1(\boldsymbol{x}), \ldots, \theta_N(\boldsymbol{x})]$ dependent on (scalar or vector) functions of a point $\boldsymbol{x}$ of multidimensional space are defined in a completely similar manner. Thus, for example, the functional $\Phi[\theta(\boldsymbol{x})]$ is said to be differentiable with respect to $\theta_l(\boldsymbol{x})$ for a given value of the function $\theta(\boldsymbol{x}) = \{\theta_1(\boldsymbol{x}), \ldots, \theta_N(\boldsymbol{x})\}$ if the equation

$$\frac{\partial}{\partial h} \Phi \left[\theta_1(\boldsymbol{x}), \ldots, \theta_l(\boldsymbol{x}) + h\theta_l^0(\boldsymbol{x}), \ldots, \theta_N(\boldsymbol{x})\right]\Big|_{h=0} = \int A_l [\theta(\boldsymbol{x}); \boldsymbol{x}_1]\, \theta_l^0(\boldsymbol{x}_1)\, d\boldsymbol{x}_1 \quad (4.47)$$

is satisfied, i.e., if the principal part of the increment of $\Phi[\theta(\boldsymbol{x})]$ for a fairly small change $\delta\theta_l(\boldsymbol{x})$ in the function $\theta_l(\boldsymbol{x})$ is linearly dependent on $\delta\theta_l(\boldsymbol{x})$. In this case, the functional

$$A_l [\theta(\boldsymbol{x}); \boldsymbol{x}_1] = \frac{\delta\Phi [\theta(\boldsymbol{x})]}{\delta\theta_l(\boldsymbol{x}_1)\, d\boldsymbol{x}_1}, \quad (4.48)$$

which is dependent on the point $\boldsymbol{x}_1$ as a parameter is called the functional derivative (or, more precisely, the partial functional derivative) of $\Phi[\theta(\boldsymbol{x})]$ with respect to $\theta_l(\boldsymbol{x})$ at the point $\boldsymbol{x} = \boldsymbol{x}_1$. In other words, this derivative may be defined as the limit

$$\lim_{\substack{|\delta_l\theta(\boldsymbol{x})| \to 0 \\ \Delta\boldsymbol{x}_1 \to 0}} \frac{\Phi[\theta(\boldsymbol{x}) + \delta_l\theta(\boldsymbol{x})] - \Phi[\theta(\boldsymbol{x})]}{\int_{\Delta\boldsymbol{x}_1} \delta_l\theta(\boldsymbol{x})\, d\boldsymbol{x}},$$

where $\delta_l\theta(\boldsymbol{x})$ is a vector function in which only the lth component is nonzero, and that only in a small neighborhood $\Delta\boldsymbol{x}_1$ of the point $\boldsymbol{x}_1$. The functional derivative (4.48) may then be differentiated with respect to $\theta_j(\boldsymbol{x})$; we then obtain the second functional derivative:

$$\frac{\delta^2\Phi [\theta(\boldsymbol{x})]}{\delta\theta_l(\boldsymbol{x}_1)\, d\boldsymbol{x}_1\, \delta\theta_j(\boldsymbol{x}_2)\, d\boldsymbol{x}_2} = A_{lj} [\theta(\boldsymbol{x}); \boldsymbol{x}_1, \boldsymbol{x}_2], \quad (4.49)$$

which is dependent on the pair of points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, and which satisfies the condition

$$\frac{\delta^2\Phi [\theta(\boldsymbol{x})]}{\delta\theta_l(\boldsymbol{x}_1)\, d\boldsymbol{x}_1\, \delta\theta_j(\boldsymbol{x}_2)\, d\boldsymbol{x}_2} = \frac{\delta^2\Phi [\theta(\boldsymbol{x})]}{\delta\theta_j(\boldsymbol{x}_2)\, d\boldsymbol{x}_2\, \delta\theta_l(\boldsymbol{x}_1)\, d\boldsymbol{x}_1}. \quad (4.5\theta)$$

The highcr-order functional derivatives

$$\frac{\delta^n \Phi[\theta(x)]}{\delta\theta_{i_1}(x_1)\,dx_1 \dots \delta\theta_{i_n}(x_n)\,dx_n}$$

are defined similarly. If $\Phi[\theta(x)]$ is taken to be the characteristic functional (3.27′) of the N-dimensional random field $u(x) = \{u_1(x), \dots, u_N(x)\}$, then by a process completely analogous to the deduction of Eq. (4.46), we obtain the equation

$$B_{i_1 \dots i_n}(x_1, \dots, x_n) = \overline{u_{i_1}(x_1) \dots u_{i_n}(x_n)} = \\ = (-i)^n \left. \frac{\delta\Phi[\theta(x)]}{\delta\theta_{i_1}(x_1)\,dx_1 \dots \delta\theta_{i_n}(x_n)\,dx_n} \right|_{\theta(x)=0} \tag{4.51}$$

(we note that here the property (4.50) of functional derivatives becomes the well-known property (4.18) of correlation functions).

The cumulants of the random field $u(x) = \{u_1(x), \dots, u_N(x)\}$ are now defined with the aid of the following equation:

$$S_{i_1 \dots i_n}(x_1, \dots, x_n) = (-i)^n \left. \frac{\delta^n \ln \Phi[\theta(x)]}{\delta\theta_{i_1}(x_1)\,dx_1 \dots \delta\theta_{i_n}(x_n)\,dx_n} \right|_{\theta(x)=0}, \tag{4.52}$$

where $\Phi[\theta(x)]$ is the characteristic functional of the field. It is easy to see that

$$S_j(x_1) = -i \left. \frac{\delta \ln \left\{ \overline{\exp\left[i \int \Sigma u_\alpha(x)\,\theta_\alpha(x)\,dx\right]} \right\}}{\delta\theta_l(x_1)\,dx_1} \right|_{\theta(x)=0} = \\ = \left. \frac{\overline{u_l(x_1) \exp\left[i \int \Sigma u_\alpha(x)\,\theta_\alpha(x)\,dx\right]}}{\overline{\exp\left[i \int \Sigma u_\alpha(x)\,\theta_\alpha(x)\,dx\right]}} \right|_{\theta(x)=0} = \overline{u_l(x_1)},$$

$$S_{lj}(x_1, x_2) = -i \frac{\delta}{\delta\theta_j(x_2)\,dx_2} \left. \left\{ \frac{\overline{u_l(x_1) \exp\left[i \int \Sigma u_\alpha(x)\,\theta_\alpha(x)\,dx\right]}}{\overline{\exp\left[i \int \Sigma u_\alpha(x)\,\theta_\alpha(x)\,dx\right]}} \right\} \right|_{\theta(x)=0} = \\ = \overline{u_l(x_l)\,u_j(x_2)} - \overline{u_l(x_1)} \cdot \overline{u_j(x_2)} = b_{lj}(x_1, x_2)$$

and, in general, that $S_{i_1 \dots i_n}(x_1, \dots, x_n)$ is identical to the cumulant of the system of variables $u_{i_1}(x_1), \dots, u_{i_n}(x_n)$, defined by the corresponding characteristic function with the aid of Eq. (4.11). From Eq. (4.52) it is easy to deduce all the properties of cumulants [property (4.50) of the functional derivative, for example, means that the correlation function of the fluctuations must have the same property (4.18) as the ordinary correlation functions].

For a functional $\Phi[\theta(x)]$ possessing functional derivatives of all orders, under certain conditions we may obtain a Taylor-series expansion

$$\Phi[\theta(x)] = \Phi[0] + \sum_{l=1}^{N} \int \frac{\delta\Phi[0]}{\delta\theta_l(x_1)\,dx_1} \theta_l(x_1)\,dx_1 + \\ + \frac{1}{2} \sum_{l=1}^{N} \sum_{j=1}^{N} \int \int \frac{\delta^2\Phi[0]}{\delta\theta_i(x_1)\,dx_1\,\delta\theta_j(x_2)\,dx_2} \theta_l(x_1)\,\theta_j(x_2)\,dx_1\,dx_2 + \dots \tag{4.53}$$

In the case of the characteristic functional of the random field $\boldsymbol{u}(\boldsymbol{x})$ this expansion becomes an expansion in correlation functions of all possible orders

$$\begin{aligned}\Phi[\theta(\boldsymbol{x})] = 1 + i\sum_{j=1}^{N}\int \overline{u_j(\boldsymbol{x})}\,\theta_j(\boldsymbol{x})\,d\boldsymbol{x} - \\ -\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\int\int B_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2)\,\theta_j(\boldsymbol{x}_1)\,\theta_k(\boldsymbol{x}_2)\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2 + \\ + \dots + \frac{i^n}{n!}\sum_{i_1=1}^{N}\dots\sum_{i_n=1}^{N}\int\dots\int B_{i_1 \dots i_n}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)\,\theta_{i_1}(\boldsymbol{x}_1)\dots \\ \dots\theta_{i_n}(\boldsymbol{x}_n)\,d\boldsymbol{x}_1\dots d\boldsymbol{x}_n + \dots\end{aligned} \tag{4.54}$$

It may be shown that for convergence of the series on the right side of Eq. (4.54), it is necessary and sufficient that its nth term tend to zero as $n \to \infty$; if we break off this series after a finite number of terms, for the remainder we may obtain an estimate similar to the estimate for the remainder of an ordinary Taylor series [see Novikov (1964)]. We note, however, that if we simply break off the expansion (4.54) of the functional $\Phi[\theta(x)]$ after any finite number of terms, without taking the remainder into account, then we arrive at a functional which possesses property (3.24) of the characteristic functional, but which certainly does not possess the necessary property (3.25). In fact this follows even from the fact that a functional equal to the sum of a finite number of terms of the right side of Eq. (4.54) does not satisfy the simple inequality $|\Phi[\theta(\boldsymbol{x})]| \leqslant 1$, which follows from the inequality (3.25) with $n = 2$ (and also from the very definition (3.27′) of the characteristic functional $\Phi[\theta(\boldsymbol{x})]$. Therefore, assuming that all the moments of some random field of order higher than some given K are identically equal to zero, we finally reach a *reductio ad absurdum* which contradicts, for example, the condition that the probability is always less than or equal to unity.

Somewhat more satisfactory from this viewpoint is the application of Taylor's expansion to the logarithm of the characteristic functional $\Phi[\theta(\boldsymbol{x})]$, i.e., the representation of $\Phi[\theta(\boldsymbol{x})]$ in the form

$$\begin{aligned}\Phi[\theta(\boldsymbol{x})] = \exp\Bigg\{i\sum_{j=1}^{N}\int \overline{u_j(\boldsymbol{x})}\,\theta_j(\boldsymbol{x})\,d\boldsymbol{x} - \\ -\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\int b_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2)\,\theta_j(\boldsymbol{x}_1)\,\theta_k(\boldsymbol{x}_2)\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2 + \\ + \dots + \frac{i^n}{n!}\sum_{i_1=1}^{N}\dots\sum_{i_n=1}^{N}\int\dots\int S_{i_1 \dots i_n}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)\,\theta_{i_1}(\boldsymbol{x}_1)\dots \\ \dots\theta_{i_n}(\boldsymbol{x}_n)\,d\boldsymbol{x}_1\dots d\boldsymbol{x}_n + \dots\Bigg\}.\end{aligned} \tag{4.55}$$

If we cut Eq. (4.55) short after a finite number of terms, we obtain a functional which satisfies not only Eq. (3.24), but also the necessary condition for a characteristic functional $|\Phi[\theta(\boldsymbol{x})]| \leqslant 1$. Moreover, if we restrict ourselves only to the linear and quadratic terms in $\theta_j(\boldsymbol{x})$, in the right side of Eq. (4.55), we obtain a functional which certainly is a characteristic functional of some random field, e.g., of the Gaussian random field with moments of the first and second orders equal to those of the initial field $\boldsymbol{u}(\boldsymbol{x})$ [cf. Eq. (4.37)]. If, however, we also take into account the third-order terms in $\theta_j(\boldsymbol{x})$ in the expansion (4.55), i.e., if we put

$$\Phi[\theta(\boldsymbol{x})] = \exp\left\{ i \sum_{j=1}^{N} \overline{u_j(\boldsymbol{x})}\, \theta_j(\boldsymbol{x})\, d\boldsymbol{x} - \right.$$
$$- \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} \int \int b_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2)\, \theta_j(\boldsymbol{x}_1)\, \theta_k(\boldsymbol{x}_2)\, d\boldsymbol{x}_1\, d\boldsymbol{x}_2 -$$
$$\left. - \frac{i}{\varphi} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} \int \int \int b_{jkl}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)\, \theta_j(\boldsymbol{x}_1)\, \theta_k(\boldsymbol{x}_2)\, \theta_l(\boldsymbol{x}_3)\, d\boldsymbol{x}_1\, d\boldsymbol{x}_2\, d\boldsymbol{x}_3 \right\}, \quad (4.56)$$

or if we cut short the series (4.55) after any other finite number of terms of order higher than the second in $\theta_j(\boldsymbol{x})$, then we obtain a functional which will satisfy Eq. (3.24) and the condition $|\Phi[\theta(\boldsymbol{x})]| \leqslant 1$, but which, generally speaking, does not possess the property (3.25) which is necessary for characteristic functionals. Thus, if we assume that all cumulants of the field $\boldsymbol{u}(\boldsymbol{x})$ of order greater than some given $K \geqslant 4$ are equal to zero, we also finally obtain a contradiction to the obvious properties of probability distributions [e.g., to the fact that the probability is nonnegative, from which condition (3.25) follows]. In Volume 2 of this book we shall have occasion to return to this fact.

4.5 Stationary Random Functions

Let us now return to the important question (cf. Sect. 3.3) of the conditions under which the time and space mean values of a random field $u(\boldsymbol{x}, t)$ will converge to the probability mean value as the averaging interval becomes infinitely great. We shall obtain some special classes of such fields, which are of particular interest for the theory of turbulence.

For definiteness, first, we shall discuss time-averaging only; then the dependence of the field $u(\boldsymbol{x}, t)$ on $\boldsymbol{x}$ will have no significance, and we can consider only the function $u(t)$ of the single variable t. We shall be interested in the question of the conditions under which the random quantity

$$\tilde{u}_T(t) = \frac{1}{T} \int_{-T/2}^{T/2} u(t+\tau)\, d\tau \quad (4.57)$$

will converge to $\overline{u(t)}$ as $T \to \infty$.[11] One very important necessary condition for convergence is deduced extremely simply. For any

[11] We are concerned here with the convergence of random variables which, in general, requires a special definition (see below, the beginning of Sect. 4.7).

bounded function $u(t)$, the difference

$$\tilde{u}_T(t)-\tilde{u}_T(t_1)=\frac{1}{T}\left\{\int_{-T/2}^{T/2} u(t+\tau)\,d\tau-\int_{-T/2}^{T/2} u(t_1+\tau)\,d\tau\right\}=$$
$$=\frac{1}{T}\left\{\int_{-\frac{T}{2}+t}^{-\frac{T}{2}+t_1} u(s)\,ds-\int_{\frac{T}{2}+t}^{\frac{T}{2}+t_1} u(s)\,ds\right\}, \quad (4.58)$$

where t and t_1 are fixed numbers (and, for example, $t_1>t$) will become infinitely small as $T\to\infty$. Therefore, the limits as $T\to\infty$ of $\tilde{u}_T(t)$ and $\tilde{u}_T(t_1)$ [provided that such exist] must become equal, i.e., the time mean value of $u(t)$, defined as $\lim \tilde{u}_T(t)$, must be independent of t. At the same time, the probability mean $\overline{u(t)}$ is, generally speaking, a function of t. Consequently, for the two means to be equal, it is necessary that the following condition hold:

$$\overline{u(t)}=U=\text{const.} \quad (4.59)$$

The case is similar when time-averaging is used to determine higher-order moments and other functions of values of $u(t)$ at several points. Thus, for example, if we use the time-mean (with respect to t) of the product $u(t)u(t_1)=u(t)u(t+s)$ [where $s=t_1-t$ is assumed to be fixed] to determine the correlation function $B_{uu}(t,t_1)$, then we obtain

$$\tilde{B}_{uu}(t,t_1)=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} u(t+\tau)\,u(t+s+\tau)\,d\tau, \quad (4.60)$$

which can depend only on $s=t_1-t$, but not on t and t_1 individually. Therefore, the time correlation function $\tilde{B}_{uu}(t_1\ t_2)$ can equal the probability mean $\overline{u(t_1)u(t_2)}=B_{uu}(t_1,t_2)$ only if

$$B_{uu}(t_1,t_2)=B_{uu}(t_2-t_1). \quad (4.61)$$

In exactly the same way, for the Nth-order moment $B_{uu\ldots u}(t_1,t_2,\ldots,t_N)$ to be definable with the aid of time-averaging, it is necessary for this moment to depend only on the $N-1$ differences $t_2-t_1,\ldots,t_N-t_1$:

$$B_{uu\ldots u}(t_1, t_2, \ldots, t_N) = B_{uu\ldots u}(t_2 - t_1, \ldots, t_N - t_1). \quad (4.62)$$

Finally, if we require that the mean values of all possible functions of $u(t_1), u(t_2), \ldots, u(t_N)$ be obtained by means of time-averaging, then we must confine ourselves to only those random functions $u(t)$ for which the N-dimensional probability density $p_{t_1 t_2 \ldots t_N}(u_1, u_2, \ldots, u_N)$ for any N and $t_1, t_2, \ldots, t_N$ depends not on the N parameters $t_1, t_2, \ldots, t_N$ but only on the $N-1$ differences $t_2 - t_1, \ldots, t_N - t_1$, i.e., they satisfy the condition

$$p_{t_1 t_2 \ldots t_N}(u_1, u_2, \ldots, u_N) = p_{t_1+h, t_2+h, \ldots, t_N+h}(u_1, u_2, \ldots, u_N) =$$
$$= p_{t_2-t_1, \ldots, t_N-t_1}(u_1, u_2, \ldots, u_N), \quad (4.63)$$

where h is any real number. Condition (4.63) is extremely general, in particular, conditions (4.59), (4.61) and (4.62) follow from it. We note, further, that in the special case of a Gaussian random function $u(t)$, the general equations (4.62) and (4.63) follow from Eqs. (4.59) and (4.61).

Thus we have obtained a special class of random functions of time for which all multidimensional probability densities satisfy Eq. (4.63), i.e., they do not change when there is a shift of a corresponding group of points t_i through any time-interval h. Such random functions are often encountered in the most diverse applied problems; they are called *stationary random functions* or *stationary random processes* (since random functions of time in scientific literature are also often called random processes). A number of special monographs or chapters in more general monographs have been devoted to the mathematical theory of stationary random functions [see, for example Doob (1953), Loève (1955), Yaglom (1962), Rozanov (1967)]. However, we shall restrict ourselves at this point to a few remarks which have a direct bearing on the subject of this book (see also Chapt. 6 in Volume 2).

The physical meaning of the condition of stationarity is perfectly clear. It means that the physical process which has the function $u(t)$ as its numerical characteristic will be steady, i.e., that all the conditions governing the process will be time-independent. With reference to turbulence characteristics, the stationarity condition means that the turbulent flow under discussion will be steady in the ordinary fluid dynamic sense, i.e., all averaged characteristics of the flow (e.g., the mean velocity distribution, the mean temperature) and also all external conditions (e.g., external forces, position of the surfaces bounding the flow) will remain unchanged with the passage of time. Flows which satisfy this condition with sufficient accuracy may be obtained comparatively simply in the laboratory; however, in the case of natural turbulent flows, it is usually difficult to ensure the invariance of all averaged characteristics of the flow (this is true particularly in the case of atmospheric turbulence, where the mean values of all variables are usually very unstable and have a clearly expressed diurnal and annual cycle). However, here too, for the values of fluid dynamic variables considered over comparatively short intervals of time (for example, of several minutes or tens of minutes), the corresponding random functions may often be considered as stationary. In all such cases the probability means of the variables of the flow may often be found by time-averaging; for this to be possible, it is necessary that the time means as $T \to \infty$ converge to the probability mean and that the means taken over time T, in the course of which the process may be assumed stationary, will already be fairly close to the limits corresponding to $T \to \infty$.

The values of any fluid dynamic variable at several points of a stationary turbulent flow or the values of several such variables at one or several points provide us with examples of multidimensional stationary random processes, e.g., of vector functions $\boldsymbol{u}(t) = \{u_1(t), \ldots, u_n(t)\}$ such that the probability density for any choice of values $u_{l_1}(t_1), u_{l_2}(t_2), \ldots, u_{l_N}(t_N)$ does not vary when all the instants of time $t_1, t_2, \ldots, t_N$ are simultaneously shifted through the same arbitrary time interval h. In this case, all the joint moments of the functions $u_j(t)$ will depend only on the differences of the corresponding instants of time [for example, all cross-correlation functions $B_{jk}(t_1, t_2) = \overline{u_j(t_1)\, u_k(t_2)}$ will depend only on the argument $\tau = (t_2 - t_1)$]. It is clear that if the statistical characteristics dependent on the values of several random functions of time may be obtained by means of time-averaging, the set of these random functions must constitute a multidimensional stationary process; in the opposite case, characteristics obtained by means of time-averaging will depend on a smaller number of variables than do the corresponding probability means.

4.6 Homogeneous Random Fields

Before proceeding to an investigation of the conditions under which the time means of functions of stationary random processes converge to the corresponding probability means, we shall consider briefly the question of the changes which must be made in our discussion if we work on the basis of space-averaging instead of time-averaging. Here, of course, we must consider random functions $u(\boldsymbol{x})$ of the point $\boldsymbol{x}=(x_1, x_2, x_3)$, i.e., random fields in three-dimensional space. The space means of such fields will be defined as

$$\tilde{u}_{A,B,C}(\boldsymbol{x})=\frac{1}{ABC}\int\limits_{-A/2}^{A/2}\int\limits_{-B/2}^{B/2}\int\limits_{-C/2}^{C/2} u(x_1+\xi_1, x_2+\xi_2, x_3+\xi_3)\times \times d\xi_1\, d\xi_2\, d\xi_3. \quad (4.64)$$

We must determine the conditions under which $\tilde{u}_{A,B,C}(\boldsymbol{x})$, as $A\rightarrow\infty$, $B\rightarrow\infty$, $C\rightarrow\infty$ (or at least one of these limiting processes occurs) will tend in some definite sense to the probability mean $\overline{u(\boldsymbol{x})}$. For this to occur, we first require that a condition analogous to Eq. (4.59) will be satisfied:

$$\overline{u(\boldsymbol{x})}=U=\text{const}, \quad (4.65)$$

i.e., the probability mean must be identical at all points of space. In exactly the same way, if it is possible to determine the correlation function $\overline{u(\boldsymbol{x}_1)u(\boldsymbol{x}_2)}=B_{uu}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ with the aid of space-averaging, then it is necessary in every case that this function depend only on the vector difference of its arguments:

$$B_{uu}(\boldsymbol{x}_1, \boldsymbol{x}_2)=B_{uu}(\boldsymbol{x}_2-\boldsymbol{x}_1). \quad (4.66)$$

Finally, if the mean values of all possible functions of the values of the field at several points of space may be obtained by space averaging, it is necessary that the N-dimensional probability density of $u(\boldsymbol{x}_1), u(\boldsymbol{x}_2), \ldots, u(\boldsymbol{x}_N)$ for any N and $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ should depend only on the differences $\boldsymbol{x}_2-\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N-\boldsymbol{x}_1$, i.e., it should not change with any parallel displacement of the system of points $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ by addition of the same vector $\boldsymbol{y}$ to each:

$$p_{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N}(u_1, u_2, \ldots, u_N)=p_{\boldsymbol{x}_1+\boldsymbol{y}, \boldsymbol{x}_2+\boldsymbol{y}, \ldots, \boldsymbol{x}_N+\boldsymbol{y}}(u_1, u_2, \ldots, u_N)= =p_{\boldsymbol{x}_2-\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N-\boldsymbol{x}_1}(u_1, u_2, \ldots, u_N). \quad (4.67)$$

For a Gaussian field $u(\boldsymbol{x})$, Eq. (4.67) follows from Eqs. (4.65) and (4.66). A field $u(\boldsymbol{x})$ which satisfies the condition (4.67) is called a *statistically homogeneous* field (or, briefly, a *homogeneous* field). Thus, for space-averaging of any functions of the values of a random field to lead to the same results as probability-averaging, the field in question must be homogeneous. When we consider more general functions of values of several fields $u_1(\boldsymbol{x}), u_2(\boldsymbol{x}), \ldots, u_n(\boldsymbol{x})$, the requirement is now that the multidimensional (vector) field $\boldsymbol{u}(\boldsymbol{x}) = \{u_1(\boldsymbol{x}), \ldots, u_n(\boldsymbol{x})\}$ should be homogeneous, i.e., that all probability densities of arbitrary collections of the values of the components $u_1(\boldsymbol{x}), \ldots, u_n(\boldsymbol{x})$ of the field $\boldsymbol{u}(\boldsymbol{x})$ at some set of points of space will not vary when all these points are displaced by the same vector $\boldsymbol{y}$.

With reference to fields of fluid dynamic variables in a turbulent flow, the assumption of homogeneity is always a mathematical idealization, and can never be satisfied. In fact, for us to be able to speak of homogeneity, the flow should fill an entire unbounded space, and this postulate, applied to real flows, is always an idealization. Further, we should require that all the mean values of the flow (mean velocity, pressure and temperature) should be constant throughout all space and that the statistical regime of the fluctuations should not change from one point of space to another. Of course, all these requirements can be satisfied with sufficient accuracy only within some finite region of space which is small in comparison with the scales of the macroscopic inhomogeneities and is sufficiently far from all rigid boundaries of the flow (or free surfaces). Thus, in practice, we may only speak of the homogeneity of fluid dynamic fields within some definite region, and not throughout all space.[12] Nevertheless, when considering turbulent flow which is homogeneous within such a region, frequently it is convenient to regard it as part of a fully homogeneous turbulence filling all space; the value of such a postulate is connected with the considerable mathematical simplicity of the homogeneous random field model, which considerably simplifies the theoretical analysis. Thus the ergodic theorem [i.e., the theorem on the convergence of the space means $\tilde{u}_{A,B,C}(\boldsymbol{x})$ to the probability mean $\overline{u(\boldsymbol{x})}$] may be applied to a random field that is homogeneous only within a bounded region provided that the dimensions of this region are sufficiently large (cf. the remarks on stationary random functions at the end of Sect. 4.5).

One very important method of artifically creating a turbulent

[12]A random field is said to be homogeneous in a region of space G if Eq. (4.67) holds for any $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ and $\boldsymbol{y}$ such that all the points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$; $\boldsymbol{x}_1 + \boldsymbol{y}, \ldots, \boldsymbol{x}_N + \boldsymbol{y}$ lie within G.

flow that is very close to homogeneous will be discussed in Chapt. 7, Vol. 2, in connection with the study of homogeneous and isotropic turbulence. Here we shall only observe that in addition to fields which are homogeneous throughout all three-dimensional space, we may also consider fields $u(\boldsymbol{x}) = u(x_1, x_2, x_3)$ that are homogeneous only in some plane (or along some axis), i.e., satisfying Eq. (4.67) for all vectors $\boldsymbol{y}$ belonging to a given plane (or axis), but not, generally speaking, satisfying Eq. (4.67) for other $\boldsymbol{y}$. The values of such a field $u(\boldsymbol{x})$ in any plane or line parallel to the direction of homogeneity will clearly form a homogeneous field in this plane or line. It is natural to expect too that space-averaging may often be used when the fluid dynamic fields are not homogeneous in all space, but are homogeneous only in some plane (or along some line); however, in these cases, it is necessary to consider averaging, not in three-dimensional space, but only with respect to the corresponding plane or straight line (i.e., the triple integral in Eq. (4.64) must be replaced by a double or single integral).

In many problems of the theory of turbulence we may assume that the corresponding fluid dynamic fields are homogeneous, at least in one direction, or that they are stationary (or that they satisfy both these conditions simultaneously). Thus if we could prove that under conditions of homogeneity or stationarity the probability means may be replaced by time or space means, this would have very great practical significance. In fact, however, neither stationarity nor homogeneity by itself is sufficient to ensure the convergence of the time or space means to the probability mean values.[13]

However, as we shall now show, the necessary conditions for convergence have a very general character. Consequently, in applied problems, they may almost always be assumed to be satisfied.

4.7 The Ergodic Theorem

First, for definiteness, we shall discuss only stationary random processes $u(t)$ and time-averaging (exactly the same discussion, of

[13]To verify this assertion, it is sufficient to consider the example of a stationary process $u(t)$ for which, with probability equal to unity, $u(t) = u(0)$ for any t (so that almost all realizations of the process will be represented by straight lines parallel to the t-axis). Here, clearly, $\widetilde{u}_T(t) = u(0)$ for all T, and if $u(0)$ has some probability distribution, then for the mean value $\overline{u(t)} = \overline{u(0)}$ to be defined, it is necessary to have a large number of different realizations of the function $u(t)$ [each of which reduces to a unique value of $u(0)$]. In applied questions, however, this will always be considered as an example of a random variable $u = u(0)$, but not a random function (incidentally, the multidimensional probability densities $p_{t_1 t_2 \ldots t_N}(u_1, u_2, \ldots, u_N)$ will not be ordinary functions in this case, but will be expressed in terms of the Dirac δ-function).

course, with the time coordinate t replaced by the space coordinate x, will apply to a homogeneous random field $u(x)$ on a straight line). We begin by defining quite clearly what we mean by the convergence of the random variables $\tilde{u}_T(t)$, defined by Eq. (4.57), as $T \to \infty$ to a constant $\overline{u(t)} = U$. It will be convenient to consider random variables $\tilde{u}_T$ to converge, as $T \to \infty$, to a limit U (generally speaking, a random variable), if

$$\lim_{T \to \infty} \overline{|\tilde{u}_T - U|^2} = 0. \tag{4.68}$$

By Chebyshev's inequality

$$P\{|\tilde{u}_T - U| > \varepsilon\} \leqslant \frac{\overline{|\tilde{u}_T - U|^2}}{\varepsilon^2} \tag{4.69}$$

it also follows from Eq. (4.68) that

$$\lim_{T \to \infty} P\{|\tilde{u}_T - U| > \varepsilon\} = 0, \tag{4.70}$$

i.e., that the probability of the deviation of $\tilde{u}_T$ from U exceeding any given ε will tend to zero as $T \to \infty$ (and hence can be as small as desired, provided that the value of T is chosen sufficiently large).[14]

Equation (4.70) gives us sufficient justification for using in practice the value of $\tilde{u}_T$ instead of U, where T is comparatively large; thus it remains only to find under what conditions Eq. (4.68) will hold.

It is easy to see that the left side of Eq. (4.68) where $\tilde{u}_T$ is the time mean (4.57) of the values of a stationary random process $u(t)$, while $U = \overline{u(t)}$–its probability mean value–may be expressed in terms of the correlation function of fluctuations $u(t)$

$$b_{uu}(\tau) = \overline{[u(t+\tau) - U][u(t) - U]} = B_{uu}(\tau) - U^2 \tag{4.71}$$

[see Eq. (4.83) below]. Therefore, the correctness of Eq. (4.68) must be determined by some properties of the function $b_{uu}(\tau)$. From the viewpoint of applications to turbulence theory the following fact is of fundamental significance: if $b_{uu}(\tau) \to 0$ as $\tau \to \infty$, then the

[14] In probability theory, Eq. (4.68) is generally called the condition of convergence in quadratic mean of $\tilde{u}_T$ to U as $T \to \infty$, and Eq. (4.70) is called the condition of convergence in probability. Thus using Chebyshev's inequality, we have shown that from the convergence in quadratic mean of $\tilde{u}_T$ to U, there follows also its convergence in probability to the same limit.

convergence of $\tilde{u}_T(t)$ to the probability mean value is bound to occur. This is a corollary of the following general theorem, first proved by Slutskiy (1938): *in order that Eq. (4.68) be satisfied for a stationary random process $u(t)$, it is necessary and sufficient that the correlation function $b_{uu}(\tau)$ satisfy the condition*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T b_{uu}(\tau)\, d\tau = 0 \ . \tag{4.72}$$

The proof of this theorem, normally called the law of large numbers or the ergodic theorem[15] for stationary random processes, is given at the end of this subsection. It is clear that if $b_{uu}(\tau) \to 0$ as $\tau \to \infty$, then the condition (4.72) will certainly be satisfied; thus it follows that in this case, Eq. (4.68) will always be correct. Since the correlation function $b_{uu}(\tau)$ of the fluctuations of the fluid dynamic variable $u(t)$ of the turbulent flow can always be assumed to tend to zero as $\tau \to \infty$ (see above, Sect. 4.2), in the theory of turbulence we may always take as our starting point the fact that for steady flows the probability mean values of any fluid dynamic fields may be determined by averaging over a sufficiently large interval of time.

If the rate of decrease of $b_{uu}(\tau)$ with increase of τ is such that

$$\int_0^\infty b_{uu}(\tau)\, d\tau < \infty, \tag{4.73}$$

then there also exists a simple estimate of the length T of those intervals, averaging over which is sufficient for the mean value U to be obtained with sufficient accuracy. It may be shown that here, with sufficiently large T, the asymptotic equation

$$\overline{|\tilde{u}_T - U|^2} \approx 2\,\frac{T_1}{T}\, b_{uu}(0) \tag{4.74}$$

holds, where

$$T_1 = \frac{1}{b_{uu}(0)} \int_0^\infty b_{uu}(\tau)\, d\tau \tag{4.75}$$

[15] To avoid misunderstanding, we should point out that in probability theory the name "ergodic theorem for stationary random processes" is sometimes given to other propositions, relating to the connection between the probability means and the time means.

is a constant with dimensions of time, which may be called the "correlation time" (or the "integral time scale") of a stationary function [Eq. (4.74)], and was first obtained by Taylor (1921). Thus, for a reliable determination of U, it is only necessary to use time-averaging over a period T much greater than the corresponding "correlation time" T_1. After selecting the desired accuracy (i.e., the greatest permissible mean square error, when U is replaced by $\tilde{u}_T$), we may use Eq. (4.74) to determine the requisite averaging time T.[16]

We will now discuss briefly the homogeneous random fields $u(\boldsymbol{x})$ and space-averaging. The case of homogeneous fields on a straight line does not differ, in general, from the case of stationary processes. For homogeneous random fields in a plane or in a space, the values of such a field on any straight line will form a homogeneous random field on that line. Thus, provided the correlation function $b_{uu}(\boldsymbol{r})$ of the fluctuations of this field is such that for at least one direction (unit vector) $\boldsymbol{r}_0$ the function $b_{uu}(\tau\boldsymbol{r}_0)$ of τ satisfies Eq. (4.72) [e.g., if $b_{uu}(\boldsymbol{r}) \to 0$ as $|\boldsymbol{r}| \to \infty$ in at least one direction], then the space mean (or plane mean) will certainly converge to a constant $\overline{u(\boldsymbol{x})} \equiv U$ (this convergence will hold even for averaging along a single parallel to $\boldsymbol{r}_0$). The general necessary and sufficient condition for the convergence in quadratic mean of $\tilde{u}_{A,B,C}(\boldsymbol{x})$ for all $\boldsymbol{x}$ to U as $A \to \infty$, $B \to \infty$, $C \to \infty$ for homogeneous fields in space, takes the form

$$\lim_{A\to\infty,\ B\to\infty,\ C\to\infty} \frac{1}{4ABC} \int_0^A \int_{-B}^{B} \int_{-C}^{C} b_{uu}(\boldsymbol{r})\, dr_1\, dr_2\, dr_3 = 0 \qquad (4.76)$$

(the changes necessary for homogeneous fields on a plane are obvious); the proof of this condition differs little from the analogous proof for the one-dimensional case. Instead of Eq. (4.74) we shall now have

$$\overline{[\tilde{u}_V - U]^2} \approx 2\frac{V_1}{V} b_{uu}(0), \qquad (4.74')$$

where $\tilde{u}_V$ is the mean over the volume $ABC = V$, and V_1 is the "correlation volume" (or "integral volume scale")

[16]We may note here that the right side of Eq. (4.74) is completely analogous to the expression for the mean square error of the arithmetic mean of N independent measurements, if we put $N = T/2T_1$. Thus, averaging a stationary function with correlation time T_1 over a period $2NT_1$ is equivalent to averaging over N independent measurements of this function, i.e., over N samples.

$$V_1 = \frac{1}{b_{uu}(0)} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b_{uu}(\boldsymbol{r})\, dr_1\, dr_2\, dr_3, \tag{4.75'}$$

which is assumed finite in Eq. (4.74′).

The results above may also be applied to the calculation of higher-order moments or other probability mean values of any functions of values of $u(t)$ or $u(\boldsymbol{x})$ using time- or space-averaging. Here we only have to replace the process $u(t)$ or the field $u(\boldsymbol{x})$ by a new process or field which is a nonlinear function of the original one. For example, in the case of a stationary process $u(t)$, for it to be possible to obtain the Nth-order moment (4.62) by averaging with respect to time τ, the variables $u(t_1 + \tau)\, u(t_2 + \tau) \ldots u(t_N + \tau)$, we require only that the $2N$th-order moment

$$\begin{aligned} B_{u \ldots uu \ldots u}(t_2 - t_1, \ldots, t_N - t_1, \tau, t_2 - t_1 + \tau, \ldots, t_N - t_1 + \tau) = \\ = \overline{u(t_1) \ldots u(t_N)\, u(t_1 + \tau) \ldots u(t_N + \tau)} - [\overline{u(t_1) \ldots u(t_N)}]^2 \end{aligned} \tag{4.77}$$

for fixed $t_1, t_2, \ldots, t_N$ be a function of τ, satisfying Eq. (4.72). Since it is normally asserted on physical grounds that the correlation coefficient between $u(t_1) \ldots u(t_N)$ and $u(t_1 + \tau) \ldots u(t_N + \tau)$ tends to zero as $\tau \to \infty$, then, in practice, the necessary condition generally may be assumed to be satisfied, and the use of time-averaging will be justified. To estimate the necessary averaging time, it is necessary to estimate the corresponding "correlation time," i.e., the integral of the correlation coefficient. For Gaussian processes $u(t)$ the moment (4.77) may be expressed in terms of the correlation function and the mean value of the process $u(t)$, using the general rule (4.28) for evaluating higher moments. In particular, in the determination of the mean square $\overline{u^2(t)} = B_{uu}(0)$ by time-averaging for a Gaussian random process $u(t)$ with zero mean, the role of the correlation function $b_{uu}(t)$ will be played by

$$\overline{[u^2(t + \tau) - \overline{u^2(t)}]\,[u^2(t) - \overline{u^2(t)}]} = 2b_{uu}^2(\tau). \tag{4.78}$$

Thus Eqs. (4.74)–(4.75) here take the form

$$\overline{[\widetilde{u_T^2} - b_{uu}(0)]^2} \approx \frac{4T_2}{T} b_{uu}^2(0), \qquad T_2 = \frac{1}{b_{uu}^2(0)} \int_0^{\infty} b_{uu}^2(\tau)\, d\tau. \tag{4.79}$$

We note that in this case, from the condition

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T [b_{uu}(\tau)]^2\,d\tau=0, \tag{4.80}$$

[obtained from Eq. (4.72) by substituting for $b_{uu}(\tau)$ from (4.78)], the initial condition (4.72) also follows. Moreover, for a Gaussian stationary random process, it will follow from Eq. (4.80) that the probability mean value of any function of the process values (having finite probability mean) will equal the limit as $T\to\infty$ of the corresponding time mean value for time T [see, for example, Grenander (1950)]. A generalization of this last result to Gaussian homogeneous random fields (and to some even more general random functions) was formulated in Tempel'man's note (1962); some weaker results on such random fields were probed in the paper by Birkhoff and Kampé de Fériet (1962).

To prove that if the condition (4.72) is satisfied Eq. (4.68) must hold, we express the mean square of the difference $\widetilde{u}_T(t)-U$ in terms of the correlation function $b_{uu}(\tau)$:

$$\begin{aligned}\overline{[\widetilde{u}_T-U]^2}&=\frac{1}{T^2}\overline{\left\{\int_{-T/2}^{T/2}[u(t)-U]\,dt\right\}^2}=\\&=\frac{1}{T^2}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2}\overline{[u(t)-U][u(s)-U]}\,dt\,ds=\\&=\frac{1}{T^2}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2}b_{uu}(t-s)\,dt\,ds=\frac{2}{T^2}\int_0^T\int_0^{\tau_1}b_{uu}(\tau)\,d\tau\,d\tau_1\end{aligned} \tag{4.81}$$

[having made the substitutions $t-s=\tau$, $t+s=\tau_1$ and taking into account that $b_{uu}(-\tau)=b_{uu}(\tau)$]. If $b_{uu}(\tau)$ satisfies Eq. (4.72) then for any $\delta>0$ there will exist $T_0=T_0(\delta)$ such that

$$\int_0^{\tau_1} b_{uu}(\tau)\,d\tau<\frac{\delta}{2}\tau_1 \quad \text{for} \quad \tau_1>T_0. \tag{4.82}$$

On the other hand, since $|b_{uu}(\tau)|\leqslant b_{uu}(0)$, for any τ_1

$$\int_0^{\tau_1} b_{uu}(\tau)\,d\tau<b_{uu}(0)\,\tau_1. \tag{4.83}$$

Integrating Eq. (4.83) with respect to τ_1 from zero to T_0 and Eq. (4.82) from T_0 to some $T > T_0$, for any $T > T_0$ we find that

$$\int_0^T \int_0^{\tau_1} b_{uu}(\tau)\, d\tau\, d\tau_1 < \frac{b_{uu}(0)\, T_0^2}{2} + \frac{\delta\left(T^2 - T_0^2\right)}{4}. \tag{4.84}$$

Consequently,

$$\int_0^T \int_0^{\tau_1} b_{uu}(\tau)\, d\tau\, d\tau_1 < \frac{\delta}{2} T^2 \qquad \text{for} \qquad \frac{b_{uu}(0)\, T_0^2}{2} < \frac{\delta T^2}{4},$$

i.e., for

$$T > \sqrt{\frac{2b_{uu}(0)}{\delta}}\, T_0.$$

Thus, for any $\delta > 0$, for $T > T_0$ and $T > \sqrt{\frac{2b_{uu}(0)}{\delta}}\, T_0$

$$\overline{[\tilde{u}_T - U]^2} = \frac{2}{T^2} \int_0^T \int_0^{\tau_1} b_{uu}(\tau)\, d\tau\, d\tau_1 < \delta, \tag{4.85}$$

i.e., Eq. (4.68) will hold.

If the limit shown on the left side of Eq. (4.72) exists but is equal to $c \neq 0$, then Eq. (4.72) will be satisfied by the function $b_1(\tau) = b_{uu}(\tau) - c$. Therefore, in this case it is clear that

$$\lim_{T\to\infty} \overline{[\tilde{u}_T - U]^2} = \lim_{T\to\infty} \frac{2}{T^2} \int_0^T \int_0^{\tau_1} b_{uu}(\tau)\, d\tau\, d\tau_1 =$$

$$= \lim_{T\to\infty} \frac{2}{T^2} \int_0^T \int_0^{\tau_1} [b_1(\tau) + c]\, d\tau\, d\tau_1 = \lim_{T\to\infty} \frac{2}{T^2} \int_0^T \int_0^{\tau_1} b_1(\tau)\, d\tau\, d\tau_1 + c = c,$$

i.e., $\tilde{u}_T$ does not converge to U as $T \to \infty$. It may be shown, however, that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T b_{uu}(\tau)\, d\tau$$

for the correlation coefficient $b_{uu}(\tau)$ will always exist (see, for example, Chapt. 6, Vol. 2 of this book). Consequently, it follows that when condition (4.72) breaks down, the time means $\tilde{u}_T$ will not converge to the corresponding probability mean $\overline{u(t)} = U$ as $T \to \infty$.

If the correlation function $b_{uu}(\tau)$ satisfies not only Eq. (4.72) but also the stronger condition (4.73), then Eq. (4.74) follows easily from Eq. (4.81).

We note that from Eq. (4.68), generally speaking, it still does not follow that for any realization of the random process the time means $\tilde{u}_T$ will actually tend, as $T \to \infty$, to a limit which is identical to $\overline{u(t)} = U$. Equation (4.68) would not be contradicted, if for any realization we sometimes encountered values of T as large as desired for which $\tilde{u}_T$ was distant from U (so that for individual realizations the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T u(t)\, dt$$

simply did not exist). However, it may be shown that for any stationary process $u(t)$ with $\overline{|u(t)|} < \infty$ for almost all realizations (i.e., for all realizations except perhaps some "singular" realizations having total probability equal to zero) the values of the time mean $\tilde{u}_T$ will tend to a definite limit as $T \to \infty$. (This is called the Birkhoff-Khinchin ergodic theorem; see, for example, Doob (1953), Chapt. XI, Sect. 2, Rozanov (1967), Chapt. IV, Sect. 5.) But it follows from this that in all cases, when Eq. (4.68) is satisfied [i.e., when the correlation function $b_{uu}(\tau)$ satisfies Eq. (4.72)] the limit of the time means formulated for individual realizations of the process $u(t)$, with probability one exist and are equal to $\overline{u(t)}$.

3 REYNOLDS EQUATIONS AND THE SEMIEMPIRICAL THEORIES OF TURBULENCE

5. TURBULENT SHEAR FLOWS IN TUBES, BOUNDARY LAYERS, ETC.

5.1 Reynolds Equations

The study of turbulent fluid flows naturally begins with the flows in a circular tube and in the boundary layer on a flat plate. This is because these are the easiest cases to reproduce in the laboratory and have great importance for many engineering problems. The abundant experimental material which has been collected on such flows permits us to consider them as a standard against which we may verify various theories and hypotheses on the nature of turbulence. The discussion of the basic data on the most important integral characteristics of flows in tubes and boundary layers, i.e., the longitudinal velocity profile, the flow rate and the skin friction law, will also occupy a central place in this section. We shall also discuss briefly the case of "free turbulence" in which no perceptible effect is produced by solid walls and in conclusion we shall consider certain hypotheses on turbulent flows which are widely used in practical calculations. First, however, we must present O. Reynolds' general considerations (1894) relating to arbitrary turbulent flows, which constitute the basis of the whole theory of turbulence.

We have already stated that the fluid dynamic fields of velocity, pressure, temperature, etc., in a turbulent flow are so complex in structure that it is practically impossible to describe them individually. Therefore, initially, we must consider the whole ensemble of such flows and study only its averaged statistical characteristics, assuming that all the fluid dynamic fields concerned are random in the sense explained in Sect. 3.2. Henceforth, we shall always assume that such an approach is possible; i.e., we shall *define as turbulent* only those flows for which there exists a statistical ensemble of similar flows, characterized by some probability distribution (with continuous density) for the values of all possible fluid dynamic variables. In this connection, we must stress that the ordinary definition of turbulent flows simply as flows accompanied by disordered fluctuations of all the fluid dynamic variables, is quite insufficient for the formulation of a mathematical theory of turbulence. On the other hand, if a statistical ensemble of flows exists, then the corresponding statistical description of all the fluid dynamic variables will not be "incomplete" even from a purely practical viewpoint. This is because knowledge of every detail of the very confused individual flow is never required for any application, and it is only the mean characteristics which are of interest. Of course, in practice, it is not ensemble averaging which is usually employed, but time- or space-averaging. Thus from a practical viewpoint, it is also necessary that the random fields of the fluid dynamic variables possess some ergodic properties. Henceforth, we shall always assume this last condition to be satisfied without making special mention of it.

The most important, and at the same time, the simplest statistical characteristics of random fluid dynamic fields are their mean values. The differences $u' = u - \bar{u}$ between individual values of the field $\bar{u}$ and its mean value u are naturally called *fluctuations* of the field u. The decomposition of all the fluid dynamic variables into mean values and fluctuations played a fundamental role in Reynolds' original considerations, and also in almost all subsequent investigations of turbulence.

The mean values of the fluid dynamic variables generally prove to be very smooth and slowly varying; on the other hand, the fluctuations are characterized by great variation in time and space. Generally speaking, it might be assumed that turbulent inhomogeneities have arbitrarily small scales and periods, down to scales comparable with the length of the mean free path of a molecule and periods comparable with the mean interval between two successive molecular collisions. If this is so, then the use in the theory of turbulence of the ordinary concepts and methods of continuum mechanics (in particular, the differential equations of fluid dynamics) will, of course, be invalid. However, by experiment, turbulent inhomogeneities never have such small space-time dimensions as these. This is explained by the fact that for inhomogeneities as small as these, the corresponding velocity gradients would be extremely large; hence for very small-scale motions the energy loss in overcoming the viscous friction would be so great as to make the existence of such motion impossible. Consequently, the minimum scales and periods of turbulent inhomogeneities will always exceed the scales and periods of molecular motion by several orders of magnitude. More precisely, the length-scales of the smallest inhomogeneities observed in real turbulent flows of air and water are of the order of several millimeters, or, in extreme cases, of tenths of a millimeter (see, e.g., Volume 2, Chapt. 8) while under normal conditions, the length of the mean free path of air molecules is of the order of 10^{-5} cm, and for molecules of water, it is of even smaller order. As far as orders of magnitude are concerned, since the velocity of fluid flows does not exceed the mean velocity of thermal motion of molecules (close to 5×10^4 cm/sec, or of the order of sound velocity), the characteristic periods of turbulent fluctuations will always exceed the mean time between two successive molecular collisions by several orders of magnitude. At distances comparable to the dimensions of minimum inhomogeneities and for time intervals comparable to the minimum periods of the fluctuations, all the fluid dynamic variables will vary smoothly, and may be described by differentiable functions. Thus it follows that the description of turbulent flows by means of the usual differential equations of fluid dynamics is completely justified.

However, it is impossible to apply these equations directly, if only because the fluid dynamic fields in turbulent flows are always unsteady and depend strongly on the finest details of the initial conditions, details which are never known with sufficient precision. Moreover, even if the initial conditions are known exactly, the solution of the problem with them is excessively cumbersome and of no practical use because of instability with respect to small disturbances of the initial data. However, this does not mean that the equations of fluid dynamics can never be used in the study of turbulence. Due to the fact that the individual realizations of the fluid dynamic fields of a turbulent flow will satisfy given differential equations, the statistical characteristics of these fields will be connected with a number of important relationships, which are of paramount importance in the theory of turbulence.

The simplest connection of this kind was established by Reynolds by means of direct averaging of the equations of fluid dynamics of an incompressible fluid. Let us take as our starting point the equation of balance of momentum (i.e., the Navier-Stokes equations (1.6) multiplied by 3) in which it will now be convenient to transform the term $u_\alpha \frac{\partial u_i}{\partial x_\alpha}$ with the aid of the continuity equation into the form $\frac{\partial}{\partial x_\alpha}(u_i u_\alpha)$. We apply the averaging operation to all terms of this equation and use the fact that this operation commutes with the operations of space and time differentiation [Eq. (3.6)] and also the equation

$$\overline{u_i u_j} = \overline{(\bar{u}_i + u'_i)(\bar{u}_j + u'_j)} = \bar{u}_i \bar{u}_j + \overline{u'_i u'_j},$$

which follows directly from Eqs. (3.3) and (3.7′). In this case we obtain the equations

$$\frac{\partial \rho \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_\alpha}(\rho \bar{u}_i \bar{u}_\alpha + \rho \overline{u'_i u'_\alpha}) = \rho \bar{X}_i - \frac{\partial \bar{p}}{\partial x_i} + \rho\nu \nabla^2 \bar{u}_i, \quad i = 1, 2, 3, \tag{5.1}$$

which are usually called the *Reynolds equations*. These equations contain only smoothly varying averaged quantities; thus their use does not give rise to any difficulties connected with the complexity and irregularity of the fluid dynamic variables of turbulent flows. However, another difficulty does arise. It is connected with the presence in the Reynolds equations of the new unknowns $\tau_{ij}^{(1)} = -\rho\overline{u'_i u'_j}$ [the reason for including the minus sign will be explained below], which characterize the fluctuating component of the velocity field. The appearance of these new unknowns is evidently due to the nonlinearity of the equations of fluid dynamics. Of course, when linear equations are averaged, no new terms arise. Thus, for example, the averaged continuity equation will take the simple form

$$\frac{\partial \bar{u}_\alpha}{\partial x_\alpha} = 0. \tag{5.2}$$

To understand the physical meaning of the additional term $-\rho\overline{u'_i u'_j}$ in Eqs. (5.1), let us consider the mean value of the momentum flux density

$$\overline{\rho u_i u_j + p\delta_{ij} - \sigma_{ij}} = \rho\bar{u}_i\bar{u}_j + \bar{p}\delta_{ij} - (\bar{\sigma}_{ij} - \rho\overline{u'_i u'_j}),$$

where $\sigma_{ij} = \rho\nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is the viscous stress tensor in an incompressible fluid. This expression shows that in relation to the averaged motion the role of the viscous stress tensor is played by the tensor $\tau_{ij} = \bar{\sigma}_{ij} - \rho\overline{u'_i u'_j} = \bar{\sigma}_{ij} + \tau^{(1)}_{ij}$; to obtain the sum here, the minus sign was introduced into the expression for $\tau^{(1)}_{ij}$. Thus, in turbulent flow, in addition to the exchange of momentum between fluid particles due to the forces of molecular viscosity (described by the viscous stress tensor) the transport of momentum from one volume of fluid to another occurs, induced by the mixing caused by the velocity fluctuations. In other words, the effect of turbulent mixing on the averaged motion is similar to that of an increase of viscosity; to emphasize this, the Reynolds equations are sometimes written as:

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_\alpha \frac{\partial \bar{u}_i}{\partial x_\alpha} = \bar{X}_i - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_\alpha}\left(\nu\frac{\partial \bar{u}_i}{\partial x_\alpha} - \overline{u'_i u'_\alpha}\right). \tag{5.1$'$}$$

The terms $-\rho\overline{u'_i u'_j}$ in Eqs. (5.1) for the averaged motion have the physical meaning of components of the tensor of the additional stresses which are caused by the turbulent fluctuations, just as in ordinary fluid mechanics, microscopic molecular motions lead to the appearance of viscous stresses. These additional stresses are called *Reynolds stresses* in turbulence theory.

The component describing the transfer of the momentum of the fluid to a solid body washed by the flow is of particular interest in studying the Reynolds stress tensor. Let Σ be some small area on the surface of such a body, which may be considered to be approximately plane and to coincide with part of the plane $x_3 = 0$. Further, we assume that the direction of the mean flow in the neighborhood of this plane is parallel to the Ox_1 axis. In this case, the frictional forces acting on the area Σ will also be directed along the Ox_1 axis. The magnitude of the frictional force acting on the unit area of the surface of the body will be equal to the density of flux of the x_1-component of momentum in the direction Ox_3 taken at a point of the surface; i.e., it will be defined by the equation

$$\tau_0 = \left(\bar{\sigma}_{13} - \rho\overline{u_1' u_3'}\right)_{x_3=0}.$$

The variable τ_0 is called the shear stress *at the wall.* Since in a rigid wall $\bar{u}_i$ and u_i' and their derivatives with respect to x_1 are evidently equal to zero, while

$$\bar{\sigma}_{13} = \rho\nu\left(\frac{\partial \bar{u}_1}{\partial x_3} + \frac{\partial \bar{u}_3}{\partial x_1}\right),$$

τ_0 may also be written in the following simpler form:

$$\tau_0 = \rho\nu \left.\frac{\partial \bar{u}_1}{\partial x_3}\right|_{x_3=0} \tag{5.3}$$

and may be called the *viscous shear stress* at the wall. However, if we choose an area, parallel to Σ, above the surface of the body, then the total stress on it will be defined by

$$\tau = \bar{\sigma}_{13} - \rho\overline{u_1' u_3'}. \tag{5.4}$$

In the important special case when the mean velocity is everywhere directed along the Ox_1 axis, τ will be equal to the *shear stress*

$$\tau = \rho\nu \frac{\partial \bar{u}_1}{\partial x_3} - \rho\overline{u_1' u_3'}. \tag{5.4'}$$

In this case, according to the ideas of Boussinesq (1877; 1897), we can write, formally,

$$-\rho\overline{u_1' u_3'} = \rho K \frac{\partial \bar{u}_1}{\partial x_3}, \tag{5.5}$$

where K is a new physical variable with dimensionality L^2T^{-1}, called the coefficient of eddy viscosity (or of *turbulent viscosity*, or of *momentum transport*). The sum $\nu + K$ may therefore be called the *total* (or *effective viscosity*) of the turbulent flow. Unlike the ordinary (molecular) viscosity ν, the eddy viscosity K does not describe any physical properties of the fluid, but characterizes the statistical properties of the fluctuating motion. Consequently, it does not have to be a constant, but may vary in space and time, and may, in principle, even take negative values (see below, Sect. 6.3).[1] It is important initially to stress that the eddy viscosity K normally is considerably greater than the molecular viscosity ν; hence, at a distance from the rigid boundaries, the total shear stress (defined by the flux of the x_1-component of momentum in the direction of the Ox_3 axis) may be evaluated from the equation

$$\tau = \tau^{(1)} = -\rho \overline{u_1' u_3'}. \tag{5.6}$$

The Reynolds equations (5.1) are equations of the balance of momentum of the mean motion; the Reynolds stresses which occur in them describe the turbulent transport of this momentum. Similar balance equations may also be obtained for an arbitrary scalar conservative quantity transferred by the fluid (e.g., for heat or a passive substance like water-vapor, carbon dioxide, smoke or dust in the atmosphere). Applying the averaging procedure to Eq. (1.72), which describes the transport of the scalar quantity ϑ in an incompressible fluid, we obtain

$$\frac{\partial \overline{\vartheta}}{\partial t} + \frac{\partial}{\partial x_\alpha}\left(\overline{u}_\alpha \overline{\vartheta} + \overline{u_\alpha' \vartheta'}\right) = \chi \nabla^2 \overline{\vartheta}, \tag{5.7}$$

or, otherwise,

$$\frac{\partial \overline{\vartheta}}{\partial t} + \overline{u}_\alpha \frac{\partial \overline{\vartheta}}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}\left(\chi \frac{\partial \overline{\vartheta}}{\partial x_\alpha} - \overline{u_\alpha' \vartheta'}\right) \tag{5.7'}$$

[cf. Eqs. (5.1) and (5.1′)]. If ϑ denotes the temperature, then the vector $c_p \rho \overline{\vartheta} \overline{u}_i$ describes the advective heat transport by the mean

[1] Here we should note that the eddy viscosity, by definition, depends also on the averaging procedure used, i.e., on the choice of the statistical ensemble of similar flows, the assignment of which must be included in the definition of a given turbulent flow. This is especially important for "natural" turbulence in the atmosphere or ocean, where usually a uniquely defined natural "ensemble of similar flows" does not exist, and where averaging over different statistical ensembles may lead to values of K that differ by several orders of magnitude.

motion $\varkappa \frac{\partial \bar{\vartheta}}{\partial x_i}$, where $\varkappa = c_p \rho \chi$ is the heat transport due to the molecular thermal conductivity, and

$$q_i = c_p \rho \overline{\vartheta' u_i'} \tag{5.8}$$

will be the density of the turbulent heat flux in the direction of the Ox_i axis. The turbulent fluctuations lead to the appearance of an additional heat flux, somewhat similar to that produced by the molecular thermal conductivity. As in Eq. (5.5) we may write, formally,

$$c_p \rho \overline{\vartheta' u_i'} = -c_p \rho K_\vartheta \frac{\partial \bar{\vartheta}}{\partial x_i}, \tag{5.9}$$

where K_ϑ plays the part of a new coefficient of thermal conductivity, called the *coefficient of eddy* (or *turbulent*) *thermal conductivity* or the *coefficient of heat transport*; generally speaking, this coefficient may differ for different coordinate axes. On the other hand, if in Eqs. (5.7) and (5.7′), ϑ represents the concentration of a passive substance, then

$$j_i = \rho \overline{\vartheta' u_i'} \tag{5.8'}$$

will denote the density of the turbulent flux of the substance in the direction of the Ox_i axis; in this case, the coefficient K_ϑ in the equation

$$\rho \overline{\vartheta' u_i'} = -\rho K_\vartheta \frac{\partial \bar{\vartheta}}{\partial x_i} \tag{5.9'}$$

is called the *coefficient of eddy* (or *turbulent*) *substance diffusivity* or *the coefficient of substance transport* [this last coefficient was introduced by Schmidt (1917; 1925)].

All the preceding remarks on eddy viscosity may, of course, be repeated for the eddy thermal conductivity and eddy diffusion coefficients. Section 6.3 discusses all these coefficients.

5.2 General Form of the Mean Velocity Profile Close to a Rigid Wall

Let us now investigate the general properties of turbulent flows close to a rigid wall, parallel to the direction of the mean flow. The results of this study will be applicable to flows in a circular tube or in a plane channel, and also to flows in a boundary layer on a flat plate (in particular, to the surface layer of the neutrally stratified atmosphere). We shall begin, first, by considering the simplified idealized case of a steady plane-parallel flow of fluid moving in the direction of the Ox axis in the half-space $z > 0$, bounded by a rigid wall, in the absence of a mean pressure gradient.

For laminar steady plane-parallel flow with zero pressure gradient, it follows from the first equation of fluid dynamics (see the first example of Sect. 1.2) that the velocity profile $u(z)$ must be linear. The first Reynolds equation for the analogous turbulent flow will take the form

$$\nu \frac{d^2\bar{u}}{dz^2} - \frac{d}{dz}\overline{u'w'} = 0, \tag{5.10}$$

which states that here, the flux of the x-component of momentum along the Oz axis (directed from the fluid to the wall) will be the same at any distance from the wall:

$$\tau(z) = \rho\nu \frac{d\bar{u}}{dz} - \rho\overline{u'w'} = \tau_0 = \text{const}, \tag{5.11}$$

where τ_0 is the viscous shear stress on the wall $z = 0$. However, this does not determine uniquely the profile of the mean velocity $\bar{u}(z)$ since, in addition to the function $\bar{u}(z)$, it contains another unknown $\overline{u'w'}$. Nevertheless, certain deductions on the possible form of the function $\bar{u}(z)$ may be obtained here by dimensional analysis. In fact, the averaged characteristics of the flow under consideration at a distance z from the wall can depend only on the shear stress τ_0, the z-coordinate, and the parameters of the fluid ν and ρ. Moreover, τ_0 and ρ can enter into the kinematic characteristics of the flow only in the combination $\frac{\tau_0}{\rho}$, which is independent of the dimension of mass.

Instead of the ratio $\frac{\tau_0}{\rho}$ it is convenient to use the quantity

$$u_* = \sqrt{\frac{\tau_0}{\rho}}, \tag{5.12}$$

which has the dimensions of velocity and is therefore a natural scale of velocity for the flow close to the wall; we shall refer to this as the *friction velocity*. Since only one dimensionless combination $\frac{zu_*}{\nu}$ can be formulated from u_*, ν, and z, the general form of the dependence of the mean velocity profile $\bar{u}(z)$ on z, τ_0, ρ, and ν may be written in the form

$$\bar{u}(z) = u_* f\left(\frac{zu_*}{\nu}\right), \tag{5.13}$$

or, equivalently, in the form

$$\bar{u}_+ = f(z_+), \quad \text{where} \quad \bar{u}_+ = \frac{\bar{u}}{u_*}, \quad z_+ = \frac{zu_*}{\nu}. \tag{5.13$'$}$$

Here $\bar{u}_+$ and z_+ are the dimensionless velocity and distance from the wall, and $f(z_+)$ is some universal function of a single variable. The important result (5.13) in turbulence theory is called the universal *law of the wall*; it was first formulated by Prandtl (1925) [see also, Prandtl (1932b)].

Equations (5.13) and (5.14) will of course be vahd only when the wall may be assumed smooth, i.e., described by the simplified equation $z = 0$. Since the only length scale in the wall turbulence will be the *friction length* $z_* = \frac{\nu}{u_*}$, we can now give a quantitative explanation of the requirement of smoothness: the wall will be dynamically smooth if the mean height h_0 of the protrusions on it satisfies the condition

$$h_0 \lesssim z_* = \frac{\nu}{u_*} \tag{5.14}$$

(for more precise data, see Sect. 5.4). Only in this case will the velocity profile close to the wall be defined by Eq. (5.13). However, for a rough wall which has protrusions with heights that do not satisfy Eq. (5.14), these protrusions will also influence the distribution of the mean velocity close to the wall. Consequently, in this case Eq. (5.13) must be replaced by the more general equation

$$\bar{u}(z) = u_* f\left(\frac{zu_*}{\nu}, \frac{h_0 u_*}{\nu}, \alpha, \beta, \ldots\right), \tag{5.15}$$

where α, β, ... are dimensionless parameters characteristic of the form of the protrusions and their distribution over the surface of the wall. Again, for extremely rough walls, the use of Eq. (5.15) causes a further difficulty connected with the fact that here it is unclear from what level the height z is to be measured; we shall return to this question later in a more detailed discussion of flows near rough walls.

Before considering the form of the function f, let us discuss briefly the applicability of Eqs. (5.13) and (5.15) to various turbulent flows encountered in practice. We begin with a turbulent flow with zero mean pressure gradient between two parallel walls moving with respect to each other. Equations (5.10) and (5.11) in this case will be exact; thus, for example, for smooth walls, for Eq. (5.13) to be valid, it is necessary only that the distance z from one of the walls be much less than the distance H between the walls.[2] However, a flow between plane moving walls is very difficult to reproduce precisely in the laboratory; consequently to verify Eqs. (5.13) and (5.15) we must examine other more easily reproducible flows, having velocity profiles which are described by these equations.

As already observed, one example of such a flow is the air motion close to the earth's surface (over land or water) in the case of neutral thermal stratification. Of course, in the earth's atmosphere the horizontal pressure gradient is always slightly different from zero and, moreover, the wind velocity here is also affected by the coriolis forces produced by the rotation of the earth (see below, Sect. 6.6). However, in the lowest layer of air, these factors play only a small part, and cause no perceptible variation of the shear stress τ with height; simple estimates, which we shall give at the end of Sect. 6.6, show that in the atmosphere, right up to a height of about 50 m, τ can usually be taken to be practically constant. Consequently, within this layer, for neutral stratification, we have sufficient grounds for using the simphfied equati›ns (5.11) and (5.15) [we note that in the real atmosphere the underlying land surface is almost always rough, since u_* is generally of the order of 10 cm/sec and, consequently, z_* does not exceed some tenths of a millimeter].

[2] Strictly speaking, for turbulent flow between moving walls, an equation of the form $\bar{u}(z) = u_* f\left(\frac{zu_*}{\nu}, \frac{z_1 u_*}{\nu}\right)$ must apply, where $z_1 = H - z$ is the distance to the sccond bounding wall. For $z_1 \gg z$, it is permissible to take $\frac{z_1 u_*}{\nu} = \infty$, and this equation then reduces to Eq. (5.13).

Let us now discuss laboratory turbulent flows in tubes, channels and boundary layers over a plate. In the simplest case of a wide plane channel, the flow may be considered approximately as a steady plane-parallel flow between two infinite planes $z = 0$ and $z = H = 2H_1$, homogeneous in the planes $z =$ const. In this case, the first and third of the Reynolds equations will take the form

$$\frac{\partial \tau}{\partial z} = \frac{\partial \bar{p}}{\partial x}, \qquad \rho \frac{\partial \overline{w'^2}}{\partial z} = -\frac{\partial \bar{p}}{\partial z} \tag{5.16}$$

where, as usual, $\tau = \rho\nu \frac{d\bar{u}}{dz} - \rho\overline{u'w'}$. From the second equation of Eq. (5.16) it follows that $\bar{p} + \rho\overline{w'^2} = \bar{p}_0$ depends only on x ($\bar{p}_0$ evidently coincides with the mean pressure on the wall). But $\overline{w'^2}$ cannot depend on x; hence, using the first equation of Eq. (5.16)

$$\tau = \frac{\partial \bar{p}_0}{\partial x} z + \text{const.}$$

We denote the shear stress at the lower wall $z = 0$ by τ_0; since by symmetry the shear stress at the center of the channel (for $z = H_1$) must equal zero, the last equation may also be written as

$$\tau = \tau_0 \left(1 - \frac{z}{H_1}\right). \tag{5.17}$$

Thus, for $z \ll H_1$, we have $\tau(z) = \tau_0 =$ const; consequently, in this case also, a layer exists, contiguous to the wall of the channel, in which the shear stress is practically constant. Within this layer, the influence of the pressure gradient may be ignored, and therefore Eqs. (5.13) and (5.15) may be used (see also the additional remarks at the end of this subsection).

The case of a steady pressure flow in a circular tube of diameter $D = 2R$ is completely analogous. Here, transforming the Reynolds equations to cylindrical coordinates (x, r, ϕ), we can rewrite the first and second equations as:

$$\frac{1}{r}\frac{\partial}{\partial r} r\tau = \frac{\partial \bar{p}}{\partial x}, \qquad \rho\left(\frac{1}{r}\frac{\partial}{\partial r} r\overline{u_r'^2} - \frac{\overline{u_0'^2}}{r}\right) = -\frac{\partial \bar{p}}{\partial r}, \tag{5.16'}$$

where

$$\tau = \rho\nu \frac{d\bar{u}_x}{dr} - \rho \overline{u'_x u'_r}\,.$$

From these equations it follows that $\frac{\partial^2 \bar{p}}{\partial x \partial r} = 0$, i.e., that the longitudinal pressure gradient does not depend on z. Therefore $\frac{\partial \bar{p}}{\partial x} = \frac{d\bar{p}_0}{dx}$ where $\bar{p}_0$ is the static pressure on the wall. Now from the first equation (5.16′) we obtain

$$\tau = \frac{d\bar{p}_0}{dx} \cdot \frac{r}{2} = \frac{d\bar{p}_0}{dx} \cdot \frac{R}{2}\left(1 - \frac{z}{R}\right) = \tau_0\left(1 - \frac{z}{R}\right), \qquad z \le R, \quad (5.17')$$

where $z = R - r$ is the distance from the wall. Consequently, with $z \ll R$, the shear stress τ may once again be assumed constant, and the mean velocity profile will be given by Eq. (5.13) [or Eq. (5.15)], as in the case of plane-parallel flow with zero pressure gradient in a half-space.

Finally, the Reynolds equations are obtained in a boundary layer on a flat plate by adding terms corresponding to the Reynolds stresses, to the usual boundary-layer equations (1.38)–(1.39) for averaged fluid dynamic fields. By considering the fact that the fluctuation velocity in the boundary layer is much less than the mean longitudinal velocity $\bar{u}$, and that the x derivative is much less than the z derivative, retaining only the leading terms we find that the third Reynolds equation for zero longitudinal pressure gradient will take the form

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{w}\frac{\partial \bar{u}}{\partial z} = -\frac{\partial}{\partial z}\overline{u'w'} + \nu\frac{\partial^2 \bar{u}}{\partial z^2}\,. \qquad (5.18)$$

From this it is possible to estimate the variation of the shear stress τ along the vertical and to show that in the boundary layer there exists a considerable layer within which the shear stress is practically constant. Within this layer the flow may be assumed plane-parallel and described by Eq. (5.10), while the mean velocity profile is given by Eq. (5.13) or (5.15).

It is interesting to note here that, according to Ludwieg and Tillmann (1949), Eq. (5.13) with the ordinary universal function f proves to be correct also for a rather thick portion of boundary layers with considerable (though not too strong) longitudinal pressure gradients, which, it would seem, must cause great variation of the shear stress within the portion of the flow under consideration. This observation was also confirmed later by several other investigators. Coles (1955) explained this by the fact that in a boundary layer with longitudinal pressure gradient, the variation of τ_0 along the wall leads to a variation of τ along the vertical, which to a great extent, compensates for the change in τ in the z-direction produced by the mean pressure gradient [cf. Rotta (1962a; b)]. However, the question is not completely clear; e.g., Mellor (1966) found with the aid of specific semiempirical arguments that the effect of the pressure gradient on the velocity profile in a boundary layer near a wall is quite considerable.

Another explanation of the comparatively small influence of the pressure gradient on the velocity profile may be related to the fact (confirmed both by experimental data and approximate theoretical calculations) that the vertical profiles of the mean dynamical variables are only slightly sensitive, even to fairly large changes in τ. Thus, for example, for a layer in which τ varies by not more than 15-20%, we may almost always use the simplified theory, considering the layer to be constant-stress (i.e., τ = const). Later, we shall see that in several cases, the results for a constant-stress layer give a fair approximation even over the total section of the channel (or tube). In the next approximation the influence of the stress variation may be taken into account by a simple perturbation procedure which will be described at the very end of Sect. 5.5.

5.3 Flow Close to a Smooth Wall; Viscous Sublayer and Logarithmic Boundary Layer

Let us still assume that the wall $z = 0$ is dynamically smooth; thus Eq. (5.13) holds. Then the form of the function $f(z_+)$ occurring in this equation may be determined explicitly for two limiting cases, for large and for small values of the argument z_+.

In the region very close to the wall, the fact that $\rho\overline{u'w'} = 0$ and $\tau_0 = \rho\nu \dfrac{d\bar{u}}{dz}$ on the wall itself [see Eq. (5.3)] plays a fundamental role. Thus for sufficiently small values of z, the viscous stresses will be considerably greater in magnitude than the Reynolds stress $-\rho\overline{u'w'}$. The layer of fluid within which $\nu\left|\dfrac{d\bar{u}}{dz}\right| \gg |\overline{u'w'}|$ is usually called the *viscous sublayer*.[3] Within this sublayer we may assume

[3]At one time the term "viscous sublayer" and the term "laminar sublayer" were both used, since it was assumed that in this layer the flow was laminar. However, the data of direct ultramicroscopic observations of the motion of small particles or bubbles in fluid close to a wall [Page and Townend (1932); Fidman (1959); Orlov (1966); Popovich and Hummel (1967); Kline, Reynolds et al. (1967); and others] and of numerous refined hot-wire anemometer measurements in the vicinity of a wall [Laufer (1951; 1954); Klebanoff (1955); Comte-Bellot (1963; 1965); Coantic (1966; 1967a); Kline, Reynolds et al. (1967); and many others] show quite definitely that although the mean velocity within the sublayer is identical to the linear velocity profile of a plane-parallel laminar flow with zero pressure gradient, the flow within it nevertheless is not laminar, but accompanied by considerable irregular fluctuations. Consequently, at present, the term "laminar sublayer" must be considered to be misleading.

that

$$\rho\nu \frac{\partial \bar{u}}{\partial z} = \tau_0 = \text{const} \tag{5.19}$$

and therefore

$$\bar{u}(z) = \frac{u_*^2}{\nu} z, \qquad f(z_+) = z_+. \tag{5.20}$$

Strictly speaking, Eq. (5.20) is only the first term of the expansion of the function $f(z_+)$ as a Taylor series in powers of z_+ in a neighborhood of the point $z_+ = 0$. The subsequent terms of this expansion may be obtained by differentiating Eq. (5.11) with respect to z at the point $z = 0$. Here, naturally, $u' = v' = w' = 0$ for $z = 0$; as a result, when $z = 0$ all the derivatives of the velocity fluctuations with respect to x and y are equal to zero and, furthermore, $\left(\frac{\partial w'}{\partial z}\right)_{z=0} = 0$ from the continuity equation.

Thus

$$\left(\frac{d^2\bar{u}}{dz^2}\right)_{z=0} = \frac{1}{\nu}\left(\frac{\partial}{\partial z}\overline{u'w'}\right)_{z=0} = \frac{1}{\nu}\left(\overline{\frac{\partial u'}{\partial z}w'} + \overline{u'\frac{\partial w'}{\partial z}}\right)_{z=0} = 0,$$

$$\left(\frac{d^3\bar{u}}{dz^3}\right)_{z=0} = \frac{1}{\nu}\left(\frac{\partial^2}{\partial z^2}\overline{u'w'}\right)_{z=0} = \frac{1}{\nu}\left(\overline{\frac{\partial^2 u'}{\partial z^2}w'} + 2\overline{\frac{\partial u'}{\partial z}\frac{\partial w'}{\partial z}} + \overline{u'\frac{\partial^2 w'}{\partial z^2}}\right)_{z=0} = 0,$$

$$\left(\frac{d^4\bar{u}}{dz^4}\right)_{z=0} = \frac{1}{\nu}\left(\frac{\partial^3}{\partial z^3}\overline{u'w'}\right)_{z=0} = \frac{3}{\nu}\left(\overline{\frac{\partial u'}{\partial z}\frac{\partial^2 w'}{\partial z^2}}\right)_{z=0}.$$

Therefore, it follows that the Taylor expansion of the mean velocity profile in powers of z has the form

$$\begin{aligned}\bar{u}(z) &= \frac{u_*^2}{\nu} z - c_4 \frac{u_*^5}{\nu^4} z^4 + c_5 \frac{u_*^6}{\nu^5} z^5 + \cdots, \\ f(z_+) &= z_+ - c_4 z_+^4 + c_5 z_+^5 + \cdots\end{aligned} \tag{5.20'}$$

Here

$$c_4 = -\frac{\nu^3}{24 u_*^5}\left(\frac{\partial^3 \overline{u'w'}}{\partial z^3}\right)_{z=0} = -\frac{\nu^3}{8 u_*^5}\left(\overline{\frac{\partial u'}{\partial z}\frac{\partial^2 w'}{\partial z^2}}\right)_{z=0}.$$

The minus sign is inserted because it seems intuitively clear that close to the wall $\overline{u'w'} < 0$ and hence $\left(\frac{\partial^3 \overline{u'w'}}{\partial z^3}\right)_{z=0} < 0$, while $c_5 = \frac{\nu^4}{120 u_*^6}\left(\frac{\partial^4 \overline{u'w'}}{\partial z^4}\right)_{z=0}$. The expansion (5.20′) was demonstrated (apparently for the first time) in a somewhat different form by Murphree (1932) [see also Townsend (1956)]. Later, certain authors [in particular, Landau and Lifshitz (1963); Deissler (1955); Elrod (1957); Levich (1962); Lyatkher (1968) et al.; cf. the closing remarks of this subsection] formulated some arguments in favor of the assumption that $c_4 = 0$, but all these are not rigorous.[4] The existing experimental data are also insufficient for reliable estimation of the coefficient c_4 and therefore give no reason to conclude that $c_4=0$. Nevertheless, the fact that the second and third derivatives of $f(z_+)$ are equal to zero when $z_+ = 0$ is sufficient for the variation of the velocity over a considerable region to be very close to linear, i.e., for the concept of a viscous sublayer to be justified.

The upper boundary of the viscous sublayer may be defined conditionally, e.g., as the value of z for which $\overline{u'w'} = 0.1\, \nu \frac{d\bar{u}}{dz}$; certain other definitions related to this one may also be used. In every case the thickness of the viscous sublayer δ_v can depend only on the parameters u_* and ν. Consequently, it must be given by the equation $\delta_v = \alpha_v \frac{\nu}{u_*} = \alpha_v z_*$, where α_v is a universal constant of the order of unity (the precise value of which will of course depend on the chosen definition), which must be found from experimental data. The fact that this constant depends on the choice of definition, i.e., has no unique value, is quite natural since the viscous sublayer does not have a sharp upper boundary, but passes smoothly into the next region of the flow, in which the viscous stress and the Reynolds stress have the same order of magnitude. Nevertheless, just as in the case of the numerical coefficient in the equation for the boundary layer thickness δ, the range of permissible values of α_v in this case proves not to be very broad. Most often the number 5 is chosen as α_v, i.e., it is assumed that $\delta_v = 5\ \frac{\nu}{u_*}$; the basis for this choice is the experimental data, given in Fig. 25.

Let us turn now to the second limiting case, to values of z considerably greater than $z_* = \frac{\nu}{u_*}$. In a developed turbulent flow at a distance from rigid walls, the turbulent stresses are many times

[4]The demonstration of the equation $c_4 = 0$ proposed by Ohji (1967) is incorrect; in fact it is based on a particular nonrigorous assumption (not formulated explicitly in the paper). Let us note also that considering that in real channel- and tube-flow $\tau(z) \neq$ const, but $\partial\tau/\partial z =$ const [cf. Eqs. (5.17) and (5.17')] we shall obtain a nonzero value of the dimensionless coefficient c_2 of the z^2-term of the Taylor series for $u(z)$; however, $c_3 = 0$ even in this case.

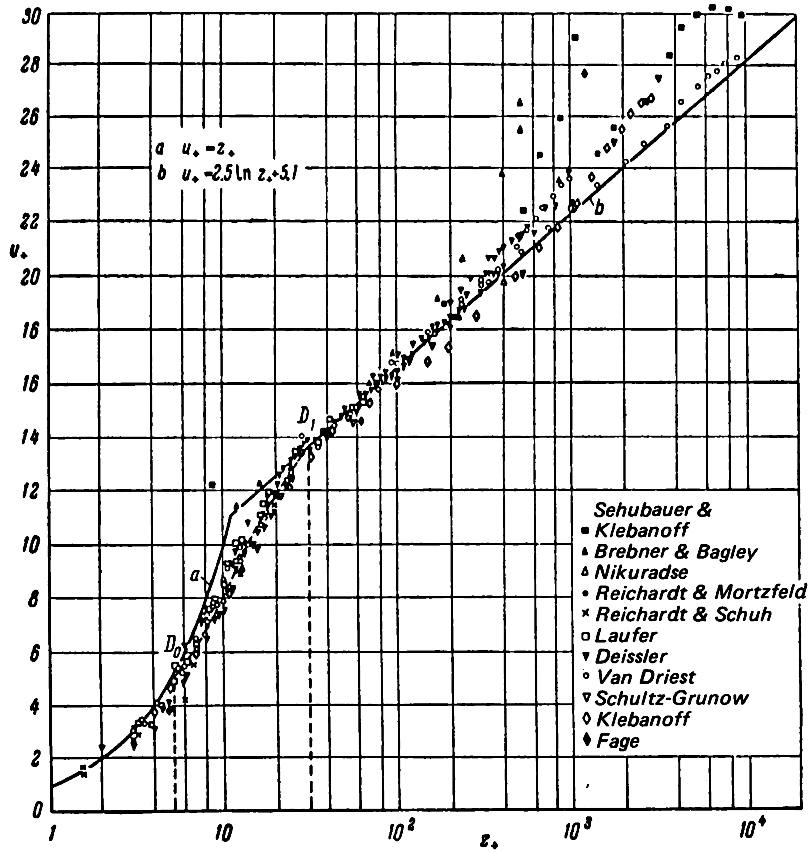


FIG. 25. Universal dimensionless mean velocity profile of turbulent flow close to a smooth wall according to the data of tube-, channel- and boundary-layer measurements [according to Kestin and Richardson (1963)].

greater in magnitude than the viscous stresses. Consequently, for sufficiently large z (let us say, for $z > \delta_l$) we may ignore the term $\rho\nu \dfrac{d\bar{u}}{dz}$ in Eq. (5.11) and assume that $\tau_0 = -\rho\overline{u'w'}$. Hence for $z > \delta_l$ the law of variation of the mean velocity must not depend on the viscosity ν but must be determined only by the values of the density ρ and of the momentum flux τ_0 passing through the fluid (and equal to the total interaction force of the fluid with the wall per unit area of the wall). Essential here is that we are speaking not of the velocity itself, but the law of variation of the mean velocity. In fact, by considering the flow within a layer of negligible viscous stresses, we can say nothing about the absolute values of the velocity $\bar{u}(z)$. However, we can study only the differences $\bar{u}(z_1) - \bar{u}(z_2)$ of the values of the velocity at two heights z_1 and z_2 within this layer, and, in particular, the increase $\bar{u}(z) - \bar{u}(\delta_l)$ of the velocity in the layer (z, δ_l), where $z > \delta_l$. This follows from the Galilean invariance of the equations of mechanics, according to which the addition of any constant to all velocities $\bar{u}(z)$ for $z \geqslant \delta_l$ cannot change the momentum flux transmitted through the fluid. Consequently, the absolute value of the velocity $\bar{u}(z)$, $z > \delta_l$ is not defined by the values τ_0 and z, but depends also on the value of $\bar{u}(\delta_l)$, i.e., on the law of variation of velocity in the layer $z \leqslant \delta_l$ which itself is influenced by the viscosity ν. However, the value $\frac{d\bar{u}}{dz}$ of the gradient of the mean velocity at height z for $z > \delta_l$ must be independent of ν; i.e., it must be a function of the parameters τ_0, ρ, and z only. As easily seen from these parameters, it is possible to formulate a unique combination $\tau_0^{\frac{1}{2}}/\rho^{\frac{1}{2}}z = \frac{u_*}{z}$ with dimension of a velocity gradient. Thus for $z > \delta_l$ the relationship

$$\frac{d\bar{u}(z)}{dz} = A\frac{u_*}{z} \tag{5.21}$$

must be satisfied, where A is a universal dimensionless constant. For the velocity profile at $z > \delta_l$ from Eq. (5.21) we obtain the logarithmic equation

$$\bar{u}(z) = Au_* \ln z + A_1, \tag{5.22}$$

where A_1 is a new constant which, by the preceding argument, may also depend on the viscosity ν. The layer of fluid in which Eq. (5.22) is satisfied is called the *logarithmic layer*, or *logarithmic boundary layer*; the existence of this layer is extremely important for a number of problems in which turbulent flows along rigid walls are encountered. The universal equation (5.22) was first deduced by von Kármán (1930) and Prandtl (1932b) using completely different (from that presented above and from each other) arguments. Later, a whole series of proofs was found for it; some of them are of considerable interest, and will be reproduced below. A simple, purely dimensional deduction of this equation was published for the first time in 1944, in the first edition of Landau and Lifshitz' book (1963); see also Squire (1948).

Instead of dimensional analysis in the deduction of Eq. (5.22), we may also use the invariance of the dynamic equations of an ideal fluid, which describe the flow for $z > \delta_l$, under the similarity transformations $x \to kx$, $y \to ky$, $z \to kz$, and $t \to kt$. Since these transformations change the half-space $z > 0$ into itself, it is natural to think that all the statistical characteristics of turbulence in this half-space (insofar as they are independent of the viscosity) will also be invariant under them. However, as we have already seen, if we ignore the viscosity (i.e., neglecting for the time being the boundary condition $u = 0$ when $z = 0$), we can consider only the relative velocities $\bar{u}(z_1) - \bar{u}(z_2)$. Since the friction velocity u_* clearly does not change under similarity transformations, then in accordance with the assumption on invariance, the dimensionless ratio $\frac{\bar{u}(z_2) - \bar{u}(z_1)}{u_*}$, for values of z_1 and z_2 that are not too small, can depend only on the ratio $\frac{z_2}{z_1}$

$$\frac{\bar{u}(z_2) - \bar{u}(z_1)}{u_*} = g\left(\frac{z_2}{z_1}\right). \tag{5.23}$$

From the definition (5.23) of the function $g\left(\frac{z_2}{z_1}\right)$ it follows that $g\left(\frac{z_3}{z_1}\right) = g\left(\frac{z_3}{z_2}\right) + g\left(\frac{z_2}{z_1}\right)$; the latter relationship, by virtue of the identity $\frac{z_3}{z_1} = \frac{z_3}{z_2} \cdot \frac{z_2}{z_1}$ may also be written in the form

$$g(\xi_1 \cdot \xi_2) = g(\xi_1) + g(\xi_2). \tag{5.24}$$

It is not difficult to show that the only continuous solution of the functional equation (5.24) is the logarithmic function $g(\xi) = A \ln \xi$; thus, we arrive at the equation

$$\bar{u}(z_2) - \bar{u}(z_1) = Au_* \ln \frac{z_2}{z_1}, \tag{5.22'}$$

which is equivalent to Eq. (5.22).

From dimensional arguments it is clear that the height δ_l of the lower boundary of the logarithmic boundary layer must be defined by the equation $\delta_l = \alpha_l \frac{\nu}{u_*}$, where α_l is another universal dimensionless constant, determined to approximately the same degree of accuracy as the constant α_v. The data of Fig. 25 show that it is permissible to take $\alpha_l = 30$. From a comparison of Eqs. (5.22) and (5.13) it follows that the constant A_1 must be of the form $A_1 = Au_* \ln \frac{u_*}{\nu} + Bu_*$, where B is a dimensionless universal constant; therefore Eq. (5.22) may be rewritten as

$$\bar{u}(z) = u_*\left(A \ln \frac{zu_*}{\nu} + B\right) \quad \text{when} \quad z > \alpha_l \frac{\nu}{u_*}. \tag{5.25}$$

Thus

$$f(z_+) = A \ln z_+ + B \quad \text{when} \quad z_+ > \alpha_l. \tag{5.25'}$$

Here it must be pointed out that instead of the coefficient A the traditional $\varkappa = \frac{1}{A}$, is used frequently; $\varkappa$ is usually called *von Kármán's constant*. Replacing A by $\frac{1}{\varkappa}$ and denoting $e^{-\varkappa B}$ by β, Eq. (5.25) may also be rewritten as

$$\bar{u}(z) = \frac{u_*}{\varkappa} \ln \frac{zu_*}{\beta\nu}. \tag{5.25''}$$

The numerical values of the constants A (or $\varkappa = 1/A$) and B (or $\beta = e^{-\varkappa B}$) in Eqs. (5.25)–(5.25″) may be determined from the experimental data. As indicated in Sect. 5.2, the measurements may be carried out in smooth tubes, in rectangular channels with smooth walls, and in the boundary layer on smooth plates. The first useful measurements for this purpose, i.e., precise measurements of the mean velocity distribution and the shear stress distribution, were made by Nikuradse (1932) in water flows in smooth, straight tubes of different radii with Re = $\frac{U_m D}{\nu}$ varying between 4×10^3 and 3.2×10^6. His data show that for a considerable part of the flow, beginning at a distance of about $30 \frac{\nu}{u_*}$ from the wall and continuing almost to the center of the tube, the mean velocity may be described with good precision by an equation of the form of Eq. (5.25). For the coefficients A and B in this equation, Nikuradse gave two choices of values: with $A = 2.4$, $B = 5.8$ (i.e., with $\varkappa = 0.417$, $\beta \approx 0.09 \approx 1/11$) the best agreement of Eq. (5.25) with the experimental data was obtained for the region $30\ \nu/u_* < z < 1000\ \nu/u_*$, which does not by any means extend to the center of the tube, while the values $A = 2.5$, $B = 5.5$ (i.e., $\varkappa = 0.40$, $\beta \approx 0.11 \approx 0.11 \approx 1/9$) prove to be the best for the application of Eq. (5.25) to the whole region from $z = 30\ \nu/u_*$ to the center of the tube. Later, similar measurements were repeated many times, both for flows in tubes (or in plane channels), and for boundary layers on flat plates; while Eq. (5.25) in all cases is quite reliably confirmed, in the values obtained for A and especially B, there is a small amount of scatter [see, for example, the survey by Hinze (1962), which contains a detailed analysis of a large amount of data on the values of A and B].[5] Many data referring to the three types of flow under consideration are collected in Fig. 25 (borrowed from Kestin and Richardson (1963), and presented here by way of example). We see that for $z < 5\ \nu/u_*$ all the observed values of $u_+ = \bar{u}/u_*$ lie fairly closely on the curve $u_+ = zu_*/\nu$ of Eq. (5.20), while for $500\ \nu/u_* > z > 30\ \nu/u_*$ they lie on the curve $u_+ = 2.5 \log zu_*/\nu + 5.1$, corresponding to Eq. (5.25) with coefficients $A = 2.5$, $B = 5.1$ (i.e., $\varkappa = 0.4$, $\beta = 0.13$) recommended by Coles (1955) which are close to the mean of the values of the coefficients proposed by all the other investigators (both earlier and later than 1955).[6]

In the intermediate region $5\ \nu/u_* < z < 30\ \nu/u_*$, the experimental values of the ratio $\bar{u}(z)/u_*$ plotted in Fig. 25 clearly deviate from both the values given by Eq. (5.20) and from those obtained from Eq.

[6]We recall that, according to the theoretical deduction, A and B must be universal constants only for an idealized plane-parallel flow with constant shear stress τ_0. Thus for real laboratory experiments, in which also the region of applicability of Eq. (5.25) is determined "by eye," the small scatter of the empirical values of these coefficients will be explained completely, even if the unavoidable errors of measurement and procedure are ignored.

[6]For example, Clauser (1956) proposed for boundary-layer flows the values $A = 2.44$, $B = 4.9$; Townsend (1956) used the values $A = 2.44$, $B = 5.85$; in the textbook by Longwell (1966) slightly differing values $A = 2.71$, $B = 3.5$ are recommended; Spalding (1964) and Escudier and Nicoll (1966) assumed that $A = 2.5$, $B = 4.7$; the measurements of Coantic (1966) in circular tube flow and of Comte-Bellot (1965) in plane channel flow gave the results $A = 2.50$, $4.40 \leqslant B \leqslant 5.50$ and $A = 2.7$ and $4.5 \leqslant B \leqslant 6.0$, respectively, etc. The suggestion of a possible slight dependence of A and B on Re (i.e., on viscosity ν) has been made [Hinze (1962); Comte-Bellot (1965)]. For a constant-stress layer such a dependence is obviously incompatible with physically convincing dimensional arguments; however, it can be explained easily by the influence of stress variation. The data on the dependence are extremely uncertain [e.g., the dependence on Re is lacking entirely in the quite accurate recent data by Lindgren (1965)]; therefore at present it seems reasonable to disregard it.

(5.25). According to the experimental data, the values of $f(z_+)$ in this region may be represented by a smooth curve, which passes easily into the curve (5.20) at the point $z_+=5$ and into the curve (5.25′) with $A=2.5$, $B=5.1$ at the point $z_+=30$ (the dotted line in Fig. 25). Further, we note that the deviation of the dotted curve from the broken curve, sohd line in Fig. 25, is not very great. Thus, in cases when great accuracy is not required, it is permissible to assume that right up to the value $z_+=11.1$ (the abscissa of the intersection point of the curves $u_+=z_+$ and $u_+=2.5\ln z_+ + 5.1$) the mean velocity profile is given by Eq. (5.20) while for $z_+>11.1$ it is given by Eq. (5.25). This assumption [accepted, e.g., in the papers of Prandtl (1919; 1928) and G. I. Taylor (1916)] means, of course, that we ignore the intermediate zone in which $\rho\nu\frac{d\bar{u}}{dz}$ and $-\rho\overline{u'w'}$ are of the same order of magnitude, and assume that outside the viscous sublayer, which we now consider as having thickness $\delta_v=\delta_l=11.1\,\nu/u_*$, immediately follows the logarithmic layer, in which the viscous stresses are negligibly small in comparison with the Reynolds stresses.

Further, we note that from the viewpoint of the turbulent viscosity K [see Eq. (5.5)] we may define the viscous sublayer as the sublayer in which it is permissible to assume that $K\equiv 0$ (i.e., that the effective viscosity is equal to $\nu=$ const). In exactly the same way the logarithmic layer may be defined as the layer in which the molecular viscosity is negligibly small in comparison with the turbulent viscosity given by the equation

$$K=\frac{u_*z}{A}=\varkappa u_*z, \qquad \varkappa\approx 0.4. \tag{5.26}$$

In these terms the dotted hne in Fig. 25 will correspond to a layer in which the effective viscosity varies smoothly in some way from a value close to ν (for $z=5\nu/u_*$) to $30\varkappa\nu\approx 12\nu$ (for $z=30\nu/u_*$).

So far, we have discussed only the mean velocity profile of a turbulent flow; however, the physical considerations which have led us to a universal law of the wall, and to the concept of a logarithmic layer, may be applied equally well to the investigation of any other one-point velocity moments close to a rigid plane wall. All these moments, within the constant-stress layer will, of course, depend only on the parameters z, τ, ν, and ρ, i.e., they must be represented by the product of some power of the friction velocity u_* and of a particular function of the dimensionless distance $z_+=zu_*/\nu$. For sufficiently large z_+, the statistical regime of turbulent fluctuations can no longer depend on the viscosity ν; thus in the case of central moments (independent of the mean velocity $\bar{u}$) the corresponding universal functions will tend to a constant as $z_+ \to \infty$. As an example we take the one-point second moments of the fluctuations u', v', and w'. There are six such moments in all; however, two of them (namely, $\overline{u'v'}$ and $\overline{v'w'}$) are identically zero due to the symmetry of the turbulence with respect to the plane Oxz. Therefore there remain only four nonzero moments: $\overline{u'^2} = \sigma_u^2$, $\overline{v'^2} = \sigma_v^2$, $\overline{w'^2} = \sigma_w^2$, and $\overline{u'w'}$. Therefore, in addition to Eq. (5.13), we shall also have four equations of the form

$$\begin{aligned} \sigma_u &= u_* f_1(z_+), \quad \sigma_v = u_* f_2(z_+), \\ \sigma_w &= u_* f_3(z_+), \quad -\overline{u'w'} = u_*^2 f_4(z_+), \end{aligned} \tag{5.27}$$

containing four new universal functions. Instead of $f_4(z_+)$ we may also use the function $f_5(z_+) = f_4/f_1 f_3$, which describes the height variation of the correlation coefficient between u' and w'

$$-r_{uw} = \frac{-\overline{u'w'}}{(\overline{u'^2}\,\overline{w'^2})^{1/2}} = f_5(z_+).$$

Within the logarithmic layer the function $f_4(z_+)$ must be identically equal to unity, while the functions f_1, f_2, f_3, and f_5 assume constant values

$$f_1(\infty) = A_1, \; f_2(\infty) = A_2, \; f_3(\infty) = A_3 \text{ and } f_5(\infty) = A_5 = (A_1 A_3)^{-1}.$$

As we approach the wall, the functions $f_1, \ldots, f_4$ tend to zero, while their Taylor-series expansions at the point $z_+ = 0$ take the form:

$$\begin{aligned} f_1(z_+) &= a_1 z_+ + b_1 z_+^2 + \ldots, \\ f_2(z_+) &= a_2 z_+ + b_2 z_+^2 + \ldots, \\ f_3(z_+) &= a_3 z_+^2 + b_3 z_+^3 + \ldots, \\ f_4(z_+) &= a_4 z_+^3 + b_4 z_+^4 + \ldots \end{aligned}$$

(since $u' = v' = w' = \frac{\partial w'}{\partial z} = 0$ when $z = 0$). The coefficients a_4 and b_4 are related in a simple manner to the coefficients c_4 and c_5 of Eq.

(5.20′) since close to the wall $f_4(z_+)=1-f'(z_+)$ [from Eq. (5.11)]. The exact form of the functions $f_1, \ldots, f_5$ cannot be determined theoretically, but their approximate form may be found from the data of Laufer (1954), Klebanoff (1955), Comte-Bellot (1965), Coantic (1966; 1967) and others (see, for example, Figs. 26 and 27 based on the data of Laufer and Klebanoff). Other examples of these two graphs were published by Dumas and Marcillat (1966); the agreement of their data with that of Comte-Bellot is not very close, but must be considered satisfactory. The data for tube flow and for boundary-layer flow agree well with each other. In particular, for constants A_1, A_2, and A_3 in both cases approximately identical values are obtained:

$$A_1 \approx 2.3, \qquad A_2 \approx 1.7, \qquad A_3 \approx 0.9.$$

(Values of A_1 and A_3 relatively close to these were also obtained by other authors including those who made measurements in the surface layer of the atmosphere; see Sect. 8.5 below.) Processing the existing data on the velocity fluctuations in proximity to a wall, we may also obtain an approximate estimate (although still with fairly low accuracy) of the values of the coefficients $a_1, \ldots, a_4$; namely,

$$a_1 \approx 0.3, \quad a_2 \approx 0.07, \quad a_3 \approx 0.008, \quad a_4 \approx 0.001$$

[see Laufer (1954); Klebanoff (1955); Townsend (1956); Coantic (1965)].

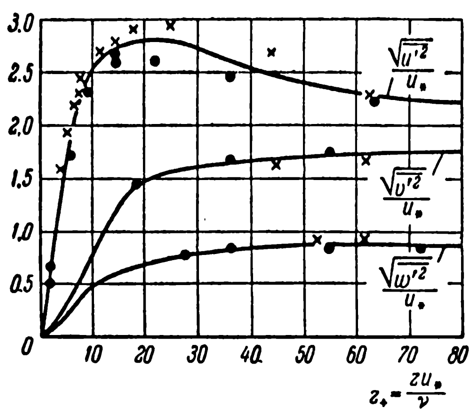


FIG. 26. Universal dimensionless profiles of the intensity of fluctuation of the three velocity components in a flow close to a smooth wall according to the data of Klebanoff (x) and Laufer (•).

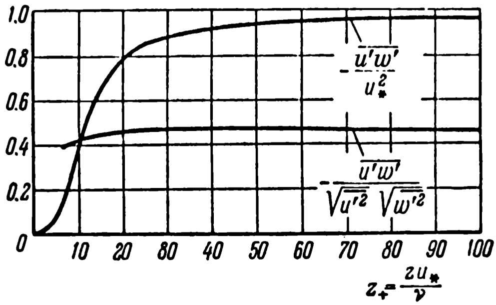


FIG. 27. Universal functions $f_4(z_+)$ and $f_5(z_+)$ according to Laufer's measurements.

For higher-order moments of the velocity fluctuations (of order three and higher) there is considerably less data than for the second-order moments defined in Eq. (5.27). Nevertheless, Comte-Bellot (1963; 1965) constructed some preliminary empirical graphs of the universal functions $f_6(z_+) = (\overline{u'^3})/(\overline{u'^2})^{3/2}$ and $f_7(z_+) = \overline{u'^4}/(\overline{u'^2})^2$ which define the skewness and excess of the longitudinal component u'.

Similarity laws related to the law of the wall may be formulated not only for the mean velocity profile and the profiles of one-point velocity moments, but for general many-point probability distributions of the velocity components at points in the constant stress layer. All these distributions must depend only on ν, u_*, and the coordinates of the points; moreover, the dependence on the viscosity ν must vanish if we consider the probability distribution of the simultaneous values of the velocity fluctuations u', v', and w' at points $(x_1, y_1, z_1), \ldots, (x_n, y_n, z_n)$ within the constant stress layer satisfying the following conditions: 1) $z_i \gg z_* = \nu/u_*$ for all $i = 1, \ldots, n$ (i.e., all the points are located within the logarithmic layer); and 2) the distances between any two of the points are much greater than z_*. Under these conditions, *the probability density of the dimensionless variables* u'/u_*, v'/u_*, w'/u_* *can depend only on the dimensionless parameters*

$$\frac{x_2 - x_1}{z_1}, \ldots, \frac{x_n - x_1}{z_1}, \frac{y_2 - y_1}{z_1}, \ldots, \frac{y_n - y_1}{z_1}, \frac{z_2}{z_1}, \ldots, \frac{z_n}{z_1}.$$

This general similarity hypothesis for the probability distributions within the logarithmic layer can be generalized also to the many-point and many-time probability distributions (cf. Sect. 7.5, where the more general case of a density-stratified turbulent boundary layer will be considered). However, at present no experimental data exist even for the simpler many-point probability distributions at a single time, suitable for the confirmation of the hypothesis, and the data on many-point velocity moments within the constant-stress layer are extremely poor.

In certain cases, primarily in the case of turbulent heat- and mass-transfer through a fluid (see below, Sect. 5.7), the solid line of Fig. 25 will clearly be insufficiently exact, while the smooth broken line will be quite inconvenient. In these cases, it is desirable to have an analytical expression which describes the behavior of the mean velocity profile in the intermediate layer $\delta_v < z < \delta_l$ and which makes a smooth juncture with the limiting expressions (5.20) and (5.25) for $z \leqslant \delta_v = \alpha_v \nu/u_*$ and $z \geqslant \delta_l = \alpha_l \nu/u_*$, respectively. A whole series of such expressions for functions $f(z_+)$ of varying complexity has been put forward at various times by different authors. One of the first of these was von Kármán's expression (1934; 1939), which uses the approximation $f(z_+) = A_1 \ln z_+ + B_1$ for the transition region $\alpha_v < z_+ < \alpha_l$, where the coefficients A_1 and B_1 (different from A and B) are such that the sum $A_1 \ln z_+ + B_1$ takes at $z_+ = \alpha_v$ and at $z_+ = \alpha_l$ the same values as the functions (5.20) and (5.25), respectively; in other words, it corresponds to the joining of the points D_0 and D_1 in Fig. 25 by a straight line. Later, Hofmann (1940), Reichardt (1940) and Levich (1962) Sect. 4, proposed other interpolation formulas for the values of $f(z_+)$, $\alpha_v < z_+ < \alpha_l$, which give a somewhat better correspondence with the results of the measurements. A different idea was used in the works of Squire (1948), Loitsyanskiy (1958), Frank-Kamenetskiy (1947), and in the very similar works of Rotta (1950), Hudimoto (1941; 1951) and Miles (1957); here the region of flow is divided into two layers only: $z_+ \leqslant \alpha_v$ and $z_+ > \alpha_v$. In the first of these regions, the usual linear equation (5.20) is used; in the second region $f(z_+)$ is given by a more complicated analytical expression than Eq. (5.25′), which tends asymptotically to Eq. (5.25′) as $z_+ \to \infty$, and makes a smooth juncture with Eq. (5.20) at $z_+ = \alpha_v$. On the other hand, Ribaud (1940), Hama (1953), Deissler (1955; 1959), Rannie (1956), Levich (1959) Sect. 25, Ts'ai Ko-yeng (1961) and Tien and Wasan (1963) divided the flow into two layers $z_+ < \alpha_l$ and $z_+ \geqslant \alpha_l$ and used the usual logarithmic equation (5.25′) in the second of these, and a more complicated expression for $f(z_+)$ than Eq. (5.20) in the first. The works of Lin, Moulton and Putnam (1953), Loytsanskiy (1960; 1962a) and Carr (1962), used a three-layer model (i.e., it was assumed that $f(z_+)$ was given by different equations for $z_+ < \alpha_v$, $\alpha_v \leqslant z_+ \leqslant \alpha_l$, and for $z_+ > \alpha_l$), but unlike von Kármán's work, in the first of these layers, a nonlinear expression for $f(z_+)$ was adopted (different such nonlinear expressions being used in all cited works). In addition, Loytsyanskiy (1960; 1962a) also proposed a single, but very complicated expression for $f(z_+)$ which is applicable for all values of z_+. Simpler, approximate expressions, describing the behavior of $f(z_+)$ over the whole constant-stress region and which are in fairly good agreement with all the data were proposed by Reichardt (1951a), van Driest (1956), and Spalding (1961); one of Spalding's two equations was also derived later by Kleinstein (1967). A survey of work along these lines is given in Hinze (1961), Sect. 7, Longwell (1966), Sect. 8.7, and in the articles of Loitsyanskiy (1962a), Rotta (1962b) and Kestin and Richardson (1963); a very detailed review (containing about forty different equations) is included in the thesis of Coantic (1966).

The question of the detailed behavior of $f(z_+)$ as $z_+ \to 0$ has evoked widespread discussion. In some of the above-mentioned works, it is assumed that $f(z_+) = z_+$ for $z_+ \leqslant \alpha_v$; i.e., all derivatives of $f(z_+)$ higher than the first are assumed equal to zero at $z_+ = 0$; this assumption was later criricized by a number of authors. According to the equation proposed by Ranney, $f(z_+) = z_+ + c_3 z_+^3 + \dots$ as $z_+ \to 0$, where $c_3 \neq 0$; this result clearly contradicts the exact equation (5.20′). At the same time, the equations of Reichardt, Ribaud, Lin, Moulton and Putnam, Carr and Tien and Wasan lead to an expansion of Eq. (5.20′) with $c_4 \neq 0$, while according to Hama, Deissler, Levich, Loytyanskiy and Ta'ai Ko-yeng, $c_4 = 0$ in this expansion. Finally, Spalding simultaneously introduced two different expressions for $f(z_+)$ with $c_4 \neq 0$ and with $c_4 = 0$. The considerable attention given to the question of the behavior of $f(z_+)$ as $z_+ \to 0$ is explained by the fact that for sufficiently large Prandtl (or Schmidt) numbers, this behavior will have a great effect on turbulent heat-transfer (or mass transfer); this will be discussed in greater detail in Sect. 5.7.

In the majority of the works we see also that it is not the function $f(z_+)$ that is chosen as the basis of the discussion, but the dependence on z_+ of the effective viscosity $\nu + K = \tau/\rho \dfrac{d\bar{u}}{dz} = \nu\,[f'(z_+)]^{-1}$. The proposed form of this dependence in some cases, is selected simply on the basis of the data, but more often it is motivated theoretically with the aid of some sort of qualitative arguments, usually supplemented by a special semiempirical hypothesis. However, none of the hypotheses used has been strictly proved; at the same time, the scatter of the curves $\bar{u}_+ = f(z_+)$proposed by the various authors has the same order of magnitude (if not greater) and has the same character as the scatter of the existing experimental points. Consequently, at present, there is no justification for choosing any one "best" equation for $f(z_+)$, and in choosing between the different variants of these equations, one has only to consider which is most suitable for solving any arbitrary problem.

The present situation concerning the equations for the functions $f_1 \ldots f_4$, etc., is considerably less satisfactory. These functions depend on the quite mysterious statistical regime of the velocity fluctuations very close to a rigid wall. One of the most general ideas on this regime is that the relative role of the nonlinear terms in the dynamic equations decreases with diminishing distance from the wall. Therefore it seems reasonable that the fluctuations u', v', and w' in the viscous sublayer could be described by linearized equations. One of the first concrete deductions from this hypothesis was made by Landau and Levich early in 1940 [cf. Landau and Lifshitz (1963) and Levich (1962)]; they conjectured that the linear equations must imply a constant time-scale for the entire viscous sublayer, and with this reasoning, obtained the proportionality of $-\overline{u'w'}$ to y_+^4. However, this deduction is not rigorous and the more recent analytic derivation of the same result by Lyatkher (1968) has even more doubtful points in it. Some other works (cited above) on the mean velocity profile near a wall may also be included in the family of investigations related to a viscous sublayer; however, we do not think that further discussion of them here will be useful.

More detailed investigations of the velocity fluctuations in the vicinity of a wall were made by Sternberg (1962; 1965) and Schubert and Corcos (1967). It was assumed in all these works that turbulent fluctuations are initiated in the outer parts of a turbulent layer and are transported to the viscous sublayer from the outside; also, these works are all based on linearized dynamic equations. The difference between each work is in the different simplifications of the equations and different selections of the boundary conditions on the outer boundary of the sublayer. In all cases the authors carried out preliminary calculations of the statistical characteristics of the fluctuations within the sublayer and obtained results, which by and large agree with the data on the velocity fluctuations found before 1965. It is interesting to note that Einstein and Li (1956) considered a completely different physical model of the process of initiation of fluctuations within the viscous sublayer (assuming the processes at the wall to be most important); nevertheless, their equations turn out to be identical with those of Sternberg (1962) [thorough comparison of the theories of Einstein and Li and Sternberg can be found in the paper by Kistler (1962)].

However, all the existing theories must be considered at present as completely preliminary since they do not take into account the latest profound experimental findings of Kline, Reynolds et al. (1967), Coantic (1967a; b), Willmarth and Bo Jang Tu (1967), Bakewell and Lumley (1967), and others. Especially important seem to be the results of

Kline and his co-authors, showing very clearly many new features of the fluctuations within the viscous sublayer (e.g., the great part played by v fluctuations and by sharp variations of the flow in the spanwise direction), which were not taken into account in all previous theories.

5.4 Flow Along a Rough Wall; Roughness Parameter and Displacement Height

We now consider a turbulent flow near a rough wall with protrusions of mean height h_0 that are not small in comparison with the friction length $z_* = \frac{\nu}{u_*}$. In this case the mean velocity profile will be given by Eq. (5.15) which contains the universal function f of z_+ and a number of other parameters. Since functions of many variables are quite inconvenient for practical applications, the use of dimensional reasoning in this case is not very fruitful. In fact, however, the situation is by no means as bad as this.

Let us assume that $h_0 \gtrsim \frac{\nu}{u_*}$. Once again, we consider the cases of small and large values of z separately. For small z, comparable to the height h_0, we must expect that the mean velocity $\bar{u}(z)$ will depend considerably on the form and mutual spacing of the irregularities of the wall and will be different above the protrusions and over the hollows between them. Thus here it is impossible to hope for any simple general rules. However, in many cases we are interested primarily in the mean velocities not too close to the wall, at distances z which are large in comparison both with the friction length $z_* = \frac{\nu}{u_*}$ and in comparison with the mean height h_0 of the protrusions. Naturally, it is considered that at such distances from the wall neither the viscosity nor the local properties of the surface will have any effect; here the essential feature will be only the presence of a constant flux of momentum in the negative direction of the Oz axis with a "sink" on the plane $z = 0$. Therefore, both the deductions of Eq. (5.22), applied to a flow in a half-space bounded by a smooth wall, for $z \gg h_0 > \frac{\nu}{u_*}$ still hold in the case of a rough wall. Consequently,

$$\bar{u}(z) = Au_* \ln z + A_1 \quad \text{for} \quad z \gg h_0. \tag{5.22a}$$

Since the constant $A = \frac{1}{\varkappa}$ in Eq. (5.22a) gives the value of the mean velocity gradient corresponding to a given momentum flux $\tau_0 = \rho u_*^2$, it must have the same value (≈ 2.5) for a rough wall, as for a smooth wall. (The same must be true also for all the characteristics of the velocity fluctuations in the logarithmic layer above a rough wall, e.g., for the values of the ratios $\sigma_u/u_* = A_1$, $\sigma_v/u_* = A_2$, and $\sigma_w/u_* = A_3$). However, as far as the coefficient A_1 is concerned, this is determined by the boundary conditions on the lower boundary of the region of applicability of Eq. (5.22a); consequently, it depends on the variation of the mean velocity in the immediate neighborhood of the wall, which differs considerably depending on whether the wall is rough or smooth. We note that the origin of the distances z for $z \gg h_0$ may be chosen arbitrarily between the bottom and top of the protrusions of the wall, without perceptibly changing the results; in practice, it is chosen usually from the condition of best agreement of the observed profile $\bar{u}(z)$ with the logarithmic equation (5.22′) [we shall return to this later in the subsection].

Equation (5.22a) for the velocity profile above a rough surface may be written in dimensionless form in various different ways. We may, of course, use Eq. (5.25), which is applicable in the case of a smooth wall, i.e., putting

$$\bar{u}(z) = u_*\left(A \ln \frac{zu_*}{\nu} + B\right) \quad \text{for} \quad z \gg h_0. \tag{5.25a}$$

However, here B will no longer be a universal constant, but will depend on the dimensionless parameters which determine the sizes, forms and positions of the irregularities of the wall: $B = B\left(\frac{h_0 u_*}{\nu}, \alpha, \beta, \ldots\right)$. Moreover, writing Eq. (5.22a) as Eq. (5.25a), although convenient for comparison of the mean velocity profiles close to smooth and rough walls, is not really natural (in any case, when $h_0 \gg \nu/u_*$). In fact, the right side of Eq. (5.25a) contains ν, while, as we would expect with $h_0 \gg \nu/u_*$, the mean velocity profile (and therefore the value of the coefficient A_1 in Eq. (5.22a) as well) is, in general, independent of the viscosity and will be determined solely by the size, form and relative position of the irregularities of the wall which completely determine the nature of the flow in the lowest layer. Thus for $h_0 \gg \nu/u_*$ it is more convenient to represent the mean velocity profile as

$$\bar{u}(z) = u_*\left(A \ln \frac{z}{h_0} + B'\right). \tag{5.28}$$

We may now expect that for sufficiently great $\frac{h_0 u_*}{\nu}$ the value of $B' = B + A \ln \frac{h_0 u_*}{\nu}$ will not change with variation of the parameter $\frac{h_0 u_*}{\nu}$, containing the viscosity ν (i.e., it will be a function of the parameters α, β, ... which describe the form and relative position of the irregularities only). It is also possible to rewrite Eq. (5.25a) in the form

$$\bar{u}(z) = u_* \left(A \ln \frac{z u_*}{\nu_e} + B_0 \right), \tag{5.25b}$$

where $B_0 \approx 5$ is the universal constant of Eq. (5.25) for a velocity profile above a smooth wall and $\nu_e = \nu \exp [(B_0 - B)/A)]$ may be called the *equivalent viscosity* [it obviously depends on the wall roughness, namely, $\nu_e = \nu \cdot f_1 (h_0 u_*/\nu, \alpha\, \beta, \ldots)$]. Finally, we can try to express the constant A_1 in Eq. (5.22a) directly in terms of the skin friction characteristics of the fluid on the wall, which depend in some unknown manner on the geometrical properties of the wall. As a dimensionless skin friction characteristic we normally use the friction coefficient (or friction factor)

$$c_f = \frac{\tau_0}{\frac{1}{2} \rho U^2} = 2\left(\frac{u_*}{U}\right)^2, \tag{5.29}$$

where U is the characteristic velocity of the flow. However, for an idealized flow in an infinite half-space, there is no characteristic velocity. Thus here we may only form the dimensionless coefficient

$$c_f(z) = 2\left[\frac{u_*}{\bar{u}(z)}\right]^2,$$

which depends on height, and is consequently of little use. But if we substitute

$$\bar{u}(z) = \frac{u_*}{\sqrt{\frac{1}{2} c_f(z)}}$$

into Eq. (5.22′) we obtain the following simple law:

$$\frac{1}{\sqrt{\frac{1}{2}c_f(z_2)}} - \frac{1}{\sqrt{\frac{1}{2}c_f(z_1)}} = A \ln \frac{z_2}{z_1}.$$

Hence we see that within the logarithmic layer

$$z_1 e^{-\frac{1}{A\sqrt{\frac{1}{2}c_f(z_1)}}} = z_2 e^{-\frac{1}{A\sqrt{\frac{1}{2}c_f(z_2)}}}.$$

Thus the quantity

$$z_0 = z e^{-\frac{1}{A\sqrt{\frac{1}{2}c_f(z)}}} = z e^{-\frac{\sqrt{2}\varkappa}{\sqrt{c_f(z)}}}, \tag{5.30}$$

which has the dimension of length, is independent of z. This quantity is an objective characteristic of the dynamic interaction of the flow with the wall, dependent, naturally, on the irregularities of the wall; it is called the *roughness parameter*, the *roughness length*, or simply the *roughness*. Now substituting

$$\sqrt{\frac{1}{2}c_f(z)} = \left[A \ln \frac{z}{z_0}\right]^{-1}$$

into

$$\bar{u}(z) = \frac{u_*}{\sqrt{\frac{1}{2}c_f(z)}},$$

we obtain

$$\bar{u}(z) = Au_* \ln \frac{z}{z_0} = \frac{u_*}{\varkappa} \ln \frac{z}{z_0}; \tag{5.31}$$

consequently,

$$z_0 = \frac{\nu}{u_*} e^{-\varkappa B} = h_0 e^{-\varkappa B'}.$$

According to Eq. (5.31) the roughness parameter may also be defined as the height at which the mean velocity of the flow would become zero if the logarithmic equation for $\bar{u}(z)$ were applicable down to this height; in fact, of course, the logarithmic equation ceases to apply at much larger values of z.

To find the dependence of B, B', ν_e, and z_0 on the height of the protrusions, we need to have results from experiments in which only their size but neither the shape nor the relative position is varied. Very careful experiments of this kind were carried out by Nikuradse (1933) in circular tubes. The walls of the tubes were coated with sand grains of a given size, which varied from experiment to experiment, and were glued on as close together as possible. The dependence relationship of the coefficient B' in Eq. (5.28) on $\frac{h_0 u_*}{\nu}$ obtained in these experiments is shown in Fig. 28. (The light and dark dots here correspond to the two different means of defining B': the first, based on direct comparison of the measured velocity profile with Eq. (5.28), the second described in Sect. 5.5.) For a homogeneous sand roughness with $\log \frac{h_0 u_*}{\nu} \leqslant 0.6$, i.e., with $\frac{h_0 u_*}{\nu} \leqslant 4$, and equation of the type

$$B' = A \ \ln \frac{h_0 u_*}{\nu} + B, \quad \text{where } A = 2.5, \quad B = \text{const}\,(\approx 5.5)$$

will hold. Thus, for $h_0 \leqslant 4 \frac{\nu}{u_*}$ we have $\nu_e = \nu$ and the velocity profile will be completely independent of h_0, i.e., the wall may be considered to be *dynamically smooth*[7]; we observe that with these values of h_0, the protrusions of the wall will be immersed entirely in the viscous sublayer; this explains the fact that they exert no effect whatsoever on the flow in the logarithmic layer. With $0.6 \leqslant \ln \frac{h_0 u_*}{\nu} \leqslant 1.7$, i.e., with $4 \leqslant \frac{h_0 u_*}{\nu} \leqslant 60$, a transition regime occurs in which the peaks of the protrustions project from the viscous sublayer and create additional disturbances, leading to a specific dependence of both the coefficients B and B' on $\frac{h_0 u_*}{\nu}$; the wall in this case must be

[7]The terms *hydraulically smooth or aerodynamically smooth* are also frequently used in the literature. However, the term adopted in the text is advantageous since it is unrelated to the specific nature of the fluid. The same remark is also true regarding the terms describing all other roughness regimes.

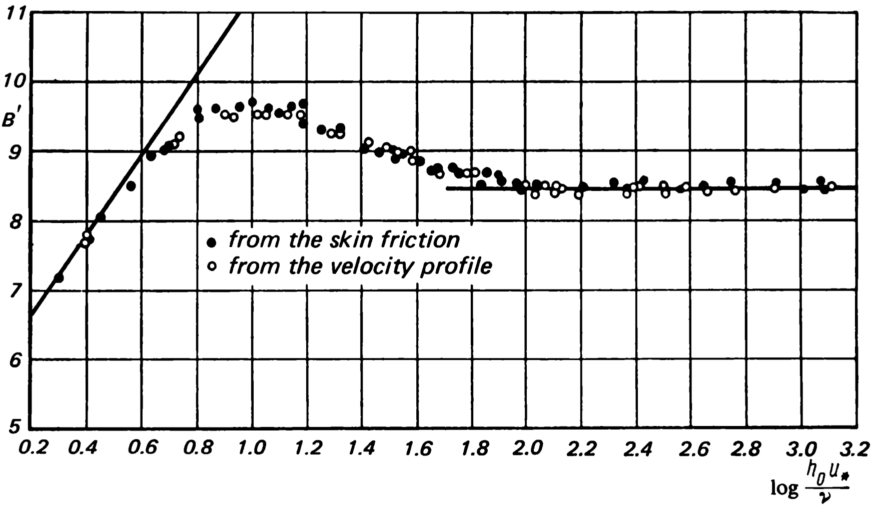


FIG. 28. Dependence of the coefficient B' on $\frac{h_0 u_*}{\nu}$ according to Nikuradse's data (1933).

considered to be *dynamically slightly rough*. Finally, with $\log \frac{h_0 u_*}{\nu} > 1.7$, i.e., with $\frac{h_0 u_*}{\nu} > 60$, the viscous sublayer practically ceases to exist, while the flow in immediate proximity to the wall consists entirely of eddies, generated by the flow around the individual protrusions; here the mean velocity profile $\bar{u}(z)$ is independent of the coefficient of viscosity ν, and B' is a constant (according to the data of Fig. 28, $B' \approx 8.5$). In this latter case the wall may be called *dynamically completely rough*. Of course, whether the wall is dynamically smooth, slightly rough or completely rough depends not only on the nature of its surface, but also on the values of u_* and ν (i.e., on the Reynolds number of the flow). The values of the three parameters B, B', and z_0 for dynamically smooth and dynamically completely rough walls according to Nikuradse's data (which refer only to a homogeneous sand roughness) are equal to

$$B \approx 5.5, \quad B' \approx 2.5 \ln \frac{h_0 u_*}{\nu} + 5.5, \quad z_0 \approx \frac{1}{9} \frac{\nu}{u_*} \quad \text{smooth wall}$$

$$B \approx -2.5 \ln \frac{h_0 u_*}{\nu} + 8.5, \quad B' \approx 8.5, \quad z_0 = \frac{h_0}{30} \quad \text{completely rough wall}$$

If we consider the data on the logarithmic layer above a homogeneous sand-roughened wall to be sufficiently complete, we may associate with roughness of any other type the *height of the* *equivalent sand roughness* h_s, which corresponds to the same logarithmic velocity profile for the same τ_0. For a number of artificial completely rough surfaces covered with geometrically regular protrusions, this height h_s was determined experimentally by Schlichting (1936) [see also Schlichting (1960), Chapt. X, Sect. 7]. In Schlichting's book, we may also find additional data and bibliographical references on the heights of the equivalent sand roughness both for artificial rough surfaces and for a number of ordinary surfaces used in engineering (concrete, cast-iron, steel, etc.) which in many cases prove to be dynamically completely rough. Similar data can be found in the book by Longwell (1966), papers by Chamberlain (1966; 1968) and some other sources.

Another procedure for describing the logarithmic velocity profile close to walls of differing roughness consists in determining the corresponding decrease in mean velocity $\bar{u}$ in comparison with the flow above a smooth wall with the same shear stress τ_0. The presence of irregularities always leads to the smoothing of the velocity profile in the immediate vicinity of the wall, i.e., to the slowing down of the rate of increase of the mean velocity for very small values of z. Therefore, in the case of a rough wall, the value of the velocity on the lower edge of the logarithmic layer, and hence throughout the layer, is less than that for a smooth wall. Using Eqs. (5.25) and (5.28) we may write

$$\frac{\delta\bar{u}}{u_*} = A \ln \frac{h_0 u_*}{\nu} + B - B', \tag{5.32}$$

where $A =$ const ≈ 2.5; $B = B_0$ = const ≈ 5. Thus $\frac{\delta\bar{u}}{u_*}$ is independent of z, and for a dynamically completely rough wall will depend linearly on $\ln \frac{h_0 u_*}{\nu}$. The dependence between $\frac{\delta\bar{u}}{u_*}$ and $\ln \frac{h_0 u_*}{\nu}$ will become nonlinear for the case of a slightly rough wall, while for values of h_0 where the wall becomes dynamically smooth, the difference $\delta\bar{u}$ is equal to zero. A considerable amount of experimental data on $\frac{\delta\bar{u}}{u_*}$, i.e., on the values of B', for different types of roughness in zero-pressure-gradient boundary layers and flumes were collected by Hama (1954) [see also Clauser (1956), Robertson (1957) and Perry and Joubert (1963)]. Figure 29, borrowed from

Clauser's survey paper (1956), gives some of the values obtained for $\frac{\delta\bar{u}}{u_*}$. We see that for sufficiently large values of $\frac{h_0 u_*}{\nu}$, $\frac{\delta\bar{u}}{u_*}$ will, in fact, be a linear function of $\ln \frac{h_0 u_*}{\nu}$, which corresponds to B' = const (case of a completely rough surface). However, transition to an all-intermediate regime, with deviation of the dependence of $\frac{\delta\bar{u}}{u_*}$ on $\ln \frac{h_0 u_*}{\nu}$ from linearity, for different types of roughness occurs at different values of $\frac{h_0 u_*}{\nu}$ (generally, between 30 and 100, according to the data of Fig. 29); moreover, the character of the dependence of $\frac{\delta\bar{u}}{u_*}$ on $\frac{h_0 u_*}{\nu}$ is completely different for different types of slightly rough surfaces. We note, further, that Fig. 29 allows us to determine very easily, for any roughness represented on it, the height h_s of the equivalent sand roughness, i.e., the sand roughness corresponding to the same value of $\frac{\delta\bar{u}}{u_*}$. For completely rough walls (but only for them) the height h_s will obviously be proportional to the height h_0;

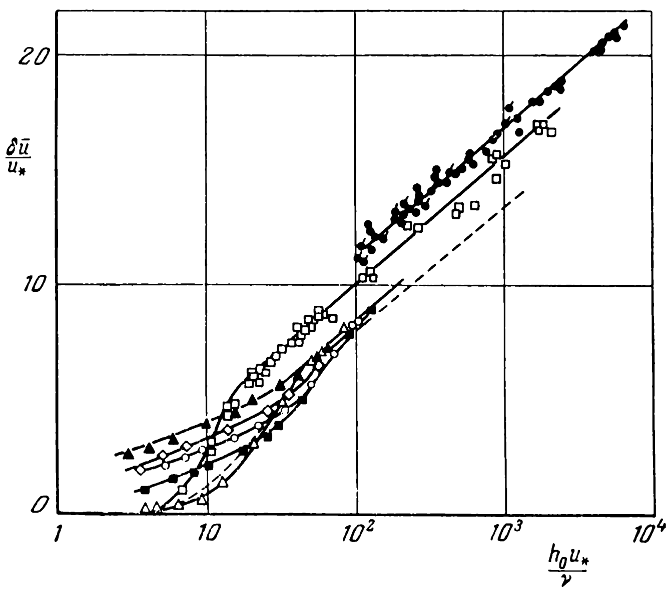


FIG. 29. Effect of wall roughness on the relative decrease of the mean velocity $\delta\bar{u}/u_*$ [according to Clauser (1956)]. The various symbols on the figure indicate data referring to different types of roughness. The dashed line corresponds to data for homogeneous sand roughness.

hence the roughness parameter $z_0 = \frac{h_s}{30}$ will in this case also be proportional to h_0.

We have already mentioned that the logarithmic dependence of the mean velocity on the height must also hold for the wind velocity in the lower layer of the atmosphere (thickness of the order of 10–100 m) for neutral (or close to neutral) thermal stratification [this fact, apparently, was first pointed out by Prandtl (1932a)]. The experimental verification of this was obtained by Sverdrup (1936), Paeschke (1936) and many other researchers. [See, e.g., Sutton (1953) and Priestly (1959a)]. In a number of cases observations on the wind profile were accompanied by direct measurements of the shear stress τ_0; for a discussion of the methods of such measurement see below, Sect. 8.3. Since $u_* = \sqrt{\frac{\tau}{\rho}}$ may be determined from the slope of the logarithmic profile, assuming $\varkappa \approx 0.4$, $A = \frac{1}{\varkappa} \approx 2.5$ is known from laboratory experiments, independent measurement of τ_0 allows the logarithmic equation (5.31) to be verified far more reliably than is possible when only observations of the vertical variation of wind velocity are available. It is only necessary to keep in mind that the accuracy of each individual measurement of τ_0 is usually not very high, thus sufficiently reliable results can only be obtained on the basis of considerable data [see, e.g., Perepelkina (1957)].

All methods of determining the friction velocity show that its values normally lie between 10 and 100 cm/sec in the atmosphere. Since for air $\nu \approx 0.15$ cm²/sec, it is clear that the underlying land surface of the atmosphere is almost always completely rough. The logarithmic profile of the wind velocity may conveniently be put in the form (5.31), using the concept of the roughness parameter z_0. Moreover, in the atmosphere, the correct choice of the origin of height z quite often is important. In fact the sizes h_0 of the irregularities of the ground here may have comparatively large values (for example, for tall grass, or fields covered with crops); at the same time increasing the height of the measurements z often proves quite frustrating, since as z increases, the role played by buoyancy effects also increases rapidly (see Chapt. 4 of this book). It is important here to consider the question of the form of the velocity profile at heights only slightly greater than the mean height z of the protrusions of the ground.

For $z \sim h_0$, the parameter h_0 (of the dimension of length) will also influence the mean flow; as already seen, the viscosity ν for a completely rough wall may be ignored. Hence $d\bar{u}/dz$ will be a function of τ_0, ρ, z, and h_0, from which we can form one dimensionless combination h_0/z. It then follows that instead of Eq. (5.21) for the mean velocity gradient, the more general equation

$$\frac{d\bar{u}(z)}{dz} = \frac{u_*}{\varkappa z} g\left(\frac{h_0}{z}\right) \tag{5.33}$$

will hold for $z \sim h_0$, where $\varkappa \approx 0.4$, and $g\left(\frac{h_0}{z}\right)$ is a correction function. The correction to the usual equation (5.28), of course, must cease to play a role at $z \gg h_0$, where it follows that $g(0) = 1$. Taking into account $\frac{h_0}{z} < 1$, we expand the function $g\left(\frac{h_0}{z}\right)$ as a power series in $\frac{h_0}{z}$, and obtain

$$\frac{d\bar{u}}{dz} = \frac{u_*}{\varkappa z}\left[1 + a\frac{h_0}{z} + b\left(\frac{h_0}{z}\right)^2 + \dots\right]. \tag{5.34}$$

We now introduce a new zero level, putting $z = z_1 + z'$, where z_1 is fixed and has the same order of magnitude as h_0, and z' is greater than both z_1 and h_0. Let us now replace z in the right side of Eq. (5.34) by z', and instead of expanding in powers of $\frac{h_0}{z}$, use an expansion in powers of $\frac{h_0}{z'}$. It is easy to see that the leading terms of this new expansion will take the following form:

$$\frac{d\bar{u}}{dz} = \frac{u_*}{\varkappa z'}\left[1 + \left(a - \frac{z_1}{h_0}\right)\frac{h_0}{z'} + b_1\left(\frac{h_0}{z'}\right)^2 + \dots\right]. \tag{5.35}$$

We now choose z_1 in such a way that the first-order terms in this expansion become zero (i.e., we put $z_1 = ah_0 = g'(0)h_0$). Then with accuracy to second-order terms we obtain

$$\frac{d\bar{u}}{dz} = \frac{u_*}{\varkappa(z - z_1)}, \tag{5.36}$$

where

$$\bar{u}(z) = \frac{u_*}{\varkappa} \ln \frac{z - z_1}{z_0}. \tag{5.37}$$

We see that when z is not too small, the effect of the finite height h_0

in the first approximation leads to the velocity profile being logarithmic, and the heights being measured from the given level $z = z_1$ and not from $z = 0$. The height z_1 may be called the *displacement height* or *zero displacement* (by analogy to the concept of the displacement thickness in boundary layer theory).

Data on the values of the roughness parameter z_0 and the displacement height z_1 for various types of natural surfaces may be found widely in the literature [see, for example, Paeschke (1936); Deacon (1949); Sutton (1953); Ellison (1956); Priestly (1959a); Tanner and Pelton (1960); Lettau (1967); Zilitinkevich (1970)]. Unfortunately, these data do not agree very well with each other (probably because both parameters z_0 and z_1 depend on fairly fine details of the underlying surface). As is obligatory for a completely rough surface, the roughness parameter z_0, determined according to the wind profile, may often be assumed approximately proportional to the height h_0 of the protrusions of the ground. However, for both ordinary grass cover and for agricultural crops, the proportionality coefficient is considerably greater than the coefficient 1/30 (which, according to Nikuradse's data, corresponds to a homogeneous sand roughness) and, generally, is close to 1/10 or even to 1/5. (Tanner and Pelton, for example, recommend $z_0 \approx h_0/7.5$ as a mean estimate for different grasses, crops, and bushes.) It is interesting that the results on h_0/z_0 obtained by Chamberlain (1966; 1968) for several forms of artificial roughness elements in a wind-tunnel are very close to the existing results for land with natural vegetation. Of course, the ratio h_0/z_0 must not be a strict constant for a set of irregularities of different forms; thus, it is not surprising that one of the empirical relationships obtained for different plant populations has the form $z_0 \approx h_0^{1.2}/11.7$ [see Lettau (1967)]. The displacement height z_1, for vegetation of not too great a height, may generally be taken equal to zero; however, for high vegetation it must often be taken between h_0 and $\frac{h_0}{2}$.

Further, we note that for high grass cover, the roughness parameter may also depend on the mean velocity of the wind, which bends the stalks and hence changes the form of the surface. For example, Deacon (1949), found that for tall grass (about 60 cm high), z_0 may vary from 9 cm in a very slight wind to about 4 cm in a strong wind. This is similar for water waves, where both the height and form of the irregularities also are clearly dependent on the wind velocity. If, in studying wind-waves, the viscosity and surface tension of the water are neglected and the roughness of the sea is defined entirely by the local atmospheric conditions (i.e., by the local value of the shear stress only), then from dimensionality arguments we obtain for the roughness coefficient z_0 an equation of the form

$$z_0 = b\frac{u_*^2}{g}. \tag{5.38}$$

Here g is the gravitational acceleration, $u_*^2 = \tau_0/\rho$ (where ρ is the air density), and the constant b may, in particular, depend on the ratio of the densities of air and water. The observations of the wind velocity profile over the sea obtained till now are still very incomplete and inexact; moreover, they do not leave the impression that the relationship (5.38) is well justified experimentally. (We note, for example, that Charnock (1958b) and Ellison (1956) who used this equation obtained very different values of b; according to Charnock, $b \approx 1/80$ and according to Ellison $b \approx 1/12$. Later, Kitaygorodskiy and Volkov (1965) [see also Kitaygorodskiy (1968) and Zilitinkevich (1970)] concluded that there is a very wide range of conditions in which Eq. (5.38) is approximately valid with $b \approx 1/30$. However, careful laboratory experiments by Hidy and Plate (1966) and Wu (1968) imply the estimate $b \approx 1/90$ and the data of field observations collected by Phillips (1966) are in agreement with the value $b \approx 1/40$.) In this connection, many researchers are inclined to consider that on the sea z_0 does not depend just on the local meteorological conditions [but, for example, on the wind fctch, which characterizes the path of the wind over the sea; cf., e.g., Hino (1966)]. At the same time, all the existing data indicate that the surface of the sea is dynamically considerably smoother than the majority of land surfaces on earth; even for a fairly strong wind on the sea $z_0 < 0.1$ cm [according to the data of several authors, the surface of the sea may even be taken to be dynamically smooth provided the wind is not very strong; see Sutton (1953); Deacon (1962)]. Therefore, in the study of winds over the sea, the concept of the roughness parameter is used comparatively rarely; more often, instead of z_0, in this case we use the values of the friction coefficient c_f, referred to the velocity $U = \bar{u}(z)$ at some fixed height z [see, for example, Wilson (1960); Deacon (1962); Deacon and Webb (1962); Roll (1965); Phillips (1966)]. In any case the question of the roughness of sea surface is, at present, far from being completely clear.

5.5 Turbulent Flows in Tubes and Channels; Skin Friction Laws

The general laws of turbulent flow along a rigid wall discussed above, refer to a wide class of flows. These include, in particular, flows in channels and tubes. We have already used some data on turbulence in channels and tubes in the two preceding subsections. However, flows in channels and tubes also possess some specific features lacking in the idealized flows in the half-space $z > 0$ with which we have dealt so far. First, in flow in a channel or tube unhke a flow in a half-space, there is a characteristic length H_1 (the half-width of the channel) or R (the radius of the tube) and a characteristic velocity U_0 (maximum mean velocity in the middle of the channel or on the axis of the tube) or U_m (average bulk velocity defined by

$$\frac{1}{H_1} \int_0^{H_1} \bar{u}(z)\, dz$$

or

$$\frac{2}{R^2} \int_0^{R} \bar{u}(r)\, r\, dr .$$

Thus, it is clear that the theory of flows in a constant stress layer close to a wall does not exhaust the whole theory of flows in channels and tubes. We shall consider this below.

We begin with general similarity considerations. For simplicity, let us assume first that the walls are dynamically smooth. We shall consider only the flow at sufficient distance from the intake into the channel or tube, for the intake conditions to have no effect. Moreover, we shall assume that the flow is steady and completely turbulent. In this case only the longitudinal component of the mean velocity $\bar{u} = \bar{u}_x$ will be nonzero, and all the statistical characteristics of the turbulence will depend on the single coordinate z, the distance from the wall of the channel or tube (for a tube $z = R - r$, where r is the distance from the axis). For a channel or tube of a given size and fluid of given density and viscosity, we shall have a one-parameter set of flows, defined by the value of the "pressure-head"—the constant longitudinal pressure gradient. The pressure gradient will also define uniquely such characteristics of the flow as the velocities U_0 and U_m and the shear stress at the wall τ_0, or, otherwise, the friction velocity $u_* = \sqrt{\frac{\tau_0}{\rho}}$. Thus, the values of the statistical characteristics of the flow at a distance z from the wall may depend on the following parameters: ρ; ν; z; H_1 (for a tube we shall now denote the radius by H_1, i.e., we put $H_1 = R$) and one of the velocities U_0, U_m or u_*. From these parameters we may formulate two dimensionless combinations: the Reynolds number $\xi = \frac{u_* H_1}{\nu}$ (it will be convenient to choose u_* as the characteristic velocity) and the dimensionless distance $\eta = \frac{z}{H_1}$.

Therefore we have

$$\bar{u}(z) = u_* \varphi\left(\frac{u_* H_1}{\nu},\ \frac{z}{H_1}\right) = u_* \varphi(\xi,\ \eta); \tag{5.39}$$

and this more general equation now replaces the law of the wall (5.13).[8]

Generally speaking, the mean velocity profile of a flow in a channel or tube is defined by a function of two variables (which, of course, can be different for both a channel and a tube). However, there are two important hmiting cases in which the function $\varphi(\xi, \eta)$ can be expressed in terms of a function of one variable.

The first of these cases (considered in detail in Sects. 5.2–5.4) is that of small z, i.e., $\eta = \frac{z}{H_1} \ll 1$. In this case the dependence of the velocity $\bar{u}(z)$ on the length H_1 becomes negligible, so that Eq. (5.39) is transformed into

$$\frac{\bar{u}(z)}{u_*} = f\left(\frac{u_* z}{\nu}\right) = f(\xi \cdot \eta) \qquad \text{for} \qquad \eta < \eta_0, \tag{5.40}$$

(where η_0 is some constant considerably less than one) which is Eq. (5.13). The second hmiting case (which we have not met before) is that of large values of $\eta = \frac{z}{H_1}$, corresponding to the "turbulent core" of a flow, close to the center of the channel or the axis of a tube. In this region of large values of η, remote from the walls, we may

[8]Equation (5.39) is, of course, equivalent to the equation $\bar{u}(z) = u_* f(zu_*/\nu)$ $(2H_1 - z)u_*/\nu)$, which was formulated in a footnote in Sect. 5.2 for a special case of a channel flow.

assume that the arguments used in Sect. 5.3 for the justification of the logarithmic velocity profile will apply. That is, since the turbulent shear stress here is many times greater than the viscous stress, there is reason to think that in the "turbulent core" the viscosity will have no direct effect on the flow (although it will affect it indirectly, since the boundary conditions on the boundary of the "core" depend on it, and also the value of u_*, which by Eqs. (5.17) and (5.17′) determines the distribution of the turbulent stresses through the whole flow). In the absence of a characteristic length scale, from here we obtain Eq. (5.21), leading to a logarithmic profile; when we have such a scale H_1, we obtain a more general equation of the form $\frac{d\bar{u}(z)}{dz} = \frac{u_*}{H_1}\varphi_1\left(\frac{z}{H_1}\right)$. Putting the unknown function φ_1 in the form $\varphi_1(\eta) = -f_1'(\eta)$ where $f_1(1) = 0$, and integrating the expression for $\frac{d\bar{u}(z)}{dz}$ from some value of z to $z=H_1$, we find

$$\frac{U_0 - \bar{u}(z)}{u_*} = f_1\left(\frac{z}{H_1}\right) = f_1(\eta) \quad \text{for} \quad \eta_1 \leqslant \eta \leqslant 1. \tag{5.41}$$

Equation (5.41) is usually called *the velocity defect law* and the region of its applicability is therefore sometimes called the *defect layer*. The law (5.41) was first formulated by von Kármán (1930) for tube flow, on the basis of the experimental data of Fritsch (1928). Later, the correctness of this law was carefully verified many times

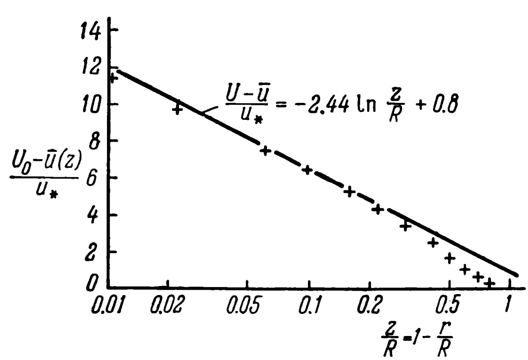


FIG. 30. Verification of the velocity defect law for turbulent flow in a tube according to Laufer's data (1954).

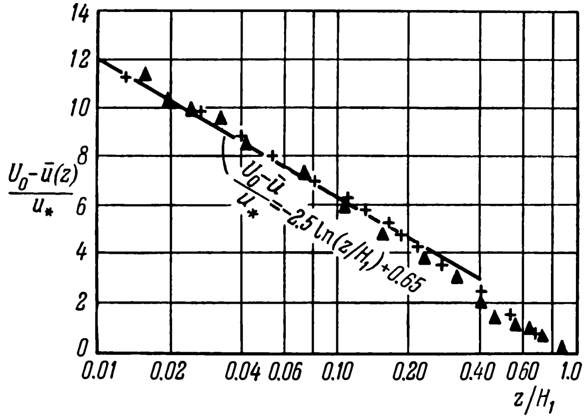


FIG. 31. Verification of the velocity defect law for turbulent flow in a rectangular channel according to the data of Laufer (▲) and (+).

and all the results were affirmative (see Figs. 30 and 31). Now this law is usually considered as a special case of the general principle of *Reynolds number similarity* which holds with considerable accuracy for a wide class of turbulent flows. According to this principle, for sufficiently large Reynolds numbers (with $\nu \lll K \sim UL$, where U and L are characteristic scales of velocity and length) for a relatively large region of turbulent flow (generally including almost the entire flow with the exception of comparatively thin layers adjacent to the walls), the mean motion will not be dependent directly on the viscosity coefficient (i.e., on the Reynolds number) which will affect the flow only through the boundary conditions and the value of U [see Townsend (1956)]. We must also note here that this Reynolds number similarity will occur not only for the mean velocity, but also for all the statistical characteristics of the flow that are not connected directly with the viscous dissipation of kinetic energy, i.e., for example, it holds for $\overline{u'^2}$ or $\overline{u'w'}$, but not for $\overline{\left(\frac{\partial u'}{\partial x}\right)^2}$. We shall return to this important assertion at the end of the present subsection.

Now we return to the mean velocity distribution and consider the important argument of Izakson (1937) [later developed in the works of Millikan (1938) and von Mises (1941)], which permits us in certain cases to establish the exact form of the functions f and f_1. Let us assume that $\eta_1 < \eta_0$, so that there exists some "overlap interval"

$\eta_1 < \eta < \eta_0$ of values of $\eta = \frac{z}{H_1}$, for which the general equation (5.29) may be put simultaneously in the form (5.40) and in the form (5.41) [this assumption, of course, cannot be justified theoretically, but must be verified by experimental data]. In this case, adding Eqs. (5.40) and (5.41), we obtain

$$f(\xi\eta) + f_1(\eta) = f_2(\xi) \qquad \text{for} \qquad \eta_1 < \eta < \eta_0, \tag{5.42}$$

where $f_2(\xi)$ denotes the value of $\frac{U_0}{u_*}$ which, clearly, can depend only on the Reynolds number $\xi = \frac{u_* H_1}{\nu}$, but not on $\eta = \frac{z}{H_1}$. From the functional equation (5.42) it follows easily that all three functions f, f_1, and f_2 are logarithmic functions. In fact, differentiating Eq. (5.42) first with respect to ξ and then with respect to η, we obtain

$$f'(\xi\eta) + \xi\eta f''(\xi\eta) = 0;$$

thus it follows that $f(\xi\eta) = A \ln \xi\eta + B$, where A and B are constants of integration. Thus we arrive once again at Eq. (5.25), starting this time from different assumptions. Substituting the expression for $f(\xi\eta)$ into Eq. (5.42), we confirm that

$$f_1(\eta) = -A \ln \eta + B_1, \quad f_2(\xi) = A \ln \xi + B_2, \quad B_2 - B_1 = B,$$

Thus the velocity defect law in the overlap region $\eta_1 < \eta < \eta_0$ must have the form

$$\frac{U_0 - \bar{u}(z)}{u_*} = -\frac{1}{\varkappa} \ln \frac{z}{H_1} + B_1. \tag{5.43}$$

If we assume, in addition, that Eq. (5.43) may be applied right up to the value $z = H_1$ (i.e., $\eta = 1$), then obviously $B_1 = 0$ (this last deduction, of course, does not follow automatically from the preceding arguments since they apply only to the region $\eta_1 < \eta < \eta_0$). Finally, the logarithmic equation for f_2 gives the following dependence of the ratio $\frac{U_0}{u_*}$ on the Reynolds number:

$$\frac{U_0}{u_*} = \frac{1}{\varkappa} \ln \frac{u_* H_1}{\nu} + B_2. \tag{5.44}$$

Introducing the friction coefficient (or friction factor)

$$c_f = \frac{\tau_0}{\frac{1}{2}\rho U_0^2} = 2\left(\frac{u_*}{U_0}\right)^2,$$

instead of $\frac{U_0}{u_*}$, we can rewrite the last equation in the form

$$\frac{1}{\sqrt{c_f}} = \frac{1}{\varkappa\sqrt{2}}\ln\left(\mathrm{Re}\sqrt{c_f}\right) + B_3; \quad \mathrm{Re} = \frac{U_0 H_1}{\nu}; \quad B_3 = \frac{B_2}{\sqrt{2}} - \frac{\ln 2}{2\varkappa\sqrt{2}}. \tag{5.45}$$

(Later, we shall see that this simple derivation of the logarithmic equation and friction law may be applied to a number of problems; see Sect. 5.7 and the end of Sect. 6.6.)

Until now, we have always assumed that the walls of the channel or tube are smooth. However, it is easy to see that the whole argument may easily be transferred to the case of a channel or tube with rough walls. In this case $\varphi(\xi, \eta)$ in Eq. (5.39) and $f(\xi\eta)$ in Eq. (5.40) may also depend on the additional arguments $h_0 u_*/\nu$ (or h_0/H_1), α, β, ..., which define the sizes, forms and relative positions of the irregularities of the wall. Nevertheless, it is natural to think that in the "core" of the flow the presence of roughness will be felt only through the boundary conditions and the value of the turbulent shear stress (depending on the value of the friction on the wall), and not directly; if this is the case, then the same equation (5.41) will be true for both smooth and rough walls. However, assuming that the regions in which Eqs. (5.40) and (5.41) are satisfied, partially overlap, we obtain again the functional equation (5.42), with the only difference being that now f and f_2 may also depend on additional parameters characterizing the roughness. Hence, as previously, it follows that for the overlap region $\eta_1 < \eta < \eta_0$ all three functions f, f_1, and f_2 must be logarithmic with common coefficient $A = 1/\varkappa$ for the logarithm; consequently, this coefficient must be a universal constant (just as the coefficient B_1 is). For B and B_2, these may contain some common summand depending on the size and nature of the roughness. If we assume that Eq. (5.43) is applicable right up to $z = H_1$, then $B_1 = 0$ and $B_2 = B$; thus on this assumption the data on the friction factor c_f allow the value of B (or B' of Eq. (5.28) which is simply connected with B) to be determined at once. The values of

B' for different $\frac{h_0 u_*}{\nu}$ represented by the dark dots in Fig. 28 were obtained in exactly the same way.

In the special case of a dynamically completely rough wall, the function

$$f\left(\xi\eta, \frac{u_* h_0}{\nu}, \alpha, \beta, \ldots\right)$$

must be independent of the viscosity ν, i.e., it must be a function only of $\xi\eta \cdot \frac{\nu}{u_* h_0} = \frac{z}{h_0}, \alpha, \beta, \ldots$. Therefore, it follows that the function $f_2\left(\xi, \frac{u_* h_0}{\nu}, \alpha, \beta, \ldots\right)$ will also depend only on $\xi \cdot \frac{\nu}{u_* h_0} = \frac{H_1}{h_0}$, $\alpha, \beta, \ldots$. Consequently, in this case Eqs. (5.44) and (5.45) take the form

$$\frac{U_0}{u_*} = \frac{1}{\varkappa} \quad \frac{H_1}{h_0} + B_2, \tag{5.44'}$$

$$\frac{1}{\sqrt{c_f}} = \frac{1}{\varkappa\sqrt{2}} \ln \frac{H_1}{h_0} + \frac{B_2}{\sqrt{2}}, \tag{5.45'}$$

where the coefficient B_2 may depend on the form and relative position of the irregularities of the walls.

Considerable experimental data confirming the correctness of Eq. (5.40) [which is the same as Eq. (5.13)] for turbulent flows in tubes and channels was collected in Fig. 25. Of course, if it is to be possible for the distances $z_+ = \frac{u_* z}{\nu}$ to be transformed into values of the variable $\eta = \frac{z}{H_1}$, we must know the number $\xi = \frac{u_* H_1}{\nu}$, which is a complicated function of the Reynolds number $\mathrm{Re} = \frac{U_0 H_1}{\nu}$, given implicitly by Eq. (5.45). As a guide, the table below is given to indicate the dependence of the lower boundary $\delta_l = 30 \frac{\nu}{u_*}$ of the logarithmic layer, for flows in smooth tubes, on the most widely used Reynolds number $\mathrm{Re}_D = \frac{U_m D}{\nu}$. (We shall discuss the connection between the maximum velocity U_0 and the bulk velocity U_m later.)

With increase of the Reynolds number, the ratio δ_l/H_1, naturally decreases, but not unboundedly. For every real tube, as the Reynolds number increases, an instant finally occurs when it appears to be completely rough. Then the lower boundary of the logarithmic layer begins to depend only on h_0 and does not vary with further increase of the mean velocity of flow.

Re_D	$5\cdot10^3$	10^4	10^5	10^6
$\frac{\delta_l}{H_1}=2\frac{\delta_l}{D}$	0.2	0.1	0.012	0.0016

For the upper boundary of the values of z for which the logarithmic equation (5.25) is applicable, it was already pointed out that, according to Nikuradse's data, this value, for both smooth and rough tubes, over a wide range of Reynolds numbers is quite close to the value $z = R$. (Thus the logarithmic layer can be extended without gross error to the center of the tube.) However, for the logarithmic equation to be apphced to the whole tube flow, it was necessary for Nikuradse to change the coefficients A and B somewhat in comparison with the values which gave the best agreement for the part of the flow next to the wall. It is clear, therefore, that the logarithmic equations proposed by Nikuradse for the whole flow cannot be considered to be in agreement with the theoretical equation (5.25) which is correct for $\delta_l/H_1<\eta<\eta_0$. In fact, the coefficients A and B in the theoretical equation (5.25) must be defined on the basis of the extensive data relating only to the part of turbulent flows adjacent to the wall; using only these coefficients, an attempt can be made to elucidate whether the equation for the velocity profile relating to the logarithmic layer will break down in the central part of the tube. Millikan (1938) was the first to attempt such an undertaking. He used almost all the scant data on flows in tubes and channels known at the time. Especially noteworthy is that for the values of A and B in Eq. (5.25), Millikan obtained $A = 2.5$, $B = 5.0$, which differ very little from what are now considered to be the most reliable values (see above, Sect. 5.3). Further, according to his estimates, the deviations of the mean velocity profile in a circular tube from the values given by Eq. (5.25) are completely unobservable up to approximately $\eta = 0.1$ (i.e., up to a 10% change in τ), while for all η they do not exceed 10% of the corresponding value of $\bar{u}(z)$. In spite of the fact that, at present, the quantity of data on the velocity profile in circular tubes has increased considerably since 1938 [see, for example, the survey article of Hinze (1962)], Millikan's deductions remain entirely valid even now.

From Nikuradse's results it is also clear that if both $A = 1/\varkappa$ and B_1 in the logarithmic form (5.43) of the velocity defect law are considered as empirical constants, then we may assume that this equation will apply right up to $z = H_1$, i.e., it may be in good agreement with the experimental data even for $B_1 = 0$. Theoretically, however, A cannot be considered as an arbitrary constant, but must be defined on the basis of measurements of the flow close to the wall. To obtain the best agreement, here we must take $B \neq 0$ [see, for example, Fig. 30 which is borrowed from Hinze (1959) and plotted on the basis of Laufer's measurements (1954); the value $A = 2.44$, i.e., $\varkappa \approx 0.41$ is chosen here in accordance with Clauser's recommendations (1956)]. Similar results are obtained for flows in a rectangular channel (see Fig. 31, borrowed from Schubauer and Tchen (1959), where A is put equal to 2.5); the deviation from the logarithmic equation (5.43) in both cases begins at about $\eta = 0.25$.

The problem of selecting an analytic equation for $f_1(\eta)$, which gives good agreement with the data over a large range of values of η, is considered in a large number of works [see, e.g., Gosse (1961)]. However, in practical problems usually it is possible simply to consider that the mean velocity profile in a tube is described by the logarithmic equation right up to the axis. Such a profile is very different from the parabolic profile of laminar Poiseuille flow. In fact due to the far more powerful radial mixing in turbulent flow, the velocity profile is everywhere (except in the thin layer on the walls) considerably more uniform than in laminar flow (see Fig. 32). Further, we note that with $\eta > 2.5$, $\ln \eta$ differs only slightly from $2.03\eta^{3/2}$; thus it is not surprising that in the first careful measurements of the turbulent velocity profile close to the axis of a tube carried out by Darcy (1858), an empirical formula was obtained which in our notation has the form

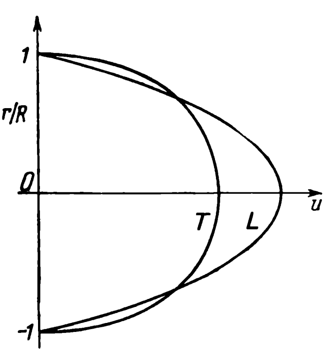


FIG. 32. Comparison of the mean velocity profiles for laminar (L) and turbulent (T) flows in a tube.

$$\frac{U_0 - \bar{u}(z)}{u_*} = 5.08\left(1 - \frac{z}{H_1}\right)^{\frac{3}{2}}. \tag{5.46}$$

Taking the velocity profile to be logarithmic right to the axis of the tube, and ignoring the thin layer on the walls of thickness $\delta_l = 30\ \nu/u_*$, to which the logarithmic equation is not applicable, we may also formulate a simple relationship between the maximum velocity on the axis of the tube U_0 and the averaged bulk velocity U_m. Integrating Eq. (5.43) with $B_1 = 0$, multiplied by $2\pi(H_1 - z)$ from $z = 0$ to $z = H_1$, and dividing the result by πH_1^2, we obtain

$$\frac{U_0 - U_m}{u_*} = -\frac{2}{\varkappa}\int_0^1 (1-\eta)\ln\eta\, d\eta = \text{const} \approx 3.75.$$

According to Eq. (5.44), for smooth tubes

$$\frac{U_m}{u_*} = \frac{1}{\varkappa}\ln\frac{u_* H_1}{\nu} + B_2', \quad B_2' = B_2 - 3.75. \tag{5.47}$$

This equation is also well confirmed by experiment.

Equation (5.47) allows us to formulate the skin friction law, i.e., the dependence on the Reynolds number $\mathrm{Re}_D = \frac{U_m D}{\nu}$ of the new dimensionless friction coefficient $\lambda = \frac{\tau_0}{\frac{1}{8}\rho U_m^2} = 8\left(\frac{u_*}{U_m}\right)^2$, which differs from the coefficient c_f in Eq. (5.45) not only by the unimportant numerical multiplier, but primarily by substituting the more easily measured bulk velocity U_m for the maximum velocity U_0. If in Eq. (5.47) we write $\frac{u_* H_1}{\nu}$ in the form of a product $\frac{U_m D}{\nu}\cdot\frac{u_*}{2U_m}$, we obtain

$$\frac{1}{\sqrt{\lambda}} = \frac{1}{\varkappa\sqrt{8}}\ln\left(\mathrm{Re}_D\sqrt{\lambda}\right) + B_3, \qquad B_3 = \frac{B_2 - 3.75 - \frac{1}{\varkappa}\ln 4\sqrt{2}}{\sqrt{8}}. \tag{5.48}$$

Here λ is determined easily from the known pressure drop $\Delta_l p = \frac{4l\tau_0}{D}$ along a section of the tube of length l and from the discharge

$Q = \frac{\pi D^2}{4} U_m$; thus the data on the dependence of λ on Re_D is considerable. Nikuradse (1932) showed that all these data are in very good agreement with Eq. (5.48) with the following values of the coefficients: $\frac{1}{\varkappa\sqrt{8}} = 0.87$ (corresponding to $\varkappa \approx 0.41$), $B_3 = -0.8$ [see Fig. 33, where together with the curve (5.48), the line $\lambda = \frac{64}{\mathrm{Re}_D}$ is also plotted corresponding, by Eq. (1.26) to the skin friction law for laminar flow in tubes]. According to Eq. (5.48), λ decreases without limit as Re_D increases; of course, this is not actually the case, since as Re_D increases, every tube will finally be completely rough. Once this happens, the friction coefficient λ will be independent of the Reynolds number and will be defined with good precision as

$$\frac{1}{\sqrt{\lambda}} = \frac{1}{\varkappa\sqrt{8}} \ln \frac{H_1}{h_0} + B_3', \tag{5.48'}$$

which follows from the assumption that Eq. (5.28) with $B' = \text{const}$ is applicable to all z for $0 < z < H_1$. This behavior of λ is illustrated clearly by the data of Fig. 34.

If great accuracy is not required, we may assume that for turbulent flow in a plane channel of width $2H_1$, the logarithmic form of the velocity defect law (5.43) will also hold right up to $z = H_1$, which corresponds to the center of the channel (in this case, of course, B_1 must equal zero). The same equation is often used in the case of flow in an open plane channel (or a flume) of depth H_1, bounded above by the free surface of the fluid; in a channel of this type, the maximum velocity U_0 will clearly be attained on the upper surface, so that here also, $B_1 = 0$. In addition, in the case of an open channel, sometimes instead of Eq. (5.43) the following equation of von Kármán (1930) is used:

$$\frac{U_0 - \bar{u}(z)}{u_*} = -\frac{1}{\varkappa}\left\{\left(1 - \frac{z}{H_1}\right)^{\frac{1}{2}} + \ln\left[1 - \left(1 - \frac{z}{H_1}\right)^{\frac{1}{2}}\right]\right\}. \tag{5.49}$$

Even better agreement with the experimental data may be obtained with the more general equation

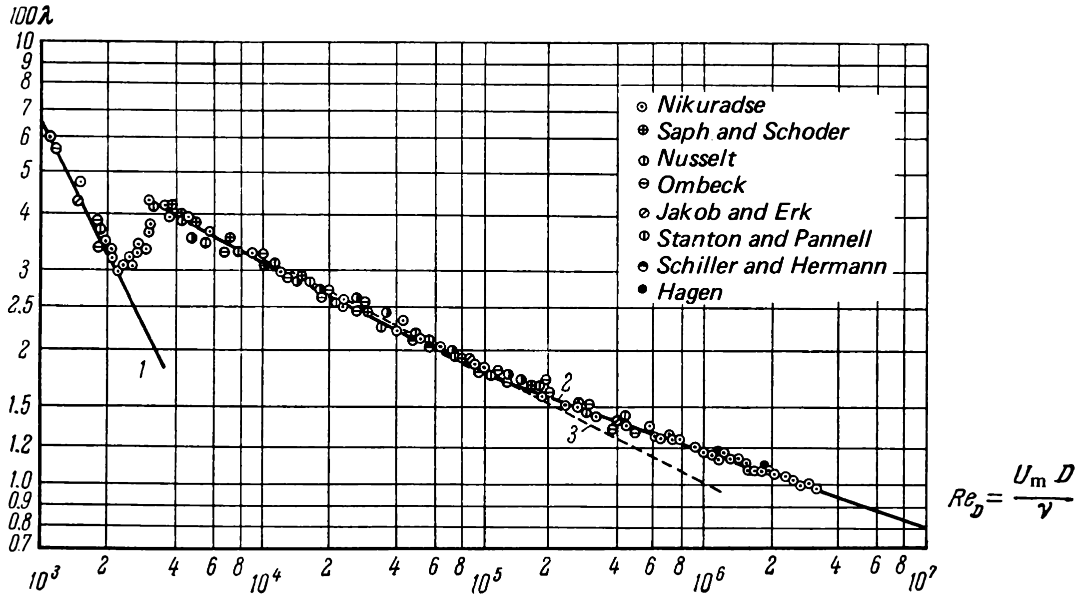


FIG. 33. Dependence of the skin friction coefficient λ on the Reynolds number Re_D according to the data of different authors. Curve 1 gives the skin friction law $\lambda = 64/Re_D$ for laminar Hagen-Poseuille flow; Curve 2 gives the law (5.48) with $\chi = 0.41$, $B_3 = -0.8$; the dotted line 3 gives the approximate law of Blasius (5.52).

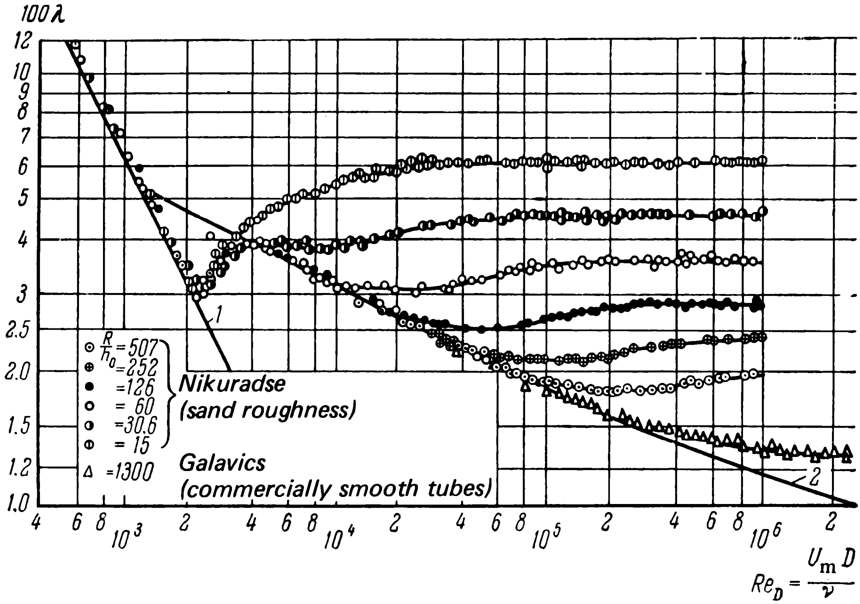


FIG. 34. The skin friction coefficient λ as a function of the Reynolds number Re_D for tubes of various roughness.

$$\frac{U_0-\bar{u}(z)}{u_*}=-\frac{1}{\varkappa}\left\{\left(1-\frac{z}{H_1}\right)^{\frac{1}{2}}+C\ln\left[\frac{C-\left(1-\frac{z}{H_1}\right)^{\frac{1}{2}}}{C}\right]\right\}, \quad (5.49')$$

where C_0 is a new numerical coefficient, which, according to Hunt's data (1964), is somewhat greater than unity. Finally, Ellison (1960), on the basis of certain theoretical considerations, proposed the use of an equation of the form

$$\frac{U_0-\bar{u}(z)}{u_*}=-\frac{1}{\varkappa}\left\{\ln\frac{1+\left(1+\frac{z}{H_1}\right)^{\frac{1}{2}}}{1-\left(1+\frac{z}{H_1}\right)^{\frac{1}{2}}}-\frac{1}{b}\ln\frac{b+\left(1-\frac{z}{H_1}\right)^{\frac{1}{2}}}{b-\left(1-\frac{z}{H_1}\right)^{\frac{1}{2}}}\right\} \quad (5.49'')$$

for open channels, instead of Eq. (5.49) or (5.49′). The right side of Ellison's equations for certain values of b, proves to be very close to the right side of Eq. (5.43) [with $B_1=0$] and to the right side of Eq. (5.49) or (5.49′). According to Ellison's indirect estimate of b (which will be discussed in greater detail in Chapt. 5), $b\approx 1.4$, which does not contradict the existing data on the mean velocity profile in open channels.

Let us consider briefly, the application of Reynolds number similarity to the moments of the velocity fluctuations in tube and channel flows. According to this similarity, e.g., all nonzero second moments, i.e., the four quantities $\overline{u'^2}$, $\overline{v'^2}$, $\overline{w'^2}$, and $\overline{u'w'}$, must be represented within almost all the flow, with the exception only of the wall layer which is quite thin for sufficiently large Re, by a formula of the form $u_*^2\psi_i(\eta)$ where $\eta = z/H_1$ and ψ_i, $i = 1, 2, 3, 4$, are universal functions (particular to each moment). There are extensive data on the four universal functions ψ_i, for example, in Townsend (1956) and Hinze (1959), in Comte-Bellot (1965), Coantic (1967a) and elsewhere. In particular, the value of the ratio $\overline{u'^2}/u_*^2$ on the tube axis must be a constant for all sufficiently great Re according to Reynolds number similarity; the correctness of this assertion is now confirmed by numerous data [the value of the constant is 0.788 ± 0.002 according to Coantic (1967a)]. Thus Eq. (5.44) shows that the relative intensity of the turbulence on the tube axis (which can be characterized quantitatively by the value of $(\overline{u'^2})^{1/2}/U_0$) must decrease continuously as Re increases; this fact is also known from experimental data (cf. the cited paper of Coantic).

Reynolds number similarity must apply also to the higher-order moments of the velocity fluctuations; however, at present, the experimental data on such moments are very scarce. Nevertheless, Comte-Bellot (1965) found that the dimensionless skewness and flatness factors of all three velocity components can be represented for almost all the flow region in a plane channel as universal functions of $\eta = z/H_1$ (in complete agreement with the principle of the Reynolds number similarity).

*For a finite range of values of Reynolds number, the general equation (5.48) may approximately be replaced by considerably simpler power skin friction laws of the form

$$\lambda = c\,(\mathrm{Re}_D)^{-m}. \tag{5.50}$$

Such power laws are obtained, in particular, with the approximation of the velocity profile $\bar{u}(z)$ by a power equation of the form

$$\frac{\bar{u}(z)}{u_*} = C\left(\frac{zu_*}{\nu}\right)^n \tag{5.51}$$

(which has often been done in the past). It is easy to see that Eq. (5.51) implies the equations

$$\frac{U_m}{U_0} = \frac{2}{(n+1)(n+2)}, \quad \frac{U_0}{u_*} = C^{\frac{1}{n+1}}\left(\frac{H_1U_0}{\nu}\right)^{\frac{n}{n+1}}, \quad m = \frac{2n}{n+1}. \tag{5.51'}$$

The best-known power friction law is Blasius' empirical law (1913)

$$\lambda = 0.3164\,(\mathrm{Re}_D)^{-\frac{1}{4}}, \tag{5.52}$$

which was formulated on the basis of data referring to Reynolds numbers Re_D not exceeding $5 \cdot 10^4$ (see Fig. 33). According to Eq. (5.51′), Blasius' law corresponds to the "one-seventh law" for the velocity profile, by which $\bar{u}(z) \sim z^{\frac{1}{7}}$; this "one-seventh law" (which agrees well with experiment at $\mathrm{Re} \approx 10^5$) has been used widely in many investigations since the 1920's.[9] However, none of the power laws is universal (i.e., suitable for all Reynolds numbers); for larger Reynolds numbers, smaller values of n must be used.

Of course, in addition to power equations, we may also choose other equations for $\lambda(\mathrm{Re})$, which approximate the implicit relationship (5.48) over a large range of values of Re. In particular, Nikuradse proposed the use of an approximate equation $\lambda = 0.032 + \dfrac{0.221}{\mathrm{Re}_D^{0.237}}$; other investigators proposed many other equations of this type (see, e.g., Kolmogorov s critical note (1952) which is devoted to several such equations).

In conclusion we must stress that Eqs. (5.48) and (5.48′) are also not rigorous because they are based on the assumption that the logarithmic equation for the velocity profile is applicable right up to the axis of the tube. More precise equations can be obtained with the aid of refined laws for the velocity profile which take into account the variability of the shear stress. One of the methods for correcting the approximate logarithmic form of the velocity distribution was developed by Townsend (1961) and A. J. Reynolds (1965) based on the general equation (5.39) supplemented by some seemingly natural but more specific hypotheses. As a result, they obtained an equation of the form

$$\frac{d\bar{u}}{dz} = \frac{u_*}{\varkappa z}\left(1 - B\frac{z}{\tau}\left|\frac{d\tau}{dz}\right|\right). \tag{5.21′}$$

Here $u_* = (\tau_0/\rho)^{1/2}$ is the usual friction velocity, $\varkappa = 1/A$ is the von Kármán constant, B is a new dimensionless constant, and $|d\tau/dz|$ for tubes and plane channels is equal to τ_0/H_1 according to Eqs. (5.17) and (5.17′). Equation (5.21′) is the refined form of the usual equation (5.21) [valid in the constant-stress layer only] which differs from the original equation by a correcting second term on the right side. It implies a special form of the velocity profile in the region outside the immediate proximity of the wall (i.e., in the region where the molecular viscosity does not play a considerable part), which transforms into the usual logarithmic profile in the thin constant-stress layer satisfying the condition $z\left|\frac{d\tau}{dz}\right| \ll \tau_0$ [see Townsend (1961)]. According to the experimental data collected and processed by A. J. Reynolds, $B \approx 1.1$ for circular tubes and $B \approx 1.7$ for two-dimensional channels; therefore the value $B = 1.5$ could be used simultaneously for both types of flows as a reasonable first approximation. (Reynolds obtained almost the same value $B \approx 1.47$ also for a boundary-layer flow with zero pressure gradient.)

A related (but more complicated) theory was developed by Kleinstein (1967) who showed that the effect of shear-stress variation could be included quite easily in the semiempirical arguments leading to Spalding's equation for the universal law of the wall

[9] More recently, Burton (1965) proposed the following simple generalization of the "one-seventh law": $z_+ = \bar{u}_+ + (\bar{u}_+/8.74)^7$. This generalized law has satisfactory accuracy in the whole law-of-the-wall layer and can be used for a refinement of Blasius' friction equation (5.52).

$\bar{u}_+ \quad f(z_+)$ [we have already mentioned that this equation was also derived by Kleinstein independently from Spalding]. However, the deviations of the logarithmic velocity profile from all the refined velocity distributions obtained by including the effect of shear-stress variation arc currently still within the range of the scatter of the existing data.

5.6 Turbulent Boundary Layer on a Flat Plate

Now we consider a turbulent boundary layer on a long flat plate with constant velocity U of the incident flow (i.e., with zero pressure gradient). The mean flow in this boundary layer will be steady and almost plane-parallel; in many ways, it will be close to the flow in a tube, the radius of which is equal to the thickness δ of the boundary layer, and the velocity of the axis equal to the velocity U outside this layer. However, there are at least two reasons for a breakdown in the analogy between the flow in a tube and in a boundary layer. First, the physical conditions on the outer edge of the boundary layer are quite different from the conditions at the center of a tube; outside the boundary layer (i.e., for $z > \delta$) there is usually no turbulence while the flow in a tube will be turbulent everywhere. Second, the characteristics of the boundary-layer flow depend not only on the coordinate normal to the plate z, but also (even if comparatively weakly) on the longitudinal coordinate x, reckoned along the plate. This leads to the fact that, theoretically, the flow in a boundary layer is considerably more complex than flows in channels or tubes.

The difference between flows in a boundary layer and in tubes and channels does not affect the thin layer of fluid in the immediate vicinity of the plate in which the universal "law of the wall" (5.13) apphes. As clearly seen from Fig. 25 where all the points lie on the same curve, this law is identical for tubes, channels and boundary layers. The value of u_* [which is found in Eq. (5.13)], in a boundary layer may, of course, vary with variation of x; however, this variation proves to be fairly slow (see below, at the end of this subsection), so that in any section x = const, the flow has time to become adjusted to local conditions, and depends only on the value of u_* for given x. The relative thickness of the layer to which the law (5.13) applies also varies with increase of x: for small $\mathrm{Re}_x = \frac{Ux}{\nu}$ not exceeding 10^6, Eq. (5.13) may be used approximately for all z almost up to the very edge of the turbulent boundary layer; for $\mathrm{Re}_x = \frac{Ux}{\nu}$ of the order of $10^7 - 10^8$, the total thickness of the viscous sublayer and logarithmic layer will no longer exceed 10–20% of the thickness δ. This last fact, of course, distinguishes boundary-layer flow from the flow in tubes and channels, where for all $\mathrm{Re} > \mathrm{Re}_{cr}$, the logarithmic equations may be used without great error right up to the axis of the tube or the middle of the channel.

The mean velocity profile in the outer part of the boundary is naturally characterized by the dependence on z of the velocity defect $U - \bar{u}(z)$. If we start from the assumption (which is confirmed well experimentally) that for an unperturbed boundary layer the velocity profile in every section $x = \text{const}$ depends only on the local conditions which refer to this section, the difference $U - \bar{u}(z)$ must be determined by U, δ, the parameters of the fluid ν and ρ and (in the case of a rough plate) the roughness parameter for given x. According to the general principle of Reynolds number similarity, we must expect that in the outer part of the flow, which is sufficiently distant from the wall, the viscosity ν and the roughness parameter will exert an effect only by means of the value of the shear stress on the wall τ_0, or the friction velocity $u_* = \sqrt{\frac{\tau_0}{\rho}}$, or the local friction coefficient $c_f = 2\left(\frac{u_*}{U}\right)^2$. Thus we obtain the following general form of the velocity defect law for a boundary layer:

$$\frac{U - \bar{u}(z)}{u_*} = f_1\left(\frac{z}{\delta}, \frac{u_*}{U}\right) \tag{5.53}$$

[see Rotta (1962a, b)]. To verify this law, it is convenient to plot a graph of $\frac{U - \bar{u}(z)}{u_*}$ as a function of $\frac{z}{\delta}$ and to attempt to explain the character of the scatter of the experimental points by dependence on the value of the ratio $\frac{u_*}{U}$. However, it is found that this scatter is masked completely by the ordinary scatter of experimental points. Therefore, within the limits of measurement errors, all the values of $\frac{U - \bar{u}(z)}{u_*}$ lie with sufficient accuracy on a single curve [see, for example, Fig. 35, borrowed from Clauser's survey article (1956)]. Thus the dependence of the ratio $\frac{U - \bar{u}(z)}{u_*}$ on $\frac{u_*}{U}$ (which varies slowly with variation of Re_x) even if it occurs, is quite weak.[10]

[10] Crocco (1965) attempted to unmask such a dependence in the experimental data and to explain it by a special semiempirical theory based on a refined form of Coles' "wake law" (this law will be discussed later). However, his attempt was not completely successful and will not be discussed in this book (cf., however, the footnote on Coles' "wake law" later in this subsection).

Consequently, in practice, the velocity defect law for a boundary layer on a flat plate may be written as

$$\frac{U-\bar{u}(z)}{u_*}=f_1\left(\frac{z}{\delta}\right), \tag{5.54}$$

which is completely analogous to the law used above for flows in tubes and channels.

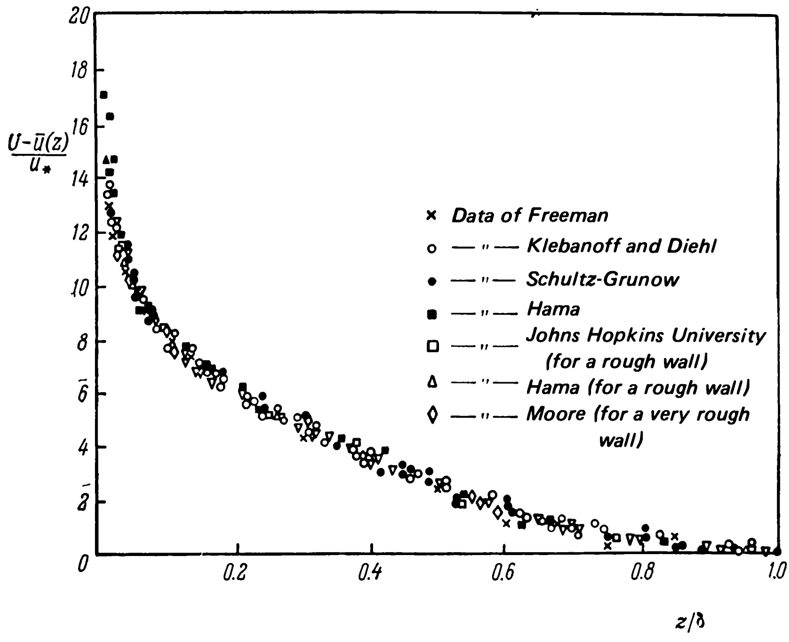


FIG. 35. Verification of the "velocity defect law" for a turbulent boundary layer, according to the data of different authors.

In comparison with the velocity defect law for tubes and channels (5.41), the law (5.54) has only one deficiency: whereas in Eq. (5.41) we have the precisely defined length H_1, in Eq. (5.54) we have the thickness of the boundary layer δ which has no precise definition and is determined only very approximately from the experimental data. At the same time, the ratio of δ to more precisely defined thicknesses of the type, e.g., of the displacement thickness δ^* in a turbulent boundary layer (as distinct from the laminar case) is variable (dependent on u_*/U). In particular, if the law (5.54) holds, then from the very definition (1.53) of the displacement thickness δ^* it follows that

$$\delta^* = C\frac{u_*}{U}\delta, \qquad C = \int_0^1 f_1(\eta)\,d\eta = \text{const}. \tag{5.55}$$

(We ignore here the thickness of the layer of nonnegligible viscous stresses to which the law (5.54) is inapplicable.) Taking this relationship into account, Rotta (1951a) proposed to use in Eq. (5.54) instead of the argument $\eta = z/\delta$, the dimensionless argument $\eta_1 = \frac{z}{\delta^*}\frac{u_*}{U}$, which does not contain δ; by Eq. (1.53), this argument will also be convenient for the normalization condition $\int_0^\infty \frac{U-\bar{u}}{u_*}\,d\eta_1 = 1$. We now assume that $\frac{U-\bar{u}}{u_*} = \varphi_1(\eta_1)$, where $\varphi_1(\eta_1) = 0$ for $\eta_1 = c$; then, clearly, $\delta = c\frac{U}{u_*}\delta^*$ and $\varphi_1(\eta_1) = f_1\left(\frac{\eta_1}{c}\right)$. Consequently, in reducing the observations it is completely permissible to replace δ by $\delta^*\frac{U}{u_*}$, and if $\frac{U-\bar{u}}{u_*}$ is a single-valued function of $\eta_1 = \frac{z}{\delta^*}\frac{u_*}{U}$, this means that the law (5.54) is also satisfied.

Of course, Eq. (5.54) cannot be satisfied right up to $z = 0$. This is because it does not contain the viscosity, and does not take the boundary conditions $\bar{u}(0) = 0$ into consideration. In the immediate vicinity of the wall the universal law (5.13) applies, and, as we have seen, in the overlap region where both these laws apply simultaneously, the functions f and f_1 are bound to be logarithmic. Thus it is not surprising that after transfer of the data of Fig. 35 to a new graph, where a logarithmic scale is used for the abscissa, the function $f_1(\eta)$ from $\eta = 0.01$ to approximately $\eta = 0.15$ will be a straight line $-A \ln \eta + B_1$ (see Fig. 36, where only a part of the points given in Fig. 35 are taken into account).[11] We note that here the coefficient $A = 1/\varkappa$ has exactly the same value as in Fig. 31 for plane channels as is required by the universality of the logarithmic law (5.25); however, B_1 differs for channels and for boundary layers. Moreover, comparing Figs. 36 and 31 makes it clear that for $\eta > 0.15$, $f_1(\eta)$ for a boundary layer diverges far more sharply from the straight line $A \ln \eta + B_1$ than the corresponding function for plane channels. According to Hama (1954), the value of $f_1(\eta)$ for boundary layers in the region $\eta > 0.15$

[11] Similar graphs of the function $\phi_1(\eta_1)$ for boundary layers on smooth and rough flat plates can be found, e.g., in the papers of Rotta (1962a, b) and Furuya and Fujita (1967).

is well described by the empirical formula $f_1(\eta) = 9.6\ (1 - \eta)^2$; for flows in plane channels, however, $f_1(\eta)$ for $\eta > 0.15$ may be described reasonably well by the equation $f_1(\eta) = A' \ln\eta + B_1'$, where A' and B_1' differ only slightly from A and B_1.

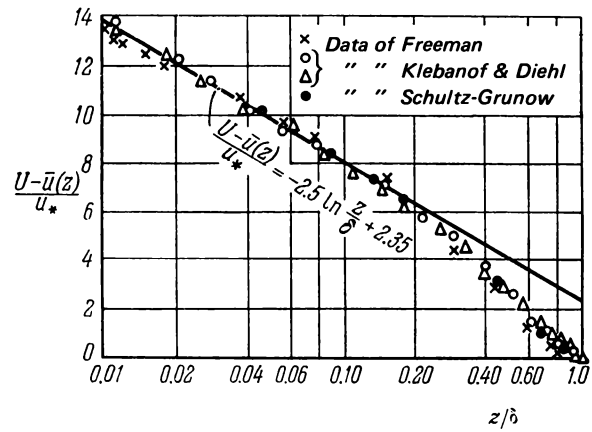


FIG. 36. Logarithmic form of the velocity defect law for a boundary layer.

The considerable difference between the velocity profile in a turbulent boundary layer close to its outer edge and the velocity profile in tubes and channels close to the axis can easily be explained by the difference between the nature of the turbulent flow itself in the outer part of the boundary layer and the central part of the tube or channel. In this respect, we observe that, unlike a laminar boundary layer which has no sharply defined outer edge, a turbulent boundary layer normally has a definite edge, outside of which there is no turbulence (i.e., the flow is actually irrotational). This edge (or interface) has an irregular outline and varies in a disordered manner with time. This is clearly visible in photographs of a turbulent boundary layer obtained by the shadow-graph method, which renders the perturbed region of flow visible (see, e.g., Fig. 37). The general diagram of the turbulent boundary layer corresponding to this photograph is given in Fig. 38. According to the experimental data of Klebanoff (1955), Corrsin and Kistler (1955), Fiedler and Head (1966) and others, the upper edge of a turbulent boundary layer at any given point will fluctuate between approximately $0.3\delta - 0.4\delta$ and 1.2δ (where δ is the distance from the wall at which

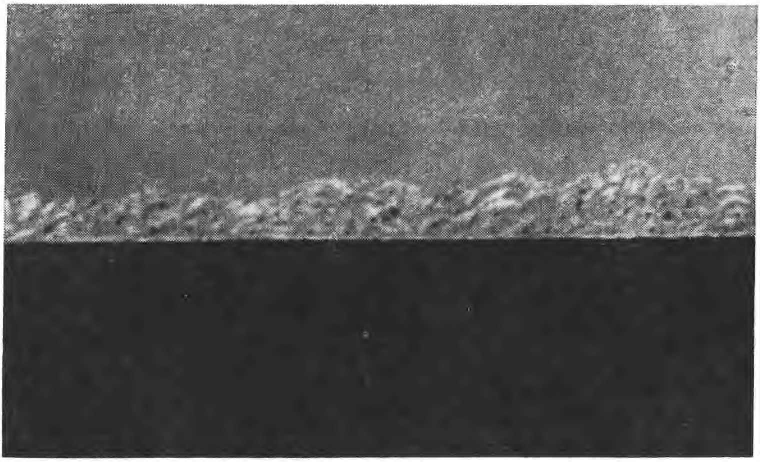


FIG. 37. Shadow graph of the turbulent boundary layer on a cylinder [from Rotta's papers (1962a, b)]. Direction of flow is from left to right.

the mean velocity equals $0.99U$) almost according to the normal distribution, with mean value $0.78\delta - 0.80\delta$. The fraction of time $\gamma(z)$ during which turbulence will be observed at a distance z from the wall decreases with increase of z, and with $z = \delta$ equals only about 0.06. Consequently, it is clear why the logarithmic profile characteristic for developed turbulence may be used with reasonable accuracy almost to the center of a tube or channel. However, in a boundary

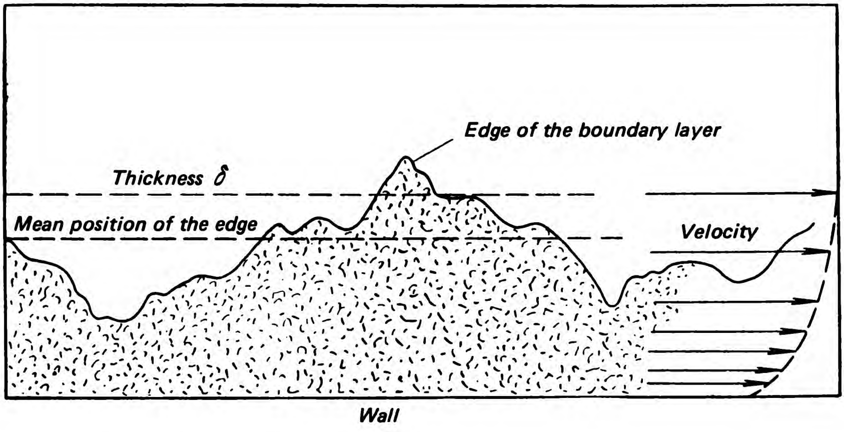


FIG. 38. General diagram of a turbulent boundary layer on a flat plate.

layer with large Re, it is satisfied comparatively accurately only for a thin layer of fluid close to the wall. Outside this region of flow, periods of the presence and the absence of turbulence follow each other alternately in rapid succession. Thus the mean velocity obtained by averaging over a sufficient time interval is actually a mean between values of the velocity corresponding to both laminar and turbulent flow. (By assuming that the logarithmic wall law in fact applies in the outer part of the boundary layer, but only when the turbulence is present, and that the irrotational velocity is that of the free stream, B. G. J. Thompson (1963; 1965) was able to obtain a two-parameter family of standard velocity profiles which gives a good agreement with measured boundary layer velocity profiles [cf. also Dvorak and Head (1967)].) Since the exchange of momentum between neighboring layers of fluid in laminar flow is considerably less than in turbulent flow (hence, the velocity gradient is sharper), it is clear that in the outer part of a turbulent boundary layer the mean velocity, with increase of distance from the wall, must increase more rapidly than by the logarithmic law. This is confirmed by the data of Fig. 36. Also, for a boundary layer occurring as a result of high-intensity turbulent flow along a flat plate, the difference between the conditions on the outer edge of this layer and at the center of a tube must be less considerable. In fact, the experiments of Wieghardt (1944) and others, who passed the flow first through a grid producing a high turbulence level, show that the form of $f_1(\eta)$ in Eq. (5.54) depends on the turbulence level of the incident flow and as this level increases, it approximates the form characteristic of flows in tubes and channels.

An interesting attempt to combine the universal law of the wall (5.13) with the velocity defect law (5.54) was made by Coles (1956). On the basis of Eq. (5.13) the general form of the mean velocity in every section x = const of the boundary layer may be written as

$$\bar{u}(z) = u_*\left[f\left(\frac{zu_*}{\nu}\right) + h(z)\right], \tag{5.56}$$

where $f(z_+)$ is the function represented in Fig. 25 (and transformed into $A \ln z_+ + B$ for $z_+ > 30$) and $h(z)$ is a correction to the universal law of the wall (5.13), which becomes zero in immediate proximity to the plate (i.e., for $z < 0.1\delta$). However, in this case $\frac{U-\bar{u}}{u_*} = A \ln \frac{\delta}{z} + h(\delta) - h(z)$ for $z > 30\frac{\nu}{u_*}$, and, hence, for the universal velocity defect law to be satisfied, $h(z)$ must depend only on $\frac{z}{\delta}$. On this basis, Coles proposed that

$$\frac{\bar{u}(z)}{u_*} = f\left(\frac{zu_*}{\nu}\right) + \frac{\Pi}{\varkappa} w\left(\frac{z}{\delta}\right), \tag{5.57}$$

where $\varkappa = \frac{1}{A} \approx 0.4$ is von Kármán's constant, Π is a new constant and $w(\eta)$ is a function of a single variable normalized by the condition $w(1) = 2$. However, a certain indeterminacy still remains connected with the fact that the thickness δ is not uniquely defined; using this fact, Coles added to $w(\eta)$ the additional normalization condition

$$\int_0^1 w(\eta)\, d\eta = 1,$$

which, as is easily seen, is equivalent to the condition $\delta^* \frac{U}{u_*} = \frac{1+\Pi}{\varkappa} \delta$. Consequently, by this condition he gave a definite form to the coefficient C in Eq. (5.55), i.e., he derived an exact connection between the scale δ and the displacement thickness δ^*, which may easily be found from experiment, the value of $\frac{u_*}{U}$ and the parameters $\varkappa$ and Π. On the basis of considerable experimental data on velocity profiles in boundary layers on a plate, for both constant pressure and in the presence of a pressure gradient, Coles found that for a broad class of two-dimensional turbulent boundary layers, the function $w(\eta)$ is the same. Thus, according to his data, the external conditions of flow, including the pressure distribution in a free flow, are reflected only in the value of the factor Π. In the case of a compound pressure distribution, this factor can be assumed to be dependent on the x-coordinate; in the simplest case of flow past a plate with zero pressure gradient (i.e., at constant velocity) Π = const ≈ 0.55 for all not too small Reynolds numbers.[12] The universal function $w(\eta)$, obtained by Coles by evaluating the experimental

[12]According to an unpublished report by Coles in 1962 [reviewed by Crocco (1965)], Π = const at $\mathrm{Re}_{\delta^{**}} = U\delta^{**}/\nu \gtrsim 6000$, while for values of $\mathrm{Re}_{\delta^{**}}$ less than about 6000, Π decreases rapidly and becomes zero at $\mathrm{Re}_{\delta^{**}} \approx 500$. According to Crocco, Π is slightly dependent on $\mathrm{Re}_{\delta^{**}}$ (i.e., on u_*/U) at all values of $\mathrm{Re}_{\delta^{**}}$.

data is shown graphically in Fig. 39. It possesses an almost exactly skew-symmetric, S-shaped form, and is fairiy well approximated by the section of the sinusoidal curve $w = 1 + \sin\frac{(2\eta - 1)\pi}{2} = 1 - \cos\pi_\eta$ for $0 \leqslant \eta \leqslant 1$. (This form of $w(\eta)$ was used extensively, e.g., by Spalding (1964) and Escudier and Nicoll (1966) for many calculations related to a broad class of flows with one plane rigid boundary.)

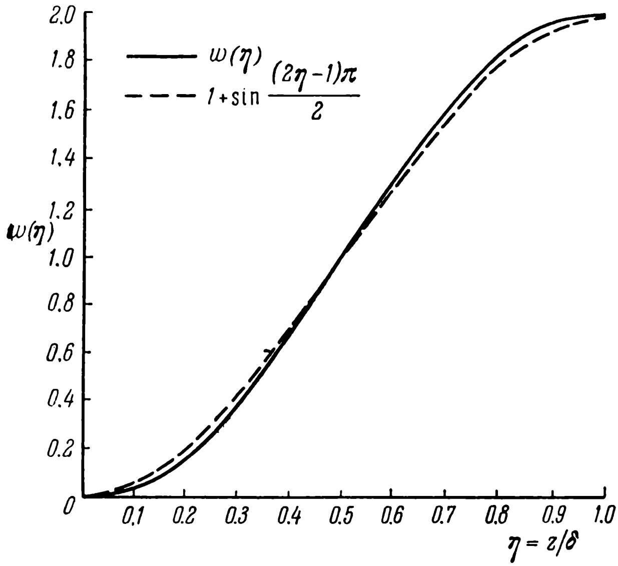


FIG. 39. Universal wake function $w(\eta)$ according to Coles (1956).

In Eq. (5.57), we may proceed to the limit formally as $u_* \to 0$ (i.e., as $\tau_0 \to 0$); with the aid of Eq. (5.55) and the condition $w(1) = 2$, we obtain the simple result

$$\bar{u}(z) = \frac{U}{2} w\left(\frac{z}{\delta}\right).$$

Thus the function $w(\eta)$ gives the form of the mean velocity profile at a point with $\tau_0 = 0$, i.e., at a point of separation of the boundary layer, at which $\frac{d\bar{u}}{dz} = 0$ (see the schematic representation in Fig. 8). On this basis, Coles called the function $w(\eta)$ the *wake function*, and the law by which the deviation of the mean velocity profile in the boundary layer from the universal law of the wall (5.13) is strictly proportional to the universal wake function, the *wake law*. The general form of the mean velocity profile $\bar{u}(z)$ satisfying this wake law is shown schematically in Fig. 40. Here the dashed line represents the distribution of the mean velocity in the upper half of a hypothetical turbulent wake. This is produced basically by the interaction of the turbulent core of this wake with the outer laminar flow, and by the fluctuations of the interface between the turbulent and nonturbulent parts of the flow in accordance with Fig. 38; the difference of the mean velocities at the center and on the boundary

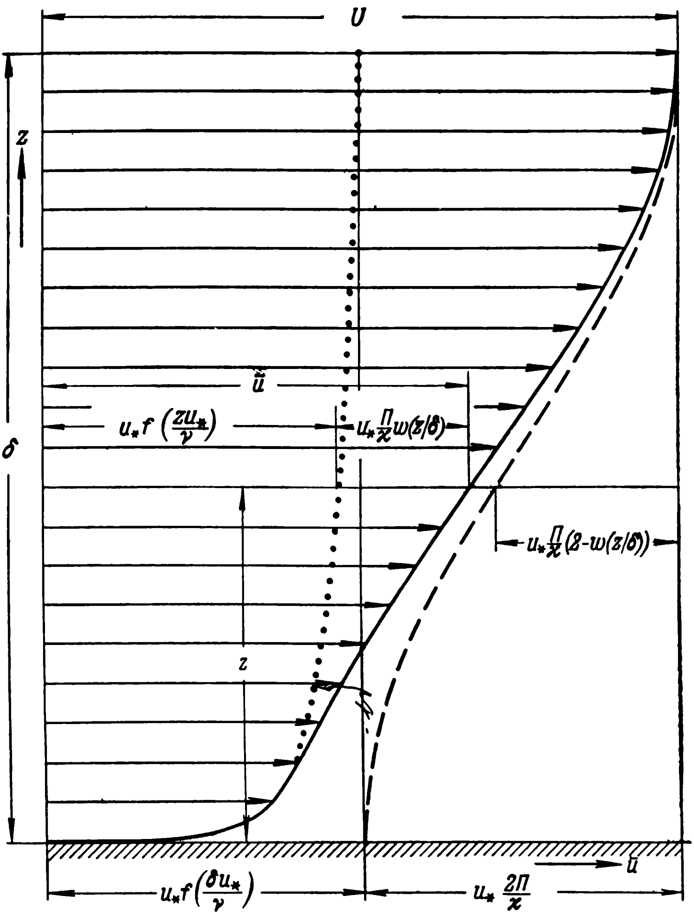


FIG. 40. Schematic representation of the velocity profile satisfying Coles' wake law.

of this wake is equal to $\frac{2\Pi}{\varkappa} u_*$. If we place a rigid plate at the center of this wake, then the velocity distribution in the wake is changed considerably. This is due to the fact that close to the surface, the flow begins to be strongly constrained by viscous effects caused by the no-slip condition $\bar{u}(z) = 0$. This additional limitation, in pure form, generates the "universal wall profile" shown in Fig. 40 by the dotted line. As a result of superimposing this "wall profile" on the wake profile, the real velocity profile in a turbulent boundary layer, represented by the solid line in Fig. 40, is obtained.

Another approach to the problem of the velocity profile in the boundary layer was developed by B. G. J. Thompson (1963; 1965). We have mentioned it earlier in this subsection; it is clearly related to Coles' approach, although it is based on quite different assumptions.

Similar to flows in a tube, from the existence of a section of the flow in which both the wall law (5.13) and the velocity defect law (5.54) are satisfied (with logarithmic functions f and f_1), it follows that the local friction coefficient

$$c_f = \frac{\tau_0}{\frac{1}{2}\rho U^2} = 2\left(\frac{u_*}{U}\right)^2$$

must satisfy an equation of the form

$$\frac{1}{\sqrt{c_f}} = \frac{1}{\varkappa\sqrt{2}} \ln\left(\mathrm{Re}_\delta \sqrt{c_f}\right) + B_3, \qquad \mathrm{Re}_\delta = \frac{U\delta}{\nu} \tag{5.58}$$

[cf. the deduction of Eq. (5.45) in Sect. 5.5]. Just as with the law (5.54), the presence in Eq. (5.58) of an inexactly defined boundary-layer thickness δ makes this equation unsuitable for practical applications; thus it is desirable to replace δ by another more easily defined scale of length. Most frequently, one uses Eq. (5.55), which allows Eq. (5.58) to be rewritten in the form

$$\frac{1}{\sqrt{c_f}} = \frac{1}{\varkappa\sqrt{2}} \ln \mathrm{Re}_{\delta^*} + B_4, \qquad \mathrm{Re}_{\delta^*} = \frac{U\delta^*}{\nu}. \tag{5.59}$$

A comparison of this relationship with the data of direct measurements of the values of the local friction coefficient at different points of the boundary layer on a smooth flat plate, obtained by

Schultz-Grunow (1940) and Smith and Walker (1959), is given in Fig. 41; in the figure, the experimental values $\frac{1}{\sqrt{c_f}}$ follow the right side of Eq. (5.58) fairly closely with $\varkappa = 0.40$ and $B_4 = 2.6$.

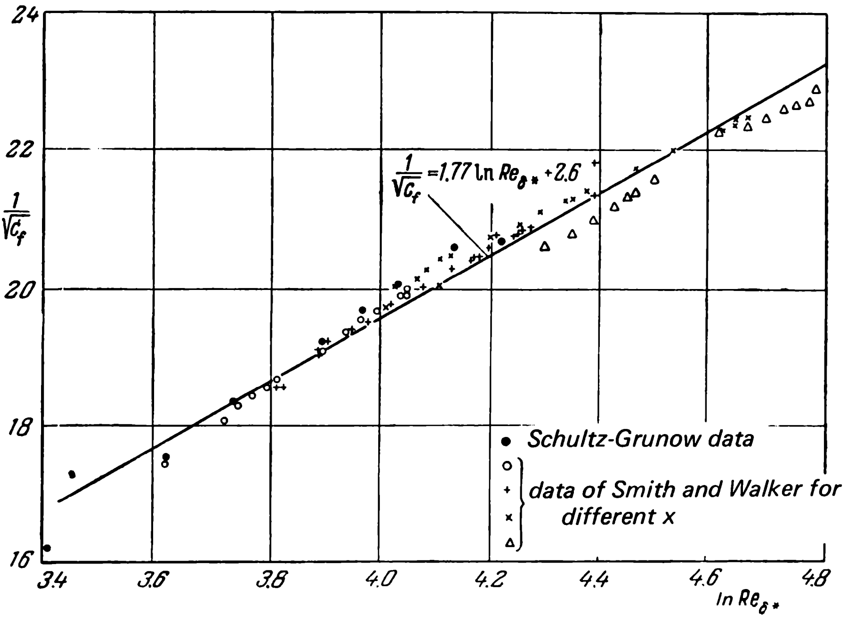


FIG. 41. Dependence of the local skin friction c_f for the boundary layer on a flat plate on the Reynolds number $Re_{\sigma*}$ according to the data of Schultz-Grunow and of Smith and Walker.

Replacing δ simply by the length x (reckoned from the origin of the plate) is somewhat more complicated. For this purpose we must express the thickness δ in terms of x, which may be carried out using the following approximate considerations. Since the streamlines of rotational turbulent flow in the boundary layer do not pass outside the layer, the mean outer edge of the boundary layer must coincide with a streamline of the mean motion at a distance δ from the plate. Therefore, it follows that the tangent of the angle of inclination of this edge to the Ox axis must be equal to the ratio of the mean vertical velocity (in the direction of the axis Oz) to the mean horizontal velocity at points of the edge. However, the mean horizontal velocity on the edge of the boundary layer is equal to U; the mean vertical velocity is independent of U and is determined only by the relative fluctuating motion connected with the transfer of momentum in the Oz direction. Consequently, by dimensionality considerations, this vertical velocity must be proportional to the friction velocity u_*, formulated with the value of τ_0 for given x. Thus follows the relationship

$$\frac{d\delta}{dx}=b_1\frac{u_*}{U}, \tag{5.60}$$

where b_1 is a numerical coefficient (of the order of unity). We now assume that the boundary layer is turbulent, originating, practically speaking, from the leading edge of the plate itself. Thus Eq. (5.60) is satisfied for all x beginning with $x=0$ (where $\delta=0$). Generally speaking, the quantity u_*/U is a function of x, but since it varies very slowly, by integrating Eq. (5.60), we may take it outside the integral sign without considerable error (changing the coefficient b_1 somewhat at the same time if need be). Therefore, we obtain the approximate equation

$$\delta=b_1\frac{u_*}{U}x. \tag{5.61}$$

Strictly speaking, Eq. (5.61) follows from Eq. (5.60) only if

$$\frac{u_*}{U}=\sqrt{\frac{c_f}{2}}$$

may be represented as a simple power function of x. However, it will be seen from the following discussion that such a representation possesses fairly high accuracy; see, for example, Eq. (5.66), a comparison of which with experimental data is given in Fig. 42.

Substituting Eq. (5.61) into Eq. (5.58), we obtain

$$\frac{1}{\sqrt{c_f}}=\frac{1}{\varkappa\sqrt{2}}\ln(\mathrm{Re}_x\cdot c_f)+B_5. \tag{5.62}$$

This result was given first by von Kármán (1930; 1934). He obtained it by means of somewhat different arguments and it is in good agreement with Kempf's (1929) older experimental data obtained with the aid of direct dynamometer measurements of the local skin friction coefficients. Von Kármán's result is also in good agreement with the later experimental data of Schultz-Grunow (1940) and Dhawan (1952) [see Fig. 42, where the coefficients $\frac{1}{\varkappa\sqrt{2}}=1.8$, i.e.,

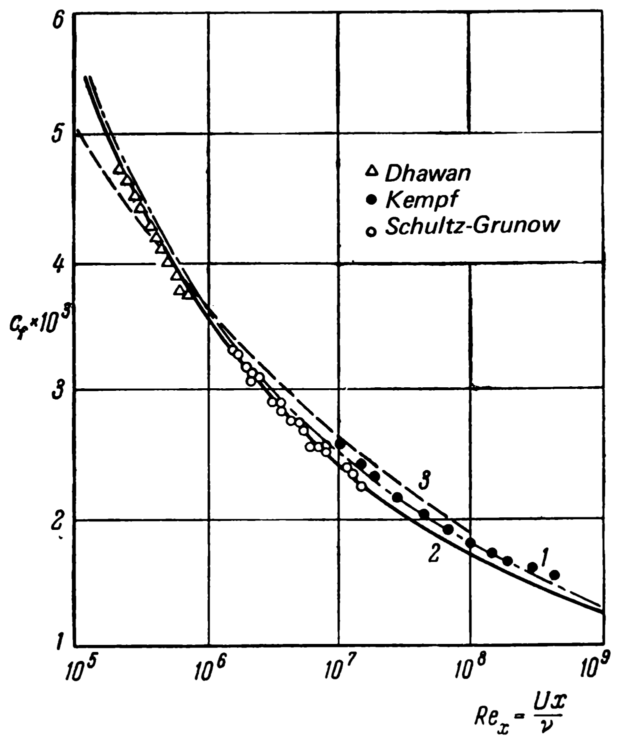


FIG. 42. Dependence of the skin friction coefficient c_f for the boundary layer on a flat plate on the Reynolds number Re_x according to the data of Kempf, Schultz-Grunow, and Dhawan. Curve 1: von Kármán's skin friction law (5.62) with $x = 0.39$ and $B_5 = 1.7$; Curve 2: Schultz-Grunow law (5.64); Curve 3: Falkner law (5.66).

$\varkappa = 0.39$ and $B_5 = 1.7$, are chosen in order to obtain the best agreement with Kempf's data]. We note further, that from the comparison of the numerical values of the coefficients B_4 and B_5 in Eqs. (5.59) and (5.62), leading to the best comparison with the experimental data, it is also possible to estimate the value of b in Eq. (5.61). In particular, if we take the values of these coefficients used in Figs. 41 and 42 (neglecting the fairly small divergence in the chosen values of $\varkappa$), for the thickness δ, defined by Eq. (5.55) with $C = \frac{1.55}{0.4} \approx 3.9$ [i.e., with the same value of C as used by Coles (1956)], we obtain $b \approx 0.3$.

For rough plates, the above equations must be changed in a manner similar to that for rough tubes; for example, Eq. (5.58), for a completely rough tube must be written in the form (5.45′) [but with

δ instead of H_1]. However, we shall not discuss this at length here [see, for example, Schlichting (1960), Chapt. 21, Sect. 4].

According to Eq. (5.62), c_f is a slowly decreasing function of x. However, the explicit definition of c_f as a function of Re_x with the aid of Eq. (5.62) is fairly complicated and, in practice, simpler interpolation or empirical equations are normally used. Thus, for example, Schlichting (1960) [proceeding, in fact, not from von Kármán's equation (5.62) but from the more general results of Prandtl, described, for example, in Schlichting's book (1960)] proposed for c_f the computational formula

$$c_f = (2 \ln \mathrm{Re}_x - 0.65)^{-2.3}, \tag{5.63}$$

which gives results close to those obtained from Eq. (5.62). Later, Schultz-Grunow (1940) used the equation

$$c_f = 0.370 (\ln \mathrm{Re}_x)^{-2.584}, \tag{5.64}$$

the results of which are compared with the data in Fig. 42. Finally, if we assume that the mean velocity profile in the boundary layer in all sections is given by the "one-seventh law" (see the closing remarks of Sect. 5.5), then the very simple formula

$$c_f = c\,(\mathrm{Re}_x)^{-\frac{1}{5}} \tag{5.65}$$

may be obtained. This formula, with $c = 0.0576$, also gives a fairly good description of the measurements for the range of Reynolds numbers $5 \times 10^5 < \mathrm{Re}_x < 10^7$. With further increase of the Reynolds number Re_x, Eq. (5.65) begins to give a reduced value of c_f, so that to obtain the best possible agreement it is necessary to replace the power index 1/5 in the formula by a smaller value. In particular, for values of Re_x up to 10^9, good agreement with the data may be obtained by using Falkner's formula (1943)

$$c_f = 0.0262\,(\mathrm{Re}_x)^{-\frac{1}{7}}. \tag{5.66}$$

In addition to the local friction coefficient c_f, we may also consider the total friction coefficient C_f of a plate of length l, given by

$$C_f = \frac{F}{\frac{1}{2}\rho U^2 \cdot 2l} = \frac{1}{l}\int_0^l c_f(x)\,dx, \qquad F = 2\int_0^l \tau_0(x)\,dx \tag{5.67}$$

[cf. Eqs. (1.51)–(1.52)]. This total friction coefficient has been measured much more frequently than the local coefficient c_f, and there is a great deal of data referring to it, the earliest of which goes back to 1793 [see Schubauer and Tchen (1959); Goldstein (1938) and Schlichting (1960)]. Using Eq. (5.62) as his starting-point, Schoenherr (1932) obtained for C_f the relationship

$$\frac{1}{\sqrt{C_f}} = 4.13 \ln (\mathrm{Re}_l \cdot C_f), \qquad \mathrm{Re}_l = \frac{Ul}{\nu}, \tag{5.68}$$

which gives good agreement with experimental data. Schlichting, using the more complicated calculations of Prandtl instead of Eq. (5.62), proposed the simple interpolation formula

$$C_f = 0.455\,(\ln \mathrm{Re}_l)^{-2.58}, \tag{5.69}$$

which gives results that differ little from those of Eq. (5.68). Schultz-Grunow (1940) used a similar formula

$$C_f = 0.427\,(\ln \mathrm{Re}_l - 0.407)^{-2.64}. \tag{5.70}$$

If we start from the "one-seventh law," then as we have already seen we obtain a dependence for C_f of the form $C_f \sim (\mathrm{Re}_l)^{-\frac{1}{5}}$; better agreement over a wide range of Reynolds numbers, however, may be obtained if, following Eq. (5.66) we put

$$C_f = 0.0306\,\mathrm{Re}_l^{-\frac{1}{7}}. \tag{5.71}$$

Comparison of this result with Eq. (1.52) for the laminar friction coefficient of a plate shows that in the turbulent case, the friction is considerably greater than in the laminar case (for example, with $\mathrm{Re}_l = 5 \times 10^5$ it is almost 2.5 times greater) and decreases far more slowly as the Reynolds number increases.

All the preceding considerations refer to the case in which the boundary layer may be assumed fully turbulent from practically the leading edge of the plate. However, if transition to a turbulent regime occurs only at the point $x = x_0$, which is sufficiently far from the leading edge, then we must introduce into the skin friction law a correction for the laminar portion of the boundary layer. If we use Prandtl's assumption (1927) that after transition to the turbulent regime, a boundary layer behaves approximately as if it had been turbulent from the leading edge of the plate, this correction will be reduced to the following: from the value of the friction force F calculated above we must subtract the difference between the friction forces for turbulent and laminar regimes for the portion of the plate from the leading edge to the point of transition $x = x_0$. If we take this correction into account, the friction coefficient will equal

$$C_f - \frac{\rho U^2 x_0\,[C_1(x_0) - C_2(x_0)]}{\frac{1}{2}\rho U^2 \cdot 2l} = C_f - \frac{\mathrm{Re}_{x_0}[C_1(x_0) - C_2(x_0)]}{\mathrm{Re}_l} = C_f - \frac{A(x_0)}{\mathrm{Re}_l}, \tag{5.72}$$

where C_f is given, say, by Eq. (5.68) [or by some of the other equations given above] and $C_1(x_0)$ and $C_2(x_0)$ are the friction coefficients for a plate of length l washed by turbulent and laminar flows, respectively. This correction permits us to describe the transitional region between the friction law (1.52) for a laminar flow and the friction laws for a purely turbulent flow which we have been discussing. The coefficient $A(x_0)$ in Eq. (5.72) will, of course, depend on the critical Reynolds number $\mathrm{Re}_{x_0} = \mathrm{Re}_{x\,\mathrm{cr}}$ at which the transition from laminar to turbulent flow occurs; according to Prandtl's data, for $\mathrm{Re}_{x_0} = 5 \times 10^5$, for example, $A(x_0) = 1700$ [the values of $A(x_0)$ for some other values of Re_{x_0} may be found in Schlichting's book (1960)].

The cited results referring to the law of variation of c_f and C_f with variation of the Reynolds numbers Re_x and Re_l may also be used in investigating the variation with increase of x of the mean velocity profile, which depends on $u_* = U\sqrt{\frac{c_f}{2}}$ and δ according to Eq. (5.57). The dependence of c_f on x has been discussed above, and the variation of δ with increase of x may be determined from Eq. (5.61) which once again contains $\frac{u_*}{U} = \sqrt{\frac{c_f}{2}}$. Since c_f decreases very slowly with increase of x, the thickness δ increases almost as the first power of x (if we use Eq. (5.66), δ is proportional to $x^{13/14}$). It is important that this increase be considerably more rapid than the increase in thickness of a laminar boundary layer, which, according to Eq. (1.33) is proportional to $x^{\frac{1}{2}}$ (cf. Fig. 4, Sect. 1.4, which contains results of the direct measurement of the thickness δ in both the laminar and the turbulent parts of the boundary layer).

5.7 Profile of Concentration of a Passive Admixture Close to a Wall; Mass- and Heat-Transfer in a Turbulent Boundary Layer

We have discussed the questions of the mean velocity distribution and of the friction in turbulent flows close to a wall. It is found that similar considerations may also be applied to the investigation of turbulent mass- and heat-transfer. Below, we shall give some basic facts about this problem; a more detailed description may be found, for example, in the engineering books of Gröber and Erk (1955), MacAdams (1954), and Eckert and Drake (1959), in Chapt. 14 of the monograph edited by Howarth (1953), Chapts. 3–4 of Levich (1962), in the survey articles of Deissler (1959), Kestin and Richardson (1963), and Spalding and Jayatillaka (1965).

As in Sects. 5.2–5.3, we consider a plane-parallel flow of fluid in the half-space $z > 0$, bounded by a rigid smooth wall, directed along the Ox axis in the absence of a longitudinal pressure gradient. Let us assume that on the bounding surface $z=0$, a constant value ϑ_0 of the concentration of the passive admixture is maintained. Then a constant flux j of the admixture will occur in the fluid, directed outward from the wall, i.e., in the positive Oz direction, and Eq. (5.7′) will have the form

$$j(z) = -\rho\chi\frac{d\bar{\vartheta}}{dz} + \rho\overline{\vartheta' w'} = j_0 = \text{const.} \tag{5.73}$$

The profile of the mean concentration will depend on the statistical characteristics of the velocity field (defined by the parameters ν and u_*) and also on the molecular diffusivity χ, the density ρ and the admixture flux j_0. Introducing the special dimension Θ for ϑ and denoting the dimension of mass by M, we have

$$[j_0] = M\Theta L^{-2}T^{-1}, \quad [\rho] = ML^{-3}, \quad [u_*] = LT^{-1}, \quad [\nu] = [\chi] = L^2T^{-1}.$$

Consequently, by dimensional considerations, we have

$$\bar{\vartheta}(z) - \bar{\vartheta}(0) = \frac{j_0}{\rho\varkappa u_*}\varphi\left(\frac{zu_*}{\nu}, \frac{\nu}{\chi}\right) = \theta_*\varphi(z_+, \mathrm{Pr}), \tag{5.74}$$

where

$$\theta_* = -\frac{1}{\varkappa u_*}\frac{j_0}{\rho} \tag{5.75}$$

is a constant of dimension Θ, giving a natural scale of the concentration [von Kármán's dimensionless constant $\varkappa = \frac{1}{A} \approx 0.4$ is included here in the definition of θ_*, since this will sometimes be found to be convenient in the subsequent argument], and $\varphi(z_+, \mathrm{Pr})$ is a new universal function of two variables, satisfying the condition $\varphi(0, \mathrm{Pr}) \equiv 0$. If ϑ is the temperature T, then Eqs. (5.73)–(5.75) must, of course, be written as

$$q(z) = -c_p\rho\chi\frac{d\bar{T}}{dz} + c_p\rho\overline{T'w'} = q_0 = \text{const}, \tag{5.73'}$$

$$\bar{T}(z) - \bar{T}(0) = -\frac{q_0}{c_p\rho\varkappa u_*}\varphi\left(\frac{zu_*}{\nu}, \frac{\nu}{\chi}\right) = T_*\varphi(z_+, \mathrm{Pr}), \tag{5.74'}$$

$$T_* = -\frac{1}{\varkappa u_*}\frac{q_0}{c_p\rho}, \tag{5.75'}$$

where the function $\varphi(z_+, \mathrm{Pr})$ is the same as in Eq. (5.74). We note that for the case of temperature, the assumption of passivity is more doubtful than for a material admixture. This is due both to the presence of buoyancy forces in a nonuniformly heated fluid (this will be discussed in greater detail in Chapt. 4), and the temperature-dependence of the fluid density, viscosity, and thermal diffusivity [for the problem of taking into account these dependences which are different for gases and for liquids, see, for example, the articles of Deissler (1959) and van Driest (1959)]; in addition, the heating due to energy dissipation is not always negligible in laboratory flows. Nevertheless, below, as a rule, ϑ will be called, for definiteness, the temperature, and equations of the form (5.73′)–(5.75′) will be used, but with ϑ replacing T. (As was explained in Sect. 1.5, the use of the symbol ϑ indicates that we are concerned throughout with a completely arbitrary passive admixture.)

Equation (5.74′) which contains the unknown function φ, may be simplified considerably for sufficiently large values of z. By sufficiently large we mean values of z which, above all, satisfy the condition $z \gg z_* = \frac{\nu}{u_*}$, so that the molecular viscosity ν has no effect on the distribution of the mean velocity and hence (for the same reason) no effect on the distribution of the mean temperature. Moreover, for the values of z under discussion, it is required that the turbulent heat flux $q^{(1)} = c_p \rho \overline{\vartheta' w'}$ be much greater than the molecular heat flux $c_p \rho \chi \frac{d\bar{\vartheta}}{dz}$; under this condition the coefficient χ will also have no effect on the variation of $\bar{\vartheta}$ with height. Since the turbulent heat flux is produced by the same eddy motion as the turbulent momentum flux $\tau^{(1)} = -\rho \overline{u'w'}$, the eddy viscosity K and the eddy thermal diffusivity K_ϑ are assumed naturally to be of the same order of magnitude. Thus, if Pr $\gtrsim 1$, i.e., $\chi \lesssim \nu$, then with $z \gg z_* = \frac{\nu}{u_*}$ the eddy thermal diffusivity will be far greater than the molecular diffusivity χ. Therefore, with $z \gg z_*$, both necessary conditions will be fulfilled for the values of z to be considered sufficiently large. However, if Pr $\ll 1$, the eddy thermal diffusivity [which, for $z \gg z_*$ according to Eq. (5.26) will be of the order of $u_* z$] will be far greater than the molecular diffusivity χ only when $z \gg \frac{\chi}{u_*} = \mathrm{Pr}^{-1} \cdot z_*$. Thus, generally speaking, we may ignore the effect of both ν and χ on the variation of the mean temperature with height only if the two conditions are satisfied: $z \gg z_*$ and $z \gg \mathrm{Pr}^{-1} \cdot z_*$ (the second of which, clearly, is only important in the case Pr $\ll 1$).

With $z \gg z_*$ and $z \gg \mathrm{Pr}^{-1} \cdot z_*$, the mean temperature gradient must be determined only by the parameters $q_0/c_p\rho$, u_*, and z. Thus, by dimensional reasoning, we have

$$\frac{d\bar{\vartheta}}{dz} = \frac{q_0}{a c_p \rho \varkappa u_* z} = \frac{\theta_*}{az}, \tag{5.76}$$

where a is a new dimensionless constant of the order of unity. Consequently,

$$\bar{\vartheta}(z) - \bar{\vartheta}(0) = \frac{\theta_*}{a} \ln z + A_1 \tag{5.77}$$

(cf. Landau and Lifshitz (1963), Part 1, Sect. 54), while the dimensional constant A_1 [the same as in the case of Eq. (5.22)] here must be determined from the matching condition of the profile (5.77) with the mean temperature profile in the lower layer, to which Eq. (5.77) is inapplicable.

From Eq. (5.77) it is clear that

$$\varphi(z_+, \mathrm{Pr}) = \frac{1}{a} \ln z_+ + C \quad \text{for} \quad z_+ \gg \max(1, \mathrm{Pr}^{-1}). \tag{5.77'}$$

Thus, for sufficiently large values of z_+, φ will differ from the function f of Eq. (5.13) only in the value of the numerical coefficients (only one of which, C, can depend on the Prandtl number). Further, we note that by Eq. (5.9) the profile (5.77) corresponds to the following value of the eddy thermal diffusivity:

$$K_\vartheta = a \varkappa u_* z. \tag{5.78}$$

Thus it is clear that the constant a has the meaning of the reciprocal of the turbulent Prandtl number for a logarithmic boundary layer:

$$a = \frac{1}{\mathrm{Pr}_t} = \frac{K_\vartheta}{K}, \quad \mathrm{Pr}_t = \frac{K}{K_\vartheta}. \tag{5.79}$$

Unfortunately, the data on the concentration profiles of a passive admixture in real flows close to a wall are considerably poorer than the data on the mean velocity profile; consequently, until now, the experimental verification of these equations has remained very incomplete. The often-repeated measurements by meteorologists of the mean temperature profile in the surface layer of the atmosphere are of very little use for this purpose, since in the atmospheric surface layer with the air temperature dependent upon height (i.e., with nonneutral thermal stratification), the buoyancy forces play a considerable role and do not allow the temperature to be considered as a passive admixture; for a more detailed discussion, see Chapt. 4. More promising would be data of careful measurements of the humidity profile (i.e., the concentration of water-vapor) in the surface layer of the atmosphere. In fact, although buoyancy will affect the vertical distribution of humidity also in the presence of vertical temperature gradients (see Chapt. 4), the humidity profile may be observed also in "neutral conditions," i.e., for an isothermal regime, whereas the temperature profile will be of interest only when the conditions are nonneutral. So far, only very little experimental data exist on this subject; nevertheless, we may mention here the results of Pasquill (1949), Rider (1954) and other investigators, which confirm that when the temperature stratification is close to neutral, the humidity profiles are well described by the logarithmic equations (see below, Sect. 8.2). In laboratory shear flows accompanied by intensive forced convection, the temperature may be considered as a passive admixture with greater justification than in the atmosphere; however, the published measurements of the temperature profiles, (or, moreover, on the admixture concentration profiles) in laboratory flows close to a wall are extremely scanty, and most of them possess very low accuracy. Thus all that may be said of them is that they do not contradict the law (5.77). However, as far back as the end of the 1920's, Elias showed that in the boundary layer on a heated flat plate, the temperature profile for $z > 0.05\delta$ is approximately logarithmic, and similar to the mean velocity profile [see Elias (1929)]. Later, Nunner (1956) also obtained in a nonisothermal flow of air along a rough tube, an almost exactly logarithmic profile of the mean temperature, commencing with $z = 0.02R$ and continuing almost to the center of the tube. The measurements of the temperature profiles by Reynolds, Kays, and Kline (1958), Brundrett, Baines, Peregrym, and Burroughs (1965) and Perry, Bell and Joubert (1966) are also in good agreement with the supposition that a comparatively thick layer having a logarithmic temperature profile exists. Finally, careful measurements by Johnk and Hanratty (1962) and by Hishida (1967) of the temperature profiles for air flow in a smooth heated tube with low values of heat flux which permit consideration of fluid properties as constant and also reduce all other sources of errors, show quite definitely that in the fully developed thermal regime and at all Re $\gg 25{,}000$, temperature profiles have a logarithmic form from $z_+ = 30$ to $z_+ = 200$-300 (see, e.g., Fig. 43 taken from Johnk and Hanratty's paper).

The results of Johnk and Hanratty have rather low scatter; their averaged data on dimensionless temperature $\overline{T}_+(z) = c_p \rho u_* [\overline{T}(z) - \overline{T}(0)]/q_0$ when $\kappa = 0.4$, can be represented by the equation

$$\phi(z_+, 0.7) = 0.9 \ln z_+ + 1.3$$

(i.e., according to the data $\mathrm{Pr}_t \approx 0.9$, $\alpha \approx 1.1$, $C(0.7) \approx 1.3$. Almost the same results were obtained by Hishida: according to his data $\mathrm{Pr}_t \approx 0.87$, $\alpha \approx 1.15$, $C(0.7) \approx 1.4$. Also, the considerably more scattered experimental data of Reynolds, Kays and Kline (1958), and Perry, Bell and Joubert (1966), were treated by the latter who obtained quite close mean estimates: $\mathrm{Pr}_t \approx 1$, $\alpha \approx 1$, $C(0.7) \approx 1.6$.

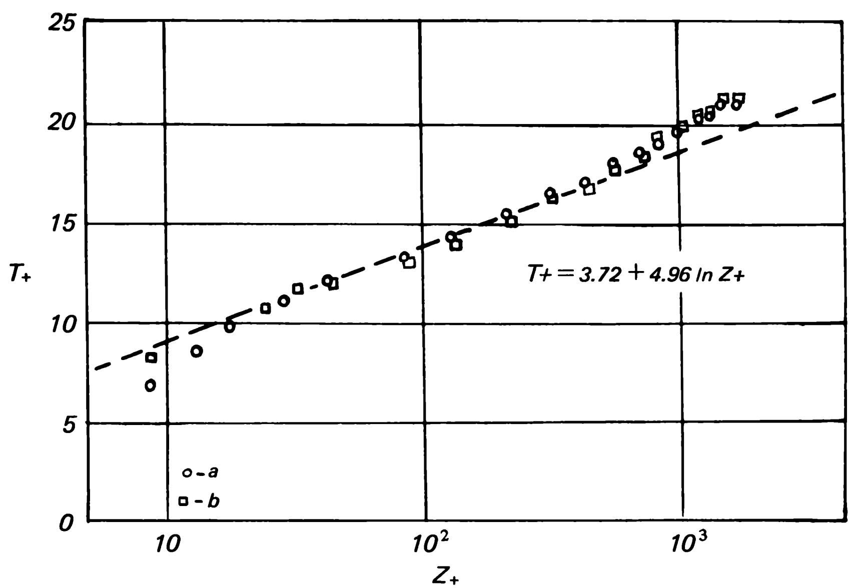


FIG. 43. The distribution of the dimensionless mean temperature $T_+(z) = c_p \rho u_* [T(0)]/q_0$ in a layer of flow in a tube according to two of Johnk and Hanratty's measurements: a) Re = 71,200; b) Re = 70,200

In the outer part of the nonisothermal flow in a tube, channel or boundary layer, an *extended Reynolds number similarity principle* (namely, a *Reynolds and Péclet number similarity principle*) apparently must be valid. According to this principle, the statistical characteristics of the temperature field are not dependent directly on the molecular viscosity and thermal diffusivity (i.e., on dimensionless combinations Re, Pe, and Pr = Pe/Re) for a region of the flow distant from rigid walls, if both Reynolds and Péclet numbers are sufficiently large. The most important result of the principle for the mean temperature distribution is a general *temperature defect law* of the form

$$\frac{\bar{\vartheta}_1 - \bar{\vartheta}(z)}{\theta_*} = \varphi_1\left(\frac{z}{H_1}\right) = \varphi_1(\eta), \tag{5.80}$$

where H_1 is a tube radius, channel half-width, or boundary-layer thickness and $\bar{\vartheta}_1 = \bar{\vartheta}(H_1)$; the function $\varphi_1(\eta)$ can of course be different for different types of flow. The data of Johnk and Hanratty for flow in a tube at various Re (but at only one Pr = 0.7) support the validity of the temperature defect law; according to these data, for a tube flow the function $\varphi_1(\eta)$ from $\eta = 0.02$ and to $\eta = 1$ can be approximated by the equation $\varphi_1(\eta) = 2.9\eta^2$ [or by the equation $\varphi_1(\eta) = 3(\eta^2 - \eta^4/12)$]. Some applications of the temperature defect law for a tube flow can be found in the paper by W. Squire (1964). Some preliminary measurements of the function $\varphi_1(\eta)$ for a nonisothermal boundary-layer flow were published by Žukauskas, Šlančiauskas, Pedišius, and Ulinskas (1968); however, their data do not agree with the assumption of independence of φ_1 on Pr (perhaps because the values of Re and Pe were not high enough in the measurements). We must also stress that in the overlap layer where both the temperature wall law (5.74′) and the temperature defect law (5.80) apply, the functions φ and φ_1 must both be logarithmic (cf. Sect. 5.5); however, the definition of the form of the φ_1 functions outside of the logarithmic layer and the accurate experimental verification of the temperature defect law strongly require further careful measurements of temperature profiles in all three types of flow at high values of Reynolds and Péclet numbers, and for a wide range of Prandtl numbers.

The existence of an overlap layer where both the temperature wall law and the temperature defect law have a logarithmic form permits a quite general *heat transfer law* to be obtained. In fact, the sum of two logarithmic laws can easily be written in the form:

$$\frac{\bar{\vartheta}_1 - \bar{\vartheta}_0}{\theta_*} = \frac{1}{\alpha} \ln \frac{H_1 u_*}{\nu} + C_1(\mathrm{Pr})$$

where $\bar{\vartheta}_0 = \bar{\vartheta}(0)$, $C_1(\Pr) = C(\Pr) + C_2$, and C_2 is the constant term in the logarithmic form of the temperature defect law (it is clearly independent of Pr, but may differ for tubes, channels and boundary layers). The last equation gives us the following general law for the main dimensionless characteristic of the heat-transfer process the heat-transfer coefficient (Stanton number)

$$c_h = \frac{q_0}{c_p \rho U(\bar{\vartheta}_0 - \bar{\vartheta}_1)},$$

where U is some typical velocity scale of the flow:

$$c_h = \frac{\sqrt{c_f/2}}{\ln(\mathrm{Re}\sqrt{c_f})/\varkappa\alpha + C_3(\Pr)}, \quad C_3 = \frac{1}{\alpha}C_1(\Pr) - \frac{\ln 2}{2\varkappa\alpha}, \qquad (5.80')$$

where $\mathrm{Re} = UH_1/\nu$ and $c_f = 2(u_*/U)^2$ is the friction coefficient (Kader and Yaglom (1970); cf. also the analogous derivation of Eq. (5.45) in Sect. 5.5).

For the case of heat transfer in air ($\Pr \approx 0.7$) we know that $C(0.7 \approx 1.3$; moreover, for the case of flow in a circular tube the constant C_2 can be roughly estimated from the data of Johnk and Hanratty (1962) on the temperature defect law as $C_2 \approx 0.3$. Therefore all the coefficients in Eq. (5.80′) for heat transfer in an air flow in a circular tube can be considered as known. However, comparison of the computations using this equation with the heat-transfer data is not straightforward since almost all the existing data on heat transfer in tubes relate not to the tube axis temperature $\bar{\vartheta}_1 = \bar{\vartheta}(H_1)$, but to the bulk temperature (i.e., to the average temperature of the fluid flowing through tubes)

$$\bar{\vartheta}_m = \int_0^{H_1} \bar{\vartheta}(z)\,\bar{u}(z)(H_1 - z)\,dz \Bigg/ \int_0^{H_1} \bar{u}(z)(H_1 - z)dz,$$

and as the typical velocity the buld velocity

$$U_m = \frac{2}{\mathrm{R}^2}\int_0^{H_1} \bar{u}(z)(H_1 - z)dz$$

is generally used. In other words, the existing data give us not the values of the coefficient c_h, but the values of the modified heat-transfer coefficients

$$C_h = q_0/c_p \rho U_m(\bar{\vartheta}_0 - \bar{\vartheta}_m) = c_h/(1 - \Delta)$$

where

$$\Delta = \frac{\bar{\vartheta}_m - \bar{\vartheta}_1}{\bar{\vartheta}_0 - \bar{\vartheta}_1}.$$

In estimating the correction Δ for practical purposes it is possible, at least for air flow in a tube, to neglect the thin wall layer of the direct influence of the molecular viscosity and thermal conductivity and to consider the velocity and temperature profiles as purely logarithmic ones. In this approximation it is easy to show that

$$\Delta \approx \frac{3.75}{\alpha} \frac{c_h}{(c_f/2)^{1/2}} [1 - (c_f/2)^{1/2}/1.2] \approx 3.4 c_h (c_f/2)^{-1/2}$$

$$\text{when } (c_f/2)^{1/2} \ll 1$$

(cf. the analogous derivation of Eq. (5.47) in Sect. 5.5). Using this equation for Δ, equation (5.80′) may be placed in the form

$$C_h = \frac{\sqrt{C_f/2}}{\ln(\mathrm{Re}_D \sqrt{C_f})/\varkappa\alpha + C_4(\mathrm{Pr})}, \quad C_4(\mathrm{Pr}) = C_3(\mathrm{Pr}) - \ln 2/\varkappa\alpha - 3.4 \tag{5.80″}$$

where $\mathrm{Re}_D = DU_m/\nu = 2H_1U_m/\nu$, $C_f = 2(u_*/v_m)^2$. Further, making use of the data from Sect. 5.5 on the dependency of $C_f = \lambda/4$ or Re_D (see Fig. 33) we can calculate from Eq. (5.80″) the dependence of the heat-transfer coefficient C_h, or, what is the same, of the so-called Nusselt number $\mathrm{Nu} = C_h \mathrm{Re}\,\mathrm{Pr}$, on Reynolds number Re_D. The comparison of the results with data from various sources is shown in Fig. A, where simultaneously the widely used purely empirical relation $\mathrm{Nu} = 0.018\,\mathrm{Re}_D^{0.8}$ [cf. for example, McAdams (1954)] is presented.

The comparison of the computations using Eqs. (5.80′) and (5.80″) with data on heat and mass transfer in tube flows with other values of Prandtl number will be considered later in this subsection.

Both the temperature law of the wall and the temperature defect law can be generalized for the characteristics of temperature fluctuations quite similar to the corresponding results for velocity fluctuations. However, at present, there are almost no reliable experimental data suitable for a verification of the relationships predicted by these generalized laws.

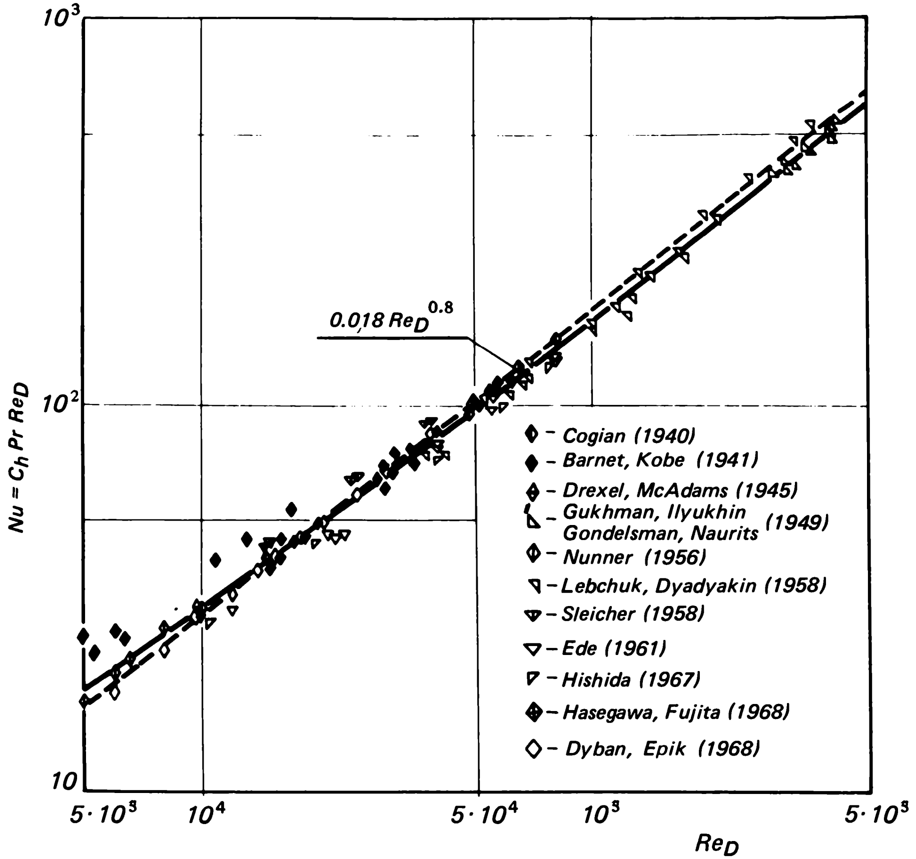


FIG. A. Comparison of data on turbulent heat transfer in air flow in tubes from various sources with theoretical dependence implied by Eq. (5.80′) [continuous line] and with the empirical formula $Nu = 0.018\,Re_D^{0.8}$ [dashed line] according to Kader and Yaglom (1970).

The lack of data considerably complicates also the reliable estimation of the coefficient α (or $Pr_t = \frac{1}{\alpha}$). In the majority of apphed heat-transfer calculations concerning both engineering devices and the atmosphere, until now, following O. Reynolds (1874), it was usually assumed that $\alpha = 1$. However, as a rule, these calculations lead to fairly good agreement with the data from direct measurements.

Nevertheless, the integral heat-transfer characteristics which are of special interest for heat technology, are usually quite insensitive to not too great variations of the parameter α and hence do not allow reliable estimates of its value to be made. Perhaps the best existing estimate of α and $\Pr_t$ for a logarithmic layer is that following from the data of Johnk and Hanratty (1962), and Hishida (1967), and which was given briefly above. Attempts to estimate α directly from the measurements of mean velocity- and temperature-profiles for flows in tubes and channels, also result generally in values of α that are somewhat greater than unity, of the order of 1.1–1.4 [i.e., to the deduction that $\Pr_t \approx 0.9–0.7 < 1$; see, for example, Corcoran et al. (1952); Cavers, Hsu, Schlinger and Sage (1953); Hsu, Sato and Sage (1956); Ludwieg (1956); Sleicher (1958); Venezian and Sage (1961); Žukauskas et al. (1968); and also the book by Longwell (1966) and the survey articles of Deissler (1959), Kestin and Richardson (1963), and Blom and De Vries (1968)]. However, sometimes (principally, in a number of experiments on heat-transfer in liquid metals) values of α slightly smaller than 1, i.e., $\Pr_t > 1$ are also obtained [see, for example, Subbotin, Ibragimov, Nomofilov (1963); Dwyer (1963) and Longwell (1966)]. The determination of α from the wind and temperature profiles in the atmosphere may be carried out only in the case of temperature stratification other than neutral; however, here, it is possible to attempt to find the limit to which α tends as the stratification approaches neutral, and to equate this limit to the ratio of the coefficients K_ϑ and K for the logarithmic boundary layer. Such attempts have been carried out many times, but with diverse results. Thus, for example, Rider (1954), Priestley (1963–1964) and Swinbank (1964) found that $\alpha \gtrsim 1$, but in the works of Swinbank (1955) and Gurvich (1965), the value $\alpha < 1$ was obtained. The results collected by Panofsky et al. (1967) give the impression that α is very close to unity (see below, Sect. 8.2). In general, we must bear in mind that, so far, all the existing data on turbulent heat-transfer (and mass-transfer are largely contradictory, and great care must be taken in using them. Thus, for example, even in works which give approximately the same mean value of α, we find obvious contradictions in the assertions regarding the variation of α with variation of the distance from the wall z. (According to Ludwieg (1956) α increases slightly with increase of z, while, on the other hand, according to Corcoran et al. (1952) and Sleicher (1958) α decreases with increase of z.) These facts may easily be explained as a result of low accuracy of all the existing measurements. This is especially dangerous because the experimental graphs must be differentiated to obtain α or Pr_t. In this connection, it is worth noting once again that according to the general considerations given above, in the logarithmic region of fully developed flow, where both the shear stress and the turbulent heat flux are constant, the coefficient α must also be *constant* (i.e., entirely independent of z). Moreover, since we are dealing here with a region of flow in which both the molecular viscosity and thermal diffusivity play no part, α must be independent of them, and hence must also be independent of the molecular Prandtl number Pr. This last assertion is in agreement with the surprisingly small variations of α values obtained for fluids ranging from machine oil (investigated by Žukauskas et al.) to liquid metals, i.e., having an enormous range of Pr-values. Further, if we believe that conditions of forced convection in which the temperature is transferred as a purely passive admixture are possible, then the values of α for heat- and for mass-transfer must be considered as completely identical. However, if these two values differ, then this can only mean that some physical mechanism (unknown to us) exists that produces a different effect on the transfer of heat and of mass.

For very small values of $z_+ = zu_*/\nu$, for a smooth wall, naturally, the equation

$$\bar{\vartheta}(z) - \bar{\vartheta}(0) = -\frac{q_0 z}{c_p \rho \chi}, \qquad \varphi(z_+, \mathrm{Pr}) = \kappa \mathrm{Pr} \cdot z_+, \qquad (5.81)$$

must hold (this is analogous to Eq. (5.20). Further terms of the expansion of the function $\varphi(z_+, \mathrm{Pr})$ as a Taylor series in powers of z_+ in the neighborhood of the point $z_+ = 0$ may be obtained by differentiating Eq. (5.73′) at the point $z = 0$. Since in the case of a wall at constant temperature $\vartheta' = 0$ at $z = 0$, then, for Eq. (5.20′), with accuracy to fourth-order terms in z_+, we will have

$$\bar{\vartheta}(z) = \bar{\vartheta}(0) - \frac{q_0 z}{c_p \rho \chi} + c_4' \frac{q_0 u_*^3 z^4}{c_p \rho \chi \nu^3}, \qquad \varphi(z_+, \mathrm{Pr}) = \kappa \mathrm{Pr}(z_+ - c_4' z_+^4) \qquad (5.81')$$

where[13]

$$c_4' = \frac{c_p \rho \nu^3}{8 q_0 u_*^3} \left(\overline{\frac{\partial \vartheta'}{\partial z} \frac{\partial^2 w'}{\partial z^2}} \right)_{z=0}.$$

From the fact that the second and third derivatives of $\varphi(z_+, \mathrm{Pr})$ with respect to z_+ become zero at $z_+ = 0$, it follows that the change in temperature will be close to linear over a considerable range of values of z; thus we may introduce the concept of a sublayer of molecular thermal diffusivity, in which the values of the function $\varphi(z_+, \mathrm{Pr})$ practically do not differ from $\kappa \mathrm{Pr} \cdot z_+$. However, the thickness of this sublayer δ_ϑ' will now be determined by the three parameters ν, χ, and u_* so that, generally speaking, we can only state that $\delta_\vartheta' = \psi(\mathrm{Pr}) \nu / u_*$. Directly beyond the sublayer of molecular thermal diffusivity there must follow some transitional region, intermediate between the region of applicability of Eq. (5.81) and the logarithmic layer. Beyond this layer begins the logarithmic temperature layer, described by Eq. (5.77). Unfortunately, at present, there are no reliable direct measurements of the temperature profile, or the profile of admixture concentration, close to a wall which would permit us to verify all these general deductions and to trace the complete source of the function $\varphi(z_+, \mathrm{Pr})$. Therefore, we must infer the values of this function for small z_+ on the basis of indirect data, obtained in the study of heat- and mass-transfer in turbulent flows.

These indirect data (of which there are a large number) consist of deductions on the values of the dimensionless integral heat-transfer characteristics, the Nusselt number $\mathrm{Nu} = \frac{-q_0 \delta}{c_p \rho \chi (\vartheta_1 - \vartheta_0)}$ or the heat-transfer coefficient (the Stanton number) $c_h = \frac{\mathrm{Nu}}{\mathrm{Re}_\delta \mathrm{Pr}} = \frac{-q_0}{c_p \rho U (\vartheta_1 - \vartheta_0)}$ for different values of $\mathrm{Re}_\delta = \frac{U\delta}{\nu}$ and $\mathrm{Pr} = \frac{\nu}{\chi}$. Here

[13] We note that unlike the exact equation (5.20′), Eq. (5.81′) for the mean temperature profile (but not for the concentration profile of a material passive admixture), is only approximate. In fact, the initial equation (5.73′) from which Eq. (5.81′) is deduced, is itself approximate, since in deducing it we drop the term describing the heating of the fluid by the dissipation of kinetic energy, which occurs in Eq. (1.70) [Sect. 1.5]. If we retain this term, then in the Taylor expansion for the function $\varphi(z_+)$, terms of the order of z_+^2 and z_+^3 occur, with coefficients depending on the statistical properties of the velocity fluctuations. The work of Tien (1964) is devoted to the estimation of these terms; he shows that in the layer $z_+ \leqslant 10$, they may play some role only if $\left| \rho u_*^3 / q_0 \right| > 10^{-3}$ (where q_0 is now assumed to be expressed in units with dimensions MT^{-3}).

$\vartheta_0 = \bar{\vartheta}(0)$ is the temperature of the wall (generally, a flat plate or a tube wall) which is assumed constant, ϑ_1 is the temperature of the flow outside the boundary layer or on the axis of the tube, δ is the thickness of the boundary layer or the radius of the tube, U is the velocity of the unperturbed flow or the velocity on the axis of the tube.[14] The same dimensionless characteristics are determined usually also in calculations on mass-transfer; in this case it is only necessary to replace q_0 by f_0, omit c_p in the denominators of Nu and c_p and take ϑ_0 and ϑ_1 to denote the corresponding concentrations. Deissler (1951; 1954), Hatton (1964) and several other investigators, showed that velocity, temperature and concentration profiles, as well as all integral characteristics of heat- and mass-transfer in tubes and boundary layers, are relatively insensitive (at least, when the Reynolds number is not too large and the Prandtl number not too small) to variation of τ and q along the Oz axis. Thus the model of a constant stress and constant flux layer may usually be used as a reasonable (or even excellent) first approximation. If we now rewrite Eq. (5.77′) as

$$\bar{\vartheta}(z) - \vartheta_0 = \theta_* \left(\frac{1}{\alpha} \ln \frac{zu_*}{\nu} + C \right), \tag{5.77''}$$

then to determine the difference $\vartheta_1 - \vartheta_0$ it is only necessary to know the parameters $\varkappa$ and α and to estimate the constant C (which depends essentially on the variation of the temperature in the thin layer close to the wall). A whole series of theories has been proposed for the estimation of this constant. They are confirmed experimentally with varying degrees of accuracy.

The simplest hypothesis permitting the estimation of the value C was proposed by O. Reynolds (1874). He proceeded from the assumption that the mechanisms of the transfer of heat and momentum in a turbulent flow are identical; thus the turbulent heat flux q according to Reynolds' theory must be connected with the shear stress τ by a relationship of the form

$$\frac{q}{c_p \rho (U_2 - U_1)(\vartheta_2 - \vartheta_1)} = \frac{\tau}{\rho (U_2 - U_1)^2}, \tag{5.82}$$

where U_1, U_2, ϑ_1, and ϑ_2 are values of the mean velocity and temperature at two arbitrary levels. According to this relationship, the mean velocity and mean temperature profiles in the whole constant flux layer (constant τ and q) must be exactly proportional (with coefficient of proportionality equal to $c_p \tau / q$). Therefore, it follows that $\varphi(z_+, \text{Pr}) = \varkappa f(z_+)$ [so that φ is independent of Pr] and $C = \varkappa B$. Comparing Eq. (5.81) with Eq. (5.20), and Eq. (5.77′) with Eq. (5.25), it is easy to ascertain that this proportionality will occur only if $\chi = \nu$ (i.e., if Pr = 1), $\alpha = 1$ (i.e., $\text{Pr}_t = 1$ in the logarithmic layers), and, moreover,

$$\frac{q}{c_p \dfrac{d\bar{\vartheta}}{dz}} = \frac{\tau}{\dfrac{d\bar{u}}{dz}}$$

(i.e., $\text{Pr}_t = 1$ also in the entire transitional region between the viscous sublayer and the

[14]We have already noted that in practical studies of heat transfer in tubes, the bulk velocity U_m and the bulk temperature ϑ_m are generally used instead of U and $\bar{\vartheta}_1$. However, we shall not dwell on this point, since if we know the values of the criteria Nu and c_h, based on $\bar{\vartheta}_1$ and U, it is easy to obtain the values of these criteria based on $\bar{\vartheta}_m$ and U_m quite similar to the development presented in Sect. 5.5 [cf. the derivation of Eq. (5.47)] and in this subsection in connection with the data of Fig. A.

logarithmic layer). We note that the condition Pr = 1 is only approximately satisfied (with accuracy not exceeding 20–30%) for air and for the majority of other gases, and is not satisfied at all for an impressive majority of liquids; the condition $\mathrm{Pr}_t = 1$, on the other hand, imposes severe restrictions on the disordered velocity and temperature fluctuations which also rarely ever hold exactly. Nevertheless, Reynolds' assumption [or as more commonly called, *the Reynolds analogy*, in view of the analogy between the flux of heat and momentum expressed by Eq. (5.82)] written in the form

$$c_h = \frac{1}{2} c_f, \tag{5.83}$$

is a surprisingly good description of many experimental results on turbulent heat exchange in gases, and, hence, is still used fairly widely in engineering calculations.

An obvious defect of the Reynolds analogy is that it ignores the effect on the heat exchange of the molecular Prandtl number, which in a number of cases plays a definite part. A simple generalization of Eq. (5.83) intended to take into account the effect of Pr in some manner, was proposed independently by Prandtl (1910; 1928), and G. I. Taylor (1916). According to their postulates, it may be assumed that $\alpha = 1$ in the logarithmic layer, but in the viscous sublayer (extending right up to $z = \alpha_v \frac{\nu}{u_*}$) we must use the strict equations (5.20) and (5.81), i.e., we must take into account the true values of the coefficients ν and χ. For the transitional region, this is neglected in the Prandtl-Taylor theory, i.e., it is assumed, in effect, that for $z > \alpha_v \frac{\nu}{u_*}$ the logarithmic equations (5.25) and (5.77) hold. Thus the function $\varphi(z_+, \mathrm{Pr})$ is defined by Eq. (5.81) for $z_+ < \alpha_v$ and by Eq. (5.77′) with $\alpha = 1$ for $z_+ > \alpha_v$; thus, in particular, it follows that in this theory, $C = \alpha_v \varkappa \mathrm{Pr} - \ln \alpha_v$. However, it is easy to verify that this value of C (and $\alpha = 1$) corresponds to the heat-transfer coefficient

$$c_h = \frac{\frac{1}{2} c_f}{1 + \alpha_v (\mathrm{Pr} - 1) \sqrt{\frac{c_f}{2}}} \tag{5.84}$$

and this is the main result of the Prandtl-Taylor theory. For Pr = 1, Eq. (5.84) becomes identical to Eq. (5.83); however, if $\mathrm{Pr} \neq 1$ (but does not differ greatly from unity) then, with a reasonable choice of α_v (Prandtl, for example, took $\alpha_v = 5.6$) it gives somewhat better agreement with the experimental data than Reynolds' formula (5.83).

A further improvement to the Prandtl-Taylor theory was proposed by von Kármán (1934; 1939) who introduced the "transitional layer" $\alpha_v < z_+ < \alpha_l$ into the calculation (where $\alpha_v = 5$, $\alpha_l = 30$; see above Sect. 5.3). As we have already observed (at the end of Sect. 5.3), von Kármán assumed that in this transitional layer the velocity profile was also given by a logarithmic equation of form (5.25), but with different values of the coefficients A and B; in particular, he took $A = \frac{2}{\varkappa} \approx 5$, and, therefore, $K = \frac{\varkappa}{2} u_* z = 0.2 \nu z_+$ for $5 < z_+ < 30$. In the calculation of the mean temperature profile [i.e., of the function $\varphi(z_+, \mathrm{Pr})$] it was assumed that in the region $z_+ < 5$, the relationship (5.81) is satisfied (i.e., only molecular thermal diffusivity is operating), in the region $5 < z_+ < 30$, there is both molecular and turbulent thermal diffusivity while $K_\vartheta = K$, that is, $\frac{d\bar{\vartheta}}{dz} = \frac{q_0}{c_p \rho (\chi + 0.2 u_* z)}$, while for $z_+ > 30$ the coefficient χ may be ignored and we may assume that $\frac{d\bar{\vartheta}}{dz} = -\frac{q_0}{c_p\rho \cdot 0.4u_*z}$ ($\Pr_t$ in this theory for $z > 5\ \nu/u_*$ is taken to be everywhere equal to unity). On this basis, it is easy to obtain the entire course of the function $\varphi(z_+, \Pr)$ [defined by three different analytical expressions] and, in particular, to find the value of C; for c_h we then obtain the equation

$$c_h = \frac{\frac{1}{2}c_f}{1+5\left[(\Pr-1)+\ln\frac{1+5\Pr}{6}\right]\sqrt{\frac{c_f}{2}}}. \tag{5.85}$$

For Pr = 1, this equation once again becomes Eq. (5.83) but with values of Pr of the order of several units of one or two tenths (i.e., for ordinary liquids), it gives considerably better agreement with experimental data than does the Prandtl-Taylor equation (5.84). However, for vary large or very small values of Pr, Eq. (5.85) also leads to sharp deviation from experiment.

Later, various authors proposed a considerable number of new semiempirical (or simply empirical) equations for c_h (or Nu); see, e.g., the works cited at the end of Sect. 5.3 and the papers by Ribaud (1961); Kapitza (1947); Sherwood (1950); Levich (1951); Reichardt (1951b); Chapman and Kester (1953); Deissler (1951; 1954), Petukhov and Kirillov (1958); van Driest (1959); Spalding (1963; 1964); W. Squire (1964); Spalding and Jayatillaka (1965); Kishenevsky (1965); and Dil'man (1967). In these works, numerous additional references may be found. Also, in a considerable part of these works, the basic improvement to the theory consists in the choice (on the basis of various intuitive or empirical considerations) of a more complicated expression for the function $\varphi(z_+, \Pr)$ which is given generally in the form of an equation for $K_\vartheta = -\frac{q_0}{c_p\rho\, d\bar{\vartheta}/dz} - \chi$. Consequently, the calculation of the profile $\bar{\vartheta}(z)$ [i.e., the function $\varphi(z_+, \Pr)$] and the determination of the coefficient c_h reduce to an integration (which for more complicated theory must be carried out numerically). The numerical procedures for solving the various heat- and mass-transfer problems are discussed, e.g., by Hatton (1964), Dvorak and Head (1967), Donovan, Hanna and Yerazunis (1967), and Patnakar and Spalding (1967). We shall not dwell here on these results, which are basically of a purely engineering character, and shall consider only the physically interesting question of the special features of heat transfer for very large or very small Prandtl numbers.

In both these limiting cases, the assumption made in the Prandtl-Taylor and von Kármán theories that the thickness of the molecular thermal diffusivity layer may be estimated as $\delta'_v = \alpha_v \frac{\nu}{u_*}$, where α_v is found from the intersection of the two asymptotes of the mean velocity profile and hence is independent of Pr, is clearly unjustified. Let us take first the case of very small Prandtl numbers, which are primarily characteristic of liquid metals (i.e., of mercury and molten metals). Since in this case χ is very large, the molecular thermal diffusivity layer is considerably thicker than the viscous sublayer; thus, neglecting the coefficient χ in the whole region of the logarithmic mean velocity profile here must lead to serious errors. To avoid them, many investigators, especially those who study the theory of turbulent heat exchange in liquid metals [for example, Martinelh (1947); Lyon (1951); Lykoudis and Touloukian (1958) et al.], did not ignore the molecular thermal diffusivity in any flow region. This is not, in fact, necessary, at least in the case of an idealized flow in the whole half-space $z > 0$; but the thickness δ'_v of the layer within which the coefficient χ still plays a role (i.e., is comparable with the eddy diffusivity K_ϑ) for $\Pr \ll 1$ must be estimated with the aid of the relationships $\delta'_v u_* \sim \chi$ (since $K \sim u_* z$ for $z \gg \nu/u_*$ and K_ϑ has the same order of magnitude as K). Thus, for example, in the simplified "two-layer"

Prandtl-Taylor model, for $\Pr \lll 1$ we must assume that $\varphi(z_+, \Pr)$ is given by Eq. (5.81) for $z_+ < \beta_v \frac{\chi}{\nu} = \frac{\beta_v}{\Pr}$ (where β_v is now independent of Pr and is of the same order as the dimensionless thickness of the viscous sublayer α_v) and is given by Eq. (5.77′) for $z_+ > \frac{\beta_v}{\Pr}$ so that $C = \varkappa\beta_v - a \ln \frac{\beta_v}{\Pr}$. On this basis it is easy to calculate the heat-transfer coefficient c_h. Such a calculation was carried out by Levich (1962), taking $\beta_v = 11.7$, $\alpha = 1.2$. In spite of the roughness of the assumptions used in the calculation, it is found that the equation thus obtained corresponds fairly well to many of the data on heat transfer for turbulent flows of liquid metals in tubes.

The expression for $C(\Pr) = C_1(\Pr) - C_2 \approx C_1(\Pr) - 0.3$ can be also substituted directly into Eq. (5.80′) for c_h. Such a procedure was used by Kader and Yaglom (1970) [who took the value $a = 1.1$ and a comparatively low value of β_v]; after introducing the correction connected with the use of $\bar{\vartheta}_m$ instead of $\bar{\vartheta}_1$ (i.e., the transformation from c_h to C_h) they obtained from Eq. (5.80′) a rather good description of numerous and relatively accurate data on heat transfer in tube flows of mercury and sodium-potassium alloy (both having $\Pr = 0.025$).

The second limiting case $\Pr \ggg 1$ is encountered in the study of heat transfer in technical oils, or in the diffusion of material admixtures in liquids; it is considerably more complex. In this case the thermal diffusivity sublayer is considerably thinner than the viscous sublayer and, consequently, even for $z \lll \nu/u_*$ the transfer of heat (or the admixture) is determined mainly by the eddy diffusivity. At the same time, the quantitative understanding of turbulence is most lacking just in the region $z \lesssim \nu/u_*$, in which the molecular viscosity still plays a significant role. Since as yet no approach is known to the problem of determining independently the behavior of the two functions $f(z_+)$ and $\varphi(z_+)$ in this region, it is generally assumed that the coefficients K and K_ϑ for $z \lesssim \nu/u_*$ either agree with each other exactly or else differ only by a constant factor α (chosen more or less arbitrarily or roughly estimated from the data). Therefore the study of turbulent heat transfer for $\Pr \ggg 1$ reduces to the investigation of the asymptotic behavior of the turbulent shear stress $-\rho\overline{u'w'}$ [or, equivalently, of only one function $f(z_+)$] for $z \lll \nu/u_*$ (i.e., for $z_+ \lll 1$), to which a number of works has been devoted (see the end of Sect. 5.3) for this very reason. Let us now assume that $K \sim K_\vartheta \sim z^m$ as $z \to 0$ (i.e., that $|\overline{u'w'}| \sim |\overline{\vartheta'w'}| \sim z^m$, and $[f(z_+) - z_+] \sim [\varphi(z_+, \Pr) - \varkappa \Pr \cdot z_+] \sim z_+^{m+1}$). Since the thickness δ_v of the viscous sublayer is defined by the condition $K(\delta_v) \sim \nu$, while the thickness δ_v' of the thermal diffusivity sublayer is defined by the condition $K_\vartheta(\delta_v') \sim \chi$, it is obvious that in the case under consideration

$$\delta_v' \sim (\Pr)^{-\frac{1}{m}} \delta_v \sim (\Pr)^{-\frac{1}{m}} \frac{\nu}{u_*}.$$

The profile of $\bar{\vartheta}(z)$ may be described satisfactorily by the linear equation (5.81) when $z > \delta_v$ only; in the layer $\delta_v' < z < \delta_v$ the solution of the equation $-q_0 = c_p \rho K_\vartheta \frac{d\bar{\vartheta}}{dz} \sim z^m \frac{d\bar{\vartheta}}{dz}$ gives a reasonable first approximation to the profile (so that $\bar{\vartheta}(z) = \text{const} - cz^{-m+1}$, while for $z > \delta_v$ it is transformed into a logarithmic profile (5.77). Analytically these results may be obtained from a simplified two-layer, Prandtl-Taylor model if we assume that $K_+ = K_\vartheta/\nu = \alpha_1 z_+^m$ for $z_+ \le \beta_v$ and $K_+ = \alpha\kappa z_+$ for $z_+ \ge \beta_v$ (where β_v is defined by the matching condition $\alpha_1\beta_v^m = \alpha\kappa\beta_v$) and take the molecular diffusivity χ into account when $z_+ \le \beta_v$ (i.e., $z < \delta_v$). The equation $-q_0 = c_p\rho(\chi + K_\vartheta)\frac{d\bar{\vartheta}}{dz}$ with $z < \beta_v\nu/u_*$ then may be transformed to

$$\frac{d\bar{\vartheta}_+}{dz_+} = \frac{\Pr}{1 + \Pr K_+} \quad \text{where} \quad \bar{\vartheta}_+ = -\frac{c_p \rho u_* \bar{\vartheta}}{q_0} = \frac{\bar{\vartheta}}{\kappa\theta_*}.$$

In fact, we shall be dealing with a three-layer model because the expansion of the solution of this equation in powers of Pr has a different form in the regions $\Pr K_+ = \alpha_1 z_+^m \Pr < 1$ and $\Pr K_+ = \alpha_1 z_+^m \Pr > 1$. Solving this equation for both layers $z_+ < (\alpha_1 \Pr)^{-1/m}$ and $z_+ > (\alpha_1 \Pr)^{-1/m}$, joining together the two solutions at $z_+ = (\alpha_1 \Pr)^{-1/m}$ and the second solution with the logarithmic profile at $z_+ = \beta_v$, and keeping only the leading terms in the expansions in powers of Pr, we obtain easily the following asymptotic equation for the heat-transfer coefficient for $\Pr \gg 1$:

$$c_h = \frac{\sqrt{c_f/2}}{B_1 \Pr^{(m-1)/m}} \sim (\Pr)^{-\frac{m-1}{m}} \tag{5.86}$$

[cf. Levich (1962); the method of derivation of the equation sketched is that of Kader (1966)]. Here B_1 is a dimensionless constant proportional to $\alpha_1^{-1/m}$; according to Kader

$$B_1 = \alpha_1^{-1/m}\left[1 - 2\sum_{k=1}^{\infty} \frac{(-1)^k}{(mk)^2 - 1}\right].$$

However, with respect to the value of the index m there is at present some divergence in views, as we have already observed. We have seen that by the continuity equation we must have $m \geq 3$. If we take $m = 3$, then according to Eq. (5.86)

$$c_h \sim (\Pr)^{-\frac{2}{3}} \quad \text{at} \quad \Pr \gg 1. \tag{5.86$'$}$$

The assumption that $m = 3$, permitted Murphree (1932) to obtain an equation for c_h which gave a good description of all the data on heat transfer in flows of gas, water and technical oils (with Prandtl numbers up to 1000); later, Ribaud (1941) came to similar conclusions. Purely empirically, the relationship (5.86′) was found by Chilton and Colburn (1934) who studied mass transfer in liquids (without any reference to the work on heat transfer); later, it was confirmed by a number of other investigators of whom we must mention, Lin, Denton et al. (1951) who carefully measured the rate of mass transfer in electrochemical reactions (with Prandtl numbers varying from 300 to 3000). At the beginning of the 1950's, the assumption that $m = 3$ once again began to be widely used in theoretical works [e.g., Reichardt (1951a), and Lin, Moulton and Putnam in their study of turbulent mass transfer (1933)]; the same assumption was used in the works on heat and mass transfer by Petukhov and Kirillov (1958), Wasan and Wilke (1964), Dil'man (1967) and some others. Nevertheless, Rannie (1956) used successfully the relationships $m = 2$ and $c_h \sim (\Pr)^{-1/2}$ for a fairly good description of the experimental data on heat transfer with $\Pr \leq 120$, and Kishinevsky (1965) made an attempt to show that these relationships can be used even for the description of mass-transfer data in a considerably wider range of Pr values; Frank-Kamenetskiy (1967) [in the second edition of his book of 1947] assumed even that the equation $c_h \sim (\Pr)^{-1}$ [which corresponds to the model of a purely laminar sublayer with $K = K_\vartheta \equiv 0$; cf. Eqs. (5.84) and (5.85)] can be applied for Pr of the order 10^3 (however, he used data for only three values of the Prandtl number as a basis for his assumption). At the same time, many others [for example, Levich (1944; 1951; 1959); Deissler (1955; 1959); Loitsyanskiy (1960); Son and Hanratty (1967)] are inclined to the proposition that $m = 4$ and, consequently,

$$c_h \sim (\Pr)^{-\frac{3}{4}} \quad \text{for} \quad \Pr \gg 1; \tag{5.86$''$}$$

they showed that this result also gives a completely satisfactory description of many existing data on turbulent heat and mass transfer with values of Pr up to 4000. We have already noted in Sect. 5.3 that in works by Spalding (1961; 1964; and others) two different equations were proposed: one with $m = 3$ and the second with $m = 4$. In the old literature on the subject, it is also possible to find equations corresponding to $m = 5$ [Reichardt (1951b)] and even to $m = 6$ [Hofmann (1937)].

The very great number of contradictory theoretical equations all having seemingly satisfactory experimental verification may be explained in several ways. First, each proposed equation contains additional unknown parameters selected so as to obtain the best agreement with experimental data used; at the same time the scatter of all the existing data is quite great because many factors make accurate experiments on heat and mass transfer rather difficult. Second, most of the researchers have used data for a comparatively restricted range of Prandtl numbers and apply diverse data processing techniques which in some cases have obvious defects. Finally, the exponents 2/3 and 3/4 in the main competing equations (5.86′) and (5.86″) differ by only 1/12, which being very small, is difficult to detect reliably in laboratory experiments. Thus it is not surprising that the same data were often used by different researchers to support various theoretical equations. It is clear that the final solution of the problem urgently requires extremely careful extensive measurements under fully controlled conditions and with a wide range of the values of Pr, permitting uniform and precise statistical treatment.

A special electrochemical technique has been developed by Hanratty and his co-workers at the University of Illinois to measure the various turbulence characteristics very close to a wall [see, e.g., Mitchell and Hanratty (1966); and Hanratty (1967), where additional references can also be found]. The technique was used in particular to measure the rate of mass transfer from a wall to fluid. All the results obtained in the range 400–2000 of Prandtl (or, more precisely, Schmidt) numbers were described by Son and Hanratty (1967) with fair accuracy by the equation $K_+ = K_\vartheta/\nu = 0.00032\, z_+^4$ (which corresponds to the assumption that $m = 4$). However, the most extensive and accurate homogeneous data on mass transfer over a very wide range of Schmidt numbers can be found in the three dissertations by Hamilton (1963) [see also, Harriott and Hamilton (1965); Hubbard (1964); Hubbard and Lightfoot (1966); Kader (1969); and Gukhman and Kader (1969)]. In the second of these works mass transfer was studied via diffusion-controlled reduction of specific ions at a cathode during an electrochemical reaction, i.e., an electrochemical technique was used which was developed by Lin, Denton et al. (1951) and later was improved considerably by Hanratty et al. The measurements were made at Schmidt numbers from 1700 to 30,000 and the results obtained agreed very well with Eq. (5.86) for $m = 3$ [i.e., with Eq. (5.86′)]. In the works of Hamilton and of Kader, a different method was used, namely, measurement of the rate of solution of special soluble tube sections made from benzoic acid or related substances when water or a a glycerine-water solution was circulated in the tube; Hamilton's observations were made at Schmidt numbers from 430 to 10^5, and Kader's observations, at Schmidt numbers from 500 to 10^6. The results of both authors are in obvious contradiction with Eq. (5.86″) and agree much more closely with Eq. (5.86′).

Following Hubbard (1964) and Hamilton (1963), Donovan, Hanna, and Yerazunis (1967) concluded that $m = 3$ must be taken in Eq. (5.86). A detailed statistical treatment of all existing data on the subject was made by Kader (1966; 1969) and Gukhman and Kader (1969). These authors have determined the nonlinear least-squares estimators of the two parameters B_1 and m in Eq. (5.86) [i.e., the values of B and m minimizing the sum of the squares of the differences between the right and the middle of Eq. (5.86)] at every particular value of Reynolds number (or precisely over many small intervals of Re-values) used in the different experiments. The results of such a treatment are collected in Fig. la and b (in Fig. la for the three most extensive studies cited and in Fig. Ib for the 26 different works in which the numerical data on heat and mass transfer at various Pr and Re can be found). We see that the results obtained definitely support the assumption that $m=3$.

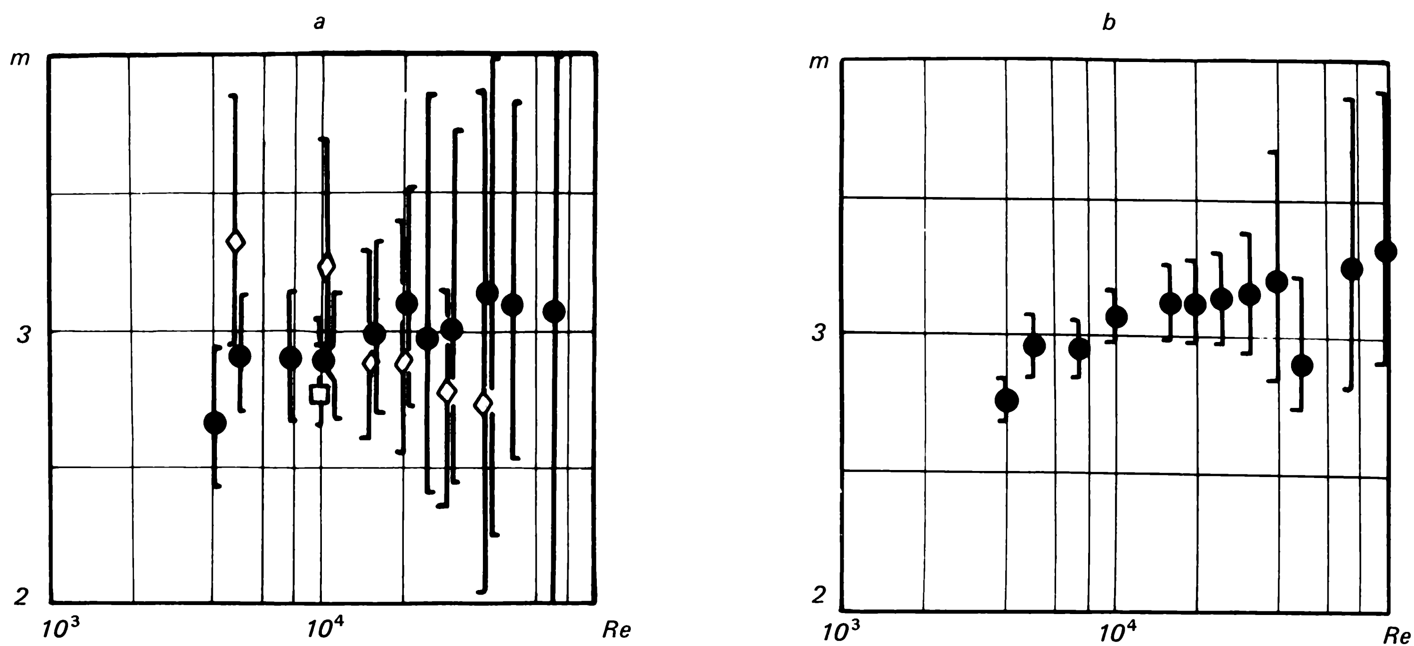


FIG. I. The empirical values of the index m (with corresponding 90% confidence intervals) obtained by Kader from heat- and mass-transfer data. a–The results obtained from the data of Hamilton (□), Hubbard (◇), and Kader (•), b–The results of the treatment of a more extensive collection of data taken from 26 different experimental works.

Equation (5.86) with $m = 3$ and $B_1 = 12.5$ is an asymptotic version (for $\mathrm{Pr} \gg 1$) of the more accurate equation (5.80′) with $C_3(\mathrm{Pr}) = 12.5\,\mathrm{Pr}^{2/3} + (\ln \mathrm{Pr})/\kappa\alpha - 5.0 - \ln 2/2\kappa\alpha$. (The last equation corresponds to a simple two-layer model and the constant term in the expression for C_3 is chosen in accordance with the data for $\mathrm{Pr} = 0.7$ on which the continuous curve in Fig. A is based). The results of the computations using Eq. (5.80′) with such $C_3(\mathrm{Pr})$ (corrected by the consideration of the transformation from the coefficient c_h to the coefficient C_h; i.e., for the change from Eq. (5.80′) to Eq. (5.80″), which is of some importance at $\mathrm{Pr} < 10$ only, and presented in the form of the dependence of $\mathrm{Nu} = C_h \mathrm{Re}_D \mathrm{Pr}$ on Re_D) are shown in Fig. B for an extensive range of values of Prandtl number and Reynolds number in comparison with numerous data on heat and mass transfer in turbulent tube flows taken from 32 different sources. The very good agreement between the calculations and the data demonstrates simultaneously the validity of the asymptotic equation (5.86) with $B_1 = 12.5$ and $m = 3$.

When the assumption $m = 3$ is used, the least-squares estimate of the the constant B_1 in Eq. (5.86) for all the existing data according to Kader is $B_1 = 12.5$. This preliminary estimate of B_1 corresponds to $\alpha_1 = 0.001$ according to the equation relating B_1 to α_1. We see that the value of α_1 obtained coincides precisely with the estimate of practically the same coefficient a_4 given on an absolutely different basis at the end of Sect. 5.3. This coincidence evidently enlarges the likelihood of both estimates.

Of course, all the above considerations relate only to the case of heat and mass transfer close to a smooth wall. In the case of the heat (or mass) transfer coefficient (i.e., the Stanton number) $c_h = \mathrm{Nu}/\mathrm{Re}\ \mathrm{Pr}$ requires the evaluation of the difference $\vartheta_1 - \vartheta_0$ to which the change of $\bar\vartheta$ in the lower layer (of thickness comparable with the mean height h_0 of the irregularities of the wall) makes a considerable contribution. To describe this contribution Owen and Thompson (1963) proposed to represent the change of $\bar\vartheta(z)$ in the lowest roughness layer (i.e., in the layer of the direct influence of wall roughness) of a completely rough flow as

$$\Delta_{h_0}\bar\vartheta = \kappa\theta_* B$$

where B is a dimensionless quantity dependent only on roughness Reynolds number $h_0 u_*/\nu$ and Prandtl (or Schmidt) number Pr for given form and arrangement of the wall irregularities. The difference $\bar\vartheta(z) - \vartheta_0$ for z in the logarithmic layer may be estimated in this case by the equation

$$\bar\vartheta(z) - \vartheta_0 = \frac{\theta_*}{\alpha}\ln\frac{z}{z_0} + \kappa B$$

where z_0 is the usual roughness parameter. Using the data by Nunner (1956), Dipprey [see Dipprey and Sabersky (1963)], their own data and that of others, Owen and Thompson obtained the empirical relation

$$B \quad c^{-1}(h_0 u_*/\nu)^{-k}(\mathrm{Pr})^{-n}$$

where the values of the constants c, k, and n, when h_0 is replaced by the equivalent sand roughness h_s, are approximately 0.5, 0.45, and 0.8, respectively. Additional information on the values of the function B can be found in the works of Chamberlain (1966; 1968), who has done special wind-tunnel experiments and has also collected data (some of it his own) on mass transfer under field conditions.

5.8 Free Turbulence

So far, we have considered turbulent flows in channels, tubes and boundary layers, i.e., flows close to rigid walls, the friction on which

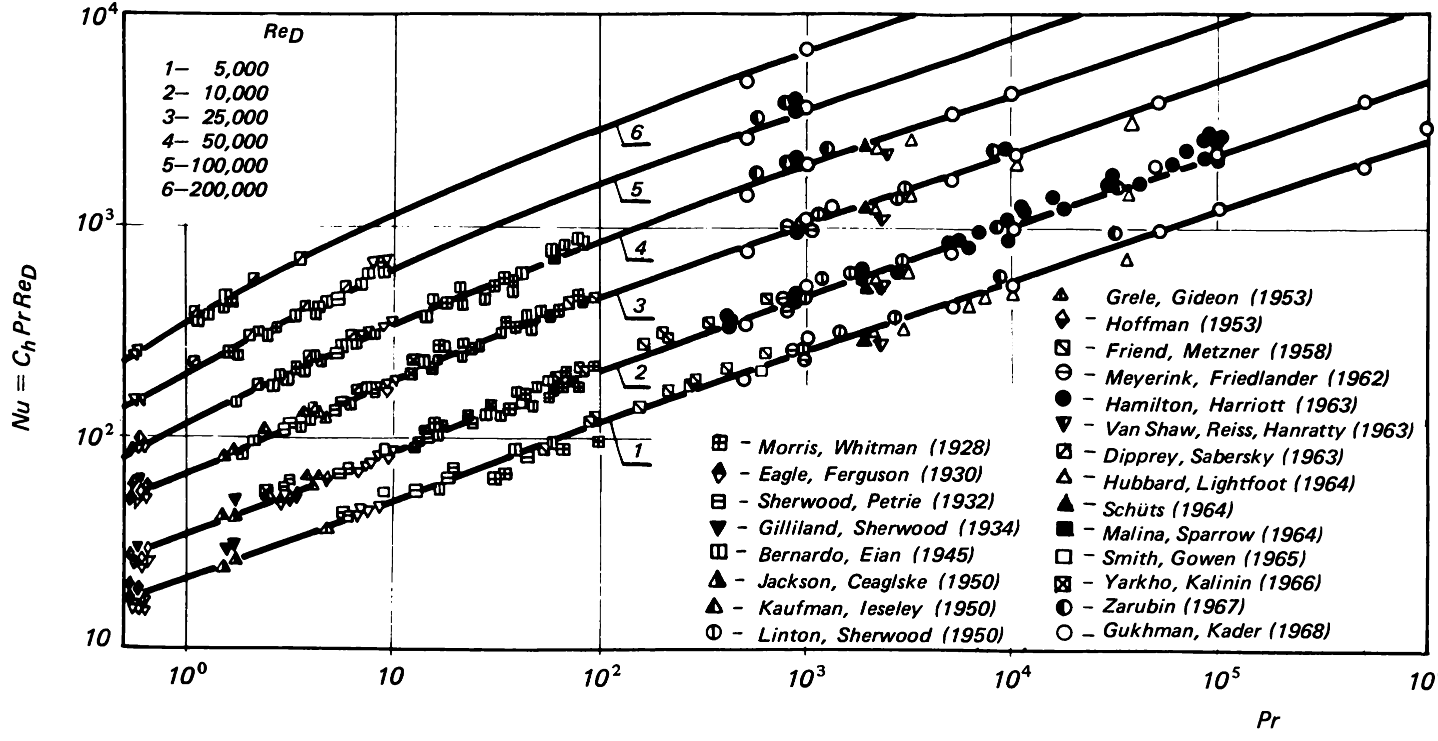


FIG. B. Comparison of data on the dependence of $Nu = C_h Pr Re_D$ on Pr and Re_D from various heat- and mass-transfer measurements in circular tubes with the results of the computations based on Eq. (5.80′) [according to Kader and Yaglom (1970)].

leads to the continuous generation of vorticity and exerts a considerable effect on the whole flow. However, in nature and in technology, we often encounter turbulent flows of a completely different kind, in which there is no direct effect of any rigid walls, and which are therefore called *free turbulent flows.* The most important kinds of free turbulent flows are turbulent wakes behind rigid bodies placed in a constant velocity flow (or moving through fluid at rest), turbulent jets, and turbulent mixing layers, occurring on the boundary between streams of different velocities without any rigid walls between them.

Free turbulent flows play a large role in many natural phenomena and in a number of engineering devices. Thus it is not surprising that a considerable number of theoretical and experimental investigations has been devoted to their study. These investigations have led to a number of results that are useful for practical calculations but which have added little to our understanding of the physical nature of turbulence. As a result, we shall limit ourselves here to a brief investigation of some general deductions relating to the basic types of free turbulent flows mentioned above.

The known methods of describing free turbulence are based, first, upon certain hypotheses of self-preservation of corresponding flows, and, second, on the use of more special semiempirical hypotheses. The self-preservation hypotheses may usually be justified with the aid of general similarity and dimensional arguments; consideration of these is of definite physical interest. However, the semiempirical theories of free turbulence use, in addition to general laws of physics, some additional hypotheses of a more speculative character; the deductions thus obtained are important, primarily, for practical applications. Henceforth, we shall confine ourselves in this section almost exclusively to the analysis of the self-preservation hypotheses for free turbulent flows.

We shall begin by attempting to explain the general idea of the self-preservation of free turbulence using the special example of a three-dimensional turbulent jet issuing in the Ox direction from the end of a thin tube of diameter D (but of arbitrary section) into an infinite space filled with the same fluid. Let us compare the fluid dynamical variables at different sections x = const. The observations show that, in general, a turbulent jet may be considered to be comparatively narrow. Thus the longitudinal component of the mean velocity in a jet considerably exceeds the transverse component, while the longitudinal variations of the statistical characteristics of the fluid dynamical variables prove to be much smaller than their transverse variations. On this basis it is natural to expect that the properties of the flow field in a given section of the jet will reproduce to some extent the corresponding properties in sections upstream. In other words, it must be expected that the statistical characteristics of the fluid dynamical variables of a jet in different sections of it (i.e., with different values of x) will be similar to each other. This means that in every section x = const, we may introduce characteristic length and velocity scales $L(x)$ and $U(x)$, respectively, such that the dimensionless statistical characteristics obtained by using these scales will be identical in all sections. Of course, this similarity will not occur immediately after the orifice from which the jet is issuing; it will become probable only at sufficiently large distances (in comparison with the diameter D) from this orifice, where its dimensions and form cease to influence the flow in the jet. Moreover, for similarity to occur it is also necessary that the Reynolds number $\mathrm{Re} = \frac{U(x)\, L(x)}{\nu}$ in the section of the jet under consideration be sufficiently large to ensure the presence of fully developed turbulence (and likewise the viscosity-independence of the statistical regime of the large-scale components of turbulence which cause interaction between the flow fields in neighboring sections of the jet, which is the physical basis of the similarity of these fields).

This postulate of similarity is yet another apphcation of the general principle of Reynolds number similarity discussed in Sect. 5.5. It may also be derived with the aid of simple considerations related to those used in Sect. 5.3 in the deduction of the logarithmic equation for the mean velocity profile close to a wall (but beyond the edge of the viscous sublayer). In fact, in the present case of a three-dimensional submerged jet, the fiow depends on the orifice diameter D, the initial velocity of efflux of the jet U_0 and the fiuid parameters ν and ρ. Thus the statistical characteristics of the flow, for example, the mean velocity $\bar{u}$ or the Reynolds stress $\rho\overline{u'w'}$ (where w' is the radial fluctuating velocity in cylindrical coordinates (r, φ, x) with axis Ox) by dimensional considerations must be given by equations of the form

$$\bar{u} = U_0 f_1\left(\frac{x}{D},\ \frac{r}{D},\ \varphi,\ \mathrm{Re}_D\right),$$

$$-\rho\overline{u'w'} = \rho U_0^2 g_1\left(\frac{x}{D},\ \frac{r}{D},\ \varphi,\ \mathrm{Re}_D\right), \quad \text{where } \mathrm{Re}_D = \frac{U_0 D}{\nu}. \tag{5.87}$$

At a sufficiently great distance from the orifice, the dependence of these characteristics on x/D practically ceases to exist. This means that when x/D is sufficiently large, we may simply write $x/D=\infty$ in Eq. (5.87); it is a corollary of the postulate that the right side of Eq. (5.87) tends to a limit as $x/D\rightarrow\infty$. However, it is natural to suppose that the "initial conditions" U_0 and D for sufficiently large x/D will, in general, be obtained only through the value of the "initial momentum" of the jet which determines the integral characteristics of the flow, slowly varying with increase of x, at the section x = const [i.e., the characteristic velocity $U_{\max}=U(x)$ on the axis of the jet and the half-width of the jet $L=L(x)$ which is equal to the value r for which $\bar{u}(r, 0, x)=\frac{1}{2}U_{\max}(x)$] ; the whole statistical flow in the neighborhood of the plane x = const will in some way be adjusted to the values of the scales U and L.. It follows, therefore, that for large values of x/D, the jet may now be considered as axisymmetric since the φ-dependence can be determined only by the form of the outlet orifice) and Eq. (5.87) in this case may be written as

$$\bar{u}=U(x)\,f_2\left[\frac{r}{L(x)},\ \mathrm{Re}_x\right],\ \overline{u'w'}=-U^2(x)\,g_2\left[\frac{r}{L(x)},\ \mathrm{Re}_x\right], \quad (5.88)$$

where

$$\mathrm{Re}_x=\frac{U(x)\,L(x)}{\nu}.$$

Finally, for sufficiently large Re_x we may put $\mathrm{Re}_x=\infty$ in Eq. (5.88) since for a developed turbulent regime the eddy viscosity always considerably exceeds the molecular, and the parameter ν will not play any part; this is precisely the principle of Reynolds number similarity. Consequently, for sufficiently large values of x/D and Re_x

$$\bar{u}=U(x)f\left[\frac{r}{L(x)}\right],\quad \overline{u'w'}=-U^2(x)\,g\left[\frac{r}{L(x)}\right], \qquad (5.89)$$

where $f(r)$ and $g(r)$ are universal functions. These relationships express the hypothesis of the self-preservation of a flow in a jet.

Further, we note that unlike turbulent flows in the presence of rigid walls, self-preservation will occur in this case for the entire region of turbulent flow. (But, as in the case of "wall" turbulence, it relates only to large-scale components of motion unconnected with the viscous dissipation of energy.)

Important connections between the functions $L(x), U(x)$, $f(r)$ and $g(r)$ may be obtained by substituting Eq. (5.89) into the Reynolds equations. In the case of steady axisymmetric motion, and neglecting terms containing the viscosity, the Reynolds equations may easily be shown to reduce to a single equation

$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{w}\frac{\partial \bar{u}}{\partial r}=-\frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{u'w'}\right)-\frac{\partial}{\partial x}\left(\overline{u'^2}-\overline{w'^2}\right) \qquad (5.90)$$

[see, for example, Townsend (1956)]. The term $\frac{\partial}{\partial x}\left(\overline{u'^2}-\overline{w'^2}\right)$ here describes the normal turbulent stress and the action of the longitudinal pressure gradient; usually it proves to be much smaller than the other terms (in any case, in the central part of the flow) and very frequently is simply neglected. Consequently, we obtain

$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{w}\frac{\partial \bar{u}}{\partial r}=-\frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{u'w'}\right), \qquad (5.90')$$

which is the basis of almost all theoretical investigations relating to turbulent jets. With the aid of the continuity equation for the averaged motion

$$\frac{\partial \bar{u}}{\partial x}+\frac{1}{r}\frac{\partial (r\bar{w})}{\partial r}=0 \qquad (5.91)$$

the radial velocity $\bar{w}$ may be ehminated from Eq. (5.90′). However, we must note first, that by multiplying Eq. (5.90) by r and integrating with respect to r from 0 to ∞, we obtain,

$$\frac{\partial}{\partial x}\int_0^\infty \bar{u}^2 r\,dr=0, \quad \text{i.e.,} \quad \int_0^\infty \bar{u}^2 r\,dr=M=\text{const}. \qquad (5.92)$$

(This equation expresses the law of conservation of momentum, i.e., the equality of the momentum flux through any sufficiently large sphere with the center close to the outlet orifice). Substituting now from the first of Eqs. (5.89), we find that

$$U^2L^2 \int_0^\infty f^2(\eta)\,d\eta = \text{const.}$$

Thus it follows that UL (and, consequently, also, the Reynolds number $\mathrm{Re}_x = UL/\nu$) cannot be dependent on x. Thus $U \sim 1/L$. Considering this result and substituting Eq. (5.89) into Eqs. (5.90′) and (5.91), it is easy to show that these equations cannot be valid if $\frac{dL}{dx} \neq$ const; the same result, of course, follows from the more general equations (5.90).

Thus, we find that self-preservation is possible only when

$$L(x) \sim x - x_0, \qquad U \sim (x - x_0)^{-1}, \tag{5.93}$$

where x_0 is the constant of integration. Here, $x = x_0$ plays the part of a virtual "origin of the jet." Since we are interested only in asymptotic laws for very large values of x, here and below we may simply assume that $x_0 = 0$. According to Eq. (5.93), at sufficiently great distances from the orifice, the breadth of the jet increases in proportion to the distance, that is, the jet takes the form of a right circular cone, while the mean velocity of flow in the jet decreases in inverse proportion to the distance. We note further that the amount of fluid passing through the cross section of the jet is $2\pi \int_0^\infty \bar{u} r\,dr$, i.e., it increases in proportion to x. This indicates that the jet is continuously taking in fluid from the surrounding, nonmoving, medium.

Under the conditions (5.93), the system of equations (5.90′)–(5.91) possesses an infinite number of solutions of the form (5.89). In fact, this system allows us to obtain only a single relationship connecting the two unknown functions $f(\eta)$ and $g(\eta)$ which is found to be of the form

$$g(\eta) = \frac{Af(\eta)}{\eta} \int_0^\eta \eta_1 f(\eta_1)\,d\eta_1, \qquad A = \text{const}$$

[see, for example, Squire (1948)]. The position is even worse if we use the more general equation (5.90). Here we obtain only one relationship containing four unknown functions [see Townsend (1956)]. A complete determination of all the unknown functions requires the introduction of still further hypotheses. In many works various semiempirical theories of turbulence are used for this purpose; see, for example, the classical papers of Tollmien (1926) and Görtler (1942), and the books of Shih-i Pai (1954), Hinze (1959), Abramovich (1963), and Vulis and Kashkarov (1965).

Of course, from the Reynolds equations it follows only that for self-preservation (5.89) to occur, the scales U and L must satisfy Eq. (5.93), but it does not follow that self-preservation must occur. Self-preservation seems probable because of the general considerations given above (which, as we shall show below, may also be obtained from standard dimensional reasoning); however, it may definitely be established only on the basis of an analysis of the data. For jets issuing into a space filled with fluid at rest, Eqs. (5.89) and (5.93) have been verified experimentally by many investigators [see, for example, Reichardt (1942); Corrsin (1943); Hinze and Van der Hegge Zijnen (1949); Corrsin and Uberoi (1950); Corrsin and Kistler (1954); Forstall and Gaylord (1955), and surveys by Hinze (1959); Abramovich (1963); and Vulis and Kashkarov (1965)]. It was found that even at fairly small distances from the orificc ($x>8D$), Eqs. (5.89) and (5.93) give a good description of the profile of the mean longitudinal velocity $\bar{u}(x, r)$. Complete self-preservation, however, of the mean velocity profiles and of the second moments of the velocity fluctuations is definitely established only at more distant cross sections of the jet ($x>50D$). A dependence of the profile on the Reynolds number $\mathrm{Re} = \frac{U_0 D}{\nu}$ is also observed, which becomes weaker as Re increases and for $\mathrm{Re} \gtrsim 10^5$ ceases to be noticeable. For example, Fig. 44 gives (according to Reichardt's data) the velocity profile in three sections of the jet, normalized by division by $U_{\max} = U$ and related to dimensionless distances obtained with a length scale L such that $\bar{u}(L) = \frac{1}{2} U_{\max}$. We see that the similarity of the profile is fulfilled in a completely satisfactory manner.

Similar hypotheses on self-preservation may be formulated also for other specific types of free turbulent flows. However, we shall consider here only a plane turbulent jet issuing into the space $x>0$

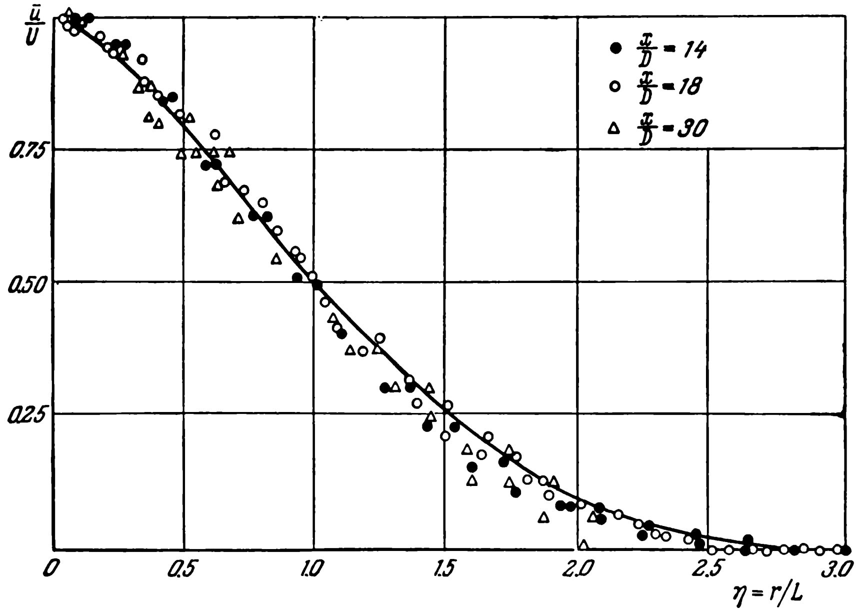


FIG. 44. Dependence of the normalized mean velocity u/U on $\eta=r/L$ in three different sections of an axisymmetric jet [according to Reichardt's data (1942)].

(which is filled with the same fluid) in the Ox direction from an infinitely long slit lying in the Oyz plane along the Oy axis; the three-dimensional turbulent wake behind a finite rigid body placed close to the origin of coordinates in a uniform stream in the Ox direction; the plane turbulent wake behind an infinitely long cylinder with its axis along the Oy direction, placed in a stream of fluid in the Ox direction; finally, the mixing layer between two plane-parallel flows in the Ox direction in the half-spaces $z>0$ and $z<0$, having constant but different velocities (say, U_1 for $z_1>0$ and U_2 for $z<0$) in the initial section Oyz. In the case of the wakes behind rigid bodies, it is natural to transform to a moving inertial system of coordinates which moves with the undisturbed flow (i.e., to consider only the deviations of the velocity in the wake from the undisturbed velocity, that are damped at great distances from the body); however, for the mixing layer, it is useful to select the coordinates in such a manner that the condition $U_2=-U_1$ is satisfied. In this case, for all the flows mentioned, it must be expected that for sufficiently large Reynolds numbers the mean velocity profile and the turbulent stress profile at sufficiently large distances x may be put in the form (5.89), where r is the transverse coordinate (the distance from the Ox axis for three-dimensional flows and from the plane $z=0$ for plane flows), and w is the transverse velocity. The scales of length $L(x)$ and velocity $U(x)$ will always be proportional to some power of the longitudinal distance from the virtual origin of x; that is,

$$L(x) \sim x^m, \qquad U(x) \sim x^{-n} \tag{5.94}$$

(so that the Reynolds number $\mathrm{Re}_x = \frac{U(x)\,L(x)}{\nu}$ is proportional to x^{m-n}). The indices m and n may be determined, as above, from the requirements for the functions to be exact solutions of the Reynolds equations (neglecting the molecular viscosity). In the case of wakes behind obstacles, it is convenient to make a preliminary simplification of the Reynolds equations by using the fact that at large distances from the body, the mean longitudinal velocity $\bar{u}$ within the wake will be only slightly less than the constant velocity U_0 of the flow (equal to the undisturbed velocity outside the wake). Thus $\bar{u}= U_0 - \bar{u}_1$ where $U_0=$ const, $\bar{u}_1 \ll U_0$ and the mean transverse velocity $\bar{w}$ will be of an order not exceeding that of $\bar{u}_1$. Therefore, for example, in the case of a three-dimensional wake, the Reynolds equation at a large distance x may be written approximately as

$$-U_0 \frac{\partial \bar{u}_1}{\partial x} = -\frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'w'}), \tag{5.95}$$

and, consequently, in this case

$$\int_0^\infty \bar{u}_1 r\,dr = N = \text{const.} \tag{5.96}$$

Further considerations for all these types of free turbulent flow are completely analogous to those for a free turbulent jet. The results obtained are given in the table below. [See also, Squire (1948); Schlichting (1960); Schubauer and Tchen (1959).] As seen from the table, the most important difference between the types of free turbulence we have mentioned consists in the types of dependence on the distance x of the Reynolds number $\mathrm{Re}_x = \frac{U(x)\,L(x)}{\nu}$. In a plane wake, as in a three-dimensional jet, Re does not vary with distance; however, the flow in a plane wake differs from that in a three-dimensional jet in the speed of self-adjustment [according to Townsend (1956), complete self-preservation in a plane wake behind a cylinder is established only at distances $x > (500-1000)D$, where D is the cyhnder diameter]. In a plane jet and in a mixing zone Re increases with increase of x. Finally, in a three-dimensional wake, Re decreases with distance, so that for sufficiently large x the turbulence must decay; at such distances the assumption of self-preservation is, of course, no longer applicable. For further discussion of such flows, see, for example, Shih-i Pai (1954); Townsend (1956); Birkhoff and Zarantonello (1957); Hinze (1959); Abramovich (1963); and Vulis and Kashkarov (1965).

	Three-dimensional jet	Two-dimensional jet	Three-dimensional wake	Two-dimensional wake	Mixing layer
m	1	1	1/3	1/2	1
n	1	1/2	2/3	1/2	0
$m-n$	0	1/2	—1/3	0	1

Attention must now be given to yet another variant of free turbulence, vertical turbulent jets of thermal origin, which occur due to the action of buoyancy above a heated body. The postulates of self-preservation of such jets were first considered by Zel'dovich (1937). If, as above, the direction of flow in the jet (i.e., the vertical direction) is taken along the Ox axis, then these postulates consist in the existence of the possibility, in every section of the jet $x=\text{const}$, of introducing scales of length $L(x)$, velocity $U(x)$ and temperature $\Theta(x)$ such that the dimensionless mean characteristics of the flow field obtained by the use of these scales will be identical in all sections. In particular, Eqs. (5.89) and similar equations for the mean temperature and the radial turbulent heat flux $c_p\rho\overline{w'T'}$

$$\overline{T}=\Theta(x)f_1\left[\frac{r}{L(x)}\right], \qquad \overline{w'T'}=-U(x)\Theta(x)g_1\left[\frac{r}{L(x)}\right] \quad (5.97)$$

will apply (where w' is the radial fluctuating velocity in the cylindrical coordinate system with axis Ox). Substituting Eqs. (5.89) and (5.97) into the Reynolds equation obtained by averaging the system of Boussinesq equations of Sect. 1.5, and neglecting terms containing the molecular viscosity and thermal diffusivity (but not terms describing the buoyancy!) we obtain the result [using a deduction similar to that of Eq. (5.93)], that

$$L(x) \sim x, \qquad U(x) \sim x^{-1/3}, \qquad \Theta(x) \sim x^{-5/3}. \tag{5.98}$$

Similar considerations may be applied to a two-dimensional convective jet arising above a heated cylinder lying in the horizontal plane $x=0$ along the Oy axis. In this case, we obtain

$$L(x) \sim x, \qquad U(x) = \text{const}, \qquad \Theta(x) \sim x^{-1}. \tag{5.99}$$

(These equations were also obtained by Zel'dovich.)

We note further, that a large part of the results collected in the table above, as well as Eqs. (5.98) and (5.99), may also be obtained simply from certain dimensional considerations. For example, for a three-dimensional jet, all the averaged characteristics at large distances x may depend, in addition to the distance x, only on the density ρ and on the total momentum $2\pi\rho M$ of the mass of fluid issuing from the orifice per unit time. Therefore, it follows that

$$\bar{u} \sim \frac{M^{1/2}}{x} f\left(\frac{r}{x}\right), \qquad -\overline{u'w'} \sim \frac{M}{x^2} g\left(\frac{r}{x}\right) \tag{5.100}$$

[which is in complete agreement with Eqs. (5.93)]. For a plane jet, instead of $2\pi\rho M$, it is necessary to consider the momentum of the mass of fluid issuing per unit time from unit length of the slit, $\rho M_1 = \rho \int_{-\infty}^{\infty} \bar{u}^2 \, dz$. Consequently, here

$$\bar{u} \sim \frac{M_1^{1/2}}{x^{1/2}} f\left(\frac{z}{x}\right), \qquad -\overline{u'w'} \sim \frac{M_1}{x} g\left(\frac{z}{x}\right) \tag{5.101}$$

(in accordance with the second column of our table). Even more simple is the application of dimensional reasoning to the plane mixing zone; here, the conditions at the section $x =$ const can depend only on x and on the given difference of the velocities

$U_1 - U_2 = 2U_1 = U_0$. Thus, for example,

$$\bar{u} \sim U_0 f\left(\frac{z}{x}\right). \tag{5.102}$$

In the case of free convective jets, all the averaged characteristics may depend on the total heat flux along the jet

$$Q = 2\pi c_p \rho \int_0^\infty \overline{u'T'}\, r\, dr,$$

or the heat flux per unit length of a heated cylinder

$$Q_1 = c_p \rho \int_{-\infty}^{\infty} \overline{u'T'}\, dz$$

and the buoyancy parameter $g\beta$ (for an ideal gas equal to g/T_0). Here Q and Q_1, naturally, may only occur in combinations $Q/c_p\rho$ and $Q_1/c_p\rho$. Hence, it follows that

$$\bar{u} \sim \left(\frac{g\beta Q}{c_p \rho x}\right)^{1/3} f_1\left(\frac{r}{x}\right), \quad T \sim \left(\frac{Q}{c_p\rho}\right)^{2/3} (g\beta)^{-1/3} x^{-5/3} g_1\left(\frac{r}{x}\right) \tag{5.103}$$

for a three-dimensional convective jet and

$$\bar{u} \sim \left(\frac{g\beta Q_1}{c_p \rho}\right)^{1/3} f_1\left(\frac{z}{x}\right), \quad T \sim \left(\frac{Q_1}{c_p\rho}\right)^{2/3} (g\beta)^{-1/3} \frac{1}{x} g_1\left(\frac{z}{x}\right) \tag{5.104}$$

for a plane convective jet. As is clear, these results coincide with Eqs. (5.98) and (5.99).

Only in the case of wakes is the position somewhat more complicated, since, in addition to the total drag force acting on the body

$$F = 2\pi\rho U_0 N = 2\pi\rho U_0 \int_0^\infty \bar{u}_1 r\, dr$$

or the drag force per unit length of the cylinder

$$F_1 = \rho U_0 N_1 = \rho U_0 \int_0^\infty \bar{u}_1 \, dz ,$$

which determines the "momentum defect" in the wake, in the conditions of this problem there occurs a further dimensional quantity, namely, the undisturbed velocity U_0. Thus in this case, equations of the type (5.100)–(5.102) cannot be deduced on the basis of dimensional considerations only. However, with the aid of somewhat more special considerations, these may be obtained, without necessitating recourse to the exact equations. Specifically, we use the fact that the tangent of the angle of inclination of a streamline of the averaged flow in the wake to the Ox axis is equal to $\frac{\bar{w}}{\bar{u}} = \frac{\bar{w}}{U_0 - \bar{u}_1} \approx \frac{\bar{w}}{U_0}$. Therefore, we may conclude that if $L(x)$ is the characteristic breadth of the wake, and $U = U(x)$ is the characteristic scale of the velocities $\bar{u}_1$ and $\bar{w}$, then

$$\frac{dL}{dx} \sim \frac{U}{U_0} . \tag{5.105}$$

But since

$$N = \int_0^\infty \bar{u}_1 r \, dr$$

is constant in the case of a three-dimensional wake and

$$N_1 = \int_{-\infty}^{\infty} \bar{u}_1 \, dz$$

is constant in the case of a plane wake, it follows that $UL^2 \sim N = \text{const}$ in the first case, and $UL \sim N_1 = \text{const}$ in the second. These results, together with the relationship (5.105) show that

$$L \sim \left(\frac{Nx}{U_0}\right)^{1/3} , \quad U \sim \left(\frac{NU_0^2}{x^2}\right)^{1/3} ,$$

that is,

$$\bar{u} \sim \left(\frac{NU_0^2}{x^2}\right)^{1/3} f\left(\frac{rU_0^{1/3}}{N^{1/3}x^{1/3}}\right) \tag{5.106}$$

for a three-dimensional wake and

$$L \sim \left(\frac{N_1 x}{U_0}\right)^{1/2}, \quad U \sim \left(\frac{N_1 U_0}{x}\right)^{1/2},$$

that is,

$$\bar{u} \sim \left(\frac{N_1 U_0}{x}\right)^{1/2} f\left(\frac{zU_0^{1/2}}{N^{1/2}x^{1/2}}\right) \tag{5.107}$$

for a plane wake.

Of course, the results concerning free convection jets must not be confused with the analogous results for free flows with forced convection, the transfer of a passive admixture ϑ (which may also be heat) from a steady continuous source by free turbulent flows of purely dynamic origin. In this case, the initial system of equations will consist of the ordinary equations of fluid dynamics, without any buoyancy terms, and the mass (or heat) transport equations (1.72). The length and velocity scales will naturally be the same as for the corresponding purely dynamic problem (i.e., they will be given by Eqs. (5.94) with the values of the indices shown in the table), but there is also an additional scale of concentration (or temperature) $\Theta(x)$. This new scale may be determined with the aid of an additional Reynolds equation, obtained by averaging Eq. (1.72) with $\chi = 0$. However, it is simpler to take as a starting point the fact that the mass of the admixture (or the heat flux) passing through any section $x =$ const per unit time is constant. This immediately gives the precise value of the index k in the equation $\Theta(x) \sim x^{-k}$ which depends on the type of the flow and the type of the source (two-dimensional or three-dimensional). The corresponding results will be discussed in detail in Chapt. 5 in connection with the study of turbulent diffusion. Here we shall note merely that knowing the distribution of the mean velocity $\bar{u}$ and the mean temperature $\bar{\vartheta}$, and the values of the turbulent fluxes $\overline{u'w'}$ and $\overline{\vartheta' w'}$, we can also determine the values of the corresponding eddy diffusivities K and K_ϑ and find their ratio $\alpha = \frac{1}{\mathrm{Pr}_t} = \frac{K_\vartheta}{K}$. Measurements permitting this to be established have been performed many times by different investigators for different kinds of flows [see, in particular, the works cited earlier in this subsection, primarily, Hinze (1959), Abramovich (1963), and Vulis and Kashkarov (1965)]. On the whole, all these results are in fairly good agreement with each other and show that to a first approximation the eddy diffusivities K and K_ϑ may be assumed to take constant values independent of the molecular coefficients ν and χ in almost every cross section of the jet or wake, and that the ratio $\alpha = K_\vartheta / K$ for free turbulence is greater than unity (namely, $\alpha \approx 1.8-2$ for a cylinder wake, and for a plane jet, and $\alpha \approx 1.2-1.4$ for an axisymmetric jet). It was also found that the values of K_ϑ for heat and mass transport are exactly the same (but different from the eddy viscosity K), so that $\mathrm{Pr}_t = \mathrm{Sc}_t \neq 1$ [cf. Kiser (1963)]. The independence of α and $\mathrm{Pr}_t = \alpha^{-1}$ on the molecular fluid properties was clearly demonstrated in the experiments of Sakipov and Tcmirbayev (1965) [see also Vulis and Kashkarov (1965)], who measured K and K_ϑ in free jets of technical oil, water and mercury (i.e., at molecular Prandtl numbers in the range from 10^3 to 10^{-2}!) and obtained identical values of α and Pr_t.

In conclusion, we must stress that all free turbulent flows possess one very important feature, the region of space occupied by the rotational turbulent flow will possess at every instant a clear but very irregular boundary (strictly speaking, this is not a surface but a very thin layer, the so-called "Corrsin's superlayer") outside which the motion is irrotational. As already explained in Sect. 2.2, fluid may be drawn into the turbulent region from outside, but cannot escape from it. Only large-scale turbulent velocity fluctuations can penetrate into the region of irrotational motion; these are damped at distances of the order of the transverse scales $L(x)$ of the region of rotational flow. These large-scale fluctuations cause the irregular form of the interface between the turbulent and the irrotational motion.

As an illustration, Fig. 45 gives an excellent shadowgraph of the turbulent wake behind a bullet in flight through the air, taken from the work of Corrsin and Kistler (1954).

Thus, every free turbulent flow has, in the words of Townsend (1956), a "double structure": it consists of intensive small-scale turbulence and relatively less intensive large-scale vortices, which

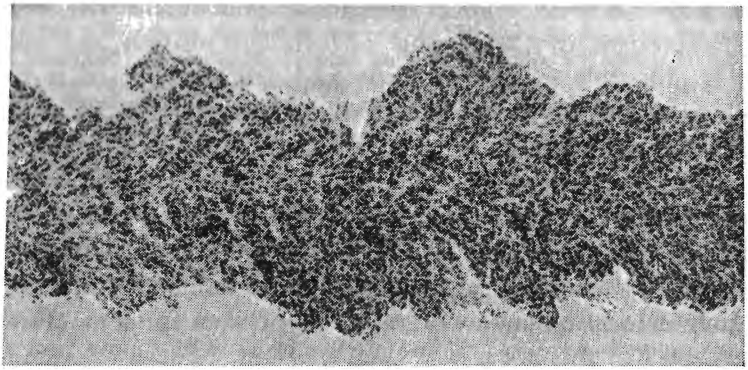


FIG. 45. Shadowgraph of the turbulent wake behind a bullet in flight.

produce irregular distortion and motion of the turbulent-irrotational interface. Due to this motion of the interface at any point of a free turbulent flow, not too remote from its axis (or, in the case of a two-dimensional flow, from its plane of symmetry), small-scale turbulence will sometimes be present and sometimes be absent, i.e., the turbulence will be intermittent.[15] In Sect. 5.6, we have already discussed the intermittency of the turbulence in the outer parts of a turbulent boundary layer. In free turbulent flows, the zone of intermittency of turbulence is, however, considerably wider than in boundary layers; if in a boundary layer, the intermittent region is observed to lie at a distance of $0.4\delta - 1.2\delta$ from the wall, then, for example, in a plane turbulent wake, according to Corrsin and Kistler (1954), it will be quite distinct in a zone from $z = 0.4L(x)$ [where z is the transverse coordinate] at least to $z = 3.2L(x)$. In addition to the cases of free turbulence and the outer part of a turbulent boundary layer (which is similar in many respects to free turbulent flows), intermittency of turbulence and quite definite irregular moving interfaces between the turbulent and nonturbulent regions of the flow are also observed. For example, in tubes and boundary-layer flows at the "transitional" Reynolds numbers which favor the formation of turbulent "slugs" and "spots" (see above, Sect. 2.1), and also, in certain other cases [see, e.g., the survey of Coles

[15] It is often assumed that intermittency will occur only beyond some definite distance from the axis (or plane of symmetry) of the flow. However, there are some indications, e.g., in the axisymmetric wake not too close to the body, that the irregular meanderings of the wake will generate considerable intermittency even on the wake axis [see Hwang and Baldwin (1966); or Gibson, Chen and Lin (1968)].

(1962)]. We may also expect that similar phenomena will, in general, be characteristic of a wide range of turbulent motion, and will play a considerable role in the initiation and development of turbulence. However, so far, the theoretical investigation of these phenomena has met with considerable difficulties and only first steps in this direction have been made (see, e.g., Coles (1962); Liepmann (1962); Townsend (1966); and Kovásznay (1967), where additional references can also be found). We shall not dwell here further on these questions; let us only note that the related subject of intermittency of small-scale turbulence will be of importance for some discussions in volume 2 of this book.

5.9 Semiempirical Theories of Turbulence

We have already shown in Sect. 5.1 that in turbulent flows the laws of mechanics are expressed by the Reynolds equations, in which the number of unknowns exceeds the number of equations. Thus, the Reynolds equations only allow us to obtain some connections between the various characteristics of turbulence, and they cannot be "solved" in the usual sense of the word. Therefore, in selecting solutions of the Reynolds equations which have a physical meaning, certain functions describing the turbulence must necessarily be given independently of these equations. In certain cases, the form of these functions may be found (accurate to a fairly small number of empirically determined constants) on the basis of dimensional considerations; more often, however, the use of dimensional considerations only leads to relationships with unknown functions that must then be found from experiment. In general, the number of these unknown functions needed to describe the various turbulent flows encountered in nature or in engineering applications, is very great. Thus it is understandable why many investigators have tried to reduce the determination of the functions to finding a small number of statistical characteristics of turbulence, which would be applicable simultaneously to many different flows. The theories of turbulence which use, in addition to the strict equations of fluid mechanics, some additional relations found purely empirically from experiment, or deduced with the aid of qualitative physical considerations and then verified experimentally, are called *semiempirical theories*. Of course, from the viewpoint of "pure" theoretical physics, all these theories must be considered as nonrigorous, but they are quite typical of investigations in the turbulent regime. In the development of our understanding of turbulence, the semiempirical theories have played a considerable part, and many of them even now are still widely used in technology. Consequently, it would seem useful here to give at least a brief outline of the fundamental ideas of the most important semiempirical theories proposed by Boussinesq (1877; 1899), Prandtl (1925), G. I. Taylor (1915; 1932) and von Kármán (1930). Further development of this approach to the theory of turbulence and some actual examples of the use of semiempirical theories will be considered in the following section.

The system of Reynolds equations for the mean values of the flow variables is not closed due to the presence in them of additional terms containing turbulent (Reynolds) stresses. Therefore, it is clear that the simplest means of closure of the system of equations consists of establishing connections between the mean flow variables, on the one hand, and the Reynolds stresses, on the other. This permits the latter values to be expressed in terms of the former. Below, we shall enumerate several of these kinds of connections proposed by different scientists at different times. We shall consider only the simplest plane-parallel flows in which only the x-component of the mean velocity $u_x = u$ is different from zero, while this depends only on the z coordinate, or, which is almost the same, flows in a circular tube in which the mean velocity u is directed everywhere along the Ox axis and depends only on the distance $z = R-r$ from the wall of the tube. In this case, the Reynolds equation will take the form (5.16) [or Eq. (5.16′)]; hence the velocity profile will be determined in this case by a single equation [the first equation of Eq. (5.16) or Eq. (5.17′)], containing a single additional unknown $\tau^{(1)} = -\rho\overline{u'w'}$. Consequently, to determine the function $u(z)$, it is only necessary to be able to express $\tau^{(1)}$ in terms of $\bar{u}(z)$.

The simplest possible assumption on the connection between the turbulent stress $\tau^{(1)} = -\rho\overline{u'w'}$ and the profile $\bar{u}(z)$ is Boussinesq's assumption (1877; 1897) on the existence of a relationship of the form (5.5) containing some coefficient of eddy viscosity K. Strictly speaking, Eq. (5.5) is not an assumption and does not formulate a new connection; it only replaces the unknown $\tau^{(1)}$ by the new unknown $K = \frac{\tau^{(1)}}{\rho\,(d\bar{u}/dz)}$. However, if it is assumed that K depends on the coordinate in some definite manner, then we immediately arrive at a semiempirical theory based on a hypothesis and capable of experimental verification.

The simplest assumption about the value of K is that it is a constant. This assumption has been applied to free turbulent flows by many authors and turns out to be a reasonable first approximation for such flows; however, as easily seen, when applied to flows in plane channels or circular tubes, it leads to completely incorrect results. In fact, with K = const, the equations for the mean velocity of turbulent flow would differ from the corresponding equations for laminar flow only in the numerical value of the viscosity. Consequently, it would follow that the tube or channel velocity profile in this case would be parabolic. However, it is known that such a profile does not correspond to experiment (see Fig. 32, Sect. 5.5).

There is, of course, nothing remarkable in this contradiction; it is well known that as we approach the wall the coefficient K must tend to zero (but not more slowly than the third power of the distance from the wall; see Sect. 5.3). Thus the assumption that K is constant is quite inapplicable to the case of flow in the presence of rigid walls. However, in certain problems, relating, for example, to turbulent jets or wakes propagated in an infinite space, or to turbulence in the free atmosphere, the assumption K = const is not entirely invalid. In these cases, substitution of Eq. (5.5) with constant K in the Reynolds equation leads to a semiempirical equation which contains the unknown parameter K, which must be determined from observation (which in various cases leads to extremely diverse values).

For flows bounded by walls, it is impossible to assume that K = const, but here in a number of cases, we may introduce other reasonable hypotheses on the coefficient of turbulent viscosity K which permit us to make the Reynolds equations determinate. Thus, for example, in the case of a plane-parallel flow close to a plane wall (but not too close, beyond the limit of the viscous sublayer) the hypothesis that K is proportional to the distance from the wall leads to good results; from it we find the logarithmic equation (5.22), which agrees with the deductions from dimensional considerations. For flows in tubes, channels and boundary layers, a considerable number of hypotheses have been proposed on the dependence of K on the distance from the wall z; as two simple examples, we may mention the note of Gosse (1961), in which fairly good agreement with experiment in tubes and boundary layers is obtained by the assumption that K is proportional to the function $1 - (1 - \eta)^2$ where $\eta = z/H_1$ and H_1 is the radius of the tube or the thickness of the boundary layer, and the work of Szablewski (1968), who obtained fair agreement for flows in tubes and plane channels with the assumption that $K = \kappa u_* z\sqrt{1-\eta}\, e^{-\eta/m}$ for $0 \leqslant \eta \leqslant m$ and $K = K(mH_1)$ = const for $m \leqslant \eta \leqslant 1$ where $m \approx 0.8$. In the immediate proximity of the wall the interaction of the processes of molecular and turbulent friction leads to a dependence of the form $K = \kappa u_* z\varphi(zu_*/\nu)$; some semiempirical hypotheses on the form of the function φ were proposed by Deissler (1955), Rannie (1956), Levich (1962) and others (see above, Sects. 5.3 and 5.7). In the surface layer of the atmosphere, in the case of stable thermal stratification, according to all the existing data as the height z increases, the coefficient first increases almost linearly, and then its rate of increase becomes slower, while for sufficiently large z, the eddy viscosity may be assumed practically constant. In this connection, several formulas for $K(z)$ have been proposed, which agree with the general rules which we have indicated [see, for example, Yudin and Shvets (1940); Dorodnitsyn (1941), Berlyand (1947)]. Each of these equations corresponds to a variant of the semiempirical theory. Thus, after determining all the parameters occurring in the chosen formula for $K(z)$ from observational data, we can calculate, to some degree of accuracy, the variation of the wind velocity with height, and solve some other problems of meteorological interest.

In the investigation of mass and heat transfer in a turbulent flow the role of the turbulent stress is played by the turbulent flux of mass $j^{(1)} = \rho\overline{\vartheta' w'}$ or heat $q^{(1)} = c_p\rho\overline{\vartheta' w'}$, and the role of the eddy viscosity K is played by the eddy diffusivity K_ϑ, defined by Eq. (5.9). For this coefficient we may repeat all that has been said above concerning K (see, in particular, Sect. 10 below, where we shall discuss atmospheric diffusion and consider certain semiempirical theories which use a definite form of the dependence of K_ϑ on z).

Often, however, the concept of the eddy viscosity (or diffusivity) does nothing to simplify the turbulence problem since the selection of an appropriate assumption concerning this quantity is very difficult and it is not clear with which such choice one should work. To facilitate this selection, some other semiempirical theories have been formulated. In many of these theories, an essential part is played by the "*mixing length*" introduced by Prandtl (1925) [this idea was put forward in slightly different form earlier by G. I. Taylor (1915)].

Use of the concept of the mixing length is best illustrated by using the example of the turbulent transfer of a conservative passive admixture ϑ. Let the mean concentration ϑ depend only on the z coordinate. As a result, we are concerned mainly with the transfer of this admixture along the Oz axis. Let us assume that the transfer is effected only by turbulent fluctuations of velocity. Clearly, this may occur in such a way that there arise in the medium small random jets which transfer the admixture from one level z = const to another. Let us assume that every such jet travels along the Oz axis a distance l', and that only after this is it mixed with the surrounding medium; in this case l' will be the mixing length. Therefore, in accordance with our assumption, only jets produced at the level $z - l'$ and moving upwards, or produced at $z + l'$ and moving downwards will impinge upon the level z. If we now assume that the mixing of the jet takes place not gradually but suddenly, then the jet produced at the level $z - l'$ will transfer to the level z its initial concentration [equal, on the average to $\vartheta(z - l')$] while the jet produced at the level $z + l'$ will transfer a concentration equal, on the average, to $\vartheta(z + l')$. Thus, arriving at the level z, the jets will contain concentrations of the admixture different from the mean concentration at that level, i.e., this will lead to fluctuations of concentration equal to $\bar{\vartheta}(z \mp l') - \bar{\vartheta}(z) \approx \mp l' \frac{d\bar{\vartheta}(z)}{dz}$. It is clear that the jets moving upwards correspond to a positive vertical fluctuating velocity w' and those moving downwards to a negative value of w'. Hence, if we take l' to have a positive sign for jets moving upward, and a negative sign for jets moving downward, then

$$j^{(1)} = \rho\overline{w'\vartheta'} = -\rho\overline{l'w'}\,\frac{d\bar{\vartheta}}{dz}. \tag{5.108}$$

Thus, as is easily seen, the turbulent diffusion coefficient K_ϑ may be expressed very simply in terms of the mixing length (which is, generally speaking, a random quantity)

$$K_\vartheta = \overline{l'w'}. \tag{5.109}$$

We may attempt to use Eq. (5.108) not only for calculating fluxes of heat or material admixtures, but also to describe the momentum flux. This is the basis of Prandtl's "*momentum transfer theory*" (1925) which proposes that for the longitudinal component of momentum ρu there also exists a definite mixing length l', and, consequently,

$$\tau^{(1)} = -\rho\overline{u'w'} = \rho\overline{l'w'}\frac{d\bar{u}}{dz}, \qquad K = \overline{l'w'}. \tag{5.110}$$

Here, the length l' in some sense is analogous to the length of the mean free path of the molecules in the kinetic theory of gases; it defines the path along which some small fluid element (lump of fluid) passes before it is mixed with the remaining fluid particles and transmits its momentum to them. In this sense, Eq. (5.110) for K is analogous to the expression for the coefficient of molecular viscosity (which derives from the kinetic theory of gases) $\nu \sim l_m u_m$, where l_m is the length of the mean free path, and u_m is the velocity of thermal motion of the molecules. Of course l' is many orders of magnitude greater than l_m; therefore, in spite of the fact that u_m generally is greater than w', Eq. (5.110) gives values of K that are considerably greater than ν.

We now note that a lump reaching the layer z from the layers $z - l'$ and $z + l'$ will have mean longitudinal velocities of $\bar{u}(z - l')$ and $\bar{u}(z + l')$, respectively. Thus, in both cases they will produce a fluctuation u' in the longitudinal velocity which is close in absolute value to $\left| l'\frac{d\bar{u}}{dz}\right|$. Then, assuming that the fluctuation of the vertical velocity w' is close in absolute value to the fluctuation of the longitudinal velocity u',

$$w' \sim u' \sim l'\frac{d\bar{u}}{dz},$$

Prandtl expressed the turbulent stress $\tau^{(1)}$ in the form

$$\tau^{(1)} = \rho l^2 \left|\frac{d\bar{u}}{dz}\right| \frac{d\bar{u}}{dz}, \tag{5.111}$$

where l is a new length of the same order as the root mean square of the random length l' (l is not, of course, simply equal to $(\overline{l'^2})^{\frac{1}{2}}$, since w' is only of the same order as $l'\dfrac{d\bar{u}}{dz}$, and does not have to be exactly equal to $l'\dfrac{d\bar{u}}{dz}$ for every lump). The absolute value sign in Eq. (5.111) is chosen so that the stress $\tau^{(1)}$ always has the same sign as $\frac{d\bar{u}}{dz}$. Thus the momentum is always transmitted from the faster-moving layers of the fluid to the slower. The length l in Eq. (5.111), which is no longer a random quantity, usually is also called the mixing length.

Equation (5.111) allows the eddy viscosity to be written in the form

$$K = l^2 \left|\frac{d\bar{u}}{dz}\right|. \tag{5.112}$$

where l is a length which depends, generally speaking, on the coordinates and characterizes the scale of turbulence (the mean dimension of the turbulent fluctuations) at a given point. Then the determination of the mean velocity profile requires only the fixing of the explicit dependence of l on the coordinates. For flows close to an infinite plane wall which are characterized by a constant value of τ, in the region beyond the viscous sublayer, there is no length scale. Therefore, every quantity with dimensions of length must be proportional to the distance from the wall. Putting $l = \varkappa z$ and $\tau^{(1)} = \tau = \rho u_*^2 =$ const, from Eq. (5.111) we obtain $\frac{d\bar{u}}{dz} = \frac{u_*}{\varkappa z}$, i.e., $\bar{u} = u_*/\kappa \ln z +$ const [see Prandtl (1932b)]. Thus, again we obtain a logarithmic law for the velocity profile, which, as we have seen, is confirmed very well by experiment. For flows in a plane channel or in a circular tube, as a first approximation we may even assume that $l=\varkappa z$ remains correct right up to the middle of the channel or the axis of the tube. Substituting this relationship into Eq. (5.111) and replacing the left side by Eq. (5.17) or (5.17′) [i.e., once again identifying $\tau^{(1)}$ with $\tau=\tau^{(1)}+\nu\frac{d\bar{u}}{dz}$], we obtain a differential equation with respect to $\bar{u}(z)$, which with an appropriate choice of the constant $\varkappa$, leads to a velocity profile which corresponds fairly well with experimental data for a considerable part of the cross section of the channel or tube [see Goldstein (1938)]. However, in immediate proximity to the center of the channel or tube, this approximation is clearly inapplicable; here it is considerably better to take the mixing length to be approximately constant. Taking, for example, $l = \beta R$ for a tube of radius R, then by Eqs. (5.17′) and (5.111), we have

$$\rho\left(\beta R\frac{d\bar{u}}{dz}\right)^2=\rho u_*^2\left(1-\frac{z}{R}\right).$$

Integrating this equation, with the boundary condition $\bar{u}(R)=U_0$., we obtain

$$\frac{U_0-\bar{u}(z)}{u_*}=\frac{2}{3\beta}\left(1-\frac{z}{R}\right)^{\frac{3}{2}}.$$

The latter result, as was shown as far back as Darcy (1858), with $\beta\approx 0.13$ is in good agreement with the data in the region $0.25<\frac{z}{R}<1$ [see Eq. (5.46), Sect. 5.5]. Finally, to obtain agreement with experiment for the flow in a circular tube for all values of $\frac{z}{R}$ outside the limits of the viscous sublayer, we must put

$$l=R\left[0.14-0.08\left(1-\frac{z}{R}\right)^2-0.06\left(1-\frac{z}{R}\right)^4\right] \tag{5.113}$$

[Nikuradse (1932)]. The latter equation, for $z/R \ll 1$, takes the form $l = 0.4z-0.44\frac{z^2}{R}+\dots\approx 0.4z$, while as $z/R\to 1$ it becomes $l\approx 0.14R$, i.e., it may be considered as an interpolation formula between the values for l close to the wall and at the center. Another equation for $l=l(z)$ for flow in a tube (or channel) was proposed by Szablewski (1968). A similar distribution of l over the cross section of the tube was proposed also by Obukhov (1942), who at the same time put forward a hypothesis on the length l permitting its dependence on the coordinates to be found for tubes of any cross section. In the immediate proximity of the wall (within the limits of the viscous sublayer) l must decrease obviously with decrease of z more rapidly than according to the linear law; hypotheses on the behavior of l in this region of flow have been proposed by Rotta (1950), Hama (1953), van Driest (1956) and others. A three-layer distribution of l across a plane turbulent boundary layer was strongly recommended by Spalding (1967) as being surprisingly satisfactory for most practical purposes; this distribution is a combination of van Driest's distribution close to the wall, a linear distribution $l = \kappa z$ (with $\kappa = 0.435$) in the logarithmic layer below $z = 0.09\delta/\kappa$ and a uniform distribution $l = 0.09\delta$ for $z \gg 0.09\delta/\kappa$.

Later, Prandtl made some generalizations of the momentum transfer theory that are applicable, for example, to three-dimensional flows [cf. Goldstein (1938), Sect. 81], or which take into account the possibility that for $\frac{d\bar{u}}{dz}=0$, the turbulent viscosity, in contradiction to Eq. (5.112) is, nevertheless nonzero [Prandtl (1942)]. However, we shall not discuss these generalizations in detail but proceed immediately to Taylor's *vorticity transfer theory*, the second fundamental semiempirical theory.

The formulation of this theory derived from an attempt to consider the effect of pressure fluctuations on moving fluid particles, which lead to variation in their momentum, thereby preventing momentum from being considered as a conservative admixture as the elements of fluid are displaced. On this basis, G. I. Taylor (1915), introduced for the first time the concept of the "mixing length." However, unlike Prandtl, he assumed that a "mixing length" must exist for the vorticity but not for the momentum; later, Taylor developed this idea in greater detail [see G. I. Taylor (1932)].

Taylor's idea is best justified for a two-dimensional flow in which, as is known, in the absence of viscosity, the vorticity is transferred without change as the fluid particles move. Thus the single component of the vorticity $\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$ of a two-dimensional flow with velocity field $\boldsymbol{u} = \{u, w\}$ is a conservative quantity everywhere outside the viscous sublayer. The turbulent shear stress $\tau = -\rho\overline{u'w'}$ of a two-dimensional flow with mean velocity $\bar{\boldsymbol{u}} = \{\bar{u}, 0\}$ everywhere parallel to the Ox axis, satisfies the equation

$$\frac{1}{\rho}\frac{\partial \tau}{\partial z} = -\frac{\partial}{\partial z}\overline{u'w'} = -\left(\overline{u'\frac{\partial w'}{\partial z}} + \overline{w'\frac{\partial u'}{\partial z}}\right) =$$
$$= -\overline{w'\left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}\right)} + \frac{1}{2}\frac{\partial}{\partial x}\left(\overline{u'^2} - \overline{w'^2}\right);$$

here the continuity equation for the velocity fluctuations $\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$ is used. In a homogeneous flow along the Ox axis, the derivatives of the averaged quantities with respect to x will be equal to zero; consequently,

$$\frac{\partial \tau}{\partial z} = -\rho\overline{w'\omega_y'}. \tag{5.114}$$

Introducing the mixing length for the vorticity l_1', such that $\omega_y' = l_1'\frac{\partial \bar{\omega}_y}{\partial z}$, $\rho\overline{w'\omega_y'} = \rho\overline{w'l_1'}\frac{\partial \bar{\omega}_y}{\partial z}$. As in Prandtl's theory, we take w' in the form $w' = l'\frac{\partial \bar{u}}{\partial z}$, where l' is the random mixing length for the velocity. In this case, we observe that $\bar{\omega}_y = \frac{\partial \bar{u}}{\partial z} - \frac{\partial \bar{w}}{\partial x} = \frac{d\bar{u}}{dz}$ and, therefore, $\frac{d\bar{\omega}_y}{dz} = \frac{\partial^2 \bar{u}}{dz^2}$. Finally, we obtain

$$\frac{d\tau}{dz} = \rho l_1^2 \frac{d\bar{u}}{dz}\frac{d^2\bar{u}}{dz^2}, \tag{5.115}$$

where $l_1 = (-\overline{l'l_1'})^{\frac{1}{2}}$ is the characteristic length which plays the same part in Taylor's theory as does the length l in Prandtl's theory. Of course, if we consider the fact that actually the velocity disturbances in a plane-parallel flow may also be three-dimensional, then the deduction given for Eq. (5.115) loses its force, in this case, Eq. (5.115) must be regarded as an empirical connection only susceptible to experimental verification. Equation (5.115) is also susceptible to further generalization [see, for example, G. I. Taylor (1935c), or Goldstein (1938), Sects. 84–85]; however, we shall not dwell on this point.

For the case l_1 = const, Eq. (5.115) turns out to be strictly equivalent to Prandtl's equation (5.111). In fact, if l_1 = const, then by integrating Eq. (5.115) with respect to z we obtain $\tau = \frac{1}{2}\rho l_1^2\left(\frac{d\bar{u}}{dz}\right)^2$. Also, the last equation agrees with Prandtl's equation if we assume that $l_1 = 1\sqrt{2}$. In general, however, the results of Taylor's theory differ slightly from those of Prandtl's theory. When applied to pressure flows in plane channels and circular tubes the

Taylor theory with $l_1 = \varkappa_1 z$ (where it is convenient to take $\varkappa_1 \approx 0.23$ for a channel flow and $\varkappa_1 \approx 0.19$ for a tube flow) permits one to obtain a form of the mean velocity profile which gives fair agreement with the existing data almost up to the middle of the channel or the axis of the tube [see, e.g., Goldstein (1938), Sects. 156–157].

Using an argument similar to the deduction of Eq. (5.111), and substituting $w' = l' \frac{d\bar{u}}{dz}$ into Eq. (5.108) we may deduce the equation

$$j^{(1)} = -\rho l_2^2 \frac{d\bar{u}}{dz}\frac{d\bar{\vartheta}}{dz}, \qquad K_\vartheta = l_2^2 \frac{d\bar{u}}{dz}, \tag{5.116}$$

where l_2 is yet another mixing length, which now characterizes the transfer of a passive conservative admixture ϑ. The relationship between the lengths l_2 and l (or l_1) determines the value of the turbulent Prandtl number $\mathrm{Pr}_t = \frac{K}{K_\vartheta}$ (or the value of $\alpha = \frac{K_\vartheta}{K}$). Thus, for example, in a region of flow in which we may put l = const, the value $l_2 = l$ corresponds to $\mathrm{Pr}_t = \alpha = 1$, while for $l_2 = l_1$ we obtain $\mathrm{Pr}_t = 1/2$, $\alpha = 2$. Theoretically, of course, the length l_2 is not bound to be similar to either l or l_1, hence it is impossible to assume (as is sometimes done) that from the momentum transfer theory follows the value $\mathrm{Pr}_t = 1$, and from the vorticity transfer theory, the value $\mathrm{Pr}_t = 1/2$, in fact, the relationship between the lengths l_2, l, and l_1 can only be sought on the basis of experiment.

Neither Prandtl's momentum transfer theory nor Taylor's vorticity transfer theory completely solves the problem of the connection between the Reynolds stresses and the mean velocity field. This is because a new quantity, the mixing length, is introduced in these theories, and to determine this at every point of the flow additional hypotheses are required. An extremely general hypothesis which allows us, in particular, to estabhsh a general connection between the length l and the mean velocity field was proposed by von Kármán (1930)—*the hypothesis of local kinematic similarity* of the field of turbulent velocity fluctuations. According to this hypothesis, the field of turbulent velocity fluctuations in the neighborhood of every point of a developed turbulent flow (excluding only the points of the thin viscous sublayer close to a rigid wall, in which the action of the viscosity is revealed directly) are similar to each other and differ only in the length and time scales, or, which is the same thing, length and velocity. To formulate this hypothesis mathematically, we introduce into the neighborhood of every point of the flow $\boldsymbol{x}_0$ a moving coordinate system, moving with a velocity equal to the mean velocity $\bar{\boldsymbol{u}}(\boldsymbol{x}_0)$ at the given point; the coordinates in this system will obviously agree with the components of the vector $\boldsymbol{x} - \boldsymbol{x}_0 - \bar{\boldsymbol{u}}(\boldsymbol{x}_0)\, t$. Also according to this hypothesis, at every point $\boldsymbol{x}_0$ may be defined a scale of length $l(\boldsymbol{x}_0)$ and a scale of velocity $U(\boldsymbol{x}_0)$ such that in the neighborhood of this point, the field of turbulent velocity fluctuations will take the form

$$\boldsymbol{u}'(\boldsymbol{x}) = U(\boldsymbol{x}_0)\, \boldsymbol{v}(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} = \frac{\boldsymbol{x} - \boldsymbol{x}_0 - \bar{\boldsymbol{u}}(\boldsymbol{x}_0)\, t}{l(\boldsymbol{x}_0)}, \tag{5.117}$$

where the function $\boldsymbol{v}(\boldsymbol{\xi})$ is universal, i.e., independent of the point $\boldsymbol{x}_0$ [and consequently of the mean velocity field $\bar{\boldsymbol{u}}(\boldsymbol{x})$]. Thus, in von Kármán's theory, the connection between the characteristics of turbulent fluctuations and the mean velocity field will occur only because of the dependence of the scales l and U on the field $\bar{\boldsymbol{u}}(\boldsymbol{x})$.

To determine the dependence of the scales l and U on the mean velocity of flow in a plane-parallel steady flow with mean velocity $\bar{u}(z)$ directed everywhere along the Ox axis and dependent only on z, we may use Eq. (1.7) for the vorticity, the y-component of which, ignoring the viscosity, will take the form

$$\frac{\partial \omega_y}{\partial t} + u_\alpha \frac{\partial \omega_y}{\partial x_\alpha} - \omega_\alpha \frac{\partial u_y}{\partial x_\alpha} = 0. \tag{5.118}$$

Substituting into this equation the value of the total velocity $\boldsymbol{u}=u(z_0)\,\boldsymbol{i}+U(z_0)\,\boldsymbol{v}(\boldsymbol{\xi})$, where i is the unit vector of the Ox axis and

$$\boldsymbol{\xi}=\{\xi_1,\ \xi_2,\ \xi_3\}=\left\{\frac{x-\bar{u}(z_0)\,t}{l(z_0)},\ \frac{y}{l(z_0)},\ \frac{z-z_0}{l(z_0)}\right\}$$

is the vector of the relative dimensionless coordinates (and it is assumed that $x_0=y_0=0$), we may pass from Eq. (5.118) to an equation in the unknowns $v_\alpha(\boldsymbol{\xi})$, $\alpha=1,2,3$, with coefficients containing the quantities $U(z_0)$, $l(z_0)$, $\bar{u}(z_0)$ and their derivatives with respect to z_0. Since according to our hypothesis $v_\alpha(\boldsymbol{\xi})$ is a universal function, it cannot be dependent on z. Thus we may obtain the relationships

$$\frac{l(z_0)}{U(z_0)}\frac{d\bar{u}(z_0)}{dz_0}=\text{const},\qquad \frac{l^2(z_0)}{U(z_0)}\frac{d^2\bar{u}(z_0)}{dz_0^2}=\text{const}$$

[see, for example, Goldstein (1938), Sect. 158, or Schlichting (1960), Chapt. 19, Sect. 5, where the corresponding deduction is given in detail for the special case of a two-dimensional field $\boldsymbol{u}'(\boldsymbol{x})$]. Therefore, it follows that

$$l=-\varkappa\frac{d\bar{u}}{dz}\Big/\frac{d^2\bar{u}}{dz^2},\qquad U=Bl\frac{d\bar{u}}{dz},\tag{5.119}$$

where $\varkappa$ and B are universal constants which, generally speaking, may be given any value, since the scales l and U are defined only up to an arbitrary multiplier.

The first of the relationships of Eq. (5.119) may also be deduced directly from dimensional considerations if it is stipulated that the length l depends only on $\frac{d\bar{u}}{dz}$ and $\frac{d^2\bar{u}}{dz^2}$.[16] The second relationship of Eq. (5.119) is obtained in the same manner from the assumption that U also depends only on $\frac{d\bar{u}}{dz}$ and $\frac{d^2\bar{u}}{dz^2}$ (or on l and $\frac{d\bar{u}}{dz}$).

The length l is clearly analogous to the mixing lengths l and l_1 in both Prandtl's and Taylor's theory. In fact, by Eqs. (5.117) and (5.119)

$$\tau^{(1)}=-\rho\,\overline{u'w'}=\rho U^2\overline{v_1(0)\,v_3(0)}=\text{const}\cdot\rho l^2\left(\frac{d\bar{u}}{dz}\right)^2,\tag{5.120}$$

where const $=B^2\overline{v_1(0)\,v_3(0)}$. Similarly,

$$\frac{\partial\tau^{(1)}}{\partial z}=-\rho\frac{\partial}{\partial z}\overline{u'w'}=\text{const}\cdot\rho l^2\frac{d\bar{u}}{dz}\frac{d^2\bar{u}}{dz^2},\tag{5.121}$$

where now the constant has a different value. Of course, if we replace the constant by unity in Eq. (5.120) or Eq. (5.121), i.e., if we identify the length l with the mixing length for momentum or vorticity, then the constant $\varkappa$ in the first equation of Eq. (5.119) cannot be chosen arbitrarily but must be determined on the basis of experiment.

[16] Since the equations of motion have Galilean invariance, the characteristic length scale of the fluctuations l will clearly be independent of the absolute value of the mean velocity $\bar{u}(z)$, and will depend only on the variation of $\bar{u}$ in the neighborhood of the given point, i.e., on the derivatives of $\bar{u}(z)$. The assumption that l depends only on the first two derivatives $d\bar{u}/dz$ and $d^2\bar{u}/dz^2$ will in this case be the simplest hypothesis relating to this quantity.

Von Kármán's hypothesis, expressed by Eq. (5.117), taken literally, imposes excessively rigid constraints on the turbulent velocity fluctuations which do not agree with the natural idea of the irregularity of the variations of the fluctuating velocity in space and time. As easily seen from the following discussion, the hypothesis of local self-preservation proves to be applicable not to individual realizations of the field of fluctuating velocity, but only to the statistical characteristics of such a field (see Chapt. 8, Volume 2 which is devoted to the similarity hypotheses proposed by A. N. Kolmogorov). It is important, however, that the fundamental results (5.119) of von Kármán's theory may be deduced even on the basis of far weaker assumptions; as we have already seen, in a certain sense they are natural consequences of dimensional concepts. Further, we shall note that as Loitsyanskiy (1935) showed, for the deduction of Eqs. (5.119) it is sufficient to apply the hypothesis of local self-preservation to the mean velocity field, with the requirement that at every point $\boldsymbol{x}_0 = (x_0, y_0, z_0)$ there will be defined a scale $l(z_0)$ such that for $z_0 \leqslant z < z_0 + l$, with accuracy to small quantities of the third order in l, the following condition holds:

$$\frac{\bar{u}(z) - \bar{u}(z_0)}{\bar{u}(z_0 + l) - \bar{u}(z_0)} = f(\zeta), \quad \zeta = \frac{z - z_0}{l}. \tag{5.122}$$

In fact, to this accuracy, we have

$$\frac{\bar{u}(z) - \bar{u}(z_0)}{\bar{u}(z_0 + l) - \bar{u}(z_0)} = \frac{(z - z_0)\left(\frac{d\bar{u}}{dz}\right)_0 + \frac{1}{2}(z - z_0)^2\left(\frac{d^2\bar{u}}{dz^2}\right)_0}{l\left(\frac{d\bar{u}}{dz}\right)_0 + \frac{1}{2}l^2\left(\frac{d^2\bar{u}}{dz^2}\right)_0} =$$

$$= \frac{\zeta + \frac{1}{2}\zeta^2\left[l\left(\frac{d^2\bar{u}}{dz^2}\right)_0 \Big/ \left(\frac{d\bar{u}}{dz}\right)_0\right]}{1 + \frac{1}{2}\left[l\left(\frac{d^2\bar{u}}{dz^2}\right)_0 \Big/ \left(\frac{d\bar{u}}{dz}\right)_0\right]}.$$

Since, according to this hypothesis, the right side of the above equation must be independent of z_0 and l, the first equation of Eq. (5.119) must also be satisfied; then, from the condition of z_0-independence of the ratio

$$\frac{\tau(z_0)}{\rho\left[\bar{u}(z_0 + l) - \bar{u}(z_0)\right]^2} \approx \frac{\tau(z_0)}{\rho l^2\left(\frac{d\bar{u}}{dz}\right)_0^2},$$

Eq. (5.120) is also deduced.

For plane-parallel flow with zero pressure gradient along an infinite plane $z=0$, with constant shear stress $\tau = \rho u_*^2$, the von Kármán relationship (5.119) for l in conjunction with Prandtl's equation $\tau = \rho l^2\left(\frac{d\bar{u}}{dz}\right)^2$ [see Eq. (5.120)] gives

$$u_* = \varkappa\left(\frac{d\bar{u}}{dz}\right)^2 \Big/ \frac{d^2\bar{u}}{dz^2}. \tag{5.123}$$

Thus, again we obtain a logarithmic equation for the mean velocity

$$\bar{u}(z) = \frac{u_*}{\varkappa}\ln(z - z_0) + B; \tag{5.124}$$

historically, this deduction was the first which led to a logarithmic equation. For pressure flow in a plane channel or circular tube, in addition to the equations $\tau = -\varkappa \frac{\bar{u}'}{\bar{u}''}$ and $\tau = \rho l^2 \left(\frac{d\bar{u}}{dz}\right)^2$ it is also necessary to use the relationship (5.17) or (5.17′). We may then obtain equations for the profile of $\bar{u}(z)$, which, with an appropriate choice of the constant $\varkappa$ (differing for a channel and a tube), may lead to satisfactory agreement with the existing data [see, e.g., Goldstein (1938)].

The semiempirical theories of Prandtl, Taylor, and von Kármán are the classical examples of approaches to the turbulence problem based on assumptions on the existence of a relation between the local Reynolds stress and the local mean velocity field. Attempts to find an improved form of such a relation are still continuing; see, for example, Lettau (1967), where other references to works of this author may also be found [a critical discussion of Lettau's approach was given by Lumley and Stewart (1965)]. The classical theories seemed to be satisfactory in the early days mainly because they were compared almost exclusively with measurements of the mean velocity distribution, and this distribution is rather insensitive to the hypotheses adopted (especially if some undetermined constants were included in the theory which must be determined from observation). When more detailed experiments were performed it became clear that any theory of this type is erroneous in principle and may be used for the description of very limited turbulence phenomena only. For example, the measurements of Schwarz and Cosart (1961), Béguier (1965) and some others, show definitely that in flows with nonsymmetric velocity profiles the point of vanishing shear stress does not agree very often with the point at which the velocity gradient vanishes; obviously, this fact contradicts all mixing length and eddy viscosity theories. It follows also from this fact that in nonsymmetrical flows, the eddy viscosity must take negative values at some points; this makes the whole concept of an eddy viscosity seem more than a little strange as applied to such flows. In fact, if we define the eddy viscosity K in a purely formal manner by $K = \frac{\tau}{\rho(d\bar{u}/dz)}$, then K proves to be negative rather often, especially when large-scale natural turbulence is considered (see, for example, a special monograph by Starr (1968) concerning this subject). At present, almost all investigators agree that the Reynolds stress is not a local property of the motion but depends on the whole flow field (including its time history). There are numerous deductions related to this idea, suggesting that a turbulent fluid must be considered as a special continuous medium with non-Newtonian viscoelastic behavior [Rivlin, (1957); Moffatt (1967); Crow (1968), and others]. Such a description of turbulence shows very clearly the limitations and inadequacies of the classical semiempirical theories; unfortunately, by itself, it does not give a unique accurate mathematical model of shear turbulence. Several very different attempts to formulate an improved non-Newtonian semiempirical theory of the Reynolds stresses (i.e., a theory according to which there is a nonlinear relation between the stress and the rate of strain of the velocity field) were made by Lumley (1967a), Phillips (1967) and some others [see also Kovásznay (1967)]. However, the results obtained by these authors are of a preliminary character only and will not be discussed further here.

6. THE ENERGY BALANCE EQUATION AND ITS CONSEQUENCES

6.1 Equation for the Reynolds Stress Tensor

Earlier, we saw that as a result of the occurrence of additional terms containing the Reynolds stress $\tau_{ij}^{(1)} = -\rho\overline{u_i' u_j'}$ in the Reynolds equations for the mean motion, the system of these equations is not closed. It is natural to attempt to close this system of equations by supplementing the Reynolds equations with new ones, which describe the variation in time of the stresses $\tau_{ij}^{(1)}$ themselves. These equations for $\tau_{ij}^{(1)}$ will be deduced in this subsection; we shall see that they in turn, also contain several additional unknowns and, hence, again, will not form a closed system. Nevertheless, these equations for $\tau_{ij}^{(1)}$ do impose new dynamic constraints, which are of definite interest, on the statistical characteristics of turbulence since they allow us to make a number of quahtative deductions on the properties of turbulent flows. Particularly useful is the equation of balance of turbulent energy, which describes the time-variation of the density of kinetic energy of the fluctuating motion (or, in short, of the turbulent energy) $E_t = \frac{1}{2}\rho\overline{u'_\alpha u'_\alpha}$; a detailed consideration of this equation, every term of which has an explicit physical meaning, shall be made in several subsequent subsections of this section.

For the deduction of equations for the Reynolds stresses, we may use the general method of formulating the equations of moments proposed by Keller and Friedmann (1924) [see also, Keller (1925)]. Let $u_1, u_2, \ldots, u_N$ be some N different or coincident fluid dynamic variables of a turbulent flow of compressible fluid, and $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ be some N different or coincident points of the flow region. In this case, the derivative with respect to time of the Nth-order moment

$$B_{u_1 u_2 \ldots u_N}(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N, t) = \overline{u_1(\boldsymbol{x}_1, t)\, u_2(\boldsymbol{x}_2, t) \ldots u_N(\boldsymbol{x}_N, t)} \quad (6.1)$$

(since the order of averaging and differentiation may be reversed) may be written as

$$\begin{aligned}\frac{\partial}{\partial t} B_{u_1 u_2 \ldots u_N}(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N, t) &= \overline{\frac{\partial u_1(\boldsymbol{x}_1, t)}{\partial t} u_2(\boldsymbol{x}_2, t) \ldots u_N(\boldsymbol{x}_N, t)} + \\ &+ \overline{u_1(\boldsymbol{x}_1, t)\frac{\partial u_2(\boldsymbol{x}_2, t)}{\partial t} \ldots u_N(\boldsymbol{x}_N, t)} + \ldots \\ &\ldots + \overline{u_1(\boldsymbol{x}_1, t)\, u_2(\boldsymbol{x}_2, t) \ldots \frac{\partial u_N(\boldsymbol{x}_N, t)}{\partial t}}. \quad (6.2)\end{aligned}$$

It is now sufficient to eliminate all time derivatives on the right side of Eq. (6.2) with the aid of the corresponding fluid dynamic equations. We obtain the equation of balance for the moment $B_{u_1} \ldots {}_{u_N}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, t)$ which express $\frac{\partial}{\partial t} B_{u_1 \ldots u_N}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, t)$ in the form of a combination of the moments of the flow variables and their space derivatives.

This method, of course, may also be applied to nonaveraged quantities. Thus, for example, substituting the Navier-Stokes equations (1.6) into the equation $\frac{\partial}{\partial t}\rho u_i u_j = \rho u_i \frac{\partial u_j}{\partial t} + \rho u_j \frac{\partial u_i}{\partial t}$ we obtain

$$\frac{\partial \rho u_i u_j}{\partial t} + \frac{\partial}{\partial x_\alpha}\left[\rho u_i u_j u_\alpha + (p u_i \delta_{j\alpha} + p u_j \delta_{i\alpha}) - (u_i \sigma_{j\alpha} + u_j \sigma_{i\alpha})\right] =$$
$$= (\rho u_i X_j + \rho u_j X_i) + p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \left(\sigma_{i\alpha}\frac{\partial u_j}{\partial x_\alpha} + \sigma_{j\alpha}\frac{\partial u_i}{\partial x_\alpha}\right), \tag{6.3}$$

where $\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is the viscous stress tensor in an incompressible fluid. Thus, in particular, for the kinetic energy density we obtain

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_\alpha}(E u_\alpha + p u_\alpha - u_\beta \sigma_{\alpha\beta}) = \rho u_\alpha X_\alpha - \rho\varepsilon, \tag{6.4}$$

where

$$\rho\varepsilon = \sigma_{\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} = \frac{\mu}{2}\sum_{\alpha,\beta}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)^2$$

denotes the specific dissipation of kinetic energy (per unit time and unit volume of fluid). The expression in parentheses on the left side of Eq. (6.4) gives the energy flux density, which is due both to the direct transfer of energy by the displacement of fluid particles, and to the action of the pressure forces and the molecular forces of internal friction. The right side of Eq. (6.4), however, indicates that the total kinetic energy of an arbitrary volume of fluid varies not only because of the mechanical influx or efflux of energy through its boundaries, and the action of the forces of pressure and viscous friction applied to its boundaries, but also due to the action of the body forces and the effect of energy dissipation which leads to the transformation of parts of the kinetic energy into heat.

Now instead of using the Navier-Stokes equations, we use the Reynolds equations (5.1),

$$\frac{\partial \rho \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_\alpha}\left(\rho \bar{u}_i \bar{u}_\alpha + \rho\overline{u'_i u'_\alpha} + \bar{p}\delta_{i\alpha} - \bar{\sigma}_{i\alpha}\right) = \rho \bar{X}_i \tag{6.5}$$

and take into account that $\rho = \text{const}$ and $\frac{\partial \bar{u}_\alpha}{\partial x_\alpha} = 0$; we obtain in a similar manner, the following equation for the tensor $\rho \bar{u}_i \bar{u}_j$:

$$\frac{\partial \rho \bar{u}_i \bar{u}_j}{\partial t} + \frac{\partial}{\partial x_\alpha} [\rho \bar{u}_i \bar{u}_j \bar{u}_\alpha + \rho \overline{u'_i u'_\alpha} \bar{u}_j + \rho \overline{u'_j u'_\alpha} \bar{u}_i +$$
$$+ (\bar{p} \bar{u}_i \delta_{j\alpha} + \bar{p} \bar{u}_j \delta_{i\alpha}) - (\bar{u}_i \bar{\sigma}_{j\alpha} + \bar{u}_j \bar{\sigma}_{i\alpha})] =$$
$$= (\rho \bar{u}_i \bar{X}_j + \rho \bar{u}_j \bar{X}_i) + \bar{p} \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) - \left(\bar{\sigma}_{i\alpha} \frac{\partial \bar{u}_j}{\partial x_\alpha} + \bar{\sigma}_{j\alpha} \frac{\partial \bar{u}_i}{\partial x_\alpha} \right) +$$
$$+ \left(\rho \overline{u'_i u'_\alpha} \frac{\partial \bar{u}_j}{\partial x_\alpha} + \rho \overline{u'_j u'_\alpha} \frac{\partial \bar{u}_i}{\partial x_\alpha} \right). \quad (6.6)$$

In particular, for the density of the kinetic energy of the averaged motion $E_s = \frac{1}{2} \rho \bar{u}_\beta \bar{u}_\beta$ we obtain

$$\frac{\partial E_s}{\partial t} + \frac{\partial}{\partial x_\alpha} (E_s \bar{u}_\alpha + \rho \overline{u'_\alpha u'_\beta} \bar{u}_\beta + \bar{p} \bar{u}_\alpha - \bar{u}_\beta \bar{\sigma}_{\alpha\beta}) =$$
$$= \rho \bar{u}_\alpha \bar{X}_\alpha - \rho \varepsilon_s + \rho \overline{u'_\alpha u'_\beta} \frac{\partial \bar{u}_\beta}{\partial x_\alpha}, \quad (6.7)$$

where

$$\rho \varepsilon_s = \bar{\sigma}_{\alpha\beta} \frac{\partial \bar{u}_\beta}{\partial x_\alpha} = \frac{\rho \nu}{2} \sum_{\alpha,\beta} \left(\frac{\partial \bar{u}_\alpha}{\partial x_\beta} + \frac{\partial \bar{u}_\beta}{\partial x_\alpha} \right)^2$$

denotes the rate of dissipation of energy of the mean motion under the action of molecular viscosity. The physical meaning of all the terms of this equation, after elimination of the last term on the right side which we shall consider in detail below, is analogous to that of the corresponding terms of Eq. (6.4). We note only that the energy flux density of the mean motion, in addition to the term $\bar{u}_\beta \bar{\sigma}_{\alpha\beta}$ describing the transfer of energy molecular viscosity forces, contains the term $\rho \overline{u'_\alpha u'_\beta} \bar{u}_\beta$ which describes the related process of energy transfer due to the action of "eddy viscosity."

By averaging Eq. (6.3) and then subtracting Eq. (6.6) from it termwise, we obtain the required equation for the Reynolds stress tensor:

$$\frac{\partial \rho \overline{u'_i u'_j}}{\partial t} + \frac{\partial}{\partial x_\alpha}\left[\rho \overline{u'_i u'_j}\,\bar{u}_\alpha + \rho \overline{u'_i u'_j u'_\alpha} + (\overline{p' u'_i}\,\delta_{j\alpha} + \overline{p' u'_j}\,\delta_{i\alpha}) - \right.$$
$$\left. - (\overline{u'_i \sigma'_{j\alpha}} + \overline{u'_j \sigma'_{i\alpha}})\right] = \rho \overline{u'_i X'_j} + \rho \overline{u'_j X'_i} + \overline{p'\left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right)} -$$
$$- \left(\overline{\sigma'_{i\alpha}\frac{\partial u'_j}{\partial x_\alpha}} + \overline{\sigma'_{j\alpha}\frac{\partial u'_i}{\partial x_\alpha}}\right) - \left(\rho \overline{u'_i u'_\alpha}\frac{\partial \bar{u}_j}{\partial x_\alpha} + \rho \overline{u'_j u'_\alpha}\frac{\partial \bar{u}_i}{\partial x_\alpha}\right). \quad (6.8)$$

This equation may also be obtained by first formulating the dynamic equation for the velocity fluctuations $u'_i = u_i - \bar{u}_i$ (equal to the difference of the ith Navier-Stokes and Reynolds equations)

$$\rho \frac{\partial u'_i}{\partial t} + \frac{\partial}{\partial x_\alpha}(\rho \bar{u}_i u'_\alpha + \rho \bar{u}_\alpha u'_i + \rho u'_i u'_\alpha - \rho \overline{u'_i u'_\alpha} + p'\delta_{i\alpha} - \sigma'_{i\alpha}) = \rho X'_i, \quad (6.9)$$

and then applying Eq. (6.2) to the central moment $\overline{u'_i u'_j} = -\frac{1}{\rho}\tau^{(1)}_{ij}$. However, we observe immediately that in addition to the mean velocity $\bar{u}_i$ and the Reynolds stresses $\rho\overline{u'_i u'_j}$, Eq. (6.8) contains several new unknowns. These new unknowns are, first, the third-order central moments $\rho\overline{u'_i u'_j u'_\alpha}$, second, multiples by ν of the second moments of the velocity fluctuations and their space derivatives, occurring in the members $\overline{u'_i \sigma'_{j\alpha}}$ and $\overline{\sigma'_{j\alpha}\frac{\partial u'_i}{\partial x_\alpha}}$ and not expressed directly in terms of the Reynolds stresses, and third, the joint second moments of the pressure and velocity fields of the form $\overline{p' u'_i}$ and $\overline{p'\frac{\partial u'_i}{\partial x_j}}$ [which, with the aid of Eq. (1.9′) may be written in the form of integrals of two-point, third-order moments of the type $\overline{u'_i(\boldsymbol{x}, t)\, u'_j(\boldsymbol{x}', t)\, u'_k(\boldsymbol{x}', t)}$]. Thus, the Reynolds equations (6.5) and the equations for the Reynolds stresses (6.8) once again do not form a closed system. Now if we try to complete this system with the aid of equations for the new unknowns occurring in Eq. (6.8) and begin with the equations for the third-order moments $\rho\overline{u'_i u'_j u'_k}$, then these equations in their turn introduce many other new unknowns [for example, the fourth-order moments $\rho\overline{u'_i u'_j u'_k u'_l}$ and third-order moments of the type $\overline{p' u'_j u'_k}$ or $\overline{\frac{\partial u'_i}{\partial x_m}\frac{\partial u'_j}{\partial x_n}u'_k}$; see Chou (1945a)] and the difference between the number of unknowns and the number of equations becomes even greater. Thus formulation of the equations for the higher moments does not permit us to obtain at any stage a closed system of equations describing the turbulent motion.

It is also easy to derive the equations for the turbulent fluxes $q_i = \rho\overline{\vartheta' u_i}$ (or $= c_p\rho\overline{\vartheta' u_i}$) which have appeared in Eq. (5.7) for $\bar{\vartheta}$; it is only necessary to use an equation for $\vartheta' = \vartheta - \bar{\vartheta}$ together with Eq. (6.9). These equations for the flux vector q_i are quite similar to Eq. (6.8); they also contain a number of new unknowns and do not allow us to obtain a closed system.

Of course, we may attempt to close the system of equations (6.5) and (6.8) with the aid of additional hypotheses of some kind or another, which permit the expression of the "new unknowns" in Eqs. (6.8) in terms of the first and second moments $\bar{u}_i$ and $\overline{u'_i u'_j}$. Such attempts, in fact, constitute new variables of the "semiempirical theory of turbulence," which differ from the theories considered in Sect. 5.9 only in the fact that the hypothetical connections here are of a somewhat more complex structure. These more complex, semiempirical theories have been proposed by many authors at different times; here we shall consider only a few of them. One of the first works devoted to the problem of the closure of the system of equations (6.5) and (6.8) was that of Zagustin (1938); then came a short note by Kolmogorov (1942) in which turbulence is characterized by two basic parameters, the intensity and scale parameters, in terms of which all the terms of the energy balance equation are then expressed. This latter equation is deduced from Eq. (6.8). [A similar idea was suggested somewhat later by Prandtl (1945).] Following Kolmogorov, Nevzglyadov (1945a; 1945b; etc.) proposed a number of different hypotheses for the closure of the system of equations (6.5) and (6.8). Similar investigations were carried out in the 1940's by Chou (1945a; 1945b; 1947). Chou, in particular, was the first to attempt to use, in addition to Eqs. (6.5) and (6.8), the equations of the third-order moments $\overline{u'_i u'_j u'_k}$; the fourth-order moments $\overline{u'_i u'_j u'_k u'_l}$ which then arise are eliminated with the aid of the hypothesis about the vanishing of the fourth-order velocity cumulants, or some related hypothesis. Finally, in the 1950's and 1960's the works of Rotta (1951b; 1951c) and Davidov (1958; 1959a; 1959b; 1961) appeared which we shall discuss in somewhat greater detail, so as to display by concrete examples the general character of this whole group of investigations.

Rotta considered Eqs. (6.5) and (6.8) for a steady turbulent flow, with mean velocity $\bar{u}_1(x_3)$ directed everywhere along the Ox_1 axis and in the absence of external forces. For simplicity, he assumed that the terms in brackets on the left side of Eqs. (6.8) describing the "transfer diffusion of Reynolds stresses," may in the first approximation be ignored, considering only those regions of the flow where the turbulence is more or less homogeneous. Consequently, the transfer of turbulent quantities from neighboring regions plays a small part. Further, as in the works of Kolmogorov (1942) and Prandtl (1945) it was assumed that turbulence may be described by its intensity (specific energy) $b = \frac{1}{2}\overline{u'_\beta u'_\beta} = \frac{E_t}{\rho}$ and the scale l, a variable of the dimension of length describing the mean size of the turbulent disturbances, and forming an analog to Prandtl's "mixing length." To evaluate the quantity $\overline{p'\left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right)}$, Rotta, following Chou (1945a; 1945b), used the fact that the pressure fluctuations satisfy the Poisson equation (1.9), and, consequently, may be put in a form related to Eq. (1.9′), where the integrand is the sum of two summands, one, dependent on the mean velocity $\bar{u}_1(x_3)$ and the other containing only velocity fluctuations. The contribution of the first of these summands in $\overline{p'\left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right)}$ may be put in the form of a series of successive derivatives of the mean velocity profile $\overline{u}_1(x_3)$ with coefficients that are integrals of the two-point, second-order velocity moments. These coefficients satisfy algebraic relationships which are deduced from the continuity equation and may be expressed approximately in terms of the Reynolds stresses and the length l. Rotta identified the contribution of the second summand, which is dependent on the two-point, third-order moments with the value of $\overline{p'\left(\frac{\partial u'_i}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)}$ in homogeneous turbulence, with constant mean velocity. Also, he proposed to evaluate it with the aid of the hypothetical relationship (justified by certain qualitative considerations; see below, the footnote in Sect. 6.2)

$$\overline{p'\left(\frac{\partial u'_i}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)}=-c_1\frac{b^{\frac{1}{2}}}{l}\left(\rho\overline{u'_iu'_j}-\frac{2}{3}\rho b\delta_{ij}\right), \tag{6.10}$$

where c_1 is a numerical coefficient, determined from the data. Rotta proposed to use a similar hypothetical relationship of the form

$$\overline{\sigma'_{i\alpha}\frac{\partial u'_j}{\partial x_\alpha}}+\overline{\sigma'_{j\alpha}\frac{\partial u'_i}{\partial x_\alpha}}=c_2\nu\frac{\rho\overline{u'_iu'_j}}{l^2}+c_3\frac{\rho b^{\frac{3}{2}}}{l}\delta_{ij}, \tag{6.11}$$

where c_2 and c_3 are new numerical constants, for the evaluation of the "dissipation terms" on the right side of Eq. (6.8) containing the viscosity. Using these formulas, Eqs. (6.5) and (6.8) permit us to determine the ratio of the various Reynolds stresses to the mean velocity gradient, and to obtain certain other results which may be verified experimentally. With an appropriate choice of the parameters c_1, c_2, c_3 all these results prove to be in fairly good agreement with the measurements of the turbulence characteristics in a channel, carried out by Reichardt (1938) and Laufer (1951). In his second paper, Rotta (1951c), in addition to the relationships (6.10) and (6.11), also proposed a differential equation for the time- and space-variations of the length l, and thereby obtained a more complete explanation of the experimental facts.

A more detailed scheme for the semiempirical closing of Eqs. (6.5) and (6.8) was developed by Davidov. In his works, unlike Rotta's works, on the left side of Eq. (6.8) only the terms $\frac{\partial}{\partial x_\alpha}\left(\overline{p'u'_i}\delta_{j\alpha}+\overline{p'u'_j}\delta_{i\alpha}\right)$ which describe the transfer of Reynolds stresses due to the pressure fluctuations are neglected. The terms of Eq. (6.8) containing the viscosity μ may be transformed into the form $\mu\Delta\overline{u'_iu'_j}+2\mu\overline{\frac{\partial u'_i}{\partial x_\alpha}\frac{\partial u'_j}{\partial x_\alpha}}$; the first term is expressed in terms of the Reynolds stresses, while Davidov proposed to take the second term equal to $\frac{2}{3}\rho\varepsilon_1\delta_{ij}$ (due to the isotropy of small-scale motion, which we shall discuss in detail in Volume 2 of this book), where $\varepsilon_1=\nu\overline{\frac{\partial u'_\beta}{\partial x_\alpha}\frac{\partial u'_\beta}{\partial x_\alpha}}$ is a new fundamental characteristic of turbulence that is subject to definition. To determine the terms on the right side of Eq. (6.8) containing the pressure, Davidov used the equation

$$\overline{p'\left(\frac{\partial u'_i}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)}=-c_1\frac{\varepsilon_1}{b}\left(\rho\overline{u'_iu'_j}-\frac{2}{3}\rho b\delta_{ij}\right)-\left(B_{ij}-\frac{1}{3}B_{\alpha\alpha}\delta_{ij}\right), \tag{6.12}$$

where c_1 is an empirical constant [related to the constant c_1 in Eq. (6.10)] and B_{ij} is a fairly complex additional tensor, expressed in terms of the mean velocity, its space derivative and the Reynolds stresses; this tensor describes the anisotropy of the velocity fluctuations in the boundary layer close to a rigid wall.[17] Further, for the third-order moments $\rho\overline{u'_i u'_j u'_k}$ equations similar to Eq. (6.8) are deduced which are then subjected to even more radical simplifications in which all terms containing the viscosity, and also the fourth cumulants of the velocity are ignored; moreover, the symmetrical combination of summands of the form $\overline{u'_i u'_j \frac{\partial p'}{\partial x_k}}$ is replaced by the expression $c_2 \frac{\varepsilon_1}{b} \overline{u'_i u'_j u'_k}$, where c_2 is another empirical coefficient. Finally, to determine ε_1, Davidov first proposed a special "transport equation" containing, in addition to ε_1, only the first, second and third moments of the velocity field; later, however, he refuted this [see Davidov (1961)] and replaced it by an equation of the form

$$\left(\frac{\partial}{\partial t} + \bar{u}_\alpha \frac{\partial}{\partial x_\alpha}\right)\varepsilon_1 + \frac{\partial \gamma_k}{\partial x_k} + c_3 \frac{\varepsilon_1}{b} \overline{u'_\alpha u'_\beta} \frac{\partial \bar{u}_\alpha}{\partial x_\beta} + 4\frac{\varepsilon_1^2}{b} = \nu \frac{\partial^2 \varepsilon_1}{\partial x_\alpha \partial x_\alpha}, \tag{6.13}$$

where c_3 is a third empirical coefficient, and $\gamma_k = \nu \overline{u'_k \frac{\partial u'_\beta}{\partial x_\alpha} \frac{\partial u'_\beta}{\partial x_\alpha}}$, $k = 1, 2, 3$, are three further additional unknowns. Now, for the closure of the system it is necessary to formulate equations for γ_k; this may be carried out on the basis of Eqs. (6.9) for u'_i with the aid of the general rule (6.2). After considerable simplification, using once again certain hypothetical relationships, these last equations are reduced to the following form:

$$\left(\frac{\partial}{\partial t} + \bar{u}_\alpha \frac{\partial}{\partial x_\alpha}\right)\gamma_k + \gamma_\alpha \frac{\partial \bar{u}_k}{\partial x_\alpha} + \overline{u'_k u'_\alpha} \frac{\partial \varepsilon_1}{\partial x_\alpha} + \frac{2}{9}\varepsilon_1 \frac{\partial \overline{u'_k u'_\alpha}}{\partial x_\alpha} + c_4 \frac{\varepsilon_1}{b}\gamma_k = 0. \tag{6.14}$$

Thus, finally, we obtain a very complex but closed system of 23 quasi-linear differential equations in 23 unknowns, $\bar{u}_i$, $\overline{u'_i u'_j}$, ε_1 and γ_k containing four empirical coefficients, the values of which may be estimated, for example, on the basis of Laufer's measurements. A more exact verification of this system requires special experiments, in which all the unknowns involved would have to be carefully measured; so far, apparently, such measurements have never been carried out.

[17]For example, in the case of a flow above a plane rigid wall $x_3 = 0$, it is reasonable to assume that $B_{ij} = d\rho\epsilon\delta_{i3}\delta_{j3}$ where $\epsilon = \bar{\epsilon_t}$ is the mean rate of dissipation of turbulent energy (which it is usually convenient to use instead of ϵ_1) and d is a dimensionless constant. Such a supposition was used in particular by Monin (1965) for a boundary-layer flow of a thermally stratified fluid. Monin has considered all the equations for the seven nonzero second moments $\overline{u_1'^2}, \overline{u_2'^2}, \overline{u_3'^2}, \overline{u'_1u'_3}, \overline{u'_1T'}, \overline{u'_3T'}$, and $\overline{T'^2}$ (based on the Boussinesq equations; in these equations all the terms were neglected which describe turbulent transfer ("diffusion") and the terms of the form $\overline{p'(\partial T'/\partial x_i)}$ were eliminated in the equations for the moments $\overline{u'_iT'}$ with the aid of the following hypothetical formula, related to Eq. (6.12):

$$\overline{p'\frac{\partial T'}{\partial x_i}} = -d_1 \frac{\epsilon}{b}\rho\overline{u'_iT'} - d_2 \frac{\epsilon}{b^{1/2}}\rho(\overline{T'^2})^{1/2}\delta_{i3}.$$

6.2 Equation of Turbulent Energy Balance

Just as Eq. (6.4) for the total kinetic energy E is derived from Eq. (6.3), we may obtain from Eq. (6.8), after summation over $i = j$, the following equation for the mean density of the kinetic energy of the fluctuating motion $E_t = \frac{1}{2}\rho\overline{u'_\alpha u'_\alpha}$:

$$\frac{\partial E_t}{\partial t} + \frac{\partial}{\partial x_\alpha}\left(E_t\bar{u}_\alpha + \frac{1}{2}\rho\overline{u'_\beta u'_\beta u'_\alpha} + \overline{p'u'_\alpha} - \overline{u'_\beta\sigma'_{\alpha\beta}}\right) =$$
$$= \rho\overline{u'_\alpha X'_\alpha} - \rho\bar{\varepsilon}_t - \rho\overline{u'_\alpha u'_\beta}\,\frac{\partial\bar{u}_\beta}{\partial x_\alpha}, \quad (6.15)$$

where

$$\rho\bar{\varepsilon}_t = \overline{\sigma'_{\alpha\beta}\frac{\partial u'_\beta}{\partial x_\alpha}} = \frac{\rho\nu}{2}\sum_{i,j}\overline{\left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right)^2}$$

is the mean rate of viscous energy dissipation of the fluctuating motion. The expression in parentheses on the left side of this equation gives the turbulent energy flux density connected with the transfer of turbulent energy by the mean motion, the pressure fluctuations, the forces of internal friction (molecular viscosity), and, finally, the "turbulent viscosity," i.e., the fluctuating component of velocity (the term $\frac{1}{2}\rho\overline{u'_\beta u'_\beta u'_\alpha}$). The term

$$A = -\rho\overline{u'_\alpha u'_\beta}\,\frac{\partial\bar{u}_\beta}{\partial x_\alpha} \quad (6.16)$$

occurring on the right side of the equations for E_s and E_t with opposite signs describes the mutual exchange of energy of the mean and fluctuating motion. We shall discuss the question of the value of this term in detail later.

Equation (6.15) is a general *energy balance equation* of turbulent flow. It shows that the density of turbulent energy at a given point of the flow may vary due to the transfer of turbulent energy from different parts of the fluid (i.e., by the diffusion of turbulent energy), the action of the fluctuations of the external forces, the viscous energy dissipation and, finally, the transformation of part of the energy of the mean motion into turbulent energy, or, conversely, the transformation of part of the turbulent energy into energy of the mean motion. The turbulent energy in this equation, of course, may also be replaced by the turbulent intensity (i.e., by the mean kinetic energy of the turbulence per unit mass of the fluid) $b = \frac{E_t}{\rho} = \frac{1}{2}\overline{u'_\beta u'_\beta}$. If we now denote by the symbol $\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{u}_\alpha \frac{\partial}{\partial x_\alpha}$ the first derivative with respect to the mean motion, then, since $\frac{\partial \bar{u}_\alpha}{\partial x_\alpha} = 0$, Eq. (6.15) may be reduced to the form

$$\frac{Db}{Dt} = -\overline{u'_\alpha u'_\beta}\frac{\partial \bar{u}_\beta}{\partial x_\alpha} - \bar{\varepsilon}_t + \frac{\partial}{\partial x_\alpha}\left[-\frac{1}{2}\overline{u'_\beta u'_\beta u'_\alpha} - \frac{1}{\rho}\overline{p'u'_\alpha} + \overline{\nu u'_\beta\left(\frac{\partial u'_\alpha}{\partial x_\beta} + \frac{\partial u'_\beta}{\partial x_\alpha}\right)}\right] + \overline{u'_\alpha X'_\alpha}. \quad (6.17)$$

The energy balance equation (6.15) or (6.17) completes the Reynolds equations in the sense that it imposes a further important constraint on the statistical characteristics of turbulence. True, it contains new unknowns $\overline{u'_\beta u'_\beta u'_\alpha}$, $\overline{p'u'_\alpha}$ and $\bar{\varepsilon}_t$ which do not occur in the Reynolds equations, therefore, a more complete description of the turbulent flow is required. However, the physical meaning of the terms in Eq. (6.17), containing these new unknowns is fairly clear. This is a great help in attempts to express them approximately in terms of simpler characteristics. Moreover, in a number of cases, for example, if the turbulence is almost spatially homogeneous, the terms describing the diffusion of turbulent energy can be ignored to a first approximation because the spatial transfer of energy here plays only a small part. The use of the energy balance equations to complete the Reynolds equations was first proposed by Kolmogorov (1942).

Proceeding to a more detailed consideration of individual terms in the turbulent energy balance equation, let us begin with the terms containing pressure fluctuations. In the general energy balance for E_t these terms are unimportant; as Eqs. (6.15) and (6.17) show, for an incompressible fluid, the pressure fluctuations lead only to an additional transfer of turbulent energy from some parts of the fluid to others. Hence, if we consider a volume of fluid, through the boundary of which there is no inward or outward flux of turbulent energy, the presence of pressure fluctuations will generally have no effect on the variation of the total turbulent energy of this volume.

Moreover, it seems reasonable to suppose that the contribution of the pressure fluctuations to the turbulent energy flux density is in many cases relatively small. Nevertheless, these fluctuations play a very important part in a turbulent flow. Specifically, consider the quantities $E_i = \frac{1}{2}\rho\overline{u_i'^2}$ (no sum over i), which are the mean densities of the fluctuating kinetic energy along the individual coordinate axes. For these quantities, on the assumption that $X_i' = 0$, Eq. (6.8) may be reduced to the form

$$\frac{\partial E_i}{\partial t} + \frac{\partial}{\partial x_\alpha}\left(E_i \bar{u}_\alpha + \frac{1}{2}\rho\overline{u_i'^2 u_\alpha'} + \overline{p'u_i'}\delta_{i\alpha} - \nu\frac{\partial E_i}{\partial x_\alpha}\right) = \\ = \overline{p'\frac{\partial u_i'}{\partial x_i}} - \rho\bar{\varepsilon}_i - \rho\overline{u_i'u_\alpha'}\frac{\partial \bar{u}_i}{\partial x_\alpha}, \qquad (6.18)$$

where

$$\rho\bar{\varepsilon}_i = \mu\sum_\alpha \overline{\left(\frac{\partial u_i}{\partial x_\alpha}\right)^2} > 0;$$

the quantity $\rho\bar{\varepsilon}_i$ may be defined as the "dissipation" of the energy E_i. (These "partial dissipations" will all be equal to $\rho\bar{\epsilon}_t/3$ if the Reynolds number is large enough, due to the isotropy of small scale motion; see Chapt. 8 in Volume 2 of the book.)

Let us now assume, for definiteness, that the mean motion is parallel to the Ox_1 direction (so that $\bar{u}_2 = \bar{u}_3 = 0$) and the turbulence is homogeneous in the Ox_1 and Ox_2 directions (i.e., all mean characteristics of the turbulence depend on x_3 only). Using the continuity equation $\frac{\partial u_\alpha'}{\partial x_\alpha} = 0$, we can rewrite Eqs. (6.18) in the form

$$\begin{aligned} &\frac{\partial E_1}{\partial t} + \frac{\partial}{\partial x_3}\left(\frac{1}{2}\rho\overline{u_1'^2 u_3'} - \nu\frac{\partial E_1}{\partial x_3}\right) = \\ &\qquad = -\overline{p'\frac{\partial u_2'}{\partial x_2}} - \overline{p'\frac{\partial u_3'}{\partial x_3}} - \rho\bar{\varepsilon}_1 - \rho\overline{u_1'u_3'}\frac{\partial \bar{u}_1}{\partial x_3}, \\ &\frac{\partial E_2}{\partial t} + \frac{\partial}{\partial x_3}\left(\frac{1}{2}\rho\overline{u_2'^2 u_3'} - \nu\frac{\partial E_2}{\partial x_3}\right) = \overline{p'\frac{\partial u_2'}{\partial x_2}} - \rho\bar{\varepsilon}_2, \\ &\frac{\partial E_3}{\partial t} + \frac{\partial}{\partial x_3}\left(\frac{1}{2}\rho\overline{u_3'^3} + \overline{p'u_3'} - \nu\frac{\partial E_3}{\partial x_3}\right) = \overline{p'\frac{\partial u_3'}{\partial x_3}} - \rho\bar{\varepsilon}_3. \end{aligned} \qquad (6.19)$$

Equations (6.19) show that $\overline{p' \frac{\partial u_2'}{\partial x_2}}$ and $\overline{p' \frac{\partial u_3'}{\partial x_3}}$ describe the mutual interchange of the energy E_1 of the longitudinal fluctuations with the energies E_2 and E_3 of the transverse fluctuations. Moreover, it is clear from these equations that the energy of the longitudinal fluctuations may be maintained from the energy of the mean flow [this process is described by the term $-\rho\overline{u_1'u_3'}\frac{\partial \bar{u}_1}{\partial x_3}$ of the first equation of Eqs. (6.19)], but that the energy of the transverse fluctuations may be increased only by the energy of the longitudinal fluctuations. In fact, if we consider a steady regime, i.e., we assume $\frac{\partial E_i}{\partial t} = 0$, and integrate all the equations of Eqs. (6.19) over a space region V, across the boundary of which there is no flux of turbulent energy, we obtain

$$
\begin{gathered}
\int_V \overline{p' \frac{\partial u_2'}{\partial x_2}}\, dV = \rho \int_V \bar{\varepsilon}_2\, dV > 0, \quad \int_V \overline{p' \frac{\partial u_3'}{\partial x_3}}\, dV = \rho \int_V \bar{\varepsilon}_3\, dV > 0, \\
-\rho \int_V \overline{u_1'u_3'}\frac{\partial \bar{u}_1}{\partial x_3}\, dV = \int_V \overline{p' \frac{\partial u_2'}{\partial x_2}}\, dV + \\
+ \int_V \overline{p' \frac{\partial u_3'}{\partial x_3}}\, dV + \rho \int_V \bar{\varepsilon}_1\, dV = \rho \int_V \bar{\varepsilon}_t\, dV > 0.
\end{gathered} \tag{6.20}
$$

Thus the energy of the mean motion, in fact, is transmitted directly only to the longitudinal fluctuations u_1', while the transverse fluctuations u_2' and u_3' obtain energy from the longitudinal velocity fluctuations due to the action of the pressure fluctuations. Consequently, the latter produce a redistribution of energy between the fluctuating motion in different directions and create a tendency towards isotropy of the fluctuating motion.

To obtain a clear picture of the energy-transfer mechanism from the longitudinal velocity fluctuations to the transverse fluctuations, we note that the negative sign of

$$
\overline{p' \frac{\partial u_1'}{\partial x_1}} = -\left(\overline{p' \frac{\partial u_2'}{\partial x_2}} + \overline{p' \frac{\partial u_3'}{\partial x_3}}\right)
$$

denotes the presence of a positive correlation between the positive (negative) pressure fluctuations and the convergence (divergence) of the longitudinal velocity fluctuation. In other words, that the convergence of u_1' leads to a predominance of positive pressure fluctuations, while the divergence of u_1' leads to a predominance of negative pressure fluctuations. At the same time, the positive pressure fluctuations produce, predominantly, a divergence of the transverse fluctuations u_2' and u_3', while negative fluctuations produce a convergence. However, this must be the case, if, for example, two neighboring elements of fluid are moving toward each other along the direction of the mean motion. Then in the region between them there will be formed at the expense of their energy a positive pressure fluctuation and this increase of pressure will lead to efflux of fluid in the transverse direction in accordance with the condition of incompressibility $\frac{\partial u_\alpha'}{\partial x_\alpha} = 0$. In general, if $\overline{u_1'^2} > \overline{u_2'^2}$ and $\overline{u_1'^2} > \overline{u_3'^2}$, so that the velocity fluctuations along the Ox_1 axis are predominant, then even in the absence of a mean velocity, the sign of the pressure fluctuations will basically be determined by whether neighboring elements of fluid converge or diverge in the Ox_1 direction; consequently, $\overline{p' \frac{\partial u_1'}{\partial x_1}}$ will here be negative and the pressure fluctuations will tend to equalize the energy of the various components of the velocity [see, e.g., Batchelor (1949a) where this role of the pressure fluctuations is worked out, using the example of homogeneous turbulence].[18]

[18]These arguments justify the assumption that $\overline{p' \frac{\partial u_1'}{\partial x_1}}$ must include the term proportional to $-\left(\overline{u_1'^2} - \overline{u_2'^2}\right) \sim \left(\overline{u_1'^2} - \overline{u_3'^2}\right)$ contained in Eqs. (6.10) and (6.12). We note further, that due to the tensorial character of $\overline{p' \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}$ it follows automatically that

$$-\overline{p' \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} \sim 3\overline{u_i'u_j'} - 2b\delta_{ij},$$

as is stated in Eqs. (6.10) and (6.12). Thus it is natural to think that the pressure fluctuations introduce a contribution to $\frac{\partial}{\partial t} \rho\overline{u_i'u_j'}$ where $i \neq j$, proportional to $-\rho\overline{u_i'u_j'}$, i.e., they lead to a decrease in the absolute value of all Reynolds stresses $\tau_{ij}^{(1)}$, $i \neq j$.

Now let us consider the term $A = -\rho \overline{u'_\alpha u'_\beta} \frac{\partial \bar{u}_\alpha}{\partial x_\beta}$ which, in the energy balance equation describes the exchange of energy between the mean and the fluctuating motion. If at a given point of space $A > 0$, then the turbulent energy density at this point increases at the expense of the energy of the mean motion. If, on the contrary, $A < 0$, then the energy density of the mean motion increases at the expense of the energy of the fluctuations. The latter possibility, at first glance, seems paradoxical; however, this requires more careful analysis.

Equation (6.15) shows that for flows of incompressible fluid in a field of nonfluctuating body forces, the only possible source of turbulent energy within a volume which has no influx of turbulent energy across its boundary is the transformation of the energy of the mean motion. Under these conditions, the initiation and development of turbulence or the maintenance of a stationary state in the given volume are possible only on the condition that the integral of A over the whole volume is positive [see, for example, Eq. (6.20)]. We encounter these conditions, in particular, in usual laboratory flows of incompressible fluid in tubes, channels and boundary layers (with low "initial turbulence" of the incident flow) where direct measurement of $\rho \overline{u'_i u'_j}$ and $\frac{\partial \bar{u}_i}{\partial x_j}$ show that, as a rule, A is positive at all points of the turbulent flow (which agrees well with the classical semiempirical theories of Sect. 5.8).

However, if the turbulence has some "external" source of energy, for example, if it is caused by artificial mixing of the fluid, or, in the case of a compressible fluid, if it arises because of the presence of density fluctuations produced by influx of heat, then the possibility of the transformation of the turbulent energy into energy of the mean motion, i.e., the possibility that $A < 0$, is not excluded. This is exactly the case for atmospheric turbulence on the scale of the general circulation of the atmosphere of the earth. In this case, by turbulence we must understand so-called *macroturbulence*, i.e., the totality of irregular large-scale motion of cyclone and anticyclone type, superimposed on the regular atmospheric circulation; the idea of the statistical description of this macroturbulence was first advanced by Defant (1921). In conditions of general circulation, the individual "turbulent disturbances" (cyclones and anticyclones) can arise entirely due to the energy introduced by local influx of heat, while later, some part of their energy may also be transferred to the mean flow of the general circulation. A long time ago, geophysicists concluded from studying the data on the energy, momentum and angular momentum balance of the whole atmosphere, that the observations cannot be explained without the assumption that in certain atmospheric regions, disturbance energy is transformed into mean flow energy [see, for example, the survey article by Rossby (1948)]. The possibility of such energy transformations was then analyzed theoretically [see, for example, the work of Lorenz (1953) in which conditions determining the sign of A are established]. Finally, we note that for the flows which make up the general circulation of the atmosphere, the actual values of A have been calculated directly by a number of authors on the basis of Eq. (6.16), according to the data of meteorological observations. Also, in many cases, it was actually found that $A < 0$; on this point, see, for example, the works of Monin (1956c), Gruza (1961), and especially the monograph by Starr (1968), all of which contain extensive additional bibliographies.

Similarly to the derivation of Eq. (6.15), we can derive the balance equation for mean square fluctuation of the concentration of a passive admixture (or of the temperature), i.e., for the variance $\sigma_\vartheta^2 = \overline{(\vartheta - \bar{\vartheta})^2} = \overline{\vartheta'^2}$. This equation has the following form:

$$\frac{\partial \overline{\vartheta'^2}}{\partial t} + \bar{u}_\alpha \frac{\partial \overline{\vartheta'^2}}{\partial x_\alpha} + \frac{\partial}{\partial x_\alpha}\left(\overline{u'_\alpha \vartheta'^2} - \chi \frac{\partial \overline{\vartheta'^2}}{\partial x_\alpha}\right) = 2\overline{\vartheta' R'} - 2\overline{u'_\alpha \vartheta'} \frac{\partial \bar{\vartheta}}{\partial x_\alpha} - 2\bar{N} \tag{6.15'}$$

where

$$2\bar{N} = 2\chi \sum_i \overline{\left(\frac{\partial \vartheta'}{\partial x_i}\right)^2}$$

is the mean rate of diminution of mean square fluctuation $\overline{\vartheta'^2}$ due to the action of molecular diffusivity (this quantity is often called the *concentration or temperature dissipation*) and R' is the fluctuation of the intensity of external sources of ϑ at a given point (e.g., of radiative influx and of temperature increase due to change of water phase state when ϑ is a temperature of the atmosphere). The term $A = -2\overline{u'_\alpha \vartheta'} \dfrac{\partial \bar{\vartheta}}{\partial x_\alpha}$ in this equation describes the production of mean square fluctuation $\overline{\vartheta'^2}$ by the interaction of turbulent flux of concentration (or temperature) and mean concentration (temperature) gradient; the term $\bar{u}_\alpha \frac{\partial \overline{\vartheta'^2}}{\partial x_\alpha}$ describes the advection of $\overline{\vartheta'^2}$ by mean motion, and the expression in parentheses gives the flux density of $\overline{\vartheta'^2}$ due to molecular and turbulent diffusion.

6.3 General Concept of the Viscosity and Thermal Diffusivity

The eddy viscosity was introduced in Sect. 5.1 for the case of a plane-parallel flow; it was used several times later in Sect. 5 in describing simple laboratory turbulent flows (see, especially, Sect. 5.9). We shall now show how similar coefficients may also be defined for arbitrary three-dimensional flows.

Let us assume that the turbulence is generated only as a result of the transfer of part of the energy of the mean flow into the energy of small-scale disturbances, i.e., due to A being positive. It is reasonable to think that in this case all the statistical characteristics of turbulence (in particular, the Reynolds stresses) must depend on the mean velocity field. However, with respect to the mean motion, the Reynolds stresses are analogous to the viscous stresses in the laminar motion of a fluid. Hence, if the mean motion has the character of the overall motion of the fluid as a rigid body, i.e., if it is not accompanied by any strain, then it is natural to assume that the Reynolds stresses acting on any selected surface element in the fluid will be directed along the normal to this element. But in that case the tensor $\rho\overline{u_i'u_j'}$ will be isotropic: $\rho\overline{u_i'u_j'} = c\delta_{ij}$, where $c = \frac{1}{3}\rho\overline{u_\alpha'u_\alpha'} = \frac{2}{3}\rho b$. Consequently, the turbulent energy $E_t = \rho b$ is in a definite sense, analogous to the pressure. However, if the mean motion results in relative displacement of the fluid particles, then the stresses $\tau_{ij}^{(1)} = -\rho\overline{u_i'u_j'}$ must apparently depend on the mean velocity derivatives. Restricting ourselves to the case of fairly small mean velocity gradients, it is possible in the first approximation to take into account only the first derivatives $\frac{\partial \bar{u}_i}{\partial x_j}$ and to assume that the dependence of $\tau_{ij}^{(1)}$ on these derivatives is linear. This idea is fundamental to attempts to introduce the general concept of eddy coefficients.

Since the stresses $\rho\overline{u_i'u_j'}$ form a symmetric tensor, their dependence on the derivatives $\frac{\partial \bar{u}_i}{\partial x_j}$ must be of a tensor nature. The quantities $\frac{\partial \bar{u}_i}{\partial x_j}$

do not form a symmetric tensor, but from them a symmetric rate of strain tensor may be formed, $\Phi_{ij} = \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}$, which characterizes the deviation of the mean fluid motion from the motion of a rigid body. Consequently, it is natural to consider the tensor $\rho\overline{u'_i u'_j}$ as a linear function of the tensor Φ_{ij} which becomes $\frac{2}{3}\rho b\delta_{ij}$ for $\Phi_{ij} = 0$. The coefficients of this linear function will be coefficients of "eddy viscosity."

It is known that the molecular viscosities ν and μ are connected by the relationships $\nu \sim u_m l_m$, $\mu \sim \rho u_m l_m$ with the molecular mean velocity u_m and mean free path l_m. We may assume that for the coefficients of eddy viscosity there will exist similar relationships, in which the role of u_m and l_m will be played by the corresponding characteristics of disordered turbulent motion. It is clear that the root mean square of the velocity fluctuations, i.e., $\sqrt{b}$ will be the characteristic analogous to u_m. Instead of l_m, however, we must use the length scale of turbulence l, which describes the mean distance to which turbulent formations can be displaced, conserving their individuality [i.e., the Prandtl "mixing length," which is of the same order of magnitude as the "correlation length," determined from the space correlation function with the aid of an equation of form (4.75)]. The turbulence may have different length scales in different directions because it may be of course nonisotropic. Hence, strictly speaking, at every point of the flow there must be defined an ellipsoid of length scales, i.e., there must be given a symmetric tensor of second rank l_{ij} (the scale tensor) of the dimension of length. Then the values $M_{ij} = \rho K_{ij} = \rho\sqrt{b}\, l_{ij}$ will be coefficients of eddy viscosity. Using the tensor l_{ij} and taking the symmetry of the stress tensor into account, we may suppose that

$$\rho\overline{u'_i u'_j} = \frac{2}{3}\rho b\delta_{ij} - \frac{1}{2}\rho\sqrt{b}\,(l_{i\alpha}\Phi_{\alpha j} + l_{j\alpha}\Phi_{\alpha i}). \tag{6.21}$$

This equation was proposed by Monin (1950b); it may be considered as one more semiempirical hypothesis, similar to those introduced in Sect. 5.9. Of course, the most general linear dependence between the tensors $\rho\overline{u'_i u'_j}$ and Φ_{ij} would have the form

$$\rho\overline{u'_i u'_j} = \frac{2}{3}\rho b\delta_{ij} - \rho K_{ij\alpha\beta}\Phi_{\alpha\beta},$$

where $K_{ij\alpha\beta}$ is a fourth-rank tensor, which is symmetrical with respect to i, j and to α, β, and which satisfies the condition $K_{jj\alpha\beta}\,\Phi_{\alpha\beta}=0$. The use of Eq. (6.21) means that the tensor $K_{ij\alpha\beta}$ is assumed to be of the specific form:

$$K_{ij\alpha\beta}=\frac{1}{2}(K_{i\alpha}\delta_{j\beta}+K_{j\alpha}\delta_{i\beta}).$$

However, if we make no additional assumptions on the scale tensor l_{ij}, then Eq. (6.21) [which is analogous to Boussinesq's equation (5.5) defining the scalar coefficient of eddy viscosity K] may be considered not as a hypothetical connection, but simply as the definition of new turbulent characteristics l_{ij} introduced instead of the Reynolds stresses $\tau_{ij}^{(1)}$. In fact, Eq. (6.21) gives a system of six equations in six unknowns l_{11}, l_{12}, l_{13}, l_{22}, l_{23}, l_{33}. Transforming to a coordinate system in which Φ_{ij} is a diagonal tensor, and taking into account that $\Phi_{11}+\Phi_{22}+\Phi_{33}=0$, it is easy to show that the determinant of this system is proportional to $(\det\|\Phi_{ij}\|)^2$. Thus, provided that $\det\|\Phi_{ij}\|\neq 0$, i.e., if Φ_{ij} is not a degenerate tensor, then the quantities l_{ij} are determined uniquely in terms of $\rho\overline{u_i'u_j'}$ and, conversely, Eq. (6.21) can always be satisfied. In this case, the arguments used above are simply an aid in the physical interpretation of l_{ij} and $K_{ij}=\sqrt{b}\,l_{ij}$, which permit, if necessary, certain hypotheses to be formulated as to their nature. If, however, $\det\|\Phi_{ij}\|=0$, then Eq. (6.21) imposes additional restrictions on the form of the Reynolds stress tensor; these restrictions may sometimes be only a fairly rough approximation to reality. For example, if the mean motion is plane-parallel, so that $\bar{u}_1=\bar{u}_1(x_3)$, $\bar{u}_2=\bar{u}_3=0$ and, accordingly, only the components Φ_{13} and Φ_{31} of the tensor Φ_{ij} are different from zero, then it follows from Eq. (6.21) that $l_{ij}=0$ when $i\neq j$ and $\overline{u_1'^2}=\overline{u_2'^2}=\overline{u_3'^2}=2b/3$; however, the last equations $\overline{u_1'^2}=\overline{u_2'^2}=\overline{u_3'^2}=0$ are not, generally speaking, satisfied in real plane-parallel turbulent flows (see above, Fig. 26, Sect. 5.3).

Nevertheless, it will still be meaningful in some cases to use b and l_{ij} instead of the tensor $\rho\overline{u_i'u_j'}$, since the tensor l_{ij} has a more obvious geometrical meaning and its principal axes may sometimes be preassigned from geometrical considerations. In many cases it is even possible as a first rough approximation simply to assume that $l_{ij}=l\delta_{ij}$ is an isotropic tensor, i.e., to use the hypothetical equation

$$\rho\overline{u_i'u_j'}=\frac{2}{3}\rho b\delta_{ij}-\rho l\sqrt{b}\,\Phi_{ij}. \tag{6.22}$$

(We note that here the tensor $\overline{u_i'u_j'}$ is not isotropic, but its principal axes are the same as those of the strain rate tensor Φ_{ij}, which is not quite true in real flows.) The hypothesis (6.22) is close to that used by Boussinesq; the quantity $l\sqrt{b} = K$ which appears in it is the scalar eddy viscosity. This hypothesis is used and discussed in many works; as examples, we can mention the papers by Harlow and Nakayama (1967), and Lumley (1967b) [in his paper, Lumley uses the time scale $t^* = l/\sqrt{b}$ instead of the length scale l].

According to the hypothesis (6.22), A has the form

$$A = \rho K \Phi_{\alpha\beta} \frac{\partial \bar{u}_\beta}{\partial x_\alpha} = \frac{1}{2} \rho K \sum_{\alpha, \beta} \Phi^2_{\alpha\beta}, \tag{6.23}$$

which is analogous to the expression for the rate of energy dissipation of the mean motion

$$\rho \varepsilon_s = \frac{\mu}{2} \sum_{\alpha, \beta} \Phi^2_{\alpha\beta},$$

but with the molecular viscosity μ replaced by the eddy viscosity K. The condition $A > 0$ is equivalent to $K > 0$. Thus the formal use of the concept of eddy viscosity for $A < 0$, indicates the introduction of "negative viscosity" (cf. the end of Sect. 5.9).

Equations analogous to Eq. (6.21) or Eq. (6.22) may also be formulated with turbulent heat and mass transfer. If ϑ is the admixture concentration, then the turbulent flux density of the admixture in the Ox_i direction is equal to $\rho\overline{\vartheta' u_i'}$. In the general case of anisotropic turbulence, we may write

$$\rho\overline{\vartheta' u_i'} = -\rho K_{\vartheta i\alpha} \frac{\partial \bar{\vartheta}}{\partial x_\alpha}, \tag{6.24}$$

where the components of the tensor $K_{\vartheta i\alpha}$ are the admixture eddy diffusivities; when ϑ is the temperature, the coefficient c_p must be inserted on both sides of Eq. (6.24) [cf. Eqs. (5.9) and (5.9′), Sect. 5.1]. When using the scale tensor l_{ij}, it may be assumed that the anisotropy of the eddy diffusivities is related to the anisotropy of the length scales, i.e., that $K_{\vartheta ij} = \alpha_\vartheta \rho \sqrt{b}\, l_{ij}$, where α_ϑ is a dimensionless parameter. If we ignore the anisotropy of the scale tensor, i.e., put

$l_{ij} \approx l\delta_{ij}$, then Eq. (6.24) becomes

$$\rho\overline{\vartheta' u_i'} = -\alpha_\vartheta \rho l \sqrt{b}\frac{\partial\bar{\vartheta}}{\partial x_i} = -\alpha_\vartheta \rho K \frac{\partial\bar{\vartheta}}{\partial x_i}. \tag{6.25}$$

We note that even for plane-parallel turbulent flows, for which Eq. (6.25) may be considered simply as the definition of a new quantity $K_\vartheta = \alpha_\vartheta K$, this relationship becomes hypothetical (and requries observational verification) as soon as we assume that α_ϑ = const (i.e., is space- and time-invariant). In a purely formal manner, we may also write $\vartheta = b' = \frac{1}{2}u_\alpha' u_\alpha'$, i.e., we may use Eq. (6.25) to calculate the third-order velocity moments occurring in Eqs. (6.15) and (6.17)

$$\frac{1}{2}\rho\overline{u_\beta' u_\beta' u_i'} = \rho\overline{b' u_i'} = -\alpha_b \rho l \sqrt{b}\frac{\partial b}{\partial x_i} = -\alpha_b \rho K \frac{\partial b}{\partial x_i}, \tag{6.26}$$

where α_b is a numerical coefficient, which need not be the same as α_ϑ [Eq. (6.26) is itself, of course, yet another semiempirical hypothesis]. Similarly, we may put $\overline{\vartheta'^2}$ instead of ϑ and evaluate the third-order moments in Eq. (6.15′) as follows:

$$\overline{u_i'\vartheta'^2} = -\alpha_c K\frac{\partial\overline{\vartheta'^2}}{\partial x_i} \tag{6.26'}$$

where α_c is a new dimensionless coefficient [see, for example, the works of Csanady (1967a; 1967b)].

Adopting the hypothetical relationships (6.22) and (6.26) we introduce into the energy balance equation, instead of the variables $\rho\overline{u_i' u_j'}$ and $\rho\overline{u_\beta' u_\beta' u_i'}$, the characteristics l (which can also be replaced by $t^* = l/\sqrt{b}$) and b (or $K = l\sqrt{b}$); i.e., we reduce the number of unknowns quite considerably. The next step in this direction will be to establish a connection between the rate of energy dissipation $\bar{\varepsilon}_t$ which also occurs as an unknown in Eq. (6.17), and l and b (or l and K, or t^* and b). If we assume that there is such a connection, then its form may be found uniquely by dimensional considerations

$$\bar{\varepsilon}_t = \frac{b^{3/2}}{c^4 l} = \frac{b}{c^4 t^*} = \frac{K^3}{c^4 l^4}, \tag{6.27}$$

where c^4 is another nonnegative dimensionless parameter (the reason why we have written it as the fourth power of another number will be seen in Sect. 6.6). The relationship (6.27) once again may be considered simply as the definition of a new quantity c, replacing $\overline{\varepsilon}_t$ (assuming that K and l are given, for example, with the aid of $\tau_{13}^{(1)} = \rho K \frac{\partial \overline{u}_1}{\partial x_3}$ and $K = l\sqrt{b}$). However, if we assume that c = const, then Eq. (6.27) becomes at once an additional semiempirical hypothesis.

Analogously, we shall obtain a semiempirical balance equation for the mean square concentration (or temperature) fluctuation if we replace the expressions $\overline{u_i'\vartheta'}$ and $\overline{u_i'\vartheta'^2}$ in Eq. (6.15′) by semiempirical relationships (6.25) and (6.26′) and use simultaneously the following hypothetical equation for $\overline{N}$:

$$2\overline{N} = \frac{\overline{\vartheta'^2}\, b^{1/2}}{c_\vartheta l} = \frac{\overline{\vartheta'^2}}{c_\vartheta t^*} = \frac{\overline{\vartheta'^2} K}{c_\vartheta l^2} \tag{6.27′}$$

where c_ϑ is a new dimensionless constant [cf. Csanady (1961a; 1967b)].

6.4 Energy Balance in a Compressible Fluid

Taking into consideration the compressibility of the fluid, and, in particular, the presence of density fluctuations ρ', a considerable complication is introduced into the above evaluations. The mean momentum density is now equal to $\overline{\rho u_i} = \overline{\rho}\,\overline{u}_i + \overline{\rho u_i'}$, where $\overline{\rho u_i'} = \overline{\rho' u_i'}$ is the mean momentum density of the fluctuating motion; in the absence of density fluctuations it becomes zero. Similarly, the mean kinetic energy density here takes the form

$$\overline{E} = \frac{1}{2}\overline{\rho u_\alpha u_\alpha} = \frac{1}{2}\overline{\rho}\,\overline{u}_\alpha \overline{u}_\alpha + \overline{\rho u_\alpha'}\,\overline{u}_\alpha + \frac{1}{2}\overline{\rho u_\alpha' u_\alpha'} = E_s + E_{st} + E_t, \tag{6.28}$$

where E_s is the energy density of the mean motion,

$$E_t = \frac{1}{2}\overline{\rho u_\alpha' u_\alpha'} = \frac{1}{2}\left(\overline{\rho}\,\overline{u_\alpha' u_\alpha'} + \overline{\rho' u_\alpha' u_\alpha'}\right)$$

is the energy density of the fluctuating motion, and $E_{st} = \overline{\rho u_\alpha'} \cdot \overline{u}_\alpha = \overline{\rho' u_\alpha'} \cdot \overline{u}_\alpha$ is an additional part of the fluctuating energy, connected with the transport of the momentum $\overline{\rho u_i'}$ by the mean motion.

In formulating the moment equations in a compressible fluid, we must take as our starting point the general equation of continuity

(1.1) and the equations of motion (1.3), which we rewrite here as

$$\frac{\partial \rho u_i}{\partial t}+\frac{\partial}{\partial x_\alpha}(\rho u_i u_\alpha+p\delta_{i\alpha}-\sigma_{i\alpha})=\rho X_i, \qquad i=1,\ 2,\ 3. \tag{6.29}$$

In Eqs. (6.29) $\delta_{i\alpha}$, as usual, denotes the unit tensor and $\sigma_{i\alpha}$ is the general viscous stress tensor, given by the equation

$$\sigma_{ij}=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\frac{\partial u_\alpha}{\partial x_\alpha}\delta_{ij}\right)+\zeta\frac{\partial u_\alpha}{\partial x_\alpha}\delta_{ij};$$

the remaining notation does not differ from that used in Eq. (1.3). Averaging Eqs. (1.1) and (6.29), we obtain

$$\frac{\partial \bar{\rho}}{\partial t}+\frac{\partial}{\partial x_\alpha}(\bar{\rho}\bar{u}_\alpha+\overline{\rho' u'_\alpha})=0, \tag{6.30}$$

$$\frac{\partial}{\partial t}(\bar{\rho}\bar{u}_i+\overline{\rho' u'_i})+\frac{\partial}{\partial x_\alpha}(\bar{\rho}\bar{u}_i\bar{u}_\alpha+\overline{\rho u'_i}\bar{u}_\alpha+\overline{\rho u'_\alpha}\bar{u}_i+\overline{\rho u'_i u'_\alpha}+\bar{p}\delta_{i\alpha}-\bar{\sigma}_{i\alpha})=$$
$$=\bar{\rho}\bar{X}_i+\overline{\rho' X'_i}, \tag{6.31}$$

which in this case play the same role as the Reynolds equations (5.1) for turbulence in an incompressible fluid. With the aid of Eq. (6.30) and the equation

$$\frac{\partial \bar{u}_i}{\partial t}+\bar{u}_\alpha\frac{\partial \bar{u}_i}{\partial x_\alpha}+\overline{u'_\alpha\frac{\partial u'_i}{\partial x_\alpha}}=\bar{X}_i-\overline{\frac{1}{\rho}\frac{\partial p}{\partial x_i}}+\overline{\frac{1}{\rho}\frac{\partial \sigma_{i\alpha}}{\partial x_\alpha}},$$

which is a corollary of Eq. (1.4) [and hence is applicable only when the viscosity fluctuations are ignored], we may also find an expression for the derivative $\frac{\partial \overline{\rho u_i}}{\partial t}$. We then obtain

$$\frac{\partial \overline{\rho u_i}}{\partial t}+\frac{\partial}{\partial x_\alpha}(\bar{\rho}\bar{u}_i\bar{u}_\alpha+\overline{\rho u'_i u'_\alpha}+\bar{p}\delta_{i\alpha}-\bar{\sigma}_{i\alpha})=\bar{\rho}\bar{X}_i+\varphi_i, \tag{6.32}$$

where

$$\varphi_i=\overline{u'_i\frac{\partial \overline{\rho u'_\alpha}}{\partial x_\alpha}}-\bar{u}_i\frac{\partial \overline{\rho u'_\alpha}}{\partial x_\alpha}+\left(\frac{\partial \bar{p}}{\partial x_i}-\bar{\rho}\overline{\frac{1}{\rho}\frac{\partial p}{\partial x_i}}\right)-\left(\frac{\partial \bar{\sigma}_{i\alpha}}{\partial x_\alpha}-\bar{\rho}\overline{\frac{1}{\rho}\frac{\partial \sigma_{i\alpha}}{\partial x_\alpha}}\right) \tag{6.33}$$

is a quantity which becomes zero for an incompressible fluid. Comparison of Eqs. (6.32) and (6.31) shows that

$$\frac{\partial \overline{\rho u_i'}}{\partial t} + \frac{\partial}{\partial x_\alpha}\left(\overline{\rho u_i' u_\alpha} + \overline{\rho u_\alpha' u_i} + \overline{\rho' u_i' u_\alpha'}\right) = \overline{\rho' X_i'} - \varphi_i. \quad (6.34)$$

Thus the φ_i have a clear physical significance: they describe the exchange of momentum between the mean and fluctuating motion.

From the dynamical equations (6.29) and the continuity equation (1.1), it is easy to obtain also an equation for the tensor $\rho u_i u_j$, which differs from the corresponding equation for an incompressible fluid [Eq. (6.3), Sect. 6.1] only in the fact that σ_{ij} must now be taken to mean the viscous stresses in a compressible fluid. In particular, the kinetic energy density $E = \frac{1}{2}\rho u_\alpha u_\alpha$ in a compressible fluid will satisfy the equation

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_\alpha}\left(E u_\alpha + p u_\alpha - u_\beta \sigma_{\alpha\beta}\right) = \rho u_\alpha X_\alpha + p\frac{\partial u_\alpha}{\partial x_\alpha} - \rho\varepsilon, \quad (6.35)$$

which outwardly differs from Eq. (6.4) only by the presence on the right side of a further term $p\frac{\partial u_\alpha}{\partial x_\alpha}$ which describes the kinetic energy variation due to compressions and rarefactions of the fluid elements (which are accompanied by changes in the internal energy). Taking the mean value of all the terms of this equation, we obtain an equation for the mean kinetic energy density $\bar{E}$. However, this equation will not be formulated here but instead we shall write down three equations for the three parts E_s, E_{st} and E_t of the density $\bar{E}$:

$$\frac{\partial E_s}{\partial t} + \frac{\partial}{\partial x_\alpha}\left(E_s \bar{u}_\alpha + \overline{\rho u_\alpha' u_\beta'}\,\bar{u}_\beta + \bar{p}\bar{u}_\alpha - \bar{u}_\beta \bar{\sigma}_{\alpha\beta}\right) = \bar{\rho}\bar{u}_\alpha \bar{X}_\alpha - \bar{\rho}\varepsilon_s + \bar{p}\frac{\partial \bar{u}_\alpha}{\partial x_\alpha} +$$
$$+ \left[\bar{u}_\alpha \varphi_\alpha + \frac{1}{2}\bar{u}_\beta \bar{u}_\beta \frac{\partial \overline{\rho u_\alpha'}}{\partial x_\alpha}\right] + \left\{\overline{\rho u_\alpha' u_\beta'}\frac{\partial \bar{u}_\beta}{\partial x_\alpha}\right\}, \quad (6.36)$$

$$\frac{\partial E_{st}}{\partial t} + \frac{\partial}{\partial x_\alpha}\left(E_{st}\bar{u}_\alpha + \overline{\rho' u_\alpha' u_\beta'}\cdot \bar{u}_\beta + \frac{1}{2}\overline{\rho u_\alpha'}\,\bar{u}_\beta \bar{u}_\beta\right) =$$
$$= \bar{u}_\alpha \overline{\rho' X_\alpha'} - \left[\bar{u}_\alpha \varphi_\alpha + \frac{1}{2}\bar{u}_\beta \bar{u}_\beta \frac{\partial \overline{\rho u_\alpha'}}{\partial x_\alpha}\right] + \left\{\overline{\rho' u_\alpha' u_\beta'}\frac{\partial \bar{u}_\beta}{\partial x_\alpha} + \overline{\rho u_\alpha'}\frac{D\bar{u}_\alpha}{Dt}\right\},$$

$$\frac{\partial E_t}{\partial t} + \frac{\partial}{\partial x_\alpha}\left(E_t \bar{u}_\alpha + \frac{1}{2}\overline{\rho u_\alpha' u_\beta' u_\beta'} + \overline{p' u_\alpha'} - \overline{u_\beta' \sigma_{\alpha\beta}'}\right) =$$
$$= \overline{\rho u_\alpha'}\bar{X}_\alpha + \overline{\rho u_\alpha' X_\alpha'} - \bar{\rho}\varepsilon_t + \overline{p'\frac{\partial u_\alpha'}{\partial x_\alpha}} - \left\{\overline{\rho u_\alpha' u_\beta'}\frac{\partial \bar{u}_\beta}{\partial x_\alpha} + \overline{\rho u_\alpha'}\frac{D\bar{u}_\alpha}{Dt}\right\},$$

where

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\bar{u}_\alpha\frac{\partial}{\partial x_\alpha},\quad \overline{\rho\varepsilon_s}=\bar{\sigma}_{\alpha\beta}\frac{\partial\bar{u}_\beta}{\partial x_\alpha}\quad\text{and}\quad\overline{\rho\varepsilon_t}=\overline{\sigma'_{\alpha\beta}\frac{\partial u'_\beta}{\partial x_\alpha}}.$$

In these equations the expression in brackets describes the mutual transformation of energies E_s and E_{st} and the expressions in braces describe the mutual transformations of E_s and E_t or E_{st} and E_t.

Equations (6.30)–(6.34) and (6.36) contain several new variables which are not contained in the corresponding incompressible equations. The most important of these new variables are the quantities $\overline{\rho u'_i}$, the components of the mean momentum density of the fluctuating motion, or, which amounts to the same thing, the mean density of the turbulent mass flux. If there is no mean turbulent mass flux in the direction of the mean motion, then $E_{st}=\overline{\rho u'_\alpha}\cdot\bar{u}_\alpha=0$ and the fluctuating energy will be described by E_t only. The most important difference between the general equation for E_t and the incompressible equation (6.15) consists of the presence of the term

$$B=\overline{\rho u'_\alpha}\left(\bar{X}_\alpha-\frac{D\bar{u}_\alpha}{Dt}\right),\tag{6.37}$$

which we shall now consider in somewhat greater detail.

In most turbulent flows encountered in practice (both in nature and in technology), the main body force acting on the fluid and performing work is the force of gravity. Taking the Ox_3 axis vertically upward, we may write $X_i=Y_i-g\delta_{i3}$, where g is the gravitational acceleration and Y_i are the components of the acceleration of the other body forces. However, since the gravitational acceleration is, in general, considerably greater than the individual acceleration of the particles in the mean motion $\frac{D\bar{u}_i}{Dt}$ and also the acceleration of all other body forces Y_i, then in the majority of problems we may write

$$B\approx-\overline{\rho u'_3}\,g.\tag{6.38}$$

The meaning of this quantity may be explained by observing that in the presence of density fluctuations ρ', a buoyancy force $-\rho' g$ acts on the turbulent elements, so that B is the mean work done by the buoyancy force in turbulent displacements of fluid elements. Thus, B describes the mutual transformation of the kinetic energy of turbulence and the potential energy of a vertical column of fluid of variable density in a gravitational field. These mutual transformations generally play a considerable role in the fluid with vertical density stratification, for example, in the atmosphere with nonneutral thermal stratification or in the sea with salinity stratification. If the vertical stratification of the fluid is stable, then the vertical displacements of the fluid elements are accompanied by consumption of energy in performing work against the buoyancy force, so that $B < 0$; we note that for stable stratification, the density decreases with height, and the density fluctuations and the vertical velocity will be positively correlated. On the other hand, for unstable stratification during vertical displacements of turbulent elements, the buoyancy forces will perform work due to the potential energy of stratification, leading to an increase of the turbulent energy; in this case $B > 0$ (the correlation of u_3' and ρ' is negative).

For atmospheric turbulence and several other important cases, to calculate $\overline{\rho u_i'}$ we may use the Boussinesq equations and in accordance with Eq. (1.73) assume that $\rho' = -\beta\bar{\rho}T'$, where β is the thermal expansion coefficient of the medium. For definiteness, we shall consider the case of a gas, and, according to the equation of state of an ideal gas, we shall assume that $\beta = 1/\bar{T}$, where $\bar{T}$ is the mean temperature. In this case $\overline{\rho u_i'} \approx -\frac{\rho}{\bar{T}}\overline{T' u_i'}$, i.e., the turbulent mass flux proves to be proportional to the turbulent heat flux. Hence, we obtain the following expression for B:

$$B = \frac{g\bar{\rho}}{\bar{T}}\,\overline{T' u_3'} = \frac{gq}{c_p\bar{T}}, \quad \text{where} \quad q = c_p\bar{\rho}\overline{T' u_3'}. \tag{6.39}$$

To calculate B approximately, we can often use the semiempirical equation

$$q = -c_p\bar{\rho}K_\vartheta \frac{\partial \bar{T}}{\partial x_3} = -c_p\bar{\rho}\alpha K \frac{\partial \bar{T}}{\partial x_3}$$

which together with Eq. (6.39) gives

$$B = -\alpha \frac{g}{\bar{T}}\bar{\rho}K\frac{\partial \bar{T}}{\partial x_3}. \tag{6.40}$$

However, sometimes we must consider the fact that as a fluid element moves vertically, its temperature T, generally speaking, varies, i.e., it is not a strictly conservative quantity. But since the radiant and molecular heat exchange between fluid elements is almost always very small, the entropy or the potential temperature

$$\theta = T\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}}$$

functionally connected with it (see the closing remarks of Sect. 2.4) may be assumed to be conservative. Due to the fact that the pressure fluctuations are small in comparison with the temperature fluctuation (which is the basis of the Boussinesq approximation) $\frac{T'}{\overline{T}} \approx \frac{\theta'}{\overline{\theta}}$; hence $B \approx \frac{g\overline{\rho}}{\overline{\theta}}\,\overline{\theta' u_3'}$. Taking this as our starting point, when Eq. (6.40) is incorrect, due to the nonconservative nature of the temperature the semiempirical relationship (6.25) can usually be applied to the potential temperature θ, writing

$$B = -\alpha\frac{g}{\overline{\theta}}\overline{\rho}K\frac{\partial\overline{\theta}}{\partial x_3}\,. \tag{6.40'}$$

(Here once again α is the ratio of the eddy diffusivities for temperature and momentum.)

6.5 The Richardson Number and the Eddy Viscosity in a Thermally Stratified Medium

In the preceding subsection we considered the energy balance equation for an arbitrary compressible medium. Henceforth, however, of all the effects connected with compressibility, we shall consider only the mutual transformation of kinetic energy and potential energy of density stratification, while, in connection with the Boussinesq approximation, the density will be assumed to depend only on the temperature fluctuations (but not on pressure fluctuations). Then the fluid may once again be assumed incompressible; i.e., the equation $\frac{\partial u_i}{\partial x_i} = 0$ may be used; however, in the equation for the vertical velocity, the buoyancy force must also be taken into account, which in the case of a gaseous medium is written in the form of the additional term $-g\beta T' = -\frac{g}{\overline{T}}T'$ on the right side (cf. Eq. (1.75), Sect. 1.5). Completely analogous results may be obtained for an incompressible fluid of variable density ρ; here it is necessary only to replace $-\frac{\bar{\rho}}{\bar{T}}T'$ by ρ' in all subsequent equations.

Thus we shall assume that the force of gravity is the only external body force, and we may proceed on the basis of the system of dynamical equations in the Boussinesq approximation, i.e., we may assume that u_3 satisfies Eq. (1.75) with $\beta = 1/\bar{T}$. In this case, again repeating the deduction of the energy balance equation, we obtain

$$\frac{\partial E_t}{\partial t} + \frac{\partial}{\partial x_\alpha}\left(E_t \bar{u}_\alpha + \frac{1}{2}\bar{\rho}\,\overline{u'_\beta u'_\beta u'_\alpha} + \overline{p'u'_\alpha} - \overline{u'_\beta \sigma'_{\alpha\beta}}\right) = \\ = -\bar{\rho}\overline{u'_\alpha u'_\beta}\frac{\partial \bar{u}_\beta}{\partial x_\alpha} + \frac{g\bar{\rho}}{\bar{T}}\overline{T'u'_3} - \bar{\rho}\bar{\varepsilon}_t, \quad (6.41)$$

where $E_t = \frac{1}{2}\bar{\rho}\overline{u'_\beta u'_\beta}$. This equation differs from its incompressible form (6.15) only by the replacement of $\rho\overline{u'_\alpha X'_\alpha}$ on the right side by $B = \frac{g\bar{\rho}}{\bar{T}}\overline{T'u'_3}$, the meaning of which was explained at the end of the previous subsection. [If we consider the corresponding equations of the form (6.19) for the "partial energies" E_1, E_2, and E_3, then the term B appears in the third of the equations.] Equation (6.41) is applicable, in particular, to atmospheric turbulence, since the air motions in the atmosphere are described usually by Boussinesq equations to high accuracy.

We now note that in developed turbulence, the viscous stresses are negligibly small in comparison with the turbulent Reynolds stresses (excluding the viscous sublayer in immediate proximity to the wall, which we shall not consider here). Thus it is natural to assume that the energy transfer due to the viscous forces, i.e., disordered molecular motions, is very small in comparison with the energy transfer by turbulent velocity fluctuations, i.e., that the last term in parentheses on the left side of Eq. (6.41) is negligibly small in comparison with the second term. Let us suppose that the turbulence is homogeneous in the direction of the Ox_1 and Ox_2 axes. In this case, all the statistical characteristics of turbulence will depend only on x_3 while, by the continuity equation $\frac{\partial \bar{u}_3}{\partial x_3} = 0$, i.e., $\bar{u}_3 = 0$. In addition to the notation x_i, u_i for the coordinates and the velocities, we shall also use the notation x, y, z and u, v, w; then ignoring the small term

$\frac{\partial}{\partial x_\alpha} \overline{u'_\beta \sigma'_{\alpha\beta}}$, we may write Eq. (6.41) in the form:

$$\frac{\partial b}{\partial t} = -\frac{1}{2}\frac{\partial \overline{u'_\alpha u'_\alpha w'}}{\partial z} - \frac{1}{\bar{\rho}}\frac{\partial \overline{p'w'}}{\partial z} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z} + \frac{g}{T}\overline{T'w'} - \bar{\varepsilon}_t, \tag{6.42}$$

where $u'_\alpha u'_\alpha = u'^2 + v'^2 + w'^2$, and $b = \frac{E_t}{\bar{\rho}} = \frac{1}{2}\overline{u'_\alpha u'_\alpha}$ is the turbulence intensity (i.e., the mean kinetic energy of the fluctuations per unit mass of the fluid).

If the mean velocity of flow has everywhere the same direction (e.g., along the Ox axis), then Eq. (6.42) becomes even more simplified; here

$$\frac{\partial b}{\partial t} = -\frac{\partial}{\partial z}\overline{\left(\frac{1}{2}u'_\alpha u'_\alpha + \frac{p'}{\rho}\right)w'} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} + \frac{g}{\bar{T}}\overline{T'w'} - \bar{\varepsilon}_t. \tag{6.43}$$

The latter equation may also be written as

$$\frac{\partial b}{\partial t} = -\frac{\partial}{\partial z}\overline{\left(\frac{1}{2}u'_\alpha u'_\alpha + \frac{p'}{\rho}\right)w'} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z}(1 - \mathrm{Rf}) - \bar{\varepsilon}_t, \tag{6.44}$$

where

$$\mathrm{Rf} = \frac{g}{\bar{T}}\frac{\overline{T'w'}}{\overline{u'w'}\frac{\partial \bar{u}}{\partial z}} = -\frac{g}{c_p\bar{T}}\frac{q}{\tau\frac{\partial \bar{u}}{\partial z}}. \tag{6.45}$$

The dimensionless quantity Rf clearly determines the relative role of buoyancy in the generation of turbulent energy, by comparison with the dynamic factors (transfer of energy from the mean motion); it is called the *flux Richardson number*. It is clear that $\mathrm{Rf} < 0$ for $q > 0$ (i.e., with unstable thermal stratification); $\mathrm{Rf} > 0$ for $q < 0$ (stable stratification); for neutral stratification $\mathrm{Rf} = 0$.

In Eq. (6.43), we have already ignored the viscous flux of energy. Also, it is often assumed that the transfer of energy by the pressure forces is small in comparison with its transfer by the velocity fluctuations, and, hence, in the energy balance equation one often ignores terms containing pressure fluctuations. This supposition seems quite plausible but it has no strict justification. In the case of a steady flow without any heat transfer (i.e., with neutral thermal stratification) close to a plane wall, a logarithmic layer is formed, within which the turbulence intensity b is constant (equal to cu_*^2, where $c \approx 5$ in accordance with the data of Fig. 26, Sect. 5.3), and the diffusion terms $\overline{u'_\alpha u'_\alpha w'}$ and $\overline{p'w'}/\rho$ both differ from u_*^3 only by a constant multiplier according to dimensional arguments. Hence the energy transfer in this layer is identically equal to zero, while in Eq. (6.44) only the terms $-\overline{u'w'}\frac{\partial \bar{u}}{\partial z}$ and $-\bar{\varepsilon}_t$, which must compensate each other, are nonzero. However, outside the logarithmic layer, and in the presence of a turbulent heat flux, the diffusion terms may in come cases be comparable in value with other terms (see below, end of Sect. 8.5). Nevertheless, they are more often than not ignored, since these terms are usually supposed to be not too large, and, principally, because they are almost always unknown. Then the energy balance equation takes the comparatively simple form

$$\frac{\partial b}{\partial t} = -\frac{1}{2}\frac{\partial \overline{u'_\alpha u'_\alpha w'}}{\partial z} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z}(1 - \mathrm{Rf}) - \bar{\varepsilon}_t. \tag{6.46}$$

Sometimes instead of dropping the diffusion term, it is assumed that this term is proportional to $\frac{g}{T}\overline{T'w'}$ [because the two become zero together; see, for example, Klug, (1963), Takeuchi and Yokoyama (1963)]. In this case, instead of Eq. (6.46) we obtain

$$\frac{\partial b}{\partial t} = -\overline{u'w'}\frac{\partial \bar{u}}{\partial z}(1 - \sigma\,\mathrm{Rf}) - \bar{\varepsilon}_t, \tag{6.46$'$}$$

which has as simple a form, but which contains the additional dimensionless parameter σ; this allows better agreement with the data to be obtained.

Let us begin with the consideration of Eq. (6.46). Since $\bar{\varepsilon}_t > 0$ always, and $-\overline{u'w'}\frac{\partial \bar{u}}{\partial z} > 0$ practically always, then, by Eq. (6.46) stationary (undamped) turbulence is possible only if

$$\mathrm{Rf} < 1. \tag{6.47}$$

Equation (6.47) is in fact the criterion for transition to turbulence in a stratified medium, obtained using the energy balance of the disturbances (see Sect. 2.9). Like the other criteria obtained by the energy method, it is apparently fairly rough, i.e., it gives an elevated value of the critical Richardson number. Thus, on this basis, we can only assert that stationary turbulence is possible for $\mathrm{Rf} < R$, where $R = \mathrm{Rf}_{\mathrm{cr}}$ is likely to be less than unity (in all probability, considerably less). In a purely formal manner, the value $1/R = 1/\mathrm{Rf}_{\mathrm{cr}}$ may be identified with that of the parameter σ in Eq. (6.46′); with such a choice of σ, the energy criterion obtained from this latter equation will give the true value of $\mathrm{Rf}_{\mathrm{cr}}$.

Now applying the semiempirical relationships (5.5), (6.26), (6.27), and (6.40) and ignoring the term containing pressure fluctuations, we may transform Eq. (6.44) into the following "semiempirical energy balance equation":

$$\frac{\partial b}{\partial t} = K\left(\frac{\partial \bar{u}}{\partial z}\right)^2 - \frac{g}{T}\alpha K \frac{\partial \bar{T}}{\partial z} - \frac{K^3}{c^4 l^4} + \frac{\partial}{\partial z}\alpha_b K \frac{\partial b}{\partial z}. \qquad (6.48)$$

Here K is the eddy viscosity i.e., eddy diffusivity for momentum, and $\alpha K = K_T$ and $\alpha_b K = K_b$ are eddy diffusivities for heat and energy, respectively. Ignoring the energy diffusion (i.e., the last term on the right side), we obtain the following criterion [which is equivalent to Eq. (6.47)] for the possibility of existence of undamped turbulence

$$\mathrm{Ri} < \frac{1}{\alpha}, \qquad \alpha = \frac{K_T}{K}. \qquad (6.49)$$

Here

$$\mathrm{Ri} = \frac{g}{\bar{T}} \frac{\partial \bar{T}/\partial z}{(\partial \bar{u}/\partial z)^2} = \frac{\mathrm{Rf}}{\alpha} \qquad (6.50)$$

is the ordinary Richardson number [see Eq. (2.3′) at the end of Sect. 2.5 in which, instead of $\frac{\partial \bar{T}}{\partial z}$, $\frac{\partial \bar{\theta}}{\partial z} = \frac{\partial \bar{T}}{\partial z} - G_a$ was used in accordance with Eq. (6.40′) which is somewhat more exact than Eq. (6.40). The criterion (6.49) was obtained first by Richardson (1920) [on the assumption that $\alpha = 1$]; it shows that $\mathrm{Ri}_{\mathrm{cr}} < \frac{1}{\alpha}$ but does not define the value of $\mathrm{Ri}_{\mathrm{cr}}$ more precisely. In Sect. 2.8 we have already seen that according to linear stability theory, a flow of stratified fluid will be stable to infinitely small disturbances if at all points of it $\mathrm{Ri} > \frac{1}{4}$;

consequently, $\mathrm{Ri}_{\mathrm{cr\,min}} \leqslant \frac{1}{4}$; of course $\mathrm{Ri}_{\mathrm{cr\,min}}$ here plays the same role as $\mathrm{Re}_{\mathrm{cr\,max}}$ in the usual hydrodynamic stability theory since large values of Ri correspond to stability of a flow. On the other hand, however, there is some justification for assuming that perhaps $\alpha \to 0$ as $\mathrm{Rf} \to \mathrm{Rf}_{\mathrm{cr}}$, and that turbulence may exist for as large a value of Ri as desired (so that $\mathrm{Ri}_{\mathrm{cr}} = \infty$, see below, Sects. 7.3 and 8.2).

Since by definition $K \frac{\partial \bar{u}}{\partial z} = \frac{\tau}{\rho} = u_*^2$, Eq. (6.48), for steady conditions and ignoring the diffusion of turbulent energy, may be put in the form

$$\frac{u_*^4}{K}(1 - \alpha\,\mathrm{Ri}) = \frac{u_*^4}{K}(1 - \mathrm{Rf}) = \frac{K^3}{c^4 l^4}.$$

Consequently, for the eddy viscosity K we obtain the equation

$$K = u_* c l (1 - \mathrm{Rf})^{1/4}. \tag{6.51}$$

According to Eq. (6.51) it seems that K would become zero (i.e., turbulent exchange would cease) for Rf = 1, that is, $\mathrm{Rf}_{\mathrm{cr}} = 1$, $\mathrm{Ri}_{\mathrm{cr}} = 1/\alpha$, which contradicts what has been said above. However, use of the semiempricial relationship (6.27) with c = const for Rf close to $\mathrm{Rf}_{\mathrm{cr}}$ becomes meaningless, and therefore Eq. (6.51) will also no longer be applicable. We now note that in the absence of thermal stratification within the logarithmic layer $cl = \varkappa z$ (since by Eq. (6.48) here $cl = K^{1/2}\left(\frac{\partial \bar{u}}{\partial z}\right)^{-1/2}$ while $K = \varkappa u_* z$, $\frac{\partial \bar{u}}{\partial z} = \frac{u_*}{\varkappa z}$); thus for arbitrary stratification, it is convenient to write

$$cl = \varkappa z\, \lambda_1(\mathrm{Rf}), \qquad K = u_* \varkappa z\, \lambda_1(\mathrm{Rf})(1 - \mathrm{Rf})^{1/4}. \tag{6.52}$$

The function λ_1 (Rf) must satisfy the conditions $\lambda_1(0) = 1$ and $\lambda_1(R) = 0$, where $R = \mathrm{Rf}_{\mathrm{cr}}$. To emphasize the fact that $\lambda_1(R) = 0$, instead of λ_1 (Rf), we may introduce into our discussion the function $\lambda(\mathrm{Rf})$ such that

$$\lambda_1(\mathrm{Rf}) = \lambda(\mathrm{Rf}) \frac{(1 - \sigma\,\mathrm{Rf})^{1/4}}{(1 - \mathrm{Rf})^{1/4}}, \qquad \sigma = \frac{1}{R}; \tag{6.53}$$

with the aid of this new function the expression for K is written in the form

$$K = u_* \varkappa z \lambda(\mathrm{Rf})(1 - \sigma \mathrm{Rf})^{1/4}, \tag{6.54}$$

which is analogous to Eq. (6.52) but with the change of Rf into σRf, $\sigma = 1/R$, in the last set of parentheses. The same result (6.54) may be obtained if we assume that $cl = \varkappa z \lambda$ (Rf). However, instead of ignoring the diffusion of turbulent energy we consider this diffusion to be proportional to $\frac{g}{T}\overline{T'w'}$; i.e., instead of Eq. (6.46) we use Eq. (6.46′) with $\sigma = 1/R$. In Eq. (6.54) we must suppose that $\lambda(0) = 1$, but now we have automatically $K(R) = 0$ and therefore we may assume that λ(Rf) has no zeros. In Sect. 7.4 we shall see, in particular, that results that are reasonable in many respects may be obtained if we simply put $\lambda(\mathrm{Rf}) \equiv 1$. It should be stressed, however, that this choice of the function λ(Rf) has no theoretical justification whatsoever and it is completely permissible to change it in order to obtain a better agreement with experiment; we shall discuss this in greater detail in the following section.

The balance equation for the mean square temperature fluctuations in a plane-parallel, thermally stratified flow contains no additional terms and may be written as

$$\frac{\partial \overline{T'^2}}{\partial t} = -\frac{\partial}{\partial z}\left(\overline{T'^2 w'}\right) - 2\overline{w'T'}\frac{\partial \bar{T}}{\partial z} - 2\bar{N} \tag{6.55}$$

[the terms $\chi \dfrac{\partial^2 \overline{T'^2}}{\partial z^2}$ and $2\overline{T'R'}$ of Eq. (6.15′) are ignored in Eq. (6.55)]. With the aid of the relationships (6.25), (6.26′), and (6.27′) we may transform the latter equation into the "semiempirical equation of the mean square temperature fluctuation balance" of the form

$$\frac{\partial \overline{T'^2}}{\partial t} = 2\alpha K\left(\frac{\partial \bar{T}}{\partial z}\right)^2 + \frac{\partial}{\partial z}\alpha_c K \frac{\partial \overline{T'^2}}{\partial z} - \frac{K\overline{T'^2}}{c_\vartheta l^2}. \tag{6.55′}$$

6.6 Turbulence in the Planetary Boundary Layer of the Atmosphere

Here and in the following subsection we shall discuss two examples of the use of the semiempirical equation for the turbulent energy balance in actual problems, which illustrate the typical features of the semiempirical approach to the study of turbulence.

As our first example, let us consider the problem of the boundary layer set up in the atmosphere close to the earth due to combined action of friction of the air on the underlying surface and of the Coriolis force caused by rotation of the earth. The layer in which the frictional forces appear to be significant is called the *planetary boundary layer* (or the *friction layer*, or the *Ekman layer*). We shall consider only the planetary boundary layer over a plane homogeneous underlying surface, which we shall take as the plane $z = 0$, for steady external conditions and shall, as a rule, assume that the thermal stratification may be assumed neutral. In addition, we use the fact that within the planetary boundary layer we may put $\bar{\rho} \approx$ const; thus, the compressibility of the air for this problem is negligible. Since all the statistical characteristics of turbulence in this layer depend only on z, we may use the form (6.42) of the energy balance equation. In this equation, we may ignore the term on the left (due to stationarity) and the term $\frac{g}{\bar{T}} \overline{T'w'}$ on the right side, since we have assumed the stratification to be neutral; moreover, as usual we also ignore the term $\frac{1}{\bar{\rho}} \frac{\partial \overline{p'w'}}{\partial z}$ which describes the transfer of turbulent energy due to pressure fluctuations; we may also assume that this term is included in the right side of Eq. (6.57) below with the aid of a change of the coefficient a_b. As a result, we obtain

$$- \overline{u'w'} \frac{\partial \bar{u}}{\partial z} - \overline{v'w'} \frac{\partial \bar{v}}{\partial z} - \bar{\varepsilon}_t = \frac{1}{2} \frac{\partial \overline{u'_\alpha u'_\alpha w'}}{\partial z}, \tag{6.56}$$

which, after application of the semiempirical hypotheses (6.22), (6.26) and (6.27) [with a_b = const], and the division of all terms by $\bar{\rho}$, takes the form

$$K\left[\left(\frac{\partial \bar{u}}{\partial z}\right)^2 + \left(\frac{\partial \bar{v}}{\partial z}\right)^2\right] - \frac{K^3}{c^4 l^4} = -\alpha_b \frac{\partial}{\partial z} K \frac{\partial b}{\partial z}, \quad K = l\sqrt{b}. \tag{6.57}$$

This equation must be considered together with the Reynolds equations (6.5) which in this case may be written as

$$\frac{\partial \rho \overline{u'_i w'}}{\partial z} + \frac{\partial \bar{p}}{\partial x_i} = \bar{\rho} \bar{X}_i, \qquad i = 1, 2, 3. \tag{6.58}$$

(We ignore the viscous vertical momentum flux in comparison with the turbulent momentum flux $\rho \overline{u'_i w'}$.) The external forces X_i acting in the boundary layer are, first, the gravity force (vertically downwards) and, second, the Coriolis force $Y = 2\,\boldsymbol{u} \times \boldsymbol{\omega}$ (which is perpendicular to the velocity $\boldsymbol{u}$ and hence does no work), where $\boldsymbol{\omega}$ is the angular velocity vector of the rotation of the earth and the sign $\times$ symbolizes the vector product. We now select the axes $Ox_1 = Ox$ and $Ox_2 = Oy$ in such a way that Ox is parallel to the surface wind. Then $\bar{X}_1$ and $\bar{X}_2$ will be equal to the corresponding components of the Coriolis acceleration, i.e., $\bar{X}_1 = Y_1 = 2\omega_z \bar{v}$, $\bar{X}_2 = Y_2 = -2\omega_z \bar{u}$, where $\omega_z = \omega \sin\varphi$ is the vertical component of the vector $\boldsymbol{\omega}$, and φ is a latitude of the observation point. Using the semiempirical equation (6.22) to evaluate $\rho \overline{u'_i w'}$, and dividing the Reynolds equations by $\bar{\rho}$, we may write the first two of these equations (which are all that are needed) in the form

$$\frac{\partial}{\partial z} K \frac{\partial \bar{u}}{\partial z} + 2\omega_z \bar{v} = \frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial x}, \qquad \frac{\partial}{\partial z} K \frac{\partial \bar{v}}{\partial z} - 2\omega_z \bar{u} = \frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial y}. \tag{6.59}$$

The horizontal pressure gradient $\left(\frac{\partial \bar{p}}{\partial x}, \frac{\partial \bar{p}}{\partial y}\right)$ in this problem will be given and we shall assume it to be independent of the coordinates x, y, z; the z-independence of the pressure gradient is the usual assumption of boundary-layer theory. Outside the friction layer the first terms in Eqs. (6.59) [which describe the frictional force] may be ignored, and the motion will be determined by the "geostrophic wind equations"

$$\bar{u} = -\frac{1}{2\rho\omega_z}\frac{\partial\bar{p}}{\partial y} = G\cos\alpha, \quad \bar{v} = \frac{1}{2\rho\omega_z}\frac{\partial\bar{p}}{\partial x} = -G\sin\alpha, \tag{6.60}$$

where

$$G = \frac{1}{2\rho\omega_z}\left[\left(\frac{\partial\bar{p}}{\partial x}\right)^2 + \left(\frac{\partial\bar{p}}{\partial y}\right)^2\right]^{\frac{1}{2}}$$

is the geostrophic wind velocity and α is the angle between the vector with components $\left(-\frac{\partial p}{\partial y}, \frac{\partial p}{\partial x}\right)$ and the surface wind (the angle of total wind rotation in the planetary boundary layer) which is subject to determination.

Equations (6.57) and (6.59) permit us to determine the distribution of the wind $\bar{u}(z)$, $\bar{v}(z)$ in the friction layer if the eddy viscosity $K(z)$ is given as a function of z or if the function $l(z)$ is assumed to be given in some way [this last assumption is advantageous in some respects relative to the assumption on the form of $K(z)$]. Different assumptions of both these types have been used in a number of works; for example, by Monin (1950a), Blackadar (1962), Lettau (1962) and Appleby and Ohmstede (1965), based on different assumptions about $l(z)$, and the detailed surveys by Zilitinkevich, Laykhtman and Monin (1967) and Zilitinkevich (1970), containing extensive bibliographies. Let us begin by estimating the variation with height in the atmospheric surface layer of the x-component of turbulent shear stress $\tau_x = \bar{\rho}K\frac{\partial\bar{u}}{\partial z} = \bar{\rho}u_*^2$. Integrating the first equation of Eq. (6.59) with respect to height, we have

$$u_*^2(0) - u_*^2(H_0) = \int_0^{H_0}\left(2\omega_z\bar{v} - \frac{1}{\rho}\frac{\partial\bar{p}}{\partial x}\right)dz < \int_0^{H_0}\left(-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x}\right)dz < 2\omega_z H_0 G.$$

Here we use the fact that omitting the term $2\omega_z\bar{v}$ leads to the strengthening of the inequality, since the Coriolis force partially compensates the action of the pressure gradient. We may select H_0 in such a way that the relative variation of u_*^2 in the layer of thickness H_0 does not exceed the tolerance a, i.e., so that the condition

$$\frac{u_*^2(0) - u_*^2(H_0)}{u_*^2(0)} \leqslant a$$

will be fulfilled. For this, it is sufficient to require that the following inequality

$$H_0 < \frac{a u_*^2(0)}{2\omega_z G} \tag{6.61}$$

be satisfied. According to the data u_*/G is of the order of 0.05. Moreover, for moderate latitudes, $2\omega_z \sim 10^{-4}$ sec^{-1}, so that for $G \sim$ 10 m/sec, we obtain $\frac{u_*^2(0)}{2\omega_z G} \sim$ 250 m. With a tolerance a = 20%, we obtain an estimate of the thickness of the surface layer $H_0 \sim$ 50 m.

Within this layer we may ignore the action of the Coriolis force, and hence the rotation of the wind with height caused by this force, and assume u_* to be constant, i.e., we may use the ordinary logarithmic equation for the wind velocity. Consequently, in this layer

$$K = \varkappa u_* z, \quad \frac{\partial \bar{u}}{\partial z} = \frac{u_*}{\varkappa z}, \quad \frac{\partial \bar{v}}{\partial z} = 0, \; b = c_1 u_*^2 = \text{const},$$

where c_1 is a constant coefficient which may be estimated, for example, from the data of Fig. 26, and, finally, by Eq. (6.57)

$$cl = \varkappa z, \qquad l = \frac{\varkappa}{c} z. \tag{6.62}$$

We now turn to Eqs. (6.59). Differentiating these equations with respect to z, we may write the result in the form of a single equation:

$$\frac{\partial^2 f}{\partial z^2} = 2 i \omega_z \frac{f}{K}, \tag{6.63}$$

where $f = K\left(\frac{\partial \bar{u}}{\partial z} + i\frac{\partial \bar{v}}{\partial z}\right)$ is a complex quantity characterizing both components of the shear stress (divided by the density $\bar{\rho}$). In the surface layer f takes the value u_*^2. Further, with the aid of f, the turbulent energy balance equation (6.57) may be written as

$$\frac{|f|^2}{K} - \frac{K^3}{c^4 l^4} = -\alpha_b \frac{\partial}{\partial z} K \frac{\partial}{\partial z} \frac{K^2}{l^2}. \tag{6.64}$$

In addition to the parameter u_* which determines the regime of turbulence in the surface layer of the atmosphere, Eqs. (6.63) and (6.64) also contain the Coriolis parameter $2\omega_z$. From these two parameters we may formulate the length scale

$$H = \frac{\varkappa u_*}{2\omega_z} \tag{6.65}$$

(The von Kármán constant $\varkappa$ is added here for convenience.)

Using dimensional considerations, we may now write

$$f = u_*^2 F\left(\frac{z}{H}\right); \qquad K = \varkappa u_* z \Phi\left(\frac{z}{H}\right); \qquad cl = \varkappa z \Psi\left(\frac{z}{H}\right), \tag{6.66}$$

where the dimensionless functions $F(\zeta)$, $\Phi(\zeta)$, $\Psi(\zeta)$ require further definition, while for $\zeta = 0$ they are all equal to unity. After the substitution of the relationships (6.66) into Eqs. (6.63)–(6.64), we obtain two equations with three unknowns; hence a further equation is needed to close the system. In the absence of reliable data on height-dependence of the length scale l, Monin (1950a) has assumed, for simplicity, that $\Psi(\zeta) \equiv 1$. Under this assumption Eqs. (6.63)–(6.64) [after substitution from Eq. (6.66)] take the form

$$\zeta\Phi \frac{\partial^2 F}{\partial \zeta^2} = iF, \qquad |F|^2 - \Phi^4 = -\delta\zeta\Phi \frac{\partial}{\partial \zeta}\left(\zeta\Phi \frac{\partial}{\partial \zeta} \Phi^2\right), \tag{6.67}$$

where $\delta = \alpha_b \varkappa^2 c^2$ is a number of the order of unity. The functions F and Φ which occur in this equation must, by reason of their physical meaning, tend to zero as $\zeta = \frac{z}{H} \to \infty$.

It may be shown that the system (6.67) has a unique damped solution if $\delta > 0$. However, for $\delta = 0$, it has no solutions which are damped as $\zeta \to \infty$, but a solution does exist which tends to zero as ζ tends to some finite value. In other words, the turbulent boundary layer in this case possesses a finite thickness. Equations (6.67) were integrated numerically by Monin for $\delta = 0$ and $\delta = \varkappa^2 \approx 0.16$. For $\delta = 0$, the thickness of the turbulent layer is close to $10H$, i.e., it is very large. With $z < 2H$ the functions $F(\zeta)$ and $\Phi(\zeta)$ for $\delta = 0$ and for $\delta = \varkappa^2$ are very close to each other.

When $F(\zeta)$ has been determined, the velocity vector of the wind $(\overline{u}, \overline{v})$ at different heights may be found from Eqs. (6.59), which for this purpose may be written conveniently in the form

$$\overline{u} + i\overline{v} = Ge^{-i\alpha} - i\frac{u_*}{\varkappa}F'\left(\frac{z}{H}\right). \tag{6.68}$$

The parameters u_* and α are unknown; we recall that u_* also occurs in the equation for the scale H. To determine them, we use the fact that the wind velocity becomes zero at some height z_0 ("roughness height") which may be assumed known. Then, equating Eq. (6.68) for $z = z_0$ to zero, we obtain

$$\varphi(\zeta_0) = \frac{F'(\zeta_0)}{\zeta_0} = \eta e^{i\left(\frac{3\pi}{2} - \alpha\right)}, \tag{6.69}$$

where $\zeta_0 = \dfrac{z_0}{H}$ and $\eta = \dfrac{\varkappa^2 G}{2\omega_z z_0}$ are known. The function $\varphi(\zeta_0)$ may be assumed known, since $F(\zeta)$ is already determined. Consequently,

$$\alpha = \frac{3\pi}{2} - \arg F'(\zeta_0). \tag{6.70}$$

Moreover, since $\eta = |\varphi(\zeta_0)|$, $\zeta_0 = \psi(\eta)$, where ψ is the inverse function of $|\varphi|$. The latter relationship may be rewritten as

$$\frac{u_*}{G} = \frac{\varkappa}{\eta\psi(\eta)}, \tag{6.71}$$

which completes the determination of the unknown parameters. This determination is simplified considerably by the fact that the roughness height z_0 is usually small enough to make it possible to put $K(z_0) = \kappa u_* z_0$, i.e., $\Phi(\zeta_0) = 1$. Therefore the function $F(\zeta)$ can be found for ζ of the order ζ_0 from the first equation (6.67) with $\Phi = 1$, which implies, together with the boundary condition $F(0) = 1$, that $F = 1 + a\zeta + i\zeta\log\zeta + \dots$, i.e., that $F'(\zeta_0) \approx i\log\kappa\zeta_0 + b$ where $b = A + iB$ is a complex constant [cf. Kazanskiy and Monin (1961)]. Since $\eta\zeta_0 = \varkappa H/u_* = \varkappa/\xi$ where $\xi = u_*/G$, Eq. (6.69) may now be rewritten in the form

$$\log\frac{\eta}{\varkappa^2} = B - \log\xi + \left(\frac{\varkappa^2}{\xi^2} - A^2\right)^{1/2}, \quad \sin\alpha = -\frac{A\xi}{\varkappa} \tag{6.72}$$

where $\zeta = u_*/G$. The constant b is found by solving the complete equations (6.67) in the entire boundary layer. With $\delta = 0$, we obtain the value $b = -1.69 + 2.49\,i$; with $\delta = 0.16$, we obtain $b = -1.81 + 1.71\,i$.

We must further note that when we approximate the wind distribution throughout the entire friction layer it is necessary to take the values of the "roughness parameter" z_0 considerably larger than those used in the approximation of the wind profile in the surface layer. This is natural since as the vertical scales of the phenomenon under consideration increase, it is necessary to increase the horizontal scales also, and, consequently, to introduce a "roughness" characteristic of large areas of the underlying surface.

Taking this remark into account, comparison of the calculation using Eq. (6.72) with the data of Lettau (1962), Blackadar (1962; 1967), Bysova and Mashkova (1966), and Kurpakova and Orlenko (1967) showed that the values of ξ and α calculated for $\delta = 0$ are clearly in disagreement with the experimental values. However, with $\delta = 0.16$ they are in qualitative agreement with experiment (see Fig. 46, Curve 2, where the *Rossby number* Ro $= \eta/\varkappa^2 = G/2\omega_z z_0$ is used instead of the parameter η).

Very similar results were also obtained by many others, using slightly different semiempirical hypotheses. For example, Blinova and Kibel (1937) used the assumption that $K(z) = \varkappa u_* z$ for all z (i.e., that $\Phi(\zeta) \equiv 1$); their results are represented in Fig. 46 as Curve 1. Blackadar (1962) used the same approach as Monin (1950a) but ignored the vertical diffusion of turbulent energy (i.e., he assumed that $\alpha_b = \delta = 0$). However, unlike Monin, in accordance with the fairly rough data of Lettau (1950), Blackadar assumed that the scale l increases linearly with height only in the lowest layer of the atmosphere, but that its rate of increase then falls, and, as $z \to \infty$, l tends asymptotically to some constant length λ/c which he chose in the form $\lambda = \beta G/2\omega_z$. The function $cl(z)$ was therefore chosen in the form

$$cl(z) = \frac{\varkappa z}{1 + \varkappa z/\lambda} = \frac{\varkappa z}{1 + \varkappa 2\omega_z z/\beta G}.$$

The numerical parameter β was chosen so that the value obtained for the angle of rotation coincided with the observational data obtained at Brookhaven (located near New York). It was found that this condition was satisfied by $\beta = 0.00027$. At the same time, Lettau (1962) has considered another variant of the semiempirical theory; he also ignored the vertical energy diffusion (i.e., supposed that $a_b = 0$), but defined the eddy viscosity K with the aid of the equation

$$K = l^2 \left[\left(\frac{\partial \bar{u}}{\partial z} \right)^2 + \left(\frac{\partial \bar{v}}{\partial z} \right)^2 \right]^{1/2}$$

[similar to Prandtl's formula (5.112)] and took the length scale l in the form

$$l = \frac{\varkappa z}{1 + 33.63\,(2\omega_z z/\varkappa u_*)^{5/4}}.$$

Let us also refer to the work of Appleby and Ohmstede (1965) who used an equation of the form

$$l = \lambda \left[1 - \exp\left(-\frac{\varkappa z}{\lambda} \right) \right]$$

together with the Prandtl-Lettau equation for the eddy viscosity K, and considered $2\omega_z \lambda/G$ to be a definite function of the Rossby number Ro. Finally, Bobyleva, Zilitinkevich and Laykhtman (1967) have taken into account the influence of the thermal stratification of the atmosphere and therefore added the term $-\frac{g}{\bar{T}}\alpha K \frac{\partial \bar{T}}{\partial z}$ on the left side of

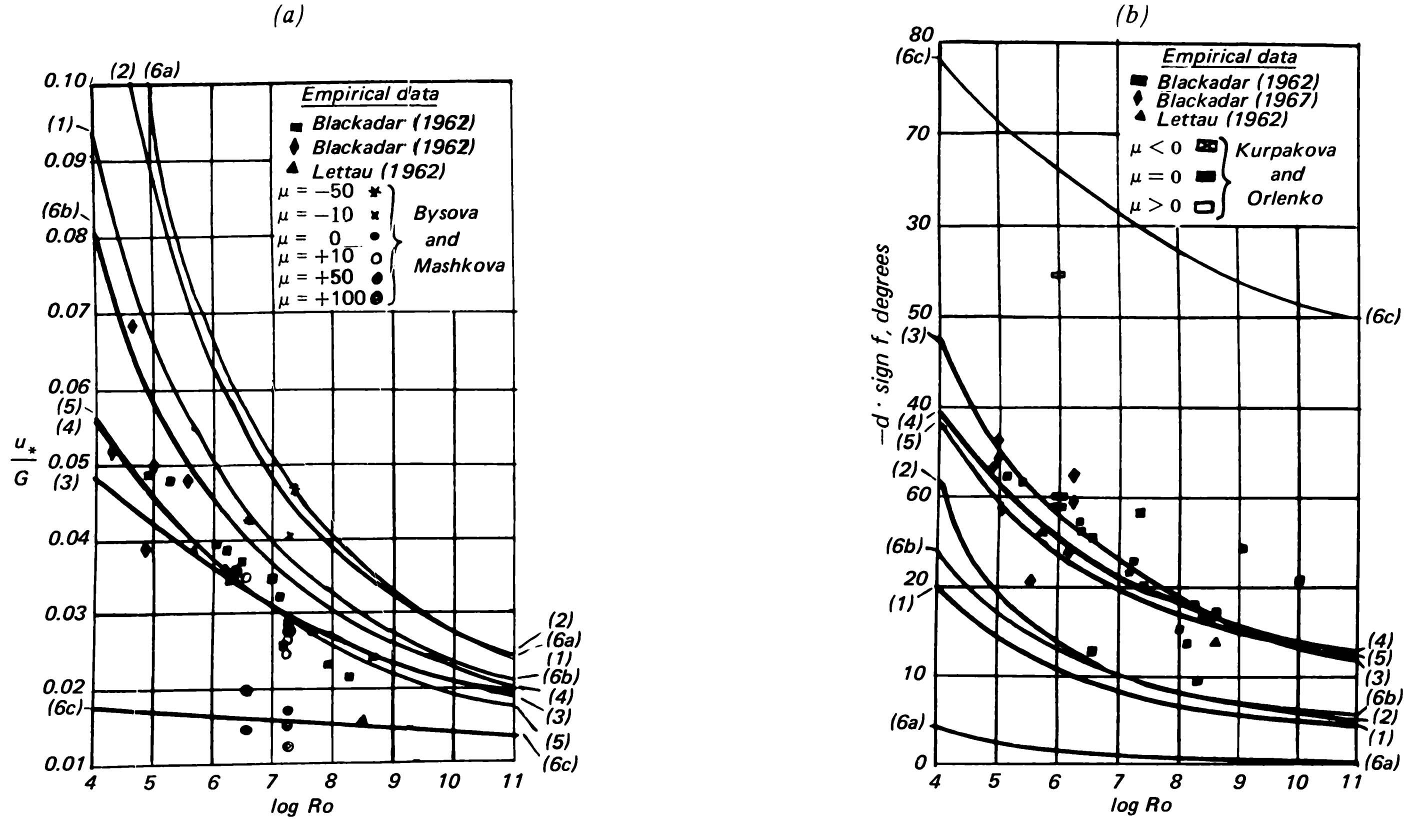


FIG. 46. The dependence of u_*/U (Fig. a) and the angle α of the wind rotation in the planetary boundary layer (Fig. b) on Rossby number Ro at different values of the stratification parameter μ; 1) Blinova and Kibel's theory (1937), $\mu = 0$; 2) Monin's theory (1950a), $\mu = 0$; 3) Blackadar's theory (1962), $\mu = 0$; 4) Lettau's theory (1962), $\mu = 0$; 5) Appleby and Ohmstede's theory (1965), $\mu = 0$; 6) Bobyleva, Zilitinkevich, and Laykhtman's theory (1967), a) $\mu = -100$, b) $\mu = 0$, c) $\mu = +100$.

Eq. (6.57); the length scale l they defined with the aid of a special modification of the von Kármán hypotheses (5.119) [this modification will be discussed in Sect. 7.4 in connection with Zilitinkevich and Laykhtman (1965)]. The results obtained were dependent, in addition to the Rossby number Ro, on the nondimensional stratification parameter $\mu = -\varkappa^2 \dfrac{gq}{2\omega_z c_p \rho \bar{T} u_*^2}$. (This new parameter is the ratio of the length scale H of Eq. (6.65) to another length $L = -c_p \rho \bar{T} u_*^3/\kappa g q$ which will play a very important part in the following section.)

As we have seen, if the law of variation of scale l with height is known, from the Reynolds equations and the energy balance equation we can obtain a closed system of three equations in three unknown functions; the fact that two of these functions were combined above into a single complex function F, does not, of course, make any difference. The corresponding equations, which were written in different works in somewhat different dimensionless forms, were solved numerically by all the authors cited above. Some of the results obtained are collected in Fig. 46 together with the corresponding data; the scatter of the experimental points is quite strong, but the general agreement between the theoretical predictions and the data can be considered as satisfactory.

The main deductions from the preceding analysis are the general equations (6.72); the particular semiempirical theories are needed only for the determination of the unknown constants A and B. (When the thermal stratification is taken into account these constants become functions of the stability parameter μ; see, for example, Zilitinkevich (1970) where the question of the dependence on μ is discussed in detail.) Following Csanady (1967c), let us now show that these equations can easily be obtained from quite simple dimensional reasoning. Let us divide the turbulent planetary boundary layer into the "wall layer" (where the Coriolis force and the pressure gradient have no influence on the flow) and the "outer layer" (where the molecular viscosity and the surface parameters are unimportant). In the wall layer the usual law of the wall will be valid which, for a dynamically rough wall, can be written in the form

$$\bar{u}(z) = u_* f\left(\frac{z}{z_0}\right)$$

where z_0 is the roughness parameter. In the outer layer it is reasonable to suppose that the velocity defect law must be valid; this law must clearly have the form

$$\frac{Ge^{-i\alpha} - \bar{u}(z)}{u_*} = f_1\left(\frac{2\omega_z z}{u_*}\right)$$

Here $Ge^{-i\alpha}$ is the complex value of the geostrophic (or "free-stream") wind velocity and the argument of the complex function f_1 on the right side is in fact the ratio of the height z to the characteristic length scale H of Eq. (6.65) which determines the thickness of our turbulent boundary layer; the arguments in favor of such a law are related to those favoring the ordinary velocity defect law of Sects. 5.5 and 5.6. Now if we suppose that an overlap layer between the two layers exists in which both the laws apply, then it is easy to show that in this layer both the functions f and f_1 must be logarithmic, i.e.,

$$\bar{u}(z) = \frac{u_*}{\kappa} \log \frac{z}{z_0}, \quad Ge^{-i\alpha} - \bar{u}(z) = -\frac{u_*}{\varkappa}\left[\log \frac{2\omega_z z}{u_*} + C\right]$$

where $C = B - iA$ is a complex constant, and $\varkappa$ and z_0 have the same meaning as in the previous subsections of this section. Equating the two expressions for $\bar{u}(z)$ in the overlap layer, we obtain the following equation:

$$\frac{1}{\varkappa}\left[\log\frac{u_*}{2\omega_z z_0} - C\right] = \frac{Ge^{-i\alpha}}{u_*}.$$

Dividing the last equation into real and imaginary parts, we obtain the two equations (6.72).

6.7 Distribution of Suspended Particles in a Turbulent Flow

Let us now consider the problem of the motion of suspended particles in a turbulent flow of incompressible fluid. The theory of this phenomenon was developed by Barenblatt (1953; 1955) [see also Kolmogorov (1954)]. The fundamental postulate of the theory is the assumption that the size of the suspended particles is small in comparison with the length scales of the turbulence. This permits us to assume that they form, as it were, a continuous distribution of admixture in the basic fluid medium. The total density of the admixture may be written in the form

$$\rho = \rho_0 (1 - s) + \rho_1 s = \rho_0 + (\rho_1 - \rho_0) s, \tag{6.73}$$

where ρ_0 and ρ_1 are the densities of the fluid and of the particles and s is the relative volume of the particles; in works on silts s is called the "turbidity"; ρ_0 and ρ_1 are physical constants, but the "turbidity" s fluctuates as a result of turbulent mixing so that

$$\rho' = (\rho_1 - \rho_0) s'.$$

The velocity u_i of the motion of the mixture at a given point (i.e., the velocity of the center of gravity of an infinitesimal volume of the mixture surrounding the given point) will be defined as the mass-weighted mean of the velocity of the basic fluid u_{0i} and the velocity of the admixture u_{1i}

$$u_i = \frac{\rho_0 (1 - s)}{\rho} u_{0i} + \frac{\rho_1 s}{\rho} u_{1i}. \tag{6.74}$$

The equation of motion for a mixture may be written as

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_\alpha}\left[\rho_0 (1 - s) u_{0i} u_{0\alpha} + \rho_1 s u_{1i} u_{1\alpha} + p\delta_{i\alpha} - \sigma_{i\alpha}\right] = -\rho g \delta_{i3}, \tag{6.75}$$

where p is the total pressure at a given point of the mixture, σ_{ij} is the sum of the viscous stress tensor in the basic fluid in the presence of admixtures, and the tensor of the additional stresses arising on account of the interaction of the suspended particles. Of the body forces, we take into account only the gravity force, which is directed along the Ox_3 coordinate in the direction of x_3 decreasing.

The equations of mass balance for the basic fluid and the admixture take the form

$$\frac{\partial \rho_0 (1 - s)}{\partial t} + \frac{\partial \rho_0 (1 - s) u_{0\alpha}}{\partial x_\alpha} = 0, \qquad \frac{\partial \rho_1 s}{\partial t} + \frac{\partial \rho_1 s u_{1\alpha}}{\partial x_\alpha} = 0.$$

Adding the equations, we obtain the equation of mass balance for the mixture

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_\alpha}{\partial x_\alpha} = 0. \tag{6.76}$$

On the other hand, dividing these equations throughout by ρ_0 and ρ_1 and then adding, we obtain the continuity equation for the mixture

$$\frac{\partial}{\partial x_\alpha}\left[(1-s)\, u_{0\alpha} + s u_{1\alpha}\right] = 0. \tag{6.77}$$

Taking into account the fact that the particles are small, and adopting the additional postulate that the fluid particle accelerations in a turbulent flow are small in comparison with the gravitational acceleration g, we may assume that the horizontal velocity of the basic fluid and the admixture are the same while the vertical velocities differ by some quantity a, the rate of gravitational sedimentation of the particles. Henceforth, we shall limit ourselves to the discussion of the case of low "turbidity," i.e., we shall assume that $s \ll 1$. Then a may be assumed independent of s, that is, constant (equal to the rate of gravitational sedimentation of a single particle in an infinite fluid which we assume to be the same for all particles). We then have

$$\begin{gathered} u_i = u_{0i} - \frac{\rho_1 s}{\rho}\, a\delta_{i3}, \\ \rho_0 (1-s)\, u_{0i} u_{0\alpha} + \rho_1 s u_{1i} u_{1\alpha} = \rho u_i u_\alpha + \frac{\rho_0 \rho_1}{\rho}\, s(1-s)\, a^2 \delta_{i3}\delta_{\alpha 3}. \end{gathered} \tag{6.78}$$

The equations of motion (6.75) now take the form

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_\alpha}\left(\rho u_i u_\alpha + p\delta_{i\alpha} - \sigma_{i\alpha}\right) = -\rho\left[g + \frac{\rho_0\rho_1}{\rho}\frac{\partial}{\partial x_3}\frac{s(1-s)\, a^2}{\rho}\right]\delta_{i3}, \tag{6.79}$$

and the continuity equations (6.77) may be written as

$$\frac{\partial u_\alpha}{\partial x_\alpha} = -(\rho_1 - \rho_0)\frac{\partial}{\partial x_3}\frac{s(1-s)\, a}{\rho}. \tag{6.80}$$

From now on, we shall use only Eqs. (6.73), (6.76), (6.79), and (6.80), and thereby consider the problem of the distribution of suspended particles in a steady flow of fluid filling the half-space $x_3 \geqslant 0$, and, in the mean, homogeneous along the Ox_1, Ox_2 axes. In this case it may be assumed that all the mean characteristics depend only on the coordinate $z = x_3$ and that $\bar{u}_2 = 0$.

Writing $u_3 = w$ and averaging the continuity equation, we obtain $\frac{\partial \overline{\rho w}}{\partial z} = 0$ or $\overline{\rho w} =$ const. Since, for a steady regime there is no overall vertical mass transfer, we must put $\overline{\rho w} = \bar{\rho}\bar{w} + \overline{\rho' w'} = 0$. Recalling Eq. (6.73), we may rewrite the latter relationship as

$$\bar{w} = -\sigma\overline{w's'}, \tag{6.81}$$

where $\sigma = \frac{\rho_1 - \rho_0}{\bar{\rho}}$. From the very existence of a steady distribution of the suspended particles, it follows that the correlation between w' and s' must be positive since in a steady regime the regular gravitational sedimentation of the particles must be balanced by their turbulent transfer upwards, while the turbulent flux of mass is equal to $\overline{\rho' w'} = (\rho_1 - \rho_0)\overline{w's'}$. Consequently, $\bar{w} < 0$, i.e., the mean motion of the mixture has a downward component; we recall that the velocity of the mixture is defined as the mass-weighted mean of the velocities of the basic fluid and the particles, and not the volume-weighted mean.

Averaging Eq. (6.80) we obtain

$$\frac{\partial \bar{w}}{\partial z} = -(\rho_1 - \rho_0)\frac{\partial}{\partial z}\left[\frac{\overline{s(1-s)a}}{\rho}\right] \approx -\sigma \frac{\partial}{\partial z}\overline{s(1-s)a},$$

whence, after integration with respect to z we find

$$\bar{w} = -\sigma\overline{s(1-s)a} + \text{const.}$$

The constant term in this equation may be taken equal to zero, since, for large z, $\bar{s}$, $\overline{w's'}$ and, consequently, $\bar{w}$, tend to zero. Hence, comparing the last equation with (6.81) we have

$$\overline{w's'} = \overline{s(1-s)a} \approx a\bar{s}. \tag{6.82}$$

If we use a semiempirical equation of the type (6.25)

$$\overline{w's'} = -\alpha_s K \frac{\partial \bar{s}}{\partial z}, \tag{6.83}$$

then the latter relationship takes the form

$$\alpha_s K \frac{\partial \ln \bar{s}}{\partial z} = -a. \tag{6.84}$$

As the mean equation of motion of the mixture, we use the Reynolds equation in the direction of the Ox_1 axis, which for values of z that are not too large, takes the following form:

$$K\frac{\partial \bar{u}}{\partial z} = u_*^2 = \text{const.} \tag{6.85}$$

Eliminating K from Eqs. (6.84) and (6.85) we obtain

$$\frac{\partial \ln \bar{s}}{\partial z} = -\omega \frac{\varkappa}{u_*}\frac{\partial \bar{u}}{\partial z},$$

where $\omega = \dfrac{a}{\alpha_s \varkappa u_*}$ is a dimensionless parameter. Integrating this equation, we obtain

$$\bar{s} = \bar{s}_0 e^{-\omega\frac{\varkappa \bar{u}}{u_*}}, \tag{6.86}$$

where $\bar{s}_0$ is the value of $\bar{s}$ at some virtual "roughness height" z_0 at which $\bar{u}(z)$ becomes zero.

Taking the approximate value of $-\rho g \delta_{i3}$ on the right side of Eq. (6.79), we see that this equation, together with the mass balance equation (6.76) coincides with the corresponding equations for a compressible fluid. Hence the energy balance equation has the same form as for a compressible fluid in a gravity field. Ignoring the effect of diffusion of the turbulent energy and other small terms, we write this equation in the form

$$\bar{\rho} l \sqrt{b}\left(\frac{\partial \bar{u}}{\partial z}\right)^2 - \frac{\bar{\rho} b^{3/2}}{c^4 l} - \overline{\rho w'}g = 0, \tag{6.87}$$

where $l\sqrt{b}=K$ and $B=-\overline{\rho w'}g$ is the work of suspending the particles by the turbulent flow per unit volume of the mixture. Using Eqs. (6.73) and (6.82) we obtain

$$B=-\sigma\bar{\rho}g\overline{s'w'}=-\sigma\bar{\rho}g\overline{s(1-s)a}. \tag{6.88}$$

This expression was first obtained by Velikanov (1946). Expressing $\overline{s'w'}$ by means of Eq. (6.83), we obtain

$$B=\alpha_s\sigma\bar{\rho}gl\sqrt{b}\,\frac{\partial\bar{s}}{\partial z}.$$

Equation (6.87) may then be reduced to the form

$$b=c^4l^2\left(\frac{\partial\bar{u}}{\partial z}\right)^2(1-\alpha_s\,\mathrm{Ri}), \tag{6.89}$$

where

$$\mathrm{Ri}=-\sigma g\,\frac{\partial\bar{s}/\partial z}{(\partial\bar{u}/\partial z)^2}>0$$

is a demensionless number analogous to the Richardson number. Eliminating the velocity gradient $\frac{\partial\bar{u}}{\partial z}$ by using Eq. (6.85) with $K=l\sqrt{b}$ we obtain

$$b=c^2u_*^2\sqrt{1-\alpha_s\,\mathrm{Ri}} \tag{6.90}$$

In the absence of suspended particles Ri = 0 and $b=c^2u_*^2$. Equation (6.90) shows that the presence of suspended particles in the flow leads to a lowering of the turbulent energy, i.e., it influences the dynamics of the flow. This result is confirmed by direct experiments. Thus the calculation of the motion of silt, on the assumption that suspended particles have no effect on the dynamics of the flow (this is called the "diffusion approximation" in sedimentation theory), is possible only with sufficiently small Ri.

The distributions of mean velocity $\bar{u}(z)$ in a flow carrying suspended particles may be found with the aid of Eqs. (6.84), (6.85), and (6.90). If we introduce dimensionless quantities, writing

$$\bar{u}=\frac{u_*}{\varkappa}F(\zeta),\quad K=\varkappa u_*z\Phi(\zeta),\quad cl=\varkappa z\Psi(\zeta),\quad \bar{s}=S(\zeta),\quad \zeta=\frac{z}{L}, \tag{6.91}$$

where $L=\dfrac{u_*^2}{\alpha_s\varkappa^2\sigma g}$, then the equations take the form

$$\zeta\Phi S'=-\omega S,\quad \zeta\Phi F'=1,\quad \Phi=\Psi\left(1+\frac{S'}{F'^2}\right)^{1/4}. \tag{6.92}$$

The function $S(\zeta)$ is of special interest; this describes the vertical distribution of the suspended particles. Eliminating F and Φ from Eq. (6.92) for $T(\zeta)=\dfrac{1}{S(\zeta)}$ we obtain

$$\frac{\omega T}{\zeta}=\Psi T'\left(1-\frac{\omega^2}{T'}\right)^{1/4}.$$

Barenblatt investigated the solution of this equation in general form, without giving a concrete form to the function Ψ. He assumed only that it satisfies the obvious condition $\Psi(0) = 1$, and is a nonincreasing function of the Richardson number since the length scale l must decrease due to the presence of the suspension. The results of the analysis show that for $\omega < 1$ and $\omega > 1$ the nature of the solution is quite different. When $\omega > 1$ (low flow velocity or large particles) the transfer of particles takes place primarily close to the bottom of the flow. However, in the bed region, the theory we have discussed (based on the assumption that s is small) is inapplicable and must be replaced by a more exact theory. Outside the bed region $\mathrm{Ri} \to 0$, so that the distribution of the suspended particles asymptotically tends to the distribution obtained according to the diffusion approximation. With $\omega < 1$ (high flow velocity or small particles), the transfer takes place in the body of the flow. The distribution of particles at great heights tends asymptotically to some limiting self-preserving distribution ($\mathrm{Ri} \to$ const) for which $\bar{s}$ is inversely proportional to z, and $\bar{u}(z) = \frac{u_*}{\varkappa\omega} \log z +$ const. Thus there exists some limiting saturation of the flow, such that at still greater saturation, this theory becomes inapplicable in the bed region.

4 TURBULENCE IN A THERMALLY STRATIFIED MEDIUM

7. GENERALIZATION OF LOGARITHMIC LAYER THEORY TO THERMALLY STRATIFIED FLOWS

7.1 Thermally Stratified Turbulent Boundary Layer as a Model of the Atmospheric Surface Layer

In Chapt. 3 we considered in detail turbulent flow in the boundary layer above an infinite flat plate. Our deductions were compared with both data from laboratory flows and observations of the air motions in the surface layer of the atmosphere. However, it was noted that for this purpose it is only valid to use those observations relating to neutral stratification, that is, to cases when the air temperature in the lower layers is practically constant with height.[1]

[1] Generally speaking, neutral stratification is taken to mean a temperature distribution in which the vertical gradient dT/dz is identical with the adiabatic gradient G_a defined in Sect. 2.4. In fact, it is only under these conditions that the vertical displacement of a fluid element will be accompanied by neither loss nor gain of potential energy. However, since the typical values of the vertical temperature gradient in the few tens of meters of the earth

(Cont'd on p. 418).

However, neutral stratification is fairly rare in nature. In fact, during the day the temperature over the earth generally decreases considerably with height, but at night, as a rule, it increases with height (i.e., so-called temperature inversion takes place). Thus neutral stratification occurs only for fairly short periods before sunset and after sunrise.[2] Consequently, the motion in the surface layer cannot, for most cases, be classified as belonging to the simple scheme of the turbulent boundary layer discussed in Chapt. 3.

Therefore, in practice, when studying turbulence in the surface layer of the atmosphere, we must take into account the presence in the atmosphere of temperature stratification which produces a systematic height variation of the density. We know that the presence in a gravitational field of density inhomogeneities leads to the appearance of buoyancy forces, producing upward displacement of the less dense fluid particles and downward displacement of the more dense fluid particles.[3] As a result, the particles that are less dense (denser) than the surrounding medium acquire additional energy during their upward (downward) motion, due to the work done by the buoyancy forces, while during downward (upward) motion, they expend part of their energy in overcoming the buoyancy forces. The potential energy of a density-stratified medium in a gravitational field can thus be directly transformed into turbulent energy, and, conversely, turbulent energy can be transformed into potential energy of the medium. This chapter will be devoted to a quantitative study of the effect of such conversions of energy on turbulence.

In Sect. 6.4 we calculated the contribution made by the work of the buoyancy forces in the balance of turbulent energy. This

atmosphere are usually of the order of some hundreds of times greater than G_a, in most cases we may take G_a to be zero. Thus it is usually possible to make no distinction between the ordinary temperature T and the potential temperature θ of Eq. (2.4). Therefore, for simplicity, we shall henceforth always speak of the ordinary temperature although, strictly speaking, in most cases the equations which we use will be exact only for the potential temperature. Also, in Sect. 8, we shall make no special mention of the fact that in certain cases concerning observations carried out in a relatively thick layer of air of the order of several tens of meters, the data used refer in fact not to the ordinary but to the potential temperature.

[2]Over the sea the situation is different and stratification comparatively close to neutral is observed more often. However, we shall not consider the marine atmosphere in detail in this book.

[3]Here and henceforth, the buoyancy force will be taken to mean the difference between the Archimedean force (described by the Archimedes law) and the body force of gravity. In a homogeneous (nonstratified) medium, these two forces balance each other exactly, and hence the gravity force may simply be ignored; see the opening paragraph of Sect. 1.2.

contribution (per unit volume) is given by the equation

$$B = -\overline{\rho' w'} g, \tag{7.1}$$

where w' is the fluctuation of the vertical velocity [see Eq. (6.38), Sect. 6.4]. From Eq. (7.1) it is clear that density stratification will affect turbulence only if the density fluctuations ρ' are correlated with the fluctuations w'. Considering the meteorological applications, we shall find it convenient to change from working with density fluctuations (which are very difficult to measure in the atmosphere) to temperature fluctuations, which are readily susceptible to direct measurement. Here, as in Sect. 6.5, we shall take into account only density fluctuations connected with fluctuations of temperature, and the considerably smaller density fluctuations produced by the fluctuations of atmospheric pressure will be ignored. In the case of not too large temperature fluctuations, we may write $\rho' \approx -(\bar{\rho}/\bar{T})T'$, where T is the absolute temperature, and, therefore, in this approximation

$$B = \frac{g}{\bar{T}} \bar{\rho} \overline{w'T'} = \frac{g}{c_p \bar{T}} q, \tag{7.2}$$

where $q = c_p \bar{\rho} \overline{w'T'}$ is the vertical turbulent heat flux (see Sect. 6.4). Thus it is clear that the buoyancy effect on turbulence in the atmosphere is connected directly with turbulent heat-transfer from the underlying surface to the atmosphere and vice-versa. This leads once again to the conclusion that this effect must vanish when the vertical temperature gradient is equal to zero; in fact, according to the fundamental equation of the semiempirical theory

$$q = -c_p \bar{\rho} K_T \frac{\partial \bar{T}}{\partial z}, \tag{7.3}$$

where K_T is the eddy thermal diffusivity and, consequently,

$$B = -\frac{g}{\bar{T}} \bar{\rho} K_T \frac{\partial \bar{T}}{\partial z}, \tag{7.4}$$

so that $B = 0$ when $\frac{\partial \bar{T}}{\partial z} = 0$.

In the theoretical analysis of turbulent phenomena in the surface layer of the atmosphere, we must take into account the presence of vertical temperature stratification and the associated vertical turbulent heat flux. On the other hand, the horizontal inhomogeneity of the underlying surface, which, to some extent, is always present in the real atmosphere, may naturally be neglected at first. In fact, for a comparatively flat underlying surface, the general character of which does not change over a comparatively large region, this inhomogeneity will not play a very important part, while taking it into consideration makes the theoretical analysis considerably more complicated. Thus we shall consider only a simplified model of turbulence in a fluid filling the half-space above an infinite homogeneous plane surface $z = 0$ of constant roughness z_0 (the displacement height h_0, which may be taken into account by a simple change of origin for z, will also, for simplicity, be ignored). As a result, we shall always assume in this chapter that all one-point averaged characteristics of the flow variables depend only on the vertical coordinate z.

If we assume that all one-point moments are dependent only on z, we assume implicitly that these moments may be determined, i.e., that the values of all flow variables in the surface layer of the atmosphere possess a determinant statistical stability. Generally speaking, this assumption may give rise to doubt, since the atmospheric conditions depend considerably upon the time of day and the time of year, while in addition to the regular diurnal and annual variations, the values of any meteorological variable at a given point of the atmosphere also undergo irregular fluctuations with very diverse periods. These irregular fluctuations may be considered to be due to the occurrence of turbulence of different three-dimensional scales from the very small (of the order of centimeters and fractions of a centimeter) to the very large (of the order of the dimensions of cyclones and anticyclones, or even of the scale of inhomogeneities of the general circulation of the atmosphere). Thus, for example, the time means of temperature or wind velocity at a given point of the atmosphere prove first, to depend considerably on the size of the averaging interval, and, second, for a given scale of averaging, to fluctuate from sample to sample under the action of components of turbulence with periods comparable to or greater than the size of the averaging interval. This phenomenon called the "drift of the mean" of meteorological variables, considerably complicates attempts to determine their statistical characteristics. Nevertheless, experience shows that if we confine ourselves to observations relating to a specified season of the year, a specified time of day, and specified synoptic conditions (i.e., specified "weather"), then averaging over a time interval τ which is considerably longer than the characteristic period of the "energy-containing eddies" (turbulent formations which contain the principal fraction of the turbulent energy), the mean values of the meteorological variables will be relatively stable. But, in this case, the relevant observations may be taken to constitute a "statistical ensemble," permitting us to carry out probability averaging. In the surface layer of the atmosphere, the time scale of the energy-containing eddies may be estimated to be of the same order of magnitude as the ratio L_0/U, where U is the characteristic value of the wind velocity and L_0 is the characteristic horizontal length scale of the energy-containing eddies, measured in tens or a few hundreds of meters. Hence L_0/U is of the order of some tens of seconds, and, averaging over a time interval of the order of ten to twenty minutes, the mean values of wind velocity, temperature, etc., prove to be relatively stable and may be considered as approximate values of the probability means for corresponding random fields. When the averaging period is extended substantially to intervals of the order of several hours or even more, the mean values change considerably, and once again may even prove to possess very low stability due to the effect of long-period "synoptic fluctuations" relating first to medium-scale turbulence and then to macroturbulence; however, we shall not deal with this type of turbulence here.

In addition to the assumption of horizontal homogeneity, we shall also adopt several simplifying assumptions relating to the dynamic equations which describe the turbulence under investigation. First, as already stated, we shall ignore variations of density produced by pressure fluctuations, and shall confine ourselves to equations which have been linearized with respect to deviations of the density, temperature and pressure from the corresponding "standard values" ρ_0, T_0 and p_0 (which are dependent only on z and satisfy the equation of statics $\partial p_0/\partial z = -\rho_0 g$ and the equation of state $p_0 = R\rho_0 T_0$). Henceforth, the values $\rho_0 = \bar{\rho}$ and T_0 will be regarded simply as constants since their variation with height in the surface layer (which is some tens of meters thick) is negligibly small. In this case, the equations of motion reduce to the well-known "free convection equations" or "Boussinesq equations" of the form

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \nu\Delta u,\\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} + \nu\Delta v,\\
&\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \nu\Delta w - \frac{g}{T_0}T'.
\end{aligned} \tag{7.5}$$

In these equations the term $-\frac{g}{T_0}T'$ describes the "buoyant acceleration" of the fluid particles depending on the gravity force. However, the Coriolis force is neglected, since, according to the estimate given in Sect. 6.6, this force has no perceptible effect on the mean motion in the bottom 50 m (or thereabouts), and, moreover, can have no effect on the fluctuating motion, the velocity of which in the lower atmosphere is approximately one order of magnitude less than the mean velocity. Of course, at heights of more than a few tens of meters above the surface of the earth, Eqs. (7.5) without the Coriolis forces no longer apply; this must be borne in mind when deductions from these equations are compared with observation.

The continuity equation for the Boussinesq approximation may be written just as in the case of an incompressible fluid:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \tag{7.6}$$

It follows from this and the horizontal homogeneity that $\bar{w}=0$. Thus the mean motion is plane-parallel and is given by the mean velocity vector $\bar{\boldsymbol{u}}(z)=\{U(z), V(z), 0\}$. It is also clear that in the absence of the Coriolis force and with horizontal pressure gradient constant with height, the direction of the vector $\bar{\boldsymbol{u}}(z)$ must be z-independent. We now take the Ox axis along the vector $\bar{\boldsymbol{u}}(z)$. In this case the equation $\bar{v}=0$ will hold; thus, the probability distribution for the flow variables may naturally be assumed invariant not only under parallel displacements in the Oxy plane, but also under reflections with respect to the Oxz plane. Consequently, in particular, it follows that $\tau_{xy}=-\rho_0\overline{u'v'}=0$ and $\tau_{yz}=-\rho_0\overline{v'w'}=0$.

If we take into account that the mean wind velocity in the surface layer varies during the 24-hour cycle by an amount of the order of 10 m/sec, it is not difficult to estimate that under ordinary conditions, in which there are no particularly rapid weather changes, the term containing $\frac{\partial \bar{u}}{\partial t}$ in the averaged first equation of Eqs. (7.5) proves to be far smaller even than the term containing the mean pressure gradient $\frac{\partial \bar{p}}{\partial x}$ or than the mean Coriolis force. Hence, when we average Eqs. (7.5), the time derivatives may also be discarded, and we may assume that in the surface layer

$$-\rho_0\overline{u'w'}+\rho_0\nu\frac{\partial \bar{u}}{\partial z}=\tau=\text{const} \tag{7.7}$$

[see Eq. (5.11), Sect. 5.2]. Thus it seems probable that the mean flow in the surface layer over a homogeneous and flat earth or sea surface may be assumed not only plane-parallel but also steady, and the shear stress $\tau = \rho_0 u_*^2$ (the vertical momentum flux) constant with height.

In addition to Eqs. (7.5) and (7.6), we must also use the heat equation, which determines the time variation of the temperature T. According to the results of Sect. 1.5, this equation may be written as:

$$c_p\rho\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=c_p\rho\chi\nabla^2 T+Q, \qquad (7.8)$$

where Q is the influx of heat due to radiation and phase changes of moisture contained in the air (other forms of heat influx and, in particular, heating due to the kinetic energy dissipation in the surface layer may be ignored completely). If we also ignore the influx of heat Q, then we obtain the ordinary thermal conductivity equation

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\chi\nabla^2 T, \qquad (7.8')$$

the averaging of which leads to the conclusion that in steady conditions

$$q = c_p\rho_0\overline{w'T'} - c_p\rho_0\chi\frac{\partial \bar{T}}{\partial z} = \text{const}, \qquad (7.9)$$

i.e., that the mean vertical heat flux caused by molecular and turbulent thermal conductivity is constant [cf. Eq. (5.73′), Sect. 5.7]. However, if we also take Q into account, the principal part of which usually consists of the radiant heat flux, then instead of Eq. (7.9) we obtain the more complex equation

$$q + q_s = \text{const}, \qquad (7.9')$$

where $\bar{Q} = -\frac{\partial q_s}{\partial z}$ so that q_s can usually be identified with the mean vertical flux of radiant heat. According to the preliminary data of Robinson (1950) [see also Haltiner and Martin (1957), Chapt. 15], significant variation of q_s with height usually takes place only in a thin layer of air close to the surface of the earth, the thickness of which does not exceed (or at least only slightly exceeds) 1 m. If this is true, then when we consider higher layers we may put $q_s = \text{const}$, i.e., we may ignore the radiant influx of heat $\bar{Q}$. This is confirmed indirectly by the fact that the theory based on Eq. (7.9), as will be seen later, gives satisfactory agreement in the majority of cases with the data of observations in the surface layer.[4] The relative smallness of the contribution of the variability of q_s to the height variation of q was confirmed also by crude estimates of Mordukhovich and Tsvang (1966). In general, however, the question of the validity of the approximate equation $q_s = \text{const}$ in the surface layer of air still can not be regarded as definitely proved, either theoretically or experimentally. Particularly doubtful from this viewpoint are conditions of very stable stratification (observed, for example, during the night), when the turbulent heat flux q is very small, and the variation with height of the flux q_s may be considerable [cf. Funk (1960)]. Hence the attempts to consider the effect of radiative heat transfer on turbulence more exactly, which were begun by Townsend (1958) and Yamamoto and Kondo (1959) should be considered most opportune. In addition, we must also note that there must exist vertical variations of the turbulent fluxes τ and q caused by nonzero horizontal gradients of mean wind velocity and temperature which may be considerable even in sites usually considered to be quite homogeneous [cf. Mordukhovich and Tsvang (1966); and Dyer (1968)]. However, since only very preliminary estimates have been made of the vertical variations of the turbulent fluxes, we shall confine ourselves here to a consideration only of the rougher theory which follows from the assumptions (7.7) and (7.9), the formulation of which, in any case, must be regarded as a necessary first step in the elucidation of the mechanism of turbulence in the surface layer.

Finally, we can now give a definite description of the idealized theoretical model to be studied in this chapter. We shall consider a plane-parallel steady fluid flow in the half-space $z > 0$ above a flat homogeneous surface $z = 0$ (characterized by given roughness z_0) described by the Boussinesq equations (7.5), (7.6) and (7.8′) and satisfying conditions (7.7) and (7.9) with given values τ and q; these conditions act as boundary conditions on the upper boundary of the layer. According to our previous remarks, we may expect that this model will correspond comparatively well to many actual flows in

[4]We must point out, however, that the logarithmic profile of the mean velocity, which follows from the theory only when $\tau = \text{const}$, is satisfied fairly well in tubes almost to the center of the tube where $\tau = 0$. Thus this indirect proof can in no way be regarded as very reliable.

the lower layer of the atmosphere (of the order of several tens of meters) above a comparatively flat and homogeneous underlying surface. We may also assume that this model will also be applicable to some other types of flow, for example, to turbulence in the bottom or surface layers of the sea, in isothermal conditions but with considerable sahnity stratification (and hence also, density stratification) and to some artificial turbulent flows set up in the laboratory. (In the case of density stratification, of course, we simply have to replace $\frac{1}{\bar{T}}\frac{\partial \bar{T}}{\partial z}$ by $-\frac{1}{\bar{\rho}}\frac{\partial \bar{\rho}}{\partial z}$ and $T'/\bar{T}$ by $-\rho'/\bar{\rho}$ everywhere.) However, henceforth, for definiteness we shall always speak of turbulence in the surface layer of the atmosphere (since most of the available data relate to this), taking "surface layer" everywhere to mean the idealized model of a constant flux layer of a density-stratified medium which we have described.

7.2 Application of Dimensional Reasoning to Turbulence in a Stratified Medium

The turbulence characteristics in a thermally stratified medium described by Eqs. (7.5), (7.6) and (7.8′) and conditions (7.7) and (7.9) can depend clearly only on a fairly small number of physical quantities; namely, on the parameters $\frac{g}{T_0}$, ρ_0, ν, and χ which occur in these equations, on the values of τ (or u_*) and q which give the momentum flux or heat proceeding from infinity to the surface $z = 0$ (or vice-versa) and characterizing the dynamic and thermal interaction of the flow with the underlying surface, and on the roughness parameter z_0, which completely describes the geometrical properties of the underlying surface. However, not all of the parameters play an equally significant part. First, we know that in regions with sufficiently developed turbulence (i.e., almost everywhere except in a very thin sublayer contiguous to the underlying surface) the fluxes of heat and momentum due to molecular motion are always very small in comparison with the turbulent fluxes of heat and momentum (see, for example, Sects. 5.1, 5.3 and 5.7). Thus in these regions the terms of Eqs. (7.5) and (7.8′) which contain the molecular coefficients ν and χ may generally simply be ignored. But then Eqs. (7.7) and (7.9) may be rewritten in the simpler form

$$-\rho_0\overline{u'w'} = \tau = \text{const}, \qquad c_p\rho_0\overline{T'w'} = q = \text{const}, \qquad (7.10)$$

which indicates that the characteristics of turbulence must be independent of ν and χ. As for the roughness parameter z_0, this determines the boundary conditions on the underlying surface and, through these boundary conditions, also affects the absolute value of the mean velocity $\bar{u}(z)$ and the difference of mean temperatures $\bar{T}(z) - \bar{T}(0)$ at considerable distances from the underlying surface. However, the vertical variation of the mean velocity and the mean temperature at distances z at which the direct effect of the underlying surface is no longer perceptible, can not depend on z_0, but must be determined only by the values τ and q of the flux of momentum and heat. In other words, variations of the roughness parameter z_0 can only lead to a shift of the curves $\bar{u} = \bar{u}(z)$ and $\bar{T} = \bar{T}(z)$ by some constant amount, but can not affect the form of the profiles $\bar{u}(z)$ and $\bar{T}(z)$. Therefore, the dependence on height of the mean velocity gradient and temperature gradient and other related characteristics of a developed turbulence in the surface layer of air must be determined by the following four parameters only: the density ρ_0, the "buoyancy parameter" g/T_0 (describing the buoyancy effect), the turbulent shear stress τ (or $u_* = \sqrt{\frac{\tau}{\rho_0}}$) and the vertical turbulent heat flux q (or $q/c_p\rho_0$).[5] Using this simple postulate, we must remember that when $q = 0$ (i.e., in the case of neutral stratification, when $\frac{\partial \bar{T}}{\partial z} = 0$) no buoyancy effect on the turbulence should be apparent, that is, the dependence on the parameter $\frac{g}{T_0}$ must vanish. In this case, we have returned to the case of a

[5] However, we should note that this natural postulate, which is fundamental for the following theory, sometimes evokes certain objections. Thus, on occasion, the idea is advanced that the temperature and wind profiles in a stratified atmosphere must depend on the molecular constants ν and χ, or at least on their ratio—the Prandtl number Pr = ν/χ [see, for example, Townsend (1962a)]. In several cases, the basis for this idea is Malkus' controversial theory of turbulent convection (1954b) discussed, for example, by Townsend (1962b), Spiegel (1962), and Lindzen (1967); according to this theory the molecular constants play a considerable role for all z. The effect of the Prandtl number Pr on turbulent convection is also the subject of Kraichnan's work (1962a), the results of which, for moderate values of Pr, do not contradict the conclusions obtained in this chapter. On the other hand, Businger (1955), for example, assumed that the roughness parameter z_0 affects the form of the wind and temperature profiles explicitly, which makes all his equations considerably more complicated. Laykhtman's assumptions (1944; 1947a; 1961) go even further: instead of the ordinary roughness parameter, he used a length parameter z_0 which affects the forms of the profiles explicitly and depends on the sizes and form of the irregularities of the surface and on the thermal stratification in a complicated manner.

homogeneous medium, for which the logarithmic layer theory developed in Sect. 5.3 is valid.

Thus, when $z \gg z_0$, the turbulence characteristics at the height z will depend only on five quantities: z, ρ_0, $\frac{g}{T_0}$, u_*, and $\frac{q}{c_p\rho_0}$. In this case, since there are four independent dimensions—length, time, mass and temperature, we can formulate only one independent dimensionless combination (to within a numerical multiplier) from these five quantities. Following Obukhov (1946), Monin (1950c) and Monin and Obukhov (1953; 1954), we shall choose this dimensionless combination to be

$$\zeta = \frac{z}{L}, \tag{7.11}$$

where

$$L = -\frac{u_*^3}{\varkappa \frac{g}{T_0} \frac{q}{c_p\rho_0}} \tag{7.12}$$

is the length scale formulated from the parameters $\frac{g}{T_0}$, u_* and $\frac{q}{c_p\rho_0}$. (The dimensionless von Kármán constant $\varkappa$ was first introduced into the equation for L by Obukhov in 1946 and has usually been preserved in all subsequent works by tradition and the sign of L is chosen so that $L > 0$ for stable thermal stratification, when $q < 0$.) Then we may affirm that the dependence on height of any mean turbulence characteristic $\bar{f}$ in the surface layer of air which is independent of the properties of the underlying surface for not too small z may be written in the form

$$\frac{\bar{f}(z)}{f_0} = F\left(\frac{z}{L}\right), \tag{7.13}$$

where f_0 is a combination with the dimensions of f, formulated from the parameters $\frac{g}{T_0}$, ρ_0, u_* and $\frac{q}{c_p\rho_0}$, while $F(\zeta)$ is a universal function. As the velocity scale we take $\frac{u_*}{\varkappa}$ and for the temperature scale we take

$$T_* = -\frac{1}{\varkappa u_*}\frac{q}{c_p\rho_0} \tag{7.14}$$

[cf. Eq. (5.75′)] where the constant $\varkappa$ is also introduced by tradition and the sign of T_* is chosen so that in the stable case $T_* > 0$. Finally, as the natural scale for the eddy exchange coefficient K we shall take the quantity $\varkappa u_*|L|$.

According to Eq. (7.13), we can describe the dependence of the turbulence characteristics on height by means of universal functions of ζ. Thus, for example, for the vertical gradients of the mean wind velocity and temperature, we obtain

$$\frac{\partial\bar{u}}{\partial z} = \frac{u_*}{\varkappa L}\, g(\zeta), \tag{7.15}$$

$$\frac{\partial\bar{T}}{\partial z} = \frac{T_*}{L}\, g_1(\zeta), \tag{7.16}$$

where $g(\zeta)$ and $g_1(\zeta)$ are two universal functions of ζ. Equations (7.15) and (7.16) were given in Monin and Obukhov (1953; 1954); they may be considered as an immediate generalization of the fundamental equations (5.21) and (5.76) of the logarithmic layer theory to the case of a thermally stratified medium. Substituting these equations into Eqs. (5.5) and (5.9), which define the eddy exchange coefficients K and K_T for momentum and heat, we find that

$$K = \frac{u_*^2}{\partial\bar{u}/\partial z} = \frac{\varkappa u_* L}{g(\zeta)}, \qquad K_T = \frac{-q/c_p\rho_0}{\partial\bar{T}/\partial z} = \frac{\varkappa u_* L}{g_1(\zeta)}. \tag{7.17}$$

The ratio of the exchange coefficients K_T and K is equal to

$$\frac{K_T}{K} = \frac{\varkappa T_*}{u_*}\frac{\partial\bar{u}/\partial z}{\partial\bar{T}/\partial z} = \frac{g(\zeta)}{g_1(\zeta)} = \alpha(\zeta); \tag{7.18}$$

generally speaking, it is some universal function of ζ. The Richardson number Ri, defined in Sects. 2.4 and 6.5 [see Eqs. (2.3′) and (6.51)] by reason of Eqs. (7.12) and (7.14)–(7.16) equals

$$\mathrm{Ri} = \frac{g}{T_0}\frac{\partial\bar{T}/\partial z}{(\partial\bar{u}/\partial z)^2} = \frac{g_1(\zeta)}{[g(\zeta)]^2} = \frac{1}{\alpha(\zeta)\, g(\zeta)}. \tag{7.19}$$

This is also a universal function of ζ. Similarly, the flux Richardson number Rf of Eq. (6.45) defined by the heat flux q, the momentum τ and the velocity profile $\bar{u}(z)$ will equal

$$\mathrm{Rf} = -\frac{g}{c_p T_0} \frac{q}{\tau \dfrac{\partial \bar{u}}{\partial z}} = \frac{1}{g(\zeta)}. \tag{7.20}$$

The flux Richardson number Rf obviously is connected simply with the eddy viscosity K:

$$K = \varkappa u_* L \cdot \mathrm{Rf}. \tag{7.21}$$

From the general form analysis of the functions $g(\zeta)$ and $g_1(\zeta)$, which will be discussed later in this section (i.e., Sect. 7.3), it may be deduced that both Ri (ζ) and Rf (ζ) are monotonic functions and hence have a single-valued inverse function. Thus it follows that Ri, Rf and ζ may all be used with equal validity as parameters characterizing the thermal stability of the air.

The situation is similar for the characteristics of the concentration field of a passive admixture in a stratified medium, when there is a constant admixture flux $j = \rho_0 \overline{w'\vartheta'}$ across the boundary $z = 0$ (in the surface layer of the air we may take this passive admixture to be, e.g., water-vapor; then j will denote the mean evaporation from unit area of the underlying surface per unit time). Here it is only necessary to add j to the set of parameters defining the mean turbulence characteristics. However, we also have the additional independent dimension of the quantity ϑ. Consequently, once again we have a single dimensionless characteristic $\zeta = z/L$. In particular, for the vertical gradient $\partial\bar{\vartheta}/\partial z$ of the mean humidity $\bar{\vartheta}$ (in the future we shall consider only this passive admixture), we obtain

$$\begin{aligned} \frac{\partial \bar{\vartheta}}{\partial z} &= \frac{\vartheta_*}{L} g_2(\zeta), \\ \vartheta_* &= -\frac{j}{\varkappa u_* \rho_0}, \end{aligned} \tag{7.22}$$

which is completely analogous to Eqs. (7.15) and (7.16); the exchange coefficient for moisture (eddy diffusivity) is then equal to

$$K_\vartheta = -\frac{j/\rho_0}{\partial\bar{\vartheta}/\partial z} = \frac{\varkappa u_* L}{g_2(\zeta)}. \tag{7.23}$$

The actual profiles of the wind velocity, temperature and humidity in the surface layer can be obtained by integrating Eqs. (7.15), (7.16), and (7.22)

$$\begin{aligned}
\bar{u}(z_2) - \bar{u}(z_1) &= \frac{u_*}{\varkappa}\left[f\left(\frac{z_2}{L}\right) - f\left(\frac{z_1}{L}\right)\right],\\
\bar{T}(z_2) - \bar{T}(z_1) &= T_*\left[f_1\left(\frac{z_2}{L}\right) - f_1\left(\frac{z_1}{L}\right)\right],\\
\bar{\vartheta}(z_2) - \bar{\vartheta}(z_1) &= \theta_*\left[f_2\left(\frac{z_2}{L}\right) - f_2\left(\frac{z_1}{L}\right)\right],
\end{aligned} \tag{7.24}$$

where

$$f(\zeta) = \int^{\zeta} g(\xi)\,d\xi; \quad f_1(\zeta) = \int^{\zeta} g_1(\xi)\,d\xi; \quad f_2(\zeta) = \int^{\zeta} g_2(\xi)\,d\xi. \tag{7.25}$$

7.3 The Form of the Universal Functions Describing Turbulence in a Stratified Medium

From the previous discussion in the surface layer of a thermally stratified medium, two qualitatively different turbulent regimes are possible. They correspond to the case of stable stratification (downward flux of heat; i.e., $q < 0$ and, accordingly, $L > 0$ and $T_* > 0$) and unstable stratification ($q > 0, L < 0, T_* < 0$). These two regimes must approach each other as the stratification approaches neutral (as $q \to 0$). Accordingly, all universal functions describing turbulence are divided into two distinct branches; for $\zeta \geqslant 0$ and for $\zeta \leqslant 0$.

In this subsection we shall discuss only the simplest turbulence characteristics, namely, the profiles of the mean velocity $\bar{u}$, the mean temperature $\bar{T}$ and the mean concentration of admixture (humidity) $\bar{\vartheta}$. Thus we shall be interested primarily in the functions $f(\zeta)$, $f_1(\zeta)$, and $f_2(\zeta)$ and their derivatives $g(\zeta) = f'(\zeta)$, $g_1(\zeta) = f_1'(\zeta)$, and $g_2(\zeta) = f_2'(\zeta)$.

Let us begin by considering the functions $f(\zeta)$ and $g(\zeta)$. It will be convenient to rewrite Eq. (7.15) in the form

$$\frac{\partial \bar{u}}{\partial z} = \frac{u_*}{\varkappa z}\zeta g(\zeta) = \frac{u_*}{\varkappa z}\varphi(\zeta); \qquad \varphi(\zeta) = \zeta g(\zeta) = \zeta f'(\zeta). \tag{7.15'}$$

We now fix the values of z, u_* and g/T_0, and let the value of the turbulent heat flux q decrease without limit, thus approaching the conditions of neutral stratification. In this case L will increase in absolute value without bound, so that $\zeta = \frac{z}{L}$ tends to zero. In the limit, as $q \rightarrow 0$, we must obtain the ordinary "logarithmic" equation $\frac{\partial u}{\partial z} = \frac{u_*}{\varkappa z}$, which contains neither q nor g/T_0. Consequently

$$\varphi(0) = \lim_{\zeta \rightarrow 0} \zeta f'(\zeta) = 1. \tag{7.26}$$

Since $\varphi(\zeta)$ is naturally assumed continuous, for fixed q and u_* but sufficiently small z (i.e., when $z \ll |L|$), the value of $\varphi(\zeta)$ will be close to unity. This means that for $z \ll |L|$ the turbulent exchange will differ very little from the exchange in a temperature-homogeneous medium, so that the turbulence in the layer $z \ll |L|$ is produced primarily by dynamic factors. Thus the scale L determines the thickness of the layer in which thermal factors do not play an important part; it may therefore be called the *height of the dynamic sublayer*.[6] Since thermal factors may be ignored in the very lowest layer, naturally it follows that we must take into account the properties of the underlying surface in the case of a stratified medium, exactly as in the case of a homogeneous medium. In other words, the roughness parameter z_0, defined as in Sect. 5.4, is in all cases an objective characteristic of the dynamic influence of the underlying surface on the flow. Since the roughness parameter z_0 may be determined according to the wind distribution only in the dynamic sublayer, it is clear that this parameter cannot depend on the thermal stratification. In practice it is most convenient to determine this parameter by using a logarithmic approximation of the velocity profiles in the vicinity of the underlying surface; in the atmospheric case, it is desirable to use the cases of neutral or near-neutral stratification, for which $|L|$ is large and the dynamic sublayer is of considerable thickness. Taking into account the fact that the logarithmic velocity profile vanishes at height z_0 and that for $z \ll |L|$ the velocity profile may always be assumed logarithmic, the first equation of Eq. (7.24) may be written as

$$u(z) = \frac{u_*}{\varkappa}\left[f\left(\frac{z}{L}\right) - f\left(\frac{z_0}{L}\right)\right]. \tag{7.24'}$$

[6]In using this name, we must not, of course, forget that the thickness of the layer in which thermal factors play no essential part is in fact equal to the length L multiplied by a constant factor—which according to the data quoted in the next section is of the order of several hundredths.

Thus it is clear that the function $f(z)$, together with the roughness parameter z_0, defines the velocity profile uniquely.

The dominant role of the dynamic factors for $z \ll |L|$ must naturally be connected with the fact that the Richardson number Ri is small for such z. In fact, limiting ourselves to the more convenient flux Richardson number $\mathrm{Rf} = \alpha \mathrm{Ri}$, it is easy to deduce from Eqs. (7.20) and (7.26) that

$$\lim_{z \to 0} \mathrm{Rf}(z) = 0 \text{ and } \left(\frac{\partial \mathrm{Rf}}{\partial z}\right)_{z=0} = \frac{1}{L}; \tag{7.27}$$

thus within the dynamic sublayer

$$|\mathrm{Rf}| \approx \frac{z}{|L|} \ll 1. \tag{7.28}$$

For fairly small values of $\zeta = \frac{z}{L}$, the function $\varphi(\zeta) = \zeta f'(\zeta)$ may be expanded in a power series

$$\varphi(\zeta) = 1 + \beta_1 \zeta + \beta_2 \zeta^2 + \ldots . \tag{7.29}$$

Therefore, for the function $f(\zeta)$ close to neutral stratification, we obtain

$$f(\zeta) = \text{const} + \ln|\zeta| + \beta_1 \zeta + \frac{\beta_2}{2} \zeta^2 + \ldots; \tag{7.30}$$

similarly, the number Rf here is represented as a power series

$$\mathrm{Rf}(\zeta) = \zeta - \beta_1 \zeta^2 - (\beta_1^2 + \beta_2) \zeta^3 - \ldots . \tag{7.31}$$

Since the data do not as yet permit reliable estimation of several coefficients β_n, only the linear approximation of the function $\varphi(\zeta)$ can be of practical value

$$\varphi(\zeta) \approx 1 + \beta\zeta, \tag{7.32}$$

$$f(\zeta) \approx \text{const} + \ln|\zeta| + \beta\zeta, \quad \bar{u}(z) \approx \frac{u_*}{\varkappa}\left(\ln \frac{z}{z_0} + \beta \frac{z - z_0}{L}\right), \tag{7.33}$$

$$\mathrm{Rf} \approx \zeta - \beta\zeta^2 = \frac{z}{L} - \beta\left(\frac{z}{L}\right)^2 \tag{7.34}$$

[here β is the coefficient β_1 of Eqs. (7.29)–(7.31)].

With regard to Eqs. (7.29)–(7.34) it must be kept in mind that since $\zeta \geqslant 0$ and $\zeta \leqslant 0$ correspond to two qualitatively different turbulent regimes, we have no grounds for assuming that the function $\varphi(\zeta)$ is analytic at the point $\zeta = 0$ (although it is continuous at this point, since as $\zeta \rightarrow 0$, the two regimes of turbulence coalesce). Consequently, $\zeta \geqslant 0$ and $\zeta \leqslant 0$ may correspond to different values of the coefficients of the expansions (7.29)–(7.31) and, in particular, to different values of the constant β. Moreover, the coefficients β_n must be considered as universal constants, of the same type as the von Kármán constant $\varkappa$, only if they are regarded as exact values of the coefficients of the Taylor series of the universal function $\varphi(\zeta)$ [limiting ourselves to one definite sign of ζ]. However, at present, we cannot determine the exact analytic form of $\varphi(\zeta)$ without recourse to some more or less arbitrary hypotheses. At the same time, this function is not determined with sufficient accuracy by measurements for the values of its derivatives at the point $\zeta = 0$ to be defined with any great precision. Thus, in practice, the coefficient β is found generally by approximating the empirical function $\varphi(\zeta)$ over some range of values of ζ by means of Eq. (7.32) or by some other analytical expression. Consequently, the value of β obtained depends considerably on the choice of the approximation interval and on the equation for $\varphi(\zeta)$; hence it is not surprising that different investigators give widely varying values of this coefficient (see below, the end of Sect. 8.2). However, all these discrepancies have no effect on the question of the sign of β, the answer to which follows directly from elementary physical considerations. In fact, for stable stratification (i.e., $q < 0$, $L > 0$ and $\zeta > 0$), turbulent exchange is impeded, and hence the mean velocity profile must be "steeper" than the logarithmic profile corresponding to neutral stratification. On the other hand, for unstable stratification ($q > 0$, $L < 0$, $\zeta < 0$), the very intensive turbulent mixing must lead to equalization of the mean velocity, so that here the wind velocity must increase more slowly with height than in the case of neutral stratification. Thus $\beta\left(\frac{z - z_0}{L}\right)$ must be positive for $L > 0$ and negative for $L < 0$; i.e., $\beta > 0$ both for $\zeta > 0$ and $\zeta < 0$.

As we have already pointed out, Eqs. (7.32)–(7.34) are significant only for fairly small absolute values of ζ. Turning now to the problem of the behavior of $\varphi(\zeta)$ and $f(\zeta)$ for very large values of $|\zeta| = \frac{z}{|L|}$, we shall begin with the case of great instability; i.e., large negative values of ζ. The asymptotic behavior of $\varphi(\zeta)$ and $f(\zeta)$ as $\zeta \to -\infty$ may be studied by considering large values of $z \gg |L|$ for given $q > 0$, u_* and g/T_0 (i.e., for given $L < 0$) or by studying the limiting process $u_* \to 0$ (that is, $L < 0$) for given $q > 0$, g/T_0 and z. The latter limiting process clearly corresponds to the approach to purely thermal turbulence in conditions of "free convection," characterized by the presence of unstable stratification with $q > 0$ and the absence of mean horizontal velocity and the associated friction on the underlying surface. In this case, the turbulence obtains its energy not from the mean motion, but from the temperature instability. In addition, it has the character of an assemblage of thermal jets arising at individual points of the underlying surface, and rather weakly intermixed. Further, if the existing horizontal mixing nevertheless is sufficient to ensure the invariance of the probability distributions of the flow variables with respect to shifts and rotations in the horizontal plane, then such turbulence is a particular case of the turbulent flows considered in this chapter. These flows are characterized by only two parameters: g/T_0 and $q/c_p\rho_0$ (since in "pure" convection, $u_* = 0$). However, from the similarity viewpoint, the turbulent regime at heights $z \gg |L|$ for given u_* cannot differ from the regime when z is fixed and not too great but u_* is very small. Thus in the unstable case, turbulence at great heights is always determined primarily by thermal factors; i.e., the statistical characteristics of the temperature field will depend in this case on g/T_0 and $q/c_p\rho_0$ but not on u_*. From this viewpoint, in the limiting case $\zeta \to -\infty$, it is simplest to consider first, the behavior of the functions $f_1(\zeta)$ and $g_1(\zeta) = f_1'(\zeta)$ which characterizes the mean temperature distribution, and only then turn to the functions $g(\zeta)$ [or $\varphi(\zeta) = \zeta g(\zeta)$] and $f(\zeta)$ [since in the study of the horizontal velocity distribution we cannot appeal to the limiting case of "pure" thermal convection, in which, generally speaking, there is no mean horizontal velocity at all].

Since it is impossible to formulate any length scale from the parameters g/T_0 and $q/c_p\rho_0$, the turbulent regime in conditions of free convection must be self-preserving (see, for example, the investigation of convective jets in Sect. 5.9). In particular, it is easy to verify that Eq. (7.16) and the second of Eqs. (7.24) will not contain u_* only if $g_1(\zeta) \sim \zeta^{-4/3}$ and $f_1(\zeta) \sim \zeta^{-1/3} +$ const. Thus

$$g_1(\zeta) = -\frac{C_1 \varkappa^{4/3}}{3} \zeta^{-4/3}, \qquad f_1(\zeta) = \text{const} + C_1 \varkappa^{4/3} \zeta^{-1/3} \qquad (7.35)$$

when $\zeta \ll -1$

(the factor $\varkappa^{1/3}$ is added here for convenience). In other words,

$$\begin{aligned}\frac{\partial \overline{T}}{\partial z} &= -\frac{C_1}{3}\left(\frac{q}{c_p\rho_0}\right)^{2/3}\left(\frac{g}{T_0}\right)^{-1/3} z^{-4/3},\\ \overline{T}(z) &= T_\infty + C_1\left(\frac{q}{c_p\rho_0}\right)^{2/3}\left(\frac{gz}{T_0}\right)^{-1/3}\end{aligned} \tag{7.36}$$

when $z \gg |L|$, $L < 0$. Equations (7.36) were first pointed out by Prandtl (1932a) and later, independently, by Obukhov (1946). They are completely analogous to the equations deduced by Zel'dovich (1937), Eqs. (5.103)–(5.104) for the temperature distribution in the center of convective ascending jets and differ from them only in that here we are concerned not with convection above a heated sphere or cylinder, but with convection above a large heated plane surface. Later, these equations were discussed in detail in several works [see, for example, Monin and Obukhov (1953; 1954), Priestley (1954; 1955; 1956), Kazanskiy and Monin (1958)]. From them it follows, in particular, that in unstable stratification, the temperature distribution at great heights tends to isothermal; this is due to the presence of very intensive mixing (which increases with height as a result of the appearance of larger and larger eddies with length scales comparable with the distance from the underlying surface), leading to equalization of the temperature. The eddy thermal diffusivity in conditions of free convection from Eq. (7.36) is equal to

$$K_T = -\frac{q}{c_p\rho_0 \dfrac{\partial \overline{T}}{\partial z}} = \frac{3}{C_1}\left(\frac{q}{c_p\rho_0}\frac{g}{T_0}\right)^{1/3} z^{4/3}; \tag{7.37}$$

with increase of the distance from the earth's surface, this increases rapidly.

It is now possible to present arguments which give quite definite asymptotic form to the functions $g(\zeta)$, $\varphi(\zeta) = \zeta g(\zeta)$ and $f(\zeta)$ as $\zeta \to -\infty$. Since the limiting process $\zeta \to -\infty$ is equivalent to $u_* \to 0$ (with the remaining parameters and z fixed), the turbulence regime with $\zeta \ll -1$ cannot depend on u_*. However, it is impossible to formulate any dimensionless combination from $q/c_p\rho_0$, g/T_0, ρ_0 and z; thus all the dimensionless characteristics defined by these parameters must have a constant (universal) value. In particular, it is natural to assume that one such dimensionless parameter is the ratio of the eddy diffusivities for heat and momentum $\alpha = K_T/K$. Thus, in free convection conditions

$$\frac{K_T}{K} = \alpha_{-\infty} = \text{const}, \qquad K = \frac{K_T}{\alpha_{-\infty}} = \frac{3}{C_1 \alpha_{-\infty}} \left(\frac{q}{c_p \rho_0} \frac{g}{T_0} \right)^{1/3} z^{4/3} \quad (7.38)$$

[where $\alpha_{-\infty} = \lim_{\zeta \to -\infty} \alpha(\zeta)$]. However, in this case, it is clear that

$$g(\zeta) = -\frac{C_2}{3} \zeta^{-4/3}, \qquad \varphi(\zeta) = -\frac{C_2}{3} \zeta^{-1/3} \quad \text{when} \quad \zeta \ll -1, \quad (7.39)$$

$$f(\zeta) = C_2 \zeta^{-1/3} + \text{const} \qquad \text{when} \quad \zeta \ll -1, \quad (7.39')$$

where $C_2 = C_1 \varkappa^{4/3} \alpha_{-\infty}$. In other words,

$$\begin{aligned} \frac{\partial \bar{u}}{\partial z} &= \frac{C_2}{3 \varkappa^{4/3}} u_*^2 \left(\frac{q}{c_p \rho_0} \frac{g}{T_0} \right)^{-1/3} z^{-4/3}, \\ \bar{u}(z_2) - \bar{u}(z_1) &= -C_2 \varkappa^{-4/3} u_*^2 \left(\frac{q}{c_p \rho_0} \frac{g}{T_0} \right)^{-1/3} \left(z_2^{-1/3} - z_1^{-1/3} \right) \end{aligned} \quad (7.40)$$

when $q > 0$, $z_1 \gg |L|$, $z_2 \gg |L|$. Equations (7.40) were also obtained first (in slightly different form) by Prandtl (1932a) [see also Obukhov (1946), Monin and Obukhov (1954)]. With increase of z, the mean velocity $\bar{u}(z)$ approaches a constant value for the same reasons which, in free convection, cause the approach to isothermal conditions as $z \to \infty$. We note that u_* now appears in Eqs. (7.40), since $\bar{u}(z) = 0$ when $u_* = 0$, and therefore the function $\bar{u}(z)$ cannot be independent of u_*.[7] The Richardson numbers Ri and Rf will also depend on u_*; in free convection conditions these numbers, by Eqs. (7.39), (7.19), and (7.20), are equal to

$$\text{Ri} = -\frac{3}{C_2 \alpha_{-\infty}} \left(\frac{z}{|L|} \right)^{4/3}, \qquad \text{Rf} = \alpha \, \text{Ri} = -\frac{3}{C_2} \left(\frac{z}{|L|} \right)^{4/3}. \quad (7.41)$$

[7]This places the derivation of Eqs. (7.38)–(7.41) on a slightly weaker footing than the derivation of Eqs. (7.35)–(7.37). In fact, there is no vertical momentum transfer under idealized free convection conditions, and, therefore, strictly speaking, there is also no eddy viscosity K. Consequently, it is theoretically possible, for example, that $K_T/K \to \infty$ as $u_* \to 0$. However, no physical reasons exist to suppose that the contribution to the eddy viscosity of the velocity fluctuations produced by purely thermal factors will vanish. Thus, at present, it seems reasonable to consider Eq. (7.38) with $0 < \alpha_{-\infty} < \infty$ as a plausible hypothesis which requires experimental verification.

Thus it is clear that with increase of z, both Ri and Rf increase in absolute value without bound (remaining negative the entire time).

Let us turn now to the second limiting case $\zeta \to +\infty$. The investigation of the asymptotic behavior of the functions $\varphi(\zeta)$ and $f(\zeta)$ for large positive ζ corresponds to the examination of the mean velocity profile $\bar{u}(z)$ for large z in the case of stable stratification (fixed $L > 0$) or to the examination of the case of very small positive L for fixed z (i.e., very sharp temperature inversions). However, in a sharp inversion with a vanishingly small wind, the turbulence becomes degenerate and the motion of the medium takes on a very special character. In fact, in conditions of highly stable stratification, the existence of large-scale turbulent fluctuations becomes impossible (since these fluctuations would have to expend too much energy in performing work against the gravity forces), and turbulence can exist only in the form of small-scale eddies (this can also be explained by the statement that in this case large-scale waves are stable and do not undergo transition to turbulence). In the case of still greater stability, even small-scale turbulence will become practically impossible, and the fluctuating motion of the medium will probably appear in the form of a random collection of internal waves. In any case, it is clear that high stability considerably impedes turbulent exchange between different layers of the fluid and hence turbulence will take on a local character; the characteristics of turbulent exchange (for example, the eddy viscosity and the number $\mathrm{Rf} = \frac{K}{\varkappa u_* L}\Big)$ for $\zeta \gg 1$ (i.e., for fixed $L > 0$ but large height $z \gg L$, or for fixed height z but very small $L > 0$) cannot depend explicitly on the distance z from the underlying surface. The latter deduction can also be confirmed by the following argument: according to Richardson's deduction (see Sect. 6.5), the flux Richardson number Rf cannot exceed unity under any steady conditions; i.e., the maximum possible value Rf_{cr} of this number $\lesssim 1$ (apparently Rf_{cr} is considerably less than unity, but, at present, this is not important for our purposes). On the other hand, it is natural to assume that the variation of Rf as $\zeta = \frac{z}{L}$ increases (i.e., for fixed z, with increase of stability) must be monotonic: physical principles which could lead to a situation where Rf began to decrease once again with increase of stability, are difficult to imagine. However, if $\mathrm{Rf}(\zeta)$ increases monotonically with increase of ζ and at the same time cannot exceed some value Rf_{cr}, as $\zeta \to \infty$, it must approach some limiting value R (which it is natural to identify with Rf_{cr}).

Thus, in the case of stable stratification, a universal value R must exist such that

$$\mathrm{Rf} \approx R = \text{const}, \quad K \approx \varkappa u_* LR \text{ when } z \gg L. \tag{7.42}$$

But from this it follows that

$$\begin{aligned} g(\zeta) &= \frac{1}{R} = \text{const}, \quad \varphi(\zeta) = \frac{\zeta}{R} = C_3\zeta, \\ f(\zeta) &= \text{const} + \frac{\zeta}{R} = \text{const} + C_3\zeta \text{ when } \zeta \gg 1, \end{aligned} \tag{7.43}$$

where $C_3 = 1/R = 1/\mathrm{Rf}_{\mathrm{cr}}$, and, of course,

$$\bar{u}(z) = \frac{u_*}{\varkappa RL} z + \text{const} = C_3 \frac{\varkappa T_*}{u_*} \frac{gz}{T_0} + \text{const}. \tag{7.44}$$

Thus, at great heights, in a stable medium, the mean velocity increases linearly with height, while the constant gradient of this velocity is determined uniquely by an "external" parameter, which varies from case to case, e.g., the parameter $\frac{\varkappa T_*}{u_*} = -\frac{q}{c_p \tau}$, which in fact is the same as the ratio of the turbulent heat flux to the turbulent shear stress. This result was pointed out by Obukhov (1946) [see also Monin and Obukhov (1954)].

According to Eqs. (7.29), (7.30), (7.39), (7.39′), and (7.43), the universal functions $\varphi(\zeta)$ and $f(\zeta) - f\left(\pm\frac{1}{2}\right)$ [the term $-f\left(\pm\frac{1}{2}\right)$ is added here to avoid indeterminacy in the choice of the reference level of the function $f(\zeta)$] must have a general form close to that given in Figs. 47 and 48. In the following section this conclusion is well supported by direct measurements in the surface layer of the atmosphere.

Now we shall discuss briefly the general form of the functions $g_1(\zeta)$, $f_1(\zeta)$ and $g_2(\zeta)$, $f_2(\zeta)$ which describe the vertical profiles of temperature and concentration of a passive admixture (humidity). According to Sect. 5.7

$$f_1(\zeta) \approx \frac{1}{\alpha_0} \ln|\zeta| + \text{const} \text{ when } |\zeta| \ll 1; \tag{7.45}$$

here $\alpha_0 = \alpha(0)$ is a constant close to unity (which was denoted simply by α in Sect. 5.7). On the other hand, as we have seen, at large negative ζ in "free convection" conditions, the behavior of the function $f_1(\zeta)$ is determined by Eq. (7.35); in this region the function $f_1(\zeta)$ differs from $f(\zeta)$ only by a constant multiplier $1/\alpha_{-\infty}$ (the exact value of which is still unknown, see below, Sect. 8.2). Later, we shall see that when $\zeta \leqslant 0$, the transition from the "forced convection" conditions, characterized by logarithmic profiles of velocity and temperature, to the "free convection" conditions defined by Eqs. (7.35)–(7.41) is comparatively sharp; i.e., it occurs over a small range of values of ζ. Thus the behavior of the function $f_1(\zeta)$ on the whole positive semiaxis is described fairly completely by the values of the constants $\varkappa$, C_1 and α_0 (or $\varkappa$, $C_2 = C_1\varkappa^{4/3}\alpha_{-\infty}$, α_0 and $\alpha_{-\infty}$). In particular, whether or not it is possible to consider the function $f_1(\zeta)$ as similar to the function $f(\zeta)$ when $\zeta \leqslant 0$ is probably determined first by how close the values of the constants α_0 and $\alpha_{-\infty}$ are to each other. The general form $f_1(\zeta)-f_1\left(-\frac{1}{2}\right)$ for $\zeta \leqslant 0$, however, certainly cannot differ from that shown in Figs. 47 and 48 as the form of the functions $f(\zeta)-f\left(-\frac{1}{2}\right)$.

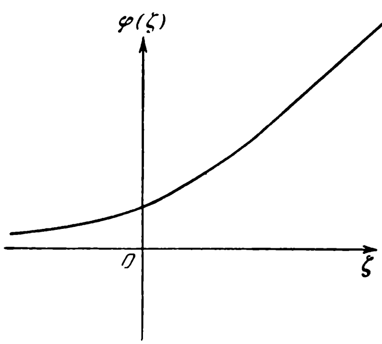


FIG. 47. Schematic form of the universal function $\varphi(\zeta)$.

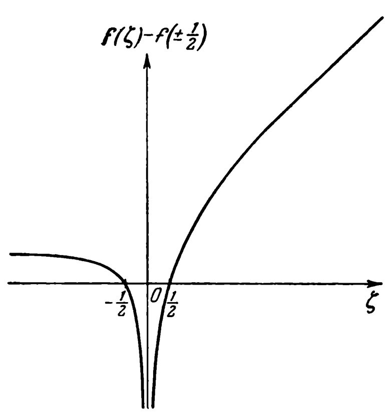


FIG. 48. Schematic form of the universal function $f(\zeta)-f\left(\pm\frac{1}{2}\right)$.

The behavior of the function $f_1(\zeta)$ when $\zeta > 0$ is somewhat more complicated. Since

$$f_1(\zeta) = \int^{\zeta} \frac{f'(\xi)}{\alpha(\xi)}\, d\xi + \text{const}, \tag{7.46}$$

the problem of the form $f_1(\zeta)$ is closely connected with that of the dependence of $\alpha(\zeta)$ on the stability parameter $\zeta = \frac{z}{L}$ (or on the numbers Rf and Ri). If we assume as is done most often in meteorological literature that $\frac{K_T}{K} = \alpha =$ const, then the function $f_1(\zeta)$ will agree with $f(\zeta)$ up to a constant multiplier; if $\alpha = 1$ (as is also assumed), then $f_1(\zeta) \equiv f(\zeta)$, up to additive constants which play no part. For small positive ζ, the function $f_1(\zeta)$ is given by the logarithmic formula (7.45); for somewhat larger ζ, by the "logarithmic + linear" equation

$$f_1(\zeta) = \text{const} + \frac{1}{\alpha_0}(\ln \zeta + \beta_1 \zeta) \tag{7.47}$$

[with coefficient β_1 which may differ from the corresponding coefficient in Eq. (7.33)]. If we assume that $\alpha(\zeta)$ varies comparatively little over a fairly wide range of positive values of ζ, including also part of the region in which there is a linear velocity profile, then, for sufficiently large values of ζ within this range, the temperature profile also will be described fairly well by a linear equation of the form

$$f_1(\zeta) = \text{const} + \frac{\zeta}{\alpha R}, \qquad \bar{T}(z) = \frac{1}{\alpha R}\left(\frac{\varkappa T_*}{u_*}\right)^2 \frac{gz}{T_0} + \text{const}, \tag{7.48}$$

where α is a constant of the order of unity. However, observations of turbulence in a highly stable sea and associated measurements in laboratory water flows definitely show that in a very stable medium $\alpha(\zeta)$ takes very small values (see below, Sect. 8.2, and, in particular, Fig. 69). In other words, for very great stability, the eddy diffusivity for heat K_T is considerably less than the eddy diffusivity for momentum K. According to Stewart (1959), physical arguments explaining the reason for this are difficult. The case of an extremely stable medium may be represented by a layer of heavy fluid (for example, water), above which there is a considerably lighter fluid (for example, air). Then turbulent motion in the lower fluid will lead to disturbances of the free surface and to the appearance of individual "splashes," penetrating into the upper fluid, and then falling again under the action of the gravity forces. The penetration of "water" into the "air" will set up fluctuations of pressure in the "air" causing a definite exchange of momentum between the two media; at the same time, heat exchange is, practically speaking, absent (we recall that we have agreed to ignore molecular thermal conductivity). Thus we may assume that for very high stability (possibly greater than that which is actually observed in atmospheric inversions), the eddy viscosity K will have a finite value, while K_T will be close to zero. However, if this is true, then it follows that at very large positive $\zeta = \frac{z}{L}$, the temperature profile $\overline{T}(z)$ will be considerably steeper than the velocity profile $\overline{u}(z)$; it follows from the fact that $K_T \to 0$ as $\zeta \to \infty$, that the steepness of the temperature profile increases without bound as z/L increases. Consequently, the general form of the functions $f_1(\zeta) - f_1\left(\frac{1}{2}\right)$ and $\varphi_1(\zeta) = \zeta f_1'(\zeta)$, at very large positive ζ will differ from the general form of the functions $f(\zeta) - f\left(\frac{1}{2}\right)$ and $\varphi(\zeta)$ [see Figs. 49 and 50]. Also connected with this is the fact that the Richardson number $\mathrm{Ri} = \frac{\mathrm{Rf}}{\alpha}$ in conditions of high stability, may apparently assume very large values (since if $\alpha \to 0$ as $\mathrm{Rf} \to \mathrm{Ri}_{cr}$, then $\mathrm{Ri} \to \infty$); thus there is no limiting "critical" value of Ri (unhke the case of Rf).

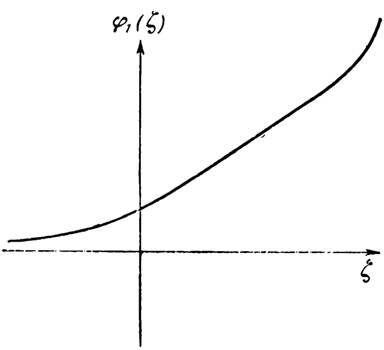


FIG. 49 Schematic form of the universal function $\varphi_1(\zeta)$.

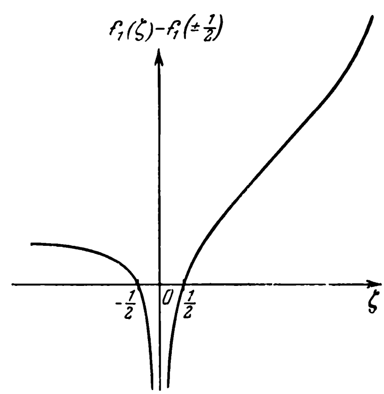


FIG. 50. Schematic form of the universal function $f_1(\zeta) - f_1\left(\pm \frac{1}{2}\right)$.

We consider now the functions $f_2(\zeta)$ and $g_2(\zeta) = f_2'(\zeta)$. As long as it was widely held that $K = K_T$ (i.e., that $\alpha \equiv 1$), it was completely natural to assume also that $K_\vartheta = K = K_T$, that is, the function $f_2'(\zeta)$ did not differ from $f'(\zeta)$ and $f_1'(\zeta)$. However, abandoning the assumption that eddy diffusivities for momentum and for heat are equal, we are immediately faced with the problem of the value of the eddy diffusivity for an admixture K_ϑ, which determines the form of the functions $f_2(\zeta)$ and $g_2(\zeta)$. Below, we shall see that the data on the values of K_ϑ and $f_2(\zeta)$ are even poorer than those on the values of K_T and $f_1(\zeta)$. Therefore, it is impossible to draw any truly reliable conclusions from them. Thus it only remains to have recourse to physical intuition. In this connection, we may note that in the paper by Priestley and Swinbank (1947), one of the first in meteorological literature in which nonequality of the different eddy diffusivities was discussed seriously, it was proposed that in view of the special role of temperature fluctuations in convective motions (expressed, for example, by the fact that particles that are warmer than the surrounding medium prefer to move upward), the eddy thermal diffusivity K_T may have one value, while the eddy diffusivities for momentum K and humidity K_ϑ will coincide at another value. However, the reasoning of these investigators is not very clear, and their viewpoint is far from being shared by all [see, for example, Robinson (1951)]; their arguments seem especially unconvincing with regard to the surface layer of the atmosphere. The more convincing view is that by reason of the similarity of the physical mechanism of heat and humidity transfer (or the transfer of some other passive admixture), which takes place only by means of direct mixing of the air masses, the eddy diffusivities K_T and K_ϑ will be equal while the eddy viscosity K is also affected by the pressure fluctuations and may differ from $K_T = K_\vartheta$. This latter view is held at present by the majority of specialists [see, for example, Ellison (1957), Charnock (1958a)]. Here, we should also take into account that for not too great temperature inhomogeneity, the selective effect of the buoyancy on the fluid elements will produce no great effect on the turbulent heat transfer. Therefore, so long as we do not know of any other physical exchange mechanism that would affect heat and humidity transfer in different ways, there is no reason to reject the assumption that $K_T = K_\vartheta$ and, consequently, that $g_2(\zeta) = g_1(\zeta)$, $f_2(\zeta) = f_1(\zeta) + \text{const}$. Of course, this assumption needs careful verification, using the data of sufficiently accurate special observations in nature and in the laboratory, which only began to appear recently.

7.4 Further Discussion of the Universal Functions; Interpolation Formulas and Semiempirical Formulas

In Sect. 7.3 we analyzed the general behavior of the universal functions $f(\zeta)$, $\varphi(\zeta) = \zeta f'(\zeta)$ and $f_1(\zeta)$, $\varphi_1(\zeta) = \zeta f_1'(\zeta)$, and displayed the asymptotic formulas which determine the behavior of these functions for values of the arguments close to zero, large and positive, and large and negative. However, in many cases, it is convenient to have explicit formulas giving the values of the functions for all $\zeta = \frac{z}{L}$. Such explicit formulas may be obtained most simply by interpolation, using the known asymptotic laws, and also taking into account the experimental data on the functions in the intermediate range of moderate values of ζ. For this purpose we may also use one of the variants of the semiempirical theory of turbulence, generalizing it to the case of a stratified medium, or we may simply attempt to select expressions for the turbulence characteristics on the basis of data only, without any reference to the theory. A considerable amount of literature has been devoted to these three approaches to the problem of describing the turbulence characteristics in a stratified medium. The literature comprises at least several hundred papers, but here we shall deal only briefly with certain of the relevent results, paying particular attention to more recent works.

First, we consider the behavior of the functions $f(\zeta)$ and $f_1(\zeta)$ in the region of negative ζ. As we have already observed, for small negative ζ, both these functions must be approximated by the logarithmic function. For values of ζ somewhat greater in absolute value, we may use the "logarithmic + linear" formula (7.33) or (7.47); while for very large negative ζ, the functions $f(\zeta)$ and $f_1(\zeta)$ will approach a constant value, their deviation from the corresponding constants being damped like $\zeta^{-1/3}$ [see Eqs. (7.35) and (7.39′)]. Above, we also recalled that according to the data, the transition from the "forced convection" regime described by the logarithmic functions $f(\zeta)$ and $f_1(\zeta)$, to the "free convection" regime corresponding to Eqs. (7.35) and (7.39′) apparently takes place without any considerable intermediate region (and, moreover, for quite small values of $|\zeta| = z/|L|$, of the order of a tenth or several hundredths; see below, Sects. 8.1 and 8.2). Following this, Kazanskiy and Monin (1958) assumed that the values of $f(\zeta)$ on the whole negative semiaxis will be described fairly well by the simplest interpolation formula:

$$f(\zeta)=\begin{cases} \ln|\zeta| & \text{when } \zeta_1 \leqslant \zeta < 0, \\ \ln|\zeta_1| + C_2(\zeta^{-1/3} - \zeta_1^{-1/3}) & \text{when } \zeta \leqslant \zeta_1. \end{cases} \tag{7.49}$$

Similarly, for $f_1(\zeta)$

$$f_1(\zeta)=\begin{cases} \frac{1}{\alpha_0}\ln|\zeta| & \text{when } \zeta_2 \leqslant \zeta < 0, \\ \frac{1}{\alpha_0}\ln|\zeta_2| + \frac{C_2}{\alpha_{-\infty}}(\zeta^{-1/3} - \zeta_2^{-1/3}) & \text{when } \zeta \leqslant \zeta_2, \end{cases} \tag{7.50}$$

where $C_1\varkappa^{4/3} = C_2/\alpha_{-\infty}$. The values of the dimensionless constants α_0, $\alpha_{-\infty}$ and ζ_2/ζ_1, which determine the extent of difference between the functions $f(\zeta)$ and $f_1(\zeta)$ must be refined further on the basis of the data. At present, it is possible to assert only that the coefficient α_0 is close to unity (but the assumption which is often allowed that all these constants are equal to unity is not justified). As for the constants C_2, $C_2/\alpha_{-\infty} = C_1\varkappa^{4/3}$ and ζ_1 which occur in Eq. (7.49), their values will be discussed in greater detail in the following section. Applied to the functions $\varphi(\zeta) = \zeta f'(\zeta)$, and $\varphi_T(\zeta) = \alpha_0\zeta f_1'(\zeta) = \alpha_0\varphi_1(\zeta)$, Eqs. (7.49)–(7.50) clearly lead to the following relationships:

$$\varphi(\zeta)=\begin{cases} 1 & \text{when } \zeta_1 \leqslant \zeta \leqslant 0, \\ -\frac{C_2}{3}\zeta^{-1/3} & \text{when } \zeta < \zeta_1 \end{cases} \tag{7.51}$$

and similarly

$$\varphi_T(\zeta)=\begin{cases} 1 & \text{when } \zeta_2 \leqslant \zeta \leqslant 0, \\ -\frac{C_2'}{3}\zeta^{-1/3} & \text{when } \zeta < \zeta_2, \end{cases} \tag{7.52}$$

where $C_1\varkappa^{4/3}\alpha_0 = C_2' = \frac{\alpha_0}{\alpha_{-\infty}} C_2$. Naturally, these functions will be discontinuous since the functions (7.49) and (7.50) are continuous but not differentiable at the points ζ_1 and ζ_2, respectively.

In spite of the presence of a discontinuity in the derivative, in many respects the functions (7.49) and (7.50) are quite sufficient to describe the wind and temperature profiles in the surface layer, with arbitrary, unstable stratification. However, when it is desirable to have functions $f(\zeta)$ and $f_1(\zeta)$ with continuous derivatives, we can, for example, following Priestley (1960a), reject the determination of ζ_1 and ζ_2 from the data, and instead write $\zeta_1 = -\left(\frac{C_2}{3}\right)^3$ and $\zeta_2 = -\left(\frac{C_2'}{3}\right)^3$ [the functions (7.51) and (7.52) will then, of course, be continuous]. The same result may also be obtained if, following Kazanskiy and Monin (1958), we retain the hnear term in the equations for $f(\zeta)$ and $f_1(\zeta)$ for small values of ζ; i.e., if we assume, for example, that $f(\zeta) = \ln|\zeta| + \beta\zeta$, $\varphi(\zeta) = 1 + \beta\zeta$ for $0 > \zeta > \zeta_1$, and $f(\zeta) = \ln|\zeta_1| + \beta\zeta_1 + C_2(\zeta^{-1/3} - \zeta_1^{-1/3})$ $\varphi(\zeta) = -\frac{C_2}{3}\zeta^{-1/3}$ for $\zeta < \zeta_1$. Here β may be chosen in such a way that the function $1 + \beta\zeta$ gives a good approximation to the experimental values of $\varphi(\zeta)$ for small ζ, and ζ_1 is then determined from the conditions $1 + \beta\zeta_1 = -\frac{C_2}{3}\zeta_1^{-1/3}$, which ensures the continuity of $\varphi(\zeta)$ at the point $\zeta = \zeta_1$. On the other hand, following Priestley (1960a) and taking both constants β and ζ_1 to be undetermined, we may choose these constants in such a way that the function

$$\varphi(\zeta) = \begin{cases} 1 + \beta\zeta & \text{when } \zeta_1 \leqslant \zeta \leqslant 0, \\ -\frac{C_2}{3}\zeta^{-1/3} & \text{when } \zeta < \zeta_1 \end{cases} \tag{7.53}$$

will have a continuous derivative at the point ζ_1. Then

$$f(\zeta) = \int^{\zeta} \frac{\varphi(\xi)}{\xi} d\xi$$

will have two continuous derivatives for all ζ. In general, writing

$$\varphi(\zeta) = \begin{cases} 1 + \beta_1\zeta + \ldots + \beta_n\zeta^n & \text{when } \zeta_1 \leqslant \zeta \leqslant 0, \\ -\frac{C_2}{3}\zeta^{-1/3} & \text{when } \zeta < \zeta_1, \end{cases} \tag{7.54}$$

we can ensure that the function (7.54) has n continuous derivatives and, consequently, that $f(\zeta)$ has an $(n+1)$th continuous derivative. However, to do this, it is necessary only to equate the values of the functions $1 + \beta_1\zeta + \ldots + \beta_n\zeta^n$ and $-\frac{C_2}{3}\zeta^{-1/3}$ and their first n

derivatives to each other at the point $\zeta=\zeta_1$, and then evaluate the $n+1$ unknowns $\zeta_1, \beta_1, \ldots, \beta_n$ from the system of $n+1$ equations thus obtained. It is not difficult to obtain explicit formulas for the solution of this system of equations for any n; according to these formulas, if $n \to \infty$, then $\zeta_1 \to \infty$, but the coefficients β_k all tend to finite values

$$\lim_{n\to\infty} \beta_k = \frac{a^{3k}}{(3k+1)\,k!\,C_2^{3k}}, \text{ where } a = 3 \lim_{n\to\infty} \frac{3^n n! \sqrt[3]{n}}{4\cdot 7 \ldots (3n+1)} \approx 2.7 \quad (7.55)$$

[see Priestley (1960a)]. The hmiting values (7.55) of the coefficients β_k correspond to an infinitely differentiable and, moreover, analytic limiting interpolation function

$$\varphi(\zeta) = \frac{C_2}{a\zeta^{1/3}} \int_0^{a\zeta^{1/3}/C_2} e^{t^3}\, dt. \quad (7.56)$$

Completely analogous interpolation formulas may also be formulated for the function φ_T, the sole difference being that the coefficient C_2 must now be replaced by $C_2' = \frac{\alpha_0}{\alpha_{-\infty}} C_2 = C_1 \varkappa^{4/3} \alpha_0$. Thus, giving n the values 1, 2, . . . and ∞ we obtain an infinite number of interpolation formulas of the form (7.54) for the functions $\varphi(\zeta)$ and $\varphi_T(\zeta)$. It is found that these interpolation formulas [including the limiting formula (7.56)] differ very little from each other and describe the existing data (which are characterized by considerable scatter) in practically the same manner. To illustrate the closeness of the various interpolation formulas to each other, Fig. 51 shows a diagram borrowed from Priestley (1960a). This diagram represents the dimensionless heat flux

$$\overset{\bullet}{q} = \frac{q}{c_p \rho_0 \left(\dfrac{g}{T_0}\right)^{1/2} \left|\dfrac{\partial \overline{T}}{\partial z}\right|^{3/2} z^2} = \frac{\varkappa^2 \alpha_0^{3/2}}{[\varphi_T(\zeta)]^{3/2} \zeta^{1/2}}$$

(which becomes constant in conditions of free convection), obtained by using interpolation formulas of the type (7.54) for $\varphi_T(\zeta)$ with n = 0, 2 and 5. In the calculations used as a basis for Fig. 51, it was assumed that $\alpha_0 = 1$ (and, in general, $\alpha(\zeta) \equiv 1$), $\varkappa = 0.4$ and $C_2' = C_2 = 0.95$ (this last value corresponds to certain data on temperature profiles in the atmosphere); here, Ri = Rf = $\frac{\zeta}{\varphi_T(\zeta)}$ is

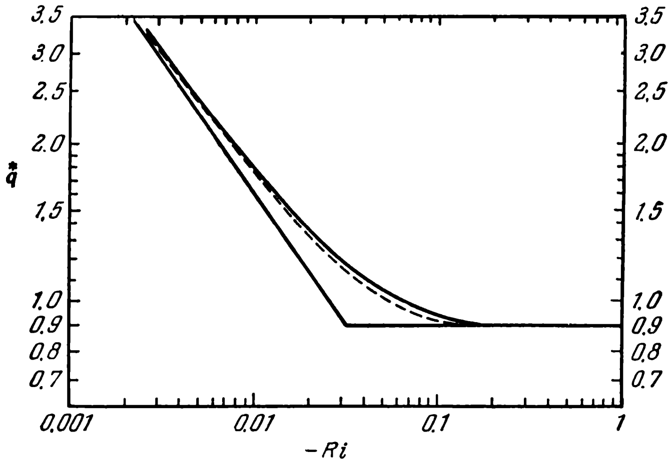


FIG. 51. Dependence of the dimensionless heat flux $\overset{*}{q}$ on the Richardson number Ri. The broken line corresponds to the function $\varphi_T(\zeta)$ of form (7.52) with $\zeta_2 = -(C_2'/3)^3$; the dotted and continuous curves correspond to interpolation formulas of the form (7.54) for $\varphi_T(\zeta)$ with $n = 2$ and $n = 5$.

plotted on the abscissa (if, instead of this, we used the argument $\zeta = \frac{z}{L}$, the divergence between the curves would be even less). In general, with $C_2 = 0.95$, the divergence between the values of $\varphi(\zeta)$ corresponding to the "zero-order" interpolation formula (7.51) [with $\zeta_1 = -\frac{C_2^3}{27}$] and the "limiting" formula (7.56), nowhere exceeds 15%, while a deviation of close to 15% is observed only for ζ close to ζ_1 and is basically connected with the presence on the curve (7.51) of the jump discontinuity at $\zeta = \zeta_1$. The first smoothing, with the aid of Eq. (7.53), immediately results in values of $\varphi(\zeta)$ which nowhere differ from the values of Eq. (7.56) by more than 5%. Thus it is clear that when dealing with observations in the atmosphere, the use of interpolation formulas of the type (7.54) that are more complicated than Eq. (7.53) is not greatly significant. From this viewpoint, there is hardly sufficient justification for Webb's proposal (1960) to use an interpolation formula of the form

$$\varphi(\zeta) = \begin{cases} 1 + \beta\zeta & \text{for} \quad \zeta_1 \leqslant \zeta \leqslant 0, \\ -\frac{C_2}{3}\zeta^{-1/3} + C\zeta^{-4/3} & \text{for} \quad \zeta < \zeta_1, \end{cases} \tag{7.57}$$

where the coefficient C_2 is determined from the data, and β, C, and

ζ_1 are chosen from the condition that the function (7.57) should have two continuous derivatives at the point $\zeta = \zeta_1$ (or else ζ_1 is found from observation and β, C_2, and C are chosen according to the aforesaid condition).

Until now, the case of positive ζ has been studied very little experimentally. Therefore, it seems reasonable to use the simplest interpolation formula for the functions $f(\zeta)$ and $\varphi(\zeta) = \zeta f'(\zeta), 0 < \zeta < \infty$ of the form

$$f(\zeta) = \ln \zeta + \beta_1' \zeta, \; \varphi(\zeta) = 1 + \beta_1' \zeta \tag{7.58}$$

which gives correct results both close to zero and for large positive values of ζ.

A different type of interpolation formula may be obtained if, following Ellison (1957), we take $\zeta = \frac{z}{L} \to 0$ (i.e., at neutral stratification) $\mathrm{Rf} \to 0$ and $K \approx \varkappa u_* z$, while for $|\zeta| \gg 1$, (i.e., for free convection, $-\mathrm{Rf} \sim |\zeta|^{1/3}$ and $K = \varkappa u_* L \cdot \mathrm{Rf} \sim \varkappa u_* z |\zeta|^{1/3}$ [if we assume that $0 < \alpha_{-\infty} < \infty$; see Eq. (7.38) and (7.41)], i.e., $\frac{K}{\varkappa u_* z} \approx (-\sigma \mathrm{Rf})^{1/4}$, where σ is some constant [equal, as may easily be seen, to $(3/C_2)^3$]. Starting from this, we may expect that for the whole range $0 \geqslant \mathrm{Rf} > -\infty$ (i.e., along the semiaxis $0 \geqslant \zeta > -\infty$) the dependence of K on Rf will be described fairly well by an equation of the form

$$K = \varkappa u_* z (1 - \sigma \mathrm{Rf})^{1/4}, \tag{7.59}$$

which gives correct results for both small and large values of Rf. At the same time, it is found that when $\sigma = 1/\mathrm{Rf}_{cr}$, Eq. (7.59) gives asymptotically correct results in the limiting case of very stable stratification as well. In fact, it is clear from Eq. (7.59) that as Rf increases, it can never exceed the value $1/\sigma$ and that $K/z \to 0$ as $\mathrm{Rf} \to 1/\sigma$ (i.e., as $\zeta = \frac{z}{L} \to +\infty$), in complete accordance with Eqs. (7.42). Thus, it may be expected that by using Eq. (7.59), which contains a single empirical constant σ, we shall be able to give an approximate description of the mean velocity profile in a turbulent stratified medium under any stratification. It is difficult to depend on the value of $\sigma = 1/\mathrm{Rf}_{cr}$ being identical to the value $\sigma = 27 C_2^{-3}$, which leads to true asymptotic behavior of K (Rf) at large negative values of Rf. However, since at present the values of Rf_{cr} and of C_2 are known only very imprecisely, we do not as yet have enough justification for using different values of σ in Eq. (7.59) for $\mathrm{Rf} < 0$ and for $\mathrm{Rf} > 0$.

Equation (7.59) may also be obtained, beginning with the simplest attempt at generalizing the semiempirical theory of turbulence to a stratified medium. In fact, as we have seen already in Sect. 6.5, the semiempirical equations for the turbulent energy balance in a stratified medium, may be transformed to

$$K = u_* l_1 (1 - \sigma \,\mathrm{Rf})^{1/4}, \tag{7.60}$$

where $l_1 = \varkappa z \lambda(\mathrm{Rf})$ is the "length scale of turbulence," suitably defined. If it is now assumed that $\lambda(\mathrm{Rf}) \equiv 1$, i.e., that the scale l_1 for any stratification is given by one and the same equation $l_1 = \varkappa z$ (which is true in the case of neutral stratification), then Eq. (7.59) follows immediately. In precisely this manner, this equation was first deduced by Obukhov (1946), who used it to investigate the asymptotic behavior of the functions $f(\zeta)$ and $\varphi(\zeta)$ at large positive and large negative values of ζ. In recent years the various methods of deducing Eq. (7.59) have been discussed in detail by a number of researchers [see, for example, Yamamoto (1959), Panofsky (1961a), Priestley (1961), Sellers (1962), Syono and Hamuro (1962), Klug (1963), and Rijkoort (1968)]. However, since all the deductions proposed use certain arbitrary hypotheses, they add little to the views of Ellison (see above), who considered Eq. (7.59) simply as an interpolation formula for $K(\mathrm{Ri})$, which gives correct results for large negative values, values close to zero and large positive values of $\zeta = \frac{z}{L}$.

Since $K = \frac{\varkappa u_* z}{\varphi(\zeta)}$ and $\mathrm{Rf} = \frac{\zeta}{\varphi(\zeta)}$, Eq. (7.59) may be rewritten as the following simple algebraic equation for the function $\varphi(\zeta)$

$$\varphi^4 - \sigma\zeta\varphi^3 - 1 = 0. \tag{7.61}$$

This equation contains a single unknown constant σ, which may be eliminated by introducing, instead of $\zeta = \frac{z}{L}$ as an independent variable, the quantity $\eta = \sigma\zeta = \frac{z}{L_*}$, where $L_* = \frac{L}{\sigma}$. Then the equation takes the form

$$\varphi^4 - \eta\,\varphi^3 - 1 = 0. \tag{7.62}$$

Equation (7.62) was solved numerically by Yamamoto (1959) and then also by Klug (1963). Yamamoto has given a table of values of the function $\varphi(\eta)$ [which is in good agreement with the schematic of Fig. 47], and some fairly simple analytical approximations of this function on the positive and negative semiaxes; graphs are plotted giving the dimensionless wind velocity $\varkappa\bar{u}/u_*$ as a function of the dimensionless height z/z_0 for different values of the stratification parameter $\eta_0 = \sigma z_0/L$. Later, Syono and Hamuro (1962) and Okamoto (1963) formulated the solution of the fourth-degree equation (7.61) in radicals, and carried out a more detailed investigation into the form of the function $\varphi(\eta)$ and the wind-velocity profiles corresponding to it. However, we shall not dwell on the detailed study of the solution of Eq. (7.61), which is of little interest to us; in any case, it is only one of the possible interpolation formulas for φ (and, moreover, is clearly not a very exact one).

Let us also note that for $\zeta \leqslant 0$, the arguments in favor of Eq. (7.59) may also be used to justify the equation

$$K = \kappa u_* z(1-\sigma' \,\mathrm{Ri})^{1/4} \tag{7.59'}$$

where σ' is another constant (namely, $\sigma' = \sigma_{-\infty}\sigma$). If we now replace the length L by $L' = u_*(\partial u/\partial z)/\kappa(g/T_0)(\partial T/\partial z) = \sigma(\zeta)L$, and the variable $\zeta = z/L'$ by $\zeta' = z/L'$, then Eq. (7.59′) may be rewritten in a form similar to Eq. (7.61), but with the replacement of ζ by ζ' [and $\varphi(\zeta)$ by $\varphi(\zeta')$] and of σ by σ'. This equation for $\varphi(\zeta')$ was used by Panofsky, Blackadar and McVehil (1960) and by several others (see Sect. 8.1).

The fact that Eq. (7.61) cannot be exact follows, in particular, from the fact noted above; i.e., this equation corresponds to the physically unlikely assumption that the scale of turbulence l_1 at height z is independent of the temperature stratification. However, if we assume that

$$l_1 = \varkappa z \lambda\left(\frac{z}{L}\right), \tag{7.63}$$

then we obtain for φ the equation

$$\varphi^4 - \sigma\zeta\varphi^3 - \frac{1}{\lambda^4} = 0, \tag{7.64}$$

which contains a further unknown universal function $\lambda(\zeta)$ [cf. Yokoyama (1962b)].

In the case of stable stratification, turbulent exchange becomes more difficult, and large turbulent formations cannot exist (and at $\mathrm{Rf} > \mathrm{Rf}_{cr} = 1\sigma$ the turbulence cannot exist at all).

Therefore, it is natural to expect that for positive ζ the function λ will decrease monotonically from $\lambda(0) = 1$ to zero as ζ increases from zero to infinity (or as Rf increases from zero to $\mathrm{Rf_{cr}}$). As a concrete example of the function $\lambda(\mathrm{Rf})$ Kazanskiy and Monin (1956) considered the case when $\lambda(\mathrm{Rf}) = (1 - \sigma\,\mathrm{Rf})^{\gamma - \frac{1}{4}}$ where $\gamma > \frac{1}{4}$ is some number. With this choice of $\lambda(\mathrm{Rf})$, Eq. (7.60) becomes

$$K = \varkappa u_* z\,(1 - \sigma\,\mathrm{Rf})^{\gamma}. \tag{7.65}$$

(This relationship, with $\gamma = 1/2$ was used earlier by Holzman (1943) as an empirical formula for K.) Expressing K and Rf once again in terms of the function $\varphi(\zeta)$, we find that Eq. (7.65) is equivalent to the equation

$$\frac{1}{\varphi(\zeta)} = \left[1 - \frac{\sigma\zeta}{\varphi(\zeta)}\right]^{\gamma}. \tag{7.66}$$

Equation (7.66) shows that when $\eta = \sigma\zeta \geqslant 0$ (i.e., in stable stratification) $\varphi \geqslant 1$, while $\varphi(0) = 1$ and $\varphi(\eta) \to \infty$ as $\eta \to \infty$. For small η, we may put $\varphi(\eta) = 1 + \delta$, where $\delta \lll 1$; we find easily that $\delta = \gamma\eta$; i.e., $\varphi \approx 1 + \sigma\gamma\zeta$. Thus the product $\sigma\gamma = \beta$ must be identical to the coefficient β in Eq. (7.32). Further, for large $\eta = \sigma\zeta$, the asymptotic relationship $\varphi \approx \eta = \sigma\zeta$ will be satisfied, showing that $\sigma = \frac{1}{R} = \frac{1}{\mathrm{Rf_{cr}}}$ [cf. Eq. (7.43)]. Kazanskiy and Monin evaluated $\varphi(\eta)$ and $f(\eta)$ for the special case $\gamma = 0.6$; the dependence of the dimensionless eddy viscosity $\sigma K/\varkappa u_* L$ on the dimensionless height $\eta = \sigma z/L$, obtained on the basis of this calculation, is very close to the exponential dependence $\sigma K/\varkappa u_* L = 1 - e^{-\eta}$, which is used in a number of meteorological works [Dorodnitsyn (1941), Matveyev (1960) and others]. The case $\gamma = 6$ was considered by Borkowski (1964) and the general equation (7.66) was investigated later in detail for several values of γ by Rijkoort (1968).

The approximation (7.65) is most suitable for positive ζ (and Rf). In fact, when applied to negative ζ (for unstable stratification), it gives the apparently valid free convection laws (7.38)–(7.41) as $\zeta \to -\infty$ and $\mathrm{Rf} \to -\infty$ only if $\gamma = 1/4$. Thus, with $\zeta < 0$ it is necessary to use either a special case (7.61) of Eq. (7.66) [i.e., to assume in advance that $\gamma = 1/4$] or else reject the assumption on the existence of a finite limit $\alpha_{-\infty}$. It is also easy to obtain a general form of the dependence of the scale of turbulence l_1 on the stratification in unstable conditions, which is compatible with the free convection laws. In fact, if we assume that $l_1 = \varkappa z \lambda(z/L)$ [so that $\lambda(\zeta) = 1$ for approximation (7.61)], then by Eqs. (7.64) and (7.39) we obtain

$$\lambda(\zeta) = \left(\frac{27}{\sigma C_2^3}\right)^{1/4}\left(1 + \frac{C_2}{3\sigma\zeta^{1/3}}\right)^{-1/4} \to \lambda(\infty) = \left(\frac{27}{\sigma C_2^3}\right)^{1/4} \quad \text{as} \quad \zeta \to -\infty. \tag{7.67}$$

(Here $\lambda(\infty) = \text{const}$, i.e., l_1 is proportional to z follows naturally from the absence in the conditions of free convection of any fixed length scale other than z.) For small values of z, using Eq. (7.32), we obtain

$$\lambda(\zeta) \approx 1 - \left(\beta - \frac{\sigma}{4}\right)\zeta, \quad l_1 \approx \varkappa z\left[1 - \left(\beta - \frac{\sigma}{4}\right)\frac{z}{L}\right]. \tag{7.67'}$$

Thus, if $\beta > \frac{\sigma}{4}$, then as ζ varies from zero to $-\infty$ the function $\lambda(\zeta) = l_1/\varkappa z$ will first increase linearly, then its rate of growth will decrease, and as $\zeta \to -\infty$, it will tend to a constant.

We may also begin by choosing in some way the function $\lambda(\zeta)$; then from the fourth-degree equation (7.64) it is easy to determine the function $\varphi(\zeta)$, i.e., to find the wind distribution with height, corresponding to a given scale of turbulence $l_1 = \varkappa z \lambda(z/L)$; one example of such a procedure may be found in the work by Takeuchi and Yokoyama (1963), where some fairly rough data was used for a crude estimate of one of the possible "scales" of turbulence and then the corresponding function $\varphi(\zeta)$ was evaluated.

Equations (7.61) [or Eq. (7.64) for given $\lambda(\zeta)$] and (7.66) are particular examples of the formulas for $\varphi(\zeta)$ obtained from semiempirical theories. As we have already seen in Chapt. 3, even for turbulence in a homogeneous medium there exist a whole series of different variants of the semiempirical theory, which lead to slightly different results. Therefore, in the considerably more complex conditions of nonneutral stratification, the number of possible semiempirical theories is very large, and these theories lead to very diverse expressions for $\varphi(\zeta)$ and the other universal functions. Of course, all these theories contain certain arbitrary assumptions, which require further verification; however, the expressions obtained for $\varphi(\zeta)$ in many cases do not even satisfy the general asymptotic relationships deduced in the previous subsection and, therefore, can be used only for some limited range of values of ζ. Thus, for example, in the first semiempirical theory of turbulence in a stratified medium proposed by Rossby and Montgomery (1935), it was assumed that $K = u_* l_1$, and a special hypothesis was used to define the dependence of the scale l_1 on the height and stratification; as a result, the authors concluded that

$$K = \frac{\varkappa u_* z}{(1 + \sigma\,\mathrm{Ri})^{1/2}}, \tag{7.68}$$

where σ = const, i.e., that

$$\varphi^3 - \varphi - \sigma\zeta = 0 \tag{7.69}$$

if $\alpha(\zeta) \equiv 1$ [cf. the deduction of Eq. (7.61) from Eq. (7.59)]. If $|\zeta| \ll 1$, Eqs. (7.68) and (7.69) lead to the reasonable "logarithmic + linear" equation for the wind profile (with $\beta = \sigma/2$); however, the asymptotic behavior of $\varphi(\zeta)$ both as $\zeta \to -\infty$ and as $\zeta \to +\infty$ proves to be incorrect here. In another semiempirical theory developed by Ogura (1952a), the initial premises are the equation of turbulent energy balance and the hypotheses that $K = \dfrac{u_* l_1^2}{\varkappa z}$ and $\alpha \equiv 1$. A formula was then obtained for the wind velocity, which in our notation may be written as

$$\varphi(\zeta) = \left(1 + \frac{\sigma^2}{4}\zeta^2\right)^{1/2} + \frac{\sigma}{2}\zeta, \tag{7.70}$$

where σ is an empirical constant. It is clear that when $|\zeta| \ll 1$, this formula also is equivalent to the "logarithmic + linear" formula (with $\beta = \sigma/2$). However, as $\zeta \to -\infty$, once again we do not obtain the relationship (7.39), which corresponds to the "1/3-power law" for the wind-velocity profile (and the temperature profile) in conditions of free convection (in the other limiting case $\zeta \to +\infty$ the correct asymptotic relationship is now satisfied if we assume that $\sigma = 1/R = 1/\mathrm{Rf}_{cr}$). Ogura's eddy viscosity hypothesis was also used by Bussinger (1955) but in conjunction with the Rossby-Montgomery hypothesis on the scale l_1 (instead of using the semiempirical equation of energy balance). The result which he obtained may be written as follows:

$$\varphi(\zeta) = \frac{1}{\frac{1}{2}\left[1 + (1 - 4\zeta)^{1/2}\right] - \zeta}. \tag{7.71}$$

According to Eq. (7.71), $\varphi(\zeta) \approx 1 + 2\zeta$ when $|\zeta| \lll 1$, but the asymptotic behavior of $\varphi(\zeta)$ for large values of $|\zeta|$ once again proves to be incorrect for either sign of ζ. (In the case of strong stability, it is found that $\varphi(\zeta)$ even takes complex values.)

With the aid of a special, quite artificial, hypothesis, Swinbank (1960; 1964) obtained the very simple formula

$$\varphi(\zeta) = \frac{\zeta}{1 - e^{-\zeta}}, \tag{7.72}$$

which, like Eq. (7.71) contains no empirical constants at all. This formula once again has the correct asymptotes [of the form of Eqs. (7.32) and (7.43), respectively] for very small and very large positive values of ζ (with coefficients $\beta = 0.5$, $R = C_3 = 1$), but as $\zeta \to -\infty$ it does not become the "1/3-power law." We note further that Eq. (7.72) leads to a simple exponential height dependence of the eddy viscosity K [proportional to $1 - \exp(-z/L)$] which seems probable in the case of stable stratification. A further variant of the semiempirical theory was proposed by Kao (1959), the main result of which may be written as

$$\varphi(\zeta) = \frac{1}{2}\{1 + (1 + 4\sigma\zeta)^{1/2}\} \quad \text{for} \quad \zeta > -\frac{1}{4\sigma}, \tag{7.73}$$

where σ is an empirical constant. Similar to all the preceding formulas, Eq. (7.73) reduces to the "logarithmic + linear" profile in near neutral conditions, but when $\zeta \to +\infty$ it has an incorrect asymptote. A more complicated relationship for $\varphi(\zeta)$, which may be written as

$$\varphi(\zeta) - \sigma\zeta = \left\{\frac{\sigma\zeta + [(\sigma\zeta)^2 + 4(\varphi(\zeta) - \sigma\zeta)^2]^{1/2}}{2\varphi(\zeta)}\right\}^{3/4}, \tag{7.74}$$

was obtained in the semiempirical theory of Businger (1959). This relationship is advantageous in that the corresponding function $\varphi(\zeta)$ exhibits the correct asymptotic behavior both for $\zeta \to +\infty$ (with $\mathrm{Rf}_{cr} = 1/\sigma$, that is, $C_3 = \sigma$) and for $\zeta \to -\infty$. The same advantages are also shown by the function

$$f(\zeta) = \int^{\zeta} \frac{\varphi(\xi)}{\xi}\, d\xi,$$

found by Zilitinkevich and Laykhtman (1965), starting from the semiempirical energy balance equation (6.48) [with $a_b = 0$, i.e., ignoring the diffusion of energy] supplemented by the following generalization of von Kármán's hypothesis (5.119) on the "mixing length" l:

$$l = -\varkappa_1 \frac{\left(\dfrac{d\bar{u}}{dz}\right)^2 - \alpha \dfrac{g}{T_0}\dfrac{d\bar{T}}{dz}}{2\dfrac{d\bar{u}}{dz}\dfrac{d^2\bar{u}}{dz^2} - \alpha\dfrac{g}{T_0}\dfrac{d^2\bar{T}}{dz^2}}$$

(where $\varkappa_1$ and α are empirical constants, $\alpha = K_T/K$). A more general semiempirical theory taking the effect of the energy diffusion into account (i.e., assuming that $a_b \neq 0$) was developed numerically by Vager (1966); it also leads to a definite form of the functions $\phi(\zeta)$ and $f(\zeta)$. At the same time, by semiempirical arguments Yamamoto and Shimanuki (1966) advocated the replacement of Eq. (7.62) by a pair of equations

$$\varphi^4 - |\eta|\varphi^3 - 2\varphi^2 + 1 = 0 \text{ for the unstable case } (\eta < 0)$$
$$\varphi^4 - |\eta|^{1-2p}\varphi^3 - 2\varphi^2 + 1 = 0 \text{ for the stable case } (\eta > 0) \tag{7.75}$$

where p is a small numerical constant; they calculated the universal functions corresponding to Eq. (7.75) with $p = 1/6$. The solution of the first equation (7.75) is quite close to the solution of Eq. (7.62) at $\eta < 0$ (and has the same asymptotic behavior when $\eta \to 0$ or $\eta \to -\infty$), but the solution of the second equation (7.75) behaves like $1 + 1/2|\eta|^{1/2-p}$ for small η and $|\eta|^{1-2p}$ for large η, i.e., has a usual behavior when $p = 0$ only.

Another approach to the determination of the universal functions of the wind and temperature profiles in the lower atmosphere for unstable stratification was used by Pandolfo (1966). He assumed that the wind velocity satisfies the "logarithmic + linear" equation in the range $0 > \zeta > \zeta_1$ of values of ζ, while the temperature profile satisfies the "1/3-power law" (7.35)–(7.36) for all $\zeta < \zeta_1$. This implies that for the full determination of the functions $\varphi(\zeta)$ and $\varphi_1(\zeta)$ on the whole semiaxis $\zeta < 0$ it is necessary to know only the function $K_T/K = a(\zeta)$. The last function was determined by Pandolfo with the aid of the questionable hypothesis that ζ = Ri for all $\zeta < 0$. It follows from the hypothesis and the stated assumptions on the wind and temperature profiles that $\alpha(\zeta) = 1/(1 + \beta\zeta)$ when $0 > \zeta > \zeta_1$ and $\alpha(\zeta) \sim |\zeta|^{1/6}$ when $\zeta < \zeta_1$. Now it is quite easy to write down the final equations for the functions $\varphi(\zeta)$ and $\varphi_1(\zeta)$ for all values $\zeta < 0$. Since $\alpha(\zeta) \to \infty$ as $\zeta \to -\infty$ according to the Pandolfo theory it is clear that the corresponding velocity profile $\bar{u}(z)$ does not satisfy the "1/3-power law."

One more set of equations for velocity and temperature profiles in the unstable case (i.e., at $\zeta < 0$) was proposed by Businger (1966), based on not very reliable purely empirical arguments. These equations have the form

$$\varphi(\zeta) = (1 - \beta\zeta)^{-1/4}, \quad \varphi_1(\zeta) = (1 - \beta\zeta)^{-1/2}, \quad \alpha(\zeta) = (1 - \beta\zeta)^{1/4}. \tag{7.76}$$

We see also that here ζ = Ri for all $\zeta < 0$ and the "1/3-power law" is not valid either for velocity or for temperature profiles.

Still further examples could be given of the use of semiempirical hypotheses for the explicit determination of the functions $\varphi(\zeta)$ and $f(\zeta)$ [see, for example, Elliott (1957; 1960), Kravchenko (1963), Naito (1964) and especially Rijkoort (1968), where a long list of the hypothetical equations for $\varphi(\zeta)$ and $f(\zeta)'$ is considered]. It must be borne in mind, however, that all the expressions for these functions obtained in such a manner must be considered only as approximate formulas, the accuracy of which in a given range of values of ζ (or of the numbers Rf, or Ri) must be established further on the basis of comparison with observation. In the next section, we shall see that the scatter of the existing observational data is still fairly great; hence choosing between the various semiempirical formulas is rather difficult.

In addition to semiempirical formulas for the wind profile, which are dimensionally correct [i.e., they reduce to certain assumptions concerning the functions $\varphi(\zeta)$ or $f(\zeta)$] we may also find in the meteorological literature a great number of works (as a rule, more than fifteen years old) which propose semiempirical or purely empirical formulas for the profiles of the meteorological variables that are not based on dimensional considerations (very frequently containing, instead of empirical constants, some empirical functions of the Richardson number or other stratification characteristics). Thus, for example, for several years, there was ardent discussion in the scientific literature as to the possibility of using the "generalized logarithmic law" (with parameters dependent on the stability) to describe the wind profile in the surface layer of the atmosphere for different conditions of temperature stratification. Among the authors who advocated this law, we can mention, for example, Rossby and Montgomery (1935), Sutton (1936; 1937), Budyko (1946; 1948), Bjorgum (1953) and many others. An obvious defect of all formulations of the "generalized logarithmic law" is that they do not take into account the actual systematic deviations of the height-dependence of the wind velocity from the logarithmic equation in conditions of nonneutral stratification. Moreover, even in the lowest layer, where temperature stratification plays a very small part, the generalized logarithmic profiles do not tend as a rule to those which are observed in the atmosphere (in particular, extrapolation of the wind profile to the value $\bar{u}(z_0) = 0$ here leads to a value z_0 which depends on the stability; i.e., is not an objective characteristic of the underlying surface.

Other authors, including Schmidt (1925), Best (1935), Laykhtman (1944; 1947a), Frost (1948), Deacon (1949) and Takeda (1951), suggested approximating the wind profile in the atmospheric surface layer by power functions of z (i.e., by a so-called "generalized power law"). Schmidt, Best and Frost used a simple formula of the $\bar{u}(z) \sim z^{\varepsilon}$, while Laykhtman, Deacon and Takeda developed a theory in which

$$\bar{u}(z) = \bar{u}(z_1)\frac{z^{\varepsilon} - z_0^{\varepsilon}}{z_1^{\varepsilon} - z_0^{\varepsilon}} \tag{7.77}$$

that is,

$$\frac{\partial \bar{u}}{\partial z} \sim z^{\epsilon - 1} \quad \text{if } z \gg z_0$$

where ϵ is a parameter dependent on the thermal stratification (positive for inversions, zero for neutral stratification, and negative in the unstable case). Equation (7.77) enables us to detect the general character of the deviations of the wind-velocity profile from the logarithmic law; as $\varepsilon \to 0$ it transforms into the ordinary logarithmic profile. However, if, following Laykhtman, we assume that the parameter z_0 also depends on the stratification, then the effect of the stratification will still be evident as $z \to 0$, which contradicts the deductions from similarity theory. In addition, the number of free parameters available in such a theory for matching the observational data in each case, is so large that it is difficult to find their value, and the accuracy of the calculations is reduced. However, if, following Deacon, we take z_0 to be constant, then we must assume that ϵ is also height-dependent [see, for example, Davidson and Barad (1956], which is entirely unsatisfactory. We must also point out that formula (7.77) cannot be consistent with the dimensional considerations of Sect. 7.2 either; in particular, $\frac{\partial \bar{u}}{\partial z}$ here depends explicitly on the function $\varepsilon = \varepsilon(\mathrm{Ri})$ [where Ri is taken at a fixed height] and on the roughness parameter z_0 (which in Laykhtman's scheme also depends on the stratification).

On the basis of the analysis of data, it was proposed, in particular, by Halstead (1943) and Panofsky (1952), to take into account the stratification effect on the wind profile by introducing a linear correction to the logarithmic law, i.e., by using a formula of the form

$$\bar{u}(z) = \frac{u_*}{\varkappa}\left(\ln\frac{z}{z_0} + bz\right), \tag{7.78}$$

where b is a parameter dependent on the stability (positive for inversions and negative in the unstable case). Formula (7.78) can be reconciled with dimensional considerations; we need only put $b = \frac{\beta}{L}$ which transforms into the Monin-Obukhov formula (7.33).

All the remarks on the wind profile may be repeated for the temperature and humidity profiles, provided that the stratification is unstable, neutral or slightly stable. However, for very high stability, as we have already observed, a special situation arises; there are grounds for believing that under these conditions the eddy diffusivity $K_T = K_\vartheta$ is considerably less than K, and hence the profiles of temperature $\overline{T}$ and admixture concentration $\overline{\vartheta}$ in this case differ in form from the mean velocity profile (cf. Figs. 48 and 50). The theoretical analysis of this phenomenon presents very great difficulties and, so far, has not had any conspicuous success; however, Ellison's attempt (1957) to estimate the dependence of $\alpha = \frac{K_T}{K}$ on the Richardson number $\mathrm{Rf} = \frac{\zeta}{\varphi(\zeta)}$ is worthy of mention, since it led to results which are in unexpectedly good agreement with those of the later experiments of Ellison and Turner (1960) [see the end of Sect. 8.2 below]. To calculate α, Ellison used the turbulent energy balance equation for a stratified medium and related equations for the second moments $\overline{T'^2}$, $\overline{w'^2}$ and $\overline{w'T'}$. As we have already seen in Sect. 6.1, these equations contain unfortunately, a large number of new unknown terms. Therefore, in order to obtain his deductions, Ellison had to make a number of additional rough assumptions (of the type of the semiempirical hypotheses) by which certain terms were assumed negligibly small and certain connections were postulated between the nonnegligibly small terms and the second moments under consideration. Using these assumptions, which implicitly contained the assumption that the heat exchange is small in comparison with the momentum exchange in the case of high stability, Ellison arrived at a relationship which may be written as follows:

$$\alpha(\zeta) = \frac{K_T}{K} = \frac{\alpha_0(1-\mathrm{Rf}/R)}{(1-\mathrm{Rf})^2} = \frac{\alpha_0\varphi(\zeta)\,[R\varphi(\zeta)-\zeta]}{R\,[\varphi(\zeta)-\zeta]^2}, \tag{7.79}$$

where α_0 and R are slowly varying functions of ζ, which in the first approximation, may simply be assumed to be empirical constants. It is clear that here α_0 is identical with the value of K_T/K for neutral stratification (i.e., in the logarithmic layer) and R is the critical value $\mathrm{Rf_{cr}}$ which Rf can never exceed and for which K_T becomes zero. Very rough intuitive estimates made by Ellison led him to conclude that the parameter R must be considerably less than unity (for example, close to 1/7 or 1/10); however, this conclusion still cannot be taken as final. If we know the way in which the number Rf approaches its limiting value $R = \mathrm{Rf_{cr}}$, i.e., the order of the small term o(1) in the asymptotic expansion $\varphi(\zeta) = \frac{\zeta}{R}[1 + o(1)]$ of the function $\varphi(\zeta)$ for large ζ; then, proceeding from Eq. (7.79) it is possible to determine also the asymptotic behavior of the function $\varphi_1(\zeta)$ [or $\varphi_T(\zeta) = \frac{1}{\alpha_0}\varphi_1(\zeta)$] as $\zeta \to$. However, at present, we still do not have the necessary data for this.

7.5 General Similarity Hypothesis for a Turbulent Regime in a Stratified Boundary Layer and Its Application to the Characteristics of Turbulent Fluctuations

So far, we have considered only profiles of the mean velocity, temperature and humidity in the surface layer of the atmosphere, and have applied the dimensional considerations discussed in Sect. 7.2 to them only. However, these considerations have a general character, and the results of Sects. 7.3–7.4 far from exhaust all the applications of similarity theory to turbulence in a stratified boundary layer. We shall now consider some further applications of this theory, relating to statistical characteristics other than the mean values of the basic flow variables.

First, we formulate the fundamental postulates on similarity in the most general form. In Sect. 7.1 we noted that for plane-parallel turbulent flow of a stratified fluid above a plane homogeneous rough surface (which is a natural model of the "atmospheric surface layer"), all the one-point moments of the flow variables will depend on the vertical coordinate z only. However, the restriction to one-point moments was introduced only because we were interested primarily in them. In fact, in such a model, all the probabihty distributions for the values of the flow variables at several points will be invariant under arbitrary translations of this set of points in the plane Oxy and its reflections in the vertical plane Oxz passing through the direction of the mean velocity $\bar{u}(z)$, and will also be stationary (independent of the time shifts). In other words, in this model, the probability distribution for the values of arbitrary flow variables at the points $(x_1, y_1, z_1, t_1), \ldots, (x_n, y_n, z_n, t_n)$ can depend only on the parameters $x_2-x_1, \ldots, x_n-x_1$; $y_2-y_1, \ldots, y_n-y_1$; $z_1, \ldots, z_n$; $t_2-t_1, \ldots, t_n-t_1$ and will not change if the direction of the Oy axis is reversed.

As flow variables we shall consider only the three components of velocity u, v, w and the temperature T (the further inclusion of the admixture concentration ϑ leads only to the appearance of obvious additional formulas, which are of little importance in the discussion. These variables may be represented as

$$u=\bar{u}(z)+u', \quad v=v', \quad w=w', \quad T=\bar{T}(z)+T', \qquad (7.80)$$

where $\bar{u}(z)$ and $\bar{T}(z)$ are ordinary (nonrandom) functions of z which were discussed in detail earlier. Thus it is sufficient to consider only the probabihty distributions for the random fluctuations $u'(x, y, z, t)$; $v'(x, y, z, t)$; $w'(x, y, z, t)$ and $T'(x, y, z, t)$. In this model of a stratified medium, the general hypothesis of similarity of the turbulent regime, applied to the surface layer of the atmosphere, may be formulated as follows: *the joint probability distribution for the values of the dimensionless fluctuations* u'/u_*, v'/u_*, w'/u_* *and* $T'/|T_*|$ *at the points* $(x_1, y_1, z_1, t_1), \ldots, (x_n, y_n, z_n, t_n)$ *can depend only on the dimensionless parameters*

$$\frac{x_2-x_1}{|L|}, \ldots, \frac{x_n-x_1}{|L|}; \quad \frac{y_2-y_1}{|L|}, \ldots, \frac{y_n-y_1}{|L|}; \quad \frac{z_1}{L}, \ldots, \frac{z_n}{L};$$
$$\frac{(t_2-t_1)u_*}{|L|}, \ldots, \frac{(t_n-t_1)u_*}{|L|},$$

provided that the following two conditions are satisfied: 1) *the* *heights* $z_1, \ldots, z_n$ *are not too great* (they lie within the layer in which it may be assumed that $u_* =$ const, $q =$ const and the effect of the Coriolis force may be ignored) *and not too small* (all much greater than the "roughness height" z_0); *and* 2) *the distance between any two different points* (x_i, y_i, z_i) *and* (x_j, y_j, z_j) *and all nonzero differences* $|t_i - t_j|$ *are not too small* (so that the interactions between fluctuations at the space-time points under consideration show no influence of molecular effects, determined by the molecular viscosity and thermal conductivity of the air) *and are not too great* (so that the conditions of horizontal homogeneity and steadiness do not break down). Throughout this chapter, L and T_* denote the same length and temperature scales as in Eqs. (7.12) and (7.14). The justification of this hypothesis is, in fact, already included in the considerations of Sects. 7.1–7.2.

As an example, we shall consider in greater detail the joint probability distribution for fluctuations (u', v', w', T') at a fixed space-time point (x, y, z, i). According to the similarity hypothesis, this distribution can depend only on the vertical coordinate z, and its probability density $\Phi(u', v', w', T')$ may be written as

$$\Phi(u', v', w', T') = \frac{1}{u_*^3 |T_*|} \Psi\left(\frac{u'}{u_*}, \frac{v'}{u_*}, \frac{w'}{u_*}, \frac{T'}{|T_*|}; \frac{z}{L}\right), \qquad (7.81)$$

where Ψ is a universal function of five variables. Of course, a function of five variables is a very complicated characteristic; thus it is essential that in the limiting cases of great instability $\left(\frac{z}{L} \to -\infty\right)$ and great stability $\left(\frac{z}{L} \to \infty\right)$ and also in the case of neutral stratification $\left(\frac{z}{L} \to 0\right)$ formula (7.81) can be simplified. In the case of neutral stratification $q \to 0$ and the buoyancy has no effect on the turbulence. Thus, here, the parameter g/T_0 must drop out of Eq. (7.81). Therefore, in this case, there is no dependence on z/L (L contains g/T_0); i.e., the probability distribution is height-independent. Moreover, in neutral stratification there are no temperature fluctuations, so that the dependence of Φ on T' is described by the multiplier $\delta(T')$ [where δ is the Dirac delta-function]. As a result, we obtain

$$\Phi(u', v', w', T') \to \frac{1}{u_*^3} \Psi_1\left(\frac{u'}{u_*}, \frac{v'}{u_*}, \frac{w'}{u_*}\right) \delta(T') \quad \text{as} \quad \frac{z}{L} \to 0, \qquad (7.82)$$

where Ψ is a universal function of three variables, describing the probability distribution of the velocity fluctuations at a point of the logarithmic layer. The limiting case of very great instability (free convection) is obtained for $q>0$, $u_* \to 0$, so that here the parameter u_* must disappear from Eq. (7.81). Consequently, it follows that

$$\Phi(u', v', w', T') \to \varkappa^{1/3}\left(\frac{g}{T_0}\right)^{-2/3}\left(\frac{q}{c_p\rho_0}\right)^{-5/3} z^{-2/3} \times$$
$$\times \Psi_2\left(\frac{u'}{\left(\varkappa\frac{g}{T_0}\frac{q}{c_p\rho_0}z\right)^{1/3}}, \frac{v'}{\left(\varkappa\frac{g}{T_0}\frac{q}{c_p\rho_0}z\right)^{1/3}},\right.$$
$$\left.\frac{w'}{\left(\varkappa\frac{g}{T_0}\frac{q}{c_p\rho_0}z\right)^{1/3}}, \frac{T'}{\varkappa^{-1/3}\left(\frac{g}{T_0}\right)^{-1/3}\left(\frac{q}{c_p\rho_0}\right)^{2/3}z^{-1/3}}\right), \quad (7.83)$$

where Ψ_2 is another universal function. Finally, it may be assumed that in the case of very great stability the turbulent fluctuations acquire a local character, i.e., that their statistical properties cease to depend on the distance z from the underlying surface. Thus it must be expected that as $\frac{z}{L} \to +\infty$, the parameter z will disappear from Eq. (7.81), i.e.,

$$\Phi(u', v', w', T') \to \frac{1}{u_*^3 |T_*|}\Psi_3\left(\frac{u'}{u_*}, \frac{v'}{u_*}, \frac{w'}{u_*}, \frac{T'}{T_*}\right) \quad \text{as} \quad \frac{z}{L} \to \infty. \quad (7.84)$$

Unfortunately, Eqs. (7.82)–(7.84) are still all very difficult to verify, since multidimensional probability distributions are very difficult to determine reliably from experimental data. Therefore, we shall further restrict ourselves to the investigation of only the simplest characteristics of the distribution (7.81), namely, to the lowest moments of the fluctuations u', v', w', T' at the fixed point (x, y, z, t). According to the general formula (7.13), any one-point moment of these fluctuations may be represented in the form of some combination of the parameters $\frac{g}{T_0}$, u_* and $\frac{q}{c_p\rho_0}$, multiplied by a universal function of $\zeta = \frac{z}{L}$ [the same result, of course, follows from Eq. (7.81)]. From the definition of the fluctuations, their mean values are zero; thus we can proceed at once to the second moments. In all there are ten such moments. However, three of these, namely,

$\overline{u'v'}$, $\overline{v'w'}$ and $\overline{v'T'}$ are identically zero. This is due to the symmetry of the turbulence with respect to the direction of the mean wind (this has already been discussed in relation to the moments $\overline{u'v'}$ and $\overline{v'w'}$ in Sect. 7.1). The moments $\overline{u'w'} = -u_*^2$ and $\overline{w'T'} = \frac{q}{c_p\rho_0}$ have constant values; thus only five moments remain to be considered—the variances of u', v', w', T' (in place of which it will be convenient to consider the standard deviations σ_u, σ_v, σ_w, σ_T) and the mixed second moment $\overline{u'T'}$. By Eq. (7.13), these quantities may be written as

$$\frac{\sigma_u}{u_*} = f_3(\zeta), \quad \frac{\sigma_v}{u_*} = f_4(\zeta), \quad \frac{\sigma_w}{u_*} = f_5(\zeta),$$
$$\frac{\sigma_T}{|T_*|} = f_6(\zeta), \quad \frac{\overline{u'T'}}{u_*T_*} = f_7(\zeta). \tag{7.85}$$

Thus, to determine all the second moments of the wind-velocity and temperature fluctuations, it is sufficient to determine the parameters u_* and q and to know the five universal functions f_3, f_4, f_5, f_6, and f_7. These may be used, in particular, to express such interesting turbulence characteristics as the anisotropy coefficients σ_v/σ_u and σ_w/σ_u and the correlation coefficients r_{uw}, r_{wT} and r_{uT}

$$\frac{\sigma_v}{\sigma_u} = \frac{f_4}{f_3}, \quad \frac{\sigma_w}{\sigma_u} = \frac{f_5}{f_3}, \quad r_{uw} = -\frac{1}{f_3 f_5},$$
$$r_{wT} = \pm \frac{\varkappa}{f_5 f_6}, \quad r_{uT} = \mp \frac{f_7}{f_3 f_6} \tag{7.86}$$

(in the last two equations, the upper sign relates to unstable stratification and the lower to stable stratification).

The functions $f_3(\zeta), \ldots, f_6(\zeta)$ are clearly nonnegative by definition for all values of ζ. The moment $\overline{u'T'}$ defined by the function $f_7(\zeta)$ describes the turbulent heat transfer in the direction of the mean wind; this transfer usually plays no essential part and often may be neglected. With neutral stratification and with extremely great instability $\overline{u'T'} = 0$ (see below); however, for other values of ζ, the function $f_7(\zeta)$ takes positive values, because with unstable stratification, the inequality $T' > 0$, as a rule, will hold when $w' > 0$. Therefore the u' fluctuations will more often than not be negative, while for stable stratification, the situation is reversed. Moreover, the ratio $-\overline{u'T'}/\overline{w'T'} = \varkappa f_7(\zeta)$ may even take values greater than 1, since

u' fluctuations are usually considerably greater than w' fluctuations [see Sect. 8.5 below and Yaglom (1969)].

The asymptotic behavior of the functions $f_3, \ldots, f_7$ as $\zeta \to -\infty$, $\zeta \to +\infty$ and as $|\zeta| \to 0$ may be established, starting with Eqs. (7.82)–(7.84), as was done above for the functions f, f_1, f_2 describing the mean profiles. With increase of instability (i.e., as $\zeta \to -\infty$), u_* must disappear from all the equations of (7.85); consequently, functions f_3, f_4, and f_5 will increase asymptotically in the manner of $|\zeta|^{1/3}$; f_6 tends to zero similar to $|\zeta|^{-1/3}$, and f_7 will tend to some constant. Moreover, it must be borne in mind that as $u_* \to 0$, $\bar{u}(z) \to 0$ also, i.e., the turbulence approaches a regime in which in the plane Oxy, no direction is especially distinguished, and the fluctuations of u' and v' play identical roles (the idealized regime of "true free convection" in the absence of mean horizontal velocity). Thus we must expect that as $\zeta \to -\infty$,

$$\begin{gathered} f_3(\zeta) \approx f_4(\zeta) \approx C_4|\zeta|^{1/3}, \quad f_5(\zeta) \approx C_5|\zeta|^{1/3}, \\ f_6(\zeta) \approx C_6|\zeta|^{-1/3}, \quad f_7(\zeta) \approx 0, \end{gathered} \tag{7.87}$$

or, in other words,

$$\begin{gathered} \sigma_u \approx \sigma_v \approx C_4'\left(\frac{q}{c_p\rho_0}\frac{gz}{T_0}\right)^{1/3}, \quad \sigma_w \approx C_5'\left(\frac{q}{c_p\rho_0}\frac{gz}{T_0}\right)^{1/3} \\ \sigma_T \approx C_6'\left(\frac{q}{c_p\rho_0}\right)^{2/3}\left(\frac{gz}{T_0}\right)^{-1/3}, \quad \overline{u'T'} \approx 0 \end{gathered} \tag{7.87$'$}$$

[formula (7.87$'$) for σ_w was actually known to Prandtl (1932a) but was then forgotten; later, it was obtained independently by Obukhov (1960), together with the formula for σ_T]. The universal constants $C_4 = C_4'\varkappa^{-1/3}$, $C_5 = C_5'\varkappa^{-1/3}$ and $C_6 = C_6'\varkappa^{4/3}$ in Eqs. (7.87) can be determined from observations. From Eqs. (7.87) it also follows that in conditions of free convection $\frac{\sigma_v}{\sigma_u} \approx 1$, $\frac{\sigma_w}{\sigma_u} = \text{const}$, $r_{wT} = \text{const}$, $r_{uw} \sim |\zeta|^{-2/3} \to 0$ and $r_{uT} \approx 0$. In the other limiting case as $\zeta \to \infty$ (i.e., with unbounded increase of stability), the turbulence characteristics by Eq. (7.84) cannot depend explicitly on z. Therefore as $\zeta \to \infty$, all the functions of Eq. (7.85) [and their combinations (7.86)] must tend to constants. Finally, as $|\zeta| \to 0$, the fluctuations of temperature vanish [see Eqs. (7.82)] and $T_* \to 0$. Consequently, $f_6(0)$ and $f_7(0)$ can only be interpreted as limits; in fact, the functions $f_6(\zeta)$ and $f_7(\zeta)$

must be considered as consisting of two separate branches (for $\zeta > 0$ and for $\zeta < 0$). For the constants $f_3(0) = A_3$, $f_4(0) = A_4$ and $f_5(0) = A_5$, these describe the intensity of velocity fluctuations in the logarithmic layer of a homogeneous fluid and have well-defined values (see above, Sect 5.3 where these constants are denoted by A_1, A_2 and A_3). For small values of $|\zeta|$, f_3, f_4, and f_5 may probably be described approximately by two terms of the Taylor series

$$f_i(\zeta) = A_i + \beta_i \zeta, \qquad i = 3,\ 4,\ 5, \tag{7.88}$$

where the coefficients β_i are all negative (since turbulent exchange becomes weaker as stability increases).

Equations similar to Eq. (7.85) [and containing new universal functions] may be written for higher-order moments of the fluctuations u' v', w', and T'. Thus, for example:

$$\begin{gathered}\frac{\overline{w'^3}}{u_*^3} = f_8(\zeta), \quad \frac{\overline{T'^3}}{|T_*|^3} = f_9(\zeta), \\ \frac{\overline{w'(u'^2 + v'^2 + w'^2)}}{u_*^3} = f_{10}(\zeta), \quad \frac{\overline{w'T'^2}}{u_* T_*^2} = f_{11}(\xi).\end{gathered} \tag{7.89}$$

Functions f_8 and f_9 describe the skewness of the probability distribution for w' and T':

$$\frac{\overline{w'^3}}{\sigma_w^3} = \frac{f_8}{f_5^3}, \quad \frac{\overline{T'^3}}{\sigma_T^3} = \frac{f_9}{f_6^3}, \tag{7.90}$$

while functions f_{10} and f_{11} describe the vertical diffusion of turbulent energy and of the intensity of temperature fluctuations. On the basis of the concept of thermal convection, as a set of ascending motions of warm air in the form of comparatively intensive narrow jets and descending motions of cool air in the form of the slow sinking of considerable masses, we may expect that $f_8, \ldots, f_{11}$ will be positive, at least for negative ζ. According to Eqs. (7.82)–(7.84), as the instability increases (as $\zeta \to -\infty$) the functions f_8 and f_{10} must increase asymptotically similar to $|\zeta|$, f_9 must decrease asymptotically similar to $|\zeta|^{-1}$ (so that the skewness factors f_8/f_5^3 and f_9/f_6^3 must tend to constants), and $f_{11}(\zeta)$ decreases similar to $|\zeta|^{-1/3}$. With increase of the stability (as $\zeta \to \infty$) all functions $f_8, \ldots, f_{11}$ and also the ratios f_8/f_5^3 and f_9/f_6^3 must tend to constants.

The statistical characteristics of the velocity and temperature derivatives at a fixed point (x, y, z, t), generally speaking, will now also depend on the molecular viscosity and thermal diffusivity ν and χ (recall that in the formulation of the similarity hypothesis it was required that the differences $|t_i - t_j|$ and the distances between different points (x_i, y_i, z_i) and (x_j, y_j, z_j) should not be too small). Thus applying dimensional reasoning to such characteristics leads to more complicated formulas which now contain universal functions of several variables. However, there are two important exceptions to this rule, relating to the quantities

$$\bar{\varepsilon} = \frac{1}{2}\nu \sum_{i,j} \overline{\left(\frac{\partial u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)^2} \tag{7.91}$$

and

$$\bar{N} = \chi\overline{(\nabla T')^2} = \chi \sum_{i} \overline{\left(\frac{\partial T'}{\partial x_i}\right)^2}. \tag{7.92}$$

In fact, these quantities occur in the turbulent energy balance equation (6.43), where $\bar{\varepsilon}$ is denoted by the symbol $\bar{\varepsilon}_t$, and in the related equation (6.55) for the balance of temperature fluctuation intensity, and with the aid of these equations can be expressed in terms of the ordinary one-point moments of the velocity and temperature fluctuations. Thus $\bar{\varepsilon}$ and $\bar{N}$ should not depend explicitly on the molecular constants ν and χ. This is no fortuitous fact; as we shall show in Chapt. 8 in Volume 2 of this book, $\bar{\varepsilon}$ and $\bar{N}$ play a considerable role in the laws of relatively large-scale motion and can be determined from the probability distributions for the fluctuations in velocity and temperature at two points sufficiently far apart. Thus, to describe the dependence of $\bar{\varepsilon}$ and $\bar{N}$ on the height z, we may use the ordinary similarity formula (7.13) which does not contain ν and χ. That is, we may put

$$\bar{\varepsilon} = \frac{u_*^3}{\varkappa z}\varphi_\varepsilon(\zeta), \quad \bar{N} = \frac{\varkappa u_* T_*^2}{z}\varphi_N(\zeta), \tag{7.93}$$

where φ_ε and φ_N are new universal functions. As $|\zeta| \to 0$, φ_ε and φ_N will tend to finite limits $\varphi_\varepsilon(0) = 1$ (since here $\bar{\varepsilon} = u_*^2 \frac{\partial \bar{u}}{\partial z}$, see above, Sect.

6.5), and $\varphi_N(0) = \text{const} = 1/\alpha_0$; moreover $\varphi_\varepsilon(\zeta) \sim |\zeta|$ and $\varphi_N(\zeta) \sim |\zeta|^{-1/3}$ as $\zeta \to -\infty$. The formulas for the movements of fluctuations at several points are considerably more complicated; these will contain, in all cases, universal functions of several variables. Thus, for example, the two-point moments of the simultaneous turbulent fluctuations at the points (x_1, y_1, z_1) and (x_2, y_2, z_2) will depend, generally speaking, on the four variables $\frac{z_1}{L}, \frac{z_2}{L}, \frac{x_2 - x_1}{L}$ and $\frac{y_2 - y_1}{L}$. Practically speaking, the experimental determination of functions of four variables is hopeless at present; thus conditions in which the number of variables in the formulas for two-point moments can be reduced are of great interest. In particular, one such condition is that $-\zeta > \zeta_1$ where ζ_1 is a positive number defining the lower bound of the range of values of $|\zeta|$ for which the turbulent regime will be a regime of free convection (we shall discuss the order of magnitude of ζ_1 in the following section). Under this condition, we may take $u_* = 0$, so that no finite length scale can be formulated from the parameters of the problem. Moreover, in this case there is no favored direction in the Oxy plane, thus all scalar (i.e., independent of the orientation of the coordinate axes) two-point moments can depend only on $r_0 = [(x_2 - x_1)^2 + (y_2 - y_1)^2]^{1/2}$, but not on $x_2 - x_1$ and $y_2 - y_1$ separately. From z_1, z_2 and r_0, two dimensionless combinations may be formulated, for example, z_2/z_1 and $r_0/\sqrt{z_1 z_2}$ (or z_2/z_1 and $r/\sqrt{z_1 z_2}$, where $r = [r_0^2 + (z_2 - z_1)^2]^{1/2}$). Thus in conditions of free convection, all scalar two-point moments will be equal to some combinations of the parameters g/T_0, $q/c_p\rho_0$ and $z = \sqrt{z_1 z_2}$, multiplied by a universal function of z_2/z_1 and r_0/z (or of z_2/z_1 and r/z). In particular,

$$\begin{aligned}
&\overline{w'(x_1, y_1, z_1, t)\, w'(x_2, y_2, z_2, t)} = \\
&\qquad = \left(\frac{q}{c_p\rho_0}\right)^{2/3} \left(\frac{g}{T_0}\right)^{2/3} (z_1 z_2)^{1/3} R_1\left(\frac{z_2}{z_1}, \frac{r}{\sqrt{z_1 z_2}}\right), \\
&\overline{u'(x_1, y_1, z_1, t)\, u'(x_2, y_2, z_2, t)} = \overline{v'(x_1, y_1, z_1, t)\, v'(x_2, y_2, z_2, t)} = \\
&\qquad = \left(\frac{q}{c_p\rho_0}\right)^{2/3} \left(\frac{g}{T_0}\right)^{2/3} (z_1 z_2)^{1/3} R_2\left(\frac{z_2}{z_1}, \frac{r}{\sqrt{z_1 z_2}}\right), \qquad (7.94) \\
&\overline{T'(x_1, y_1, z_1, t)\, T'(x_2, y_2, z_2, t)} = \\
&\qquad = \left(\frac{q}{c_p\rho_0}\right)^{4/3} \left(\frac{g}{T_0}\right)^{-2/3} (z_1 z_2)^{-1/3} R_3\left(\frac{z_2}{z_1}, \frac{r}{\sqrt{z_1 z_2}}\right),
\end{aligned}$$

where R_1, R_2, and R_3 are universal functions of two variables [these formulas were presented by Obukhov (1960)].

The experimental determination of functions of two variables is also fairly complicated, but not hopeless. However, it is noteworthy that in certain cases the form of the dependence on one of the variables has been predicted successfully in a theoretical manner, while for a number of statistical characteristics it has proved possible, in general, to reduce all the indeterminacy existing in the theoretical formulas to an indeterminacy in the choice of a numerical coefficient. For this it is only necessary to use some additional similarity considerations. These considerations are related to a completely different class of turbulent flows, which includes atmospheric turbulence as a special case. Chapter 8 in Volume 2 will be devoted mainly to this type of similarity; therefore, we shall postpone any further analysis of formulas of the type of Eq. (7.94) to Volume 2 of this book.

As in the case of f, f_1, f_2, a number of additional nonrigorous results concerning the functions $f_3, \ldots, f_{10}$, φ_8 and φ_N may be obtained using different variants of the semiempirical theory of turbulence. Thus, for example, Kazanskiy and Monin (1957) [see also Monin (1959a)], on the basis of the simplified energy balance equation (6.46) supplemented by some further semiempirical hypotheses, obtained for $f_5(\zeta)$ an approximate equation of the form

$$f_5(\zeta) = A_5\left[1 - \frac{1}{f'(\zeta)}\right]^{1/4}, \tag{7.95}$$

where $A_5 = f_5(0)$ is an empirical constant. The same considerations, applied to the balance equation in the form (6.46′), leads to the relationship

$$f_5(\zeta) = A_5\left[1 - \frac{\sigma}{f'(\zeta)}\right]^{1/4}, \tag{7.95′}$$

which contains a further empirical constant σ. Later, Panofsky and McCormick (1960) advanced a hypothesis which led them to conclude that

$$f_5(\zeta) = A_5\,[\zeta f'(\zeta) - B\zeta]^{1/3}, \tag{7.96}$$

where B is a new empirical constant, while Businger (1959) and Munn (1961), using two different assumptions on the mixing length in a stratified medium, obtained the formulas

$$f_5(\zeta) = \left\{\frac{\sigma + [\sigma^2 + 4\,(f'(\zeta) - \sigma)^2]^{1/2}}{2f'(\zeta)}\right\}^{1/2} \tag{7.97}$$

[where σ is the same coefficient as in Eq. (7.74)] and

$$f_5(\zeta) = -A_5\,\frac{f''(\zeta)}{[f'(\zeta)]^2}, \tag{7.98}$$

respectively. Equation (7.98) with $A_5 = 1$ (which contradicts the data) was also used by Yokoyama (1962a). For $f_6(\zeta)$, Priestley (1960b) and Panofsky (1961b) used the semiempirical relationship

$$f_6(\zeta) = A_6 \alpha_0 \zeta f_1'(\zeta), \quad A_6 = f_6(0). \tag{7.99}$$

Monin (1965) considered the complete system of dynamic equations for the one-point, second-order moments of velocity and temperature fluctuations, and neglected the terms in this system which describe the vertical transfer of these quantities (i.e., in particular, all terms containing third-order moments), while he eliminated the pressure fluctuations with the aid of semiempirical hypotheses of the type (6.12). He thus obtained the approximate relationship

$$\left[f_3^2(\zeta) - f_4^2(\zeta)\right] f_5^2(\zeta) \approx 2, \tag{7.100}$$

connecting the values of the functions f_3, f_4, and f_5 (for $\zeta = 0$, it is supported by the data given in Sect. 5.3). Further, using certain additional semiempirical hypotheses, Monin was able to express the anisotropy coefficients $f_3(\zeta)/f_4(\zeta)$ and $f_5(\zeta)/f_4(\zeta)$ in terms of $\varphi(\zeta) = \zeta f'(\zeta)$ and the constants $A_3 = f_3(0)$, $A_4 = f_4(0)$ and $A_5 = f_5(0)$ which are known from experiment (while it was shown that on these assumptions, the inequality $C_2 = 1/R \geqslant 5.5$ must be valid). A further semiempirical relationship was proposed by Pandolfo (1936), connecting the functions $\varphi(\zeta)$, $f(\zeta)$, and $f_5(\zeta)$. By using some fairly rough hypotheses, Panchev (1961) obtained the following connections between the coefficients C_1, C_6', and C_6' of Eqs. (7.36) and (7.87'):

$$C_6' = \frac{1}{C_5'} = \left(\frac{C_1}{6}\right)^{1/4} \tag{7.101}$$

(from which, in particular, it follows, that in free convection $r_{wT} = 1$, which seems quite doubtful).

The connection between the universal functions $\varphi_\varepsilon(\zeta)$, $f(\zeta)$ and $f_{10}(\zeta)$ may be obtained from the energy balance equation (6.43); according to this equation, it seems probable that

$$\varphi_\varepsilon(\zeta) = \zeta f'(\zeta) - \zeta - \frac{\varkappa}{2} \zeta f_{10}'(\zeta) \tag{7.102}$$

[see Monin (1958)]. Equation (7.102) is exact to the degree in which it is permissible to ignore the vertical transfer of turbulent energy due to the pressure fluctuations and viscous friction (it becomes quite exact if we include the summand $2\overline{p'w'}/\rho_0 u_*^3$ in the definition of the function f_{10}). If we completely ignore the vertical diffusion of energy [i.e., we adopt Eq. (6.46)], then we obtain

$$\varphi_\varepsilon(\zeta) = \zeta f'(\zeta) - \zeta, \tag{7.103}$$

while if we consider this diffusion to be proportional to $\dfrac{g}{T_0} \dfrac{q}{c_p \rho}$ (i.e., if we adopt Eq. (6.46'), then

$$\varphi_\varepsilon(\zeta) = \zeta f'(\zeta) - \sigma\zeta. \tag{7.103'}$$

Of course, both Eqs. (7.103) and (7.103') are, in general, less exact than Eq. (7.102) [but they are equivalent to Eq. (7.102) in the limiting case of neutral stratification and Eq. (7.103') is equivalent to Eqs. (7.102) and (6.43) in the case of free convection, since then

$f_{10}(\zeta) \sim \zeta]$. Remembering the asymptotic formulas (7.33) and (7.39′) for the function $f(\zeta)$ and taking as starting point, for example, Eq. (7.103′) we find

$$\varphi_\varepsilon(\zeta) = \begin{cases} -\sigma\zeta = \dfrac{\sigma z}{|L|} & \text{for} \quad \zeta \to -\infty, \\ 1 + (\beta - \sigma)\zeta & \text{for} \quad |\zeta| \ll 1. \end{cases} \tag{7.104}$$

Thus, according to Eq. (7.103′) in nearly neutral stratification, $\bar{\varepsilon} \approx \dfrac{u_*^3}{\varkappa z}$; and in the case of intense instability $\bar{\varepsilon} \approx \sigma \dfrac{g}{T_0}\left|\dfrac{q}{c_p\rho_0}\right|$, i.e., it ceases to depend on the height (the same results, but with undetermined numerical coefficients in both cases are obtained by dimensional considerations only).

Similar relationships may also be deduced for $\varphi_N(\zeta)$ if we use as our starting point the balance equation for intensity of temperature fluctuations (6.55). Using this equation we easily obtain the relationship:

$$\varphi_N(\zeta) = \zeta f_1'(\zeta) - \frac{1}{2\varkappa}\zeta f_{11}'(\zeta). \tag{7.105}$$

Moreover, if we also ignore the vertical diffusion of the intensity of temperature fluctuations [first term on the right side of Eq. (6.55)], then we obtain

$$\varphi_N(\zeta) = \zeta f_1'(\zeta). \tag{7.106}$$

Function (7.106) possesses the properties

$$\varphi_N(\zeta) = \begin{cases} \dfrac{\varkappa^{4/3}}{3} C_1 |\zeta|^{-\frac{1}{3}} & \text{as} \quad \zeta \to -\infty, \\ \dfrac{(1 + \beta\zeta)}{\alpha_0} & \text{as} \quad |\zeta| \ll 1. \end{cases} \tag{7.107}$$

Consequently, in this approximation for near-neutral stratification, $\bar{N} \approx \dfrac{\varkappa u_* T_*^2}{\alpha_0 z}$ while for great instability, $\bar{N}$ decreases with height according to the law

$$\bar{N} \approx \frac{C_1}{3}\left(\frac{q}{c_p\rho_0}\right)^{\frac{5}{3}}\left(\frac{g}{T_0}\right)^{-\frac{1}{3}} z^{-\frac{4}{3}}. \tag{7.108}$$

The result for near-neutral stratification does not depend on ignoring the diffusion term, while the effect of diffusion may lead to a change of the numerical coefficient in Eq. (7.108). The behavior of $\varphi_N(\zeta)$ when $\zeta \to +\infty$ cannot be stated without knowing how the function $\alpha(\zeta) = f'(\zeta)/f_1(\zeta)$ behaves for large positive ζ.

Of course, we may repeat for formulas (7.95)–(7.101), (7.103), (7.103′), and (7.108) all that has been said previously about the analogous semiempirical formulas relating to f, f_1, and f_2; they are all only approximations, the accuracy of which and their regions of applicability must be found by comparing them with good-quality experimental results. At present, the data on fluctuations are still very incomplete; nevertheless, the semiempirical formulas can be of considerable value, since in many cases they can help us to estimate with fair accuracy the order of magnitude of the corresponding statistical characteristics, and serve as a general guide in working up the results of the existing (comparatively rough) observations.

8. COMPARISON OF THE THEORETICAL DEDUCTIONS WITH THE DATA

8.1 Wind-Velocity Profiles in the Surface Layer of the Atmosphere

Here we compare data from meteorological measurements in the atmospheric surface layer with the theoretical deductions of Sect. 7. We take first the simplest measurements of the mean wind velocity at different heights z, repeated many times at different points on the surface of the earth with the aid of various types of anemometers [many of which are described, for example, in the books of Middleton and Spilhaus (1953); Kedrolivanskiy and Sternzat (1953) and Lettau and Davidson (1957)]. It was already noted in Sect. 5.4 that when the underlying surface is comparatively flat and the temperature stratification is close to neutral, the data on the wind-velocity profile in the lower atmosphere are described fairly well by the logarithmic formula (5.31). However, in the diabatic case (i.e., with considerable temperature gradients), the dependence of the wind velocity $\bar{u}(z)$ on $\ln z$ [data on which may be found, for example, in Thornthwaite and Kaser (1943), Shcherbakova (1949), Deacon (1949), Pasquill (1949), Monin (1953) and in many dozens of more recent works] exhibits regular deviations from the simple linear relationship. For unstable stratification, the increase of $\bar{u}(z)$ with increase of $\ln z$ is always slower than linear, while for stable stratification it is faster than linear. As a typical example, Fig. 52 [borrowed from Monin (1953)], gives mean wind-velocity profiles obtained in 1951, from meteorological measurements in the Kazakhstan steppe at heights of 0.5, 1, 2, 4, 8, and 15 m. The six curves in Fig. 52 were obtained by averaging 61 individual wind-velocity profiles in six groups, each characterized by nearly constant values of the stability parameter

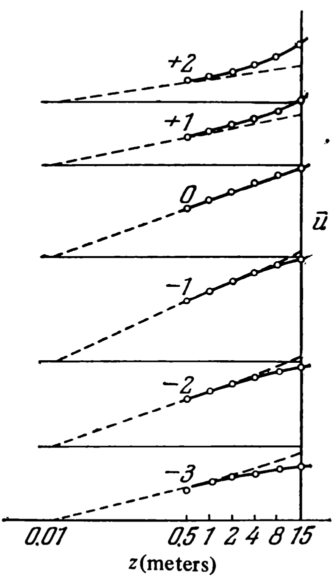


FIG. 52. Empirical profiles of the mean wind velocity in the surface layer for different thermal stratifications.

$$B_1 = \frac{g}{T_0} \frac{\bar{T}(2) - \bar{T}(0.5)}{[\bar{u}(1)]^2} \tag{8.1}$$

where T_0 is the standard temperature, and the figures in parenthesis indicate the height (in meters) of the observation point.

The six stability groups indicated by the numbers +2, +1, . . . , −3 in Fig. 52, correspond to stabihty classes ranging from moderate stability (group +2) through slight stability (+1), near-neutral stratification (0), slight instability (−1) and moderate instability (−2) to intense instability (−3).

We see that only in the neutral stratification does the logarithmic formula give a good representation of the whole wind-velocity profile (i.e., it is represented by a straight line in Fig. 52). In all other cases, at sufficiently great heights, deviations from the logarithmic law are observed and are in qualitative agreement with the theoretical predictions of the preceding section. However, at the same time, at low heights the wind-velocity profiles for all groups will be given approximately by a formula of the type $\bar{u}(z) = A \ln \frac{z}{z_0}$ with the same value of the "roughness height" z_0 (which is, of course, established most reliably using profiles with near-neutral stratification); in the conditions of Fig. 52, this height is close to 1 cm.

According to the similarity theory of Sect. 7, the height variation of the wind velocity in all cases must be determined by the universal function $f(\zeta)$ of the dimensionless height $\zeta = \frac{z}{L}$, where $L = -c_p\rho_0T_0u_*^3/\varkappa gq$. To verify this result and to establish the exact form of the function $f(\zeta)$, in addition to the values of $\bar{u}(z)$, we require the values of $u_* = \left(\frac{\tau}{\rho_0}\right)^{1/2}$ and q, which permit us to evaluate the length scale L. However, direct measurement of the turbulent momentum and heat fluxes τ and q in the surface layer of the atmosphere (which is the subject of a special discussion in Sect. 8.3) began to be carried out only comparatively recently, and, so far, are fairly infrequent and rather inaccurate. Thus the first empirical verification of Eq. (7.24) containing the function $f(\zeta)$, which permitted a typical graph of this function to be drawn for the first time, was carried out by Monin and Obukhov (1953; 1954) by a quite different method. As a basis for this verification, they used extensive data (collected during four expeditions in 1945, 1947, 1950 and 1951 in different regions of the U.S.S.R.) relating to wind and temperature profiles under different meteorological conditions. Further, it was assumed that formulas (7.24) hold and also some additional assumptions were adopted as to the form of the functions $f(\zeta)$ and $f_1(\zeta)$. The satisfactory agreement between the results of data processing on the basis of these assumptions, confirmed their admissibility as a reasonable first approximation, and showed that the general form of the function $f(\zeta)$ must be consistent with its empirical values found in this manner.

The additional assumptions on the wind and temperature profiles made by Monin and Obukhov consisted of assuming the validity of the "logarithmic + linear" approximation (7.33) of $f(\zeta)$ in describing the wind profile in the lowest four meters, and of assuming the similarity of the wind and temperature profiles within this layer. On this basis, all these wind profiles, within the lowest four meters, were approximated by a formula of the form

$$u_i(z) = A_i(\log z + D) + C_i z, \tag{8.2}$$

where i is the number of the profile, and A_i, C_i, and D are empirical coefficients (determined by the least squares method). Since all measurements refer to heights $z \gg z_0$, the coefficient D is assumed to be the same for all profiles of the same observation point and equal to $-\log z_0$, where z_0 is the roughness parameter. The values of the coefficients A_i and C_i then allow us to find for every profile the values of $\frac{u_*}{\varkappa}$ and $\frac{\beta}{L}$ according to the formulas

$$\frac{u_*}{\varkappa} = \frac{A_i}{\ln 10}; \qquad \frac{\beta}{L} = \frac{C_i \ln 10}{A_i}. \tag{8.3}$$

Further, for each profile, the empirical value of the stability parameter $B_1 = \frac{g}{T_0} \frac{\overline{T}(2) - \overline{T}(0.5)}{[\overline{u}(1)]^2}$ is evaluated which, by Eq. (7.33) and the assumption of the similarity of the $\overline{u}(z)$ and $\overline{T}(z)$ profiles, is taken to be equal to the quantity

$$B_1 = \frac{g}{T_0} \frac{\varkappa^2 T_*}{u_*^2} \frac{\ln 4 + \frac{3}{2}\frac{\beta}{L}}{\left(\ln \frac{1}{z_0} + \frac{\beta}{L}\right)^2} = \frac{1}{\beta}\Phi\left(\frac{\beta}{L}\right), \qquad \Phi(t) = \frac{t\left(\ln 4 + \frac{3}{2}t\right)}{\left(\ln \frac{1}{z_0} + t\right)^2}.$$

Knowing that $z_0 = 10^{-D}$, we may also define β as the regression coefficient of the set of values $\Phi\left(\frac{\beta}{L}\right)$ [where $\frac{\beta}{L}$ is taken from Eq. (8.3)] on the empirical values of B_1. It is then found that $\beta \approx 0.6$, while the correlation between the values of $\Phi\left(\frac{\beta}{L}\right)$ and B_1 was fairly high. The values of $\frac{u_*}{\varkappa}$ and $L = \frac{0.6}{\beta/L}$ together with all the data on $\bar{u}(z)$ were used to formulate the empirical function

$$f(\zeta) - f\left(\pm \frac{1}{2}\right) = \frac{\varkappa}{u_*}\left[\bar{u}(z) - \bar{u}\left(\frac{|L|}{2}\right)\right], \tag{8.4}$$

where the plus sign denotes stable stratification and the minus sign, unstable stratification [the value of $\bar{u}\left(\frac{|L|}{2}\right)$ is obtained by interpolation of the measured values of $\bar{u}(z)$]. The obtained results shown in Fig. 53 allow us to formulate the empirical function $f(\zeta)$ for the range $-3 < \zeta < 6$. In spite of lack of precision in measuring the wind velocity and the comparatively rough approximate method used to determine the values of $\frac{u_*}{\varkappa}$ and L, the empirical points lie on a smooth curve (consisting of two branches) with small scatter (except, perhaps, for cases of large positive ζ, for which the scatter becomes fairly considerable). The graph thus obtained confirms the existence of the universal function $f(\zeta)$ which determines the dependence of the wind velocity on height. Also, the empirical function $f(\zeta)$, taken as a whole, is in agreement with the schematic graph in Fig. 48, i.e., it is found to have a number of asymptotic properties predicted in the previous section: in near-neutral stratification (small $|\zeta|$) it is close to $\ln |\zeta| + \text{const}$, for great instability ($\zeta \ll -1$) it tends asymptotically to a constant, while for great stability ($\zeta \gg 1$), it increases approximately similar to a linear function (in Fig. 53, these three asymptotes are represented by dotted lines). As later investigations showed, the value of the function $f(\zeta)$ found by Monin and Obukhov in 1953–1954 proved to be fairly accurate for $\zeta \leqslant 0$, but for $\zeta > 0$ they were substantially too low (in fact, the graph of $f(\zeta)$ for ζ increasing apparently rises considerably more steeply than the graph shown in Fig. 53).

Graphs similar to that in Fig. 53 have been plotted frequently by different authors on the basis of other experimental evidence. Thus,

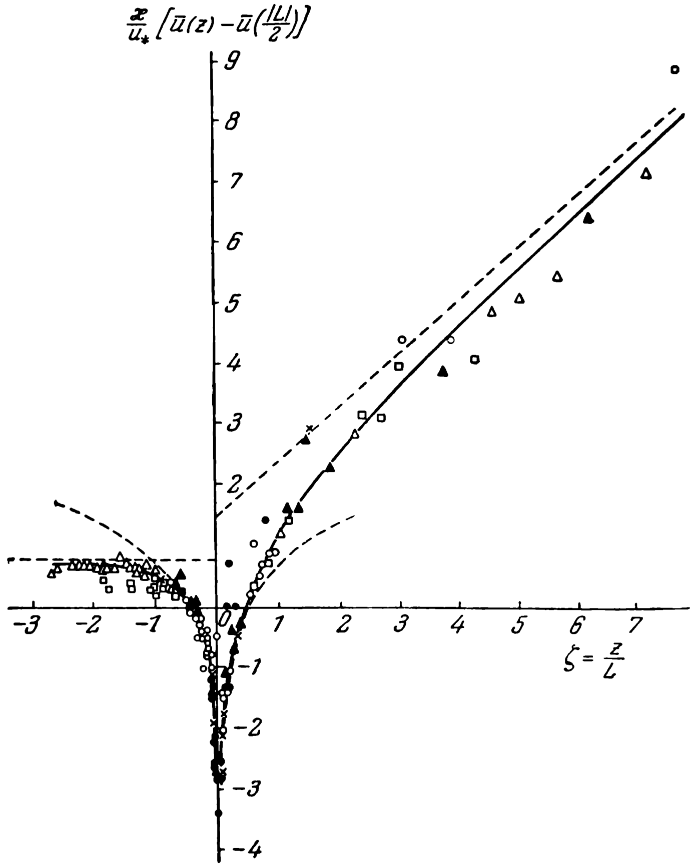


FIG. 53. Empirical graph of the function $f(\zeta) - f\left(\pm \frac{1}{2}\right)$ according to Monin and Obukhov (1953; 1954). The various symbols on the graph denote data obtained in different expeditions.

Perepelkina (1959b) used the data on the wind and temperature profiles for $z \leqslant 2$ m, by Pasquill (1949) and Rider (1954), for an empirical determination of the values of $f(\zeta) - f\left(\pm \frac{1}{2}\right)$ for $|\zeta| < \frac{1}{2}$. Later, she also worked out the data on the wind and temperature profiles up to a height of 70 m, carried out in the summer and autumn of 1960–1961, from a 70-m meteorological mast in the south Russian steppe close to the town of Tsimlyansk. Still later, Shiotani (1962) published a graph of the function $f(\zeta) - f\left(\pm \frac{1}{2}\right)$ for

$\frac{1}{4} < |\zeta| < \frac{3}{2}$ similar to Fig. 53, and also a separate detailed graph of $f(\zeta) - f\left(\pm \frac{1}{10}\right)$ for $|\zeta| < 0.3$, obtained using a method similar to that described above from the data of observations on wind and temperature profiles carried out in 1960–1962 on a 45-meter television tower near Tokyo. As soon as data of the direct measurement of u_* and q appeared, attempts were made to plot a graph similar to Fig. 53, in which values of L would be determined according to the results of such measurements (made simultaneously with profile measurements), without recourse to any special hypotheses. Such graphs were plotted, for example, by Gurvich (1965) and by Zilitinkevich and Chalikov (1968a) with reference to the data obtained from 1962 to 1965 in the region of Tsimlyansk (see below, Figs. 54 and 55).

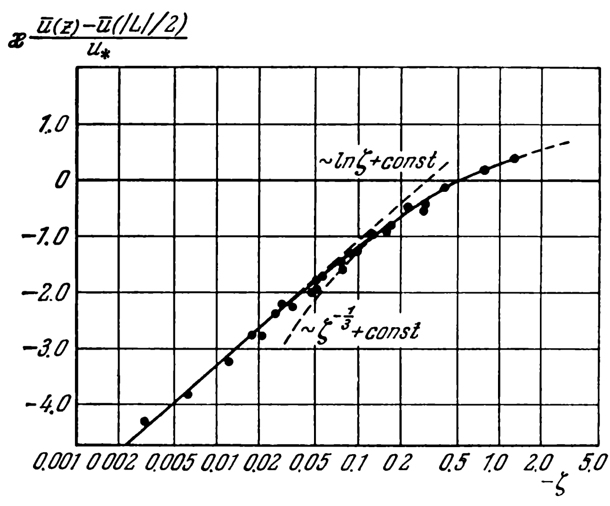


FIG. 54. Empirical graph of the function $f(\zeta) - f\left(-\frac{1}{2}\right)$ for $\zeta < 0$ according to the data of Gurvich (1965).

The graphs of the function $f(\zeta) - f\left(\pm \frac{1}{2}\right)$ given in Fig. 53 are intuitively clear and give a good description of the general pattern of the dependence of the wind velocity on height for unstable and stable stratification. However, for the quantitative estimation of the

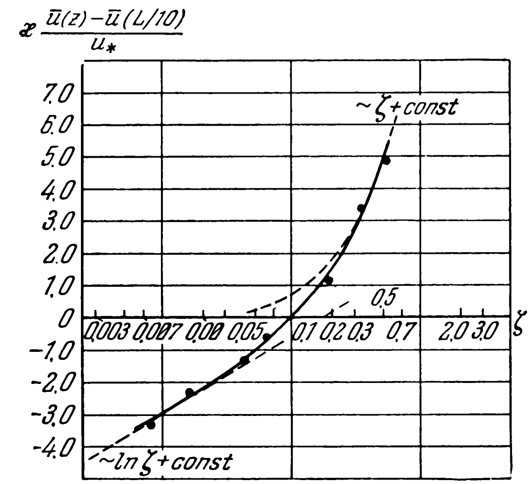


FIG. 55. Empirical graph of the function $f(\zeta) - f\left(\frac{1}{10}\right)$ for $\zeta > 0$ according to the data of Gurvich (1965).

deviations of the function $f(\zeta)$ from the logarithmic function for small $|\zeta|$ and its deviations from the constant value for large negative ζ, these graphs are not very suitable, since corresponding parts of the empirical curves on such graphs are compressed greatly. As a result, several researchers have either used modifications of the graph of Fig. 53, or modifications of the empirical function $f(\zeta)$ itself, which are more convenient in certain respects. Thus, for example, Priestley (1959a) modified the section of Fig. 53 corresponding to negative ζ (unstable stratification) by replacing the linear scale of the dimensionless heights $\zeta = \frac{z}{L}$ by a logarithmic scale. The deviations of $f(\zeta)$ from the logarithmic function for small $|\zeta|$ then became much more marked, and, in particular, it became evident that even for $\zeta \approx -0.05$, these deviations are fairly considerable (although in Fig. 53 it seems that the continuous and dotted lines do not differ from each other for $|\zeta| < 0.5$). Similar graphs [but with $\bar{u}\left(\frac{|L|}{2}\right)$ replaced by $\bar{u}\left(\frac{|L|}{10}\right)$, which in many cases is determined more easily from the measured values of $\bar{u}(z)$] were later reported by R. J. Taylor (1960a), who used the measurements of Rider (1954) and Swinbank (1955), and by Takeuchi (1961), who obtained extensive data from observations made on a prairie near O'Neill, Nebraska (U.S.A.), in 1953 and 1956, and collected in the monographs of Lettau and Davidson (1957) and of Barad (1958). R. J. Taylor determined the values of $\frac{u_*}{\varkappa}$ and L according to direct measurements of $\tau = \rho_0 u_*^2$ and q made by Rider and Swinbank, while for this purpose Takeuchi used an approximation of the wind and temperature profiles by functions of the form (8.2) and formulas (7.33) and (7.47) [with $\alpha_0 = 1$, $\beta_1 = \beta$]. At the same time, Takeuchi compared the empirical values of L obtained by this method with the values obtained by the Monin-Obukhov method (1954) which is similar in principle, but different in certain details, and uses measured values of u_* and q. According to his results, all three methods of determining L agree satisfactorily with each other. Later, Gurvich (1965) plotted the same graphs using the data of simultaneous measurements of the wind profile, turbulent heat flux and shear stress made in the summer of 1962 and 1963 in the steppe close to the town of Tsimlyansk (see Figs. 54 and 55, the points of which correspond to data averaged over a series of measurements). Additional results of treatment of the wind profile data based on the similarity theory of Sect. 7 may be found in scores of works including those by Panofsky, Blackadar and McVehil (1960), Panofsky (1963; 1965), Lumley and Panofsky (1964), Panofsky, Busch, Prasad et al. (1967), Webb (1960; 1965), J. J. O'Brien (1965), Swinbank (1964; 1966; 1968), Swinbank and Dyer (1968), Busch (1965), Busch, Frizzola and Singer (1968), Bernstein (1966), Charnock (1967a,b), Paulson (1967), Businger, Miyake, Dyer and Bradley (1967), Zilitinkevich and Chalikov (1968a), Rijkoort (1968), Fichtl (1968), and others. Similar results have also been obtained recently by investigators who have made laboratory measurements in special wind-tunnels with a portion of the floor that can be heated or cooled to produce thermal stratification [cf., for example, Malhotra and Cermak (1963); Cermak, Sandborn et al. (1966); Cermak and Chuang (1967) and Chuang and Cermak (1967)]. Although the scatter of the experimental points is considerable in all the measurements, the values of $f(\zeta)$ found by different investigators are, on the whole, in satisfactory agreement with each other. (Some of the discrepancies discovered will be pointed out in the subsequent discussion of the results.)

In several cases, when the experimental results are obtained, the basis adopted is not the wind profile $\bar{u}(z)$ itself, but the values of the wind-velocity gradient $\frac{\partial \bar{u}(z)}{\partial z}$. According to Eq. (7.15′), to obtain the universal relationship, we must consider the dimensionless quantity

$$\frac{\varkappa z}{u_*}\frac{\partial \bar{u}}{\partial z}=\varphi\left(\frac{z}{L}\right), \qquad \varphi(\zeta)=\zeta f'(\zeta), \tag{8.5}$$

or, which is the same, the quantity

$$\overset{*}{K}=\frac{K}{u_* z}=\frac{u_*}{z\dfrac{\partial \bar{u}}{\partial z}}=\frac{\varkappa}{\varphi(\zeta)}. \tag{8.6}$$

However, since the evaluation of $\frac{z}{L}=\zeta$ requires the values of the turbulent fluxes $\tau=\rho u_*^2$ and q, (on the latter of which there exist, so far, relatively few reliable data), the values of $\overset{*}{K}$ have often been determined according to the Richardson number

$$\mathrm{Ri}=\frac{g}{T_0}\frac{\partial \bar{T}}{\partial z}\bigg/\left(\frac{\partial \bar{u}}{\partial z}\right)^2=\frac{1}{\alpha(\zeta)f'(\zeta)}$$

which is a single-valued function of ζ. Graphs of the function $\overset{*}{K}(\mathrm{Ri})$ were reported by Rider (1954), Deacon (1955), Priestley (1959a), Ellison and Turner (1960), Gurvich (1962), Deacon and Webb (1962), and Webb (1965). Figure 56 shows the graphs of Gurvich (1962) relating to the range $-0.005>\mathrm{Ri}>-4$, and also taking into account Deacon's data (1955). The broken line in the graph gives the relationship

$$\overset{*}{K}(\mathrm{Ri})=a\,|\,\mathrm{Ri}\,|^{\frac{1}{4}}; \qquad a=\left(\frac{3}{C_2}\right)^{\frac{3}{4}}\varkappa\alpha_{-\infty}^{\frac{1}{4}}=\frac{3^{\frac{3}{4}}}{C_1^{\frac{3}{4}}\alpha_{-\infty}^{\frac{1}{2}}}, \tag{8.7}$$

corresponding to the asymptotic "1/3-power law" (7.39) for conditions of free convection, with coefficient $a=0.97$, found by the least squares method, applied to the aggregate of data with $R<-0.05$. The continuous line represents the asymptotic value $\overset{*}{K}(0)=\varkappa\approx 0.4$ of the function $\overset{*}{K}(\mathrm{Ri})$ in the logarithmic sublayer, which is determined very accurately from the graph of $\overset{*}{K}(\mathrm{Ri})$ plotted using a linear scale of Ri numbers (in exactly this way, an extremely reliable demonstration was given that in the atmosphere also $\varkappa\approx 0.4$). There are no data for $\mathrm{Ri}>0$ (stable stratification) in Fig. 56, but, in general, observations with large positive Ri are very rare in meteorology. Therefore, the most extensive data on the values of $\overset{*}{K}(\mathrm{Ri})$ for Ri > 0 were obtained by Ellison and Turner (1960) in laboratory experiments in which the role of the temperature gradient was played by a density gradient produced by the salinity gradient of water (see below, Sect. 8.2). The data of Ellison and Turner are quite scattered, but overall they demonstrate convincingly that on the positive semiaxis $\overset{*}{K}(\mathrm{Ri})$ decreases with increase of Ri, as would be expected.

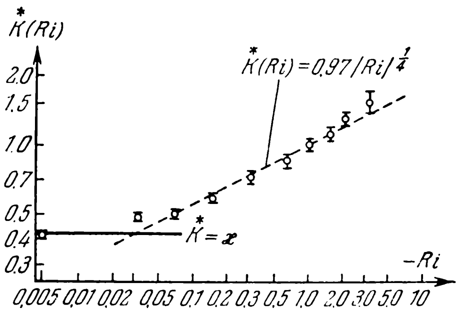


FIG. 56. Empirical graph of the function $\overset{*}{K}$ (Ri) according to Gurvich (1962).

Let us consider now in more detail the results for unstable stratification. The dependence of $\varphi = \frac{\varkappa z}{u_*}\frac{\partial \bar{u}}{\partial z}$ on the parameter $\zeta' = \frac{z}{L'}$ [where $L' = u_* \frac{\partial \bar{u}}{\partial z} \Big/ \varkappa \frac{g}{T_0}\frac{\partial \bar{T}}{\partial z} = \alpha(\zeta) L$ so that $\zeta' = \mathrm{Ri}\,\phi(\zeta')$] which is also connected single-valuedly with ζ, was studied empirically by Panofsky, Blackadar and McVehil (1960) among others. According to their results, for all negative ζ' a good approximation to $\varphi(\zeta')$ is given by the solution of an equation of the form (7.61) with the coefficient $\sigma' = \sigma a_{-\infty} = 18$ [i.e., by the relationship $\varphi(\zeta) = (1 - \sigma'\mathrm{Ri})^{-1/4}$ with $\sigma' = 18$; cf. Sect. 7.4, Eq. (7.59')]. Later, the same equation was tested on extensive data from quite different locations in several other works, including those by Kondo (1962), Lumley and Panofsky (1964), Panofsky (1965), Swinbank (1966), Panofsky, Busch et al. (1967), Bernstein (1966), Charnock (1967a), Paulson (1967), Fichtl (1968) and Rijkoort (1968). The results show some small discrepancies between them, but it is generally concluded that an equation of the form $\varphi(\zeta) = (1 - \sigma'\mathrm{Ri})^{-1/4}$ describes the existing data with sufficient accuracy for all negative values of Ri if σ' is chosen between 10 and 20 (for example, Paulson considers $\sigma' = 11$ as the best guess, while Kondo recommends the value $\sigma' = 13$ and according to the works of Panofsky and his group, $\sigma' \approx 18$ for all geographical locations). Nevertheless, some investigators prefer the original Ellison equation (7.59) [or, what is the same thing, Eq. (7.61)], which is equivalent to Eq. (7.59′) only if $\alpha(\zeta)$ is considered constant; however, in such an approximation of $\varphi(\zeta)$ the coefficient σ is usually found to be considerably smaller than σ' (with the exception of the works by Panofsky et al. (1960), and Bernstein (1966) where the value $\sigma = 14$ was recommended, and by Zilitinkevich and Chalikov (1968a), where it was found that $\sigma = 12$ for a restricted range of Ri values). Thus Panofsky (1965) and Klug (1967) recommend the value $\sigma = 7$, Pasquill [see Swinbank (1966)] finds that a value in the range from 4 to 6 is appropriate and Charnock (1967a) uses the value $\sigma = 4$ in his reexamination of Bernstein's results (in all these works the data of Swinbank's observations in Australia were used). The relative smallness of σ agrees well with the finding that $\alpha_{-\infty}$ is apparently considerably greater than unity (see below, Sect. 8.2). Let us also note that both Eqs. (7.59) and (7.59′) are in agreement with the "1/3-power law" (7.39) and $\sigma = (3/C_2)^3$, $\sigma' = \sigma\alpha_{-\infty} = (a/\kappa)^4$ where σ' is the Gurvich constant of Eq. (8.7). Therefore the data of Fig. 56 correspond to a value of σ of the order of 30 (while $\sigma' = 18$ corresponds to $a \approx 0.82$).

Webb (1960) approximated the empirical values of $\varphi(\zeta')$ on the semiaxis $\zeta' \leqslant 0$ by an equation of the form (7.57) which leads to values of $\varphi(\zeta')$ close to those obtained from the equation $\varphi(\mathrm{Ri}) = (1 - 18\mathrm{Ri})^{-1/4}$ for all not too great values of $-\zeta'$ [see Panofsky (1963) or Lumley and Panofsky (1964)], but in the limit as $\zeta' \to -\infty$ corresponds to a "1/3-power law" of the form $\varphi(\zeta') \approx -(30\zeta')^{-1/3}$ (i.e., to the equation $\varphi \approx (-30\mathrm{Ri})^{-1/4}$). Later, Webb (1965) proposed another form of the universal wind-profile equation which disagrees with the "1/3-power law" in the free convection limit. Some transformations and an approximation of the most widely used equation $\varphi(\mathrm{Ri}) = (1 - \sigma'\mathrm{Ri})^{-1/4}$ may be found in Fichtl (1968). In Zilitinkevich and Laykhtman (1965) it was stated that their semiempirical equation for $\varphi(\zeta)$ gives a satisfactory description of the data (however, the data used in this work were rather early). Swinbank (1964; 1966) found that his exponential equation (7.72) [which contradicts the "1/3-power law"] approximates the existing data quite well, but some other authors do not share this opinion [see, for example, Barad (1963); Panofsky (1965); Swinbank (1966);

cf. also Bernstein (1966) and Charnock (1967a)]. The discrepancy with the theoretically very attractive "1/3-power law" for the velocity profile makes doubtful the empirical equations for $\varphi(\zeta)$ and $\varphi_1(\zeta)$ proposed by Businger (1966) [and independently also by Dyer] and the equations of Pandolfo (1966) [see Sect. 7.4; let us recall that both these sets of equations prescribe that $\zeta = \mathrm{Ri}$]. Nevertheless, both these sets of equations found some satisfactory empirical justification and were used in certain applications [see, for example, Paulson (1967); Businger, Miyake, Dyer and Bradley (1967); Deardorff (1968); Krishna (1968); Yoshihara (1968); cf. also the discussion in Kapoor and Sundararajan (1968)]. Purely empirical graphs of the function $\varphi(\zeta)$ or $\varphi(\zeta')$ were plotted by Gurvich (1962; 1965) [see also Fig. 57, taken from the latter of these works; Busch (1965), Swinbank (1968)]; in this work, the dependence of $\zeta^{-1/3}\varphi(\zeta)$ on Ri is given. All these graphs are in agreement with the assumption of a universal dependence, but show certain small deviations from each other. Knowing the function $\varphi(\zeta)$ [or $f(\zeta)$], it is easy to find the dependence of the flux Richardson number $\mathrm{Rf} = \zeta/\varphi(\zeta)$ on ζ as well. In exactly the same way, if we know $\varphi(\zeta')$, where $\zeta' = z/L' = \zeta/\alpha(\zeta)$, it is easy to find the dependence of the ordinary Richardson number $\mathrm{Ri} = \mathrm{Rf}/\alpha = \zeta'/\varphi(\zeta')$ on ζ'. Finally, if we know the functions $\alpha(\zeta)$ [its determination will be discussed later] and $\varphi(\zeta)$ [or $\varphi(\zeta')$], then we can find the dependence of both Richardson numbers Rf and Ri on any of the variables ζ and ζ'. As an example, Fig. 58 shows the dependence of ζ on Ri which follows from the data of Mordukhovich and Tsvang (1966). Empirical graphs of the dependence of

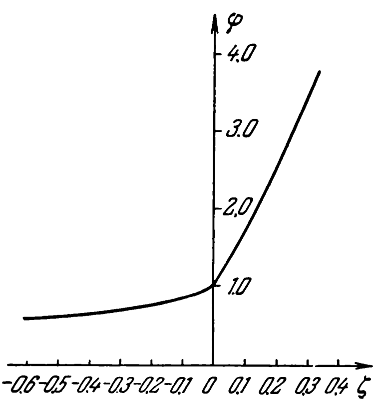


FIG. 57. Empirical graph of the function $\varphi(\zeta)$ according to the data of Gurvich.

Ri on ζ (or ζ') were also plotted by Panofsky (1963) [cf. Lumley and Panofsky (1964), Gurvich (1962), Webb (1965), Pandolfo (1966), where an artificial method is used to show that $\mathrm{Ri} = \zeta$, Swinbank (1968) and some other authors]. An empirical graph of the dependence of R_f on ζ was constructed by Cramer (1967). As $\zeta \to 0$ and $\mathrm{Ri} \to 0$, of course, $\mathrm{Rf} \approx \zeta$ and $\mathrm{Ri} \approx \zeta' \approx \zeta/\alpha_0$ [see Eq. (7.28)]; therefore it follows from the data of Fig. 58 (and from all the other data on dependence of Ri on ζ) that α_0 is close to unity.

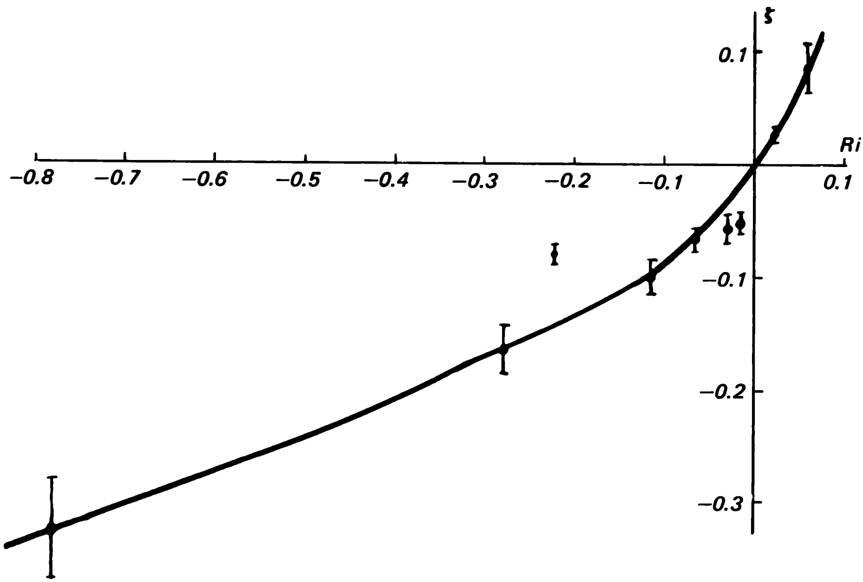


FIG. 58. The empirical dependence of ζ on Ri according to the data of Mordukhovich and Tsvang (1966).

We see that the precision of the existing data on profiles and turbulent fluxes at present is insufficient to reliably distinguish between the various expressions proposed for universal wind-profile functions. Therefore it does not seem worthwhile to discuss in detail the pros and cons of various formulations. At present, it only seems reasonable to use the physically justified laws describing the asymptotic behavior of the universal functions and to verify their agreement with experimental results; if possible it is also desirable to obtain estimates of the numerical coefficients in these laws. This program was carried out by Zilitinkevich and Chalikov (1968a) [see also Zilitinkevich (1970)] using extensive data obtained by the researchers at the Moscow Institute of Atmospheric Physics during the summers of 1963 to 1965 in the steppe near Tsimlyansk. As to unstable conditions (the stable case will be considered later), it was assumed that the simple interpolation formula (7.53) is valid, i.e., that

$$f(\zeta) = \begin{cases} \ln|\zeta| + \beta\zeta & \text{when } \zeta_1 \leqslant \zeta < 0 \\ a_1 + C_2\zeta^{-1/3} & \text{when } \zeta < \zeta_1 \end{cases} \tag{8.8}$$

where the unknown constants a_1 and β may be eliminated by the continuity conditions of the functions $f(\zeta)$ and $\varphi(\zeta) = \zeta f'(\zeta)$ at the point $\zeta = \zeta_1$. Therefore only two undetermined coefficients C_2 and ζ_1 were included in this formulation of the universal wind-profile functions on the negative semiaxis $\zeta < 0$. Further, the values of these coefficients (and of the constant κ entering into the definition of the universal functions) were determined by the least squares method from all the data of the simultaneous measurements of wind and temperature profiles and turbulent fluxes τ and q. As a result, the following least squares estimates were obtained: $C_2 \approx 1.25$, $\zeta_1 \approx -0.16$, $\kappa \approx 0.43$ which imply that $\beta \approx 1.45$ and $a_1 \approx 0.24$. The 95% confidence intervals of all the estimates proved to be rather narrow and the measured points are concentrated along the curve (8.8) with relatively small scatter; thus the asymptotic behavior of the functions $f(\zeta)$ and $\varphi(\zeta) = \zeta f'(\zeta)$ given by Eq. (8.8) agree with the experimental results treated by Zilitinkevich and Chalikov (unfortunately it is restricted to the range $0 \geqslant \zeta \geqslant -1.2$ only). It was also stated by Zilitinkevich and Chalikov that the same data may also be described without noticeable increase in the scatter by the equation

$$f(\zeta) = \begin{cases} \ln|\zeta| & \text{when } -0.07 < \zeta < 0 \\ 1.2\,\zeta^{-1/3} + 0.25 & \text{when } \zeta < -0.07 \end{cases}$$

of the form (7.49), or by Eq. (7.61) when $\sigma = 12$, or by a semiempirical equation related to one proposed by Zilitinkevich and Laykhtman (1965).

An empirical verification of the "1/3-power law" for wind velocity profile was also given (based on various meteorological data by R. J. Taylor (1960a, b) and by Gurvich (1962; 1965) [cf. Figs. 54 and 56]. In all cases the data confirm the law, but the estimates obtained for the coefficient m display some scatter: R. J. Taylor gives values of C_2 ranging from 0.9 to 1.5, while $C_2 \approx 0.93\alpha_{-\infty}^{1/3}$ according to the data of Fig. 56 and $C_2 \approx 1.4$ according to the data of Fig. 54. Let us also recall that $C_2 = 3/\sigma^{1/3}$; thus the value $\sigma = 7$ [recommended by Panofsky (1965)] corresponds to $C_2 \approx 1.55$.

It is important to note that all existing data show that the "1/3-power law" (which refers theoretically only to the case of very large negative ζ) begins to be valid at unexpectedly small values of ζ (or ζ', or Rf, or Ri) of the order of -0.1 (or even of several hundredths). Since Rf is equal to the ratio of the rates of production of turbulent energy by thermal and by dynamical factors, this latter result apparently shows that convection produces vertical turbulent mixing much more effectively than wind shear. As a result *the thickness of the dynamic sublayer* (in which the thermal factors play no significant role) *in fact comprises only a small part of* $|L|$.[8]

The measurements of velocity profiles and turbulent fluxes in thermally stratified flows generated in special meteorological wind-tunnels are, at present, even less reliable than meteorological field measurements. However, the data presented by Cermak, Sandborn et al. (1966) and by Chuang and Cermak (1967) are quite consistent with the deductions of the similarity theory of Sect. 7 and show that there are no significant discrepancies between the universal relationships observed in wind-tunnels with artificial thermal stratification and in the atmospheric surface layer (both in unstable and in stable conditions).

Now we consider the case of stable stratification. The observations in stable conditions are much less numerous than those in the unstable case, and the corresponding data are, in general, rather scattered and do not agree so well with the assumption of universal similarity, defined by only one dimensionless height $\frac{z}{L}$. (Hence, Gurvich's graph given in Fig. 55 must be regarded as merely tentative without additional confirmation.) It is possible that the scatter of the experimental points for positive values of ζ is explained partially by the considerable intermittency of turbulence in stable conditions, leading to great variability for the time-means, in view of which it is necessary here to increase considerably the averaging period in order

[8]From this viewpoint, the traditional inclusion, in the denominator of expression (7.12) for the scale L, of the von Kármán constant $\varkappa$,–leading to an additional increase in this scale of 2.5 times, is very unfortunate.

to obtain reliable results. Some part may also be played by the fact that strong inversions in the surface layer of the atmosphere are generally accompanied by strong radiation cooling, during which the meteorological conditions are quite unsteady and q varies considerably with height. Nevertheless, we may note that even at the beginning of the 1950's Rider and Robinson (1951) and Halstead (1952) found from observations that in the case of very stable stratification, the wind profile is usually approximately linear. The same deduction may be made from the observations of Liljequist (1957), in the Antarctic, where strong inversions are customary. According to McVehil, (1964) and Webb [see Lumley and Panofsky (1964), p. 117)] for $0 \leqslant \zeta' \lesssim 0.3$, where $\zeta' = z/L'$ (see above) the formula $\varphi(\zeta') = 1 + \beta_1'\zeta'$ gives a farily good representation of $\varphi(\zeta')$ where $\beta_1' = 7$ according to the data of McVehil (who worked out the results of observations made near O'Neill, Nebraska and at the "South Pole" station in the Antarctic) and $\beta_1' \approx 4.5$ according to Webb's data. Finally, the treatment of the various meteorological observations in temperature inversions carried out more recently by McVehil (1964), Webb (1965) [see also Lumley and Panofsky (1964), p. 117], J. J. O'Brien (1965), Gurvich (1965), Busch (1965), Panofsky, Busch et al. (1967), Högström (1967b), Chalikov (1968), Hoeber (1968), and Zilitinkevich and Chalikov (1968a) lead to very similar conclusions which tends to lend them weight. According to all the data where is no abrupt change of regime when the stability increases (similar to transition from the forced to the free convection egime in the unstable case) and the wind profile can be described by he unique "logarithmic + linear" equation (7.58) in an extensive range of positive values of ζ (or ζ', or Rf, or Ri). As an example, we give in the figure (A) below, comparison of the prediction from the equation $f(\zeta') = \ln \zeta' + 7\zeta' + \text{const}$ or, what is practically the same thing, $\varphi(\zeta') = 1 + 7\zeta'$ (the continuous line on the figure) with McVehil's observations (made near O'Neill, Nebraska and at the "South Pole" station in the Antarctic). The figure shows very good agreement; according to Panofsky et al. (1967) the equation $\varphi(\zeta') = 1 + 7\zeta'$ in this stability range is approximately equivalent to the equation $\varphi(\zeta) = 1 + 10\zeta$ of the form (7.58). The same value $\beta' = 10$ of the coefficient β_1' in Eq. (7.58) was obtained by Gurvich (1965) [see Fig. 55] and by Zilitinkevich and Chalikov (1968a) [see also Zilitinkevich (1970)] who treated by the least squares method the extensive data of several expeditions in the steppe near Tsimlyansk. At the same time J. J. O'Brien (1965) obtained the averaged estimate

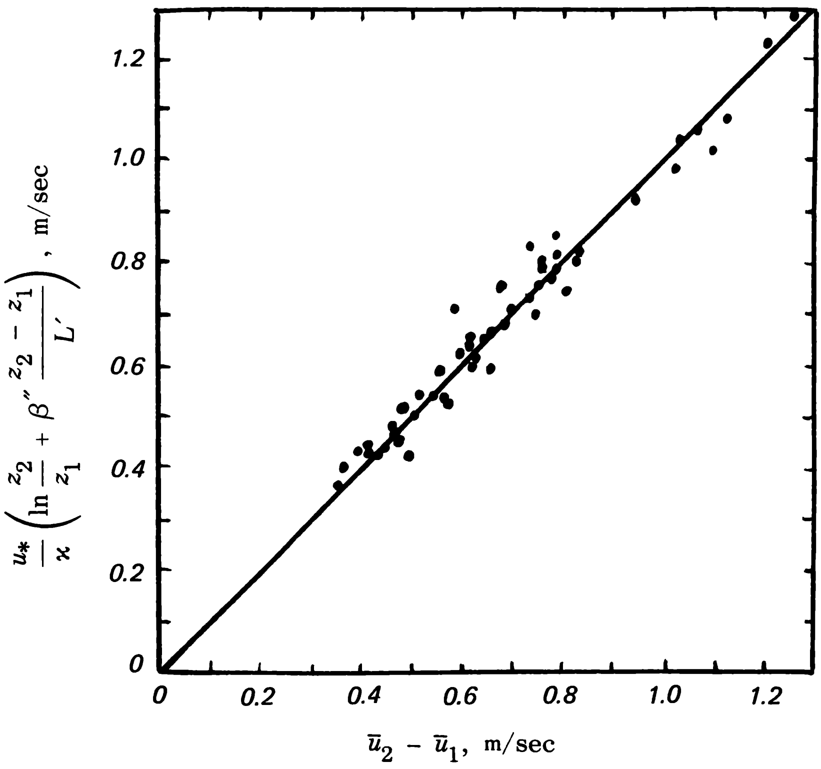


FIG. A. The comparison of the predictions from the "logarithmic + linear" law with $\beta'' = 7$ related to the difference $\bar{u}_2 - \bar{u}_1$ of wind velocities at 1 m and 2 m with McVehil's observations (1964) for stable stratifications (at $0 < z/L' < 0.3$).

$\beta'' \approx 5.0$ (with great scatter) of the coefficient in the equation $\varphi(\zeta') = 1 + \beta''\zeta'$; Fichtl (1968) used a two-layer equation of the form $\varphi(\zeta') = 1 + 4.5\zeta'$ when $0 < \mathrm{Ri} < 0.01$ and $= 1 + 7\zeta'$ when $0.01 \leqslant \mathrm{Ri} \leqslant 0.1$ and Chalikov (1968) found that $\beta_1' \approx 7$ according to data of the Soviet Antarctic Expedition. The value $\beta_1' \approx 7$ was also obtained by Cermak et al. (1966) from laboratory measurements in a wind-tunnel with a stable thermal stratification. Thus we must conclude that Eq. (7.58) with β_1 close to 10 satisfactorily describes almost all existing data from stable stratifications. The coefficient C_3 in the asymptotic equations (7.43) and (7.44) apparently must be of the same order (near 10). It is interesting to note in this connection that the data of Ellison and Turner (see below, Sect. 8.2) also lead to the conclusion that the value of the constant $R = Rf_{cr}$ apparently lies between 0.01 and 0.15, so that $C_3 = 1/R \approx 7 - 10$. However, for a more precise determination of this constant [and for a more complete elucidation of the conditions under which the "linear law" (7.44) holds] it is still necessary to organize special careful meteorological observations in strong inversions and to carry out additional laboratory tests.

It is easy to see that the equation $\varphi(\zeta) = 1 + \beta'_1\zeta$ is equivalent to the relationship $\varphi(Rf) = (1 - \beta'_1 Rf)^{-1}$. The dependence of ζ on Rf in this case also takes a very simple form: here $\zeta = Rf/(1 - \beta'_1 Rf)$, $Rf = \zeta/(1 + \beta'_1\zeta)$ [cf. Eq. (7.20)]. Similarly if $\varphi(\zeta') = 1 + \beta''\zeta'$ then $\zeta' = Ri/(1 - \beta'' Ri)$, $Ri = \zeta'/(1 + \beta''\zeta')$. If $\alpha(\zeta)$ is considered constant (and is known), then these relations also give the dependence of Ri on ζ. The empirical graph of the last dependence is shown in Fig. 59 based on the data of Gurvich (1965).

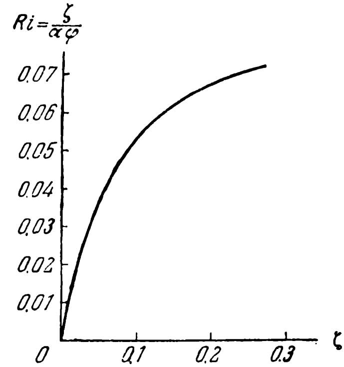


FIG. 59. Empirical dependence of Ri on ζ in the case of stable stratification [according to Gurvich's data (1965)].

Finally, the value of the constant β in formulas (7.32)–(7.33), which determines the behavior of $\varphi(\zeta)$ and $f(\zeta)$ for small $|\zeta|$, proves to be very indeterminate, it depends to a great extent on the selection of the range of values of ζ for which this value is found. Let us consider the case $\zeta < 0$ only (since the stable case was discussed above). We have pointed out already that Monin and Obukhov (1954) found that $\beta \approx 0.6$ leads to a fairly good approximation of the wind profile in the lowest four-meter layer by the "logarithmic + linear" function. This result was also confirmed later by Panofsky, Blackadar and McVehil (1960), J. J. O'Brien (1965), Busch (1965), Bernstein (1966), and others who used an approximation of the form (7.32) for a comparatively wide range of negative values of ζ. At the same time, certain researchers who determined β from data referring to a more restricted range of values of ζ or even with the aid of the formula

$$\beta = \left[\frac{\partial}{\partial\zeta}\left(\frac{\varkappa z}{u_*}\frac{\partial \bar{u}}{\partial z}\right)\right]_{\zeta=0},$$

obtained considerably higher values of β, of the order of several units [cf. the value $\beta \to 1.45$ obtained by Zilitinkevich and Chalikov (1968a)] or even of the order of 10 [see, for example, Webb (1960; 1965); Taylor (1960a); or Kondo (1962a)]. Thus the determination of β from the condition of the best approximation of the values of ζ over a limited range is dependent upon the range selected and does not coincide with the determination of β from the derivative of $\varphi(\zeta)$ at zero. It is not surprising, therefore, that in R. Taylor (1960a, b), Takeuchi (1961), Kondo (1962a), and O'Brien (1965) a whole series of different values of β was obtained. Consequently, it follows that the "logarithmic + linear" approximation must be used, in general, only with considerable caution, while for $\zeta < 0$, it is not advisable to apply it widely. As we have seen in the case of unstable stratification, the transition from the logarithmic law to the limiting "1/3-power law" of the theory of free convection takes place in a very thin layer; thus formulas (7.32) and (7.33) which in fact refer only to this transitional layer, have little meaning here.

8.2 Data on Temperature and Humidity Profiles

In addition to observations on the wind-velocity profile $\bar{u}(z)$, the most common meteorological observations also include observations on the mean temperature profile $\bar{T}(z)$ and, to a lesser extent, of the mean humidity $\bar{\vartheta}(z)$ in the surface layer of the atmosphere. For temperature measurements, various types of thermometers are used. They are fitted with special devices to shield them from the direct action of solar radiation. Electrical instruments are most convenient for measuring the profile $\bar{T}(z)$. These include, for example, resistance thermometers and thermistors, which register the current variation due to the variations of resistance of a conductor or of a special semiconductor (thermistor), induced by variations of the air temperature, and thermocouples, with one junction maintained at a constant temperature, and the others exposed to the atmosphere [see, for example, Laykhtman and Chudnovskiy (1949), McIlroy (1955), Krechmer (1957), Lettau and Davidson (1957)]. The electrical instruments permit the direct measurement of small temperature differences $\bar{T}(z_1) - \bar{T}(z_2)$, which allows the precision of such measurements to be increased to a few hundredths or even to one hundredth of a centigrade degree. To measure the humidity, the most common method is the comparison of the readings of wet- and dry-bulb thermometers. At various times, many other principles have also been suggested as a basis for these measurements [see, for example, Laykhtman and Chudnovskiy (1949), Middleton and

Spilhaus (1953), Lettau and Davidson (1957); International Conference, 1965]; so far, however, none of these principles has led to the development of a simple and reliable device which would give values of ϑ with high relative accuracy. Thus it is still very difficult to obtain high-quahty data on the profiles of $\bar{\vartheta}(z)$.

The general form of the temperature profile $\bar{T}(z)$ for both stable and unstable stratification is well known at present from many observations. A typical example illustrating this form is the data of Monin (1962) in Fig. 61, obtained by averaging 61 individual temperature profiles observed during the summer of 1951 in open-steppe conditions (the averaging carried out over the same six stability-homogeneous groups mentioned at the beginning of Sect. 8.1 in connection with Fig. 52). From Fig. 61 it is clear that in unstable stratification, the rate of decrease of $\bar{T}(z)$ as ln z increases is slower than the linear law, while for stable stratification, the rate of increase of $\bar{T}(z)$ with increase of ln z is faster than linear. A similar rule has already been noted above with regard to the stratification-dependence of the law of increase of wind velocity with height (see Fig. 52). Comparing Figs. 52 and 61, we may even assume that in all stratifications the profiles of wind velocity and temperature are similar to each other, i.e., that for any stability of the atmosphere the ratio

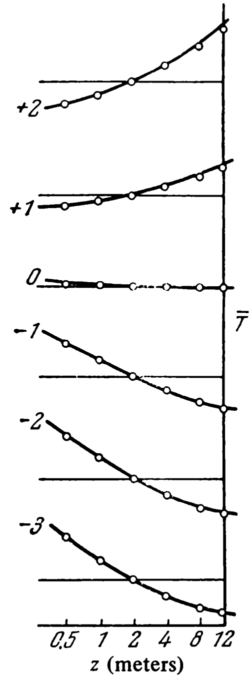


FIG. 61.† Empirical profiles of the mean temperature in the surface layer for different stratifications.

$$\frac{\bar{T}(z_1) - \bar{T}(z_0)}{\bar{u}(z_1) - \bar{u}(z_0)} \tag{8.9}$$

for any two heights z_1 and z_0 within the surface layer will assume the same value (positive for inversions, zero for neutral stratification, and negative for unstable stratification). To verify this assumption it is sufficient to compare the two shape functions

$$\frac{\bar{u}(z) - \bar{u}(z_0)}{\bar{u}(z_1) - \bar{u}(z_0)} \quad \text{and} \quad \frac{\bar{T}(z) - \bar{T}(z_0)}{\bar{T}(z_1) - \bar{T}(z_0)}, \tag{8.10}$$

where the heights z_0 and z_1 are fixed, and z is a variable, and to check

†Due to Author's revision Fig. 60 was deleted.

whether or not they are the same; another method consists of the verification of the constancy of the temperature to velocity gradient ratio with z. Both types of verifications have been carried out by many researchers at various times and to various degrees of accuracy [in particular, by Pasquill (1949), Rider and Robinson (1951), Panofsky (1961b), Swinbank (1964; 1968), McVehil (1964), Gurvich (1965), Busch (1965), Swinbank and Dyer (1967), and Charnock (1967b)]. In the older works it was always found that the wind and temperature profiles in the surface layer were particularly similar when the deviations from neutral stratification were not too great; often it was even asserted that this similarity held exactly for any stratification, or at least for any unstable stratification. However, the recent results of the Australian group (which are not yet reliably confirmed by the data of other researchers) indicate the existence of a considerable shape difference of the wind-velocity and temperature profiles for unstable stratifications (cf., for example, Fig. 62 below).

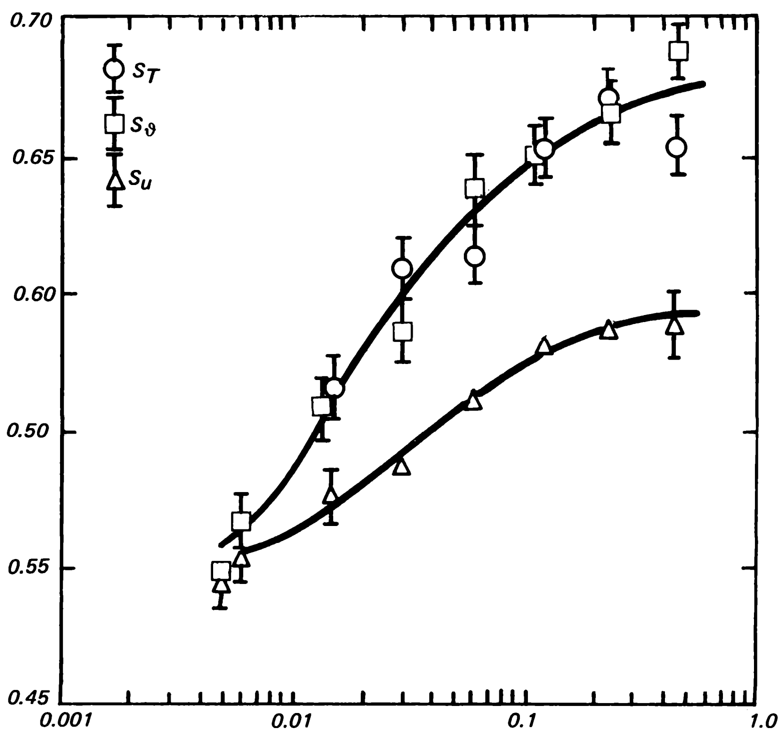


FIG. 62. Shape functions $S_X = [X(4) - X(1)]/[X(16) - X(1)]$ for mean velocity (u), temperature (T), and humidity (ϑ) vs. Richardson number at 2 m with corresponding standard errors of the mean [according to Swinbank and Dyer (1967)].

Thus, we must conclude that the profiles of $\bar{u}(z)$ and $\bar{T}(z)$ are not similar to each other although a completely reliable quantitative determination of the degree of dissimilarity for a wide stability range requires further careful investigation (cf. the discussion of the function $\alpha(\zeta)$ below).

The situation is even worse for humidity profiles $\bar{\vartheta}(z)$. The comparatively old measurements of these profiles, for example, those by Pasquill (1949), and Rider (1954), all showed that within the lowest few meters, the shape function $\frac{\bar{\vartheta}(z) - \bar{\vartheta}(z_0)}{\bar{\vartheta}(z_1) - \bar{\vartheta}(z_0)}$ is close to both the functions (8.10). In other words, the early measurements of humidity profiles $\bar{\vartheta}(z)$ for various stratifications did not contradict the statement that these profiles are similar to those of the wind and the temperature; let us recall that the wind and the temperature profiles are quite similar according to early data. However, more recent measurements by Crawford (1965), Högström (1967a, b), and Swinbank and Dyer (1967; 1968) [see also Dyer (1967)] show that humidity profiles are apparently similar to the temperature profiles, but, in general, are dissimilar to wind profiles. A more complete discussion of these results will be given at the end of this subsection; here we reproduce Fig. 62 [taken from Swinbank and Dyer (1967)] where the dependence of the three shape functions $S_X = [X(4) - S(1)]/[X(16) - X(1)]$ on the Richardson number at 2 m is shown (for negative Ri only) according to all the data from the four Australian expeditions. The arguments in the definition of the shape function are the heights in meters; the mean wind velocity $\bar{u}$, the mean temperature $\bar{T}$ and the mean humidity $\bar{\vartheta}$ are used as the quantity X (the corresponding functions are indicated as S_u, S_T, and S_ϑ); the standard errors of the mean values over the stability-homogeneous groups presented in Fig. 62 are also shown (except, however, the case of the wind-function over the middle of the stability range where the errors are too small to show). According to the data in Fig. 62, the wind and humidity shape functions take approximately the same value, namely, 0.5, which corresponds to a logarithmic profile for close to neutral stratification; there are no values for S_T with $|\mathrm{Ri}| < 0.01$ in the diagram because the accuracy of temperature difference measurements is clearly insufficient for determination of S_T in almost isothermal conditions. However, with increasing instability the shape functions for all three variables depart increasingly from the neutral value; moreover, S_T and S_ϑ are practically indistinguishable over the whole range of Ri, but they both depart considerably from S_u values. The results shown in Fig. 62 are very significant and agree completely with the deductions from the intuitive physical reasoning on the similarity of the mixing mechanism for heat and passive material admixture advanced at the end of Sect. 7.3. We must keep in mind that the accuracy of humidity profile measurements is rather poor at present and that even the data of the temperature measurements by the Australian group show certain regular departures from some data from other sources [e.g., Zilitinkevich and Chalikov (1968a)]. Hence the results in Fig. 62 must be regarded at present as merely preliminary and thus require further careful verification.

By the general formulas of similarity theory (7.24), the ratio (8.9) may be written as

$$\frac{\overline{T}(z_1)-\overline{T}(z_0)}{\overline{u}(z_1)-\overline{u}(z_0)}=\frac{T_*\varkappa}{u_*}\,\frac{f_1(\zeta_1)-f_1(\zeta_0)}{f(\zeta_1)-f(\zeta_0)}, \tag{8.9'}$$

where $\zeta_1=\frac{z_1}{L}$, $\zeta_0=\frac{z_0}{L}$, and $f(\zeta)$ and $f_1(\zeta)$ are universal functions describing the dependence of the wind and the temperature profiles on the dimensionless height. Thus it is clear that the constancy of this ratio for all z_1 and z_0 means that the functions $f(\zeta)$ and $f_1(\zeta)$ differ only by a constant multiplier; but

$$\frac{f'(\zeta)}{f_1'(\zeta)}=\alpha(\zeta)=\frac{K_T}{K};$$

thus the similarity of the wind and temperature profiles is equivalent to the height-independence, and independence of Richardson number of the ratio α of the eddy diffusivities for heat and momentum. The data shown in Fig. 62 and other similar data give definite grounds for assuming that the ratio of the eddy diffusivities K_T/K is not strictly constant. However, as yet no well-established data exist which provide a completely reliable quantitative determination of its dependence on $\zeta = z/L$; for a more detailed discussion of this point see below. With regard to the eddy diffusivity for humidity, at present we can say only that all existing data on the humidity profiles are in agreement with the assumption that $K_\vartheta/K_T=$ const [and even with the assumption that $K_\vartheta/K_T = 1$)].

A more complete investigation of the temperature profile from the viewpoint of similarity theory requires the formulation of the empirical dependence of the quantity $[\overline{T}(z) - \overline{T}(a|L|)]/T_*$ or $\frac{z}{T_*}\frac{\partial \overline{T}}{\partial z}$

on the argument $\zeta = \frac{z}{L}$. Here a is a given coefficient, for example, 1/2 or 1/10, and L and T_* must be determined from the values of u_* and q, obtained from direct measurement or by estimation using some reliable indirect method, on the basis Eqs. (7.12) and (7.14). The function $[\bar{T}(z) - \bar{T}(a|L|)]/T_*$ was formulated by R. J. Taylor (1960a) [using the data of Rider (1954) and Swinbank (1955)], Takeuchi (1961) [from the data of extensive observations of the wind and the temperature profiles, carried out in 1956 near O'Neill, Nebraska], and Gurvich (1955), and Zihtenkevich and Chalikov (1968a) [using the data of extensive measurements near Tsimlyansk (U.S.S.R.), in 1962–1965; see Figs. 63 and 64 taken from Gurvich]. Since Takeuchi determined the values of L and T_* using an approximate indirect method on the assumption that $K_T = K$, his data cannot be used for determination of $\alpha(\zeta)$. The old data treated by R. J. Taylor show that even for $\zeta \approx -0.03$, a transition occurs in the atmosphere to a "free convection regime," characterized by the "1/3-power law." Moreover, these data were also used by Taylor for an approximate estimation of $\alpha = f'(\zeta)/f_1'(\zeta)$. He found that for $|\zeta| \lesssim 0.03$, i.e., in conditions of "forced convection," $\alpha(\zeta) = \alpha_0 \approx 1.2$ according to Rider's data, and $\alpha_0 \approx 0.8$ according to Swinbank's data; for $\zeta < -0.03$ (for free convection), $\alpha(\zeta) = \alpha_{-\infty} \approx 1.7$ according to

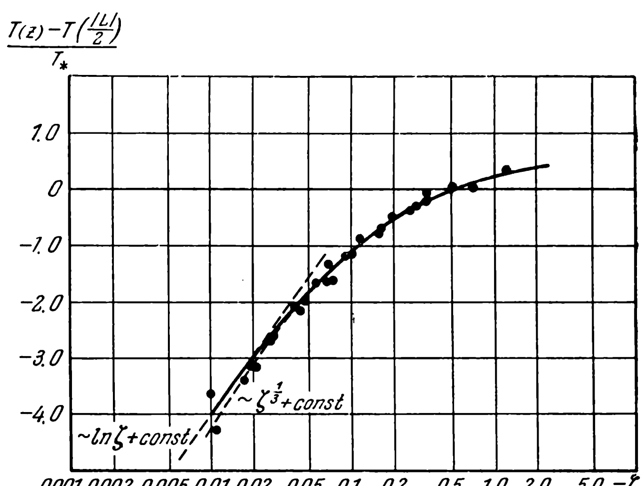


FIG. 63. Empirical graph of the function $f_1(\zeta) - f_1\left(-\frac{1}{2}\right)$ for $\zeta < 0$ [according to Gurvich's data (1965)].

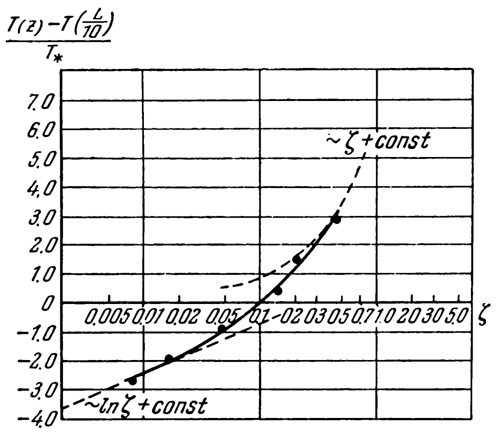


FIG. 64. Empirical graph of the function f_1 (ζ) — f_1 (0, 1) for $\zeta > 0$ [according to the data of Gurvich (1965)].

Rider, and $\alpha_{-\infty} = 1.3$ according to Swinbank [see also, Deacon (1959)]; finally, in inversions (for $\zeta > 0$), $\alpha \approx 1.8$ according to Rider and $\alpha \approx 1.0$ by Swinbank's data. The considerable discrepancy between the data of Rider and Swinbank is entirely natural if we consider the poor accuracy of their turbulent flux measurements and that the quantity of observations which could be used was very small in both cases. Later, numerous attempts were made to estimate the values of the function $\alpha(\zeta) = K_T/K$ from various existing observations in the atmospheric surface layer. For example, Swinbank (1964), Gurvich (1965), Busch (1965), Dyer (1965; 1967), J. L. H. Sibbons [see Swinbank (1966)], Mordukhovich and Tsvang (1966), Pandolfo (1966), Record and Cramer (1966), Charnock (1967b), Zubkovskiy (1967), Businger, Miyake et al. (1967), Prasad and Panofsky [see Panofsky, Busch, Prasad et al. (1967)], Kapoor and Sundararajan (1968), Zilitinkevich and Chalikov (1968a), and some others. However, the scatter of all the existing estimates is very great at present, as can be seen, for example, in Fig. B where the data obtained in Australia, U.S.S.R., and the U.S. are shown. The dotted line in the figure gives the results of Charnock's treatment (1967b) of Australian data, the broken and continuous lines were proposed by Zubkovskiy (1967) and Zilitinkevich and Chalikov (1968a) based on the observations near Tsimlyansk covering a much more restricted stability range (according to the last authors, $\alpha(\zeta) \approx 1$ and $\varphi(\zeta) \approx$

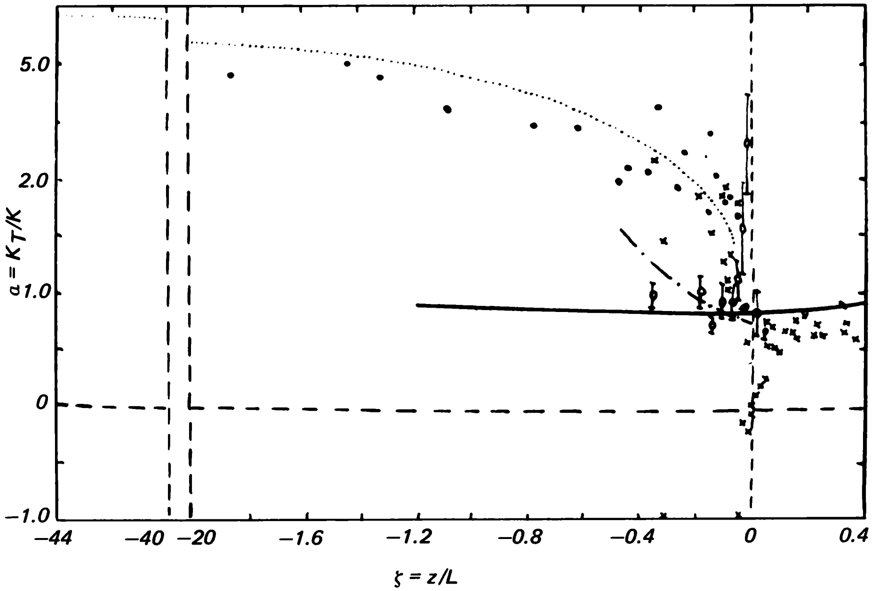


FIG. 8. Data on the dependence of $\alpha = K_T/K$ on ζ. •–Kerang, Australia [Swinbank (1964)]; ○–Tsimlyansk, U.S.S.R. [Mordukhovich and Tsvang (1966)]; ×–Round Hill, Mass., U.S.A. [Panofsky et al. (1967)]. The dotted line was proposed by Charnock (1967a), the broken line by Zubkovskiy (1967), the continuous line by Zilitinkevich and Chalikov (1968a).

$\varphi_1(\zeta)$ for all values of z/L observed in Tsimlyansk). The same degree of scatter is also shown by all the other data on $\alpha(\zeta)$. The scatter may be explained naturally by the necessity to differentiate empirical wind and temperature profiles of low accuracy and to use values of the turbulent fluxes τ and q which are now measured with quite low precision; we must also remember that the time and space variability of these fluxes is almost unknown for the atmospheric surface layer; however, the preliminary data of Mordukhovich and Tsvang (1966), and Dyer (1968), on height variability of τ and q show that it is quite large. Thus the general deduction from all the existing data is that a reliable estimate of $\alpha(\zeta)$ requires considerably more accurate measurements of the wind and temperature profiles and of the simultaneous values of τ and q at the same observation point than those presently available. On the whole the existing data show only that $\alpha(0) = \alpha_0$ is close to unity, i.e., they are in agreement with the best laboratory measurements mentioned in Sect. 5.7, according to which $\alpha_0 \approx 1.1$; with increase of instability, the ratio α increases and with increase of stability it seems to be slightly decreasing. However, the estimates of the limiting value $\alpha_{-\infty}$ are presently quite uncertain: the

Australian observations imply the value $\alpha_{-\infty} \approx 3-3.5$. Nevertheless, some investigators are inclined to use considerably lower estimates (close to 2 or even between 1 and 1.5). The data for atmospheric inversions are even more scattered, and much less numerous than those for unstable stratification; according to the treament by Prasad and Panofsky [Panofsky et al. (1967); see also Busch and Panofsky (1968)], the mean value of the eddy diffusivity ratio α in surface layer inversions is close to 0.7; the data from nonatmospheric sources on values of α in very stable conditions will be discussed at the end of this subsection.

Many of the existing determinations of $\alpha(\zeta)$ were based on the evaluation of the universal function $\varphi_1(\zeta) = \frac{z}{T_*}\frac{\partial \bar{T}}{\partial z}$ connected with $\alpha(\zeta)$ by the relationship $\alpha(\zeta) = \varphi(\zeta)/\varphi_1(\zeta)$. The graphs $\varphi_1(\zeta)$ were plotted, for example, by Gurvich (1965); see Fig. 65, Dyer (1965; 1967), Charnock (1967b), and Swinbank (1968); here the graph of $\zeta\varphi_1(\zeta)$ is given. In the discussion of these results the question of the validity of the "1/3-power law" (7.35)–(7.36), i.e., of the relation $\varphi_1(\zeta) \sim |\zeta|^{-1/3}$, and of the range of its applicabihty to unstable atmospheric surface layers, are of primary importance. We have already mentioned the data of R. J. Taylor (1960a) which shows that in an unstable atmosphere the transition to the "free convection regime," i.e., to the regime characterized by the "1/3-power law," occurs at unexpectedly small values of $-\zeta$ of the order of several hundredths. However, the first indications of the vahdity of this were

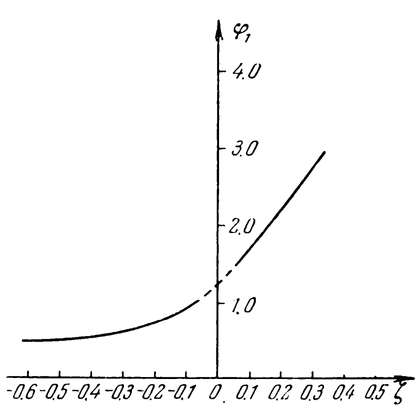


FIG. 65. Empirical graph of the universal function $\varphi_1(\zeta)$ [according to the data of Gurvich (1965)].

obtained even earlier, in the works of Priestley (1955; 1956) and R. J. Taylor (1956a) where the "dimensionless heat flux" $\overset{*}{q}$ was first introduced; we shall define this quantity later. For very small negative Richardson numbers and values of ζ, the transition from the "forced convection regime" characterized by a near-logarithmic temperature profile to the "free convection regime" with its "1/3-power law" follows from Webb's result. Webb (1958) obtained data on temperature-profile observations made near Edithvale, Australia, over a period of several years. Thus, for a series of observations with unstable stratification, he calculated the quantity

$$\Gamma = \frac{1}{1.5 \log 4} \frac{\overline{T}(8) - \overline{T}(2)}{\left(\frac{\partial \overline{T}}{\partial z}\right)_{z=1.5}}$$

[the numbers in parentheses denote the height in meters] and plotted the dependence of this quantity on the Ri number at a height of 1.5 m (see Fig. 66). The horizontal continuous lines in Fig. 66 denote the values of $\Gamma = 1$, corresponding to the logarithmic profile $\overline{T}(z) \sim$ const $-\ln z$, and $\Gamma = 0.73$, corresponding to the "1/3-power law" $\overline{T}(z) \sim$ const $+ z^{-\frac{1}{3}}$. For $-\mathrm{Ri}_{1.5}$, the smallest observed values of which are close to 0.015 or even less, Γ is actually close to unity, however, it then begins to decrease, and for Ri ≈ -0.03 it takes a value close to 0.73. This value is maintained thereafter for a considerable range of Richardson numbers. The dotted line in Fig. 66 denotes the gradual transition from $\Gamma = 1$ to $\Gamma = 0.73$, which would have been observed if observation of the relationship $\frac{\partial \overline{T}}{\partial z} \sim z^{-1}$;

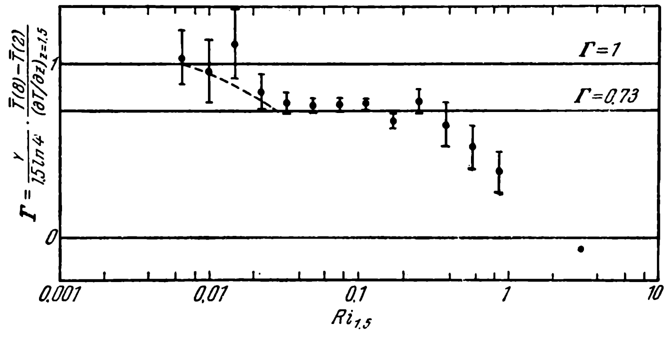


FIG. 66. Dependence of Γ on Ri in the case of unstable stratification, according to measurements.

which corresponds to the logarithmic profile, had continued right up to the height z at which Ri $= -0.03$, while above this, the relationship $\frac{\partial \bar{T}}{\partial z} \sim z^{-\frac{4}{3}}$ was satisfied.

An interesting and unexpected peculiarity of Fig. 66 is that beginning from approximately $\mathrm{Ri}_{1.5} = 0.4$, the value of Γ once again begins to decrease. If this result is correct and has a universal character (this is still quite doubtful) then it follows that with increasing instability, but fixed height, or with increasing height, but fixed unstable stratification, an instant occurs when the dimensional arguments which imply the "1/3-power law," cease to operate. This agrees somewhat with the fact that the "1/3-power law" is not confirmed by the rather rough laboratory convection experiments of Thomas and Townsend (1957) and Townsend (1959), where the horizontal velocity $\bar{u}(z)$ was equal to zero. However, Townsend's results may be explained by the single fact that with zero (or very small) horizontal velocity, horizontal mixing is greatly reduced, and as a result, the turbulence at a given height z is horizontally homogeneous only over a very large averaging time. There are also some other explanations for the apparent inapplicability of the "1/3-power law" to Townsend's experiment. For example, Deardorff and Willis (1967) concluded that existing laboratory data neither contradict nor confirm the assumption that at sufficiently large Rayleigh numbers artificial convection in a box in the complete absence of a horizontal mean velocity will lead to a $-1/3$-power temperature profile with a coefficient C_1 close to that observed in the atmosphere. On the other hand, there are some slight grounds for assuming that in the absence of horizontal mixing due to a horizontal mean velocity, the condition of generation of rising convective jets within the sublayer of molecular thermal conductivity will affect the whole turbulent regime. Thus the statistical characteristics of turbulence at all heights will depend in this case on the molecular diffusivity χ [cf. Malkus' convection theory (1954b)]. This question was discussed at some length by Townsend (1962a), Priestley (1962) and Webb (1962), who, however, did not obtain completely definite conclusions. Meanwhile, new measurement data were obtained by Dyer (1965; 1967) and Swinbank (1968); their results also suggest that with very unstable stratification (for $\zeta < -0.6$ approximately) some deviations from the free convection "1/3-power law" may occur in the atmosphere, but these deviations now appear to be considerably smaller than those found earlier. (Dyer and Swinbank suggest that

$\varphi_1(\zeta) \sim |\zeta|^{-1/2}$ for $\zeta < -0.6$ and that the law $\varphi_1(\zeta) \sim |\zeta|^{-0.44}$ is in excellent agreement with all the data where $-0.1 > \zeta > -5$ in contrast with the law $z(\partial\bar{T}/\partial z) \sim z^{-1}$ which follows from Malkus' theory and was advocated in the 1962 discussion.) The same problem was discussed in the above-mentioned paper by Deardorff and Willis (1967), but their deductions are also quite uncertain. Because of the obscurity of the question, in further consideration of unstable stratification, we shall, as a rule, have in mind only a layer with modest values of $|\zeta|$, where the "1/3-power law" undoubtedly is well satisfied, beginning from a height z of the order of several hundredths of $|L|$.

To determine the numerical value of the coefficient C_1 in the "1/3-power law" (7.36), it is convenient to formulate the dimensionless heat flux

$$\overset{*}{q} = \frac{q}{c_p \rho_0 \left(\frac{g}{T_0}\right)^{\frac{1}{2}} \left|\frac{\partial \bar{T}}{\partial z}\right|^{\frac{3}{2}} z^2} = \frac{\varkappa^2}{\zeta^2 \left| f_1'(\zeta) \right|^{\frac{3}{2}}}, \tag{8.11}$$

which is a single-valued function of ζ and of Ri. In conditions of free convection, this quantity will take a constant value, namely, $\overset{*}{q}_{-\infty} = \left(\frac{3}{C_1}\right)^{\frac{3}{2}}$; its asymptotic behavior for small ζ and Ri is given by the formula

$$\overset{*}{q} = \alpha_0^{\frac{3}{2}} \varkappa^2 |\zeta|^{-\frac{1}{2}} = \alpha_0 \varkappa^2 |\mathrm{Ri}|^{-\frac{1}{2}}.$$

The schematic form of the dependence of $\overset{*}{q}$ (Ri) on Ri is shown in Fig. 51, Sect. 7.4, with logarithmic scales on both axes. This dependence was also investigated empirically by Priestley (1955; 1956), R. J. Taylor (1956a), Perepelkina (1959a), Busch (1965), Mordukhovich and Tsvang (1966), and Swinbank (1968), using various data from simultaneous measurements of $\bar{T}(z)$, $\bar{u}(z)$, and q. Figure 67 gives an empirical graph of $\overset{*}{q}$ (Ri), plotted according to Swinbank (1955), R. Taylor (1956a), Perepelkina (1959a) and Mordukhovich and Tsvang (1966); the vertical segments in Fig. 67 indicate the mean square deviations within various stability-homogeneous groups, with the exception of the data of Perepelkina, who gave only the mean values. The individual values of $\overset{*}{q}$ are characterized in all cases by considerable scatter, which is connected with the comparatively low accuracy of all the existing

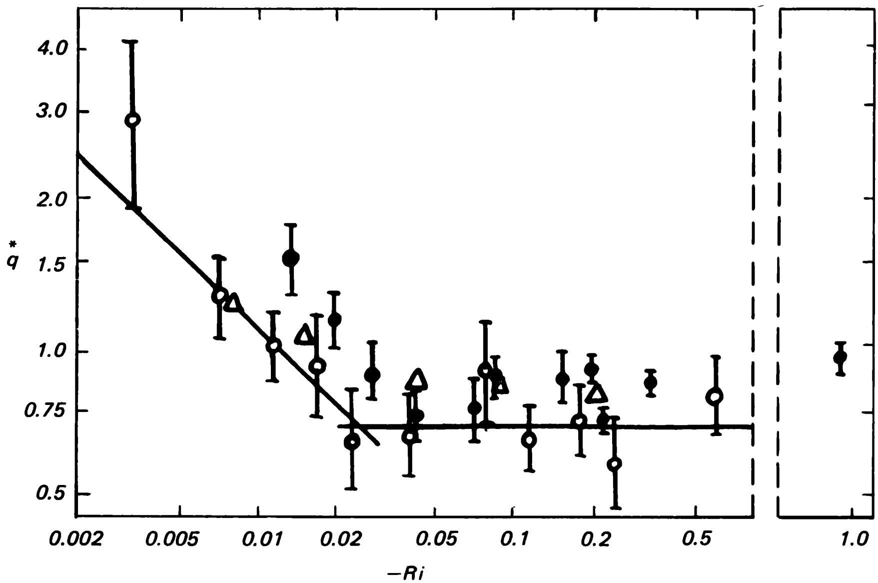


FIG. 67. The dependence of $\overset{*}{q}$ on Ri according to the data of Swinbank and R. J. Taylor (○), Perepelkina (Δ), and Mordukhovich and Tsvang (●).

measurements; however, the mean values in Fig. 67 agree well with the assumption of the existence of two limiting regimes $\overset{*}{q} \sim |\mathrm{Ri}|^{-\frac{1}{2}}$ and $\overset{*}{q} = \overset{*}{q}_{-\infty} = \mathrm{const}$, denoted by continuous lines in the figure. The transition region between the limiting regimes is extremely narrow according to the early data of Swinbank and R. J. Taylor; the data of Perepelkina, Mordukhovich and Tsvang, and all the other existing data on $\overset{*}{q}$ and $\varphi_1(\zeta)$ [including those of Panofsky, Blackadar and McVehil (1969); Gurvich (1965); Busch (1965); Dyer (1965; 1967); and Zihtinkevich and Chalikov (1968a)] indicate a somewhat smoother transition, but confirm that it is quite sharp in comparison with other transitional regimes encountered in physics. According to all the data, with the exception of the recent results of Swinbank (1968) which will be mentioned a little later, the free convection regime and the "1/3-power law" begin to be valid at unexpectedly small values of $|\mathrm{Ri}|$ and $|\zeta|$ in the interval 0.02–0.05; this fact was already noted above. The empirical value of the proportionality coefficient in the asymptotic relationship $\overset{*}{q}(\mathrm{Ri}) \sim |\mathrm{Ri}|^{-\frac{1}{2}}$ is close to 0.17 [Deacon (1959)], which agrees well with the generally accepted values $\varkappa \approx 0.4$, $\alpha_0 \approx 1$. However, the data on the constant $\overset{*}{q}_{-\infty}$ are not in satisfactory agreement with each other. According to the early

Australian data of Swinbank and R. J. Taylor, after introducing an instrumental correction, $\overset{*}{q}_{-\infty} \approx 0.9$ [Priestley (1959a); Deacon (1959)]; according to Perepelkina, $\overset{*}{q}_{-\infty} \approx 0.8$; according to Gurvich, and Zilitinkevich and Chalikov (who determined the value of $C_1 = 3/(\overset{*}{q}_{-\infty})^{2/3}$ from measurements in a restricted range of negative values of Ri), $\overset{*}{q}_{-\infty} \approx 0.7-0.75$; according to fairly rough data of Cramer and Record, and of Panofsky, $\overset{*}{q}_{-\infty} \approx 0.95$ and $\overset{*}{q}_{-\infty} \approx 0.75$, respectively, while according to Mordukhovich and Tsvang, $\overset{*}{q}_{-\infty} \approx 0.8$. The agreement among all these estimates clearly is quite good. Unfortunately the more recent Austrahan measurements of Swinbank and Dyer (1968) do not support the results of the other investigations. According to Dyer (1967) the best value of $\overset{*}{q}_{-\infty}$ [corrected for the error caused by the use of the difference ratio $\Delta\overline{T}/\Delta z$ instead of the derivative $\partial\overline{T}/\partial z$] is close to 1.15 for the range $0.02 \leqslant |z/L| \leqslant 0.6$. (In his earlier paper where no corrections was made, Dyer (1965) gave an even slightly greater estimate; a greater value of $\overset{*}{q}$ also corresponds to Dyer's data for $|z/L| > 0.6$.) The analysis of the same data by Swinbank (1968) showed that in fact $\overset{*}{q}$(Ri) does not take on a strictly constant value according to the Australian measurements of 1962–1964. Dyer's estimate corresponds in Swinbank's graph to a flat minimum of $\overset{*}{q}$ (Ri) at Ri ≈ -0.1; however, with increase of $-$Ri the value of $\overset{*}{q}$ also increases and at the greatest value of $-$Ri considered by Swinbank (close to 1) $\overset{*}{q}$ shows only a slight tendency to "flatten-out" to a constant value (slightly greater than 1.5), but a strictly constant value has not been established even at such strong instability. The explanation of the obvious discrepancy among these results is still unclear.

Knowing $\overset{*}{q}_{-\infty}$, it is easy to evaluate the coefficient $C_1 = 3/(\overset{*}{q}_{-\infty})^{2/3}$ and vice versa. Therefore all the determinations of C_1 and of $\overset{*}{q}_{-\infty}$ are closely connected with each other. For $\overset{*}{q}_{-\infty} = 0.75$, 1, 1.15, and 1.5, we obtain the following values of C_1: $C_1 \approx 3.6$, 3.0, 1.7, and 2.3, respectively.

Of course, many dimensionless combinations of $\partial\overline{T}/\partial z$, $\partial\overline{u}/\partial z$, u_*, $q/c_p\delta$, g/T_0, and z also exist, different from $\overset{*}{q}$ (and $\overset{*}{K}$; see Eq. (8.6), Sect. 8.1); they may all be represented by the principal wind velocity and temperature functions $f(\zeta)$ and $f_1(\zeta)$ [or $\varphi(\zeta)$ and $\varphi_1(\zeta)$] and may also be determined from experimental data. Thus, for example, Pasquill (1949), Rider (1954) and Perepelkina (1959a) considered the empirical formulation of the function

$$F_1(\mathrm{Ri}) = -\frac{q}{c_p\rho_0 \frac{\partial \bar{u}}{\partial z} \frac{\partial \bar{T}}{\partial z} z^2} = \frac{\varkappa^2}{\zeta^2 f'(\zeta) f_1'(\zeta)}. \tag{8.12}$$

If $f(\zeta)$ is known, then the values of $F_1(\mathrm{Ri})$ also allow us to determine the universal function $f_1(\zeta)$ which describes the temperature profile. However, the accuracy of the data on $F_1(\mathrm{Ri})$ in the comparatively early works cited is completely insufficient for use in this manner.

Swinbank (1968) constructed the empirical graphs of the functions

$$F_3(\zeta) = F_4(\mathrm{Ri}) = -\frac{g}{T_0}\frac{z}{u_*}\frac{\partial \bar{T}}{\partial \bar{u}} = \frac{\zeta}{\varkappa^2}\frac{\varphi_1(\zeta)}{\varphi(\zeta)}.$$

His graphs are characterized by fairly small scatter; the data on $F_3(\zeta)$ permit the conclusion that $K_T/K = \alpha(\zeta) \approx 2.7\,|\zeta|^{1/4}$ over a wide range of negative stabilities. However, we have already noted that there are some unusual discrepancies between the data of Swinbank (1968) and those of certain other investigators.

The function analogous to $F_1(\mathrm{Ri})$, but for the humidity profile,

$$F_2(\mathrm{Ri}) = -\frac{j}{\rho_0 \frac{\partial \bar{u}}{\partial z} \frac{\partial \bar{\vartheta}}{\partial z} z^2} = \frac{\varkappa^2}{\zeta^2 f'(\zeta) f_2'(\zeta)} \tag{8.13}$$

was formulated empirically by Pasquill (1949) and Rider (1954) from measurements of the wind and humidity profiles and measurements of evaporation with the aid of weighing lysimeters. Clearly, since

$$F_1(\mathrm{Ri}) = K_T\Big/z^2\frac{\partial \bar{u}}{\partial z}, \quad F_2(\mathrm{Ri}) = K_\vartheta\Big/z^2\frac{\partial \bar{u}}{\partial z} \text{ and } \overset{*}{K}{}^2(\mathrm{Ri}) = K\Big/z^2\frac{\partial \bar{u}}{\partial z}$$

[cf. Eq. (8.6) in Sect. 8.1], by comparing the values of F_2, F_1 and $\overset{*}{K}$, we could in principle also draw definite conclusions about the behavior of the ratios K_ϑ/K_T and K_ϑ/K. However, due to the considerable inaccuracy of the data of Pasquill and Rider on humidity profiles and evaporative fluxes, they are useless for such a comparison.

Later, Charnock and Ellison (1959) found a strong correlation between fluctuations of humidity and temperature above sea level and considered this to be rather uncertain indirect confirmation of the equality between the eddy diffusivities K_ϑ and K_T for water vapor and heat. However, much more definite (although also not quite rigorous) proofs of the hypothesis that $K_\vartheta = K_T$ were obtained still later by Crawford (1965), Dyer (1967), Swinbank and Dyer (1967) and Högström (1967b). One of the proofs consists of the diagram shown in Fig. 62, which was discussed above. Another method is based on the investigation of the dependence on thermal stability of the dimensionless flux of water vapor

$$\overset{*}{j} = \frac{j}{\rho_0 (g/T_0)^{1/2} |\partial \bar{T}/\partial z|^{1/2} |\partial \bar{\vartheta}/\partial z| z^2} = \frac{\varkappa^2}{\zeta^2 |f_1'(\zeta)|^{1/2} |f_2'(\zeta)|}$$

which is quite analogous to the dimensionless heat flux $\overset{*}{q}$. The first empirical graph of the function $\overset{*}{j}(\mathrm{Ri})$ in the range $-0.0003 \geqslant \mathrm{Ri} \geqslant -2.5$ was constructed by Crawford (1965). He used the data on evaporation and on the profiles of wind velocity, temperature and absolute humidity, obtained at a special site during 1962–1963 from measurements with a weighing lysimeter, an infrared hygrometer (the air was drawn through it from different heights), and a thermo-couple-anemometer mast. The resulting graph is quite similar to that in Fig. 67; for $-\mathrm{Ri} \leqslant 0.025$ the points are concentrated near the straight line $\overset{*}{j} = 0.2|\mathrm{Ri}|^{-1/2}$, with the coefficient 0.2 close to the coefficient 0.17 in the similar relation for $\overset{*}{q}$ in the forced convection regime, whereas for $-\mathrm{Ri} > 0.025$ they correspond with relatively small scatter to a constant value $\overset{*}{j} \approx 1.4$. The last value is slightly greater than the majority of estimates of $\overset{*}{q}$ for the free convection regime, but is quite close to the estimate $\overset{*}{q} \approx 1.32$ obtained by Dyer (1965). Later, Dyer (1967) calculated the quantities $\overset{*}{q}$ and $\overset{*}{j}$ and also $\varphi_1(\zeta) = z/T_* \, \partial \bar{T}/\partial z$ and $\varphi_2(\zeta) = z/\Theta_* \, \partial \bar{\vartheta}/\partial z$ from the data of the 1962–1964 Australian expeditions where the evaporative water-vapor flux was measured by a method described in the next subsection, and the humidity profile by a special crystal dew point hygrometer with air sampled from various heights. It was found here that the functions $\varphi_1(\zeta)$ and $\varphi_2(\zeta)$ are practically identical over the whole range of unstable stratifications investigated, although the φ_2 data are somewhat more scattered. In the range $-0.02 \leqslant \zeta \leqslant -0.6$ both functions satisfy the "1/3-power law" very well and the value $\overset{*}{j}$ evaluated for this range is close to 1.10 when corrected for the small error due to the approximation of the derivatives $\partial\bar{T}/\partial z$ and $\partial\bar{\vartheta}/\partial z$ by the corresponding finite difference expressions; this error, which leads to an overestimate of $\overset{*}{j}$ and q, was neglected by Crawford and Dyer in 1965. The last estimate practically coincides with the estimate $\overset{*}{q} \approx 1.15$ obtained by Dyer in the same work. Finally, Högström (1967b) used the comparatively rough data of meteorological and hydrological observations in the plains of southern Sweden during 1961–1963, where humidity fluctuations were measured by an Australian instrument and the humidity profile was measured with poor accuracy by the usual meteorological hygrometers. He obtained an estimate $\overset{*}{j} \approx 1.25$ (without a finite difference correction), which agrees with those of Crawford and Dyer within the limits of the experimental errors. Since $\overset{*}{j}/\overset{*}{q} = K_\vartheta/K_T$, all the above-mentioned data clearly support the hypothesis that $K_\vartheta = K_T$ for all neutral and unstable stratifications.

So far, we have considered only data referring to negative (or, at least, only small positive) values of Ri and ζ. Data referring to strong inversions have until now been obtained only rarely in atmospheric measurements and display great scatter. Consequently, further careful investigation of this subject is necessary. Nevertheless, Liljequist's observations (1957) in the Antarctic seem to confirm in a number of cases the presence of a linear temperature profile in strong inversions, as predicted by similarity theory under some additional assumptions (see Sect. 7.3). On the other hand, if we adopt the hypothesis that the eddy diffusivities for heat and for a material admixture are the same also for stable stratifications, then in the investigation of values of $\alpha = K_T/K = K_\vartheta/K$ in extremely stable conditions, we may use the data from oceanographic observations and laboratory experiments, for flows of liquid with very stable density stratification caused by a negative vertical salinity gradient. The first and best-known data of this type were obtained by G. I. Taylor from oceanographic observations in the Kattegat. These data showed that considerable turbulent exchange may exist even when Ri is of the order of 10, and that $\alpha = K_\vartheta/K$ is then so small that $\mathrm{Rf} = \alpha\,\mathrm{Ri}$ remains much smaller than unity [see Proudman (1953) and the discussion by Stewart (1959)]. The evaluation of these data in Proudman's book show that $\alpha \approx 0.03$–0.05 for $4 < \mathrm{Ri} < 10$ (black squares in Fig. 68). Values of $\alpha = K_T/K \approx 0.02$–$0.05$ in close agreement were established by direct measurements of K and K_T in a layer of water under the ice on Lake Baikal [see Kolesnikov (1960)]. Later, comparatively rough data on the dependence of α on Ri for $\mathrm{Ri} > 0$ were obtained in the laboratory experiments of Ellison and

Turner (1960). In these experiments a turbulent flow of salt water with negative vertical salinity gradient was set up in a tube of rectangular cross section (with glass side walls; in one section of the tube the profiles of velocity and density were measured, and also, by indirect methods, the eddy diffusivities K and K_ϑ were estimated. Figure 68 gives the results obtained and the oceanographic data of Taylor and Proudman. In this figure, the scatter of the experimental points is very considerable, and, moreover, there is a systematic deviation of the data referring to $z = 1$ cm and $z = 1.5$ cm; however, the general tendency of α to decrease with increase of Ri can be seen fairly easily. The continuous lines in Fig. 68 correspond to the values of α obtained on the basis of the semiempirical formula

$$\alpha(\mathrm{Rf}) = \frac{\alpha_0 (1 - \mathrm{Rf}/R)}{(1 - \mathrm{Rf})^2}, \quad \mathrm{Rf} = \alpha \, \mathrm{Ri}, \tag{8.14}$$

proposed by Ellison (1947). It is assumed here that $\alpha_0 = 1.4$ which is apparently too high; see Sect. 5.7; the values of $R = \mathrm{Rf_{cr}}$ which correspond to each curve are written above it. The experimental points of Fig. 68 more or less agree with the deductions drawn from Eq. (8.14); according to these data, the best estimate of $R = \mathrm{Rf_{cr}}$ lies somewhere between 0.10 and 0.15. Another rough estimate of the value of $\mathrm{Rf_{cr}}$ was obtained by Webster (1964) from measurements in a wind-tunnel. In these measurements, the velocity shear and thermal

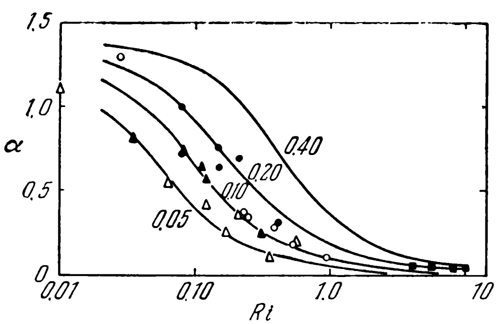


FIG. 68. Comparison of the data of G. I. Taylor and of Ellison and Turner concerning the dependence of α on Ri with the formula of Ellison (8.14): ●○–data of Ellison and Turner, relating to height $z = 1.5$ cm; ▲△–data relating to $z = 1$ cm; ■–Taylor's data. The open symbols refer to less reliable experimental data.

stratification were produced by two grids, one with variable heating of the bars and the other with variable bar diameter. According to Webster's result, $\mathrm{Rf_{cr}} \approx 0.35$. Some other estimates of $\mathrm{Rf_{cr}}$ were reported by Högström (1967b) and by Webster (1964); none of them is very reliable and most of them assert that $\mathrm{Rf_{cr}}$ is somewhere between 0.08 and 0.4.

8.3 Methods of Measuring Turbulent Fluxes of Momentum, Heat, and Water Vapor

As already observed, a complete verification of the deductions from similarity theory discussed in Sect. 7, requires a knowledge of not only the profiles $\bar{u}(z)$, $\bar{T}(z)$, and $\bar{\vartheta}(z)$, but also of the turbulent momentum flux (shear stress) $\tau = \rho u_*^2$, the heat flux q and the water vapor flux (rate of evaporation or condensation) j. Moreover, since the quantities τ (or u_*), q and j are very important characteristics of atmospheric turbulence, describing most directly the interaction between the atmosphere and the underlying surface, it is natural that their determination be one of the central problems in the physics of the atmospheric surface layer. Thus, at this point it is relevant to give at least a brief description of the principal methods of measuring turbulent fluxes.

The most universal direct method of measuring u_*, q, and j is the fluctuation (or eddy correlation) method, which consists in recording the time variation of the fluctuations u', w', T', and ϑ' at a fixed point, and subsequently evaluating the time-means of the products $u'w'$, $w'T'$, and $w'\vartheta'$. Since

$$u_* = (-\overline{u'w'})^{\frac{1}{2}}, \quad q = c_p \rho_0 \overline{w'T'}, \quad j = \rho_0 \overline{w'\vartheta'}, \tag{8.15}$$

these means determine all the quantities of interest to us. However, the instruments used for fluctuation measurements must record without distortion all details of the micrometeorological fluctuations, which contribute substantially to the values of u_*, q, and j. The latter requirement imposes important limitations on the parameters of the instruments which are difficult to determine accurately without advance knowledge of the details of the fluctuations that affect the turbulent fluxes. In fact the information required concerns the spectral composition of the turbulent fluxes which still is studied very inadequately; we shall discuss this in greater detail in Chapt. 8 of Volume 2 of this book. Moreover, when the fluctuation method is used to determine turbulent fluxes, we need to know how to choose the time-averaging interval in such a way that the mean values obtained will be done close to the mean taken according to some sensibly determined statistical ensemble, and so that the gradual "drift of the mean" mentioned in Sect. 7.1 will not occur. All these facts make the measurement process far more complicated, and in the final analysis lead to results that are not always sufficiently accurate. Nevertheless, at the present time, the fluctuation method remains the only general means of determining all three quantities τ, q, and j directly.

For measuring the fine details of the wind-velocity fluctuations u' and w', until 1958, most researchers used the hot-wire anemometer, which had already been successfully used for many years in the measurement of velocity fluctuations in wind-tunnel flows. Göedecke (1935) was one of the first to succeed in measuring in the atmosphere using the hot-wire technique. Later, from 1947 to 1949, suitable instruments of this type were developed in the U.S.S.R. by Obukhov and Krechmer [see, for example, Krechmer (1954)]. At about the same time, hot-wire anemometers also began to be used for atmospheric research in Australia and Great Britain [see, for example, Swinbank (1951a; 1955); Deacon (1955); McIlroy (1955); Jones and Pasquill (1955)]. A particularly sensitive hot-wire anemometer for atmospheric measurements was developed by McCready (1953). A considerable number of works on measurements by atmospheric hot-wire anemometers was published in the late 1950's and in the 1960's; however, we shall not even attempt to mention all of them here.

The general principles of hot-wire anemometry are well known, for example, in works by Corrsin (1963) and Kovásznay (1966), and therefore will not be repeated here in detail.

The sensors of all hot-wire anemometers are thin metal wires which are heated by a current to a temperature of several hundred degrees during the measurements. The temperature variations of the wire, and consequently the variations of its resistance, are determined by the heat transfer to the air. This heat transfer depends on the flow velocity component normal to the wire, and is practically independent of small air-temperature variations, which are generally of the order of fractions of a degree. By measuring the fluctuations of the wire resistance, that is, the current through the wire, the fluctuations of the wind-velocity component normal to the wire may be determined. When the wire of the hot-wire anemometer is placed vertically, the current fluctuations correspond to fluctuations of the horizontal wind velocity. If we now set up two wires in the vertical plane passing through the direction of the mean wind, which is found from an inertial wind vane, and incline them at angles of $+45^\circ$ and -45° to the vertical, then the difference in the resistances of these wires will be proportional to the vertical component of the wind velocity, with the proportionality coefficient dependent on the mean wind velocity. Hot-wire anemometers facilitate the recording of fluctuations of wind velocity with an accuracy of up to 1 cm/sec, with a time-lag (time-constant) of the order 0.01 sec; instruments of this type having smaller time-lag and which are considerably more sensitive, are also possible.

The nonlinearity of the dependence of the resistance and the current of the wire on the wind velocity is a great disadvantage. For example, when an analog computer is used instead of a digital computer, the calculations of the mean values of the fluctuation products become very comphcated. One method of linearizing a hot-wire anemometer is to pass the current fluctuations through an additional nonlinear device having a characteristic which corresponds to the anemometer calibration curves in such a way that the output signal will be proportional to the measured velocity fluctuations [McCready (1953), R. J. Taylor (1958), Dyer and Maher (1965a)]. To avoid such complications, Bovsheverov and Gurvich [Gurvich (1959); Bovsheverov and Voronov (1960)] constructed a linear instrument at the Moscow Institute of Atmospheric Physics in 1958 for measuring fine-scale velocity fluctuations. Their instrument, an ultrasonic microanemometer, was based on the dependence of the transit time of sound from transmitter to microphone on the velocity of air in its path.

Instruments of the same type but less precise were also designed in the U.S. and Japan by Businger and Suomi. [See Lettau and Davidson (1957), Barad (1958), Kaimal and Businger (1963), Kaimal, Cramer et al. (1964), and Kaimal, Wyngaard and Haugen (1968).] The most important parts of the Bovsheverov-Gurvich instrument are the miniature cylindrical sound transmitters and the microphones, which are 2 mm in diameter and about 5 mm long. The instrument has two transmitters (sources) S_1 and S_2, and two receivers (microphones) R_1 and R_2; the pair S_1 and R_2 is 2.5 cm from the pair R_1 and S_2 (see Fig. 69). The sound, that is, the ultrasound, is propagated along practically identical paths from S_1 to R_1 and from S_2 to R_2 in the opposite direction. If the transmitter signals have identical phases, the phase difference of the microphone signals will be proportional to the wind-velocity component in the direction of the base of the instrument and almost independent of the air temperature. This instrument also enables the velocity fluctuations to be measured with an accuracy of up to 1 cm/sec, with a time lag of less than 0.001 sec.

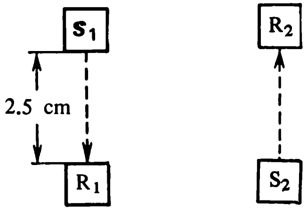


FIG. 69. Scheme of the arrangement of emitters and receivers in the acoustic microanemometer.

In addition to hot-wire anemometers and acoustic anemometers, sometimes propeller anemometers and special microvanes, constructed on the principle of an ordinary wind vane, are used to measure fine-scale fluctuations of wind velocity [see, e.g., Kaimal, Cramer et al. (1964) and Dyer, Hicks and King (1947)]. Microvanes are especially useful for recording fluctuations of the wind direction which, together with data obtained by the hot-wire anemometer, permit instantaneous values of all three components of velocity u', v', and w' to be determined (see, for example, Cramer and Record (1953), and Lettau and Davidson (1957) Sect. 5.2). However, this type of apparatus has greater time lag than the instruments discussed above, and the possibility of automating the measurements and their subsequent processing is smaller.

Measurement of the temperature fluctuations T' may be made using resistance-thermometer instruments, with conductor or semiconductor sensors, or with the aid of a miniature thermocouple. A typical example of a resistance thermometer for measuring T' is Krechmer's microthermometer (1954) or similar Australian instruments [McIlroy (1955), Dyer (1961), Dyer and Maher (1965a)]. When the resistance element is used to measure temperature fluctuations, a very weak current is passed through it, and the overheat relative to the surrounding air does not exceed 0.01°. Under these conditions there is practically no heat transfer into the air, and the wire temperature, and consequently the wire resistance, depend only on the air-temperature fluctuations but are independent of the wind-velocity fluctuations. As a result, the fluctuations of the current passing through the resistance clement are proportional to the air-temperature fluctuations, with the proportionality coefficient determined only by the parameters of the instrument.

To measure the humidity fluctuations ϑ', wet- and dry-bulb resistance thermometers, or wet and dry thermocouples, may be used. They give the instantaneous values of the two temperatures T' and T_1', from which ϑ' is determined with the aid of the "psychrometric equation" [see, for example, Middleton and Spilhaus (1953), or Kedrolivanskiy and Sternzat (1953)]. Then T_1' may be found numerically from the values of T' and T_1' [McIlroy (1955), or by feeding the fluctuations T' and ϑ' into a special electrical analog device which automatically carries out an operation equivalent to solving the psychrometric equation [Swinbank (1951b); R. J. Taylor (1956b); Taylor and Dyer (1958); Dyer and Maher (1965a);

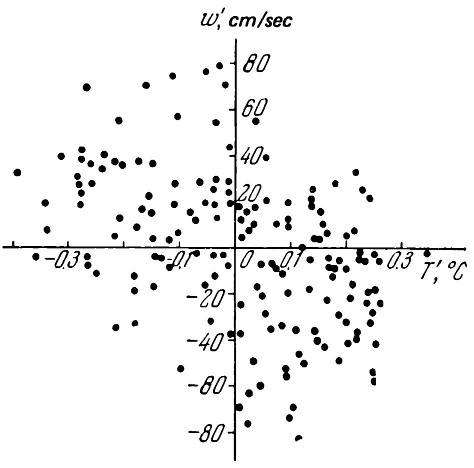


FIG. 70. Example of a correlation graph for the fluctuations of temperature and vertical velocity.

Polavarapu and Munn (1967)]. In either case, estimating the accuracy of the obtained values of ϑ' and, especially, $j = \rho\overline{w'\vartheta'}$, is considerably more difficult than determining T' and q' [see R. J. Taylor (1963)]; consequently, the possibility of using the fluctuation method to obtain completely reliable values of j still has a bright future. Prospects in this direction are enhanced by substituting for the two thermometers, an optical infrared hygrometer which is practically free of time lag; this method of hygrometry is based on the dependence of the air-refraction coefficient on the water-vapor content [see Elagina (1962)]. However, the infrared hygrometer has not yet been developed to a stage where it can be used systematically under field conditions.

In principle, fluctuation measurements permit simultaneous values of u' and w', or of w' and T' or w' and ϑ' to be determined for a series of successive instants. To illustrate the results obtained with this method, Fig. 70 [which is borrowed from Monin (1953)] shows a correlation graph which characterizes the dependence between the fluctuations w' and T' from measurements at a height of 1.5 m under inversion conditions. The graph corresponds to a correlation coefficient r_{wT} between the values of w' and T' equal to -0.24; the turbulent heat flux is equal here to $q = -0.03$ cal/cm^{2}min. Figure 71, also taken from Monin, shows a correlation graph which characterizes the relation between the fluctuations u' and w' from measurements also at a height of 1.5 m. Here the correlation coefficient r_{uw} is equal to -0.65 and $u_* = 0.33$ m/sec.

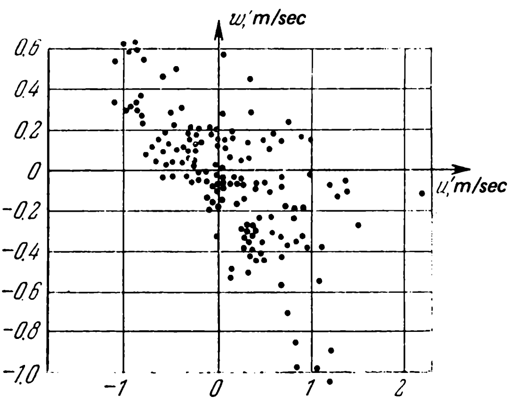


FIG. 71. Example of a correlation graph for the fluctuations of the horizontal and vertical components of the wind velocity.

Of course, the numerical calculation of the mean values of the fluctuation products from traces of the type shown in Fig. 1 is very laborious and tedious. However, if linear instruments are used, or instruments made almost linear by a special linearizing circuit, then the mean products (or squares) of the fluctuations may be obtained automatically by a special correlometer. This is an analog integrating device which calculates the integral of the product of the two input signals. Such automatic correlometers were developed in the U.S.S.R. by Bovsheverov, Gurvich, Tatarskiy and Tsvang (1959); see also Bovsheverov, Gurvich, Mordukhovich and Tsvang (1962). They were used to determine the turbulent heat flux [Bovsheverov, Gurvich, and Tsvang (1959)]. In Australia a similar apparatus was developed by Taylor and Webb (1955) and Dyer (1958). [See also Dyer and Maher (1965a, b) and Dyer, Hicks, and King (1967).] Fluctuations may also be recorded on analog magnetic tape, then converted to digital form and, finally, computed by digital computer [cf. Kaimal, Haugen and Newman (1966)]. Businger, Miyake et al. (1967) made a comparison of different heat-flux measurement methods, and showed that the horizontal variability of turbulent heat flux is quite large even for homogeneous underlying surfaces.

In addition to the fluctuation method for measuring turbulent shear stress, the dynamometric method may also be used. This was first proposed by Sheppard (1947) and then used by Pasquill (1950), Rider (1954), several American researchers [see Lettau and Davidson (1957), Sect. 3.2, and the references in Bradley (1968), Gurvich (1961), and Bradley (1968)]. This method consists in the use of a block of soil, with a grass cover characteristic of the locality. This block is set flush with the surface of the ground on a moving platform; suspended by wires in the experiments of Gurvich and Bradley, and floating on fluid in the experiments of other researchers; this platform is held in place by a special spring. The wind drag on the platform is determined by measuring either the tension in the spring, as done by Sheppard, Pasquill et al., or the displacement of the platform. The instruments used to measure τ and u_* by this method are very capricious, and their use requires great caution. However, the accuracy of the best of these instruments is comparable to that of the best existing fluctuation instruments designed for the same purpose.

In the field of meteorology, the turbulent water vapor flux (or the evaporation rate) j is most often measured by means of various types of weighing lysimeters (soil evaporators). In these instruments, a large block of soil is cut, with care taken to preserve its structure. The hole made when this block is removed, is deepened and special scales are placed inside. The block of soil is then returned to its original position and weighed at regular intervals. In comparison to these superior lysimeters, the use of other types requires a special transfer of the block of soil for each weighing. In either case, the rate of evaporation of moisture from the block can be determined; cf., for example, McIlroy and Angus (1963), and Crawford (1965). Unfortunately, it is still difficult to determine the extent to which the rate of evaporation of an excised block corresponds to the evaporation rate of the undisturbed soil. As a result, some researchers are inclined to think that all the data obtained from weighing lysimeters are quite questionable. To refute this opinion, it is necessary for comparison to have abundant material from simultaneous measurements of the rate of evaporation using lysimeters and the fluctuation method. The first attempts of this type were made by Swinbank and Dyer (1967; 1968). So far, however, completely reliable material of this nature does not exist. This is because the present fluctuation measurements of the evaporation rate are neither well developed nor particularly accurate.

To determine the turbulent heat flux q and evaporation rate j, we may also use the energy balance equation, i.e., the heat budget equation, of the underlying surface which has the form

$$B = q + Q + mj. \tag{8.16}$$

Here B is the radiation balance of the surface of the earth, specifically, the sum of the incident long- and short-wave radiation, from which the outgoing radiation is subtracted, Q is the heat flux in the soil, and m is the latent heat of vaporization. In certain cases of very dry soil, for example, in a desert, the last term on the right side may be assumed to be small in comparison with the other two terms. Then, measuring B and Q we can calculate q. However, when q is measured by the fluctuation method, Eq. (8.16) and the data from measurements of q, B, and Q, can be used to find the evaporation rate j. If q is unknown, but the values of j obtained by using lysimeters are considered sufficiently reliable, then Eq. (8.16) may be used to find the value of q; sec Rider (1954). On the other hand, if q and j arc measured directly, then Eq. (8.16) may be used as an additional verification of the data obtained [Dyer (1961); Dyer and Maher (1965a, b)]. Finally, if neither q nor j is measured directly but we have data on the temperature and humidity profiles, then we may use the assumption that the eddy diffusivities for heat and moisture are equal. It follows, therefore, that the so-called *Bowen ratio* $\gamma = q/mj$, a dimensionless quantity, satisfies the equation

$$\gamma = \frac{q}{mj} = \frac{c_p}{m} \frac{\bar{T}_1 - \bar{T}_0}{\bar{\vartheta}_1 - \bar{\vartheta}_0} \tag{8.17}$$

where $\bar{T}_1$ and $\bar{T}_0$ are the mean temperatures at two fixed heights z_1 and z_0, and $\bar{\vartheta}_1$ and $\bar{\vartheta}_0$ are the mean specific humidities at the same heights. From Eqs. (8.16) and (8.17) it follows that

$$q = \frac{\gamma(B-Q)}{1+\gamma}, \quad j = \frac{B-Q}{m(1+\gamma)}; \tag{8.18}$$

these last equations allow us to calculate the values of q and j from the measured values of B, Q, $\bar{T}_1$, $\bar{T}_0$, $\bar{\vartheta}_1$, and $\bar{\vartheta}_0$. Similar equations may be deduced on the basis of certain other postulates concerning the relation between the different eddy diffusivities [see, for example, Lettau and Davidson (1957), Sect. 7.3]. It must be noted, however, that Q and $\bar{\vartheta}_1 - \bar{\vartheta}_0$ can only be measured with very large error; therefore the accuracy of the energy balance method is considerably less, for example, than that of the fluctuation method.

8.4 Determination of Turbulent Fluxes from Data on the Profiles of Meteorological Variables

Above we gave several methods of measuring u_*, q, and j. However, all these methods are fairly complicated and require special equipment, which is only available at present in a few research organizations. At the same time, it would be of interest to make a systematic determination of turbulent fluxes at a large number of points on the earth's surface. It is quite useful therefore to give methods of determining u_*, q, and j from simpler measurements, primarily from the ordinary measurements of the profiles of the meteorological variables. The profile measurements are often called gradient measurements because they differ from the standard meteorological measurements at a single fixed height. In these measurements, we also determine the values of the gradients of the meteorological variables. Therefore, the method of determining fluxes from profiles is often called "the method of interpreting gradient measurements" or "the establishment of the flux-gradient relation."

We shall not dwell here on all the old speculative semiempirical formulas which express the fluxes in terms of the profiles, because few of them have been compared with the fairly recent data from direct measurements, and they are very unreliable. Instead, we shall use the similarity theory developed in Sect. 7 for a turbulent regime in the surface layer of the atmosphere. According to this theory, the wind-velocity profile and the temperature profile (omitting, for now the humidity) are determined by the general formulas (7.24) which contain the parameters u_*, q, z_0 [on the latter of which depends the mean velocity $\bar{u}(z)$], $\bar{T}_0 = \bar{T}(z_0)$ [$\bar{T}(z)$ depends on $\bar{T}_0$] von Kármán's universal constant $\varkappa$ and the two universal functions $f(\zeta)$ and $f_1(\zeta)$. Here $\varkappa$ is close to 0.4; some information on f and f_1 is given in Sects. 7, and 8.1–8.2. For the time being, we shall assume that these functions are exactly known. In this case any four measurements of the values of the velocity and temperature will permit formulation of four equations which are sufficient in principle, for the determination of the four parameters u_*, q, z_0, and $\bar{T}_0$. The required number of measurements may also be reduced without great difficulty, for example, by not considering the values $\bar{T}(z)$ at all, but only the differences $\bar{T}(z_2) - \bar{T}(z_1)$ which are independent of $\bar{T}_0$. However, this is complicated by the fact that at present, f and f_1 are known only approximately and are not given by any simple analytical formulas; data given by different researchers are contradictory, and the measurement results always contain some errors. Therefore, in practice, approximate expressions must be used for f and f_1, and different methods, using different sets of original data, will lead to somewhat different results.

The Monin-Obukhov method in Sect 8.1 of treating the measurements of the wind and temperature profiles included as a component part, the determination of the values of $\frac{u_*}{\varkappa}$ and L (which also permit us to find q), in terms of the values of $\bar{u}(z)$ and $\bar{T}(z)$. In spite of the fact that this method was based on the use of the later-criticized "logarithmic + linear" formula for the profiles and on the assumption that $\alpha \equiv 1$, its accuracy proves to be higher than might be expected. This was shown by comparing the values of u_* and q, obtained by this method, with the results of direct measurement. However, it is impossible now to recommend this method for direct practical use in its original form, since it is rather rough and very cumbersome; in particular, it includes the determination of some parameters by the least squares method from the results of several measurements.

Takeuchi (1961) and Shiotani (1962) proposed closely related methods of determining the values of u_* and q from wind- and velocity-profile measurements. These methods also use the "logarithmic + linear" approximation of the universal functions, and the postulate that $K_T = K$. Specifically, Takeuchi gave practical recommendations for simplifying the calculation, and also compared the calculated values of u_* and q with some results of direct measurement of the fluxes. Nevertheless, the methods of finding u_* and q are still unjustifiably complicated; at the same time, they use only a small part of the available information on $f(\zeta)$.

Methods of determining the turbulent fluxes according to profile measurements were also discussed by many researchers, for example: Kazanskiy and Monin (1956; 1958; 1962), Priestley (1959b), Webb (1960; 1965), Panofsky, Blackadar, and McVehil (1960), Monin (1962a), Kondo (1962b), Deacon and Webb (1962), Panofsky (1963), Klug (1963; 1967), Dyer (1965; 1967), Businger, Miyake et al. (1967), Zilitinkevich and Chalikov (1968b) and Busch et al. (1968).

Following Kazanskiy and Monin, a general method is given here which represents in a suitable form the data on the universal functions for practical calculation of the turbulent fluxes; first, however, a few general remarks shall be made.

The initial problem is to select the most suitable quantities for determining the fluxes. In principle, the values of the wind velocity at three arbitrary heights should be sufficient to find the parameters u_*, q, and z_0; consequently, the impression may be given that it is quite superfluous to use temperature observations. However, as Priestley (1956b) showed, even with the use of the values of the wind-velocity $\bar{u}(z)$ at more than three heights, z does not allow us to determine q reliably. This is because the values thus obtained are quite different from the data of direct measurements and differ considerably for different methods of treatment. Therefore, the supplementary use of $\bar{T}(z)$ would appear to have great practical significance. This deduction in fact follows also from Panofsky, Blackadar and McVehil (1960), who experimented with a method of determining q from the values of $\bar{u}(z)$ at three heights (and the value of z_0) only, but did not obtain good results. Nevertheless, it is worth noting that Klug (1967) obtained satisfactory agreement between the data of the direct flux measurements of Swinbank (1964) and the values computed from wind observations at six heights by a rather complicated least squares method based on the form of the universal wind function $f(\zeta)$ proposed by Panofsky (1965).

Since the values of u_* and q are of primary interest, it might be considered convenient to eliminate z_0 at the very beginning, and to consider only the differences $\bar{u}(z_2) - \bar{u}(z_1)$, ignoring $\bar{u}(z)$ itself. However, in this case it is easy to see that there could be a great loss in accuracy, since the differences in wind velocity are determined considerably less accurately than the wind itself, especially if the heights z_1 and z_2 are comparatively close together. Moreover, it may be shown that when z_0 is eliminated, even a fairly small error in the velocity differences will produce a great effect on u_* and q. Since z_0 at a given point and time of year is to be determined only once (it does not vary), and since it is not difficult to find, it is more convenient to find z_0 first, and then use this value throughout.[9] At the same time, $\bar{T}_0 = \bar{T}(z_0)$ is not a constant and therefore may be eliminated; thus it is reasonable to

[9] Of course, sometimes it is very difficult to determine z_0, for example, in the marine atmosphere. Therefore, with this in mind, Zilitinkevich and Chalikov (1968b) gave nomograms for determining u_* and q from the values of $\delta u = \bar{u}(2) - \bar{u}(0.5)$ and $\delta T = \bar{T}(2) - \bar{T}(0.5)$, where the numbers indicate the height in meters.

use the temperature-profile data only in the form of the differences $\bar{T}(z_2) - \bar{T}(z_1)$. Then the data of only two measurements are required to determine u_* and q.

Consequently, it is recommended that, to find u_* and q from data on $\bar{u}(z)$ and $\bar{T}(z)$, one should begin by determining the roughness parameter z_0, and then proceed from the wind velocity U at some fixed height H and from the difference δT between the mean temperatures at two different heights, for example, at $2H$ and $H/2$. According to the similarity hypothesis for a turbulent regime in the surface layer of the atmosphere, the quantities $U = \bar{u}(H)$ and $\delta T = \bar{T}(2H) - \bar{T}\left(\frac{H}{2}\right)$ may be expressed in terms of H, $\frac{g}{T_0}$, u_*, $\frac{q}{c_p \rho_0}$, and z_0. Consequently, u_* and $\frac{q}{c_p \rho_0}$, and also L, which depends on them, may, in turn, be expressed in terms of H, $\frac{g}{T_0}$, U, δT, and z_0. From these five terms we may formulate two independent dimensionless combinations, which are written as

$$B = \frac{\alpha g H}{T_0} \frac{\delta T}{U^2}, \quad \zeta_0 = \frac{z_0}{H}, \tag{8.19}$$

where α is an arbitrary dimensionless coefficient; we shall say more about the suitable choice of α later. Thus we obtain

$$\frac{u_*}{U} = F_1(B, \zeta_0), \quad \left| \frac{q/c_p \rho_0}{\alpha U \delta T} \right| = F_2(B, \zeta_0), \quad \frac{L}{H} = F_3(B, \zeta_0), \tag{8.20}$$

where F_1, F_2, F_3 are some universal functions of B which also depend on ζ_0. If graphs of these functions were available, they could be used to determine u_*, q, and L from the data of gradient measurements. Graphs of F_i, $i = 1, 2, 3$, may be plotted either empirically using the data of independent simultaneous measurements of U, δT, u_*, and q, or else theoretically by fixing the form of the universal functions $f(\zeta)$ and $f_1(\zeta)$. The second of these methods permits a progressive improvement in the accuracy of determining fluxes compared to the accuracy of our information on the universal functions. Therefore, we shall discuss this method in somewhat greater detail.

As our starting point, we take the general formula (7.24) which was used to determine the universal functions $f(\zeta)$ and $f_1(\zeta)$. Then the first equation of Eqs. (8.19) and the first two equations of Eq. (8.20) may be written as

$$\begin{aligned} B &= \frac{\alpha}{F_3} \frac{f_1(2/F_3) - f_1(1/2F_3)}{[f(1/F_3) - f(\zeta_0/F_3)]^2}, \\ F_1 &= \frac{\varkappa}{f(1/F_3) - f(\zeta_0/F_3)}, \\ F_2 &= \frac{\varkappa^2}{\alpha [f(1/F_3) - f(\zeta_0/F_3)][f_1(2/F_3) - f_1(1/2F_3)]}. \end{aligned} \tag{8.21}$$

The first of these equations gives the form of $F_3(B, \zeta_0)$ implicitly. If the functions $f(\zeta)$ and $f_1(\zeta)$ are known, then, for example, this equation may be solved graphically for F_3 and the two remaining equations of Eq. (8.21) will give $F_1(B, \zeta_0)$ and $F_2(B, \zeta_0)$ parametrically.

Kazanskiy and Monin (1962), and Zilitinkevich and Chalikov (1968b) assumed that the wind-velocity profile and the temperature profile are similar, i.e., that $f_1(\zeta) = \frac{1}{\alpha} f(\zeta)$, where $\alpha = \frac{K_T}{K} =$ const. Under this assumption, the coefficient α in Eqs. (8.19) and (8.20) also may conveniently be taken equal to K_T/K. Then Eq. (8.21) takes the simpler form

$$\begin{aligned} B &= \frac{1}{F_3} \frac{f(2/F_3) - f(1/2F_3)}{[f(1/F_3) - f(\zeta_0/F_3)]^2}, \\ F_1 &= \frac{\varkappa}{f(1/F_3) - f(\zeta_0/F_3)}, \\ F_2 &= \frac{\varkappa^2}{[f(1/F_3) - f(\zeta_0/F_3)][f(2/F_3) - f(1/2F_3)]}. \end{aligned} \tag{8.21'}$$

In fact in the works cited, an even more special assumption was used, namely, that $\alpha = 1$. However, retaining the coefficient α in B and F_2 does not change the calculation scheme at all; thus no special purpose is served by assuming a particular numerical value. It is also possible to assume in the next approximation that α is different, for example, in neutral, unstable, and stable stratifications; under this assumption Eqs. (8.21′) can also be used.

On the basis of Eq. (8.21′), Kazanskiy and Monin (1962), and Zilitinkevich and Chalikov (1968b) drew up nomograms of the functions F_1, F_2, and F_3 based on two different assumptions concerning the form of the universal functions $f(\zeta) = f_1(\zeta)$. Figures 72–74 give the Zilitinkevich-Chalikov nomograms, which correspond to the function $f(\zeta)$ satisfying Eqs. (7.58) and (8.8) with the numerical parameters recommended by Zilitinkevich and Chalikov (1968a), and two supplementary assumptions: $\varkappa = 0.43$; and $\alpha \equiv 1$. These nomograms were checked by Zilitinkevich and Chalikov using extensive data of field measurements in the U.S.S.R., Australia and the U.S.; the corresponding values of the fluxes were also compared with results obtained from some other assumptions concerning the form of $f(\zeta)$ [see Zilitinkevich and Chalikov (1968b), and Zilitinkevich (1970)].

Of course, the nomogram for F_2 may not only be used to determine the turbulent heat flux q, but also to determine the evaporation rate j, provided that, in addition to U and δT, the difference between the specific humidities at two heights $\delta\vartheta = \bar{\vartheta}(2H) - \bar{\vartheta}\left(\frac{H}{2}\right)$ is measured. According to the general formulas (7.24) and the empirically discovered similarity between the profiles $\bar{T}(z)$ and $\bar{\vartheta}(z)$

$$\left|\frac{j/\rho_0}{\alpha_1 U\,\delta\vartheta}\right| = \left|\frac{q/c_p\rho_0}{\alpha U\,\delta T}\right| = F_2(B, \zeta_0), \tag{8.22}$$

where $\alpha_1 = K_\vartheta/K$ ($= \alpha$ if $K_\vartheta = K_T$).

With the aid of Eqs. (8.21′) it is not difficult to establish the form of the function (8.20) for near-neutral, very unstable and very stable stratifications. For near-neutral stratification $|B|$ is small and $|F_3|$ is large while in Eq. (8.21′) we may assume that $f(\zeta) \approx \ln|\zeta|$. Then Eqs. (8.21′) become considerably simpler, and have explicit solutions of the form

$$F_1 \approx \frac{\varkappa}{\ln(1/\zeta_0)}, \qquad F_2 \approx \frac{\varkappa^2}{\ln 4 \cdot \ln(1/\zeta_0)}, \qquad F_3 \approx \frac{\ln 4}{\left(\ln\frac{1}{\zeta_0}\right)^2}\,\frac{1}{B}. \tag{8.23}$$

In the case of extreme instability, B will be large and negative and F_3 small and negative. The asymptotic behavior of the functions (8.20) is found here by introducing $f(\zeta) \approx c_2 - C_2|\zeta|^{-\frac{1}{3}}$ in Eqs. (8.21′). Then we obtain

$$F_1 \approx \frac{\varkappa}{\left(2^{\frac{1}{3}} - 2^{-\frac{1}{3}}\right)^{\frac{1}{4}} C_2^{\frac{3}{4}}}\,\frac{|B|^{\frac{1}{4}}}{\left(\zeta_0^{-\frac{1}{3}} - 1\right)^{\frac{1}{2}}},$$

$$F_2 \approx \frac{\varkappa^2 |B|^{\frac{1}{2}}}{\left(2^{\frac{1}{3}} - 2^{-\frac{1}{3}}\right)^{\frac{3}{2}} C_2^{\frac{3}{2}}}, \tag{8.24}$$

$$F_3 \approx -\frac{\left(2^{\frac{1}{3}} - 2^{-\frac{1}{3}}\right)^{\frac{3}{4}}}{\left(\zeta_0^{-\frac{1}{3}} - 1\right)^{\frac{3}{2}} C_2^{\frac{3}{4}}}\,|B|^{-\frac{3}{4}}.$$

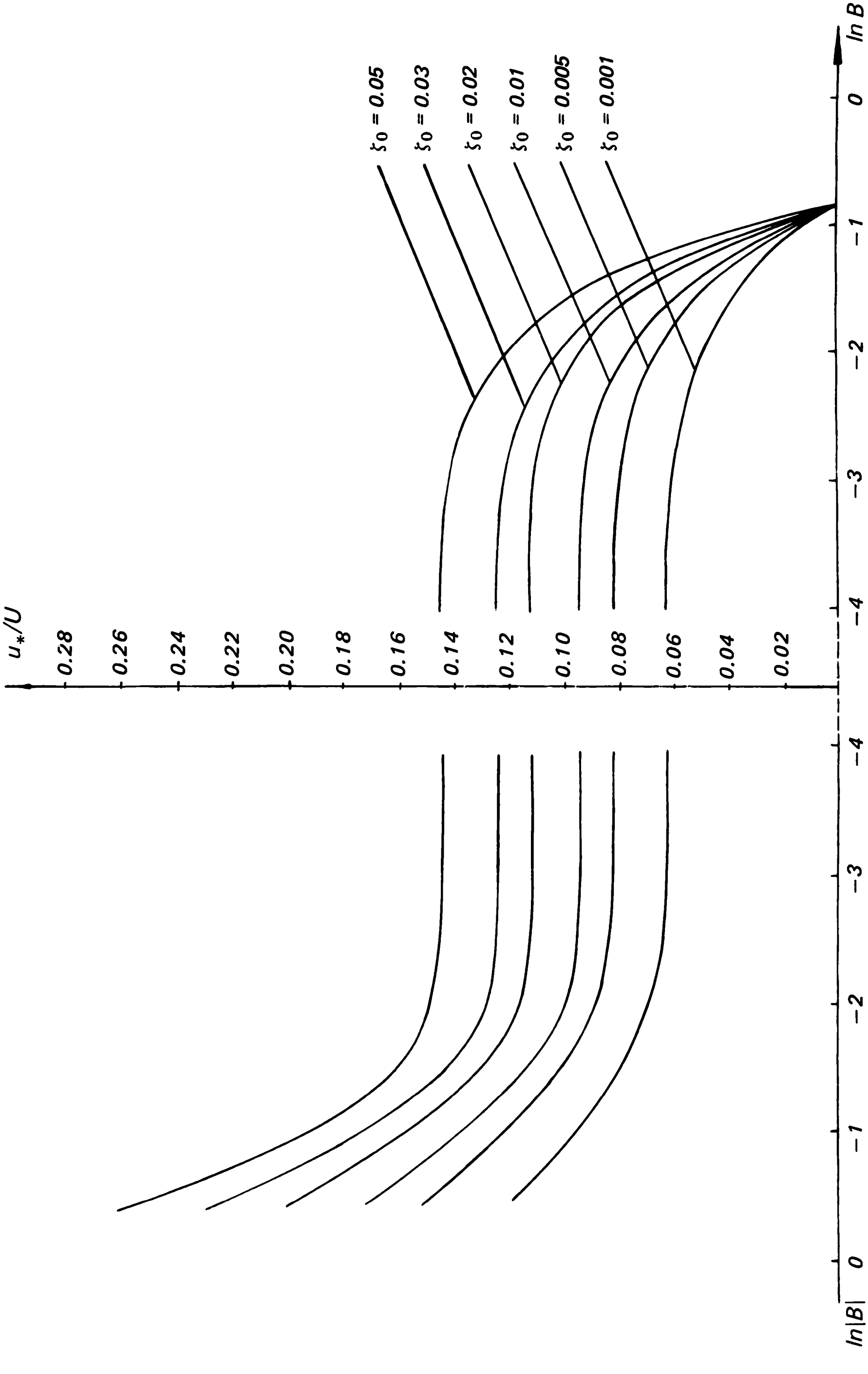


FIG. 72. A nomogram for the determination of the dimensionless friction velocity $u_*/U = F_1$ using B and ζ.

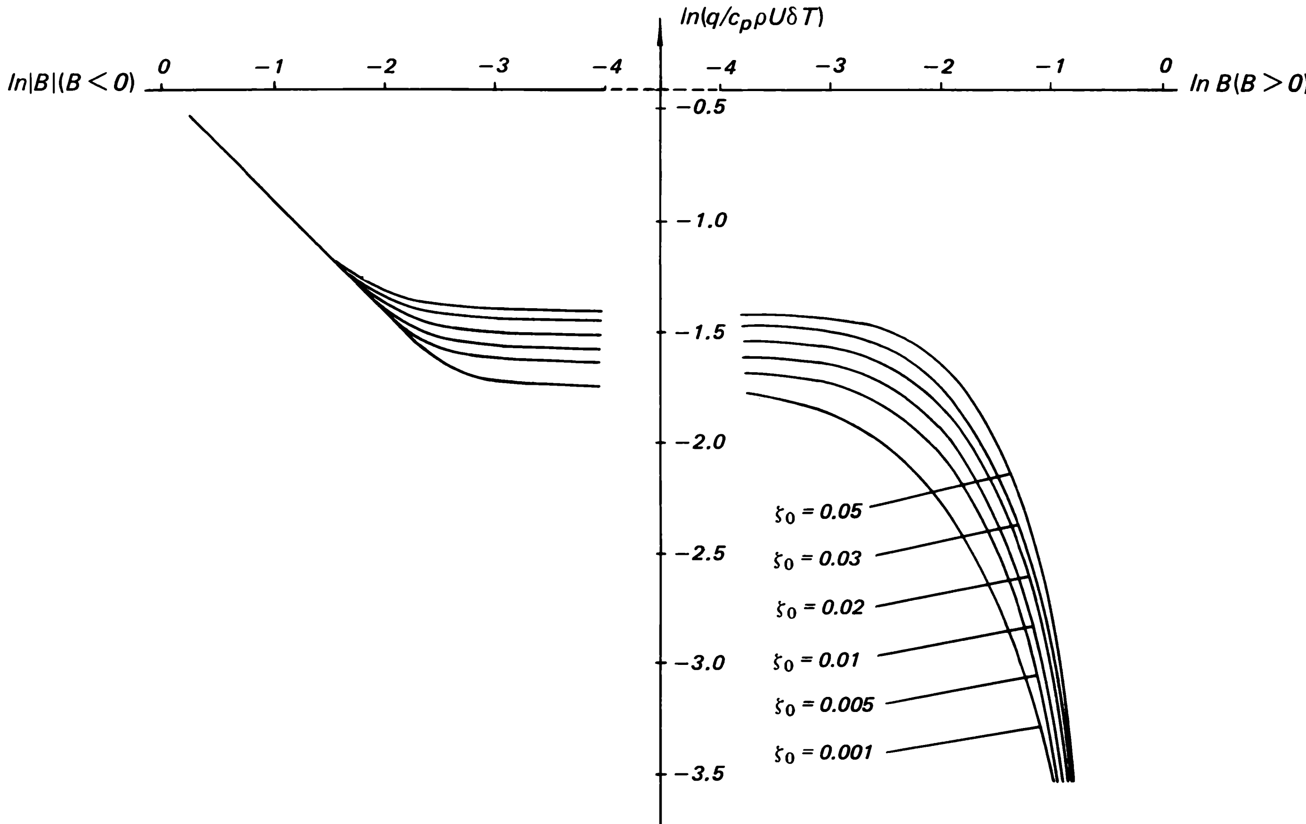


FIG. 73. A nomogram for the determination of the dimensionless heat flux F_2 using B and ζ.

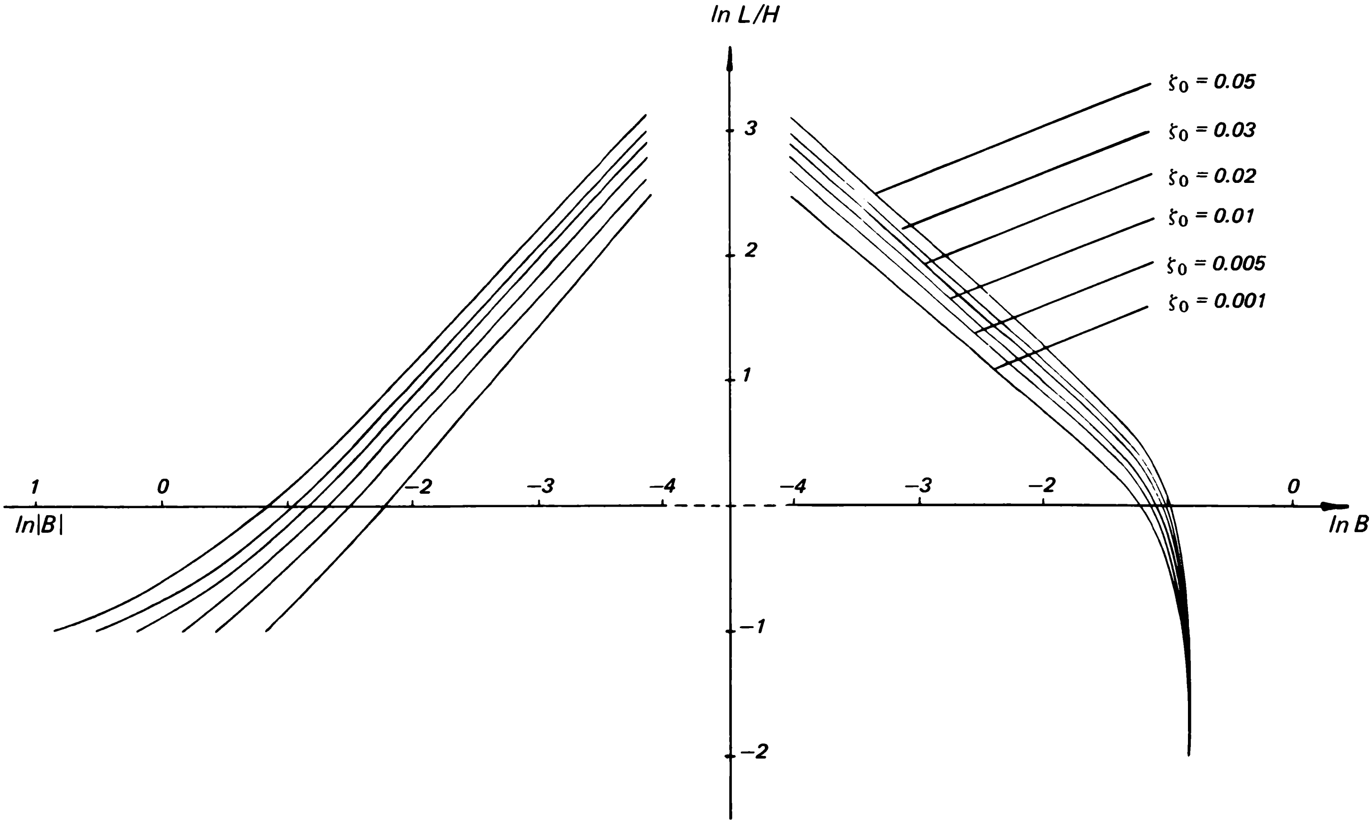


FIG. 74. A nomogram for the determination of $L/H = F_3$ using B and ζ.

Finally, for great stability, B will be large and positive and F_3 small and positive. The asymptotic behavior of the function (8.20) in this case may be determined by introducing $f(\zeta) \approx c_3 + C_3\zeta$ in Eq. (8.22′). Then we find that B cannot increase without bound, but will tend to the limit $\frac{3}{2C_3(1-\zeta_0)^2}$ as $F_3 \to 0$. Thus for stable stratification, B cannot exceed $3/2C_3$. When F_3 is small, for F_1 and F_2 we obtain

$$F_1 \approx \frac{\varkappa F_3}{C_3(1-\zeta_0)}, \qquad F_2 \approx \frac{2\varkappa^2 F_3^2}{3C_3^2(1-\zeta_0)}. \tag{8.25}$$

The flux-gradient relation assumes an especially simple form for great instability. In this case, if all the measurement heights exceed $\zeta_1 L$, which is the lower boundary of the "1/3-power-law" layer (for this it is sufficient that $H/2 \gg \zeta_1 L$), then to interpret the measurement results we can use the formulas which are correct in the limiting case of free convection. The criterion $H/2 \gg \zeta_1 L$ which ensures this possibility may be rewritten as $|F_3| \ll \frac{1}{2|\zeta_1|}$. Using formula (8.24) for F_3, we may write this criterion in the form

$$|B| \gg \frac{2\left(2^{\frac{2}{3}}-1\right)|\zeta_1|^{\frac{4}{3}}}{C_2\left(\zeta_0^{-\frac{1}{3}}-1\right)^2} \approx \frac{0.01}{\left(\zeta_0^{-\frac{1}{3}}-1\right)^2},$$

where, as in the formulation of the nomogram, it is assumed that $C_2 \approx 1.25$ and $|\zeta_1| \approx 0.03$. The value of this criterion, in round figures, varies from 0.0005 for $\zeta_0 = 0.005$, to 0.004 for $\zeta_0 = 0.05$. Thus the right side of this inequality may be written approximately in the form $0.1\,\zeta_0$ for this range of ζ_0. The proposed criterion then reduces to the form

$$\left|\frac{\delta T}{U^2}\right| \gg \frac{3z_0}{H^2}, \tag{8.26}$$

where z_0 and H are measured in meters, U in m/sec, and δT in °C.

If the criterion (8.26) is satisfied, then for $z \gg H/2$, we have a regime of free convection, where

$$q = \left(\frac{3\alpha_{-\infty}}{C_2}\right)^{\frac{3}{2}} c_p\rho_0\varkappa^2\left(\frac{g}{T_0}\right)^{\frac{1}{2}}\left|\frac{\partial\overline{T}}{\partial z}\right|^{\frac{3}{2}} z^2 \tag{8.27}$$

[see Eqs. (7.36) and (7.39)]. It is easy to show from this that here

$$q = c_p\rho_0\varkappa^2\left[\frac{\alpha_{-\infty}}{C_2\left(2^{\frac{1}{3}}-2^{-\frac{1}{3}}\right)}\right]^{\frac{3}{2}}\left(\frac{g}{T_0}\right)^{\frac{1}{2}} H^{\frac{1}{2}}|\delta T|^{\frac{3}{2}} \tag{8.28}$$

which may also be obtained from Eqs. (8.20) and (8.24). Substituting this value of q into Eq. (7.38) for the eddy viscosity $K = \alpha_{-\infty}^{-1} K_T$ at height $z = H$, we obtain

$$K(H) = \frac{3\varkappa^2}{C_2}\left[\frac{\alpha_{-\infty}}{C_2\left(2^{\frac{1}{3}}-2^{-\frac{1}{3}}\right)}\right]^{\frac{1}{2}}\left(\frac{g}{T_0}\right)^{\frac{1}{2}} H^{\frac{3}{2}}|\delta T|^{\frac{1}{2}}. \tag{8.29}$$

Then q and $K(H)$ differ from $H^{\frac{1}{2}} \mid \delta T \mid^{\frac{3}{2}}$ and $H^{\frac{3}{2}} \mid \delta T \mid^{\frac{1}{2}}$, respectively, only by universal, but not dimensionless, multipliers. Expressing H in meters, δT in degrees, q in cal/cm^2 min, and $K(H)$ in m^2/sec, and putting $\varkappa \approx 0.43$, $\alpha_{-\infty} \approx 1$, $C_2 \approx 1.25$ as recommended by Zilitinkevich and Chalikov (1968a, b), and $T_0 \approx 300^\circ$K, we obtain the approximate equations

$$q \approx 0.17\, H^{1/2} |\delta T|^{3/2}, \qquad K(H) = 0.12\, H^{3/2} |\delta T|^{1/2}. \tag{8.30}$$

These formulas enable q and $K(H)$ to be estimated very quickly from temperature differences in the case of unstable stratification. However, the numerical coefficients in the last equation are not completely reliable at present. This is because an unexplained discrepancy exists between the various data; for example, according to Dyer (1967), the first of Eqs. (8.30) has the form $q \approx 0.26\, H^{\frac{1}{2}} \mid \delta T \mid^{\frac{3}{2}}$.

8.5 Characteristics of Wind Velocity and Temperature Fluctuations in the Atmospheric Surface Layer

The turbulent fluxes of momentum and heat are only special cases of the statistical characteristics of wind velocity and temperature fluctuations. Many other characteristics of these fluctuations have been analyzed from the viewpoint of similarity theory in Sect. 7.5; they may all be found experimentally with the same apparatus used in the fluctuation method of determining u_* and q. If data exist concerning simultaneous measurements of u_* and q and of any statistical characteristic of the fluctuations, or if we have data on the measurements of this statistical characteristic, and estimate u_* and q by the method described in Sect. 8.4, the deduction of similarity theory relating to this characteristic may be verified and the corresponding universal function of $\zeta = \frac{z}{L}$ formulated empirically. Thus the data of fluctuation measurements permit us, in principle, to determine all the functions $f_3, \ldots, f_{11}$, φ_ε and φ_N introduced in Sect. 7.5.

Data for moments of the turbulent fluctuations in the atmospheric surface layer can be found in Swinbank (1955), Deacon (1955), Perepelkina (1957; 1962), Lettau and Davidson (1957), Barad (1958), Gurvich (1960), Tsvang (1960), Panofsky and McCormick (1960), Monin (1962b), Zubkovskiy (1962; 1967), Lumley and Panofsky (1964), Klug (1965), Panofsky and Prasad (1965), Mordukhovich and Tsvang (1966), Panofsky, Busch et al. (1967), Cramer (1967) and in many other works. Unfortunately, all these data are still fairly inexact, strongly scattered and do not agree very well with each other; in addition, they are far from being complete. Both the scatter and the disagreement among the data can possibly be explained by the fact that in all cases, there is considerable error in determining the turbulent fluxes which have quite large variability, both vertical and horizontal [cf., for example, Mordukhovich and Tsvang (1966), and Businger, Miyake et al. (1967)], even above a relatively very homogeneous surface. However, many of the measurements were carried out above a clearly inhomogeneous terrain. In every case the existing data allow us to draw only certain preliminary conclusions about the statistical characteristics of the fluctuations. It is important, however, that in all cases these conclusions agree fairly well with the predictions of the similarity theory and give a definite representation of the general form of certain universal functions.

Data on the dependence of the dimensionless standard deviations of the velocity fluctuations σ_u/u_*, σ_v/u_*, and σ_w/u_* on thermal stratification, that is, on the universal functions f_3, f_4, and f_5 are given in Swinbank (1955), Deacon (1955), Gurvich (1960), Panofsky and McCormick (1960), and in all the above-mentioned works later than 1960. For example, Fig. 75 shows the functions $f_3(\zeta)$, $f_4(\zeta)$, and $f_5(\zeta)$ obtained by Cramer (1967) from observations at Round Hill, Mass., at heights of 16 m (circles) and 40 m (triangles) for averaging time of 1.2 min; such comparatively small averaging times may be advantageous for eliminating the low-frequency fluctuations which are dependent on large-scale features of the terrain and do not follow similarity theory. A survey of other data on σ_u/u_*, σ_v/u_* and σ_w/u_* can be found in Panofsky and Prasad (1965), and Panofsky, Busch et al. (1967). The graph of the function $f_3(\zeta)$ in Fig. 75a shows almost no dependence on the stability parameter ζ although σ_u/u_* may increase somewhat with increasing instability. More definite indications concerning the slight monotonic increase of σ_u/u_* with decrease of ζ along both semi-axes are seen on the plots given by Panofsky, Busch et al. The values of $f_3(0)$ and $f_4(0)$, corresponding to the values of the ratios σ_u/u_* and σ_v/u_* at neutral stratification, can be determined from atmospheric observations only with relatively low precision, but in general they agree quite well with the estimates $\sigma_u/u_* = A_1 \approx 2.3$ and $\sigma_v/u_* = A_2 \approx 1.7$ obtained from measurements in the logarithmic layer of laboratory flows [see the end of Sect. 5.3]. The relationship $f_3(\zeta) = f_4(\zeta) \sim |\zeta|^{1/3}$ predicted by similarity theory for the "free convection regime," or for large negative values of ζ, is neither confirmed nor unconfirmed by the existing data.

Since there is no doubt that the large-scale features are unimportant, the data on $f_5(\zeta)$ are more numerous and more reliable than

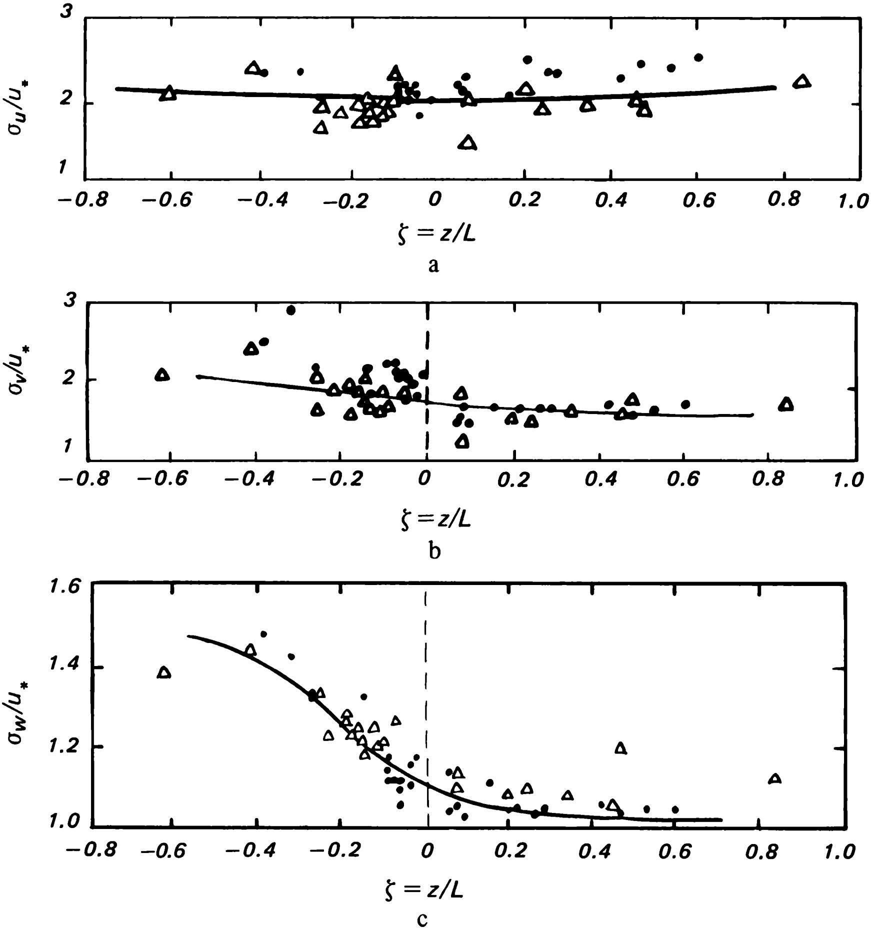


FIG. 75. Data of H. E. Cramer (1967) on the universal functions: a—$\sigma_u/u_* = f_3(\zeta)$; b—$\sigma_v/u_* = f_4(\zeta)$; c—$\sigma_w/u_* = f_5(\zeta)$.

those on $f_3(\zeta)$ and $f_4(\zeta)$. In addition to Cramer's figure (Fig. 75c), we give here also the summary figure (Fig. C) of Panofsky, Busch et al. (1967), where observations from many sites are collected. The equations $f_5(\zeta) = 1.25\ [\varphi(\zeta) - 1.8\ \zeta]^{1/3}$ and $f_5(\zeta) = 1.25\ [1 - \zeta/\varphi(\zeta)]^{1/4}$ (where $\varphi(\zeta)$ is taken as recommended by Panofsky, Busch et al.) of the forms (7.96) and (7.95) are also plotted in Fig. C; they turn out to be almost indistinguishable from each other over the whole range of values of ζ in Fig. C, and agree with the data quite well. The problem of the exact value of $f_5(0)$ according to observations in the atmospheric surface layer has stimulated considerable discussion. The data of the comparatively old Russian observations of Gurvich

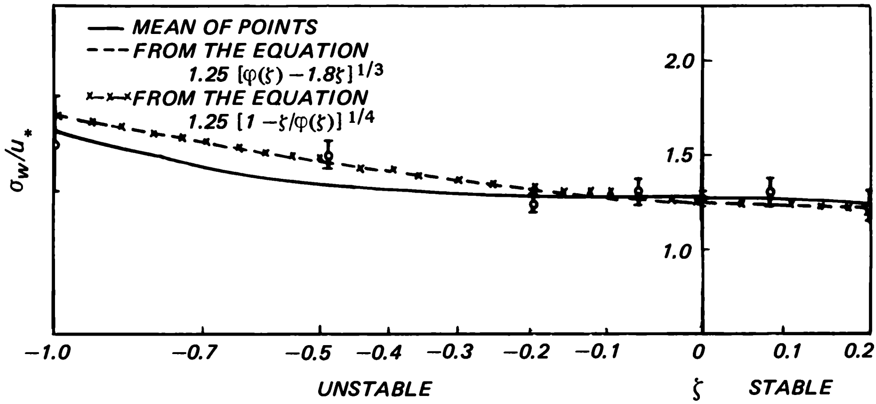


FIG. C. The ratio σ_w/u_* as a function of ζ according to many sources. The segments represent the standard deviation of the means, assuming all observations are independent of each other.

(1960) [see also Perepelkina (1957)] gave the estimate $f_5(0)=A_3 \approx 0.7-0.8$, which is slightly below the laboratory estimate $A_3 \approx 0.9$ of Sect. 5.3, but agrees very well with the measurements of Cermak, Sandborn et al. (1966), and Cermak and Chuang (1967), in a meteorological wind-tunnel at Colorado State University in the U.S. However, most of the other investigators [for example, Panofsky and McCormick (1960), Pasquill (1962a), Panofsky and Prasad (1965), Kaimal and Izumi (1965), Mordukhovich and Tsvang (1966), Businger, Miyake et al. (1967), and Cramer (1967)], came to the conclusion that $f_5(0)$ is actually slightly greater than unity. The mean value of all the estimates of $f_5(0)=A_3$ from atmospheric observations lies in the range from 1.2 to 1.3 (the value 1.25 is used in the summary Fig. C); although this value is higher than the laboratory estimate, the discrepancy is not serious. The theoretical prediction (7.87), according to which $f_5(\zeta) \approx C_5|\zeta|^{1/3}$, i.e., $\sigma_w \approx C_5'\,(qgz/c_p\rho_0 T_0)^{1/3}$ at large negative ζ, agrees fairly well with the data; this was also confirmed by the special measurements of Myrup (1967) from an instrumented aircraft flying over a desert-dry lake bed in California, at heights of 7.5 m to 93 m. His results showed that the equation $\sigma_w = cz^{1/3}$ is valid in strong convection with rather high accuracy. The values of the coefficients C_5' and $C'_5 = C'_5\kappa^{-1/3}$ were estimated by Perepelkina (1962) from Gurvich's fluctuation measurements in the steppe near Tsimlyansk, on the assumption that Eq. (7.87) is correct for Ri < -0.04. Perepelkina found that $C'_5 \approx 1.4$, i.e., $C_5 \approx 1.9$. However, bearing in mind that Gurvich's data on σ_w are probably underestimated, it is not surprising that Perepelkina obtained a higher estimate $C'_5 \approx 1.8$, or $C_5 \approx 2.4$, on the basis of the data from Swinbank's old and quite rough measurements (1955). The treatment of the raw data of Mordukhovich and Tsvang (1966) gives an estimate which is even slightly higher than the last one: $C'_5 \approx 2.1$, $C_5 \approx 2.8$ if all the points with $\mathrm{Ri} < -0.04$ are used. However, a similar treatment of the Mordukhovich-Tsvang measurements corresponding to $\mathrm{Ri} < -0.3$ leads to a lower estimate: $C'_5 \approx 1.6$, $C_5 \approx 2.2$.

Figure 76 gives data on $\sigma_T/|T_*| = f_6(\zeta)$ from various sources. Although the scatter is considerable, especially for near-neutral conditions where the ratio $\sigma_T/|T_*|$ becomes indeterminate because both the numerator and the denominator become very small, it is possible to plot a smooth mean curve on the basis of the most reliable measurements. According to this curve, the function $f_6(\zeta)$ takes a value close to unity for near-neutral stratification (it is clear of course that $f_6(0)$ can be defined only as a limit of $f_6(\zeta)$ as $\zeta \to 0$ either from positive or negative values), is almost constant under stable conditions and drops off somewhat, taking values less than unity, with increasing instability. The question of the validity of the limiting "$-1/3$-power law" (7.87), according to which $f_6(\zeta) \approx C_6|\zeta|^{-1/3}$, i.e., $\sigma_T \approx C'_6(q/c_p\rho_0)^{2/3}(gz/T_0)^{-1/3}$ was specially examined by Priestley (1960b) according to data from Australian measurements, those in the U.S. at O'Neill, Neb., and by Myrup (1967) from measurements above a desert-dry lake bed in extremely unstable conditions. Both Priestley and Myrup found that the predicted relation $\sigma_T = cz^{-1/3}$ is confirmed rather well by existing data. Nevertheless, the large scatter of points in Fig. 76 indicates that additional careful investigation will be very useful. Priestley also

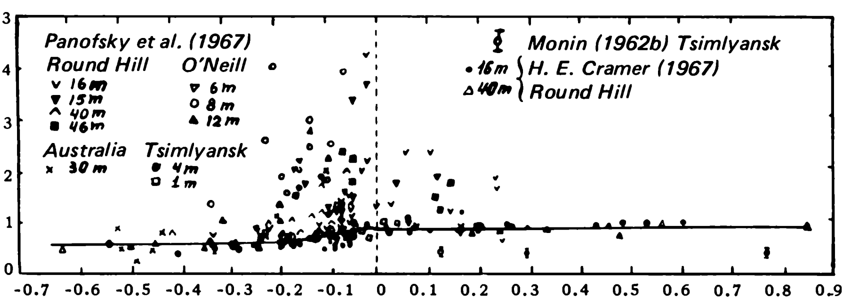


FIG. 76. Data on the universal function $\sigma_T/|T_*| = f_6(\zeta)$.

made an attempt to estimate the values of the numerical coefficients C'_6 and $C_6 = C'_6\kappa^{4/3}$ from observations. According to his quite rough estimate, $C_6 \approx 0.4$, i.e., $C'_6 \approx 1.3-1.4$. Later, Perepelkina (1962) used the data of Tsvang's measurements of temperature fluctuations at Tsimlyansk in 1958-1959 and independently from Priestley obtained a fairly close estimate $C'_6 \approx 1.1$, i.e., $C_6 \approx 0.3-0.35$. Finally, Mordukhovich and Tsvang (1966) evaluated the constant C'_6 C_2 in their notation from extensive measurements at Tsimlyansk in 1964 at heights of 1 m and 4 m, and also found that $C'_6 \approx 1.0-1.1$, that is, $C_6 \approx 0.3$. The data from Swinbank's old measurements (1955) were improved by Perepelkina (1962); they led to the conclusion that $C'_6 \approx 1.2, C_6 \approx 0.35$.

The dependence of the correlation coefficients $r_{uw} = -1/f_3 f_5$ and $r_{wT} = \pm\varkappa|f_5 f_6|$ on the stability was studied by Swinbank (1955), Mordukhovich and Tsvang (1966), and Zubkovskiy and Tsvang (1966). The existing data show that the typical absolute values of both correlation coefficients r_{uw} and r_{wT} in the atmospheric surface layer are in the range 0.3–0.5. Moreover, r_{wT} increases considerably with increase of instabihty from about 0.35 for near-neutral conditions, to about 0.6 for Ri in the range −0.3 to −0.8. On the other hand, r_{uw} is approximately independent of stability, and close to 0.4 for all neutral and moderately unstable conditions.

The horizontal heat flux measurements $q_h = c_p\rho_0\overline{u'T'}$ in the atmospheric surface layer led to interesting results. Apparently the first measurements of this type were carried out by Shiotani (1955) who published results of three measurements of the value of q_h, for an averaging time of 1 min; the values of q_h obtained in all three cases turned out to be greater than the simultaneous values of vertical heat flux q. However, these results were forgotten, or overlooked, by almost all scientists and for many years most researchers assumed, and even explicitly formulated, that horizontal flux q_h must be equal to zero or at least be very small. However, the data of the direct measurements of Zubkovskiy and Tsvang (1966), and Zubkovskiy (1967) [see Fig. D] show that the ratio $-q_h/q = \varkappa f_7(\zeta)$ is close to 3 for near-neutral conditions and even grows with increasing stability. With increasing instability the ratio drops off but not too rapidly, taking a value near 1.5 at Ri $\approx$ −0.8. These data also agree with those of similar measurements made in England, above the surface of a lake, under the direction of P. A. Sheppard (unpublished). The absolute value of the correlation coefficient $r_{uT} = \overline{u'T'}/\sigma_u\sigma_T$ is close to the absolute value of the coefficient r_{wT} ($\approx$ 0.35) for near-neutral conditions, and even exceeds $|r_{wT}|$ in inversions; however, with increasing instability $|r_{wT}|$ grows and $|r_{uT}|$ decreases, so that, $r_{wT} \approx 0.6$ and $r_{uT} \approx -0.2$ when Ri = −0.7 according to the data of Zubkovskiy and Tsvang. The theoretical deductions of Sect. 7.5 imply also that $\overline{u'T'} \to 0$ and $r_{uT} \to 0$ as Ri $\to -\infty$. Of course, it must be borne in mind that it is natural to compare the horizontal turbulent heat flux q_h with the horizontal advective heat flux $c_p \rho_0 \bar{u}\bar{T}$ and not with the vertical heat flux q. In comparison with the advective flux, the flux q_h will usually be rather small and often can be neglected. However, some problems exist which quite definitely require due consideration of the flux q_h [the semiempirical approach of Monin (1965) is one example; cf. also Yaglom (1969)]; moreover, the nonvanishing of q_h implies important results on the form of the eddy diffusivity tensor, which will be considered in Sects. 10.3 and 10.5.

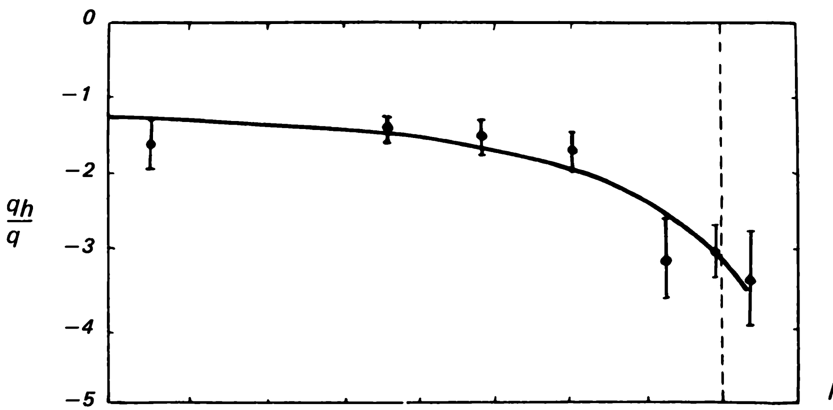


FIG. D. The dependence of q_h/q on Ri according to the data of Zubkovskiy and Tsvang (1965).

The data on higher-order moments of the turbulent fluctuations in the atmospheric surface layer are so scanty that they are clearly insufficient for a graph of any of the corresponding universal functions to be plotted. However, all the available data are in agreement with the qualitative predictions of Sect. 7.5. Thus, according to Deacon (1955), the values of the vertical velocity skewness $\overline{w'^3}/\sigma_w^3$ for unstable stratification lie between 0.24 and 0.81, that is, are strongly positive as was predicted in Sect. 7.5; moreover, these values increase considerably with increase of instability. Similarly, Gurvich (1960) obtained an estimate of the order 0.4–0.6

for $\overline{w'^3}/\sigma_w^3$ at heights of 1 m and 4 m above the steppe at midday in the summer. Finally, Pries and Appleby (1967), and Holland (1968), analyzed the skewness of w, together with that of u, v, and T, for a series of observations in the U.S. at Round Hill, Mass., over a relatively inhomogeneous surface. They also found that w' has positive skewness in unstable conditions, increasing with the instability parameter $-\zeta = z/|L|$. Even fewer data exist for the values of temperature skewness $\overline{T'^3}/\sigma_T^3$, but the data of Pries and Appleby, and that of Holland, indicate that these values are positive under both unstable and stable conditions which agrees also with the very preliminary results of Perepelkina (1957) for the unstable case. Some data on the skewness of the horizontal velocity components u and v, and on the flatness factors δ, i.e., on the ratios of the fourth moments to the square of variances, or, what is the same, on the coefficients of excess $\delta - 3$ of the fluctuations u', v', w', and T' in the atmospheric surface layer (based on Round Hill observations), were analyzed by Pries and Appleby (1967) and Holland (1968); however, all the results obtained are very preliminary and will not be discussed here.

From the dependence of the correlation coefficients r_{uw}, r_{uT}, and r_{wT} on ζ, we may conclude that the multidimensional probability density for the normalized fluctuations u'/σ_u, v'/σ_v, w'/σ_w, and T'/σ_T varies with change in the stratification, specifically, that the form of the corresponding probability distribution is not universal. Even the form of the one-dimensional distributions of w'/σ_w and T'/σ_T is not universal, being essentially non-Gaussian. In addition, the probability distributions of u'/σ_u and v'/σ_v in general are not strictly Gaussian, but they are probably close to the Gaussian distribution in many cases. Some data on one-dimensional probability distributions of turbulent fluctuations in the atmospheric surface layer are given in Pries and Appleby (1967), and Holland (1968). Holland (1967; 1968) has also developed a special computational technique for evaluating a great variety of different statistical characteristics of the four-dimensional distribution of u', v', w', and T', and of the corresponding filtered variables for several digital filters. Among these are the two-dimensional joint probability density functions for many pairs of variables, the joint moments, and the conditional mean values of different variables at fixed values of a pair of other variables. However, the possibility of analyzing all these results within the framework of the general similarity theory for turbulence in a stratified fluid is very remote.

Several methods exist of determining the values of the variables $\bar{\epsilon}$ and $\bar{N}$, but they will be discussed in Chapt. 8 of Volume 2 of this book. Nevertheless, at present, almost no data exist on the values of $\bar{N}$ in the atmospheric surface layer and on the corresponding universal function $\varphi_N(\zeta)$. There are considerable data on $\bar{\epsilon}$ in the surface layer and on the universal function $\varphi_\epsilon(\zeta)$ [for example: R. J. Taylor (1952); Takeuchi (1962); Panofsky (1962); Lumley and Panofsky (1964); Hess and Panofsky (1966); Record and Cramer (1966); Zubkovskiy (1967); Myrup (1967); Panofsky, Busch et al. (1967); Busch and Panofsky (1968); and Fichtl (1968)]. However, all the available empirical estimates of $\varphi_\epsilon(\zeta)$ display a very great scatter; moreover, some of the most extensive measurements were carried out in insufficiently homogeneous locations (Brookhaven, Round Hill, and Cape Kennedy) and at relatively great heights. Therefore all attempts to verify the turbulent energy balance equation (7.102) are presently inconclusive and lead to contradictory results. We must also bear in mind that all the existing direct measurements of turbulent energy diffusion and the corresponding universal functions $f_{10}(\zeta)$ and $\zeta f'_{10}(\zeta)$ [Record and Cramer (1966); Myrup (1967); Panofsky et al. (1967)] are extremely rough, do not include the pressure term which is usually assumed without proof to be small, and are so scattered that it is almost impossible to use them. There are two main hypothetical simplifications of the balance equation (7.102) which have been suggested (and advocated) by different authors: 1) the assumption of an approximate balance between the rate of energy dissipation $\bar{\epsilon}$ and the rate of production of mechanical energy $-\overline{u'v'}(\partial\bar{u}/\partial z)$, for example, on the approximate relationship $\varphi_\epsilon(\zeta) = \varphi(\zeta) = \zeta f'(\zeta)$, which implies for nonneutral conditions that the diffusion of turbulent energy be approximately equal to the rate of buoyant energy production or absorption $\overline{w'T'}(g/T_0)$ in absolute value, but has the opposite sign, so that $\zeta + (\varkappa/2)\zeta f'_{10}(\zeta) \approx 0$ [Lumley and Panofsky (1964); Hess and Panofsky (1966); Record and Cramer (1966); Fichtl (1968)]; and 2) the assumption of the negligible smallness of the energy diffusion term and the validity of the approximate equation $\varphi_\epsilon(\zeta) \approx \varphi(\zeta) - \zeta$ of the form (7.103) [Takeuchi (1962); Panofsky, Busch et al. (1967); Busch and Panofsky (1968)]. The contradiction between these two opinions is not yet explained satisfactorily. It is also worth noting that Myrup (1967) has found some preliminary indications that in extremely strong convection, where the mechanical energy production plays no substantial role and the term $\zeta f'(\zeta)$ may be neglected in Eq. (7.102), energy diffusion is important and cannot be neglected (that is, Eq. (7.103) here is incorrect and $\sigma \neq 1$). However, this result also needs further confirmation.

5 PARTICLE DISPERSION IN A TURBULENT FLOW

9. THE LAGRANGIAN DESCRIPTION OF TURBULENCE

9.1 The Lagrangian Dynamic Equations of an Incompressible Viscous Fluid

In the preceding chapters the Eulerian description of the fluid motion is used throughout. In such a description, the motion of an incompressible fluid (to which we shall also limit ourselves in this chapter is characterized at time t by the velocity field $\boldsymbol{u}(\boldsymbol{X}, t)$, that is, by the values of the velocity vector at all possible space points $\boldsymbol{X} = (X_1, X_2, X_3)$. Here, for reasons that will be clarified from what follows, it will usually be convenient to designate the coordinates as X_i instead of x_i as was done in foregoing chapters. In this case, the dynamic equations, from which pressure may be excluded with the help of Eq. (1.9), permit at least in principle, determination of the values of the Eulerian variables $\boldsymbol{u}(\boldsymbol{X}, t)$ at any time $t > t_0$ in terms of the given initial values $\boldsymbol{u}(\boldsymbol{X}, t_0) = \boldsymbol{u}_0(\boldsymbol{X})$. However, it is inconvenient to use Eulerian variables to study such phenomena as turbulent transport, that is, the spread of admixtures in a turbulent flow, or the deformation of material surfaces and lines in a turbulent flow, which consist of fixed fluid elements. It is more convenient to use the Lagrangian description of the flow which consists of following the motion of fixed "fluid particles" beginning from some initial time $t = t_0$ rather than the velocities of the fluid at fixed space points $\boldsymbol{X}$. By fluid particle we mean a volume of fluid having linear dimensions which are very large compared to the average distance between molecules. Therefore, for such volumes it is reasonable to speak of their velocity, and remain within the framework of the mechanics of a continuous medium. Nevertheless, the linear dimensions are so small that the velocity and pressure inside the volume may be considered practically constant, and during the time interval under discussion these volumes may be considered moving "as a whole," that is, without noticeable deformation. In other words, a "fluid particle" is an identifiable "point" of the volume of fluid which is moving within this volume according to the equations of fluid mechanics. The Lagrangian description is related most directly to the motions of the individual fluid elements, producing in sum the fluid flow; therefore, it is physically more natural than the Eulerian description. At the same time the use of Lagrangian variables, which refer to individual particles of the fluid, turns out to be much more awkward analytically than the use of Eulerian variables $\boldsymbol{u}(\boldsymbol{X}, t)$. As a result, the fluid dynamic equations in Lagrangian form are rarely used for specific calculations. Although for an ideal, nonviscous fluid such equations are presented in some well-known textbooks [see for example, Lamb (1932)], for a viscous fluid the corresponding equations are comparatively unknown at present. A derivation of these equations which permits them to be written in a relatively compact form was first reported only rather recently by Gerber (1949) [see also Monin (1962c)]. Although the use of the viscous Lagrangian equations in turbulence theory is still a matter for the future [cf. Lyubimov (1969)], we shall nevertheless present here the derivation of these equations, and at the same time introduce several concepts and symbols that will be used constantly later in the chapter.

The complete Lagrangian characteristic of an incompressible fluid flow is the function $\boldsymbol{X}(\boldsymbol{x}, t)$ which gives, for any time t, the coordinates $\boldsymbol{X}$ of all possible "fluid particles" identified by the values of some parameter $\boldsymbol{x}$. In principle, the fluid dynamic equations permit the value of $\boldsymbol{X}(\boldsymbol{x}, t)$ to be determined for any $t > t_0$ in terms of the given initial values of the "fluid particle" velocities $V(\boldsymbol{x}, t) = \frac{\partial X(\boldsymbol{x}, t)}{\partial t}$, that is, in terms of the values of

$$V(\boldsymbol{x}, t_0) = \left[\frac{\partial X(\boldsymbol{x}, t)}{\partial t}\right]_{t=t_0}.$$

The relationship between the Lagrangian and the Eulerian characteristics is given by the expression

$$\frac{\partial X(\boldsymbol{x}, t)}{\partial t} = \boldsymbol{u}\,[X(\boldsymbol{x}, t), t]. \tag{9.1}$$

The transformation from the Eulerian dynamic equations to the Lagrangian equations includes replacement of the independent variables $(\boldsymbol{x}, t)$ by (X, t) and transformation from the unknown function $\boldsymbol{u}(X, t)$ to the new unknown $X(\boldsymbol{x}, t)$ related to $\boldsymbol{u}(X, t)$ by Eq. (9.1).

Furthermore, in this section, we shall always use as the Lagrangian parameters of the fluid particles $\boldsymbol{x}$, the initial values of their spatial coordinates X at the time $t = t_0$; that is, we assume

$$\boldsymbol{x} = X(\boldsymbol{x}, t_0). \tag{9.2}$$

In this case the function of two variables $X(\boldsymbol{x}, t)$ describes the family of trajectories of "fluid particles" found at the initial time $t = t_0$ at all possible points $\boldsymbol{x}$ of the volume occupied by the fluid. Thus at any time $t > t_0$, the points $X = X(\boldsymbol{x}, t)$ corresponding to all possible values of $\boldsymbol{x}$ continuously fill the entire volume occupied by the fluid. Therefore, the Lagrangian description consists of assigning a fluid flow by a family of trajectories, differing from each other by the values of $\boldsymbol{x}$, and in each of which the role of the parameter is played by the time t. According to the above, the "fluid particles" which correspond to these trajectories are actually mathematical points flowing along with the fluid.

Let us now derive the Lagrangian dynamic equations. We shall use the Cartesian components of the vectors X and $\boldsymbol{x}$, and designate them as (X_1, X_2, X_3) and (x_1, x_2, x_3), respectively. As indicated above, the transformation from the Eulerian dynamic equations to Lagrangian equations includes first the replacement of the independent variables (X_1, X_2, X_3, t) by (x_1, x_2, x_3, t). In making this change of variables, we emphasize the transform from Cartesian coordinates to nonstationary curvilinear and nonorthogonal coordinates which follow the motion of the fluid. Actually, at all times each coordinate surface $x_i =$ const consists of the same "fluid particles"; initially such surfaces are planes, but as time passes they become distorted due to mixing with the fluid.

Furthermore, with respect to the variables (x_1, x_2, x_3), we shall use for the Jacobians the abbreviated notation

$$\frac{\partial (A, B, C)}{\partial (x_1, x_2, x_3)} = [A, B, C] \tag{9.3}$$

and without further comment the fact that the quantity $[A, B, C]$ does not vary under an even permutation of the variables A, B, C, and changes sign under an odd permutation.

In transforming from Eulerian coordinates X_α to Lagrangian coordinates x_3 a major role is played by the matrix

$$M = \left\| \frac{\partial X_\alpha}{\partial x_\beta} \right\|; \qquad \det M = |M| = [X_1, X_2, X_3]. \tag{9.4}$$

According to Eq. (9.2), at the initial time $\frac{\partial X_\alpha}{\partial x_\beta} = \delta_{\alpha\beta}$; that is, the matrix M is a unit matrix and $|M| = 1$. The variables $\frac{\partial x_\alpha}{\partial X_\beta}$ are elements of the inverse matrix M^{-1}, or the co-factors of the elements $\frac{\partial X_\beta}{\partial x_\alpha}$ in the matrix M, divided by $|M|$. Consequently, we obtain the following formula for calculation of the derivatives with respect to the Eulerian coordinates X_i:

$$\frac{\partial f}{\partial X_i} = \frac{1}{|M|} [X_j, X_k, f]. \tag{9.5}$$

Here and henceforth by (i, j, k) we mean an even permutation of the indexes (1, 2, 3). It is easy to prove the correctness of formula (9.5), by writing the left side as $\frac{\partial f}{\partial x_\alpha} \frac{\partial x_\alpha}{\partial X_i}$, where the repeated index α, as usual, denotes summation; the same expression is obtained on the right side of the formula by expansion of the determinant which enters into it with respect to the elements of the column f.

From Eqs. (9.1) and (9.5) is derived the following expression for the velocity divergence:

$$\operatorname{div} \boldsymbol{u} = \frac{\partial u_\alpha}{\partial X_\alpha} = \frac{\partial}{\partial X_\alpha} \frac{\partial X_\alpha}{\partial t} =$$

$$= \frac{1}{|M|} \left\{ \left[\frac{\partial X_1}{\partial t}, X_2, X_3 \right] + \left[X_1, \frac{\partial X_2}{\partial t}, X_3 \right] + \left[X_1, X_2, \frac{\partial X_3}{\partial t} \right] \right\} = \frac{1}{|M|} \frac{\partial |M|}{\partial t}$$

where u_α are the Cartesian components of the velocity field. For an incompressible fluid the velocity divergence is exactly equal to zero; thus $\frac{\partial |M|}{\partial t} = 0$ and consequently $|M|$ does not vary with time. Since at the initial time $|M| = 1$, this equation remains correct also for any time. Recalling expression (9.4) for $|M|$, we have

$$[X_1, X_2, X_3] = 1. \tag{9.6}$$

This equation also acts as the continuity equation for an incompressible fluid, either viscous or ideal, in the Lagrangian formulation.

Furthermore, we shall use formula (9.5), setting $|M| = 1$ on its right side. Using this formula twice, we obtain the expression for the Laplacian with respect to the Eulerian coordinates X_α:

$$\nabla^2_X f = \frac{\partial}{\partial X_\alpha} \frac{\partial f}{\partial X_\alpha} =$$

$$= [X_2, X_3, [X_2, X_3, f]] + [X_3, X_1, [X_3, X_1, f]] + [X_1, X_2, [X_1, X_2, f]]. \tag{9.7}$$

Now we can easily proceed to the Lagrangian formulation of the Navier-Stokes equations

$$\frac{du_l}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial X_l} + \nu \nabla^2_X u_l. \tag{9.8}$$

Using Eqs. (9.1), (9.5), and (9.7), we obtain

$$\frac{\partial^2 X_i}{\partial t^2} = -\frac{1}{\rho} [X_j, X_k, p] + \nu \left\{ \left[X_2, X_3, \left[X_2, X_3, \frac{\partial X_i}{\partial t} \right] \right] + \right.$$
$$\left. + \left[X_3, X_1, \left[X_3, X_1, \frac{\partial X_i}{\partial t} \right] \right] + \left[X_1, X_2, \left[X_1, X_2, \frac{\partial X_i}{\partial t} \right] \right] \right\}. \tag{9.9}$$

Equations (9.6) and (9.9), in the unknowns $X_i(\boldsymbol{x}, t), i = 1, 2, 3$ and

$p(\boldsymbol{x}, t)$ form the complete system of Lagrangian dynamic equations of an incompressible viscous fluid.

Forces exist corresponding to the nonlinear terms in the equations of motion. These forces describe the interaction between the components of a mechanical system. Thus, in the Navier-Stokes equations (9.8) nonlinear, that is, quadratic terms in the variables u_i are found in the expression for the acceleration $\frac{du_i}{dt}$; these represent the forces of the inertial interaction between the spatial inhomogeneities of the velocity field $\boldsymbol{u}(\boldsymbol{X}, t)$ in terms of which the pressure gradient may be expressed with the help of Eq. (1.9′). (Let us emphasize that the viscous forces are described in Eq. (9.8) by a linear expression.) However, the inertial interactions have a relative nature; they are eliminated in the transformation to the particle attached reference system. In the Lagrangian equations of motion (9.9) the expressions which are nonlinear in the variables X_i describe only the real forces of interaction between the fluid particles, that is, the pressure and viscous forces; the viscous interactions are described here by nonlinear terms of the fifth degree in the variables X_i. The ratio of the typical values of the nonlinear and linear terms in the equations of motion may be called the interaction constant. Thus, for the Navier-Stokes equations (9.8), the inertial interaction constant is the ratio of the typical values of the inertia forces and the viscous forces, that is, the Reynolds number Re; for sufficiently large Re, which is characteristic of developed turbulence, the inertial interactions are quite strong. In the Lagrangian description the viscous itneraction constant is the ratio of the typical values of the viscous forces to the typical acceleration, for example, 1/Re; for large Re the viscous interactions turn out to be quite weak.

Let us now formulate the Lagrangian dynamic equations for a two-dimensional, plane-parallel flow. Let the motion of the fluid take place only in the planes $x_3 = \text{const}$, so that $X_3 \equiv x_3$, where X_1 and X_2 do not depend on x_3. Then, using designations of the following type for the Jacobians with respect to (x_1, x_2):

$$\frac{\partial (A, B)}{\partial (x_1, x_2)} = [A, B], \tag{9.10}$$

Eqs. (9.6) and (9.9) may easily be reduced to the form

$$[X_1, X_2] = 1,$$
$$\frac{\partial^2 X_1}{\partial t^2} = -\frac{1}{\rho}[p, X_2] + \nu\left\{\left[X_1, \left[X_1, \frac{\partial X_1}{\partial t}\right]\right] + \left[X_2, \left[X_2, \frac{\partial X_1}{\partial t}\right]\right]\right\}, \tag{9.11}$$
$$\frac{\partial^2 X_2}{\partial t^2} = -\frac{1}{\rho}[X_1, p] + \nu\left\{\left[X_1, \left[X_1, \frac{\partial X_2}{\partial t}\right]\right] + \left[X_2, \left[X_2, \frac{\partial X_2}{\partial t}\right]\right]\right\}.$$

Here the viscous forces are described by nonlinear expressions of the third degree in the variables X_i.

In concluding this subsection, let us note that in addition to the use of expression (9.1), another means exists of relating the Lagrangian and Eulerian descriptions of fluid flow. This is done by investigating arbitrary conservative characteristics of "fluid particles," that is, characteristics that have values for a fixed fluid particle which do not vary during its motion. In the Lagrangian description each such characteristic may be written in the form $\Psi(\boldsymbol{x})$ since for fixed $\boldsymbol{x}$ it does not depend on time t. In the Eulerian description, however, the values ψ of such a characteristic at a fixed space point $\boldsymbol{X}$ may vary with time t, that is, $\psi = \psi(\boldsymbol{X}, t)$. The relationship between the Lagrangian and Eulerian descriptions will be given by the expression

$$\Psi(\boldsymbol{x}) = \psi[\boldsymbol{X}(\boldsymbol{x}, t), t]. \tag{9.12}$$

Since $\frac{\partial \Psi(\boldsymbol{x})}{\partial t} = 0$, then the total derivative with respect to time of the right side must also be equal to zero, that is,

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial X_\alpha}\frac{\partial X_\alpha}{\partial t} = \frac{\partial \psi}{\partial t} + u_\alpha \frac{\partial \psi}{\partial X_\alpha} = 0,$$

where we have used Eq. (9.1). Using also the continuity equation for an incompressible fluid $\frac{\partial u_\alpha}{\partial X_\alpha} = 0$, we obtain

$$\frac{\partial \psi}{\partial t} + \frac{\partial u_\alpha \psi}{\partial X_\alpha} = 0. \tag{9.13}$$

This equation is the differential analog of expression (9.12). Thus, the Eulerian field of any conservative characteristic of the fluid particles satisfies the *advection equation* (9.13).

Consequently, the conservative characteristic $\Psi(\boldsymbol{x}) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$ which differs from 0, and is equal to infinity only for one fluid particle, namely, the initial coordinate $\boldsymbol{x}_0$, corresponds to the Eulerian field

$$\psi(X, t) = \delta[X - X(\boldsymbol{x}_0, t)], \tag{9.14}$$

which is the solution of the advection equation (9.13) for the initial condition

$$\psi(X, t_0) = \delta(X - \boldsymbol{x}_0). \tag{9.15}$$

The general notion of a "conservative characteristic" represents, of course, a mathematical idealization. To make it possible to actually follow the motion of the fluid particle it must be "marked," that is, have some properties distinguishable from those of the surrounding medium. Such marking is achieved most simply by the addition of some distinctive passive admixture which is transported by the fluid motion without exerting an influence on the flow. In such a case the admixture concentration $\vartheta(X, t)$ acts as the Eulerian conservative characteristic $\psi(X, t)$. However, the real concentration field is also subjected to molecular diffusion; consequently, it satisfies the diffusion equation

$$\frac{\partial \vartheta}{\partial t} + \frac{\partial u_\alpha \vartheta}{\partial X_\alpha} = \chi \nabla^2 \vartheta. \tag{9.13$'$}$$

This equation differs from the advection equation (9.13) for the idealized conservative characteristic $\psi(X, t)$ by the proportionality of the right side to molecular diffusivity. Now the Lagrangian form of the diffusion equation may be deduced in a manner quite similar to that of the deduction of the Lagrangian dynamic equations [Corrsin (1962); cf. also Okubo (1967)]. Here the Lagrangian concentration field

$$\vartheta(x, t) = \vartheta[X(x, t] \tag{9.12$'$}$$

would be the main variable; this variable depends on t and satisfies the equation

$$\frac{\partial\vartheta(\mathrm{x},t)}{\partial t} = \chi\Big\{\big[X_2, X_3, [X_2, X_3, \vartheta]\big] + \big[X_3, X_1, [X_3, X_1, \vartheta]\big] + \big[X_1, X_2, [X_1, X_2, \vartheta]\big]\Big\}, \tag{9.9'}$$

which implies that ϑ does not depend on t when $\chi = 0$, where $X_i(\mathrm{x}, t), i = 1, 2, 3$ are the solutions of the dynamic equations (9.6) and (9.9).

9.2 Lagrangian Turbulence Characteristics

For a statistical description of turbulence it is necessary first to indicate which of its characteristics are assumed to have probability distributions, that is, which are realizations of some random field. In the discussion of turbulent flows in previous chapters, we always assumed the random field to be the Eulerian velocity field $\boldsymbol{u}(\boldsymbol{X}, t)$. However, in that case, the Lagrangian variables

$$V(\boldsymbol{x}, t) = \boldsymbol{u}[\boldsymbol{X}(\boldsymbol{x}, t), t] \quad \text{and} \quad X(\boldsymbol{x}, t) = \boldsymbol{x} + \int_{t_0}^{t} V(\boldsymbol{x}, t')\,dt'$$

will also be random functions of the arguments $\boldsymbol{x}$ and t. Moreover, for any finite number n of fluid particles, identified by the fact that at the time $t = 0$ they are at the points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, multidimensional joint probability densities will exist for their coordinates $\boldsymbol{X}$ and velocities $\boldsymbol{V}$ at arbitrary times $t_1, \ldots, t_m$. These multidimensional densities are functions of $3n+m$ variables $x_{11}, x_{12}, x_{13}, \ldots, x_{n1}, x_{n2}, x_{n3}$ $t_1, \ldots, t_m$, where x_{i1}, x_{i2}, x_{i3} are three components of the vector $\boldsymbol{x}_i$, and are the basic Lagrangian statistical characteristics of turbulence. Let us note also that joint probability distributions will exist for sets of variables, that is, for both the Lagrangian coordinates or velocities of fixed fluid particles, and the Eulerian flow velocities at fixed points; the densities of these "mixed probability" distributions are sometimes also of interest.

We shall designate probability densities by the following type symbols:

$$p(\boldsymbol{X}, \boldsymbol{V}, \ldots \mid \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots; t_1, t_2, \ldots),$$

where those values to the left of the vertical line in parentheses denote the random variables in question, designated by the same letters as the corresponding variables themselves, and those values after the vertical line, the parameters on which these variables depend. If the parameters corresponding to several variables coincide, and if there is no danger of confusion, we designate them following the straight line only once.

The various Lagrangian and mixed statistical characteristics of turbulence satisfy many general relationships, some of which will be pointed out here. We shall begin with the relations resulting from the "advection equation" (9.13). By taking expression (9.14) to be the solution of this equation, and according to Eq. (9.1)

$$u_\alpha(X, t)\,\delta[X - X(x, t)] = \\ = u_\alpha[X(x, t), t]\,\delta[X - X(x, t)] = V_\alpha(x, t)\,\delta[X - X(x, t)],$$

we obtain

$$\frac{\partial\delta[X - X(x, t)]}{\partial t} + \frac{\partial V_\alpha(x, t)\,\delta[X - X(x, t)]}{\partial X_\alpha} = 0.$$

However, from the definition of probability averaging [see Eq. (3.12)]

$$\overline{\delta[X - X(x, t)]} = p(X \mid x, t),$$
$$\overline{V_\alpha(x, t)\,\delta[X - X(x, t)]} = \int V_\alpha p(X, V \mid x, t)\,dV.$$

Consequently, averaging the equation for $\delta[X - X(x, t)]$, we have

$$\frac{\partial p(X \mid x, t)}{\partial t} + \frac{\partial}{\partial X_\alpha}\int V_\alpha p(X, V \mid x, t)\,dV = 0, \qquad (9.16)$$

which is the statistical analog of the advection equation (9.13).

Another type of relationship between the statistical characteristics may be obtained along with the Lagrangian velocity $V(x, t)$, which represents the value of the random function $u(X, t)$ at the random point $X(x, t)$, by introducing the random variable $V(X_1, x, t)$ depending on the fixed point $X_1 = X(x, t)$. The values of this variable are the velocities of those fluid particles which at the time $t = t_0$ were found at the point x, and at the time t, turned out to be at the fixed point X_1; its probability density $p(V|X_1, x, t)$ is the conditional probability density of the variable $V(x, t)$, under the condition that $X(x, t) = X_1$. Therefore, strictly speaking, the probability distribution of $V(X_1, x, t)$ is neither purely Lagrangian nor purely Eulerian. However, it may be anticipated that with an increase in $t - t_0$, the dependence of this probability distribution on the value of x will generally become less and less essential. For sufficiently large values of $t = t_0$, by comparison with the *Lagrangian correlation time*, or the *Lagrangian integral time scale,*

$$T = \int_{t_0}^{\infty} \frac{\overline{V_i'(x, t) V_i'(x, t_0)}}{[\overline{V'^2(x, t)}\,\overline{V'^2(x, t_0)}]^{1/2}} dt,$$

this dependence may often be completely neglected. Thus, the random variable $V(X_1, x, t)$, for such $t - t_0$, may frequently be considered equivalent to the Eulerian random variable $u(X_1, t)$. Nevertheless, let us emphasize that replacement of $V(X_1, x, t)$ by $u(X_1, t)$ may not always be proper for any values of $t - t_0$. For example, we shall see later that in a turbulent boundary layer along a plate at $x_3 = 0$, $\overline{V_3(X_1, x, t)} > 0$ for all X_1 and t, particularly in the logarithmic layer $\overline{V_3(X_1, x, t)} = \text{const} \approx bu_*$, where b is a universal numerical constant, while $\overline{u_3(X_1, t)} = 0$.

It is not difficult to see that the joint probability density for the n fluid particle velocities $V(x_1, t_1), \ldots, V(x_n, t_n)$, taken at different times $t_1, \ldots, t_n$), may be represented in the form

$$\begin{aligned} p(V_1, \ldots, V_n | x_1, \ldots, x_n; t_1, \ldots, t_n) = \\ = \int \cdots \int p(V_1, \ldots, V_n | X_1, \ldots, X_n; x_1, \ldots, x_n; t_1, \ldots, t_n) \times \\ \times p(X_1, \ldots, X_n | x_1, \ldots, x_n; t_1, \ldots, t_n) dX_1 \ldots dX_n, \end{aligned} \tag{9.17}$$

where the first factor under the integral sign is the conditional probability density for the indicated velocities of the fluid particles under the condition that their coordinates at the corresponding times assume fixed values $X(x_1, t_1) = X_1, \ldots, X(x_n, t_n) = X_n$. If all the fluid particles are different, that is, among the initial points $x_1, \ldots, x_n$ no two are identical, then this first factor is the joint probability density for the random variables $V(X_1, x_1, t_1), \ldots, V(X_n, x_n, t_n)$ introduced above. The second factor under the integral sign in Eq. (9.17) is the joint probability density for the Lagrangian random variables $\boldsymbol{X}(\boldsymbol{x}_1, t_0), \ldots, \boldsymbol{X}(\boldsymbol{x}_n, t_n)$. Equation (9.17), which like its corollaries is taken from Monin (1960), is a special case of the general "theorem of total probability" of probability theory. If all the fluid particles under discussion are different and all the differences $t_1 - t_0, \ldots, t_n - t_0$ are sufficiently large, then the dependence of the first factor under the integral sign in Eq. (9.17) on the arguments $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ may be neglected in several cases. This factor may then be identified with the probability density of the velocities $\boldsymbol{V}_1 = \boldsymbol{u}(\boldsymbol{X}_1, t_1), \ldots, \boldsymbol{V}_n = \boldsymbol{u}(\boldsymbol{X}_n, t_n)$ at fixed points so that Eq. (9.17) here becomes an approximate relation between the Lagrangian and the Eulerian statistical characteristics of turbulence.

Let us give special consideration to the particular case of Eq. (9.17) when $\boldsymbol{x}_1 = \boldsymbol{x}_2 = \ldots = \boldsymbol{x}_n (= \boldsymbol{x})$. Here the formula contains the probability densities for the coordinates and velocities of a single fluid particle at different times. When $n = 1$ we obtain

$$p(\boldsymbol{V} \mid \boldsymbol{x}, t) = \int p(\boldsymbol{V} \mid \boldsymbol{X}, \boldsymbol{x}, t)\, p(\boldsymbol{X} \mid \boldsymbol{x}, t)\, d\boldsymbol{X}. \tag{9.18}$$

As already noted, for sufficiently large $t - t_0$ the function $p(\boldsymbol{V} \mid \boldsymbol{X}, \boldsymbol{x}, t)$ may sometimes be identified with the probability density of Eulerian velocity $\boldsymbol{V} = \boldsymbol{u}(\boldsymbol{X}, t)$ at a fixed space-time point $(\boldsymbol{X}, t)$; in this sense Eq. (9.18) may be considered as the statistical analog of the basic expression (9.1) relating the Lagrangian and Eulerian velocities. Now substituting Eq. (9.18) into the expressions for the mean value of $\overline{\boldsymbol{V}(\boldsymbol{x}, t)}$ which is derived from the general equation (3.12), we obtain

$$\overline{\boldsymbol{V}(\boldsymbol{x}, t)} = \int \overline{\boldsymbol{V}(\boldsymbol{X}, \boldsymbol{x}, t)}\, p(\boldsymbol{X} \mid \boldsymbol{x}, t)\, d\boldsymbol{X}. \tag{9.19}$$

When the dependence $\boldsymbol{V}(\boldsymbol{X}, \boldsymbol{x}, t)$ on $\boldsymbol{x}$ disappears for $t - t_0 \to \infty$, for sufficiently large $t - t_0$ this equation may be rewritten in the form

$$\overline{\boldsymbol{V}(\boldsymbol{x}, t)} = \int \overline{\boldsymbol{u}(\boldsymbol{X}, t)}\, p(\boldsymbol{X} \mid \boldsymbol{x}, t)\, d\boldsymbol{X}. \tag{9.20}$$

For the joint probability distribution of the velocities $\boldsymbol{V}_1 = \boldsymbol{V}(\boldsymbol{x}, t_0)$ and $\boldsymbol{V}_2 = \boldsymbol{V}(\boldsymbol{x}, t)$ of a single fluid particle at two sequential times t_0 and $t > t_0$, Eq. (9.17) leads to the expression

$$p(V_1, V_2 \mid x, t_0, t) = \int p(V_1, V_2 \mid X; x, t_0, t)\, p(X \mid x, t)\, dX, \quad (9.21)$$

where the first factor under the integral sign is the joint probability density for the variables $V_1 = V(x, t_0) = u(x, t_0)$ and $V_2 = V(X, x, t)$. Thus from this, for the *Lagrangian velocity correlation function*, that is, the mixed second moment $\overline{V_{1i} V_{2j}}$ of the distribution (9.21), where the subscripts i and j are the numbers of components of the vectors V_1 and V_2, we obtain the following relation:

$$\overline{V_i(x, t_0)\, V_j(x, t)} = \int \overline{u_i(x, t_0)\, V_j(X, x, t)}\, p(X \mid x, t)\, dX. \quad (9.22)$$

For sufficiently large values of $t - t_0$, that is, in comparison to the Lagrangian correlation time, the velocity correlation function under the integral sign may be identified with the simpler Eulerian space-time correlation function $\overline{u_i(x, t_0)\, u_j(X, t)}$ on the basis of the above arguments. However, for these $t - t_0$, the values of $V_i(x, t_0)$ and $V_j(x, t)$, and also the values of $u_i(x, t_0)$ and $u_j(X, t)$ may already be considered practically uncorrelated; therefore, in this case the asymptotic form of the equation is not of great interest. The left side of Eq. (9.22) is not the most general Lagrangian velocity correlation function since at one of the times, namely, t_0, the coordinate of the corresponding "fluid particle" has a fixed value equal to x. The Lagrangian velocity correlation function $\overline{V_i(x, t_1)\, V_j(x, t_2)}$ where $t_2 > t_1 > t_0$ is more general; using Eq. (9.17), we can express it as

$$\overline{V_i(x, t_1)\, V_j(x, t_2)} = \int \int \overline{V_i(X_1, x, t_1)\, V_j(X_1, X_2, x, t_1, t_2)} \times \\ \times p(X_1, X_2 \mid x, t_1, t_0)\, dX_1\, dX_2, \quad (9.23)$$

where $V(X_1, X_2, x, t_1, t_2)$ indicates the random velocity of the fluid particle at the time t_2 under the condition that at the times t_0, t_1, and t_2, this particle is found at the fixed points x, X_1, and X_2, respectively. For sufficiently large $t_1 - t_0$, the value of $V(X_1, x, t_1)$ may often be considered approximately equal to $u(X_1, t_1)$; in exactly the same way, for sufficiently large $t_2 - t_1$, the value of $V(X_1, X_2, x, t_1, t_2)$ frequently may be considered approximately equal to $u(X_2, t_2)$.

9.3 Displacement Characteristics of a Single Fluid Particle; the Case of Homogeneous Turbulence

The motion of a fluid particle, that is, one which at the time $t=t_0$ is at the point $\boldsymbol{x}$, is described completely by the vector function $\boldsymbol{X}(\boldsymbol{x}, t)$ which gives the position of this particle at an arbitrary time t. In the place of the vector $\boldsymbol{X}(\boldsymbol{x}, t)$ one may also use the displacement vector of the particle during a time interval τ:

$$\boldsymbol{Y}(\tau) = \boldsymbol{X}(\boldsymbol{x}, t_0+\tau) - \boldsymbol{x} = \int_{t_0}^{t_0+\tau} \boldsymbol{V}(\boldsymbol{x}, t)\, dt; \tag{9.24}$$

this usually proves quite convenient. In this subsection we shall study the statistical characteristics of the random vector $\boldsymbol{Y}(\tau)$.

A complete description of the variable $\boldsymbol{Y}(\tau)$ requires assignment of its three-dimensional probability density $p(\boldsymbol{Y} \mid \tau;\ \boldsymbol{x}, t_0)$, which depends on τ, $\boldsymbol{x}$, and t_0 as parameters. It is easily seen that for very small τ, compared with the time scale T given by the typical Lagrangian correlation time, this probability density may be expressed in terms of the Eulerian statistical characteristics of turbulence. Indeed, when $\tau \ll T$, the Lagrangian velocity practically does not vary in time τ; therefore Eq. (9.24) may be rewritten here as

$$\boldsymbol{Y}(\tau) \approx \boldsymbol{V}(\boldsymbol{x}, t_0)\tau = \boldsymbol{u}(\boldsymbol{x}, t_0)\tau.$$

Therefore, the probability density for $\boldsymbol{Y}(\tau)$ in this case may be transformed to

$$p(\boldsymbol{Y} \mid \tau;\ \boldsymbol{x}, t_0) \approx \tau^{-3} P\left(\frac{\boldsymbol{Y}}{\tau} \,\middle|\, \boldsymbol{x}, t_0\right), \tag{9.25}$$

where $P(\boldsymbol{u} \mid \boldsymbol{x}, t_0)$ is the probability density of the Eulerian velocity $\boldsymbol{u}(\boldsymbol{x}, t_0)$ at the fixed point $\boldsymbol{x}$ at time t_0. If the turbulent flow is steady, then the density $P(\boldsymbol{u} \mid \boldsymbol{x}, t_0)$ may easily be determined by analyzing velocity observations at the point $\boldsymbol{x}$ during a long time interval. Thus, for example, in turbulent flow in a wind-tunnel having a square-mesh grid at its opening, thereby creating strong turbulence downstream, the measurement data of Simmons and Salter (1938) and Townsend (1947) show convincingly that the distribution $P(\boldsymbol{u} \mid \boldsymbol{x})$, and consequently also $p(\boldsymbol{Y} \mid \tau, \boldsymbol{x})$ when $\tau \ll T$, is quite close to normal.

For not very small τ the distribution $p(\boldsymbol{Y}|\tau;\ \boldsymbol{x}, t_0)$ may not be expressed in terms of the Eulerian statistical characteristics. However, if $\tau \gg T$, then one may use the fact that here the right side of Eq. (9.24) may be represented as a sum of integrals taken over nonintersecting time intervals of length of order T, and that are weakly dependent random variables. Therefore, it is possible to apply to this sum the so-called *central limit theorem for weakly dependent random variables*, according to which the probabihty distribution of the sum of a large number of such variables, under some broad conditions, turns out to be very close to normal. Recently, the central limit theorem was also proved directly, although with some mild restrictions, for integrals of the type (9.24); see, for example, Rozanov (1967) where an integral of a stationary random function is considered; similar theorems occur also for integrals of several nonstationary random functions. Unfortunately, direct use of these proofs is still impossible since the mild conditions imposed on the random functions which figure in them cannot be directly verified for real processes.[1] Nevertheless, these conditions are so natural that it would be extremely remarkable if the probability distribution for the displacement $\boldsymbol{Y}(\tau)$ when $\tau \gg T$ differed essentially from a normal distribution. In some cases the distribution for $\boldsymbol{Y}(\tau)$, or at least for some components of this vector, may be determined approximately by measuring the concentration distribution at various cross sections of the "mantle" of contaminated fluid created by a source of admixture placed in the flow, for example, the distribution of temperature in various cross sections of the thermal wake behind a hot body. Thus, experiments have successfully shown that in many turbulent flows the distribution $\boldsymbol{Y}(\tau)$ for large τ actually is very close to normal, while in the special case of wind-tunnel turbulence behind a grid, it turns out that it is almost normal for all values of τ [see, for example, Collis (1948); Townsend (1951); Uberoi and Corrsin (1953)]. The fact that the distribution of $\boldsymbol{Y}(\tau)$ is almost normal for all values of τ is not surprising, since as we have seen, it must be normal both for small and for large values of τ; its closeness to normal for all τ indicates only that for intermediate

[1] Thus, for example, several proofs use the so-called *strong mixing conditions*; as applied to the function $\boldsymbol{V}(\boldsymbol{x}, t)$, this reduces, roughly speaking, to the requirement of the existence of a positive function $\alpha(\tau)$, such that $\alpha(\tau) \to 0$ as $\tau \to \infty$ and that the correlation coefficient between an arbitrary functional of the values $\boldsymbol{V}(\boldsymbol{x}, t')$, $-\infty < t' < t_0$, and that an arbitrary functional of the values of $\boldsymbol{V}(\boldsymbol{x}, t'')$, $t_0 + \tau < t'' < \infty$, be less than $\alpha(\tau)$ in modulus.

values of τ no sharp readjustment of the form of the distribution occurs. In general, however, there is no reason to expect that the distribution of $\boldsymbol{Y}(\tau)$ will also be close to normal for moderate values of τ.

Let us now investigate the most important numerical characteristics of the random vector $\boldsymbol{Y}(\tau)$, its first and second moments. The mean value of this vector equals

$$\overline{\boldsymbol{Y}(\tau)} = \int_{t_0}^{t_0+\tau} \overline{\boldsymbol{V}(\boldsymbol{x}, t)}\, dt. \tag{9.26}$$

According to this, it may be shown that in several important special cases $\overline{\boldsymbol{Y}(\tau)} = \boldsymbol{U}\tau$ where $\boldsymbol{U}$ is the suitably defined average flow velocity (see examples below). Then it remains only to investigate the characteristics of the displacement fluctuations

$$\boldsymbol{Y}'(\tau) = \boldsymbol{Y}(\tau) - \overline{\boldsymbol{Y}(\tau)} = \int_{t_0}^{t_0+\tau} \boldsymbol{V}'(\boldsymbol{x}, t)\, dt.$$

The second moment tensor of this random vector will be given by the equation

$$D_{ij}(\tau) = \overline{Y_i'(\tau) Y_j'(\tau)} = \int_{t_0}^{t_0+\tau} \int_{t_0}^{t_0+\tau} \overline{V_i'(\boldsymbol{x}, t_1) V_j'(\boldsymbol{x}, t_2)}\, dt_1\, dt_2; \tag{9.27}$$

it is called the *fluid particle displacement covariance* (or *variance*) *tensor*. Let us note that when the probability distribution for $\boldsymbol{Y}(\tau)$ may be considered normal, the vector (9.26) and the tensor (9.27) characterize this distribution completely.

For sufficiently small τ, for which $\boldsymbol{Y}'(\tau) \approx \boldsymbol{u}'(\boldsymbol{x}, t_0)\tau$, from Eq. (9.27) we obtain

$$D_{ij}(\tau) \approx \overline{u_i'(\boldsymbol{x}, t_0)\, u_j'(\boldsymbol{x}, t_0)} \cdot \tau^2 = B_{ij}\tau^2. \tag{9.28}$$

The coefficients $B_{ij} = \overline{u_i'(\boldsymbol{x}, t_0)\, u_j'(\boldsymbol{x}, t_0)}$ for steady flows will depend only on $\boldsymbol{x}$ and, with the additional assumption of homogeneity of the random field $\boldsymbol{u}(\boldsymbol{x}, t)$, will be strictly constant. Specific information about the form of the functions $D_{ij}(\tau)$ for not too small τ may be obtained, however, only for some special turbulent flows.

We consider first the idealized case of stationary homogeneous turbulence in which all flow variables are homogeneous random fields in three-dimensional space and at the same time, stationary random functions of t. In this case the mean velocity $\bar{\boldsymbol{u}}$ is constant both in space and in time, and consequently

$$\overline{V(\boldsymbol{x},\ t)} = \overline{u\,[X(\boldsymbol{x},\ t),\ t]} = \bar{u},\quad \overline{Y(\tau)} = \bar{\boldsymbol{u}}\,\tau.$$

Furthermore, here the fluctuating velocity of the fluid particle $\boldsymbol{V}'(\boldsymbol{x},\ t)$ will have the same statistical characteristics for all $\boldsymbol{x}$ and will be a stationary random function of t, so that

$$\overline{V_i'(\boldsymbol{x},\ t_1)\,V_j'(\boldsymbol{x},\ t_2)} = B_{ij}^{(L)}(t_2 - t_1) = \left(\overline{u_i'^2}\,\overline{u_j'^2}\right)^{\frac{1}{2}} R_{ij}^{(L)}(t_2 - t_1), \qquad (9.29)$$

where the letter L indicates that the corresponding functions are Lagrangian correlation functions. Changing now to the new variables $s = t_2 - t_1$ and $t = \frac{t_1 + t_2}{2}$, Eq. (9.27) may be rewritten as

$$D_{ij}(\tau) = \int_0^{\tau} \int_{t_0+\frac{s}{2}}^{t_0+\tau-\frac{s}{2}} \left[B_{ij}^{(L)}(s) + B_{ji}^{(L)}(s)\right] dt\, ds =$$

$$= \left(\overline{u_i'^2}\,\overline{u_j'^2}\right)^{\frac{1}{2}} \int_0^{\tau} \int_{t_0+\frac{s}{2}}^{t_0+\tau-\frac{s}{2}} \left[R_{ij}^{(L)}(s) + R_{ji}^{(L)}(s)\right] dt\, ds \qquad (9.30)$$

or after integrating with respect to t

$$D_{ij}(\tau) = \int_0^{\tau} (\tau - s)\left[B_{ij}^{(L)}(s) + B_{ji}^{(L)}(s)\right] ds =$$

$$= \left(\overline{u_i'^2}\,\overline{u_j'^2}\right) \int_0^{\tau} (\tau - s)\left[R_{ij}^{(L)}(s) + R_{ji}^{(L)}(s)\right] ds. \qquad (9.30')$$

A result similar to Eq. (9.30) for the variance of one component of $\boldsymbol{Y}(\tau)$ was first obtained in the classical work of G. I. Taylor (1921); it was presented by Kampé de Fériet (1939) in the form of Eq. (9.30') for the case where $i = j$, and by Batchelor (1949b) in the general case. When $i = j$, Eq. (9.30′) becomes

$$D_{ii}(\tau) = 2\int_0^\tau (\tau - s)\, B_{ii}^{(L)}(s)\, ds = 2\overline{u_i'^2}\int_0^\tau (\tau - s)\, R_{ii}^{(L)}(s)\, ds. \quad (9.31)$$

(Here, just as below, summation with respect to i is not assumed!) Let us now make the natural assumption that the Lagrangian correlation function $R_{ii}^{(L)}(s)$ approaches zero when $s \to \infty$, and so rapidly that the following correlation time will exist:

$$T_i = \int_0^\infty R_{ii}^{(L)}(s)\, ds. \quad (9.32)$$

As the Lagrangian time scale T mentioned above, the maximum or average value of the three variables T_i, $i = 1, 2, 3$ may be taken. Assuming also that the integral

$$\int_0^\infty sR_{ii}^{(L)}(s)\, ds = S_i \quad (9.33)$$

is finite, it is possible for sufficiently large τ, that is, when $\tau \gg T_i$, to replace Eq. (9.31) with the asymptotic expression

$$D_{ii}(\tau) \approx 2\overline{u_i'^2}\int_0^\infty (\tau - s)\, R_{ii}^{(L)}(s)\, ds = 2\overline{u_i'^2}\,(T_i\tau - S_i). \quad (9.34)$$

For very large τ, the basic role in the right side of Eq. (9.34) will be played by the term which is linear with respect to τ. Thus, Eq. (9.34) may be rewritten as

$$D_{ii}(\tau) \approx 2\overline{u_i'^2}\, T_i\tau, \quad (9.35)$$

which, as is easily seen, may also be obtained without assuming finiteness of the integral (9.33). Similarly, for very large τ the following expression is obtained for $D_{ij}(\tau)$:

$$D_{ij}(\tau) \approx \left(\overline{u_i'^2}\,\overline{u_j'^2}\right)^{\frac{1}{2}} T_{ij}\tau; \quad T_{ij} = \int_0^\infty \left[R_{ij}^{(L)}(s) + R_{ji}^{(L)}(s)\right] ds. \quad (9.36)$$

Equation (9.35) indicates that the variance of particle displacement after sufficiently large time τ becomes proportional to τ. This result is completely analogous to the basic law of Brownian motion according to which the mean square displacement of the Brownian particle, or any particle that participates in the molecular diffusion, is proportional to the time of motion (diffusion). For very small τ the displacement variance according to Eq. (9.28) depends quadratically on τ, which must be the case for any motion having finite velocity. For intermediate values of τ the dependence of $D_{ii}(\tau)$ on τ turns out to be more complex and depends on the form of the correlation function $R_{ii}^{(L)}(s)$; see, for example, the graphs of the functions $D_{ij}(\tau)$ corresponding to several special forms of the function $R_{ij}^{(L)}(s)$ plotted by Frenkiel (1952; 1953) and by Pasquill (1926b). Also, according to the data of Frenkiel and Pasquill, large variations in the values of $D_{ii}(\tau)$, for variation in the form of the function $R_{ii}^{(L)}(s)$, occur only if we assume that the function $R_{ii}^{(L)}(s)$ may take on negative values and change sign frequently with increase in the argument s. For functions $R_{ii}^{(L)}(s)$ which remain positive everywhere, the dependence of $D_{ii}(\tau)$ on the specific form of the function $R_{ii}^{(L)}(s)$ is very weak and the asymptotic formula (9.28) here is satisfied rather well for all $\tau \leqslant T_i$, and the asymptotic formula (9.35), for all $\tau \geqslant 5T_i$. Thus, for nonnegative functions $R_{ii}^{(L)}(s)$ the variance $D_{ii}(\tau)$ turns out to be greatly dependent on the intensity of the turbulence $\overline{u_i'^2}$ and its time scale T_i, but only very weakly dependent on the specific form of the correlation function $R_{ii}^{(L)}(s)$.

It may be assumed that the results arising from the model of stationary homogeneous turbulence have no practical significance. This is because homogeneous turbulence in an unbounded space is generally only a mathematical idealization, and the assumption of stationarity worsens the matter since the energy dissipation implies that a stationary flow of viscous fluid must have external sources of energy, and therefore may not be homogeneous. Indeed, however, it is easy to see that the above derivation for formula (9.31) requires only that the flow be homogeneous with respect to the Ox_i axis. Consequently, we may indicate several completely realistic flows to which the results obtained above may be applied. In particular, following G. I. Taylor (1954a) [whose work will be discussed in detail later], Batchelor noted that these results may be applied directly to the simplest turbulent flow in a sufficiently long straight tube of arbitrary constant cross section [Batchelor and Townsend (1956); Batchelor (1957)]. Indeed, if the direction of the tube coincides with the Ox_1 axis, then along this direction the flow will be homogeneous, although here the distribution of the mean velocity $\bar{u}_1(x_2, x_3)$ in the Ox_2x_3 plane may be quite complex. Let us now investigate the component $Y_1(\tau)$ of displacement of the fluid particle in the time τ in the Ox_1 direction. The corresponding Lagrangian velocity $\frac{dY_1(\tau)}{d\tau} = V_1(\boldsymbol{x}, t_0+\tau)$ will, generally speaking, be a nonstationary random function of τ which depends on the initial position of the particle $\boldsymbol{x}$ in the plane Ox_2x_3. It is natural to expect, however, that at some time after the starting time t_0 of the particle under discussion, the influence of the initial position $\boldsymbol{x}$ will almost cease to be felt. Thus the function $V_1(\boldsymbol{x}, t_0+\tau)$ may be considered independent of $\boldsymbol{x}$ and stationary. Here, the mean longitudinal particle velocity $\overline{V_1(\boldsymbol{x}, t_0+\tau)} = U_1$ for sufficiently large τ will depend on neither τ nor $\boldsymbol{x}$, that is, it will be constant with respect to time and the same for all fluid particles which occupy a fixed position at the time t_0 or earlier. It is obvious, and may be strictly proved with the help of the arguments in Sect. 9.5, that U_1 will also coincide with the mean longitudinal velocity of all the fluid which intersects the fixed cross section of the tube during the given time interval; that is, with the averaged bulk velocity U_{ave} defined as the ratio of the volume of the fluid which comes from the tube in a unit time to the area of its transverse cross section. Thus,

$$\overline{Y_1(\tau)} \approx U_{\text{ave}} \cdot \tau \tag{9.37}$$

for sufficiently large τ. Furthermore, on the basis of Eq. (9.35)

$$\overline{[Y_1(\tau) - \overline{Y_1(\tau)}]^2} \approx 2\overline{u_1'^2}\, T_1 \tau \tag{9.38}$$

for sufficiently large τ; for slightly smaller values of τ it is possible to use the more exact equation (9.34) with $i = 1$. The characteristics $\overline{u_1'^2}$ and T_1 in Eq. (9.38) must be determined by the parameters on which the statistical regime of turbulence in a tube depends. Since in a round tube, the viscous sublayer fills an insignificant part of the cross section for a sufficiently large Reynolds number (see above, Sect. 5.5), it follows that in this case $\overline{u_1'^2}$ and T_1 will depend only on the tube radius R and the friction velocity $u_* = \sqrt{\tau_0/\rho}$, where τ_0 is the shear stress on the wall. On the basis of dimensional arguments

$$\overline{u_1'^2}\, T_1 = cRu_*, \tag{9.39}$$

where c is a universal constant the value of which may be estimated from data (see below, Sect. 10.4). A formula of this type will be correct also for turbulent flow in a straight tube or channel of noncircular cross section for sufficiently large Re; here, for R it is only necessary to use a typical linear dimension of the cross section of the tube or channel, and u_* is determined by the mean value of the wall shear stress.

9.4 Fluid Particle Displacements in Grid Turbulence and in Turbulent Shear Flows

The Lagrangian Characteristics of Self-Preserving Turbulent Flows

The derivation of Eqs. (9.37)–(9.39) depends considerably on the fact that in a tube flow the motion of the fluid particles in the transverse plane Ox_2x_3 proceeds at all times only within the limits of a fixed bounded part thereof. It is clear that this latter condition is satisfied only for certain special turbulent flows. For example, in the case of a turbulent boundary layer along a plane wall or a turbulent jet, the average distance of a particle from the wall or from the jet axis will increase without limit with an increase in τ; in such a flow, therefore, the Lagrangian velocity $\frac{dY_1(\tau)}{d\tau}$ cannot be considered a stationary random function of time for any τ. Thus, the class of real flows to which Eqs. (9.30)–(9.36) can be applied directly is quite narrow.

However, Batchelor (1957) noted that there is also a class of flows, including several practically important ones, to which the above formulas may be applied after some simple modifications. This class consists of the stationary, self-preserving flows in which the mean velocity is directed mainly along the Ox_1 axis, and the turbulence structure for various values of the coordinate x_1 is similar, that is, differs only by the values of the characteristic length scale $L(x_1)$ and velocity scale $U(x_1)$. In other words, in such flows all the Eulerian statistical characteristics of turbulence in the plane $x_1 =$ const, reduced to a dimensionless form by division by the corresponding combination of scales L and U, do not depend on the values of x_1. In such a case a particular fluid particle will be found at all times under practically identical conditions, but with a variable velocity scale $U(\tau) = U[X_1(\boldsymbol{x}, t_0+\tau)]$ and a variable time scale

$$T(\tau) = \frac{L[X_1(\boldsymbol{x}, t_0+\tau)]}{U[X_1(\boldsymbol{x}, t_0+\tau)]}.$$

Therefore, to study the motion of this particle it is expedient to measure time by the scale $T(\tau)$, that is, to use in place of the time τ the variable $\eta(\tau)$ related to τ by the expression $d\eta=\frac{d\tau}{T(\tau)}$, and velocity by the scale $U(\tau)$. Here it is natural to assume that *for sufficiently large values of* τ, such that the initial position $\boldsymbol{x}$ has ceased to have an influence, *the dimensionless Lagrangian velocity fluctuation* $V'(\boldsymbol{x}, t_0+\tau)/U(\tau)$ *will be a stationary random function* $F(\eta)$ *of the variable* η:

$$\frac{V'(\boldsymbol{x}, t_0+\tau)}{U(\tau)}=F(\eta), \quad d\eta=\frac{U(\tau)\,d\tau}{L(\tau)}. \tag{9.40}$$

True, the strict proof of this proposition, and generally the fact that self-preservation of the Eulerian properties of turbulence implies self-preservation of its Lagrangian properties, is still missing. However, it is a very plausible hypothesis, the results of which agree well in a number of cases with the existing, although still very incomplete data concerning the Lagrangian statistical characteristics. Also, when self-preservation of the Eulerian turbulence characteristics is established with the help of dimensional considerations, the same considerations will generally be used to form the basis for self-preservation of the Lagrangian characteristics; the results thus obtained agree in all cases with the conclusions from the assumption in italics.

One of the simplest cases to which the hypothesis formulated here can be applied is that of turbulence in the central part of a wind-tunnel behind a turbulence-producing grid. In this case, the mean velocity $\overline{\boldsymbol{u}}=U_0\boldsymbol{i}_1$, where $\boldsymbol{i}_1$ is the unit vector along the axis Ox_1, is strictly constant and consequently

$$\overline{Y_1(\tau)}=U_0\tau, \quad \overline{Y_2(\tau)}=\overline{Y_3(\tau)}=0.$$

However, the turbulence here still is not completely homogeneous but is decaying since because of viscosity the intensity of the velocity fluctuations decreases slowly with increase in the distance from the grid. As the measurements of grid turbulence show [see Chap. 7, Volume 2 of this book], during an initial period of decay, specifically, at comparatively small distances from the grid, the turbulence behind the grid usually is approximately homogeneous in the planes $x_1=$ const, and differs for various values of x_1 only by the velocity scale, proportional to $(x_1-x_1^0)^{-1/2}$ and the length scale proportional to $(x_1-x_1^0)^{1/2}$, where x_1^0 is some virtual origin of reference on the Ox_1 axis. Since the mean velocity U_0 in such a flow will always be large compared to the velocity fluctuations, one may assume that $X_1(\tau)-x_1^0=U_0\tau$, where τ is the time of motion of the fluid particle reckoned from the virtual time t_0 where it intersected the plane $x_1=x_1^0$. For the velocity scale $U(\tau)$ in the plane $x_1=x_1^0+U_0\tau$, it is possible to assume any constant velocity multiplied by $\tau^{-1/2}$; for example, it is convenient for investigating the motion of a fluid particle in the plane $x_1 = a$ at the initial time to assume

$$U(\tau)=U_a\left(\frac{\tau_a}{\tau}\right)^{\frac{1}{2}}$$

where U_a is the characteristic value of the velocity fluctuations when $x_1=a$, and $\tau_a=\dfrac{a-x_1^0}{U_0}$. In this case the variable η of Eq. (9.40) is equal to $\ln\tau+\text{const}$; therefore, one may take $\eta=\ln\dfrac{\tau}{\tau_a}$. It follows from this that proposition (9.40) here reduces to the assumption that the Lagrangian velocity correlation function has the form

$$\begin{aligned}\overline{V_i'(x,\ t_0+\tau_1)\ V_j'(x,\ t_0+\tau_2)}=\\ =U_a^2\tau_a(\tau_1\tau_2)^{-\frac{1}{2}}S_{ij}\left(\ln\frac{\tau_2}{\tau_a}-\ln\frac{\tau_1}{\tau_a}\right),\end{aligned} \tag{9.41}$$

where S_{ij} is the cross-correlation function of the stationary processes

$$F_i(\eta)=\frac{V_i'(x,\ t_0+\tau)}{U(\tau)}\quad\text{and}\quad F_j(\eta)=\frac{V_j'(x,\ t_0+\tau)}{U(\tau)}.$$

Substituting Eq. (9.41) into the general equation (9.27) and replacing the integration with respect to t_1 and t_2 by integration with respect to τ_1 and τ_2 between the limits τ_a and τ, in a manner similar to the derivation of Eq. (9.30′) we obtain

$$D_{ij}(\tau)=U_a^2\tau_a^2\int\limits_0^{\ln\frac{\tau}{\tau_a}}\left(\frac{\tau}{\tau_a}e^{-\frac{\theta}{2}}-e^{\frac{\theta}{2}}\right)[S_{ij}(\theta)+S_{ji}(\theta)]\,d\theta. \tag{9.42}$$

If we make the natural assumption of sufficiently rapid decrease of the correlation function (9.41) with an increase in $\tau_1 - \tau_2$, from Eq. (9.42) it follows that for sufficiently large values of τ

$$D_{ij}(\tau) = 2U_a^2\tau_a\Xi_{ij}\tau, \quad \Xi_{ij} = \frac{1}{2}\int_0^\infty [S_{ij}(\theta) + S_{ji}(\theta)]\, e^{-\frac{\theta}{2}}\, d\theta. \quad (9.43)$$

At the same time we have shown that the asymptotic law $D_{ij}(\tau) \sim \tau$ may also be approximately applied to decaying turbulence behind a grid in a wind-tunnel.

A similar argument may be made for a turbulent wake produced in a free stream of uniform velocity by a long cylinder of arbitrary cross section located along the Ox_2 axis, or by some bounded solid body with its center at the origin. As already seen in Sect. 5.9, at a sufficiently large distance from the wake-producing body, where the turbulent velocities in the wake become small by comparison with the free-stream velocity U_0, the turbulent structure in the wake may be considered self-preserving and different for various values of the longitudinal coordinate x_1, only in the length scale $L(x_1)$ and the velocity scale $U(x_1)$. Moreover,

$$L(x_1) \sim x_1^{\frac{1}{2}}, \quad U(x_1) \sim x_1^{-\frac{1}{2}}$$

for a two-dimensional wake behind a cylinder, and

$$L(x_1) \sim x_1^{\frac{1}{3}}, \quad U(x_1) \sim x_1^{-\frac{2}{3}}$$

for a three-dimensional wake behind a bounded body. In the case under discussion $X_1(\tau) \approx U_0\tau$ for the appropriate choice of the origin of the time coordinate, and sufficiently large τ; consequently,

$$d\eta = \frac{d\tau}{\tau}, \quad \eta = \ln\frac{\tau}{\tau_a}$$

for both the two- and the three-dimensional wakes. Since the turbulent fluid does not leave the wake, at all times a given fluid particle will wander within the limits of the volume occupied by the wake, not leaving its boundaries. Therefore, it is natural to expect that for sufficiently large τ, hypothesis (9.40) here will also be correct with

$$\text{(with } U(\tau) = U_a\left(\frac{\tau_a}{\tau}\right)^{\frac{1}{2}} \text{ or } U(\tau) = U_a\left(\frac{\tau_a}{\tau}\right)^{\frac{2}{3}} \text{ respectively).}$$

respectively. However, this indicates that

$$D_{ij}(\tau) = U_a^2 \tau_a^2 \int_0^{\ln\frac{\tau}{\tau_a}} \left(\frac{\tau}{\tau_a} e^{-\frac{\theta}{2}} - e^{\frac{\theta}{2}}\right) [S_{ij}(\theta) + S_{ji}(\theta)]\, d\theta \tag{9.44}$$

for a fluid particle in a two-dimensional turbulent wake and

$$D_{ij}(\tau) = \frac{3}{2} U_a^2 \tau_a^2 \int_0^{\ln\frac{\tau}{\tau_a}} \left[\left(\frac{\tau}{\tau_a}\right)^{\frac{2}{3}} e^{-\frac{\theta}{3}} - e^{\frac{\theta}{3}}\right] [S_{ij}(\theta) + S_{ji}(\theta)]\, d\theta \tag{9.45}$$

for a three-dimensional wake where $S_{ij}(\theta)$ in both cases has the same meaning as in Eq. (9.42). Thus, for sufficiently large values of τ

$$D_{ij}(\tau) \sim \begin{cases} \tau & \text{for a two-dimensional wake} \\ \tau^{\frac{2}{3}} & \text{for a three-dimensional wake.} \end{cases} \tag{9.46}$$

In the case of a two-dimensional wake behind a long cylinder located along the Ox_2 axis, $\overline{Y_2(\tau)} \equiv 0$. In regards to the variable $\overline{X_3(\tau)}$ for a two-dimensional wake, and the variables $\overline{X_2(\tau)}$ and $\overline{X_3(\tau)}$ for a three-dimensional wake, at a large distance from a wake-producing body, the two-dimensional wake becomes practically symmetric with respect to the plane Ox_1x_2, and the three-dimensional wake becomes practically axisymmetric. Thus, one may anticipate that for sufficiently large τ, they will be close to zero for any initial position of the given fluid particle.

We now investigate the motion of a fluid particle in a plane or round turbulent jet which extends along the Ox_1 axis in a fluid-filled space. Here also for sufficiently large x_1, the statistical state may be considered completely characterized by the length scale $L(x_1)$ and the velocity scale $U(x_1)$ as was done in Sect. 5.9, where

$$L(x_1) \sim x_1, \quad U(x_1) \sim x_1^{-\frac{1}{2}}$$

for a two-dimensional plane jet and

$$L(x_1) \sim x_1, \quad U(x_1) \sim x_1^{-1}$$

for a three-dimensional, axisymmetric jet. However, in contrast to the case of a turbulent wake, no basic motion with a constant velocity $U_0 \gg u_1'$ exists in a turbulent jet; therefore, it is impossible here to assume that $X_1(\tau) \approx U_0\tau$. However, since the mean flow in the jet is also self-preserving, one may hope that in this case hypothesis (9.40), with

$$U(\tau) = U[\overline{X_1(\tau)}], \quad L(\tau) = L[\overline{X_1(\tau)}]$$

for sufficiently large τ will be applicable not only to Lagrangian velocity fluctuations, but also to the Lagrangian velocity itself $\boldsymbol{V}(\boldsymbol{x}, t_0 + \tau)$. Consequently,

$$\frac{d\overline{X_1(\tau)}}{d\tau} = \overline{V}_1 = \overline{F}_1 \cdot U[\overline{X_1(\tau)}],$$

where $\overline{F}_1 = \overline{F_1(\eta)} = \text{const}$, since $F_1(\eta)$ is a stationary random process. Taking into account that $U(x_1) \sim x_1^{-\alpha}$ where $\alpha = 1/2$ or 1, respectively, we obtain from this that for an appropriate selection of the time origin

$$\begin{aligned} &\overline{X(\tau)} \sim \tau^{\frac{2}{3}}, \quad U(\tau) \sim \tau^{-\frac{1}{3}}, \quad L(\tau) \sim \tau^{\frac{2}{3}} \quad \text{for a two-dimensional jet} \\ &\overline{X(\tau)} \sim \tau^{\frac{1}{2}}, \quad U(\tau) \sim \tau^{-\frac{1}{2}}, \quad L(\tau) \sim \tau^{\frac{1}{2}} \quad \text{for a three-dimensional jet.} \end{aligned} \tag{9.47}$$

Consequently, in both cases, $d\eta = \frac{d\tau}{\tau}$, $\eta = \ln\frac{\tau}{\tau_a}$. Furthermore, similar to the derivation of Eqs. (9.42), (9.44), and (9.45), we find

$$D_{ij}(\tau) = \frac{3}{4} U_a^2 \tau_a^2 \int_0^{\ln\frac{\tau}{\tau_a}} \left[\left(\frac{\tau}{\tau_a}\right)^{\frac{4}{3}} e^{-\frac{2}{3}\theta} - e^{\frac{2}{3}\theta}\right] [S_{ij}(\theta) + S_{ji}(\theta)]\, d\theta \tag{9.48}$$

for a two-dimensional jet and

$$D_{ij}(\tau)=U_a^2\tau_a^2\int_0^{\ln\frac{\tau}{\tau_a}}\left(\frac{\tau}{\tau_a}e^{-\frac{\theta}{2}}-e^{\frac{\theta}{2}}\right)[S_{ij}(\theta)+S_{ji}(\theta)]\,d\theta \qquad (9.49)$$

for a three-dimensional jet where $S_{ij}(\theta)=\overline{F_i'(\eta+\theta)F_j'(\eta)}$. For very large values of τ these equations become the asymptotic expressions

$$D_{ij}(\tau)\sim\begin{cases}\tau^{\frac{4}{3}} & \text{for a two-dimensional jet}\\ \tau & \text{for a three-dimensional jet.}\end{cases} \qquad (9.50)$$

The reasoning used for the investigation of fluid particle motion in turbulent jets is also applicable to the two-dimensional plane and three-dimensional convective jets over heated bodies, and also to the turbulent mixing layer between two plane-parallel flows of different velocities U_1 and U_2. As is known from Sect. 5.9, turbulent motion both in the convective jets and in the mixing layer is self-preserving for sufficiently large x_1, where the Ox_1 axis is considered to be vertical for a convective jet and parallel to the velocity of both flows for a mixing layer. The length scale $L = L(x_1)$ will be proportional in all cases to x_1; the velocity scale U, for a two-dimensional convective jet over a heated cylinder and of a mixing layer, turns out to be the same in all cross sections x_1 = const; for a three-dimensional convective jet $U(x_1) \sim x_1^{-1/3}$. It follows from this that the average longitudinal Lagrangian velocity $\overline{V}_1=\frac{d\overline{X_1(\tau)}}{d\tau}$ in the two-dimensional convective jet and in the turbulent mixing layer is constant; in the case of the mixing layer it is equal to $(U_1 + U_2)/2$, and in the three-dimensional convective jet, proportional to $\overline{X}_1^{-1/3}$. Therefore, hypothesis (9.40), with $U(\tau) = U[\overline{X_1(\tau)}]$ and $L(\tau) = L[\overline{X_1(\tau)}]$ for a two-dimensional convective jet and a mixing layer reduces to the simple statement that $\boldsymbol{V}'(\boldsymbol{x}, t_0+\tau)$ is a stationary random function of the variable $\eta = \ln\frac{\tau}{\tau_a}$. In the case of a three-dimensional convective jet, for which $\frac{d\overline{X}_1}{d\tau}\sim\overline{X}_1^{-\frac{1}{3}}$, that is,

$$\overline{X_1(\tau)}\sim\tau^{\frac{3}{4}},\quad L(\tau)\sim\overline{X_1(\tau)}\sim\tau^{\frac{3}{4}},\quad U(\tau)\sim\tau^{-\frac{1}{4}},$$

this hypothesis indicates that the variable

$$V'(x, t_0+\tau)\, U_a^{-1}\left(\frac{\tau}{\tau_a}\right)^{\frac{1}{4}}$$

is a stationary random function of $\eta=\ln\frac{\tau}{\tau_a}$. In the usual way, we find from this that

$$D_{ij}(\tau)=\frac{1}{2}U^2\tau_a^2\int\limits_0^{\ln\frac{\tau}{\tau_a}}\left[\left(\frac{\tau}{\tau_a}\right)^2 e^{-\theta}-e^{\theta}\right][S_{ij}(\theta)+S_{ji}(\theta)]\,d\theta \tag{9.51}$$

for the two-dimensional convective jet and the mixing layer and

$$D_{ij}(\tau)=\frac{2}{3}U_a\tau_a^2\int\limits_0^{\ln\frac{\tau}{\tau_a}}\left[\left(\frac{\tau}{\tau_a}\right)^{\frac{3}{2}} e^{-\frac{3}{4}\theta}-e^{\frac{3}{4}\theta}\right][S_{ij}(\theta)+S_{ji}(\theta)]\,d\theta \tag{9.52}$$

for the three-dimensional convective jet. In particular, for very large values of τ

$$D_{ij}(\tau)\sim\begin{cases}\tau^2 \text{ for a two-dimensional convective jet and a mixing layer}\\ \tau^{\frac{3}{2}} \text{ for a three-dimensional convective jet.}\end{cases} \tag{9.53}$$

[The asymptotic equations (9.43), (9.46), (9.50), and (9.53) first appeared in Batchelor and Townsend (1956), Batchelor (1957) and Yaglom (1965).]

Let us now show that Eqs. (9.50) and (9.53) may be derived also from purely dimensional considerations without any use of special hypotheses, if we accept the general *Lagrangian similarity hypothesis* which states that the physical parameters on which the Eulerian statistical characteristics of turbulence depend, also determine the Lagrangian characteristics; in other words, they completely determine the entire turbulent state. Indeed, according to Sect. 5.9, for a three-dimensional jet of dynamic origin the defining physical parameters are the fluid density ρ and the total momentum of the fluid injected per unit time $2\pi\rho M$; for a two-dimensional dynamic jet, the density ρ and the momentum ρM_1 of the fluid injected per unit time from unit length of the slit; for the mixing layer between two plane parallel flows, ρ and the velocity $U_0 = U_2 - U_1$; for the three-dimensional convective jet, ρ, c_p, the total heat flux along the jet Q and the buoyancy parameter g/T_0; for the two-dimensional convective jet, ρ, c_p, g/T_0 and the specific heat flux Q_1 per unit length of the heated cylinder. If, for example, the probability distribution for displacement $\boldsymbol{Y}(\tau)$ of the fluid particle in the time τ, for a sufficiently large τ, depends only on these parameters and on τ as implied by the Lagrangian similarity hypothesis, then on the basis of dimensional arguments the corresponding probability density $p(\boldsymbol{Y}) = p(Y_1, Y_2, Y_3)$ will have the form:

$$p(\boldsymbol{Y}) = \frac{1}{M^{3/4}\tau^{3/2}} P^{(1)}\left(\frac{Y_1}{M^{1/4}\tau^{1/2}}, \frac{Y_2}{M^{1/4}\tau^{1/2}}, \frac{Y_3}{M^{1/4}\tau^{1/2}}\right) \tag{9.54}$$

in the case of a three-dimensional dynamic jet;

$$p(\boldsymbol{Y}) = \frac{1}{M_1\tau^2} P^{(2)}\left(\frac{Y_1}{M_1^{1/3}\tau^{2/3}}, \frac{Y_3}{M_1^{1/3}\tau^{2/3}}, \frac{Y_3}{M_1^{1/3}\tau^{2/3}}\right) \tag{9.54'}$$

in the case of a two-dimensional dynamic jet;

$$p(\boldsymbol{Y}) = \frac{1}{U_0^3\tau^3} P^{(3)}\left(\frac{Y_1}{U_0\tau}, \frac{Y_2}{U_0\tau}, \frac{Y_3}{U_0\tau}\right) \tag{9.54''}$$

in the case of a plane mixing layer;

$$p(\boldsymbol{Y}) = \frac{1}{\left(\frac{Q}{c_p\rho}\frac{g}{T_0}\right)^{3/4}\tau^{9/4}} P^{(4)}\left(\frac{Y_1}{\left(\frac{Q}{c_p\rho}\frac{g}{T_0}\right)^{1/4}\tau^{3/4}}, \frac{Y_2}{\left(\frac{Q}{c_p\rho}\frac{g}{T_0}\right)^{1/4}\tau^{3/4}}, \frac{Y_3}{\left(\frac{Q}{c_p\rho}\frac{g}{T_0}\right)^{1/4}\tau^{3/4}}\right) \tag{9.54'''}$$

in the case of a three-dimensional convective jet; and

$$p(\boldsymbol{Y}) = \frac{1}{\frac{Q_1}{c_p\rho}\frac{g}{T_0}\tau^3} P^{(5)}\left(\frac{Y_1}{\left(\frac{Q_1}{c_p\rho}\frac{g}{T_0}\right)^{1/3}\tau}, \frac{Y_2}{\left(\frac{Q_1}{c_p\rho}\frac{g}{T_0}\right)^{1/3}\tau}, \frac{Y_3}{\left(\frac{Q_1}{c_p\rho}\frac{g}{T_0}\right)^{1/3}\tau}\right) \tag{9.54''''}$$

for a two-dimensional convective jet. Here $P^{(1)}, \ldots, P^{(5)}$ are five universal functions characterizing the five indicated self-preserving turbulent flows. Expressions (9.50) and (9.53) follow immediately from Eqs. (9.54)–(9.54''''). In general, asymptotic expressions for arbitrary moments of the random vector $\boldsymbol{Y}(\tau) = (Y_1, Y_2, Y_3)$ are easily obtained from them. Equations similar to (9.54)–(9.54'''') may be written for the probability density of the Lagrangian velocity $\boldsymbol{V}(\tau) = \frac{d\boldsymbol{Y}(\tau)}{d\tau}$ and the other Lagrangian variables; however, we shall not discuss this here.

Particle Dispersion in a Shear Flow with Constant Velocity Gradient

It is possible to determine the asymptotic behavior of the variances and covariances $D_{ij}(\tau)$ in idealized turbulent flow in the entire unbounded three-dimensional space such that its Eulerian velocity fluctuations $\boldsymbol{u}'(\boldsymbol{x}, t)$ are stationary and statistically homogeneous, and the mean velocity $\bar{\boldsymbol{u}}(\boldsymbol{x})$ does not vary with time but depends linearly on the spatial coordinates. The latter condition is necessary so that the field of velocity fluctuations may be considered homogeneous, since the gradient of the mean velocity has an essential influence on the structure of turbulent flow; however, in homogeneous turbulence, the velocity gradient must be constant. Let us assume that the mean velocity is directed along the Ox_1 axis and varies only with respect to the Ox_3 direction so that, for example, $\bar{u}_1 = \Gamma x_3, \bar{u}_2 = \bar{u}_3 = 0$ where $\Gamma =$ const. Without loss of generahty, it may be assumed that at the time $t = 0$ the fluid particle was at the point $\boldsymbol{x} = 0$. We designate the coordinates of this particle at the time t as $X_i(t)$ and its velocity as $\boldsymbol{V}(t)$. In this case,

$$\boldsymbol{V}(t) = \bar{\boldsymbol{u}}[\boldsymbol{X}(t)] + \boldsymbol{u}'[\boldsymbol{X}(t), t] \tag{9.55}$$

or in the projections on the coordinate axis

$$\begin{gathered} V_1(t) = \Gamma X_3(t) + V_1'(t), \\ V_2(t) = V_2'(t), \quad V_3(t) = V_3'(t), \end{gathered} \tag{9.55'}$$

and

$$\begin{gathered} X_1(t) = \int_0^t [\Gamma X_3(t) + V_1'(t)]\, dt, \\ X_2(t) = \int_0^t V_2'(t)\, dt, \qquad X_3(t) = \int_0^t V_3'(t)\, dt. \end{gathered} \tag{9.56}$$

Using these equations, it is not difficult to also express $D_{ij}(\tau)$ in terms of the statistical characteristics of the field $\boldsymbol{u}'(\boldsymbol{x}, t)$, where in constrast to the cases examined above, no special hypotheses are required concerning Lagrangian self-preservation.

Since $\overline{\boldsymbol{u}'(\boldsymbol{x}, t)} \equiv 0$, then also $\overline{V'(t)} \equiv 0$; consequently, also $\overline{\boldsymbol{X}(t)} = 0$ on the basis of Eq. (9.56). Since the Eulerian velocity $\boldsymbol{u}'(\boldsymbol{x}, t)$ is homogeneous and stationary, the Lagrangian velocity $V'(t)$ will also be a stationary random function; its correlation tensor will have the form

$$\overline{V_i'(t_1)\,V_j'(t_2)} = B_{ij}^{(L)}(t_1 - t_2).$$

Now let us investigate the covariance tensor $D_{ij}(\tau) = \overline{X_i(\tau)X_j(\tau)}$. The presence of the mean velocity directed along the Ox_1 axis in no way influences the particle displacement along the Ox_2 and Ox_3 directions. Therefore, here the variances $D_{22}(\tau)$, $D_{33}(\tau)$ and $D_{23}(\tau)$ take the form (9.30′), which is usual for homogeneous turbulence, and their asymptotic behavior for small and large values of τ is described by Eqs. (9.28) and (9.35). This is not the case for the most interesting components of the covariance tensors $D_{11}(\tau)$ and $D_{13}(\tau)$. For the former, we may obtain from the first formula of Eq. (9.56) the expression

$$D_{11}(\tau) = \overline{X_1^2(\tau)} = \int_0^\tau \int_0^\tau \{\Gamma^2\, \overline{X_3(t_1)\,X_3(t_2)} +$$
$$+ \Gamma\left[\overline{X_3(t_1)\,V_1'(t_2)} + \overline{X_3(t_2)\,V_1'(t_1)}\right] + B_{11}^{(L)}(t_1 - t_2)\}\, dt_1\, dt_2.$$

On the basis of the third equation of Eq. (9.56), the first term in the braces may here be transformed into the following:

$$\overline{X_3(t_1)\,X_3(t_2)} = \int_0^{t_1} \int_0^{t_2} \overline{V_3'(\theta_1)\,V_3'(\theta_2)}\, d\theta_1\, d\theta_2 =$$
$$= \int_0^{t_1} \int_0^{t_2} B_{33}^{(L)}(\theta_1 - \theta_2)\, d\theta_1\, d\theta_2 = \int_0^{t_1} (t_1 - \theta)\, B_{33}^{(L)}(\theta)\, d\theta +$$
$$+ \int_0^{t_2} (t_2 - \theta)\, B_{33}^{(L)}(\theta)\, d\theta - \int_0^{|t_1 - t_2|} (|t_1 - t_2| - \theta)\, B_{33}^{(L)}(\theta)\, d\theta.$$

The second term in the braces is calculated similarly:

$$\overline{X_3(t_1)\,V_1'(t_2)}+\overline{X_3(t_2)\,V_1'(t_1)}=$$

$$=\int_0^{t_1} B_{13}^{(L)}(t_2-\theta)\,d\theta+\int_0^{t_2} B_{13}^{(L)}(t_1-\theta)\,d\theta=$$

$$=\int_0^{t_1} B_{13}^{(L)}(\theta)\,d\theta+\int_0^{t_2} B_{13}^{(L)}(\theta)\,d\theta+\int_0^{|t_1-t_2|}\left[B_{31}^{(L)}(\theta)-B_{13}^{(L)}(\theta)\right]d\theta.$$

Using these expressions, the formula $D_{11}(\tau)$ after repeated integration by parts may be reduced to the form

$$D_{11}(\tau)=\frac{\Gamma^2}{3}\int_0^{\tau}(2\tau^3-3\tau^2\theta+\theta^3)\,B_{33}^{(L)}(\theta)\,d\theta+\Gamma\int_0^{\tau}(\tau-\theta)^2\,B_{31}^{(L)}(\theta)\,d\theta+$$

$$+\Gamma\int_0^{\tau}(\tau^2-\theta^2)\,B_{33}^{(L)}(\theta)\,d\theta+2\int_0^{\tau}(\tau-\theta)\,B_{11}^{(L)}(\theta)\,d\theta. \quad (9.57)$$

Quite similar calculations lead to the following formula for D_{13}:

$$D_{13}(\tau)=\Gamma\tau\int_0^{\tau}(\tau-\theta)\,B_{33}^{(L)}(\theta)\,d\theta+\int_0^{\tau}(\tau-\theta)\left[B_{13}^{(L)}(\theta)+B_{31}^{(L)}(\theta)\right]d\theta. \quad (9.57')$$

From Eqs. (9.57) and (9.57′) the following asymptotic equations are obtained for large τ:

$$D_{11}(\tau)\approx\frac{2}{3}\,\Gamma^2\overline{u_3'^2}T_3\cdot\tau^3;\qquad D_{13}(\tau)\approx\Gamma\overline{u_3'^2}T_3\cdot\tau^2. \quad (9.58)$$

Thus, the variance of the fluid particle displacement along the Ox_1 axis in the direction of the mean flow for large τ is asymptotically proportional to τ^3; that is, it increases with time significantly more rapidly than the variances of the transverse displacements, asymptotically proportional to τ. In addition, the displacements along the Ox_1 and Ox_3 axes turn out to be mutually correlated. Using the asymptotic equation (9.35) for $D_{33}(\tau)$, we obtain the following limiting value for the correlation coefficient between the variables $X_1(\tau)$ and $X_3(\tau)$:

$$r_{X_1X_3}=D_{13}(\tau)\,[D_{11}(\tau)\,D_{33}(\tau)]^{-\frac{1}{2}}\to\frac{\sqrt{3}}{2}=0.866\ldots. \quad (9.59)$$

This limiting value turns out to be universal, that is, it does not depend on the parameters Γ, $\overline{u_3'^2}$ and T_3.

The asymptotic equations (9.58) and (9.59) were first reported by Corrsin (1959b). However, the equivalent expressions were known even earlier from the semiempirical theory of turbulent diffusion which uses parabolic partial differential equations to describe the fluid particle dispersion. We shall consider this problem in the following section (see Sect. 10.4).

The result by Högström (1964) was quite similar to the first equation (9.58). He considered particle dispersion in a flow with strictly constant velocity u_2 along the axis OX_2, which makes it possible to use the coordinate $X_2 = u_2\tau$ instead of time τ, a linear profile $u_1 = \Gamma X_3$ of the component u_1, and turbulent velocity fluctuations u_3' in the direction of the OX_3 axis alone. Moreover, assuming $D_{33}(X_2) = CX_2^{2\alpha}$, Högström found the relation

$$D_{11}(X_2) = \Gamma^2 D_{33}(X_2) X_2^2/2(\alpha + 1)$$

which coincides with Eq. (9.58) if $\alpha = 1/2$; he then used this relationship in an attempt to explain some atmospheric diffusion phenomena. Later, Smith (1965) [see also the discussion of this paper] investigated the same problem. He considered all three components of the velocity fluctuation and gave special attention to the conditional characteristics for the particles to reach the given height X_3 at a given time $\tau (= X_2/u_1)$. However, his results will not be dwelled upon here.

The Lagrangian Characteristics of the Turbulent Boundary Layer

Let us now consider the problem of the fluid particle displacements in a turbulent boundary layer. As in Chapts. 3 and 4, the flow will be considered as filling the half-space $z > 0$ where, specifically, the wall $z = 0$ will be taken to be dynamically completely rough, with a roughness parameter z_0. The Ox axis will be taken in the direction of the mean flow. Without loss of generality, one may assume that the given fluid particle at the initial time $t = t_0$ will be found at the point with coordinates $\boldsymbol{x} = (0, 0, H)$. Let $[X(\tau), Y(\tau), Z(\tau)]$ be the coordinates of this particle at the time $t_0 + \tau$, and $\boldsymbol{V}(\tau) = [U(\tau), V(\tau), W(\tau)]$, its velocity at this time. The random function $\boldsymbol{V}(\tau)$ obviously is not stationary; thus, for example, for a sufficiently large time interval τ, the particle will most probably rise to a significant height $Z(\tau)$, thereby strongly increasing its horizontal velocity $U(\tau)$. Generally speaking, there is no reason to expect that the function $V(\tau)$ may be transformed into a stationary function with the help of a simple transformation to new scales of length and time. However, it is natural to assume that in addition to the parameters τ and H, the Lagrangian statistical characteristics of turbulence in a boundary layer will depend only on a small number of "external" parameters which determine the turbulent state; that is, those which enter into the expressions for the Eulerian statistical characteristics. This assumption, essentially simplifying the study of the Lagrangian characteristics, was used in implicit form by Kazanskiy and Monin (1957) [see also Monin (1959a)] to calculate the form of a smoke plume in the atmospheric surface layer under various stratification conditions. Later, it was precisely formulated and investigated in detail by Ellison (1959) and Batchelor (1959) [see also Batchelor (1964) and Chatwin (1968)] for the special case of a neutral or nonstratified boundary layer. Still later, Gifford (1962), supplementing this assumption by some semiempirical hypotheses, and Yaglom (1965) derived from it a series of corollaries concerning the general case of thermally stratified fluids. Gifford's conclusions were compared with the existing data by Gifford himself, Malhotra and Cermak (1963) and Cermak (1963). Mention should also be made of the works of Panofsky and Prasad (1965), Pasquill (1966), E. E. O'Brien (1966), Mandell and O'Brien (1967), and Klug (1968). These works were all devoted to subsequent development of the Lagrangian similarity treatment of boundary-layer flows and its applications to the problem of diffusion in the atmospheric surface layer. These apphcations will be considered in Sect. 10.5.

Since the proposition stated earlier includes the statement that the boundary-layer flow is described by a few parameters, when using it, it is expedient to limit ourselves to cases where the fluid-particle motion does not exceed the upper boundary of the constant stress region in which $\tau = \rho u_*^2$ = const where τ is the shear stress, or in the case of a thermally stratified fluid of the constant flux region in which $\tau =$ const and $q =$ const, where q is the vertical heat flux. Following Ellison (1959) and Batchelor (1959; 1964), we begin with the case of a neutral boundary layer for which $q = 0$. To be specific, it may be considered, for example, that we are dealing with the atmospheric surface layer under conditions of neutral stratification. In this case, the turbulent state is defined by the parameters u_* and

z_0. However, it is essential that outside the very thin fluid layer, which has a thickness of the same order of magnitude as z_0, and directly adjacent to the wall $z = 0$, only one parameter u_* plays an important role. Variation in the value of z_0, that is, replacement of it by z_0', leads only to additional horizontal displacement of the entire mass of fluid along the Ox axis with a constant velocity of

$$\frac{u_*}{\varkappa} \ln \frac{z_0}{z_0'}, \text{where } \varkappa \approx 0.4$$

is von Kármán's constant (see above, Chapt. 3, Sect. 5.4). In addition, it is obvious that after a sufficiently large time τ the particle must "forget" its initial height H, that is, for a large τ the value of H almost ceases to influence the statistical characteristics of the fluid-particie motion. It follows from this that although the statistical characteristics of the random vector $\boldsymbol{V}(\tau)$ may generally depend on four parameters, τ, H, u_*, and z_0, the effect of the second and fourth is quite limited. The influence of the initial height H will be felt only during a finite time, the duration of which, on the basis of dimensional considerations, must be of the order of the ratio H/u_*. Also, the value of the roughness parameter z_0 will really be essential only if $H \lesssim z_0$, and even in this case only during a time interval of order z_0/u_*. If $H \gg z_0$, or $H \lesssim z_0$ but $\tau \gg \frac{z_0}{u_*}$, then z_0 will influence the statistical characteristics of $\boldsymbol{V}(\tau)$ only through an additional constant summand of the form $-\frac{u_*}{\varkappa} \ln z_0$ in the expression for $\overline{U(\tau)}$.

Let us now investigate the fluid particle mean velocity

$$\overline{\boldsymbol{V}(\tau)} = (\overline{U(\tau)}, \overline{V(\tau)}, \overline{W(\tau)})$$

at the time $t_0+\tau$. The symmetry of the boundary-layer flow with respect to the Oxz plane implies that $\overline{Y(\tau)} = 0$ for all τ and consequently $\overline{V(\tau)} = \frac{d\overline{Y(\tau)}}{d\tau} = 0$. For the components $\overline{U(\tau)}$ and $\overline{W(\tau)}$, according to the above when $\tau \gg H/u_*$, or when $\tau \gg z_0/u_*$, if $H = 0$ or, generally speaking, $H \lesssim z_0$, the second of these components may depend only on u_* and τ, and the first must equal the sum of some function of u_* and τ, and the constant velocity $-\frac{u_*}{\varkappa} \ln z_0$. On the basis of dimensional considerations, we obtain

$$\overline{U(\tau)} = \frac{d\overline{X(\tau)}}{d\tau} = \frac{u_*}{\varkappa}\left(\ln\frac{u_*\tau}{z_0} + d\right) = \frac{u_*}{\varkappa}\ln\frac{cu_*\tau}{z_0}, \tag{9.60}$$

$$\overline{W(\tau)} = \frac{d\overline{Z(\tau)}}{d\tau} = bu_*, \tag{9.61}$$

where b and $c = e^d$ are dimensionless universal constants. Integrating these equations with respect to τ, we find that for sufficiently large τ

$$\overline{X(\tau)} \approx \frac{u_*\tau}{\varkappa}\left(\ln\frac{cu_*\tau}{z_0} - 1\right) = \frac{u_*\tau}{\varkappa}\ln\frac{cu_*\tau}{ez_0}, \tag{9.60'}$$

$$\overline{Z(\tau)} \approx bu_*\tau. \tag{9.61'}$$

With an increase in the initial height H, Eqs. (9.60′) and (9.61′) become applicable later, but the values of the parameters b and c for any H remain the same. The fact that the particle experiences an asymptotically constant average vertical velocity $\overline{W(\tau)}$ in spite of the zero value of the average vertical Eulerian velocity $\overline{w(\boldsymbol{X})}$ at all space points of the flow, so that, in particular, also $\overline{W(0)} = \overline{w(0, 0, H)} = 0$, is related to the inability of the particle to drop below this level due to the wall at $z = 0$, while nothing prohibits its unlimited ascent. Therefore, the probability distribution for $Z(\tau)$ with an increase in τ extends upward more and more. Consequently, the average displacement of the particle $\overline{Z(\tau)}$ increases, which means $b > 0$. It follows from this that for any layer $0 \leqslant z \leqslant h$ of fixed thickness h we have a constant outflow of mass to the upper layers of the fluid. This must be compensated for by an equivalent influx of mass from the upper layers downward. (Let us recall that for the Eulerian velocity $w(\boldsymbol{X})$ the inequality $\overline{w(\boldsymbol{X})} \neq 0$ would mean that conservation of mass is violated!) It seems natural to assume, however, that the velocity $\overline{W(\tau)}$ will be less than the characteristic value of the Eulerian vertical velocity fluctuations, for example, less than the value of $\sigma_w = (\overline{w^2})^{1/2}$. Therefore the constant b will apparently be appreciably less than unity; several values in the range 0.1 to 1 have in fact been proposed by several researchers; see below. An effort may be made to evaluate the constant c beginning with the approximate equation

$$\overline{U(\tau)} \approx \bar{u}\,[\overline{Z(\tau)}], \tag{9.62}$$

where $\bar{u}(Z)$ is the mean horizontal Eulerian velocity at a height Z. Let us emphasize that in Eq. (9.62) it is not possible to substitute the sign of exact equality. This is because the average on the left side is taken with respect to the set of fluid particles which at the time $\tau = 0$ are found at the given height H, and on the right side, with respect to quite another set of all possible fluid particles which regardless of the time, are at a fixed height $\overline{Z(\tau)}$. If the fluid particles in question have started at a given height H, but $\tau \gg \frac{H}{u_*}$, then H will have no significant effect on the value of $\overline{U(\tau)}$. Even in this case, however, the mean Lagrangian horizontal velocity will not be exactly equal to the mean Eulerian horizontal velocity at a height $Z(\tau)$, since $\overline{W(\tau)} > 0$. Consequently in the expression

$$\overline{U(\tau)} = \overline{U[X(\tau),\ Y(\tau),\ Z(\tau)]}$$

averaging mainly considers particles which come to the height $Z(\tau)$ from below. In addition, even if it could be considered that

$$\overline{U(\tau)} = \bar{u}[\overline{X(\tau)}, \overline{Y(\tau)}, \overline{Z(\tau)}]$$

where for fixed values of $X(\tau)$, $Y(\tau)$, and $Z(\tau)$ the average over the values of u on the right side is the usual average over the values of the Eulerian velocity, Eq. (9.62) still would not be exact. In fact, $\bar{u}(Z)$ increases more slowly than Z, and consequently

$$\overline{\bar{u}[Z(\tau)]} \neq \bar{u}[\overline{Z(\tau)}].$$

Let us also note that both of these reasons must lead to the left side of Eq. (9.62) being less than the right side. However, it is hoped that when $\tau \gg \frac{H}{u_*}$, the difference between both sides of Eq. (9.62) will be relatively small, so that the expression $c \approx b$ which is derived from Eqs. (9.62), (9.60), and (9.61′) could be used as an acceptable or at least a rough first approximation. Nevertheless, in some cases, it may be appropriate to consider c as being somewhat less than b. (See Sect. 10.5 where a semiempirical estimate of c will be given, according to which $c \approx 0.66$.)

Using dimensional considerations, it is possible also to derive expressions for the second- and higher-order moments of the vectors

$$\boldsymbol{V}'(\tau) = \boldsymbol{V}(\tau) - \overline{\boldsymbol{V}(\tau)} = (U'(\tau), V'(\tau), W'(\tau))$$

and

$$X'(\tau) = X(\tau) - \overline{X(\tau)} = (X'(\tau), Y'(\tau), Z'(\tau)).$$

However, it is simpler to write down immediately the general equations for the probability densities of these vectors, from which the expressions for all their statistical characteristics follow. As we have seen, when $\tau \gg \frac{H}{u_*}$ and $\tau \gg \frac{z_0}{u_*}$, the probability distribution for $V'(\tau)$ may depend only on the parameters u_* and τ. Consequently, the corresponding probability density must have the form

$$p(V') = \frac{1}{u_*^3} P_1\left(\frac{U'}{u_*}, \frac{V'}{u_*}, \frac{W'}{u_*}\right), \tag{9.63}$$

where $P_1(u, v, w)$ is a universal function of three variables. In a similar manner the probability density of the vector $X'(\tau)$ is determined by a formula of the type

$$p(X') = \frac{1}{u_*^3\tau^3} P_2\left(\frac{X'}{u_*\tau}, \frac{Y'}{u_*\tau}, \frac{Z'}{u_*\tau}\right), \tag{9.64}$$

where $P_2(x, u, z)$ is another universal function; in particular, it follows immediately from here that in the logarithmic boundary layer $D_{ij}(\tau) \sim \tau^2$, namely, $= d_{ij}u_*^2\tau^2$ where d_{ij} are universal dimensionless constants. According to Eq. (9.61′) the length of $u_*\tau$ in Eq. (9.64) may be replaced by the length $\overline{Z} = \overline{Z(\tau)}$ proportional to it; therefore, for example, the probability density for the vector $X(\tau)$ may be written as

$$p(X) = \frac{1}{\overline{Z}^3} P_3\left(\frac{X - \overline{X}}{\overline{Z}}, \frac{Y}{\overline{Z}}, \frac{Z - \overline{Z}}{\overline{Z}}\right), \tag{9.65}$$

where the function P_3 is obtained from P_2 by a simple change in scale on the coordinate axes.

Let us now proceed to the more complex case of the fluid-particle motion in a thermally stratified boundary layer; for example, the atmospheric surface layer with nonneutral stratification. Here, in addition to u_*, we must consider among the "external parameters" of the problem, the parameters $q/c_p\rho_0$ and g/T_0. From these three variables it is possible to make up a combination having the dimension of length

$$L = -u_*^3/\varkappa \frac{g}{T_0} \frac{q}{c_p \rho_0},$$

which plays an important role in the deductions of Chapt. 4. The roughness parameter z_0 also in this case will be essential only if $H \lesssim z_0$ and simultaneously $\tau \lesssim \frac{z_0}{u_*}$. If $H \gg z_0$ or $H \lesssim z_0$, but $\tau \gg \frac{z_0}{u_*}$, then replacing z_0 by z_0' only leads to an additional motion of the entire mass of air in the Ox direction with a constant velocity

$$\frac{u_*}{\varkappa}\left[f\left(\frac{z_0'}{L}\right) - f\left(\frac{z_0}{L}\right)\right]$$

where $f(\zeta)$ is a universal velocity profile function in Chapt. 4. Consequently, when $\tau \gg \frac{H}{u_*}$ and $\tau \gg \frac{z_0}{u_*}$ we must have the expressions

$$\overline{U(\tau)} = \frac{d\overline{X(\tau)}}{d\tau} = \frac{u_*}{\varkappa}\left[\psi_1\left(\frac{u_*\tau}{L}\right) - f\left(\frac{z_0}{L}\right)\right], \tag{9.66}$$

$$\overline{W(\tau)} = \frac{d\overline{Z(\tau)}}{d\tau} = u_*\psi_2\left(\frac{u_*\tau}{L}\right), \tag{9.67}$$

where ψ_1 and ψ_2 are universal functions of the variable $\xi = \frac{u_*\tau}{L}$. Integrating the second of these expressions, we obtain

$$\frac{\overline{Z(\tau)}}{L} = \psi_3\left(\frac{u_*\tau}{L}\right), \quad \frac{u_*\tau}{L} = \Psi\left(\frac{\overline{Z(\tau)}}{L}\right), \tag{9.68}$$

where

$$\psi_3(\xi) = \int_0^\xi \psi_2(\eta)\, d\eta,$$

and Ψ is the function which is the inverse of ψ_3. With the help of Eq.

(9.68), Eqs. (9.66)–(9.67) may be rewritten as

$$\frac{d\overline{X(\tau)}}{d\tau} = \frac{u_*}{\varkappa}\left[\varphi_1\left(\frac{\overline{Z(\tau)}}{L}\right) - f\left(\frac{z_0}{L}\right)\right], \tag{9.69}$$

$$\frac{d\overline{Z(\tau)}}{d\tau} = bu_*\varphi\left(\frac{\overline{Z(\tau)}}{L}\right), \tag{9.70}$$

where φ_1 and φ are two universal functions, and the constant b is introduced so that we may consider $\varphi(0) = 1$; under this condition, b will have the same value as in Eq. (9.61). On the basis of the approximate equation (9.62) we may also expect that $\varphi_1(\zeta) \approx f(\zeta)$; therefore, as a first approximation we may assume

$$\frac{d\overline{X(\tau)}}{d\tau} \approx \frac{u_*}{\varkappa}\left[f\left(\frac{\overline{Z(\tau)}}{L}\right) - f\left(\frac{z_0}{L}\right)\right]. \tag{9.71}$$

However, if greater accuracy is desired, it must be considered that indeed $\varphi_1(\zeta) < f(\zeta)$ for all ζ; but as yet, there are no data on the value of the difference $f(\zeta) - \varphi_1(\zeta)$. Expressions (9.70) and (9.71) were used by Gifford (1962), and later by Malhotra and Cermak (1963), and Cermak (1963). In addition, they all propose without firm proof that the function $\varphi(\zeta)$ differs only by a constant factor from the function $f_5(\zeta)$ of Eq. (7.85) which describes the root mean square value of the Eulerian vertical velocity. Subsequently, the function $\varphi(\zeta)$, according to the semiempirical formula (7.95) of Kazanskiy and Monin, was assumed to satisfy the equation

$$\varphi(\zeta) = \left[1 - \frac{1}{f'(\zeta)}\right]^{1/4}.$$

Of course, this equation is not exact, but its use may be justified somewhat by the fact that the condition $\varphi(0) = 1$, and the asymptotic laws describing the behavior of $\varphi(\zeta)$ as $|\zeta| \to \infty$ are fulfilled here.

For the probability density $p(\boldsymbol{X})$ of the three-dimensional vector

$$\boldsymbol{X}(\tau) = (X(\tau), Y(\tau), Z(\tau)) \quad \text{for} \quad \tau \gg \frac{z_0}{u_*} \text{ and } \tau \gg \frac{H}{u_*},$$

we must have on the basis of dimensional considerations, a formula of the type

$$p(X)=\frac{1}{\overline{Z}^3}P\left(\frac{X-\overline{X}}{\overline{Z}},\ \frac{Y}{\overline{Z}},\ \frac{Z-\overline{Z}}{\overline{Z}};\ \frac{\overline{Z}}{L}\right), \tag{9.72}$$

where $\overline{X}=\overline{X(\tau)}$ and $\overline{Z}=\overline{Z(\tau)}$ are determined from Eqs. (9.69), or (9.71), and (9.70), and $P(x, y, z; \zeta)$ is a universal function of four variables, the last of which behaves as a parameter determining the form of the probability distribution. Gifford, and Malhotra and Cermak cited above, used in a simpler formula in place of this:

$$p(X)=\frac{1}{\overline{Z}^3}P\left(\frac{X-\overline{X}}{\overline{Z}},\ \frac{Y}{\overline{Z}},\ \frac{Z-\overline{Z}}{\overline{Z}}\right); \tag{9.72'}$$

that is, it was implicitly proposed that the dependence of the form of the probability distribution on the stratification parameter $\zeta=\frac{\overline{Z}}{L}$ is quite weak, and to a first approximation, may be neglected. However, this assumption does not follow from dimensional considerations; it is justified only by some very preliminary experimental results [see Cermak (1963)].

When $q>0$, that is, for unstable stratification, it is also possible to obtain several simple asymptotic results for the variables $\overline{W(\tau)}$ and $\overline{Z(\tau)}$ based on the limiting laws of the free convection regime [cf. Yaglom (1965)]. Indeed, when $\tau \gg \frac{|L|}{u_*}$ and $\tau \gg \frac{H}{u_*}$, and quite possibly even when $\tau \gg \frac{0.1\,|L|}{u_*}$ as follows from some of the empirical results of Sect. 8, Chapt. 4, not only will the initial height H cease to influence the vertical motion of the fluid particle, but also the variable u_*. In fact, under these conditions, the particle will spend most of its time in the region of flow in which the condition of "pure" free convection is predominant, and which does not depend on the friction velocity. Therefore if $q>0$, or $L<0$, then with sufficiently great $-\xi$, perhaps even with $-\xi>0.1$, the functions $\psi_2(\xi)$ and $\psi_3(\xi)$ of Eqs. (9.67) and (9.68) must have the following asymptotic form:

$$\psi_2(\xi)\sim(-\xi)^{1/2} \quad \text{and} \quad \psi_3(\xi)\sim(-\xi)^{3/2}.$$

In exactly the same way, the function $\varphi(\zeta)$ of Eq. (9.70) probably differs little from its asymptotic expression $\varphi(\zeta) \sim (-\zeta)^{1/3}$ even for $-\zeta > 0.1$. In this case, when

$$\tau \gg 0.1|L|/u_* \quad \text{and} \quad \tau \gg \frac{H}{u_*},$$

the following relationships will be valid:

$$\overline{Z(\tau)} \approx a\left(\frac{q}{c_p\rho_0}\frac{g}{T_0}\right)^{1/2}\tau^{3/2}, \tag{9.73}$$

$$\overline{W(\tau)} = \frac{d\overline{Z(\tau)}}{d\tau} \approx \frac{3}{2}a\left(\frac{q}{c_p\rho_0}\frac{g}{T_0}\right)^{1/2}\tau^{1/2} \approx \frac{3}{2}a^{2/3}\left(\frac{q}{c_p\rho_0}\frac{g}{T_0}\right)^{1/3}[\overline{Z(\tau)}]^{1/3}, \tag{9.74}$$

where a is a universal constant the value of which, in principle, may be determined by experiment. Equations (9.73) and (9.74) are quite analogous to the equations for $\overline{X_1(\tau)}$ and $\overline{V_1(\tau)}$ obtained above for the two- and three-dimensional convective jets. The right side of Eq. (9.74) is also very similar to the right side of Eq. (7.87′) for σ_w, but this is explained simply by the fact that the variables σ_w and $\overline{W(\tau)}$ have the same dimensions and depend on almost the same dimensional parameters, except for replacement of the coordinate z by $\overline{Z(\tau)}$. Consequently, the analogy of Eqs. (9.74) and (7.87′) hardly implies that the coefficients C_5' and $(3/2)\,a^{2/3}$ will be close in value. On the contrary, it is natural to expect that the second of these coefficients will be noticeably less than the first, that is, that a will have a value of several tenths.

With respect to the mean horizontal Lagrangian velocity

$$\overline{U(\tau)} = \frac{d\overline{X(\tau)}}{d\tau}$$

under free convection conditions, the situation is more complex. This is because this velocity may not be independent of the parameter u_*; when $u_* = 0$ obviously also $\overline{U(\tau)} = 0$. However, it is clear that the function

$$\varphi_1\left(\frac{\overline{Z(\tau)}}{L}\right) = \varphi_1(\zeta)$$

in Eq. (9.69) in this case will approach a constant value of $f(-\infty)$

with an increase in ζ, that is, with an increase in τ. This is explained by the fact that with time the particle rises into the fluid layer where the horizontal velocity is practically constant. For a more precise evaluation of $\overline{U(\tau)}$ the approximate formula (9.71) may be used with $f(\zeta) = C_2\zeta^{-1/3} + \text{const}$; it is also possible that in some cases it will be expedient to assume that $\varphi_1(\zeta) = C_2'\zeta^{-1/3} + \text{const}$, where C_2' assumes a smaller value than the coefficient C_2 in Eq. (7.39). However, it is natural to assume that the parameter u_* when $\overline{Z(\tau)} > 0.1|L|$ will influence only the average horizontal displacement velocity of the fluid particles, and not the turbulent velocity fluctuations. Therefore, we may expect that for sufficiently large τ, for unstable thermal stratification, we will have the expressions

$$p(U', V', W') =$$
$$= \frac{1}{\left(\frac{q}{c_p\rho_0}\frac{g}{T_0}\tau\right)^{3/2}} P_4\left(\frac{U'}{\left(\frac{q}{c_p\rho_0}\frac{g}{T_0}\tau\right)^{1/2}}, \frac{V'}{\left(\frac{q}{c_p\rho_0}\frac{g}{T_0}\tau\right)^{1/2}}, \frac{W'}{\left(\frac{q}{c_p\rho_0}\frac{g}{T_0}\tau\right)^{1/2}}\right) =$$
$$= \frac{1}{\frac{q}{c_p\rho_0}\frac{g}{T_0}\overline{Z}} P_4'\left(\frac{U'}{\left(\frac{q}{c_p\rho_0}\frac{g}{T_0}\overline{Z}\right)^{1/3}}, \frac{V'}{\left(\frac{q}{c_p\rho_0}\frac{g}{T_0}\overline{Z}\right)^{1/3}}, \frac{W'}{\left(\frac{q}{c_p\rho_0}\frac{g}{T_0}\overline{Z}\right)^{1/3}}\right), \quad (9.75)$$

and

$$p(X', Y', Z') = \frac{1}{\overline{Z}^3} P_5\left(\frac{X'}{\overline{Z}}, \frac{Y'}{\overline{Z}}, \frac{Z'}{\overline{Z}}\right), \quad (9.75')$$

which are related to the expressions (9.54‴) and (9.54⁗) which are satisfied for the turbulent convective jet. From Eqs. (9.73) and (9.75′), in particular, it follows that asymptotically, as $\tau \to \infty$, for unstable stratification $D_{ij}(\tau) \sim \tau^3$ or, more precisely, $D_{ij}(\tau) = \delta_{ij}(qg/c_p\rho_0\tau_0)\tau^3$, where δ_{ij} are universal constants. (It is clear that all these conclusions assume that the conditions for the validity of the "1/3-power law" for the Eulerian mean velocity and temperature are retained at least up to the height $\overline{Z(\tau)}$.)

9.5 The Lagrangian Velocity Correlation Function and Its Relationship to the Eulerian Statistical Characteristics

In the previous section, we have seen that when investigating the motion of a given fluid particle, the Lagrangian velocity correlation function plays a very important role:

$$B_{ij}^{(L)}(t_1, t_2; \boldsymbol{x}) = \overline{V_i(\boldsymbol{x}, t_1) V_j(\boldsymbol{x}, t_2)}.$$

In the following section, data relating to this function will be discussed. Meanwhile, let us note only that these data are quite sparse and inaccurate because of the absence of reliable methods for measuring the Lagrangian statistical characteristics of turbulence. Consequently, it is worthwhile to discuss at least briefly the problem of the possible methods of determining these characteristics theoretically, either directly or beginning with their relationship to the Eulerian statistical characteristics of the random field $\boldsymbol{u}(\boldsymbol{X}, t)$ which have been studied more thoroughly.

The similarity considerations of the present subsection can also be apphed to the statistical characteristics of the relative motion of two fluid particles having fixed coordinates $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ at the initial time t_0. The relative two-particle statistical characteristics are closely related to the process of relative diffusion of a cloud of marked particles in turbulent flow, relative to one particle or to the cloud center of mass, which will be discussed in detail in Sect. 24 of Volume 2 of this book. Here, we shall restrict ourselves to several remarks concerning the characteristics of the two-particle relative displacement vector $l(\tau) = X_2(X_2, t_0 + \tau) - X_1(x_1, t_0 + \tau)$ for a boundary-layer flow. The probabihty distribution of the random vector $l(\tau)$ depends on parameters of the turbulence regime, time τ and the initial coordinates x_1 and x_2, more precisely, on $l_0 = x_2 - x_1$ and z_1 only. However, it is natural to expect that the statistical dependence on the details of the initial position of the particles, i.e., on l_0 and z_1, will be lost if τ is sufficiently large. Thus the asymptotic probability density function and the moments of the vector $l(\tau)$ as $\tau \to \infty$ will satisfy the same equations as the vector $X'(\tau) = [X'(\tau), Y'(\tau), Z'(\tau)]$; in particular, $\overline{l_i(\tau) l_j(\tau)} \sim u_*^2 \tau^2$ in the nonstratified boundary layer, and $\overline{l_i l_j} \sim (qg/c_p \rho_0 T_0)\tau^3$ in the unstable boundary layer [cf. E. E. O'Brien (1966) and Mandell and O'Brien (1967)].

Mandell and O'Brien (1967) have also considered some higher approximations to the asymptotic results obtained with the aid of Taylor series expansions of the universal functions introduced in this subsection. However, the possibility of Taylor expansion of the universal functions about the zero value of their arguments does not follow from dimensional considerations and depends on the selection of a specific form of the arguments. Therefore, these results cannot be considered to have the same rigor as purely dimensional deductions.

The problem of determining Lagrangian statistical characteristics from Eulerian statistical characteristics will clearly not be simple. Indeed, from the basic equations

$$V(\boldsymbol{x},\ t) = \boldsymbol{u}\,[X(\boldsymbol{x},\ t),\ t], \qquad X(\boldsymbol{x},\ t) = \boldsymbol{x} + \int_{t_0}^{t} V(\boldsymbol{x},\ t')\,dt', \tag{9.76}$$

which are actually a definition of the variables $V(\boldsymbol{x}, t)$ and $X(\boldsymbol{x}, t)$, the Lagrangian variables at time t depend on the values of the Eulerian field $\boldsymbol{u}\,(X, t)$ at all points of the random trajectory $X(\boldsymbol{x}\,,\ t')$, $t_0 \leqslant t' \leqslant t$. This trajectory is defined as the solution of a system of integral equations

$$X_i(\boldsymbol{x},\ t') = x_i + \int_{t_0}^{t'} u_i\,[X(\boldsymbol{x},\ t''),\ t'']\,dt'', \qquad i = 1,\ 2,\ 3, \tag{9.77}$$

in turn containing the field $\boldsymbol{u}\,(X,\ t)$. Therefore, generally speaking, the probability distribution for each Lagrangian variable will be dependent on the entire infinite-dimensional probability distribution for the values of $\boldsymbol{u}(X, t)$ in a function space of all possible vector fields. Moreover, even the relationship between the function space distributions of the fields $\boldsymbol{u}\,(X,\ t)$ and $V(\boldsymbol{x},\ t)$, or $\boldsymbol{u}\,(X, t)$ and $X(\boldsymbol{x}, t)$, turns out to be very complex and efforts to explicitly describe it do not lead to results that may be applicable to any specific turbulent flows [see Lumley (1962b)].

Investigation of the probability distributions in function space may be avoided, assuming that the functions $u_i(X,\ t)$, $i = 1, 2, 3$ are analytic in all variables and, therefore, may be expanded in Taylor series. It may be concluded from this that all the Lagrangian variables may be expressed in terms of the Eulerian velocity and all its partial derivatives at a single point in space-time. Expressing $X(\boldsymbol{x},\ t)$ and $V(\boldsymbol{x},\ t)$ as power series

$$X(\boldsymbol{x},\ t) = \sum_{n=0}^{\infty} \frac{1}{n!}\,\frac{\partial^n X(\boldsymbol{x},\ t)}{\partial t^n}\bigg|_{t=t_0}(t - t_0)^n =$$
$$= \boldsymbol{x} + \sum_{n=1}^{\infty} \frac{1}{n!}\,\frac{\partial^n X(\boldsymbol{x},\ t)}{\partial t^n}\bigg|_{t=t_0}(t - t_0)^n, \tag{9.78}$$

$$V(x, t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n V(x, t)}{\partial t^n}\bigg|_{t=t_0} (t - t_0)^n =$$
$$= u(x, t_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n V(x, t)}{\partial t^n}\bigg|_{t=t_0} (t - t_0)^n, \qquad (9.78)\ \text{(cont'd)}$$

it is possible using Eq. (9.76) to transform the coefficients of these series to the form

$$\begin{aligned} &\frac{\partial^n X(x, t)}{\partial t^n}\bigg|_{t=t_0} = \frac{\partial^{n-1} V(x, t)}{\partial t^{n-1}}\bigg|_{t=t_0}, \qquad n = 1, 2, \ldots \\ &\frac{\partial^n V(x, t)}{\partial t^n}\bigg|_{t=t_0} = \left(\frac{\partial}{\partial t} + u_i (X, t) \frac{\partial}{\partial X_i}\right)^n u(X, t)\bigg|_{X=x,\, t=t_0}, \\ &\qquad n = 0, 1, 2, \ldots, \end{aligned} \qquad (9.79)$$

that is, to express them in terms of the Eulerian variables at the point (x, t_0). From Eqs. (9.78)–(9.79) some important general conclusions follow; for example, as was noted by Lumley (1962a), it follows directly that *if the random field $u(X, t)$ is statistically homogeneous, then the random fields $Y(x, t) = X(x, t) - x$ and $V(x, t)$ are also statistically homogeneous*; this fact has actually already been used in Sect. 9.3. However, if we use these formulas to determine the Lagrangian statistical characteristics, the results turn out to be of little importance. For the moments of the fields $Y(x, t)$ and $V(x, t)$, very complex representations are obtained as sums of an infinite number of single-point Eulerian statistical characteristics multiplied by various powers of $t - t_0$. The use of such representations for specific calculations of the Lagrangian characteristics is possible only for very small values of $t - t_0$, for which all but the first few terms of the series (9.78) may be neglected. For example, as applied to the Lagrangian velocity correlation function, only terms of order no higher than $(t - t_0)^2$ have been evaluated successfully, and even for terms of the order $(t - t_0)^2$, the estimate obtained turns out to be quite complex.

Only in the case of terms of zero order with respect to $t - t_0$, that is, at the time t_0, is the situation simple since by definition $V(x, t_0) = u(x, t_0)$. Therefore, for example,

$$B_{ij}^{(L)}(t_0, t_0; x) = \overline{u_i(x, t_0)\, u_j(x, t_0)} = B_{ij}(t_0, t_0; x).$$

Thus, for an incompressible fluid and a statistically homogeneous field $\boldsymbol{u}(\boldsymbol{X}, t)$, and consequently also the field $\boldsymbol{V}(\boldsymbol{x}, t)$, a similar result occurs also for all single-point and single-time characterisitcs of the Lagrangian velocity. Here let us briefly outline its proof according to Lumley (1962a). In the space of points $\boldsymbol{x}$, we select the volume R and investigate an integral of the type

$$\int_R \varphi(\boldsymbol{V}(\boldsymbol{x}, t))\, d\boldsymbol{x},$$

where

$$\varphi(\boldsymbol{V}) = \varphi(V_1, V_2, V_3)$$

is some function of three variables. In this integral we transform from the variables x_1, x_2, x_3 to the new variables X_1, X_2, X_3, where $\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t)$. In the case of an incompressible fluid, and only in this case, such a transformation is considerably simplified by the fact that the corresponding Jacobian

$$\frac{\partial(X_1, X_2, X_3)}{\partial(x_1, x_2, x_3)} = [X_1, X_2, X_3],$$

as we know, is identically equal to unity [see Eq. (9.6)]. Consequently, in this case

$$\int_R \varphi(\boldsymbol{V}(\boldsymbol{x}, t))\, d\boldsymbol{x} = \int_{R_t} \varphi(\boldsymbol{u}(\boldsymbol{X}, t))\, d\boldsymbol{X}, \tag{9.80}$$

where R_t is the space region which is filled at the time t by the fluid filling the region t_0 at the time R. In addition, if the fields $\boldsymbol{V}(\boldsymbol{x}, t)$ and $\boldsymbol{u}(\boldsymbol{X}, t)$ are statistically homogeneous, then the probability mean values of the functions under the integral sign on both sides of Eq. (9.80) do not depend on the coordinates. Nevertheless, we cannot simply average both sides of Eq. (9.80) and take the mean values of the function φ from under the integral sign. This is because the region of integration R_t on the right side is random and depends on the field $\boldsymbol{u}(\boldsymbol{X}, t)$. However, since the space filled by the liquid is unbounded (otherwise it would not be homogeneous), for any fixed t we can select as R, for example, a sphere which has as large a radius as desired. In particular, we can select its radius so large that the region R_t differs from R only in regions of irregular shape located comparatively close to the boundary of R and having a volume which, with high probability, will not exceed a sufficiently small part of the total volume of the sphere R. Thus, with very high probability, we shall introduce only a very small relative error by replacing the integral with respect to R_t on the right side of Eq. (9.80) by a similar integral over the fixed region R. Now, averaging both sides of the approximate equation obtained and dividing the result by the volume R, we find

$$\overline{\varphi(\boldsymbol{V}(\boldsymbol{x},\ t))}=\overline{\varphi(\boldsymbol{u}(\boldsymbol{X},\ t))}, \tag{9.81}$$

where we write the sign of exact equality, since here the relative error may be made as small as desired by selecting a sufficiently large region R. In particular, if $\varphi(V) = e^{i\lambda_\alpha V_\alpha}$, then $\varphi(\boldsymbol{V}(\boldsymbol{x}, t)$ is the characteristic function of the Lagrangian velocity $\boldsymbol{V}(\boldsymbol{x}, t)$, and $\varphi(\boldsymbol{u}(\boldsymbol{X}, t)$ is the characteristic function of the Eulerian velocity $\boldsymbol{X}(\boldsymbol{x}, t)$. Therefore, *for homogeneous turbulence in an incompressible fluid, the probability distributions for the Eulerian and the Lagrangian velocities will coincide with each other at all times.* Thus, in homogeneous turbulence, paradoxes of the type encountered in the case of a boundary-layer flow where

$$\overline{u_3(\boldsymbol{x},\ t)}\equiv 0, \text{ but } \overline{V_3(\boldsymbol{x},\ t)}\neq 0$$

are impossible.

Clearly, all of our arguments refer only to one-point and one-time probabihty distributions; for two-time probability distributions, the statement in italics will, generally speaking, be invalid. Therefore, for the Lagrangian velocity correlation function $B_{ij}^{(L)}(t_1, t_2; \boldsymbol{x})$ we may conclude only that for homogeneous turbulence

$$B_{ij}^{(L)}(t,\ t;\ \boldsymbol{x})=B_{ij}\ (t)=\overline{u_i\ (\boldsymbol{X},\ t)\, u_j\ (\boldsymbol{X},\ t)},$$

but nothing follows regarding the dependence of $B_{ij}^{(L)}(t_1, t_2)$ on t_2-t_1. However, if the velocity field $\boldsymbol{u}(\boldsymbol{X}, t)$ is not only homogeneous but also stationary, then the function $\boldsymbol{V}(\boldsymbol{x}, t)$ will be stationary with respect to t; we have actually also used this circumstance above. Indeed, representing

$$V_j\,(\boldsymbol{x},t+\tau)=u_j[\boldsymbol{X}\,(\boldsymbol{x},t+\tau),\ t+\tau]$$

in the form of a Taylor series (9.78)–(9.79), replacing t_0 by t, and τ by $t+\tau$, it is possible to write the correlation function as the sum of single-point Eulerian characteristics evaluated at the Lagrangian point $(\boldsymbol{X}(\boldsymbol{x}, t), t)$, and multiplied by powers of t. Therefore, if all the Eulerian single-point characteristics do not depend on t due to the stationarity and homogeneity of the field $\boldsymbol{u}(\boldsymbol{x}, t)$, then

$$B_{ij}^{(L)}(t, t+\tau)=B_{ij}^{(L)}(\tau).$$

Equation (9.81) was proved above for the idealized case of stationary homogeneous turbulence in unbounded space and is approximately valid for atmospheric surface layer turbulence; cf. Lumley and Panofsky (1964), pp. 142-144. However, an equality of the same type may also be established for a turbulent flow in an infinitely long straight tube. Thus it is necessary only to take as R a sufficiently long segment of the tube between the cross sections $x_1=a$ and $x_1=b$. For any fixed $\tau=t-t_0$ we can select $b-a$ so large that the region R_t differs from R only by small irregularly shaped regions close to the edges $x_1=a$ and $x_1=b$, the net volume of which is very small compared to the total volume R. After dividing both sides of Eq. (9.80) by the volume R and averaging, we obtain a value on the right side which is essentially the volumetric average of the function $\overline{\varphi(\boldsymbol{u}(\boldsymbol{X}, t))}$ over the cylindrical volume bounded by the cross sections $x_1=a$ and $x_1=b$ and the tube walls. On the left side will be the variable $\overline{\varphi(\boldsymbol{V}(\boldsymbol{x}, t))}$ which for sufficiently large $t-t_0$ may generally be considered as independent of $\boldsymbol{x}$ (compare with the end of Sect. 9.3). In particular, with $\varphi(\boldsymbol{u})=u_1$ in this way we obtain $\overline{V}_1=U_{\mathrm{av}}$ which was discussed before Eq. (9.37).

Let us again consider the simple case of homogeneous and stationary turbulence. In this case $\overline{\boldsymbol{u}}=\boldsymbol{U}=$ const, and without loss of generality, one may even consider, as we shall do in the future, that $\boldsymbol{U}=0$; this is equivalent to a simple transformation to a new inertial system of coordinates. If $B_{ij}^{(L)}(\tau)=\overline{V_i(\boldsymbol{x}, t)V_j(\boldsymbol{x}, t+\tau)}$ is the Lagrangian velocity correlation tensor of this turbulence, and

$$B_{ij}(\xi_1, \xi_2, \xi_3, \tau)=$$
$$=\overline{u_i(x_1, x_2, x_3, t)u_j(x_1+\xi_1, x_2+\xi_2, x_3+\xi_3, t+\tau)}$$

is the time-space Eulerian correlation tensor, then as we know $B_{ij}^{(L)}(0)=B_{ij}(0, 0, 0, 0)$. However, the values of $B_{ij}^{(L)}(\tau)$ for $\tau>0$, generally speaking, cannot be expressed in terms of the values of B_{ij}

$(\xi_1, \xi_2, \xi_3, \tau)$. Due to the absence of reliable data on the function $B_{ij}^{(L)}(\tau)$, some investigators have tried to assume that $B_{ij}^{(L)}(\tau) = B_{ij}(0, 0, 0, \tau)$ [see, for example, Baldwin and Mickelsen (1962) who also compared the results of this assumption with experiment]; however, this assumption has no theoretical basis. It is more natural to assume that the Lagrangian correlation coefficients

$$R_{ii}^{(L)}(\tau) = B_{ii}^{(L)}(\tau) / \overline{V_i'^2}$$

[not summed with respect to i!] which describe the correlation between the ith component of the velocity of the given fluid particle at time t, and at $t+\tau$ will decrease with an increase in τ more slowly than the Eulerian correlation coefficients

$$R_{ii}(\tau) = \frac{B_{ii}(0, 0, 0, \tau)}{\overline{u_i'^2}},$$

describing the correlation between the velocities of the various fluid particles which at time t and at time $t+\tau$ turn out to be at the same point in space. If this is so, then when $\tau > 0$, the following inequality must be fulfilled for all $i = 1, 2, 3$:

$$R_{ii}^{(L)}(\tau) > R_{ii}(\tau).$$

From the last inequality, in particular, it follows that the Lagrangian correlation time, or the Lagrangian time macroscale, or the Lagrangian integral time scale

$$T_i = \int_0^\infty R_{ii}^{(L)}(\tau)\, d\tau$$

must be greater than the Eulerian correlation time, or time macroscale, or integral time scale

$$T_i' = \int_0^\infty R_{ii}(\tau)\, d\tau,$$

and the Lagrangian time microscale, or differential time scale

$$\theta_i = \left[-2 \Big/ \left\{ \frac{\partial^2}{\partial \tau^2} R_{ii}^{(L)}(\tau) \right\}_{\tau=0} \right]^{\frac{1}{2}}$$

in exactly the same way must be greater than the Eulerian time microscale, or differential time scale

$$\theta_i' = [-2/R_{ii}''(0)]^{1/2}.$$

Unfortunately, a direct test of these conclusions is not yet possible due to the absence of data from simultaneous measurement of the Eulerian and Lagrangian time scales under conditions close to homogeneity and stationarity. However, as Corrsin (1963) noted, comparing the measurements of the Eulerian time-space velocity correlations by Favre, Gavigho and Dumas (1953) in a wind-tunnel behind a turbulence-producing grid, with data from the diffusion experiments of Uberoi and Corrsin (1953) carried out under similar conditions, makes it reasonable to assume that under such conditions $\frac{\theta_i'}{\theta_i} \approx 0.7$. When the mean velocity $|\bar{\boldsymbol{u}}| = \bar{U}$ is different from zero and comparatively large, the Eulerian correlation coefficient

$$R_{ii}(\tau) = B_{ii}(0, 0, 0, \tau)/\overline{u_i'^2}$$

may be expected to decrease, with an increase in τ, considerably faster than when $U = 0$. The former describes the relationship between the velocities of fluid particles which are carried with great velocity at times t and $t + \tau$ past a given space point, that is, which are actually at large distance from each other. Actually, assuming that

$$R_{22}(\tau) = R_{22}^{(L)}(\beta\tau) \tag{9.82}$$

where the direction of the Ox_2 axis coincides with the horizontal direction perpendicular to the mean wind, Hay and Pasquill (1959) found from the data on horizontal diffusion in the atmosphere that in the first, quite rough, approximation, $\beta \approx 4$. Subsequent investigations in this area by many researchers [see, for example, Lumley and Panofsky (1964), pp. 197–201; N. Thompson (1965); and Haugen (1966)] show that the empirical estimates of β are extremely scattered, but confirm its mean value [see below, following Eq. (10.11)]. From Eq. (9.82) it follows that

$$T_2 = \beta T_2' \quad \text{and} \quad \theta_2 = \beta\theta_2';$$

therefore, the value of $\beta = 4$ corresponds to the case in which the

Lagrangian time scales significantly exceed the Eulerian time scales.

Equation (9.82), concerning coincidence of the form of the Lagrangian and Eulerian time correlation functions, is frequently used in investigations of turbulent diffusion. It may be considered permissible only because the exact form of the functions $R_{ii}^{(L)}(\tau)$ in many cases turns out to be not particularly important; see, for example, the discussion following Eq. (9.36). In fact, there is no theoretical foundation for this proposition. Moreover, in Chapt. 8, Volume 2 of this book, it will be shown that for very large values of the Reynolds number

$$R_{ii}^{(L)}(\tau) \approx 1 - C_i\tau \quad \text{for} \quad \theta_i^2/T_i \ll \tau \ll T_i;$$

while the Eulerian time correlation function $R_{ii}(\tau)$ for very large Re and $\bar{u} \gg \left(\overline{u'^2}\right)^{1/2}$, which were satisfied in the experiments of Hay and Pasquill, for such τ has the form

$$R_{ii}(\tau) \approx 1 - C_i'\tau^{\frac{2}{3}}$$

but again is linear with respect to τ if $\bar{u} = 0$; compare Corrsin (1963), where from the dependence of the various time-scale ratios on the Reynolds number is evaluated approximately. Thus, when $\bar{u} \gg \left(\overline{u'^2}\right)^{1/2}$ Eq. (9.82) cannot be satisfied for all τ. In general, nothing can be said concerning the behavior of the Lagrangian and Eulerian correlation functions for values of τ which are comparable to the corresponding correlation times. Therefore, the theoretical determination of the form of the functions $R_{ii}^{(L)}(\tau)$ is possible only with special purely empirical or semiempirical hypotheses.

Saffman (1963) made an approximate calculation of the values of the function $R_{ii}^{(L)}(\tau)$ for a specific form of the Eulerian time-space correlation function $R_{ij}(\xi_1, \xi_2, \xi_3, \tau)$, assuming that in expression (9.22) the velocity $V_j(\boldsymbol{X}, \boldsymbol{x}, t)$ for all t may be replaced by the Eulerian velocity $u_j(\boldsymbol{X}, t)$. However, since we have already noted that this assumption is unfounded, we shall not discuss this further. Inoue (1950–1951) and Ogura (1952b) investigated a model of turbulent motion in the form of a set of disturbances of the velocity field of various scales in which the Lagrangian correlation functions $R_{ii}^{(L)}(\tau)$, $i = 1, 2, 3$, all turned out to be given by the following universal formula:

$$R_{ii}^{(L)}(\tau) = 1 - \frac{\tau}{2T_i} \quad \text{for} \quad 0 \leqslant \tau \leqslant 2T_i \quad \text{and} \quad R_{ii}^{(L)}(\tau) = 0 \quad \text{for} \quad \tau > 2T_i.$$

However, Inoue (1952; 1959) later came to the conclusion that another model, in which $R_{ii}^{(L)}(\tau) = e^{-\tau/T_i}$, is more convenient. Grant (1957) used a special mixing-length hypothesis to calculate the function $R_{ii}^{(L)}(\tau)$ and obtained a more complex, but also universal, formula

$$R_{ii}^{(L)}(\tau) = 1 - \frac{\tau}{4T_i} + \frac{\tau}{4T_i} \ln \frac{\tau}{4T_i} \text{ for } 0 \leqslant \tau \leqslant 4T_i \text{ and } R_{ii}^{(L)}(\tau) = 0 \text{ for } \tau > 4T_i.$$

The same result was then obtained by Matsuoka (1960) with a slight modification of Inoue's first model mentioned above. Another model of turbulence was proposed by Wandel and Kofoed-Hansen (1962; 1967). In their model, a universal formula was not obtained for the Lagrangian correlation function, but this function turned out to be expressed quite simply via the Eulerian time correlation function. Still another hypothesis for the relation between Eulerian and Lagrangian characteristics was proposed by Philip (1967). Other relations follow also from a simplified vortex model of turbulence as a system of circular eddies superimposed on the mean flow [cf. for example, Jones (1966)]. Indeed, however, the Lagrangian correlation function does not have the same form in all homogeneous and stationary turbulent flows and may not be uniquely expressable via the Eulerian time or space-time correlation function. Therefore, all the results enumerated here may be considered only as approximations to the real case, the accuracy of which still cannot be estimated due to the absence of reliable data.

10. TURBULENT DIFFUSION

10.1 Problem of the Description of Turbulent Diffusion

In the preceding section the motion of individual "fluid particles" in turbulent flow was investigated. By "fluid particle" we mean here simply a volume of fluid which is so small that within the framework of continuum theory it may be identified with a point moving along with the surrounding fluid. However, it is clear that the results obtained cannot be compared with observation if there are no particles in the fluid which are somehow "tagged" so that their motion may be followed. In other words, the theory developed in the preceding section will be relevant only when certain elements of the fluid are marked; that is, have some properties which are distinguishable from the properties of the surrounding fluid. This difference in properties will occur most frequently due to a difference in chemical composition, or to the presence in the individual volume elements of "foreign substances" which differ from the fluid itself; however, it may also be as simple, for example, as a temperature difference. In any case, to provide "tagged" particles we shall say that there is some admixture in the flow. Here, by admixture we mean those particles of the fluid having special properties which permit their motion to be traced.

If we introduce an admixture only into specific parts of the turbulent flow, then as a result of its transport in disordered intermixed filaments, the ensemble of which constitutes such a flow, it rapidly spreads to the entire volume occupied by the fluid. This phenomenon, called *turbulent diffusion*, is the definitive characteristic of turbulent flows; it is not by chance that in Reynolds' classical experiments the occurrence of turbulence was defined precisely as the phenomenon whereby, for the addition of a small quantity of dye, the entire fluid rapidly becomes colored. It is clear that in addition to turbulent diffusion the admixture usually will participate also in molecular diffusion not related to turbulence. However, this process is comparatively much slower and therefore in the turbulent flows, it plays only a relatively small role. Turbulent diffusion dominates such all-important and well-known phenomena as the spreading of plant pollen in the atmosphere, the spread of bacteria and viruses, radioactive substances, volcanic ash and oceanic salt, air pollution especially in cities due to industrial and transportation smoke and gases, the transfer of moisture evaporated from the surface of the earth, and from all types of water reservoirs, the dispersion of floating objects on the surface of water reservoirs, etc. Thus, it is not surprising that turbulent diffusion has been studied widely; see, for example, Sutton (1953), Frenkiel and Sheppard (1959), and Pasquill (1962b), the review articles by Batchelor and Townsend (1956), Ellison (1959), Monin (1959c), and others.

For the marked fluid particles to be taken as "fluid particles" in the sense of Sect. 9 above, it is only necessary that the marking admixture be passive, or have no influence on the fluid motion, and move with a velocity that is practically coincident with the instantaneous flow velocity at the corresponding point. In particular, it follows that the admixture particles must be sufficiently fine, or smaller in linear dimension than those distances over which the velocity $\boldsymbol{u}(X, t)$ may change appreciably, and so close in density to the surrounding fluid that neither gravitational settling of the admixture nor its buoyant floating up will be significant. Even here the admixture particles still can not be completely identified as ideal "fluid particles." Indeed, any admixture may be dispersed also by molecular diffusion or Brownian motion due to the thermal motion of the fluid molecules while the molecular motion has no influence on the "fluid particles," which are actually "mathematical points" of a continuous medium subject to the fluid dynamic equations. Although this is discussed in more detail in Sect. 10.2, it is not of primary importance and in most cases may be neglected completely.

If the admixture particles may be observed individually, then it is possible by tracing their motion, to determine the individual Lagrangian trajectory $X = X(t)$. Attempts may then be made to evaluate the Lagrangian statistical characteristics of turbulence by averaging the data obtained from a series of such trajectories. This method has gained wide popularity in meteorology in connection with the use of the so-called no-lift balloons, of specially selected weight which enables them to float in the air without going up or down, and the usual air balloons. [See, for example, Pasquill (1962b) where references to the original work may also be found. Experiments with other individual tracers, e.g., soap bubbles, thistledown, etc., have also been carried out; cf. the surveys by Pasquill (1962b), and Lumley and Panofsky (1964).] However, the results thus obtained permit only very preliminary estimates of the Lagrangian velocity correlation function and other Lagrangian characteristics of atmospheric turbulence. This is because the tracers usually can only be followed for a short time and frequently do not follow the air motion precisely. In this last respect, the balloons are most suspect.

As a rule, the admixture is introduced into the flow in the form of a fluid or gaseous additive, or as a large number of fine solid particles. Usually, it may be considered with complete confidence as continuously distributed in space and characterized by the Eulerian field of volumetric concentration $\vartheta(X, t)$. (In a compressible fluid a more convenient characteristic would be the mass specific concentration ϑ/ϱ, but here we shall consider only diffusion in an incompressible fluid.) By the description of turbulent diffusion we shall understand the statistical description of the field $\vartheta(X, t)$ for given initial and boundary conditions including assignment of all the sources of the admixture. In the presence of admixture sources the concentration field $\vartheta(X, t)$ will, generally speaking, be inhomogeneous, and its mathematical expectation, the *mean concentration* $\overline{\vartheta(X, t)}$, will be some function of X and t. The determination of this function is the most important, although not unique, problem of the theory of turbulent diffusion.

To describe turbulent diffusion one may begin with the fact that in each realization of turbulent flow the concentration field $\vartheta(X, t)$ in regions not containing admixture sources, satisfies the molecular diffusion equation

$$\frac{\partial \vartheta}{\partial t} + \frac{\partial u_\alpha \vartheta}{\partial X_\alpha} = \chi \nabla^2 \vartheta \tag{10.1}$$

with the given boundary conditions. Since the admixture is passive, that is, the field $\boldsymbol{u}$ does not depend on the variable ϑ, Eq. (10.1) is linear with respect to ϑ. As a rule, the boundary conditions are also linear with respect to ϑ; usually they have the form

$$\chi \frac{\partial \vartheta}{\partial n} + \beta \vartheta = f(t), \tag{10.2}$$

where n is the normal to the boundary, and β is some constant. For solid walls which bound the flow, the boundary conditions are homogeneous, namely, $f(t)=0$; here the value of $\beta=\infty$ corresponds to the wall completely absorbing the admixture, $\beta=0$ corresponds to the wall being completely impenetrable to the admixture, and values of $0<\beta<\infty$ correspond to the case of partial absorption and partial repulsion of the admixture at the boundary. In the recent works of W. Feller, A. D. Wentzel and several other mathematicians on the general theory of Markov random processes, more general boundary conditions or, in a certain sense, the most general ones, have been investigated. These conditions consider the possibility of the temporary stopping of the admixture as it reaches the boundary, and its diffusion along the boundary [see, for example, Dynkin (1965)]. However, since effects of this type have almost no real significance in the spread of admixtures, we shall not discuss the corresponding boundary conditions which are also linear. For a flow which is unbounded in any direction, the boundary conditions at infinity are usually taken in the form of the requirement $\vartheta \to 0$; that is, they again have the form of Eq. (10.2), with $f(t)=0$ and $\beta=\infty$. The instantaneous sources of the admixture are usually described by initial conditions for $\vartheta(\boldsymbol{X}, t)$; inhomogeneous boundary conditions of the type (10.2) with $f(t) \neq 0$ correspond to continuously active sources. (More details on these conditions for various types of sources will be presented below.)

For homogeneous boundary conditions the evolution of the field $\vartheta(\boldsymbol{X}, t)$ in the given region will be determined exclusively by the transport of the admixture by the velocity field $\boldsymbol{u}(\boldsymbol{X}, t)$ and molecular diffusion. In principle, the velocity field is uniquely determined by the initial field $\boldsymbol{u}(\boldsymbol{X}, t_0) = \boldsymbol{u}_0(\boldsymbol{X})$ with the help of the fluid dynamic equations. Consequently, the solution of Eq. (10.1) for the given initial concentration field $\vartheta(\boldsymbol{X}, t_0) = \vartheta_0(\boldsymbol{X})$ may be written as

$$\vartheta(\boldsymbol{X}, t) = A[\boldsymbol{u}_0(\boldsymbol{X}), t]\, \vartheta_0(\boldsymbol{X}), \tag{10.3}$$

where A is some operator which depends on the initial velocity field $\boldsymbol{u}_0(\boldsymbol{X})$, the parameter t and the form of the boundary conditions. Let us emphasize that due to the linearity of Eq. (10.1) and the boundary conditions, this operator is linear.

For a statistical description of turbulence the initial velocity field $\boldsymbol{u}_0(\boldsymbol{X})$ is considered as random, that is, it is assumed that some corresponding probability distribution exists in the function space of its realizations (all possible solenoidal vector fields). However, in this case, the operator $A[\boldsymbol{u}_0(\boldsymbol{X}), t]$, which depends on a random field $\boldsymbol{u}_0(\boldsymbol{X})$, will also be a random operator determined by some probability distribution in the space of the linear operators which are permissible reahzations of the operator A. Consequently, for a fixed $\vartheta_0(\boldsymbol{X})$, the concentration field $\vartheta(\boldsymbol{X}, t)$ when $t > t_0$ will be random, since it depends on A, that is, on $\boldsymbol{u}_0(\boldsymbol{X})$. Here the mean value of the field $\vartheta(\boldsymbol{X}, t)$ will be defined by the equation

$$\overline{\vartheta(\boldsymbol{X}, t)} = \overline{A[\boldsymbol{u}_0(\boldsymbol{X}), t]\,\vartheta_0(\boldsymbol{X})} = \bar{A}(t)\,\vartheta_0(\boldsymbol{X}), \tag{10.4}$$

where the operator $\bar{A}(t) = \overline{A[\boldsymbol{u}_0(\boldsymbol{X}), t]}$ is obtained from the random operator $A[\boldsymbol{u}_0(\boldsymbol{X}), t]$ by means of probabihty averaging, or integration with respect to the probability measure given in the space of the field $\boldsymbol{u}_0(\boldsymbol{X})$. Since the operators $A[\boldsymbol{u}_0(\boldsymbol{X}), t]$ are linear for any $\boldsymbol{u}_0(\boldsymbol{X})$, and averaging is a linear operation, the resultant mean operator $\bar{A}(t)$ is also linear. Thus, for a fixed initial concentration field $\bar{A}(t)$ the mean concentration $\overline{\vartheta(\boldsymbol{X}, t)}$ satisfies some linear equation. Using Eq. (10.3), equations may also be obtained for the correlation function and for higher-order moments of the field $\vartheta_0(\boldsymbol{X})$; however, these equations will be nonlinear and much more complex, and, moreover, they are rarely investigated [cf., however, Csanady (1967a,b) where the variance of $\vartheta(\boldsymbol{X}, t)$ is studied]. Most of the existing methods for describing turbulent diffusion reduce to the construction of the linear equation for the mean concentration $\overline{\vartheta(\boldsymbol{X}, t)}$ only or, equivalently, to finding the theoretical, semiempirical or purely empirical linear operator $\bar{A}(t)$.

When it is assumed that to determine the mean concentration we may neglect molecular diffusion by comparison with turbulent diffusion (the conditions under which this assumption is valid will be investigated in the following subsection), the operator $\bar{A}(t)$ may be represented in much more specific form than has been done previously. Indeed, let us first investigate the case where at the initial time $t = t_0$, the entire admixture (of quantity Q) is concentrated in one "fluid particle" which is at the point $\boldsymbol{x}$. In this case

$$\vartheta_0(\boldsymbol{X}) = Q\delta(\boldsymbol{X} - \boldsymbol{x}).$$

Since molecular diffusion is negligible, the admixture will at all times remain in the same fluid particle as at the beginning; that is, at time t it will be at the point $\boldsymbol{X}(\boldsymbol{x}, t)$. Consequently,

$$\vartheta(\boldsymbol{X}, t) = Q\delta[\boldsymbol{X} - \boldsymbol{X}(\boldsymbol{x}, t)].$$

Considering that

$$\overline{\delta[\boldsymbol{X} - \boldsymbol{X}(\boldsymbol{x}, t)]} = p(\boldsymbol{X}|\boldsymbol{x}, t),$$

the probability density for the coordinates of the indicated "fluid particle" at the time t, we obtain

$$\bar{A}(t)\delta(\boldsymbol{X} - \boldsymbol{x}) = p(\boldsymbol{X} \mid \boldsymbol{x}, t).$$

Due to the linearity of the operator $\bar{A}(t)$, for an arbitrary initial field $\vartheta_0(\boldsymbol{X})$ we can now use the principle of superposition; consequently, it follows that

$$\overline{\vartheta(\boldsymbol{X}, t)} = \bar{A}(t)\vartheta_0(\boldsymbol{X}) = \int p(\boldsymbol{X}|\boldsymbol{x}, t)\vartheta_0(\boldsymbol{x})\,d\boldsymbol{x}. \qquad (10.5)$$

Thus, finding the mean concentration $\overline{\vartheta(\boldsymbol{X}, t)}$, while neglecting the molecular diffusion, reduces to determining the probability density for the coordinate $\boldsymbol{X}(\boldsymbol{x}, t)$ of one "fluid particle." Let us note that the function $p(\boldsymbol{X}\ \boldsymbol{x}, t)$, on the basis of Eq. (10.5), may itself be interpreted as the concentration field $\overline{\vartheta(\boldsymbol{X}, t)}$, for the presence at the initial time $t = t_0$ at the point $\boldsymbol{x}$ of an instantaneous point source of admixture of unit concentration. Thus derives the possibility of empirically determining $p(\boldsymbol{X}\ \boldsymbol{x}, t)$ from the data of diffusion experiments which was mentioned in Sect. 9.3.

If the operator $\bar{A}(t)$ is known, that is, if it is known how to calculate the mean concentration field $\overline{\vartheta(\boldsymbol{X}, t)}$ corresponding to a given initial field $\vartheta_0(\boldsymbol{X})$, it is also easy to determine the mean concentration corresponding to the different types of admixture sources encountered in practice. Again, let us examine, for example, the case of "purely turbulent" diffusion defined by Eq. (10.5) where the velocity field $\boldsymbol{u}(\boldsymbol{X}, t)$ will, for simplicity, be considered stationary; this corresponds to steady turbulent flow. In this case, the probability density $p(X|\boldsymbol{x}, t_0+\tau)$ for the coordinates of the "fluid particle" at the time $t_0+\tau$ under the condition that at the time t_0 it was at the point $\boldsymbol{x}$, may conveniently be designated by the symbol $p_1(X|\boldsymbol{x}, \tau)$; then in contrast to p_1, the function p will not depend on the parameter t_0. The mean concentration from the instantaneous point source of output Q, i.e., producing Q units of mass of admixture, will equal $Qp_1(X|\boldsymbol{x}, t-t_0)$. For a continuously active stationary point source at the point $\boldsymbol{x}$ of output Q, that is, producing Q units of mass of the admixture per unit time, the mean concentration $\overline{\vartheta(X, t)} = \overline{\vartheta(X)}$, will not depend on time and will be given by the equation

$$\overline{\vartheta(X)} = Q \int_{-\infty}^{t} p_1(X|\boldsymbol{x}, t-t_0)\,dt_0 = Q \int_{0}^{\infty} p_1(X|\boldsymbol{x}, \tau)\,d\tau. \quad (10.6)$$

In practice it is often necessary also to deal with diffusion in a flow having constant mean velocity $\bar{u} = U$ directed along the OX_1 axis; in this case, it is convenient first to transform to a system of coordinates which moves together with the mean flow. Here the mean concentration from the continuously active stationary point source at a point $\boldsymbol{x}$ assumes the form

$$\overline{\vartheta(X)} = Q \int_{0}^{\infty} p_1(X_1-U\tau, X_2, X_3|\boldsymbol{x}, \tau)\,d\tau, \quad (10.6')$$

where $p_1(X_1, X_2, X_3|\boldsymbol{x}, \tau)$ is the probability density for the coordinates of the fluid particle in the absence of a mean velocity. In the case of a linear source located along the OX_2 axis, the mean concentration $\overline{\vartheta(X, t)}$ is obtained by integration, with respect to x_2, of the concentration from the point sources at $(0, x_2, 0)$. Consequently, for stationary turbulence, which is homogeneous in the direction of the OX_2 axis, and in the presence of a mean flow velocity U along the OX_1 axis, the mean concentration from an instantaneous linear source on the OX_2 axis will have the form

$$\overline{\vartheta(X_1, X_3, t)} = Q \int_{-\infty}^{\infty} p_1(X_1-U\tau, X_2, X_3|0, t-t_0)\,dX_2 =$$
$$= Qp_1(X_1-U\tau, X_3|0, t-t_0). \quad (10.7)$$

Here Q is the amount of admixture produced per unit length of the OX_2 axis and is considered constant, that is, independent of X_2, and

$$p_1(X_1, X_3|0, \tau) = \int_{-\infty}^{\infty} p_1(X_1, X_2, X_3|0, \tau)\,dX_2$$

is the two-dimensional probabihty density for the coordinates X_1 and X_3 of the fluid particle at the time $t_0+\tau$, under the condition that at the time t_0 it was at the point $\boldsymbol{x}=0$, in the flow with a zero mean velocity. In exactly the same way, for a continuously active stationary linear source on the OX_2 axis of output Q, that is, producing Q units of mass of admixture per unit time for each unit of length, the mean concentration $\overline{\vartheta(X_1, X_3)}$ equals

$$\overline{\vartheta(X_1, X_3)} = Q\int_0^{\infty} p_1(X_1-U\tau, X_3|0, \tau)\,d\tau. \tag{10.8}$$

Using equations for the moments $\overline{X_i}$ and $\overline{X_iX_j}$ of the coordinates of the "fluid particle" presented in the preceding section, we can now express via the Lagrangian characteristics of turbulence the values of

$$\int X_i\overline{\vartheta(X, t)}\,dX, \quad \int X_iX_j\overline{\vartheta(X, t)}\,dX$$

and other similar quantities. On the other hand, these quantities may be determined also using the mean concentration $\overline{\vartheta(X, t)}$ found from diffusion experiments. This makes it possible to obtain definite empirical information about the Lagrangian characteristics. Ks an example, let us consider the simplest case of a continuously active linear source on the OX_2 axis in a flow with mean velocity U which substantially exceeds the typical values of the fluctuating velocity. In this case the basic contribution to the integral with respect to $d\tau$ on the right side of Eq. (10.8) will be made by the values of τ close to X_1/U. Therefore, with high accuracy we may assume here that

$$\overline{\vartheta(X_1, X_3)} = \text{const}\cdot p_1\left(X_3|0, \frac{X_1}{U}\right), \tag{10.9}$$

where

$$p_1(X_3|0, \tau) = \int_{-\infty}^{\infty} p_1(X_1, X_3|0, \tau)\, dX_1$$

is the one-dimensional probability density for the coordinate X_3 of the fluid particle. On the basis of Taylor's equation (9.31) we may obtain in particular that for homogeneous turbulence with $U \gg \left(\overline{u_1'^2}\right)^{1/2}$

$$\frac{\int_{-\infty}^{\infty} X_3^2 \overline{\vartheta(X_1, X_3)}\, dX_3}{\int_{-\infty}^{\infty} \overline{\vartheta(X_1, X_3)}\, dX_3} = \overline{[X_3(X_1)]^2} = 2\overline{u_3^2} \int_0^{X_1/U} \left(\frac{X_1}{U} - \tau\right) R_{33}^{(L)}(\tau)\, d\tau, \quad (10.10)$$

where, as in Sect. 9,

$$\overline{u_3^2} R_{33}^{(L)}(\tau) = \overline{V_3'(0, t)\, V_3'(0, t+\tau)}.$$

Consequently, the average width of the wake behind a stationary linear admixture source, which naturally is identified with $(\overline{X_3^2})^{1/2}$), at small distances from the source, increases in proportion to this distance since

$$\overline{X_3^2} \approx \frac{\overline{u_3^2} X_1^2}{U^2} \quad \text{for} \quad X_1 \ll UT_3 = U \int_0^{\infty} R_{33}^{(L)}(\tau)\, d\tau.$$

This is well confirmed by the data from the diffusion experiments of Kalinske and Pien (1944), Uberoi and Corrsin (1953), Townsend (1954), and others. Similarly, it has been shown that under the same conditions the wake behind a stationary point source must first have the form of a cone, and then a paraboloid of revolution. This has also been well known for a long time from laboratory experiments and from observations of smoke plumes from factory smokestacks in a strong wind [cf. G. 1. Taylor (1921) and also, the more recent experiments by Orlob (1959)].

According to Eq. (10.10),

$$\frac{d^2\overline{X_3^2}}{dX_1^2} = 2\frac{\overline{u_3^2}}{U^2} R_{33}^{(L)}\left(\frac{X_1}{U}\right), \quad (10.11)$$

so that in principle it is possible also to obtain the function $R_{33}^{(L)}(\tau)$ from the empirical values of $\overline{X_3^2}(X_1)$. Indeed, however, the accuracy of this determination of $R_{33}^{(L)}(\tau)$ turns out to be extremely low. This is due to the necessity of differentiating the experimental curve twice, which, in practice, is always quite poorly defined. Nevertheless, some researchers, for example, G. I. Taylor (1935b), Collis (1948), Barad (1959), and Panofsky (1962) have tried to obtain in this way at least approximate values of the Lagrangian velocity correlation function. Other investigators, specifically, Kalinske and Pien (1944), Uberoi and Corrsin (1953), Mickelsen (1955), Hay and Pasquill (1959), Baldwin and Mickelsen (1962), Högström (1964), N. Thompson (1965), and Haugen (1966), used the measurements of the variable $\overline{X_3^2}$ in the wake behind a stationary linear or point source primarily for obtaining more detailed information on the Lagrangian statistics, and to estimate the Lagrangian integral time scale

$$T = \int_0^\infty R_{33}^{(L)}(\tau)\, d\tau.$$

Thus, assuming that the Eulerian and Lagrangian correlation functions of the lateral wind component are similar to each other, Hay and Pasquill, by this method, deduced the crude estimate $\beta \approx 4$ for the corresponding proportionality coefficient β in the atmospheric surface layer [see text following Eq. (9.32) where this was already mentioned]. Kalinske and Pien (1944), and Uberoi and Corrsin (1953), made laboratory measurements of the values of the function $\overline{X_3^2}(X_1)$ in the wake behind a stationary linear source. From this they estimated the time scale T and some other parameters of the function $R_{33}^{(L)}(\tau)$, and then used these estimates to refine the values of $R_{33}^{(L)}(\tau)$ found by Eq. (10.11); however, even then the form of the function $R_{33}^{(L)}(\tau)$ was poorly defined. Let us also note that in most of the experiments mentioned, the measurements refer to the diffusion of heat (Taylor, Collis, Uberoi, and Corrsin) or of a foreign gas (Mickelsen, Barad, Thompson, Hauger) in air, so that neglecting molecular diffusion here requires special justification, which is absent in all of these works. Similarly in the experiments of Hay and Pasquill, and also in those of Högström where the atmospheric diffusion of lycopodium spores and of smoke were investigated, the Brownian motion of the particles could, in principle, play some part.

In the Kalinske-Pien experiments where diffusion of fine drops of a liquid admixture in water (of the same density as the water) was studied, molecular diffusion could hardly have had any influence, but the assumption of homogeneity of the turbulence appears not to have been very accurate in either case.

As already indicated in Sect. 9.3, for homogeneous turbulence there are substantial reasons for assuming that the probability distribution for the coordinates of the "fluid particle" for any diffusion time will be quite close to a Gaussian distribution. In other words, if we take as coordinate axes the principal axes of the covariance tensor of fluid-particle displacement

$$D_{ij}(\tau) = \overline{[X_i(\tau) - x_i][X_j(\tau) - x_j]},$$

then for any τ we may take

$$p_1(X|x, \tau) = \frac{1}{(2\pi)^{\frac{3}{2}}[D_{11}(\tau)D_{22}(\tau)D_{33}(\tau)]^{1/2}} \times$$
$$\times \exp\left\{-\frac{(X_1-x_1)^2}{2D_{11}(\tau)} - \frac{(X_2-x_2)^2}{2D_{22}(\tau)} - \frac{(X_3-x_3)^2}{2D_{33}(\tau)}\right\}, \quad (10.12)$$

where the variances $D_{ii}(\tau)$ are expressed with the help of Eq. (9.31) in terms of the normed Lagrangian velocity correlation function $R_{ii}^{(L)}(\tau)$. Substituting this equation into the general expressions (10.5)–(10.9), under the not always completely justified assumption that the direction of the mean velocity coincides with one of the principal axes of the tensor $D_{ij}(\tau)$, it is possible to make these expressions substantially more specific, to consider their asymptotic behavior for various limiting cases, to obtain for some of them simpler approximate formulas, and to carry out detailed calculations for specific choice of functions $R_{ii}^{(L)}(\tau)$ [Frenkiel (1952; 1953), Fleischman and Frenkiel (1954); compare also Hinze (1959), Sect. 5.5 and the beginning of Sect. 10.4 of this book].

Often the admixture may be considered not only as a substance continuously distributed in space, but also as a set of discrete particles; sometimes this latter view turns out to be even more convenient. Even though we shall not use such a model in the future, for completeness let us present here some results which touch on the mathematical description of the field of a discrete admixture. Let us designate the total number of admixture particles as N, and their total mass will be taken as unity. We shall consider particles which do not differ from each other and are statistically equivalent in the sense that for any $k \leqslant N$, any spatial regions $V_1, \ldots, V_k$ and any various numbers $i_1, \ldots, i_k$, $1 \leqslant i_s \leqslant N$, the probability

$$P\{X_{i_1} \in V_1, \ldots, X_{i_k} \in V_k\},$$

where $X \in V$ designates that the point X belongs to the region V, does not depend on the numbers $i_1, \ldots, i_k$. From this it follows, in particular, that all the random vectors X_l are identically distributed and that

$$P\{X_{i_1} \in V_1, \ldots, X_{i_k} \in V_k\}$$

is a symmetric function of the regions $V_1, \ldots, V_k$.

The distribution of the discrete admixture at a fixed time will be characterized completely by the random function of the region $\mu(V)$, the value of which is equal to the mass of the admixture contained at the given time in the spatial region V. This random function plays the same role here as the random field $\vartheta(X)$ for a continuously distributed admixture, more exactly, $\mu(V)$ as an analog of the variable $\int\limits_V \vartheta(X)\,dX$. The function $\mu(V)$ obviously is an additive function of the region in the sense that $\mu(V_1 + V_2) = \mu(V_1) + \mu(V_2)$ for nonoverlapping regions V_1 and V_2. Instead of the multidimensional probability distributions which determine the random field $\vartheta(X)$ we now have the probabilities

$$P_{n_1 \ldots n_k}(V_1, \ldots, V_k) = P\left\{\mu(V_1) = \frac{n_1}{N}, \ldots, \mu(V_k) = \frac{n_k}{N}\right\}, \tag{10.13}$$

given for all possible finite sets of the spatial regions $V_1, \ldots, V_k$. In the remainder of this subsection we shall, for simplicity, consider only probabilities (10.13) for regions $V_1, \ldots, V_k$ which are not nonoverlapping. In the special case of independent statistically equivalent admixture particles the distribution (10.13) degenerates into the so-called polynomial distribution

$$P_{n_1 \ldots n_k}(V_1, \ldots, V_k) = C_N^{n_1 \ldots n_k}[Q_1(V_1)]^{n_1} \ldots [Q_1(V_k)]^{n_k} \times \\ \times [1 - Q_1(V_1) - \ldots - Q_1(V_k)]^{n_0},$$

where

$$n_0 = N - n_1 - \ldots - n_k, \quad C_N^{n_1 \ldots n_k} = \frac{N!}{n_1! \ldots n_k!\, n_0!}, \quad Q_1(V) = P\{X_1 \in V\}.$$

Thus, in the case of independent particles the function $\mu(V)$ is given completely by the value of the probability $Q_1(V)$ for all regions V. In the general case of dependent statistically equivalent particles the distribution (10.13) will represent some generahzation of the polynomial distribution; to assign it, it is necessary to know all the probabilities

$$Q_{n_1 \ldots n_k}(V_1, \ldots, V_k) = \\ = P\{X_1 \in V_1, \ldots, X_{n_1} \in V_1; \ldots; X_{n_1 + \ldots + n_{k-1} + 1} \in V_k, \ldots, X_{n_1 + \ldots + n_k} \in V_k\}. \tag{10.14}$$

It is possible to show that any moment of the random variables $\mu(V_1), \ldots, \mu(V_k)$ may be expressed in terms of the probabilities (10.14) with the help of the formula

$$\prod_{s=1}^{k} [\mu(V_s)]^{m_s} = \frac{N!}{N^m} \sum_{n_1+\ldots+n_k \leqslant m} \frac{a_{n_1}^{m_1} \ldots a_{n_k}^{m_k}}{n_0!} Q_{n_1 \ldots n_k}(V_1, \ldots, V_k), \tag{10.15}$$

where $m = m_1 + \ldots + m_k$ is the order of the moment, and a_n^m are the so-called Stirling numbers which differ from zero only when $0 < n \leqslant m$ and satisfy the difference equation $a_n^m = a_{n-1}^{m-1} + na_n^{m-1}$ with the boundary conditions $a_m^m = a_1^m = 1$. Thus, in particular, it follows that the mth-order moment of the random function $\mu(V)$ is defined completely by the joint probability distribution for the coordinates of m particles, in other words, that the probabilities (10.4) with $n_1+\ldots+n_k \leqslant m$ define the random function $\mu(V)$ with accuracy up to moments of the mth order.

When $m = 1$, from Eq. (10.15) we obtain

$$\overline{\mu(V)} = Q_1(V), \tag{10.16}$$

which is analogous to Eq. (10.5), more exactly, to the equation obtained from Eq. (10.5) after integration in X of both sides of it over the region V. When $m = 2$, from Eqs. (10.15) and (10.16) we obtain

$$\overline{[\mu(V) - \overline{\mu(V)}]^2} = Q_2(V) - [Q_1(V)]^2 + \frac{1}{N}[Q_1(V) - Q_2(V)], \tag{10.17}$$

$$\overline{[\mu(V_1) - \overline{\mu(V_1)}][\mu(V_2) - \overline{\mu(V_2)}]} = Q_{11}(V_1, V_2) - Q_1(V_1)Q_1(V_2) - \frac{1}{N}Q_{11}(V_1, V_2). \tag{10.18}$$

The first of these equations shows that in the presence of statistical interaction between particles, or when $Q_2(V) \neq [Q_1(V)]^2$), the fluctuations of the amount of admixture in the fixed volume V do not decrease indefinitely (in quadratic mean) even when $N \to \infty$. In other words, in the case of mutually dependent particles, even for a very large number of particles N, the values of $\mu(V)$ turn out to be quite different for different realizations. Equation (10.18) shows that, in addition, in this case the fluctuations in the amount of admixture in two different, nonoverlapping regions are mutually correlated. Significantly, mutual dependence between the particles is typical of turbulent diffusion; in fact, here the motions of neighboring particles are influenced considerably by the same disturbances in the velocity field, and therefore are statistically related to each other. Thus for turbulent diffusion the concentration of the admixture in each volume fluctuates noticeably, and for adjacent, but nonoverlapping volumes may turn out to be dependent on each other for any N and even when $N = \infty$. This last case is also described by the model of a continuously distributed admixture with a concentration $\vartheta(X)$ which is continuously dependent on X.

10.2 Interaction Between Molecular and Turbulent Diffusion

The admixture propagation in a turbulent fluid due to its transport by the moving fluid particles, that is, turbulent diffusion, usually takes place more rapidly than its spread caused by molecular motion, that is, molecular diffusion. Thus, for example, in the atmospheric surface layer the virtual "eddy diffusivity" for passive admixtures, namely, the ratio of the turbulent flux of the admixture to the concentration gradient, usually has values on the order of 10^4-10^5 cm^2 sec^{-1}. In other words, it turns out to be 10^5-10^6 times greater than the molecular diffusivity, which for most gases is on the order of 10^{-1} cm^2 sec^{-1}. Due to turbulent diffusion, cigarette smoke in concentrations sufficient to affect the human sense of smell, is spread around a room in several seconds, while the effect of only molecular duffusion would lead to the same result after only several days. However, does this indicate that for a description of turbulent diffusion it is possible to completely neglect molecular diffusion?

The transport of the admixture by the moving fluid particles obviously leads to a situation in which the volume first occupied by the admixture is extended in certain directions, and in others is compressed, being distorted with considerable confusion. (This will be discussed in more detail in Chapt. 8 of Volume 2.) However, the total volume of admixture does not change with this deformation. As a result, in each realization of turbulent flow the specific concentration $\vartheta(\boldsymbol{X}, t)$ at each point and any time will be equal to either the density of the admixture ρ_0 or 0, depending on whether or not at the initial time $t=t_0$ the corresponding fluid particle contains the admixture. [See Fig. 79 for a schematic diagram borrowed from the review article by Corrsin (1959a).] Thus turbulent diffusion leads to the formation in the fluid of highly distorted and twisted layers with sharply varying values of admixture concentration. The equalization

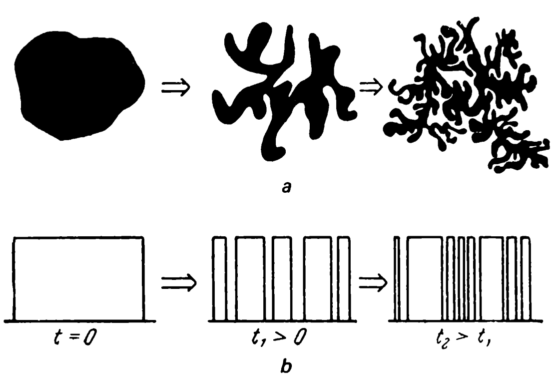


FIG. 79. a–Variation in form of the volume occupied by an admixture due to turbulent diffusion; b–schematic form of distribution of admixture concentration along a straight line intersecting this volume.

of the concentration in adjacent layers, which is accompanied by an increase in the volume occupied by the admixture, and a smoothing of the concentration field, as a result of which the values of the function $\vartheta(X, t)$ as a rule turn out to be intermediate between ρ_0 and 0, occurs only as a result of molecular diffusion, and is the slower, the smaller the diffusivity χ. It is clear from this that for the description of the small-scale statistical structure of the field $\vartheta(X, t)$ it is impossible to neglect molecular diffusion; otherwise when t is large, we obtain a rather unnatural distribution of concentration of the type depicted on the right side of Fig. 79. However, the problem of whether it is impossible to neglect the molecular diffusion, in comparison with turbulent diffusion, when finding the mean concentration $\overline{\vartheta(X, t)}$, as was done in the preceding subsection for derivation of Eq. (10.5), is quite complex. We shall now deal with this problem.

To investigate the relative role of molecular and turbulent diffusion in the formation of the mean concentration field $\overline{\vartheta(X, t)}$, for simplicity, we shall limit ourselves to the problem of diffusion of an admixture from an instantaneous point source of unit output in a field of homogeneous turbulence with zero mean velocity. We shall consider this source as being placed at the time $t = t_0$ at the point $X = 0$. In other words, we shall consider the concentration $\vartheta(X, t)$ corresponding to the solution of Eq. (10.1) for the initial condition $\vartheta(X, t_0) = \delta(X)$. Let us restrict our consideration to the admixture diffusion in some single fixed direction, for example, in the direction of the OX_1 axis. As the characteristic of the role of molecular diffusion, we shall use the relative contribution of this diffusion to the dispersion of the mean concentration distribution $\overline{\vartheta(X, t)}$. Here it is essential to distinguish, on the one hand, *the dispersion of the mean concentration distribution with respect to the location of the sources* $X = 0$

$$D_0^2(t) = \int X_1^2 \overline{\vartheta(X, t)}\, dX \tag{10.19}$$

and, on the other, *the dispersion* (also with respect to the location of the source $X = 0$) *of the center of gravity*

$$X_c(t) = \int X_1 \vartheta(X, t)\, dX$$

of the concentration distribution $\vartheta(X, t)$, that is, the variance

$$\overline{X_c^2(t)} = \overline{\left[\int X_1 \vartheta(X, t)\, dX\right]^2}. \tag{10.20}$$

In addition, *the dispersion of the concentration distribution* $\vartheta(X, t)$ *with respect to its center of gravity* is also of interest:

$$D_c^2(t) = \overline{\int [X_1 - X_c(t)]^2 \vartheta(X, t)\, dX}. \tag{10.21}$$

However, using the normalization condition

$$\int \vartheta(X, t)\, dX = 1$$

which expresses the law of conservation of the admixture mass, it is not difficult to prove that

$$D_c^2(t) = D_0^2(t) - \overline{X_c^2(t)}, \tag{10.22}$$

so that it is sufficient for us to calculate only $D_0^2(t)$ and $\overline{X_c^2(t)}$.

Further, it will be convenient for us to transform from the stationary system of coordinates X to the nonstationary, noninertial system of coordinates $Y = X - X(0, t)$, the origin of reference of which, at each time coincides with the point where the fluid particle is found, which at the time t_0 was at the point $X = 0$. For the concentration field in this new system of reference, we shall adopt the notation

$$\theta(Y, t) = \vartheta[Y + X(0, t), t] = \vartheta(X, t). \tag{10.23}$$

It is clear that

$$\frac{\partial\theta(Y, t)}{\partial Y_i} = \left.\frac{\partial\vartheta(X, t)}{\partial X_i}\right|_{X = Y + X(0, t)}.$$

On the other hand,

$$\frac{\partial\theta(Y, t)}{\partial t} = \frac{d}{dt}\vartheta[Y + X(0, t), t] = \frac{\partial\vartheta}{\partial t} + V_\alpha(0, t)\frac{\partial\vartheta}{\partial X_\alpha}, \tag{10.24}$$

where $V(0, t) = \frac{dX(0, t)}{dt}$ is the Lagrangian velocity of the fluid particle which at the time t_0 was at the point $X = 0$. Consequently, Eq. (10.1) in this mobile system of coordinates takes the form

$$\frac{\partial\theta}{\partial t}+\{u_i[Y+X(0,\ t),\ t]-V_i(0,\ t)\}\frac{\partial\theta}{\partial Y_i}=\chi\nabla^2\theta, \quad (10.25)$$

where ∇^2 is the Laplace operator with respect to the variables $(Y_1,\ Y_2,\ Y_3)$, and $V(0,\ t)$ is the velocity of the coordinate origin of our mobile system with respect to the initial stationary system of coordinates. The concentration $\theta(Y,\ t)$ also satisfies the normalization condition

$$\int\theta(Y,\ t)\,dY=1$$

which we shall use without further comment. Furthermore, Eqs. (10.19) and (10.20), after transformation to the new system of coordinates, assume the forms, respectively,

$$\begin{aligned}D_0^2(t)&=\overline{\int[Y_1+X_1(0,\ t)]^2\,\theta(Y,\ t)\,dY}=\\&=\overline{X_1^2(0,\ t)}+2\overline{X_1(0,\ t)\int Y_1\theta(Y,t)\,dY}+\overline{\int Y_1^2\theta(Y,\ t)\,dY},\end{aligned} \quad (10.26)$$

and

$$\begin{aligned}\overline{X_c^2(t)}&=\overline{\left\{\int[Y_1+X_1(0,\ t)]\,\theta(Y,\ t)\,dY\right\}^2}=\\&=\overline{X_1^2(0,\ t)}+2\overline{X_1(0,\ t)\int Y_1\theta(Y,t)\,dY}+\overline{\left[\int Y_1\theta(Y,t)\,dY\right]^2}.\end{aligned} \quad (10.27)$$

We shall limit ourselves to the calculation of the values of $D_0^2(t)$ and $\overline{X_c^2(t)}$ for small values of $t-t_0$. We shall assume that all the moments

$$\langle Y_1^{k_1}Y_2^{k_2}Y_3^{k_3}\rangle=\int Y_1^{k_1}Y_2^{k_2}Y_3^{k_3}\,\theta(Y,\ t)\,dY \quad (10.28)$$

(where k_1, k_2, k_3 are nonnegative exponents) may be represented as Taylor series in integral powers of $t-t_0$. Then the dispersions $D_0^2(t)$ and $\overline{X_c^2(t)}$ will also be representable in the form of a series in powers of $t-t_0$, where for calculation of the first few terms of the series, it is necessary to know only the first few terms of the series for the moments $\langle Y_1 \rangle$ and $\langle Y_1^2 \rangle$ which enter into Eqs. (10.26) and (10.27).

Let us use Eq. (10.25) to calculate the moments (10.28). Here we have $u_i[X(0, t), t] = V_i(0, t)$, and, consequently, the velocity difference in the braces on the left side of Eq. (10.25) may be represented in the form

$$u_i[Y+X(0, t), t] - V_i(0, t) =$$
$$= Y_\alpha \frac{\partial u_i}{\partial X_\alpha} + \frac{1}{2} Y_\alpha Y_\beta \frac{\partial^2 u_i}{\partial X_\alpha \partial X_\beta} + \frac{1}{6} Y_\alpha Y_\beta Y_\gamma \frac{\partial^3 u_i}{\partial X_\alpha \partial X_\beta \partial X_\gamma} + \cdots, \quad (10.29)$$

where all the derivatives of $u_i(X, t)$ are taken at the point $X = X(0, t)$, that is, are functions only of t. Let us now substitute the expansion (10.29) into Eq. (10.25), then multiply all of its terms by $Y_1^{k_1} Y_2^{k_2} Y_3^{k_3}$ and integrate with respect to Y. Taking into account that $\frac{\partial u_i}{\partial X_i} \equiv 0$, after integration by parts of the terms containing the spatial derivatives of $0(Y, t)$, we obtain

$$\frac{\partial}{\partial t} \langle Y_1^{k_1} Y_2^{k_2} Y_3^{k_3} \rangle = \chi \sum_{i=1}^{3} k_i (k_i - 1) \left\langle \frac{Y_1^{k_1} Y_2^{k_2} X_3^{k_3}}{Y_i^2} \right\rangle +$$
$$+ \sum_{i=1}^{3} k_i \frac{\partial u_i}{\partial X_\alpha} \left\langle \frac{Y_\alpha Y_1^{k_1} Y_2^{k_2} Y_3^{k_3}}{Y_i} \right\rangle + \frac{1}{2} \sum_{i=1}^{3} k_i \frac{\partial^2 u_i}{\partial X_\alpha \partial X_\beta} \left\langle \frac{Y_\alpha Y_\beta Y_1^{k_1} Y_2^{k_2} Y_3^{k_3}}{Y_i} \right\rangle +$$
$$+ \frac{1}{6} \sum_{i=1}^{3} k_i \frac{\partial^3 u_i}{\partial X_\alpha \partial X_\beta \partial X_\gamma} \left\langle \frac{Y_\alpha Y_\beta Y_\gamma Y_1^{k_1} Y_2^{k_2} Y_3^{k_3}}{Y_i} \right\rangle + \ldots, \quad (10.30)$$

where the moments which contain negative powers of Y_i are taken identically equal to zero.

It is obvious from Eq. (10.28) that the zero-order moment (with $k_1 = k_2 = k_3 = 0$) is exactly equal to one; since $0(Y, t_0) = \delta(Y)$ all the remaining moments approach zero as $t \to t_0$. Therefore, terms linear in $t - t_0$ will appear only in the expressions of the moments for which the right side of Eq. (10.30) contains a zero-order moment. It is not difficult to see that the only moments which have this property are the moments $\langle Y_i^2 \rangle$ and their leading terms have the form

$$\langle Y_i^2 \rangle = 2\chi(t - t_0) + \cdots. \quad (10.31)$$

However, to determine the subsequent terms of the power series for the moment $\langle Y_1^2\rangle$ which we need below, we must also write out all the moments which are of order $(t-t_0)^2$ when $t \to t_0$.

It is clear that those moments of nonzero order, different from $\langle Y_i^2\rangle$ for which the moments $\langle Y_i^2\rangle$ may be found among the summands on the right side of Eq. (10.30), will be of second order in $t-t_0$. It is easy to prove that only the moments

$$\langle Y_i\rangle,\ \langle Y_i^4\rangle \quad \text{and} \quad \langle Y_iY_j\rangle,\ \langle Y_i^2Y_j^2\rangle \quad \text{for} \quad i \neq j$$

are of this sort. Moreover, their leading terms have the form

$$\begin{aligned}
\langle Y_i\rangle &= \frac{1}{2}\chi\,\Delta u_i\,(t-t_0)^2+\ \ldots,\\
\langle Y_iY_j\rangle &= \chi\left(\frac{\partial u_i}{\partial X_j}+\frac{\partial u_j}{\partial X_i}\right)(t-t_0)^2+\ \ldots,\\
\langle Y_i^4\rangle &= 12\chi^2\,(t-t_0)^2+\ \ldots,\\
\langle Y_i^2Y_j^2\rangle &= 4\chi^2\,(t-t_0)^2+\ \ldots,
\end{aligned} \tag{10.32}$$

where all the derivatives of $\boldsymbol{u}(\boldsymbol{X}, t)$ are taken at the space-time point $\boldsymbol{X}=0, t=t_0$.

Let us consider the evaluation of $D_0^2(t)$ and $\overline{X_c^2(t)}$, with accuracy up to terms of order $(t-t_0)^3$. Since

$$\boldsymbol{X}(0, t)=\boldsymbol{u}(0, t_0)(t-t_0)+\ \ldots, \tag{10.33}$$

for this we must know the moment $\langle Y_1\rangle$ with accuracy up to terms of order $(t-t_0)^2$, and the moment $\langle Y_1^2\rangle$ with accuracy up to terms of order $(t-t_0)^3$. Thus, Eq. (10.32) for $\langle Y_1\rangle$ is sufficient for our purposes, but the expansion (10.31) for $\langle Y_1^2\rangle$ must be refined. To determine the missing terms, let us write out Eq. (10.30) separately for $k_1=2$, $k_2=k_3=0$:

$$\begin{aligned}
\frac{\partial}{\partial t}\langle Y_1^2\rangle = 2\chi+2\frac{\partial u_1}{\partial X_\alpha}\langle Y_\alpha Y_1\rangle+\frac{\partial^2 u_1}{\partial X_\alpha\,\partial X_\beta}\langle Y_\alpha Y_\beta Y_1\rangle+\\
+\frac{1}{3}\frac{\partial^3 u_1}{\partial X_\alpha\,\partial X_\beta\,\partial X_\gamma}\langle Y_\alpha Y_\beta Y_\gamma Y_1\rangle+\ \ldots\,.
\end{aligned}$$

Considering on the right side of this formula only terms of order not greater than $(t-t_0)^2$, with Eq. (10.32) we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}\langle Y_1^2\rangle &= 2\chi + 2\frac{\partial u_1}{\partial X_1}\langle Y_1^2\rangle + 2\frac{\partial u_1}{\partial X_2}\langle Y_1 Y_2\rangle + 2\frac{\partial u_1}{\partial X_3}\langle Y_1 Y_3\rangle + \\
&+ \frac{1}{3}\frac{\partial^3 u_1}{\partial X_1^3}\langle Y_1^4\rangle + \frac{\partial^3 u_1}{\partial X_1\,\partial X_2^2}\langle Y_1^2 Y_2^2\rangle + \frac{\partial^3 u_1}{\partial X_1\,\partial X_3^2}\langle Y_1^2 Y_3^2\rangle + \ldots = \\
&= 2\chi + 2\frac{\partial u_1}{\partial X_1}\langle Y_1^2\rangle + 2\chi\left[\frac{\partial u_1}{\partial X_2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) + \right. \\
&+ \left.\frac{\partial u_1}{\partial X_3}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right)\right](t-t_0)^2 + 4\chi^2\frac{\partial \nabla^2 u_1}{\partial X_1}(t-t_0)^2 + \ldots .
\end{aligned}
$$

Since $\frac{\partial u_1}{\partial X_1} = \left.\frac{\partial u_1}{\partial X_1}\right|_{t=t_0} + \left.\frac{d}{dt}\frac{\partial u_1}{\partial X_1}\right|_{t=t_0}(t-t_0) + \ldots$, we have

$$
\begin{aligned}
\langle Y_1^2\rangle &= 2\chi(t-t_0) + 2\chi\frac{\partial u_1}{\partial X_1}(t-t_0)^2 + \frac{2}{3}\chi\left\{2\frac{d}{dt}\frac{\partial u_1}{\partial X_1} + 2\left(\frac{\partial u_1}{\partial X_1}\right)^2 + \right. \\
&+ \left.\frac{\partial u_1}{\partial X_2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) + \frac{\partial u_1}{\partial X_3}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right) + 2\chi\frac{\partial \nabla^2 u_1}{\partial X_1}\right\}(t-t_0)^3 + \ldots,
\end{aligned}
\tag{10.34}
$$

where all the derivatives of $\boldsymbol{u}(\boldsymbol{X}, t)$ are taken at the point $\boldsymbol{X} = 0$, $t = t_0$.

Further, we need only the mean value of the moment $\langle Y_1^2\rangle$. Let us recall that we are considering homogeneous turbulence where mean values of all variables which depend on only a single space-point $\boldsymbol{X}$ are constant. In particular,

$$
\overline{\frac{d}{dt}\frac{\partial u_1}{\partial X_1}} = \frac{\partial}{\partial X_\alpha}\overline{\frac{\partial u_1}{\partial t}} + \frac{\partial}{\partial X_\alpha}\overline{u_\alpha\frac{\partial u_1}{\partial X_1}} = 0.
$$

Thus, in homogeneous turbulence

$$
\begin{aligned}
\overline{\langle Y_1^2\rangle} = 2\chi(t-t_0) + \frac{2}{3}\chi\left\{2\overline{\left(\frac{\partial u_1}{\partial X_1}\right)^2} + \overline{\frac{\partial u_1}{\partial X_2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right)} + \right. \\
+ \left.\overline{\frac{\partial u_1}{\partial X_3}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right)}\right\}(t-t_0)^3 + \ldots .
\end{aligned}
$$

The expression in braces may easily be converted to the following form:

$$
\overline{\frac{\partial u_1}{\partial X_\alpha}\left(\frac{\partial u_1}{\partial X_\alpha} + \frac{\partial u_\alpha}{\partial X_1}\right)} = \overline{(\nabla u_1)^2} + \frac{\partial}{\partial X_\alpha}\overline{u_1\frac{\partial u_\alpha}{\partial X_1}} = \overline{(\nabla u_1)^2},
$$

and we obtain finally

$$\overline{\langle Y_1^2\rangle} = 2\chi(t-t_0) + \frac{2}{3}\chi\overline{(\nabla u_1)^2}(t-t_0)^3 + \cdots. \tag{10.35}$$

Let us also note that according to Eqs. (10.22), (10.26), (10.27), and (10.32), the variable $D_c^2(t)$ with accuracy up to terms of order $(t-t_0)^3$ coincides with $\overline{\langle Y_1^2\rangle}$; therefore

$$D_c^2(t) = 2\chi(t-t_0) + \frac{2}{3}\chi\overline{(\nabla u_1)^2}(t-t_0)^3 + \cdots. \tag{10.36}$$

In an unmovable fluid, molecular diffusion transforms unit mass of admixture, concentrated at a point at time $t = t_0$, into a cloud described by a spherically symmetrical concentration distribution with dispersion $2\chi(t-t_0)$. It is obvious from Eq. (10.36) that when $t-t_0$ is so small that the second term on the right side of this equation may be neglected by comparison with the first, the average dispersion of the admixture distribution $\vartheta(\boldsymbol{X}, t)$ about its center of gravity $X_c(t)$ may also be assumed to be equal to $2\chi(t-t_0)$. In other words, one may consider (in any case, to the extent to which this has to do with dispersion) that in the first stage of turbulent diffusion from a point source, spreading of the admixture under the action of molecular diffusion is superimposed on its transport by the corresponding fluid particle, but does not interact with this transport. However, the case is different for somewhat larger values of $t-t_0$ because of the presence of the second term on the right side of Eq. (10.36). As a result, the dispersion $D_c^2(t)$ begins to increase more rapidly than for molecular diffusion in an unmoving fluid; the additional term here depends not only on the fluid molecular diffusivity χ, but also on the mean square velocity gradient of the turbulent motion. Thus, an interaction between molecular and turbulent diffusion occurs which causes acceleration of the molecular diffusion; that is, more rapid expansion of the volume occupied by the admixture, and, correspondingly, more rapid decrease in the maximum concentration. The actual cause of this phenomenon is obviously the sharp deformation of the volume occupied by the admixture during turbulent flow (of the type shown in Fig. 79), essentially increasing the role of molecular diffusion. Therefore, it is not surprising that the "acceleration of molecular diffusion" is not felt immediately, but only some time after the ordinary molecular diffusion transforms the point mass of admixture into a cloud, the volume of which is sufficient for deformation to occur under the influence of the instantaneous velocity inhomogeneities characterized by the gradient of $\boldsymbol{u}(X, t)$.

The effect of molecular diffusion acceleration due to turbulence was first noted by Townsend (1951). He gave preliminary estimates for this effect and presented results from special experiments (measurement of the gradual drop in the maximum temperature of thermal spots produced by a pulse discharge of current in a turbulent flow) confirming its existence. Moreover, Townsend indicated that equations of the type (10.36) provide an adequate description for significantly larger values of $t - t_0$ than was supposed earlier. More complete equations for diffusion from an instantaneous linear source [containing some minor errors in the values of the numerical coefficients; see Saffman (1960)] were obtained later by Townsend (1954), and Batchelor and Townsend (1956). Finally, the second term on the right side of Eq. (10.35) strongly resembles Eq. (9.58) for the variance of the longitudinal coordinate of a fluid particle in a homogeneous turbulent shear flow with a constant mean velocity gradient. This similarity is not accidental; for small $t - t_0$ the entire admixture is within a small neighborhood of the source, in which the velocity $\boldsymbol{u}(X, t)$ may be assumed to be linearily dependent on the coordinate $\boldsymbol{X}$, limiting ourselves to the first term of the corresponding Taylor series, and Eq. (10.35) corresponds to this representation of the velocity field.

So far, we have considered only the dispersion $D_\vartheta^2(t)$ of the distribution $\vartheta(\boldsymbol{X}, t)$ about its center of gravity. Proceeding to the consideration of the dispersion $D_\vartheta^2(t)$ of an admixture cloud about the position of the source, coinciding with the mean center of gravity location $\overline{X_c(t)}$ and $\overline{X_\vartheta^2}(t)$ of the center of gravity itself, then we find that here the interaction between molecular and turbulent diffusion leads to the opposite effect of "deceleration of diffusion." Indeed, to calculate the quantities $\overline{D_\vartheta^2}(t)$ and $\overline{X_c^2}(t)$ according to Eqs. (10.26) and (10.27), it is necessary only to determine the mean value $\overline{X_1(0, t)\langle Y_1\rangle}$. Using Eqs. (10.33) and (10.32) for $X_1(0, t)$ and $\langle Y_1\rangle$, we obtain

$$\overline{X_1(0, t)\langle Y_1\rangle} = \frac{1}{2}\chi\overline{u_1\nabla^2 u_1}(t - t_0)^3 + \ldots$$

or, since in homogeneous turbulence $\overline{u_1\nabla^2 u_1} = -\overline{(\nabla u_1)^2}$,

$$\overline{X_1(0, t)\langle Y_1\rangle} = -\frac{1}{2}\chi\overline{(\nabla u_1)^2}(t - t_0)^3 + \ldots . \tag{10.37}$$

From this and from Eqs. (10.26), (10.27), (10.35), and (10.32) it follows that

$$D_0^2(t) = \overline{X_1^2(0,\ t)} + 2\chi(t - t_0) - \frac{1}{3}\chi\overline{(\nabla u_1)^2}(t - t_0)^3 + \ldots, \tag{10.38}$$

$$\overline{X_c^2(t)} = \overline{X_1^2(0,\ t)} - \chi\overline{(\nabla u_1)^2}(t - t_0)^3 + \ldots. \tag{10.39}$$

These equations were first obtained by Saffman (1960); see also Okubo (1967) who deduced them with the aid of the Lagrangian diffusion equation (9.9′). The second of these equations shows that the variance of the admixture center of gravity for small $t - t_0$ turns out to be less than the variance $\overline{X_1^2}(0,\ t)$ of the coordinate of the fluid particle, determined almost exclusively for small $t - t_0$ by the variance of the initial velocity $V_1(0,\ t_0) = u_1(0,\ t_0)$ of the fluid particle, that is, close to $\overline{u_1^2}(t - t_0)^2$. In other words, due to molecular diffusion, the admixture particles on the average stray away from the fluid particle with which they first coincided, and, therefore, the molecular diffusion slows down the turbulent diffusion. The mechanism for this deceleration is easily seen: the quantity $\overline{X_1^2}(0,\ t)$ is determined by the variance of the "average rate of transport of the fluid particle"

$$\frac{1}{t-t_0}\int_{t_0}^{t} u_1[\boldsymbol{X}(t'),\ t']\,dt',$$

and molecular diffusion leads to spread of the point fluid particle into the entire admixture cloud. Therefore, here, the rate of transport is the velocity $u_1(\boldsymbol{X},\ t')$ averaged with respect to some volume, which as a rule is less than the instantaneous velocity at the point $\boldsymbol{X}(0,\ t')$. Since $\overline{X_c^2}(t)$ turns out to be less than $\overline{X_1^2}(0,\ t)$, the total dispersion $D_0^2(t)$ turns out to be less than the sum $\overline{X_1^2}(0,\ t) + 2\chi(t - t_0)$ of the individual contributions from the turbulent and molecular diffusion without taking into account their interaction.

The deceleration of turbulent diffusion due to molecular diffusion may also be described with the help of the "admixture correlation function" $B^{(\vartheta)}(t - t_0)$ introduced by Saffman (1962a). The function $B^{(\vartheta)}(t - t_0)$ may be defined as the average covariance of the fluid velocities at the points occupied by the admixture at the times $t > t_0$ and t_0:

$$B^{(\vartheta)}(t - t_0) = \overline{\langle u_1(0,\ t_0)\,u_1[\boldsymbol{Y} + \boldsymbol{X}(0,\ t),\ t]\rangle}, \tag{10.40}$$

where the angular brackets, as above, symbolize averaging with respect to the values of Y with the weight function $\vartheta(Y, t)$. Saffman showed that the dispersion $D_0^2(t)$ may be expressed in terms of $B^{(\vartheta)}(t-t_0)$ with the aid of the following equation:

$$D_0^2(t) = 2\chi(t-t_0) + 2\int_{t_0}^{t}(t-t_0-\tau)\, B^{(\vartheta)}(\tau)\, d\tau. \tag{10.41}$$

Equation (10.41) is similar to Taylor's equation (9.31) which has the form

$$\overline{X_1^2(0, t)} = 2\int_{t_0}^{t}(t-t_0-\tau)\, B^{(L)}(\tau)\, d\tau. \tag{10.42}$$

In addition, just as $B^{(L)}(t-t_0)$ may be represented as

$$B^{(L)}(t-t_0) = \int \overline{u_1(0, t_0)\, u_1(X, t)\, \psi(X, t)}\, dX, \tag{10.43}$$

where $\psi(X, t) = \delta[X - X(0, t)]$ is the solution of the advection equation $\frac{\partial\psi}{\partial t} + u_\alpha \frac{\partial\psi}{\partial X_\alpha} = 0$ for the initial condition $\psi(X, t_0) = \delta(X)$, $B^{(\vartheta)}(t-t_0)$ obviously may be written as

$$B^{(\vartheta)}(t-t_0) = \int \overline{u_1(0, t_0)\, u_1(X, t)\, \psi^{(\vartheta)}(X, t)}\, dX, \tag{10.44}$$

where $\psi^{(\vartheta)}(X, t)$ is the solution of the diffusion equation

$$\frac{\partial\psi}{\partial t} + u_\alpha \frac{\partial\psi}{\partial X_\alpha} = \chi\nabla^2\psi$$

for the same initial condition $\psi(X, t_0) = \delta(X)$. if we assume $B^{(\vartheta)}(\tau)$ to be nonnegative for all τ, the effect of deceleration of the turbulent diffusion as applied to the function $B^{(L)}(\tau)$ is expressed by the expectation that $B^{(\vartheta)}(\tau) < B^{(L)}(\tau)$ for all $\tau > 0$. In other words, one may say that as a result of molecular diffusion the admixture particle "forgets" its initial velocity more quickly than the fluid particle because of the additional averaging with respect to Y in Eq.

(10.40). On the basis of this, Hinze (1959) made the following purely qualitative proposal:

$$B^{(\vartheta)}(\tau) = B^{(L)}(\tau) \cdot f(\tau)$$

where $f(\tau)$ is a monotonically decreasing function which depends on the parameter χ such that $f(0) = 1$. Following a suggestion of Burgers, Hinze made a special investigation of the model $f(\tau) = e^{-\alpha\tau}$, (where $\alpha = \frac{c\chi}{L^2}$, c is a dimensionless constant, and L is a turbulence length scale. However, this assumption has no theoretical basis, except perhaps for the special case of turbulence for very small Péclėt number

$$\mathrm{Pe} = \frac{L\,(\overline{u'^2})^{1/2}}{\chi};$$

see Saffman (1962a). For very small $\tau = t - t_0$ for which the solution of the diffusion equation may be approximated by the linear term of a series in powers of τ, it is possible to deduce from Eq. (10.44) that the function $B^{(\vartheta)}(\tau)$ is representable in the form

$$B^{(\vartheta)}(\tau) = B^{(L)}(\tau) - \chi\,\overline{(\nabla u_1)^2}\,\tau + \ldots,$$

from which Eq. (10.38) follows directly. It is natural to expect that with an increase in $t - t_0$, or with an increase in volume occupied by the admixture, the effect of deceleration of the turbulent diffusion is amplified. Namely, that the ratio

$$[B^{(L)}(\tau) - B^{(\vartheta)}(\tau)]/B^{(L)}(\tau)$$

increases with an increase in τ. Assuming that when $\tau \to \infty$, this ratio approximates some constant A, Saffman (1960; 1962a) obtained the asymptotic formula

$$D_0^2(t) = \overline{X_1^2(0,\ t)} + 2\chi\,(t - t_0) - A\overline{X_1^2(0,\ t)}, \qquad (10.45)$$

which is valid for large $t - t_0$. Saffman also presented some arguments according to which

$$A \sim (\mathrm{Pr} \cdot \mathrm{Re})^{-1} = (\mathrm{Pe})^{-1}$$

where Re is the Reynolds number of the turbulence constructed, for example, of the root-mean-square velocity fluctuation $\left(\overline{u_1'^2}\right)^{1/2}$ and the so-called Taylor length microscale

$$\lambda = \left[\overline{u_1'^2} \Big/ \overline{\left(\frac{\partial u_1'}{\partial x_1}\right)^2}\right]^{1/2}.$$

This prediction turned out to agree qualitatively with Mickelsen's data (1959), in which the distribution of the helium and carbon dioxide mean concentration was measured downstream from a point source in a grid turbulence. However, neither accurate quantitative confirmation of Saffman's hypothetical asymptotic equation (10.45) nor qualitative confirmation of Eq. (10.38) for small values of $t - t_0$ has yet been obtained.

To understand the reason for this latter situation, one must estimate the relative role of the molecular and turbulent diffusion, and the interaction between them, in the formation of the mean concentration field $\overline{\vartheta(X, t)}$. For this it is necessary to compare the orders of magnitude of the three terms on the right side of Eqs. (10.38) and (10.45). For very small $t - t_0$ Eq. (10.38) must be used, and on its right side, the largest term is the second [of order $t - t_0$], and the smallest, the third [of order $(t - t_0)^3$]. However, with an increase in $t - t_0$ the contribution of turbulent diffusion increases more rapidly than the contribution of molecular diffusion, and for values of $t - t_0$ which exceed some value of τ_1, the contribution of turbulent diffusion exceeds the value of $2\chi(t - t_0)$. Considering τ_1 to be small by comparison with the Lagrangian integral time scale T_1 (i.e., the time of significant variation of the fluid particle velocity), it is possible to evaluate τ_1 by means of Eq. (10.33) for $\overline{X_1^2}(0, t)$; in this case, we find that $\tau_1 = 2\chi/\overline{u_1^2}$. In most real turbulent flows, this value of τ_1 is infinitesimally small; for example, in the atmospheric surface layer $\overline{u_1^2} \sim 10^2$ to 10^3 cm^2 · sec^{-2}, so that $\tau_1 \sim 10^{-3}$ sec when $\chi \sim 10^{-1}$ cm^2 · sec. For large values of $t - t_0$, the value of $\overline{X_1^2}(0, t)$ may be estimated with the help of the asymptotic formula (9.35); however, the ratio of the contributions of the molecular and turbulent diffusion here is of order

$$\frac{\chi}{\overline{u_1^2}T_1} = \frac{\tau_1}{2T_1}.$$

In other words, this ratio may be estimated as having the order of $(\text{Pr}\cdot\text{Re}^2)^{-\frac{1}{2}}$ since it is possible to show that usually $T_1 \sim \lambda^2/\nu$; both these estimates show that under real conditions this ratio is always extremely small.

Moreover, for sufficiently small $t - t_0$, such that the contribution to the dispersion $D_0^2(t)$ of the terms deriving from the interaction between the molecular and turbulent diffusion may be considered proportional to $(t - t_0)^3$, this contribution will be much less than the contribution of the molecular diffusion if

$$(t - t_0)^2 \ll 6[\overline{(\nabla u_1)^2}]^{-1}.$$

However, for stationary homogeneous turbulence

$$\overline{(\nabla u_1)^2} = \overline{\left(\frac{u_1 \nabla u_1}{u_1}\right)^2} \sim \overline{\left(\frac{du_1/dt}{u_1}\right)^2} \sim \tau_\lambda^{-2},$$

where

$$\tau_\lambda = \left\{ B^{(L)}(0) \Big/ \left[\frac{d^2}{d\tau^2} B^{(L)}(\tau)\right]_{\tau=0} \right\}^{1/2}$$

is the Lagrangian time microscale. Thus, when $t - t_0 \ll \tau_\lambda$, the ratio of the contribution to dispersion due to the interaction between the molecular and turbulent motion to the contribution of the molecular diffusion is of small magnitude $\left[\text{of order } \left(\frac{t - t_0}{\tau_\lambda}\right)^2\right]$. On the other hand, when $t - t_0 \gg \tau_\lambda$, the asymptotic formula (10.45) may be used to prove that this ratio has the order

$$2A \frac{T_1}{\tau_1} \sim \frac{T_1}{\text{Pe}\cdot\tau_1};$$

that is, it is not necessarily small. However, the ratio of the contribution made by the interaction to the contribution of the pure turbulent diffusion is of order $A \sim (\text{Pe})^{-1}$; that is, it is very small for large Re and $\text{Pr} \sim 1$.

In summary, we may state that molecular diffusion does change the dependence of the average diameter of the diffusing admixture cloud on time $t - t_0$ and, therefore, makes it more difficult to use the functions $\overline{X_i^2} = \overline{X_i^2}(t)$ in determining the Lagrangian velocity correlation function by equations of the type (10.11). However, the relative change in the value of $D_0^2(t)$ itself, caused by this diffusion, for $t - t_0$ which is not very small and sufficiently large Reynolds numbers, usually is negligibly small. Therefore, we may conclude that for practical purposes, molecular diffusion may be neglected in calculating the mean concentration $\overline{\vartheta(X, t)}$. Thus we shall proceed in this manner in the subsequent subsections of this section.

10.3 Semiempirical Equation of Turbulent Diffusion

Let us return now to the investigation of turbulent diffusion of a continuously distributed admixture, neglecting the molecular diffusion. Here the mean admixture concentration $\overline{\vartheta(X, t)}$ at the time t is related to the initial concentration $\vartheta_0(X)$ by Eq. (10.5). It is obvious from this that if the probability density for the coordinates of the fluid particle $p(X|x, t)$ satisfies some linear equation in the variables X, t, then the same equation will be satisfied by the function $\overline{\vartheta(X, t)}$. Let us now try to understand what form this latter equation may have.

Neglecting molecular diffusion, the admixture concentration $\vartheta(X, t)$ satisfies the advection equation

$$\frac{\partial \vartheta}{\partial t} + \frac{\partial u_\alpha \vartheta}{\partial X_\alpha} = 0,$$

which is obtained from the diffusion equation (10.1) by equating its right side to zero. Averaging all the terms of this equation, we obtain

$$\frac{\partial \overline{\vartheta}}{\partial t} + \frac{\partial \overline{u}_\alpha \overline{\vartheta}}{\partial X_\alpha} = -\frac{\partial S_\alpha}{\partial X_\alpha}, \qquad S_\alpha = \overline{u'_\alpha \vartheta'}, \tag{10.46}$$

where $S = (S_1, S_2, S_3)$ is the turbulent flux vector of the diffusing admixture. The simplest and earliest theory of turbulent diffusion proposed by G. L Taylor (1915) and Schmidt (1917; 1925), but in essence belonging to Boussinesq (1877; 1887), is based on the assumption that the flux S is proportional to the gradient of the mean admixture concentration; that is,

$$S_i = -K \frac{\partial \overline{\vartheta}}{\partial X_i}, \tag{10.47}$$

where K is the eddy diffusivity K_ϑ of Chapts. 3 and 4. (In the present chapter we shall be dealing only with it, and, therefore, the subscript ϑ will conveniently be omitted.) In the more general anisotropic case, instead of the assumption (10.47), the existence of a linear dependence between the vectors S_i and $\frac{\partial \overline{\vartheta}}{\partial X_i}$ is assumed:

$$S_i = -K_{ij} \frac{\partial \overline{\vartheta}}{\partial X_j}, \tag{10.48}$$

where the *eddy diffusivity tensor* K_{ij} is in general a function of $\boldsymbol{X}$ and t.

From the semiempirical hypothesis (10.48) the semiempirical equation of turbulent diffusion is obtained directly

$$\frac{\partial \overline{\vartheta}}{\partial t} + \frac{\partial \overline{u}_\alpha \overline{\vartheta}}{\partial X_\alpha} = \frac{\partial}{\partial X_\alpha} K_{\alpha\beta} \frac{\partial \overline{\vartheta}}{\partial X_\beta}. \tag{10.49}$$

Since this equation will play a major role in the remainder of this section, it may be well to begin by examining a few additional considerations which are relevant to the derivation of this equation, and which make more evident its range of applicability. We begin with the simplest case of diffusion in a field of stationary homogeneous turbulence, all the statistical characteristics of which are invariant with a change in the time origin or a shift in the zero point of the spatial coordinate system. In this case the covariance tensor of fluid particle displacement in time τ, that is, the tensor

$$D_{ij}(\tau) = \overline{Y_i'(\tau) Y_j'(\tau)},$$

is determined by the generalized Taylor formula (9.30′). Moreover, the mean displacement is equal to $\overline{Y_i(\tau)} = \overline{u}_i\tau$, where $\overline{\boldsymbol{u}} = (\overline{u}_1, \overline{u}_2, \overline{u}_3)$ is a constant (because of stationarity and homogeneity) mean flow velocity. In addition, as we know from Sect. 9.3, substantial grounds exist in this case for assuming that the joint probability distribution for the variables (Y_1, Y_2, Y_3) both for small and large τ, by comparison with the typical Lagrangian time scale T, will be very close to a Gaussian distribution and apparently will, in general, not be markedly different from Gaussian for all values of τ. Therefore, the probability density $p(\boldsymbol{X}|\boldsymbol{x}, t)$ here may be approximately represented as a three-dimensional Gaussian density of the form (4.23) with mean values $a_j = x_j + \bar{u}_j(t - t_0)$ and covariance matrix $\|D_{ij}(\tau)\|$, inverse to the matrix $\|g_{ij}\|$ in the expression for the density, determined by Eq. (9.30′). (In particular, in the absence of a mean velocity and under the assumption that $D_{ij}(\tau) = 0$ when $j \neq i$, we again obtain Eq. (10.12).) It is not difficult to verify that such a function $p(\boldsymbol{X}|\boldsymbol{x}, t)$ is a solution of Eq. (10.49) with coefficients K_{ij} of the type

$$K_{ij} = \frac{1}{2}\frac{dD_{ij}(t-t_0)}{dt} = \frac{1}{2}\int_0^{t-t_0} \left[B_{ij}^{(L)}(\tau) + B_{ji}^{(L)}(\tau)\right] d\tau \qquad (10.50)$$

[see Batchelor (1949b)]. This same equation will also be satisfied in this case by the mean concentration $\overline{\vartheta(\boldsymbol{X}, t)}$. Thus, the assumption of an equation of type (10.49) [but with the coefficients of K_{ij} defined by Eq. (10.50), that is, depending on the time $\tau = t - t_0$ of admixture propagation] for diffusion in a field of stationary homogeneous turbulence is exactly equivalent to the assumption that the probability distribution for the displacement vector of the fiuid particle for any τ will be normal (Gaussian). If, however, in the spirit of the semiempirical hypothesis (10.48) we wish the distribution of the mean concentration for all $t = t_0 + \tau$ to be described by an equation of type (10.49) with coefficients K_{ij} independent of τ, then according to Eq. (10.50), it is necessary to limit ourselves to values of τ which are large compared to the Lagrangian time scale T. Here, on the basis of Eq. (10.50)

$$K_{ij} = \frac{1}{2}\left(\overline{u_i'^2}\,\overline{u_j'^2}\right)^{\frac{1}{2}} T_{ij}, \quad T_{ij} = \int_0^{\infty} \left[R_{ij}^{(L)}(\tau) + R_{ji}^{(L)}(\tau)\right] d\tau \qquad (10.51)$$

in complete agreement with the semiempirical representation of the eddy diffusivities as the product of a turbulent velocity scale by a length scale [or, what is the same thing, the square of a velocity scale multiplied by a time scale].

We have seen that for diffusion in a field of homogeneous stationary turbulence the semiempirical equation (10.49), with constant diffusion coefficients K_{ij}, is satisfied only when $t \gg t_0 + T$, but for these t may be justified quite convincingly according to the Gaussian form of the probability distribution for $\boldsymbol{Y}(\tau)$ which is very likely on the basis of the central limit theorem; see above, Sect. 9.3.

Let us note, however, that in this case the usefulness of Eq. (10.49) becomes quite limited since the general expression for $\overline{\vartheta(X, t)}$ here may be written out directly, independently of this equation, for example, beginning with Eqs. (10.5) and (10.12). Therefore, the basic value of the semiempirical theory consists in the possibility of its application to the more general case of inhomogeneous, or nonstationary, turbulence which we shall consider.

First, let us show that, in general, Eq. (10.49) may also, strictly speaking, be apphcable only if the diffusion time $\tau = t - t_0$ significantly exceeds the Lagrangian time scale. For this we consider the probabilistic meaning of this equation. Equation (10.49) is a first-order equation in t; consequently, its solution is uniquely determined by the initial value $\overline{\vartheta(X, t_0)} = \vartheta_0(X)$ where the boundary conditions will be considered fixed in advance. Let us designate temporarily the probability density $p(X|x, t)$ by the symbol $p(X, t\ x, t_0)$ in order to emphasize its dependence on the initial time t_0; in this case expression (10.5) may be rewritten as

$$\overline{\vartheta(X, t)} = \int \overline{\vartheta(x, t_0)}\, p(X, t\,|\,x, t_0)\, dx.$$

Assuming here that $\overline{\vartheta(X, t_0)} = \delta(X - x)$, we obtain $\overline{\vartheta(X, t)} = p(X, t\ x, t_0)$ for any $t > t_0$. We shall use this result for the time $t_1 < t$, and then again solve Eq. (10.49) for the time interval from the time t_1 to t, for the initial condition $\overline{\vartheta(X, t_1)} = p(X, t_1|x, t_0)$. Then we obtain

$$p(X, t\,|\,x, t_0) = \int p(X, t\,|\,X_1, t_1)\, p(X_1, t_1\,|\,x, t_0)\, dX_1, \tag{10.52}$$

where $t_0 < t_1 < t$. However, this equation for the transition probability density $p(X, t|x, t_0)$ of the random function $X(t)$ will be correct only if it is a *Markov process*. This means that the conditional probability distribution for the value of the function at the time t, under the condition that its values are known at arbitrary times $t_n < t_{n-1} < \ldots < t_0$ (where $t_0 < t$), will depend only on the last of these values $X(t_0)$, but not on the values of $X(t_i)$ with $t_i < t_0$. (In the theory of Markov random processes, Eq. (10.52) is often called the Smoluchowski equation.) Indeed, for an arbitrary but not necessarily Markov random function $X(t)$, when $t_0 < t_1 < t$, the following obvious equation will occur:

$$p(X, t\,|\,\boldsymbol{x}, t_0)=\int p(X, t\,|\,X_1, t_1; \boldsymbol{x}, t_0)\,p(X_1, t_1\,|\,\boldsymbol{x}, t_0)\,dX_1,$$

where the first factor under the integral sign is the conditional probability density for $X(t)$, under the condition that the values of $X(t_1)=X_1$ and $X(t_0)=\boldsymbol{x}$ are fixed; this equation is a special example of the theorem of total probability. The latter formula can be transformed into Eq. (10.52) only if the indicated probability density does not depend on the value of $X(t_0)=\boldsymbol{x}$, and this indicates that the random function $X(t)$ is Markovian. Let us reemphasize that our conclusion does not use the explicit form of Eq. (10.49), but is based only on the fact that this equation is of first order in t.

On the other hand, if the random function $X(\boldsymbol{x}, t)$ is Markovian, then under very general conditions a differential equation of the type (10.49) may be obtained for the probability density

$$p(X|\boldsymbol{x}, t) = p(X, t\,|\,\boldsymbol{x}, t_0)$$

and, therefore, according to Eq. (10.5), also for the mean concentration $\overline{\vartheta(X, t)}$. This important mathematical theorem was proved by Kolmogorov (1931; 1933) [his special cases were established still earlier by the physicists A. Einstein, A. D. Fokker, and M. Planck]. Specifically, Kolmogorov proved that under some broad regularity conditions, imposed on the transition probability $p(X|\boldsymbol{x}, t)$ and ensuring that the Markov random function $X(t)$ in question will be continuous in a certain sense, the following derivatives exist:

$$\begin{aligned}\overline{V(\boldsymbol{x}, t_0)}&=\left[\frac{\partial}{\partial t}\overline{Y(\boldsymbol{x}, t)}\right]_{t=t_0},\\ 2K_{ij}(\boldsymbol{x}, t_0)&=\left[\frac{\partial}{\partial t}\overline{Y'_i(\boldsymbol{x}, t)Y'_j(\boldsymbol{x}, t)}\right]_{t=t_0}\end{aligned}\qquad(10.53)$$

where as usual $Y(\boldsymbol{x}, t)=X(\boldsymbol{x}, t)-\boldsymbol{x}$ and $Y'=Y-\overline{Y}$, and the probability density $p(X|\boldsymbol{x}, t)$ satisfies the equation

$$\frac{\partial p}{\partial t}+\frac{\partial}{\partial X_\alpha}[\overline{V_\alpha(X, t)}\,p]=\frac{\partial^2}{\partial X_\alpha\,\partial X_\beta}[K_{\alpha\beta}(X, t)\,p].\qquad(10.54)$$

[See also the textbook by Gendenko (1962).] This equation obviously coincides with Eq. (10.49) if we assume $\overline{V_i}-\frac{\partial K_{ij}}{\partial X_j}=\overline{u}_i$.

This latter equation means that the variable V_i, being the mean velocity of the admixture particle, according to the first formula of Eq. (10.53), is generally equal to the sum of the convection velocity $\bar{u}_i$ and the additional velocity $\frac{\partial K_{ij}}{\partial X_j}$, related to the fact that in a field of inhomogeneous turbulence the admixture particles may have a tendency to move in a certain direction even in the absence of a mean flow. Let us emphasize that here only the concept of mean, and not instantaneous, particle velocity is introduced; the instantaneous velocity in the "Markov model" does not exist. indeed, the limit

$$\lim_{t \to t_0} \overline{\left| \frac{Y(x, t)}{t - t_0} \right|^2},$$

which must denote the mean square instantaneous velocity of the particle at the time t_0, cannot be finite when $K_{ii}(\boldsymbol{x}, t_0) \neq 0$ on the basis of the second of Eqs. (10.53). This is because the mean square path length traversed by the particle in the time between t_0 and $t_1 = t_0 + \tau$, when $\tau \to 0$, according to Eqs. (10.53), is of order τ and not τ^2. Let us also note that the second of Eqs. (10.53) gives a statistical interpretation of the "eddy diffusivities": K_{ij} differs only by the factor of 1/2 from the rate of change of the particle displacement covariance which we considered in Sect. 9.3.

The nonexistence of the instantaneous velocity $\frac{\partial}{\partial t} \boldsymbol{X}(\boldsymbol{x}, t)$ indicates that the random function $\boldsymbol{X}(\boldsymbol{x}, t)$ is not differentiable. Thus, in the semiempirical theory of turbulent diffusion, the trajectory $\boldsymbol{X}(\boldsymbol{x}, t)$ of a given "fluid particle" is interpreted as a Markovian function which is not differentiable anywhere. The trajectories of the real fluid particles in a turbulent flow clearly do not have such properties; the function $\boldsymbol{X}(\boldsymbol{x}, t)$ is everywhere differentiable with respect to t and, moreover, satisfies the Lagrangian dynamic equations (9.6) and (9.9) which contain the first and second derivatives of $\boldsymbol{X}(\boldsymbol{x}, t)$ with respect to x_i and with respect to t. Knowing the statistical properties of the derivative

$$\frac{\partial \boldsymbol{X}(\boldsymbol{x}, t)}{\partial t} = \boldsymbol{V}(\boldsymbol{x}, t)$$

or, the Lagrangian velocity, is important for many problems concerning particle dispersion in turbulent flow; see, for example, Eq. (9.27) for the covariance tensor of fluid particle displacement. The fact that the variance of fluid particle displacement in small time τ actually is proportional not to τ but to τ^2 [see Eq. (9.28)] is a related phenomenon. Finally, the function $\boldsymbol{X}(\boldsymbol{x}, t)$ also is not a Markovian random function; for example, the probability densities $p(\boldsymbol{X}, t|\boldsymbol{x}, t_0)$ and $p(\boldsymbol{X}, t|\boldsymbol{x}, \boldsymbol{V}_0, t_0)$ for the variable $\boldsymbol{X}(t)$ for a fixed value of $\boldsymbol{X}(t_0) = \boldsymbol{x}$ and fixed values of

$$\boldsymbol{X}(t_0) = \boldsymbol{x} \text{ and } \left.\frac{\partial \boldsymbol{X}}{\partial t}\right|_{t=t_0} = \boldsymbol{V}_0,$$

which for a Markovian function must coincide, will indeed be different, particularly for small $\tau = t - t_0$ where the second of these probability densities will be essentially different from zero only when $\boldsymbol{X} \approx \boldsymbol{x} + \boldsymbol{V}_0\tau$, but the first will not have this property.

However, if we consider the function $\boldsymbol{X}(\boldsymbol{x}, t)$ only at a discrete set of times $t = t_n = t_0 + n\tau$ where the time step τ is large compared to the Lagrangian integral time scale T, then it is possible in practice to regard the random sequence $\boldsymbol{X}(\boldsymbol{x}, t_n)$ as Markovian. The increments of the function $\boldsymbol{X}(\boldsymbol{x}, t)$ in nonoverlapping time intervals of length $\tau \gg T$ are, in fact, practically uncorrelated, and it is natural to expect that they will also be almost independent; but a random sequence with independent increments is certainly Markovian. Let us also note that the variances of the increments of the function $\boldsymbol{X}(\boldsymbol{x}, t)$ in time intervals of length $\tau \gg T$, according to Eq. (9.35), are proportional to τ, as should be the case for a Markovian function according to Eqs. (10.53). Thus, if we considered instead of the differential equation (10.49) a related difference equation in time describing the Markovian sequence $\boldsymbol{X}(\boldsymbol{x}, t_n)$, it would closely correspond with the real properties of fluid particle motion in a turbulent flow. Consequently, the semiempirical equation of turbulent diffusion (10.49) may be expected to be significant, but only as the differential analog of a difference equation in time with a step $\tau \gg T$. Thus, Eq. (10.49) may be used to describe the field $\overline{\vartheta(\boldsymbol{X}, t)}$ at a fixed time t where $t - t_0 \gg T$. In other words, equations for the mean admixture concentration $\overline{\vartheta(\boldsymbol{X}, t)}$ obtained from the solution of Eq. (10.49) may be considered asymptotic and correct when $t - t_0 \gg T$. In practical problems related, for example, to admixture diffusion in the atmosphere, it is usually of interest to describe the diffusion on scales which significantly exceed the Lagrangian integral time scale T, which in the atmospheric surface layer usually has a value on the order of a second. Therefore, it is usually possible in these problems to use the semiempirical equation of turbulent diffusion. Incidentally, even for large values of $t - t_0$, the semiempirical equation may lead to incorrect results at very large distances from the admixture source. This is because this theory actually assumes that the rate of propagation of the admixture is infinitely large (see below, Sect. 10.6).

In the following we shall usually deal with the case of stationary turbulence homogeneous in planes $X_3 = Z = \text{const}$ with a mean velocity $\bar{u} = U$ directed everywhere along the axis $OX_1 = OX$. In this case, the eddy diffusivities K_{ij}, like all other turbulence characteristics, may depend only on the coordinate Z. In addition, from a cursory examination of the problem, it seems natural to assume that the coordinate axes $OX_1 = OX$ $OX_2 = OY$ and $OX_3 = OZ$ coincide with the principal axes of the tensor K_{ij}, since OX, OY, and OZ are clearly the preferred axes of the flow. Thus it may be assumed that the semiempirical equation of turbulent diffusion (10.49) in this case takes the following form:

$$\frac{\partial\bar{\vartheta}}{\partial t}+\bar{u}(Z)\frac{\partial\bar{\vartheta}}{\partial X}=K_{xx}(Z)\frac{\partial^2\bar{\vartheta}}{\partial X^2}+K_{yy}(Z)\frac{\partial^2\bar{\vartheta}}{\partial Y^2}+\frac{\partial}{\partial Z}K_{zz}(Z)\frac{\partial\bar{\vartheta}}{\partial Z}. \tag{10.55}$$

This form is quite standard. It is expounded and used in many textbooks and monographs [see, for example, Sutton (1953), Haltiner and Martin (1957), or Pasquill (1962b)] and in hundreds of papers. However, there have been some recent arguments that indicate that the standard and generally adopted form of the diffusion equation (10.55) cannot be exact [see, e.g., Calder (1965) and Yaglom (1969), where additional references can be found]. We shall discuss these arguments in Sect. 10.5 below; however, we shall first consider some results obtained with the aid of Eq. (10.55).

If gravitational settling of the relatively heavy diffusion particles must be considered, which takes place with a constant velocity w, then it is necessary to add the term $-w\frac{\partial\bar{\vartheta}}{\partial Z}$ to the left side of Eq. (10.55). In exactly the same way, the possibility of radioactive or some other decay of the admixture [with a half-life of $(\ln 2)/\alpha$] leads to the occurrence in the left side of an additional term $\alpha\bar{\vartheta}$. If the flow region extends to infinity in some direction, then the corresponding boundary condition usually will be that $\bar{\vartheta}(\boldsymbol{X}, t)$

decreases sufficiently rapidly as the point $\boldsymbol{X}$ becomes infinitely distant from all the admixture sources. The boundary conditions on the solid walls must be selected from an analysis of the physical processes which take place on these walls, but the solution obtained must correspond in every case to the Markov random function $\boldsymbol{X}(\boldsymbol{x}, t)$ [the enumeration of all these conditions is the subject of the works by Feller and Wentzel mentioned after Eq. (10.2)]. In accord with statements made at the beginning of Sect. 10.1, a boundary condition on a solid wall $Z = 0$, sufficiently general to describe practically all situations encountered, is a condition of the following type:

$$K_{zz}\frac{\partial\bar{\vartheta}}{\partial Z} + w\bar{\vartheta} = \beta\bar{\vartheta} \quad \text{when} \quad Z = 0, \tag{10.56}$$

where β is a constant having the dimensions of velocity. (For generality, here we even assume the presence of gravitational settling with a velocity w which leads to the addition of the flux $-w\bar{\vartheta}$ to the vertical turbulent admixture flux $-K_{zz}\dfrac{\partial\bar{\vartheta}}{\partial Z}$.) The case $\beta = 0$ corresponds to reflection of the admixture from the wall, the case $\beta = \infty$, admixture absorption, and the case $0 < \beta < \infty$, the intermediate situation of partial reflection and partial absorption. For more detailed discussion of the condition (10.56) see Monin (1956a; 1959b) and Calder (1961).

10.4 Diffusion in a Field of Homogeneous Turbulence and in Simple Shear Flows

Diffusion from Sources in Homogeneous Turbulence

We shall now consider the simplest case of diffusion in an idealized model of homogeneous and stationary turbulence with a constant mean velocity $\bar{u} = U$, which, as usual, we shall assume to be directed along the axis $OX_1 = OX$. Let us direct the coordinate axes $OX_1 = OX$, $OX_2 = OY$ and $OX_3 = OZ$ along the principal axes of the covariance tensor $D_{ij}(\tau) = \overline{Y'_i(\tau)Y'_j(\tau)}$ considering the OX_1 axis as one of the principal axes, and neglect molecular diffusion by comparison with turbulent diffusion. Finally, we use the experimental fact that the probability density $p(\boldsymbol{X}|\boldsymbol{x}, t)$, in the case of homogeneous turbulence, is close to the multidimensional Gaussian density for all $\tau = t - t_0$, where, in addition, this circumstance has a theoretical justification for large values of τ; thus, we can write the equation

$$\overline{\vartheta(X, t)} = \frac{1}{[8\pi^3 D_{xx}(\tau) D_{yy}(\tau) D_{zz}(\tau)]^{1/2}} \int\int\limits_{-\infty}^{\infty}\int \vartheta_0(x, y, z) \times$$

$$\times \exp\left\{-\frac{(X-x-U\tau)^2}{2D_{xx}(\tau)} - \frac{(Y-y)^2}{2D_{yy}(\tau)} - \frac{(Z-z)^2}{2D_{zz}(\tau)}\right\} dx\,dy\,dz \tag{10.57}$$

for the field of mean concentration $\overline{\vartheta(X, t)}$, corresponding to the initial value of $\overline{\vartheta(X, t_0)} = \vartheta_0(X)$. Consequently to obtain a solution for the mean concentration $\overline{\vartheta(X, t)}$ in any problem, it is necessary to know only the variances $D_{xx}(\tau)$, $D_{yy}(\tau)$ and $D_{zz}(\tau)$ of the components of fluid particle displacement in the time τ, which are expressed by Eq. (9.31) in terms of the corresponding Lagrangian velocity correlation functions $B_{xx}^{(L)}(\tau)$, $B_{yy}^{(L)}(\tau)$ and $B_{zz}^{(L)}(\tau)$. In particular, when

$$t - t_0 = \tau \gg T = \max(T_x, T_y, T_z),$$

where T_x, T_y and T_z are the three Lagrangian integral time scales, it is possible to assume that

$$D_{xx}(\tau) = 2\overline{u'^2}T_x\tau = 2K_{xx}\tau,\ D_{yy}(\tau) = 2\overline{v'^2}T_y\tau = 2K_{yy}\tau$$

$$D_{zz}(\tau) = 2\overline{w'^2}T_z\tau = 2K_{zz}\tau.$$

Then the right side of Eq. (10.57) will coincide with the solution of the parabolic partial differential equation

$$\frac{\partial\bar{\vartheta}}{\partial t} + U\frac{\partial\bar{\vartheta}}{\partial X} = K_{xx}\frac{\partial^2\bar{\vartheta}}{\partial X^2} + K_{yy}\frac{\partial^2\bar{\vartheta}}{\partial Y^2} + K_{zz}\frac{\partial^2\bar{\vartheta}}{\partial Z^2} \tag{10.58}$$

which is a special case of Eq. (10.55), for the initial condition $\overline{\vartheta(X, t_0)} = \vartheta_0(X)$. If $\vartheta_0(X) = Q\delta(X) = Q\delta(X)\delta(Y)\delta(Z)$ (the case of an instantaneous point source at the point $X=0$), then the distribution (10.57) will describe an ellipsoidal admixture cloud which decreases in concentration with distance from the center according to the Gaussian law. The center of this cloud is at the point $(U\tau, 0, 0)$; that is, it is displaced with the mean velocity of the flow, and the maximum concentration at the center is proportional to

$$[D_{xx}(\tau)D_{yy}(\tau)D_{zz}(\tau)]^{-1/2}.$$

(Specifically, when $\tau \gg T$, it decreases with increasing τ, proportional to $\tau^{-\frac{3}{2}}$.) If $\vartheta_0(\boldsymbol{X}) = Q\delta(X)\delta(Z)$, the case of an instantaneous linear source along the OY axis, then for any τ the admixture cloud will have the form of a cylinder of elliptical cross section with an axis on the line $X = U\tau$, $Z = 0$. In this case the maximum concentration is proportional to $[D_{xx}(\tau)D_{zz}(\tau)]^{-1/2}$, or, for the large values of τ, it decreases proportional to τ^{-1}. For a steady continuously active point source of constant output Q at the point $\boldsymbol{X} = 0$, Eqs. (10.6′) and (10.12) must be used, in which case, when $X \gg UT$, we can replace $D_{ii}(\tau)$ by $2K_{ii}\tau$ in the latter without substantial error. With this substitution the integral with respect to τ in Eq. (10.6′) is easily and explicitly carried out, and as a result we obtain

$$\overline{\vartheta(X, Y, Z)} \approx \frac{Q}{4\pi(K_{yy}K_{zz}X^2 + K_{xx}K_{zz}Y^2 + K_{xx}K_{yy}Z^2)^{1/2}} \times \\ \times \exp\left\{-\frac{U}{2K_{xx}}\left[K_{xx}^{1/2}\left(\frac{X^2}{K_{xx}} + \frac{Y^2}{K_{yy}} + \frac{Z^2}{K_{zz}}\right)^{1/2} - X\right]\right\}. \tag{10.59}$$

Equation (10.59) describes the admixture jet which has the form of an elliptical paraboloid oriented along the OX axis. When $X \gg \sqrt{Y^2 + Z^2}$ this formula may be transformed to

$$\overline{\vartheta(X, Y, Z)} \approx \frac{Q}{4\pi(K_{yy}K_{zz})^{1/2}X} \exp\left\{-\frac{U}{4X}\left(\frac{Y^2}{K_{yy}} + \frac{Z^2}{K_{zz}}\right)\right\}. \tag{10.60}$$

Equation (10.60) shows that for fixed Y and Z at a large distance from the source, the admixture concentration decreases approximately proportional to X^{-1}, and for fixed X and increasing distance from the jet axis, it decreases exponentially. In exactly the same way, for a stationary linear source on the OY axis of output Q (per unit length), Eqs. (10.8) and (10.12) for $X \gg UT$, that is, when it is possible to assume that

$$D_{xx} = 2K_{xx}\tau, \quad D_{zz} = 2K_{zz}\tau,$$

lead to the equation

$$\overline{\vartheta(X, Z)} \approx \frac{Q}{2\pi (K_{xx}K_{zz})^{1/2}} K_0\left[\frac{U}{2}\left(\frac{X^2}{K_{xx}^2} + \frac{Z^2}{K_{xx}K_{zz}}\right)^{1/2}\right]\exp\left(\frac{UX}{2K_{xx}}\right), \qquad (10.61)$$

where $K_0(z)$ is the so-called Basset (or Macdonald) function (the modified Bessel function of the third kind). When $UX \gg K_{xx}$ and $Z \ll X$, Eq. (10.61) may be simplified, taking the form

$$\overline{\vartheta(X, Z)} \approx \frac{Q}{2(\pi K_{zz}UX)^{1/2}} \exp\left(-\frac{UZ^2}{4K_{zz}X}\right). \qquad (10.62)$$

It is obvious from this that for a steady linear source, the admixture concentration in a field of homogeneous turbulence with a constant mean velocity decreases when $X \to \infty$ (for fixed Z) approximately proportional to $X^{-1/2}$.

Longitudinal Admixture Dispersion in a Straight Tube or Channel

Let us now proceed to the significantly more complex problem of turbulent diffusion in a flow with a velocity gradient. Interesting results may be obtained here only for a few specific types of flows. We begin with the practically important problem of *longitudinal diffusion of the admixture in a long straight tube (or straight channel)*. The importance of such diffusion is connected in particular with the fact that measurement of the velocity of the longitudinal transport of the admixture in a tube frequently proves to be the most acceptable method of measuring the average flow velocity for turbulent flows and even for laminar ones. Thus longitudinal dispersion came under scrutiny rather long ago; for example, the study of dispersion during laminar flow in blood vessels and in glass capillaries by G. I. Taylor (1953), and the comparatively early work by Allen and E. A. Taylor (1923) on dispersion during turbulent tube flow. The experiments described in these works were such that at some time t_0, a certain mass of admixture was introduced into the tube at some point, or, in other words, an instantaneous point source was simulated. Then at some distance $\boldsymbol{X}$ downstream from this point the change with time in the average admixture concentration over a cross section of the tube, $\vartheta_m(t)$ was measured. Naturally, for some time after the introduction of the admixture into the tube its concentration at the point $\boldsymbol{X}$ will equal 0. Let $t_0' = t_0 + \tau_0$ be the time at the point X at which the admixture is first detected; then the ratio $u_{\max} = X/\tau_0$ will be equal to the *maximum* rate of flow in the tube, or, in the case of axially symmetric tubes, the flow velocity on the axis of the tube. The *average* rate of flow u_{av} may be defined by the $u_{av} = X/\tau_1$ where $t_0 + \tau_1 = t_1$ is the time corresponding to some preferred point of the function $\vartheta_m(t)$.

According to data obtained in straight circular tubes, the longitudinal admixture dispersion has the following distinctive features: 1) the curves $\vartheta_m(t)$ which correspond to sufficiently large distances X, by comparison with the tube radius R, are found to be approximately symmetric with respect to their maxima, although the spatial distribution of the longitudinal flow velocity $\bar{u}_x(r)$ [where r is the radial coordinate] over the cross section of the tube is not symmetrical with respect to the average velocity u_{av}; 2) if the time $t_1 = t_0 + \dfrac{X}{u_{av}}$ corresponds to the maximum of the curve $\vartheta_m(t)$, then u_{av} coincides with the bulk average velocity U_{av} defined as the ratio of the volume of fluid flowing out of the tube per unit time to the area of its transverse cross section. In other words, the center of gravity of the longitudinal concentration distribution is displaced downstream with a velocity equal to the average bulk velocity U_{av}; 3) the width of the curve $\vartheta_m(t)$, defined as the square root of the ratio

$$\sigma^2_{\vartheta_m} = \int_{t_0}^{\infty} (t - t_0)^2 \vartheta_m(t)\, dt \Bigg/ \int_{t_0}^{\infty} \vartheta_m(t)\, dt,$$

increases with an increase in the distance X, or, equivalently, the time $\tau_1 = X/U_{av}$, in proportion to $X^{1/2}$ (or $\tau_1^{1/2}$). That is, the variance $\sigma^2_{\vartheta_m}$ has the form $\sigma^2_{\vartheta_m} = 2K_1\tau_1$, where K_1 is some constant; 4) the variable $K = K_1 U_{av}^2$ which obviously plays the role of an "effective longitudinal eddy diffusivity" has a value which exceeds many times that of the radial (turbulent or molecular) diffusivity of the admixture. The second feature is of particular interest since it shows that the definition of the maximum of the curve $\vartheta_m(t)$ enables the measurement of the bulk average velocity U_{av}, which is most significant for most hydraulic and other engineering applications; also the third feature is important since it shows that the length of the tube segment, symmetric with respect to the point $X = U_{av}\tau_1$ and containing, e.g., 90% (or 95%, or 99%) of the diffusing admixture, increases with time as $\tau_1^{1/2}$ so that its ratio to the average distance $X = U_{av}\tau_1$ from the admixture source continuously decreases. Since the section filled by the admixture moves with an average velocity U_{av}, the fluid upstream from this section on the axis of the tube, and consequently moving with a velocity exceeding U_{av}, overtakes and passes this section with time.

The special features of longitudinal admixture dispersion in tubes were theoretically analyzed and experimentally checked in the brilliant works by G. I. Taylor (1953; 1954a, b), two of which are devoted to laminar flows, and the other (1954a), to turbulent flows. These works stimulated interest in these phenomena and promoted several new theoretical and experimental investigations [cf. Batchelor, Binnie and Phillips (1955); Aris (1956); Elder (1959); Ellison (1960); Saffman (1962b); Philip (1963); Tyldesley and Wallington (1965), and others] which refined Taylor's results and expanded their range of applicability. In particular, it was discovered that the above-mentioned general properties (1)–(4) of longitudinal admixture dispersion may be explained simply according to the general properties of the fluid particle motion in striaght tubes and channels. As noted at the end of Sect. 9.3, longitudinal dispersion of a fluid particle in a straight tube, or a straight channel, with sufficiently large time of motion $\tau = t - t_0$, is subject to the same laws as are in force for homogeneous turbulent flow with a constant mean velocity $\bar{u} = U_{av}$. Consequently, the mean value of the longitudinal coordinate $\overline{X(t_0 + \tau)}$ of the fluid particle which at the time t_0 leaves from the point $X = 0$ will be determined quite accurately for large τ by the expression $\overline{X(t_0 + \tau)} \approx U_{av}\tau$, and the variance of the coordinate $X(t_0 + \tau)$ will have the form

$$\overline{(X - \overline{X})^2} \approx 2K\tau, \quad \text{where} \quad K = \text{const.}$$

(In Sect. 9.3 this constant was written in the form $\overline{u_1'^2}T_1$.) For the probability density $p_1(X|\tau)$ for values of $X(t_0 + \tau)$, the central limit theorem of probability theory gives substantial grounds for assuming that for large τ this probability density will be close to normal. Thus it follows that the longitudinal distribution of the admixture cloud from an instantaneous point source after a long time τ will have a bell-shaped Gaussian form with a mean value $U_{av}\tau$ and a variance $2K\tau$. If instead of $\tau = t - t_0$, we fix the value of X, and $\tau(X)$ is the time of passage of a fluid particle through a cross section at distance X from the source, then $\tau(X) = \dfrac{X}{V_{\tau(X)}}$, where $V_{\tau(X)}$ is the mean

Lagrangian velocity of the particle in the time from t_0 to $t_0 + \tau(X)$. Since, with an increase in X, the fluctuations in the passage time $\tau(X)$ become relatively less important, and $\overline{\tau(X)}$ approximates X/U_{av}, for sufficiently large X the probability distribution of the mean Lagrangian velocity $\tau(X)$ will, with high accuracy, coincide with the probabihty distribution of the mean Lagrangian velocity

$$V_{X/U_{av}} = X\left(t_0 + \frac{X}{U_{av}}\right) \cdot \frac{U_{av}}{X},$$

which is averaged over the time interval X/U_{av}. However,

$$\overline{X\left(t_0 + \frac{X}{U_{av}}\right)} = X, \text{ that is } X\left(t_0 + \frac{X}{U_{av}}\right) = X + X';$$

consequently, for sufficiently large X

$$\tau(X) \approx \frac{X}{U_{av}} \frac{X}{X + X'} = \frac{X}{U_{av}}\left[1 - \frac{X'}{X} + \left(\frac{X'}{X}\right)^2 - \ \ldots\right]. \quad (10.63)$$

Neglecting the component $\left(\frac{X'}{X}\right)^2$ having a mean value of order $1/X$, and the subsequent higher-order terms, we find that for large values of X

$$\tau(X) \approx \frac{X}{U_{av}} - \frac{X'}{U_{av}}, \quad (10.64)$$

where X' is a normally distributed random variable with zero mean value and variance $2K\frac{X}{U_{av}}$. Consequently, the variable $\tau(X)$ will also have a normal probability distribution, with a mean value $\bar{\tau} = \frac{X}{U_{av}}$ and a variance $2K_1\bar{\tau}$ where $K_1 = \frac{K}{U_{av}^2}$. These deductions are clearly in full agreement with the empirically estabhshed properties (1)–(3) of longitudinal admixture dispersion. Property (4) mentioned above may also be explained simply. In this case the longitudinal dispersion of the admixture cloud is due largely to the influence of a transverse mean velocity gradient; specifically, it is generated by the velocity differences $\bar{u}(r) - U_{av}$ which substantially exceed both the turbulent fluctuations of the longitudinal velocity $u' = u(r) - \bar{u}(r)$, producing turbulent diffusion, and, even more, molecular fluctuations which cause molecular diffusion.

However, in estimating the numerical values of the coefficients K and K_1, the arguments above concerning the Lagrangian description of the flow are of very little help. Indeed, the eddy diffusivity K is expressed in terms of the Lagrangian integral time scale T_1; however, this scale is determined by the Lagrangian velocity correlation function which is very difficult to relate to the directly measured Eulerian statistical characteristics, and is usually evaluated only from the data of diffusion experiments. In the case of longitudinal admixture dispersion in a tube or channel, the value of K may be obtained quite accurately using the semiempirical diffusion equation for the Eulerian concentration field $\overline{\vartheta(X, t)}$; this method was also used in the influential works of Taylor. Since we are interested in the study of asymptotic laws for large τ (or X), we may expect that the semiempirical equation will be quite accurate. Its use also has the important advantage that the same theory may be applied to dispersion both in turbulent and in laminar flows; for the latter, the approximate semiempirical equation is replaced by an exact molecular diffusion equation (10.1). Following G. I. Taylor (1953; 1954a), we assume that the mean concentration $\bar{\vartheta}$ does not depend on the angular variable φ, and we may write Eq. (10.55) or (10.1) for diffusion in a straight round tube in the form

$$\frac{\partial\bar{\vartheta}}{\partial t} + \bar{u}_x(r)\frac{\partial\bar{\vartheta}}{\partial X} = K_{xx}\frac{\partial^2\bar{\vartheta}}{\partial X^2} + \frac{1}{r}\frac{\partial}{\partial r} rK_{rr}(r)\frac{\partial\bar{\vartheta}}{\partial r}. \tag{10.65}$$

Furthermore, it will be convenient to transform to a system of coordinates which moves with the mean velocity of the flow, that is, to replace the longitudinal coordinate X by the coordinate $x = X - U_{av}(t - t_0)$; in addition, instead of r we shall introduce the dimensionless radial variable $z = \frac{r}{R}$ where R is the tube radius. On the basis of the fourth feature above concerning longitudinal admixture dispersion in tubes (established by experiment), we can attempt to neglect in Eq. (10.65) the term $K_{xx}\frac{\partial^2\bar{\vartheta}}{\partial X^2}$ describing the axial diffusion of the admixture. (Consideration of this term hardly changes the arguments, but leads only to a small correciton in the final results which will be indicated below.) Then Eq. (10.65) in the variables x, z, t assumes the form

$$\frac{\partial\bar{\vartheta}}{\partial t}+[\bar{u}_x(z)-U_{\mathrm{av}}]\,\frac{\partial\bar{\vartheta}}{\partial x}=\frac{1}{R^2}\,\frac{1}{z}\,\frac{\partial}{\partial z}\,zK_{rr}(z)\,\frac{\partial\bar{\vartheta}}{\partial z}. \qquad (10.66)$$

We shall use this equation to describe the concentration distribution $\bar{\vartheta}(x, z, t)$ which corresponds to the presence at the time $t = t_0$, at the point $x = z = 0$, of an instantaneous point source of admixture. Let us first compare the time intervals τ_c and τ_d during the course of which the admixture concentration at a given point in the tube will change appreciably due to the mean rate of flow only (convection), and to transverse diffusion alone. We can put $\tau_c = \sigma_{\vartheta_m}$ where $\sigma_{\vartheta_m}(t)$ is the standard deviation of the admixture concentration distribution ϑ_m averaged over the cross section of the tube. On the other hand, τ_d is the time during the course of which transverse diffusion smoothes the inhomogeneities in the concentration distribution over the cross section of the tube, and one may assume $\tau_d = \frac{R^2}{K_0}$, where K_0 is a typical value (a scale) of the transverse diffusivity. Clearly, the value of σ_{ϑ_m} and, therefore, τ_c increases with the propagation time of the admixture τ, according to the theoretical arguments and experimental data, asymptotically proportional to $\tau^{1/2}$; however, τ_d does not vary with time. Therefore, for a sufficiently large τ, or x, the following inequahty will be satisfied: $\tau_c \gg \tau_d$, that is,

$$\sigma_{\vartheta_m} \gg \frac{R^2}{K_0}. \qquad (10.67)$$

Let us limit our investigation to sufficiently large τ for which the inequality (10.67) is fulfilled, and calculate the value Q_1 of the total admixture flux through a tube cross section $x =$ const. (We must recall that $x = X - U_{\mathrm{av}}\tau$ so that the cross section $x =$ const moves along the tube with a velocity of U_{av}.) Obviously,

$$Q_1=\int_S\int[\bar{u}_x(r)-U_{\mathrm{av}}]\,\bar{\vartheta}\,dS=2\pi R^2\int_0^1[\bar{u}_x(z)-U_{\mathrm{av}}]\,\bar{\vartheta}\,z\,dz, \qquad (10.68)$$

where S is the cross section of the tube; consequently, Q_1 may be different from zero only in the presence of radial inhomogeneities of the concentration $\bar{\vartheta}$. Under condition (10.67) these radial inhomogeneities are smoothed out quite strongly; therefore, only very weak inhomogeneities may exist after a sufficiently long time, that is, those which are constantly created by the inhomogeneous convective transfer of the admixture, and which adapt to this transfer in the sense that the effects of convection and transverse diffusion are balanced for them. Such inhomogeneities satisfy the equations

$$[u_x(z) - U_{\text{av}}] \frac{\partial \vartheta_m}{\partial x} = \frac{1}{R^2} \frac{1}{z} \frac{\partial}{\partial z} z K_{rr}(z) \frac{\partial \bar{\vartheta}}{\partial z}. \qquad (10.69)$$

(Here on the left side we have replaced $\frac{\partial \bar{\vartheta}}{\partial x}$ by $\frac{\partial \vartheta_m}{\partial x}$ since the radial inhomogeneities $\bar{\vartheta} - \vartheta_m$ are quite small.) From Eq. (10.69) we obtain

$$\bar{\vartheta}(z) = \bar{\vartheta}(0) + R^2 \frac{\partial \vartheta_m}{\partial x} \int_0^z \frac{dz'}{z' K_{zz}(z')} \int_0^{z'} [\bar{u}_x(z'') - U_{\text{av}}] z''\, dz''.$$

Using this equation, (10.68) is easily reduced to the form

$$Q_1 = -\pi R^2 K \frac{\partial \vartheta_m}{\partial x},$$

where

$$K = -2R^2 \int_0^1 [\bar{u}_x(z) - U_{\text{av}}] z\, dz \int_0^z \frac{dz'}{z' K_{zz}(z')} \int_0^{z'} [\bar{u}_x(z'') - U_{\text{av}}] z''\, dz'' =$$

$$= 2R^2 \int_0^1 \frac{dz}{z K_{zz}(z)} \left[\int_z^1 [\bar{u}_x(z') - U_{\text{av}}] z'\, dz' \right]^2. \qquad (10.70)$$

Substituting the expression obtained for Q_1 into the equation

$$\pi R^2 \frac{\partial \vartheta_m}{\partial t} + \frac{\partial Q_1}{\partial x} = 0,$$

which expresses the law of admixture mass conservation, and which may also be derived, for example, by integration of all the terms of the diffusion equation (10.66) over the cross section of the tube, we obtain

$$\frac{\partial \vartheta_m}{\partial t} = K \frac{\partial^2 \vartheta_m}{\partial x^2}.$$

Thus it is obvious that except for a diffusion coefficient K defined by Eq. (10.70), the longitudinal dispersion of the admixture with respect to a plane which moves along the tube with an average velocity U_{av} is quite similar to molecular diffusion, and therefore leads to a Gaussian distribution of the concentration in the Ox direction. Consequently, the spatial variance of the longitudinal admixture distribution will equal $2K(t - t_0)$, which means

$$\sigma^2_{\vartheta_m} = 2K_1\tau, \quad \text{where} \quad K_1 = K/U^2_{av}, \quad \tau = t - t_0.$$

In addition, considering that according to Eq. (10.70) K is of order $\frac{U^2_{av} R^2}{K_0}$, we can reduce condition (10.67), for which our arguments are correct, to the form

$$\tau \gg \frac{R^2}{K_0}. \tag{10.71}$$

It only remains to calculate the values of K for both laminar and turbulent flows. In the former case in Eq. (10.70) for K, according to Eq. (1.23), we must assume that $\bar{u}_x(z) = 2U_{av}(1 - z^2)$ and equate $K_{rr} = \chi$ where χ is the molecular diffusivity of the particular admixture. Then Eq. (10.70) for K is easily reduced to the form

$$K = \frac{R^2 U^2_{av}}{48\chi}, \tag{10.72}$$

so that

$$\frac{K}{\chi} \approx \left(\frac{1}{7} \frac{RU_{av}}{\chi}\right)^2.$$

With the exception of extraordinarily slow flows in especially fine capillaries, the number $\frac{1}{7}\frac{RU_{av}}{\chi}$ significantly exceeds unity, so that

$K \gg \chi$. It is not difficult to see that by also taking into account molecular diffusion, that is, preserving the term $\chi \frac{\partial^2 \bar{\vartheta}}{\partial X^2}$ in Eq. (10.65), instead of Eq. (10.72) we would obtain

$$K = \frac{R^2 U_{av}^2}{48\chi} + \chi,$$

which usually is practically indistinguishable from Eq. (10.72).

Clearly, if the velocity $u(z)$ were constant, then the longitudinal diffusion would be determined only by the molecular diffusivity χ; thus, the presence of the transverse velocity gradient causes a sharp increase in the longitudinal admixture dispersion. It is interesting to note that, on the other hand, the transverse diffusion delays the longitudinal admixture dispersion; according to Eq. (10.72), with an increase in χ the value of K decreases. If there were no transverse diffusion, then the distribution of the admixture first concentrated in a pipe section of very small length δX, and uniform in the cross section of the tube, in a time τ is converted into a uniform distribution for which $\vartheta_m(X) =$ const in a section of length $U_{max}\tau$ and $\vartheta_m(X) = 0$ outside this section. [This result follows easily from the advection equation (9.13).] In the presence of transverse diffusion it is converted into a Gaussian distribution in which almost all the admixture is concentrated in a tube section of length $L \sim \sqrt{K\tau}$, for sufficiently large τ, which is very small compared with $U_{max}\tau$.

Since the value of K is not difficult to determine experimentally, using measurements of the concentration $\vartheta_m(t)$ or $\vartheta_m(X)$, Eq. (10.72) creates the possibility of obtaining values of the molecular diffusivities χ from such data. This method for measuring the value of χ was tested by G. I. Taylor (1953), who made experimental studies of the diffusion of potassium permanganate in glass capillaries, and thus obtained a value of the molecular diffusivity χ of this substance which agrees well with the data from other measurements. Taylor (1954b) also gave an extensive discussion of this method for measuring χ.

In turbulent flow with sufficiently large Reynolds numbers that the thickness of the viscous sublayer is a negligibly small fraction of the radius R, the average velocity $\bar{u}_x(r)$ may be given by the velocity defect law (5.41):

$$\frac{\bar{u}_x(0) - \bar{u}_x(r)}{u_*} = f_1\left(1 - \frac{r}{R}\right),$$

where u_* is the friction velocity defined by the shear stress on the tube wall. Considering that the turbulent shear stress in a round tube has the form $\tau = \rho u_*^2 \frac{r}{R}$ [wee Eq. (5.17')], and assuming that the eddy diffusivity is exactly equal to the eddy viscosity, using the Reynolds analogy in the terminology of Sect. 5.7, it is possible to represent the radial eddy diffusivity in the form

$$K_{rr}(r) = \frac{\tau}{-\rho \frac{\partial \bar{u}_x}{\partial r}} = \frac{u_* r}{-f_1'\left(1 - \frac{r}{R}\right)}.$$

Using the values of the function $f_1(\eta)$obtained by processing the data from Nikuradse's measurements and those of other investigators, G. I. Taylor (1954a) using Eq. (10.70) obtained a value of $K = 10.06$ Ru_* [so that $c \approx 10$ in Eq. (9.36)]. It is not difficult to estimate the correction to this value of K produced by longitudinal turbulent diffusion by calculating the corresponding admixture flux through a cross section of the tube

$$Q_1' = -\iint_S K_{xx}(r) \frac{\partial \bar{\vartheta}}{\partial x} dS = -2\pi R^2 \frac{\partial \vartheta_m}{\partial x} \int_0^1 K_{xx}(z)\, z\, dz. \quad (10.73)$$

For a rough estimate one may make the simple assumption that

$$K_{xx}(r) = K_{rr}(r) \quad \text{then} \quad Q_1' = -\pi R^2 K' \frac{\partial \vartheta_m}{\partial x},$$

where for K' numerical integration gives a value$K' = 0.052 Ru_*$. The quantity K' is quite small compared to K, but since only an approximate numerical value may be obtained in any case for K/Ru_*, it is reasonable to add K' to the value of K obtained above, that is, to use the formula

$$K = 10.1 Ru_*. \quad (10.74)$$

(The inaccuracy in the hypothesis used for calculating K' will have little significance due to the relative smallness of the entire summand K'.)

The result (10.74) is in good agreement with the data for longitudinal propagation of an admixture in a sufficiently long straight round tube. Thus, for example, the data of Allen and E. Taylor (1923) concerning the diffusion of salt in a water flow in a straight, round, but not very long tube, lead to the value of $K/Ru_* = 10.6$ for one series of experiments and $K/Ru_* = 11.7$ for another [see G. I. Taylor (1954a)]. Taylor in collaboration with Ellison conducted similar experiments in which a significantly longer smooth tube was used. A value of $K/Ru_* = 10.0$ was obtained under almost exactly the same conditions as in the above theory. When using a tube with very rough walls, it turned out that $K/Ru_* = 10.5$. The dependence of the concentration ϑ_m on time in all these experiments, at a sufficiently large distance from the admixture source, corresponds closely to the Gaussian curve; see, for example, Fig. 80 in which the data from one of Taylor's experiments is reproduced. In the case of bent tubes or, for example, industrial oil pipes, which are usually not exactly straight, the values of K/Ru_* naturally turn out to be somewhat different from the theoretical values given above for straight pipes; in several cases here, K/Ru_* assumes values close to 20 [G. I. Taylor (1954a)].

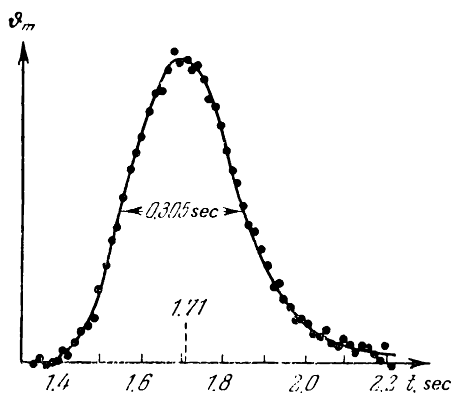


FIG. 80. Salt concentration in a fixed cross section of a straight tube as a function of time from Taylor's data (1954a).

Longitudinal admixture dispersion in a wide, straight channel is bounded by the walls $Z=0$ and $Z=H$. (The role of the upper wall may be played by the free surface of the liquid.) This problem was investigated by Elder (1959), Ellison (1960), Saffman (1962b), and Tyldesley and Wallington (1965). Elder used Taylor's method (discussed above), while Saffman used the elegant method of Aris (1956). Tyldesley and Wallington solved the semiempirical diffusion equation (10.55) numerically.[2] In Aris' method one first derives the equations for the moments from the semiempirical diffusion equation (10.55), or the equation of molecular diffusion (10.1):

$$\theta_{nm}(Z, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{\vartheta}(X, Y, Z, t) X^n Y^m \, dX dY. \qquad (10.75)$$

For example, when $m=0$, $n = 0$, 1 or 2, and Eq. (10.55) is used, it is easy to see that these equations will have the following form:

$$\frac{\partial \theta_{00}}{\partial t} - \frac{\partial}{\partial Z}\left(K_{zz} \frac{\partial \theta_{00}}{\partial Z}\right) = 0, \qquad (10.76)$$

$$\frac{\partial \theta_{10}}{\partial t} - \frac{\partial}{\partial Z}\left(K_{zz} \frac{\partial \theta_{10}}{\partial Z}\right) = \bar{u}(Z)\,\theta_{00}, \qquad (10.76')$$

$$\frac{\partial \theta_{20}}{\partial t} - \frac{\partial}{\partial Z}\left(K_{zz} \frac{\partial \theta_{20}}{\partial Z}\right) = 2K_{xx}(Z)\,\theta_{00} + 2\bar{u}(Z)\,\theta_{10}; \qquad (10.76'')$$

the equations for the moments θ_{01}, θ_{11}, θ_{02} and higher-order moments look the same. The resulting equations may be solved sequentially for the initial and boundary conditions obtained from the initial and boundary conditions for the concentration $\overline{\vartheta(X, t)}$. The asymptotic behavior of the solutions obtained will describe the mean admixture distribution for large values of t in the planes $Z=$ const, or specifically, in a thin layer which surrounds the height Z, with accuracy to moments of the same order as that at which the system of equations for the variables (10.75) was cut off. Similarly, the asymptotic behavior of the variables

[2] Tyldesley and Wallington obtained numerical solutions for an equation of the form (10.55), with an additional term $v(z)\partial\vartheta/\partial y$ on the left side, but without the term proportional to K_{xx}. They described diffusion from a continuous steady point or linear source and from an instantaneous linear source in a fluid layer $0 \leqslant Z \leqslant H$ for several specific models of the vertical velocity and diffusivity distributions. The results obtained were compared with data from various atmospheric diffusion experiments. However, we shall not give further attention to their work here.

$$\Theta_{nm}(t) = \int_0^H \theta_{nm}(Z, t)\, dZ$$

will, with the same accuracy, describe the mean admixture distribution in the entire three-dimensional volume occupied by the fluid. The final results obtained in this way will be the same as those when applying the Taylor method. According to these results, if the admixture is introduced into the flow at the time $t=t_0$ at the point $(0, 0, Z)$ where $0 \leqslant Z \leqslant H$, and is not absorbed by the walls, then the center of gravity of the admixture cloud at the time $t=t_0+\tau$ asymptotically, for large values of τ, will have the coordinates

$$\left(U_{\text{av}}\tau,\ 0,\ \frac{H}{2}\right),\ \text{where } U_{\text{av}} = \frac{1}{H}\int_0^H \bar{u}(Z)\, dZ,$$

that is, it will be horizontally displaced along the X axis with a velocity U_{av}. Along the vertical the admixture will be distributed asymptotically uniformly, in the sense that any fluid layer of fixed thickness ΔZ for large τ will contain the same amount of admixture, where the center of gravity of the admixture in each layer also will be shifted with the asymptotically constant velocity U_{av}. However, for different layers the coordinates of the center of gravity of the admixture will, for large τ, differ by a constant which depends on the corresponding values of Z and the form of the profiles $\bar{u}(Z)$ and $K_{zz}(Z)$. The admixture distribution in the planes X = const at the time $t_0 + \tau$ will, for large τ, be close to a Gaussian distribution with zero mean value and variance

$$\sigma^2_{\vartheta_y} = 2K_y\tau + \text{const}, \qquad K_y = (K_{yy})_{\text{av}} = \frac{1}{H}\int_0^H K_{yy}(Z)\, dZ. \qquad (10.77)$$

The admixture distribution in the planes Y = const will be close to a Gaussian distribution with mean value $\bar{X} \sim U_{\text{av}}\tau$ and variance

$$\sigma^2_{\vartheta_x} = 2K_x\tau + \text{const}, \qquad K_x = K + (K_{xx})_{\text{av}}, \qquad (10.78)$$

where

$$K = \frac{1}{H} \int_0^H \frac{dZ}{K_{zz}(Z)} \left[\int_Z^H [\bar{u}(Z') - U_{\mathrm{av}}]\, dZ' \right]^2,$$

$$(K_{xx})_{\mathrm{av}} = \frac{1}{H} \int_0^H K_{xx}(Z)\, dZ \tag{10.79}$$

[see Eq. (10.70)]. The dispersion of the admixture distribution in the OX direction in a thin layer close to a fixed height Z will also have the asymptotic form (10.78), but here the constant component will, in general, depend on Z.

Saffman investigated longitudinal diffusion in a two-dimensional channel between two solid walls, considering it as a possible model of admixture propagation in the atmospheric surface layer beneath an inversion layer which prohibits further rise of the admixture at some fixed height H. Since this model is quite rough, no attempt was made to describe wind velocity profile $\bar{u}(Z)$ more than schematically; in addition to the simplest case of laminar Couette flow where

$$\bar{u}(Z) = \frac{2Z}{H} U_{\mathrm{av}}, \quad K_{zz} = \chi = \mathrm{const}$$

and consequently according to Eqs. (10.78)–(10.79),

$$K_x = \frac{U_{\mathrm{av}}^2 H^2}{30\chi} + \chi, \tag{10.80}$$

Saffman analyzed only the case in which

$$\bar{u}(Z) = (a+1)\left(\frac{Z}{H}\right)^a U_{\mathrm{av}}, \text{ where } a = \mathrm{const},$$

and, according to the Reynolds analogy,

$$K_{zz}(Z) = \frac{u_*^2}{d\bar{u}/dZ}.$$

In this last case the integral entering into the first formula of Eq. (10.79) may be carried out explicitly. We obtain

$$K_x = \frac{a^3}{(a+2)(3a+2)} \frac{U_{\mathrm{av}}^3 H}{u_*^2} + (K_{xx})_{\mathrm{av}}. \tag{10.81}$$

Elder analyzed diffusion in a channel with a free surface at a height $Z=H$ and compared the results obtained with the data from experiments in which he investigated the spread of a drop of potassium permanganate solution on a slightly inclined channel along which water was freely flowing. Here it was required that a form of the mean velocity profile be selected which would correspond well to a real flow on an inclined channel; the result turned out to be quite sensitive to even small variations of this form. Elder proposed that the velocity profile in an open channel be given by the logarithmic form of the velocity defect law (5.43) with von Kármán's constant $\varkappa = 0.41$; then, if K_{zz} is defined using the Reynolds analogy and it is assumed that $K_{xx}=K_{yy}=K_{zz}$, one obtains

$$\begin{gathered}(K_{yy})_{\text{av}} = (K_{xx})_{\text{av}} = 0.068Hu_*, \qquad K = 5.86Hu_*, \\ K_x = (5.86+0.068)\,Hu_* = 5.93Hu_*. \end{gathered} \tag{10.82}$$

According to the data from Elder's measurements, $\frac{K_y}{Hu_*} = \frac{(K_{yy})_{\text{av}}}{Hu_*} \approx 0.23$, which exceeds approximately twice the corresponding value in Eq. (10.82); one explanation of this discrepancy may be that the assumption of equality of the three eddy diffusivities K_{xx}, K_{yy} and K_{zz} is incorrect. If, instead of this, we assume that $K_{xx}=K_{yy}>K_{zz}$, and recall also that $\overline{w'^2}$ usually turns out to be less than $\overline{u'^2}$ and $\overline{v'^2}$ [see the discussion following Eq. (5.27)], then the effective longitudinal diffusivity becomes

$$K_x = (5.86+0.23)\,Hu_* = 6.1Hu_*.$$

This value agrees quite well with that of $\frac{K_x}{Hu_*} = 6.3$ obtained from experiment. In fact, the agreement is even better than one might expect, considering that the experiments were carried out for a comparatively small value of the Re number, small τ, and under conditions which do not permit high accuracy in the data. According to Ellison (1960), the agreement obtained must be considered largely accidental since near a free surface the logarithmic form of the velocity profile cannot serve as a good approximation. Consequently, Ellison repeated Elder's calculations, assuming the velocity profile to be given by Eq. (5.49″). According to his results, the value of K/Hu_* depends strongly on the value of the coefficient b which enters into

Eq. (5.49″); the value of $b = 1.4$ corresponds best to Elder's experimental data.

Diffusion in Free Turbulent Flow

The case just examined, of longitudinal diffusion of an admixture in a straight tube or straight channel, is rather special since several properties are retained which are characteristic of diffusion in a field of homogeneous turbulence. From this viewpoint, diffusion in turbulent jets or turbulent wakes behind solid bodies are more typical examples of diffusion in a shear flow. However, theoretical study of turbulent diffusion in jets or wakes using the semiempirical diffusion equation is impossible without adoption of additional semiempirical hypotheses to determine the dependence of the eddy diffusivities, and also the mean velocity $\bar{u}$, on the spatial coordinates. These additional hypotheses are usually based on the mixing-length approach discussed in Sect. 5.9. The results obtained in this way frequently turn out to be quite satisfactory from a practical viewpoint [see, for example, Hinze (1959); Abramovich (1963)], but theoretically are not very well founded, and thus shall not be dwelled upon here.

In investigating an admixture distribution in a plane $X =$ const located downstream from a continuous steady point or linear source at a small distance from it, compared with the length scale $L = UT$ where U is the typical mean velocity scale and T is the Lagrangian integral time scale, several results may be obtained by transforming to Lagrangian variables, if we expand the Lagrangian velocity in a Taylor series (9.78). In particular, as noted by Hinze (1951; 1959), in many cases the first approximation turns out to be useful, according to which $\boldsymbol{V}(\boldsymbol{x}, t) = \bar{\boldsymbol{u}}(\boldsymbol{x}, t_0) + \boldsymbol{u}'(\boldsymbol{x}, t_0)$. Thus, $X(\boldsymbol{x}, t) = x + (\bar{u} + u') \cdot (t - t_0)$ and $Z(\boldsymbol{x}, t) = w' \cdot (t - t_0)$ where the values of $\bar{u}$, u' and w' refer to the space-time point $(\boldsymbol{x}, t_0)$. It follows from this that in a steady flow with a mean velocity directed along the OX axis, the coordinate $Z(t)$ of the fluid particle which starts from the point $\boldsymbol{x} = 0$ at the time t_0 will be equal in first approximation to

$$Z(X) \approx \frac{w'}{\bar{u} + u'} X \tag{10.83}$$

at the time that it reaches the plane $X =$ const, where $X \ll L$. If

$|u'| \ll \bar{u}$, then according to this equation $Z(X) \approx (w'/\bar{u})X$; therefore, the probability distribution for $Z(X)$ differs here by a constant factor from the probability distribution for $w'(0)$. However, if the values of u' are less than $\bar{u}$, but not negligibly small, then

$$\frac{Z(X)}{X} \approx \frac{w'}{\bar{u}} - \frac{u'w'}{\bar{u}^2} + \frac{u'^2 w'}{\bar{u}^3} - \dots . \tag{10.84}$$

Consequently, it is easy to express the moments of the random variable $Z(X)$ in terms of the moments of the Eulerian velocity fluctuation $\boldsymbol{u}' = (u', w')$ at the point $\boldsymbol{x} = 0$, where usually we may limit ourselves to only the first few terms of the series (10.84). When the equations obtained have fourth-order moments of $\boldsymbol{u}'$, apparently it is also permissible to replace them by combinations of second moments, assuming that the corresponding cumulants are negligibly small [see Batchelor and Townsend (1956), and Hinze (1959)]. In spite of the obvious roughness of such approximations they satisfactorily describe the main characteristic peculiarities of the temperature distribution in the planes X = const of a turbulent jet along the OX direction, into which a heated conductor is placed in the fixed plane $X = 0$ [Hinze and Van der Hegge Zijnen (1951), and Corrsin and Uberoi (1950; 1951)]. As an example, Fig. 81 shows the temperature distribution obtained by Hinze and Van der Hegge Zijnen. They introduced a thin wire heated by an electric current to a temperature of 400°C into a plane jet flowing from a narrow nozzle, at a distance of 100 mm from the nozzle and 6 mm below the axis of the jet. The OX axis in Fig. 81 was taken parallel to the direction of the mean velocity at the wire, and passing through the wire from which the distance X was measured; different symbols were used to designate the data from the measurements for various values of X. In contrast to analogous data for a homogeneous turbulence, here the temperature distribution is asymmetrical, or more elongated toward the region of positive values of Z, which correspond to greater values of the mean velocity than negative Z. The results obtained agree with Eq. (10.84) since in this case one may take $\overline{w'} = 0$ and $\overline{u'w'} < 0$. The mean value of the distribution shown in Fig. 81 is equal to 0.036; in this experiment $-\frac{\overline{u'w'}}{\bar{u}^2}$ has the same value. Thus, to calculate the mean value

$$\overline{Z(X)} = \frac{\int \overline{\vartheta(X, Z)}\, Z\, dZ}{\int \overline{\vartheta(X, Z)}\, dZ},$$

we may neglect the third term on the right side of Eq. (10.84). Let us note also that if we try to describe the temperature distribution in the jet by a semiempirical diffusion equation with a *constant* coefficient K_{zz}, then for the conditions of Fig. 81 we will obtain

$$\int_{-\infty}^{\infty} \overline{\vartheta(X, Z)}\, Z\, dZ < 0,$$

which contradicts the data.

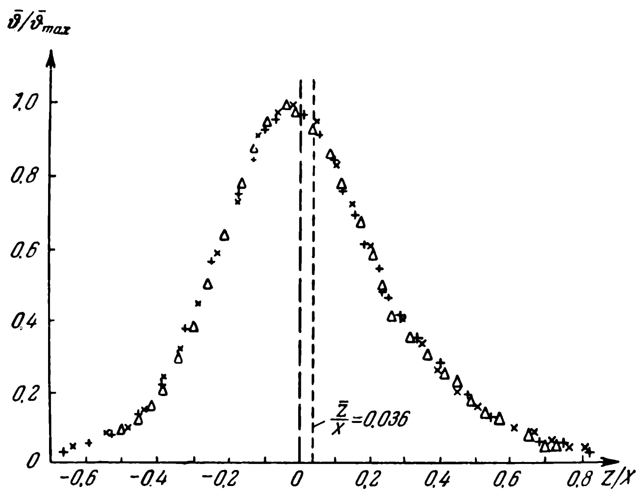


FIG. 81. Temperature distribution in several cross sections of a two-dimensional turbulent jet in which a heated wire has been placed (according to Hinze and Van der Hegge Zijnen).

In the other extreme case, that is for very large X, or, in other words, a very large diffusion time $\tau = t - t_0$, some results may be obtained using the self-preservation arguments discussed in Sects. 5.8 and 9.4. From these arguments it follows immediately that for a stationary continuous admixture source located near the body producing the wake, or the nozzle emitting the jet, the mean concentration distribution $\bar{\vartheta}$ in any two planes $X =$ const, for sufficiently large values of X, where X is calculated from the source in the direction of the mean flow velocity, will be similar to each other. In exactly the same manner, for an instantaneous source also located near the body or nozzle and acting at the time $t=t_0$, the mean concentration distribution in three-dimensional space will be similar at two times t sufficiently removed from the time t_0. On the basis of symmetry arguments it is natural to assume that the corresponding distributions will be symmetrical, their maxima being located on the axis of the jet or wake in the plane $X=$ const for a continuous source located sufficiently close to the axis or near the point $(\overline{X(\tau)}, 0, 0)$, where $\overline{X(\tau)}$ is given by the equations from Sect. 9.4 for an instantaneous source. Here the results of Sects. 5.8 and 9.4 permit simple determination of the dependence of the corresponding maximum concentrations $\vartheta_m(X)$ or $\vartheta_m(\tau)$ on X or τ although they do not offer the possibility of evaluating the exact form of the distribution $\overline{\vartheta(Y, Z)}$ or $\overline{\vartheta(X, Y, Z)}$.

Let us begin with the case of a continuously active steady admixture source, and use the fact that, due to the stationarity of the diffusion process, the admixture flux through any plane $X=$ const must be constant. In a turbulent wake the velocity of admixture transfer along the OX direction far from the body will obviously be nearly equal to the free stream velocity. At the same time the dimensions of the area in the plane $X=$ const where a noticeable admixture flux occurs from a point source at the point $\boldsymbol{x}=0$ will increase in proportion to $L^2 \sim X^{2/3}$ in the case of a three-dimensional wake behind a finite body, and in proportion to

$$L(X)\,[D_{22}(\tau)]^{1/2} \sim X^{1/2}\left[D_{22}\left(\frac{X}{U}\right)\right]^{1/2} \sim X$$

for a two-dimensional wake behind a cylinder.

In a third case of a wake behind a cylinder and a linear admixture source parallel to the axis of the cylinder, the admixture flux per unit length of the axis OY must be constant; in the plane $X=$const a unit segment of the OY axis corresponds to a rectangle in which the concentration $\bar{\vartheta}$ differs significantly from 0, having an area proportional to $L(X) \sim X^{1/2}$. However, since the admixture concentration in the plane $X=$ const is proportional to $\vartheta_m(X)$, we obtain

$$\vartheta_m(X) \sim \begin{cases} X^{-2/3} & \text{for a point source in a three-dimensional wake,} \\ X^{-1} & \text{for a point source in a two-dimensional wake,} \\ X^{-1/2} & \text{for a linear source in a two-dimensional wake.} \end{cases} \qquad (10.85)$$

Similar arguments are also applicable to diffusion in a plane mixing layer of two plane parallel flows and in turbulent jets, including the diffusion of a passive admixture in jets of convective origin. However, now the mean rate $\bar{u}$ of admixture transfer through the plane $X=$ const will not be equal everywhere to the fixed velocity U_0, but will be a function of X, Y, and Z. It is significant, however, that when the parameter X is changed, the function $\bar{u}(X, Y, Z)$ of Y and Z remains similar, or self-preserving, its maximum value remains constant for a plane mixing layer and two-dimensional convective jet, and decreases in proportion to X^{-1} for a dynamical three-dimensional jet (issuing into a space filled with the same fluid), in proportion to $X^{-1/2}$ for a dynamical two-dimensional jet, and to $X^{-1/3}$ for a three-dimensional convective jet. Furthermore, the area of that part of the plane $X =$ const where the concentration $\overline{\vartheta(X, Y, Z)}$ is noticeably different from zero increases in proportion to $L^2(X)$ for a point admixture source in a three-dimensional jet, in proportion to $L(X)$ for a linear source in a two-dimensional jet or plane mixing layer, and in the case of a point source in a two-dimensional jet, in proportion to $L(X)[D_{22}(\tau_x)]^{1/2}$ where τ_x is defined by Eqs. (9.47) or by related equations with the aid of the equality $\overline{X(\tau_x)} = X$. Since the admixture flux is proportional to the product of the concentration, the velocity, and the area, the arguments above, leading to Eqs. (10.85), now lead to the following relationships:

$$\vartheta_m(X) \sim \begin{cases} X^{-1/2} & \text{for a linear source in a two-dimensional jet,} \\ X^{-1} & \text{for a point source in a three-dimensional jet and for a linear source in a mixing layer or a two-dimensional convective jet,} \\ X^{-3/2} & \text{for a point source in the two-dimensional jet,} \\ X^{-5/3} & \text{for a point source in the three-dimensional convective jet,} \\ X^{-2} & \text{for a point source in a mixing layer or a two-dimensional convective jet.} \end{cases} \tag{10.85'}$$

Here and below, if there is no indication that the jet is convective, then we assume it to be an ordinary dynamical jet which issues into a space filled with the same fluid.

In the case of an instantaneous admixture source, the total mass of admixture must remain constant, or if the source is linear, the mass of the admixture per unit length of axis OY must remain constant. However, in the time τ, the dimension along the OX_1 axis of the admixture cloud from an instantaneous source is characterized by the length scale $[D_{ii}(\tau)]^{1/2}$, naturally coinciding with $L[X(\tau)]$ to within a constant. According to the results of Sect. 9.4, we obtain from this the following equations for the maximum concentration $\vartheta_m(\tau)$ of the admixture cloud from an instantaneous source formed in the time τ after release:

$$\vartheta_m(\tau) \sim \begin{cases} \tau^{-1} & \text{for a point source in a three-dimensional wake and a linear source in a two-dimensional wake,} \\ \tau^{-4/3} & \text{for a linear source in a two-dimensional jet,} \\ \tau^{-3/2} & \text{for a point source in a two-dimensional wake and in a three-dimensional jet,} \\ \tau^{\ 2} & \text{for a point source in a two-dimensional jet and for a linear source in a two-dimensional convective jet and in a mixing layer,} \\ \tau^{-9/4} & \text{for a point source in a three-dimensional convective jet,} \\ \tau^{-3} & \text{for a point source in a two-dimensional convective jet and in a mixing layer.} \end{cases} \tag{10.86}$$

Most of the results of Eqs. (10.85)–(10.86) belong to Batchelor (1957); the remainder were formulated by Yaglom (1965).

Diffusion in an Unbounded Homogeneous Turbulent Shear Flow

We conclude this subsection by considering admixture diffusion in an unbounded shear flow with a constant transverse mean velocity gradient $du_X(Z)/dZ = \Gamma$ and a homogeneous and stationary field of fluctuations $\boldsymbol{u}'(\boldsymbol{X}, t)$. In Sect. 9.4 we have seen that the interaction of a velocity gradient with transverse "vertical" turbulent diffusion leads to a qualitative change in the laws of longitudinal dispersion instead of a simple increase in the effective horizontal diffusivity caused by such interaction in tube or channel flows. Specifically, the interaction implies that the longitudinal dispersion $D_{xx}(\tau)$ becomes asymptotically proportional to τ^3, and not to τ, as is usually the case. Therefore, an initially spherical admixture cloud in such a flow will assume the form of an ellipsoidal spindle elongated along the OX axis, the major axis of which is slightly inclined with respect to the plane $Z = 0$. Since we are assuming the field of the fluctuations $\boldsymbol{u}'$ to be homogeneous and stationary, there is no difficulty here with the use of the semiempirical diffusion equation concerning the dependence of the eddy diffusivities on the coordinates; these diffusivities are naturally considered constant in space and in time. Therefore, one has reason to expect that in this case all the basic features of the diffusion process will be explained by investigating the solution of the differential equation

$$\frac{\partial\bar{\vartheta}}{\partial t}+\Gamma Z\frac{\partial\bar{\vartheta}}{\partial X}=K_{xx}\frac{\partial^2\bar{\vartheta}}{\partial X^2}+K_{yy}\frac{\partial^2\bar{\vartheta}}{\partial Y^2}+K_{zz}\frac{\partial^2\bar{\vartheta}}{\partial Z^2} \tag{10.87}$$

for the initial condition $\bar{\vartheta}(\boldsymbol{X}, t_0)=\delta(\boldsymbol{X})$. It is not difficult to see that the desired solution of Eq. (10.87) at the moment $t=t_0+\tau$ will be a three-dimensional Gaussian distribution with zero mean and second-order moments

$$\left.\begin{aligned} &D_{xx}=\frac{2}{3}\Gamma^2K_{zz}\tau^3+2K_{xx}\tau, \quad D_{yy}=2K_{yy}\tau, \quad D_{zz}=2K_{zz}\tau,\\ &D_{xy}=D_{yz}=0, \qquad D_{xz}=\Gamma K_{zz}\tau^2, \end{aligned}\right\} \tag{10.88}$$

i.e., that it will have the form

$$\bar{\vartheta}(\mathbf{X}, t_0+\tau)=\frac{1}{(4\pi\tau)^{3/2}[(K_{xx}+\Gamma^2K_{zz}\tau^2/12)K_{yy}K_{zz}]^{1/2}}$$
$$\exp\left\{-\frac{(X-\Gamma Z\tau/2)^2}{4K_{xx}\tau+\Gamma^2K_{zz}\tau^3/3}-\frac{y^2}{4K_{yy}\tau}-\frac{Z^2}{4K_{zz}\tau}\right\}. \tag{10.88'}$$

This result and some of its generalizations were obtained by Novikov (1958); later, it was also discussed in somewhat more detail by Elrick (1962). The moment method of Aris (1956), based on examination of Eqs. (10.76), (10.76′), (10.76″), etc., was applied to the same problem, but with $K_{xx}=0$, by Smith (1965). He found that the admixture distribution at a given height Z has, for large $\tau=t-t_0$, its center at the point ($\Gamma Z\tau/2.0$) and the variances $\sigma_{\vartheta_x}^{\ 2}=\Gamma^2K_{zz}\tau^3/6$ and $\sigma_{\vartheta_y}^{\ 2}=2K_{yy}\tau$; these results are in full agreement with Eq. (10.88′).

Equations (10.88) show the great difference between $\Gamma=0$ and $\Gamma\neq 0$; they also agree with Eqs. (10.53), according to which the diffusivities K_{ij} are equal to one-half the derivative of the covariances $D_{ij}(\tau)$ at $\tau=0$. Comparison of these equations with the exact Lagrangian equations (9.57), (9.57′), and (9.58) demonstrated quite clearly that our new results are an asymptotic form of the exact equations for the variances $D_{11}=D_{xx}$ and $D_{13}=D_{xz}$ when $\tau\gg T_3$, where, as usual,

$K_{zz}=\overline{w'^2}T_3, K_{xx}=\overline{u'^2}T_1$. However, the semiempirical diffusion theory does not permit evaluation of the covariances D_{ij} for moderate values of τ. Thus, once more we have verified that this theory describes only the asymptotic laws which are valid for large diffusion times.

The maximum of the multidimensional Gaussian density function is obviously inversely proportional to the square root of the determinant of the covariance matrix. Thus the concentration $\vartheta_m(t)$ in the center of an admixture cloud created at the time $t=t_0$ by an instantaneous point source will decrease asymptotically in proportion to $\tau^{-3/2}$ with an increase in $\tau = t - t_0$, in a field of homogeneous turbulence with a velocity gradient, and not in proportion to $\tau^{-5/2}$ as was the case for homogeneous turbulence with constant mean velocity. In exactly the same way, the maximum admixture concentration from an instantaneous linear source on the OY axis will decrease in proportion to τ^{-2} instead of τ^{-1}. Knowing the solution corresponding to the instantaneous point source, and using Eqs. (10.6) and (10.8), it is also possible to find the concentration distribution corresponding to stationary point and linear sources; however, we shall not discuss this here.

We now consider an unbounded turbulent flow having a more complex mean velocity profile $\bar{u}(Z)$ than the linear profile, and having eddy diffusivities that are dependent on the coordinate Z. In this case, the exact solution of the diffusion equation (10.55), which corresponds to an instantaneous point source, cannot be given explicitly. However, if the dependence of the mean velocity and the diffusivities on Z is given by sufficiently simple equations, or if the functions $\bar{u}(Z)$, $K_{xx}(Z)$, $K_{yy}(Z)$ and $K_{zz}(Z)$ are power functions, then the basic features of the diffusion process described by the semiempirical equation (10.55) may be investigated using Eqs. (10.76), (10.76′), (10.76″), etc., for the moments $\theta_{nm}(Z, t)$. Here also the interaction of the mean velocity gradient with the vertical turbulent diffusion, described by the diffusivity K_{zz}, leads to horizontal dispersion which, as a rule, for large diffusion time $\tau=t-t_0$ substantially exceeds the ordinary horizontal turbulent diffusion in the absence of shear. Moreover, compared to the horizontal dispersion in tubes and channels, in an unbounded space the additional horizontal dispersion usually does not reduce to a simple increase to some new value $K>K_{xx}$ in the horizontal diffusivity, but leads to the proportionality of the horizontal variance $\sigma^2_{\vartheta_x}$ to a higher power of τ than the first. This will be discussed in detail in the following subsection which investigates the practically more important case of diffusion in a half-space $Z > 0$ where all the special features of this type of diffusion in an unbounded space are retained.

10.5 Diffusion in the Atmospheric Surface Layer

The most important examples of turbulent diffusion mentioned at the beginning of Sect. 10.1 are undoubtedly those connected with diffusion of admixtures in the atmosphere. It is not surprising then, that in turbulent-diffusion literature the works devoted to atmospheric diffusion are predominant; among such works, in particular, are all the references mentioned at the beginning of Sect. 10.1.

Of greatest interest in the study of atmospheric diffusion is diffusion in the surface layer of the air most directly related to the hfe and vital activity of man. Here, the problem consists mainly in the fact that the diffusion takes place in a turbulent boundary layer that is thermally stratified in general, and fills the half-space over a solid or fluid underlying surface, which we shall assume to be homogeneous and take as the plane $Z=0$. Therefore, the results of the preceding subsection in which it was supposed that the fluid either flows between walls in a tube or a channel, or fills all unbounded space, are not directly applicable to atmospheric diffusion. However, in passing let us note that the model of diffusion in a plane channel between the walls $Z=0$ and $Z=H$ may with some justification be applied to diffusion in the atmospheric surface layer in the presence of a strong inversion at a height H. In this inversion the temperature increases with height, thereby leading to a sharp decrease in the vertical exchange at H; see, for example, Rounds (1955); Saffman (1962b); Tseytin (1963); Tyldesley and Wallington (1965). Clearly, our main interest in the theory of atmospheric diffusion is not in this model, but rather in diffusion in the entire half-space $Z > 0$ which we shall investigate below.

Solutions of the Semiempirical Diffusion Equation for the Case of Uniform Wind

We begin with the simplest approach using the semiempirical diffusion equation. As usual we shall make the nonrigorous assumption that the axes of the coordinate system $OXYZ$ [where the OX-direction coincides with the direction of the mean wind $\bar{u}=\bar{u}(Z)$] are the principal axes of the eddy diffusivity tensor $K_{ij}=K_{ij}(Z)$. In this case, the semiempirical diffusion equation assumes the form of Eq. (10.55); consequently, to reduce all the basic problems of diffusion theory to clearly stated problems of mathematical physics, we need only give somehow, the height dependence of the wind velocity $\bar{u}$ and of the diffusivities K_{xx}, K_{yy} and K_{zz}.

Equation (10.55) may be solved explicitly only for a few special assumptions regarding the height dependence of its coefficients. The qualitative features of the solutions corresponding to different types of admixture sources may be found in the simplest example of Eq. (10.58), that is, an equation of the form (10.55) with constant coefficients $\bar{u}$, K_{xx}, K_{yy}, and K_{zz}. This simplest semiempirical diffusion equation was investigated by Roberts (1923). Here the solution for an unbounded space corresponding to the presence of an instantaneous point source of output Q at the point $\boldsymbol{X}=\boldsymbol{x}$ is obtained from Eq. (10.12) on replacing X_1 by $X_1 - \bar{u}\tau$ and $D_{ii}(\tau)$ by $2K_{ii}\tau$. Using this fact it is easily seen that for diffusion in the half-space $Z>0$, with a boundary condition of "reflection" $\frac{\partial\bar{\vartheta}}{\partial Z}=0$ when $Z=0$, the solution which corresponds to an instantaneous point source at the point $(0, 0, H)$ at the time $t=t_0$ is given by the formula

$$\bar{\vartheta}(X, Y, Z, t)=\frac{Q}{[4\pi(t-t_0)]^{3/2}(K_{xx}K_{yy}K_{zz})^{1/2}}\times$$
$$\times\exp\left\{-\frac{[X-\bar{u}(t-t_0)]^2}{4K_{xx}(t-t_0)}-\frac{Y^2}{4K_{yy}(t-t_0)}\right\}\left[e^{-\frac{(Z-H)^2}{4K_{zz}(t-t_0)}}+e^{-\frac{(Z+H)^2}{4K_{zz}(t-t_0)}}\right]. \tag{10.89}$$

For the boundary condition of "absorption," that is, when $\bar{\vartheta}(X, Y, 0, t)=0$, the solution of Eq. (10.58) will differ from Eq. (10.89) only in that the second component in the brackets will have a minus sign and not a plus sign. Similarly, Eqs. (10.59) and (10.61) may be used to obtain the solutions of Eq. (10.58) in the half-space $Z>0$, with absorption or reflection conditions at $Z>0$, corresponding to a stationary point or linear source at a height $Z=H$. Usually, these solutions may be simplified by using the fact that $\bar{u}\,\bar{\vartheta}\gg\overline{u'\vartheta'}$ and therefore the convection term $\bar{u}\frac{\partial\bar{\vartheta}}{\partial X}$ almost always significantly exceeds the corresponding diffusion term

$$-K_{xx}\frac{\partial^2\bar{\vartheta}}{\partial X^2}=\frac{\partial}{\partial X}\overline{u'\vartheta'};$$

see, for example, the estimates of Karol' (1960), Walters (1964) and Yordanov (1967), concerning certain special diffusion problems. Therefore, $K_{xx}\frac{\partial^2\bar{\vartheta}}{\partial X^2}$ may usually be neglected in Eq. (10.55). In Eqs. (10.59) and (10.61) this corresponds to the limit as $K_{xx}\to 0$. Under this condition the solution corresponding to a stationary point source of output Q at the point $(0, 0, H)$ and the reflection boundary condition at $Z=0$, assumes the form

$$\bar{\vartheta}(X, Y, Z)=\frac{Q}{4\pi X\sqrt{K_{yy}K_{zz}}}e^{-\frac{\bar{u}Y^2}{4K_{yy}X}}\left[e^{-\frac{\bar{u}(Z-H)^2}{4K_{zz}X}}+e^{-\frac{\bar{u}(Z+H)^2}{4K_{zz}X}}\right]. \tag{10.90}$$

For an absorption boundary condition $Z=0$, again we need only replace the plus sign in the brackets of Eq. (10.90) by the minus sign. According to Eq. (10.90), the admixture distribution along the line X = const, Z = const transverse to the direction of the wind, for any X and Z, is given by a Gaussian function with standard deviation $\left(\frac{2K_{yy}X}{\bar{u}}\right)^{1/2}$ proportional to $X^{1/2}$ and inversely proportional to $\bar{u}^{1/2}$. When X increases and Y and Z are fixed, the concentration $\bar{\vartheta}$ decreases asymptotically in proportion to X^{-1}, similar to diffusion in a field of homogeneous turbulence in an unbounded space. For reflection of the admixture, the maximum surface concentration

$$\bar{\vartheta}_{\max}=\frac{Q}{\pi e\bar{u}H^2}\left(\frac{K_{zz}}{K_{yy}}\right)^{1/2}$$

is inversely proportional to the square of the source height; it is reached when $Y=0$ at a distance from the source $X=\frac{\bar{u}H^2}{4K_{zz}}$, directly proportional to H^2. For the absorption boundary condition, the rate of the admixture absorption by the underlying surface

$$\sigma(X, Y)=K_{zz}\frac{\partial\bar{\vartheta}}{\partial Z}\bigg|_{Z=0},$$

as is easy to prove, decreases asymptotically in proportion to X^{-2}. In this case the maximum rate of absorption

$$\sigma_{\max}=\frac{4QK_{zz}}{\pi e^2\bar{u}H^3}\left(\frac{K_{zz}}{K_{yy}}\right)^{1/2}$$

is inversely proportional to the cube of the source height and is achieved when $Y = 0$ and $X = \frac{\bar{u}H^2}{8K_{zz}}$. The solution for a stationary linear source along the straight line $X=0$, $Z=H$ producing per unit time a mass Q of admixture per unit length, for the reflection boundary condition, will have the form

$$\bar{\vartheta}(X, Z) = \frac{Q}{2(\pi\bar{u}XK_{zz})^{1/2}}\left[e^{-\frac{\bar{u}(Z-H)^2}{4K_{zz}X}} + e^{-\frac{\bar{u}(Z+H)^2}{4K_{zz}X}}\right] \qquad (10.91)$$

(In the case of absorption of the admixture by the earth's surface we must again replace the plus sign by the minus sign.) According to Eq. (10.91), the surface concentration from a linear source decreases in proportion to $X^{-1/2}$ at large distances. The solutions of Eq. (10.58) which correspond to the more general boundary conditions

$$K_{zz}\frac{\partial\bar{\vartheta}}{\partial Z}\bigg|_{Z=0} = \beta\bar{\vartheta}\,|_{Z=0}, \quad \text{where } 0<\beta<\infty,$$

may also be given explicitly, but they have a more complex form, and hence are not discussed here.

At first glance the above-enumerated results obtained using Eq. (10.58) for the concentration $\bar{\vartheta}$ appear quite likely; however, they do not agree at all with the data obtained from various field investigations of diffusion in the atmospheric surface layer. Thus, for example, experiments show that the surface concentration of admixture, produced by a stationary point source in a neutrally stratified atmosphere, decreases with distance from the source approximately proportional to X^{-2} [according to Sutton (1952), proportional to $X^{-1.8}$], but in no way proportional to X^{-1}; the surface concentration from a stationary linear source in a neutrally stratified atmosphere decreases in proportion to X^{-1} or $X^{-0.9}$ [Sutton (1953); Pasquill (1962b)], but not to $X^{-1/2}$. The data on stable or unstable stratification are fewer than for neutral stratification, but they also show that in no case does the decrease in the surface concentration follow the simple rules derived from Eqs. (10.90) and (10.91) [see, for example, Deacon (1949)]. Thus, one must conclude that Eq. (10.58) is unsuitable for quantitative description of atmospheric diffusion in the surface layer.

The lack of correspondence obtained may be explained in principle by the fact that the semiempirical diffusion equation cannot be rigorously derived and is not exact. Clearly, this is insufficient basis for concluding that the semiempirical diffusion equation is inapplicable to atmospheric admixture dispersion. Indeed, all the preceding conclusions were derived on the basis of a very rough model in which the height dependence of both the wind velocity and the eddy diffusivities was completely neglected. However, it is well known that in the real atmosphere both the wind velocity and the diffusivities increase with height (see above, Chapt. 4). Therefore, it is natural first to try to generalize the model, making some reasonable assumptions concerning the form of the functions $\bar{u}(Z), K_{xx}(Z), K_{yy}(Z)$ and $K_{zz}(Z)$. (Another type of generalization connected with the inclusion of nondiagonal components of the tensor K_{ij} will be considered later.) These efforts have been made by many investigators; since the solution of the diffusion equation with variable coefficients encounters significant analytical difficulties, these efforts have generated extensive literature, chiefly of applied interest.

Let us note first that no single explicit analytical solution of a nonstationary diffusion problem in a half-space $Z > 0$, which is easily reduced to finding equations for the diffusion from an instantaneous point source, has apparently yet been obtained in any case of a wind velocity which varies with height $\bar{u}(Z)$. Therefore, to solve the problem of diffusion of the admixture cloud from an instantaneous source, some approximate assumptions are always used. Usually, one first considers only the vertical diffusion process, that is, the time dependency of the total concentration at a height Z is determined, which is equal to

$$\theta_{00}(Z, t) = \int\int_{-\infty}^{\infty} \bar{\vartheta}(X, Y, Z, t)\, dX\, dY$$

[see above, Eq. (10.75)] and only then is the horizontal diffusion also approximately considered. The function $\theta_{00}(Z, t)$, as we know, satisfies Eq. (10.76) as

$$\frac{\partial \theta_{00}}{\partial t} = \frac{\partial}{\partial Z}\left[K_{zz}(Z)\frac{\partial \theta_{00}}{\partial Z}\right].$$

The corresponding initial and boundary conditions, for an instantaneous source at a height $Z = H$ at the time $t = t_0$, and reflection of the admixture at the boundary $Z = 0$, will have the form

$$\theta_{00}(Z, t_0) = Q\delta(Z - H), \quad K_{zz}\frac{\partial \theta_{00}}{\partial Z}\bigg|_{Z=0} = 0, \quad \theta_{00}(Z, t) \to 0 \quad \text{for} \quad Z \to \infty. \tag{10.92}$$

Thus, the admixture distribution between the various horizontal planes Z = const depends on neither the wind profile nor the horizontal diffusivities, and is determined uniquely by the function $K_{zz}(Z)$ and the source height H. Solutions of Eqs. (10.76) under conditions

(10.92) for functions $K_{zz}(Z)$, suitably approximating the profile of the vertical eddy diffusivity in the atmospheric surface layer under various meteorological conditions, were found, for example, by Laykhtman in the 1940's; later, they were also investigated by many others [see, for example, Monin (1956a)]. When $K_{zz} = K_1 Z$ this solution has the form

$$\theta_{00}(Z, t) = \frac{Q}{K_1(t-t_0)} e^{-\frac{Z+H}{K_1(t-t_0)}} I_0\left[\frac{2\sqrt{HZ}}{K_1(t-t_0)}\right], \tag{10.93}$$

where I_0 denotes a Bessel function of an imaginary argument, or a modified Bessel function of the first kind of zero order. Since for neutral thermal stratification $K_{zz} = \alpha \varkappa u_* Z$, where $\alpha = \frac{K_\vartheta}{K} = \text{const} \approx 1.1$ [see Eq. (5.29) and Sect. 5.7], it is natural to use Eq. (10.93) in the absence of a noticeable vertical temperature gradient. In the case of a surface source, or when $H = 0$, Eq. (10.93) is greatly simplified. Here

$$\theta_{00}(Z, t) = \frac{Q}{K_1(t-t_0)} e^{-\frac{Z}{K_1(t-t_0)}}. \tag{10.93'}$$

In the more general case of a power-law height dependency of the eddy diffusivity when $K_{zz} = K_1 Z^n$, where $0 < n < 2$,[3] the solution of the problem (10.92) for Eq. (10.76) is given by the following:

$$\theta_{00}(Z, t) = \frac{Q}{(2-n) K_1 (t-t_0) H^{n-1}} \left(\frac{Z}{H}\right)^{\frac{1-n}{2}} e^{-\frac{Z^{2-n}+H^{2-n}}{(2-n)^2 K_1 (t-t_0)}} \times$$
$$\times I_{-\frac{1-n}{2-n}}\left[\frac{2(ZH)^{\frac{2-n}{2}}}{(2-n)^2 K_1 (t-t_0)}\right]. \tag{10.94}$$

When $H \to 0$ we obtain

$$\theta_{00}(Z, t) = \frac{Q}{(2-n)^{\frac{n}{2-n}} \Gamma\left(\frac{1}{2-n}\right) [K_1(t-t_0)]^{\frac{1}{2-n}}} e^{-\frac{Z^{2-n}}{(2-n)^2 K_1 (t-t_0)}}. \tag{10.94'}$$

Even more awkward formulas for $\theta_{00}(Z, t)$ are obtained when using functions $K_{zz}(Z)$ containing a "break," that is, made up of two analytical expressions covering different height intervals. Nevertheless, when

$$K_{zz}(Z) = \begin{cases} K_1 Z & \text{for} \quad 0 < Z < H_1, \\ K_1 H_1 = \text{const} & \text{for} \quad Z \geqslant H_1, \end{cases} \tag{10.95}$$

the complicated analytical representation of the function $\theta_{00}(Z, t)$, in the form of an

[3] It can be shown that Eq. (10.76) has no solution satisfying the condition $\theta_{00}(Z, t) \to \delta(Z - H)$ as $t \to t_0$ when $K_{zz} = K_1 Z^n$, $n \geqslant 2$. However, this fact is not important in atmospheric diffusion theory because the case $n \geqslant 2$ is not met in atmospheric applications.

integral of Bessel functions was pointed out and examined numerically by Monin (1956a). Yordanov (1966; 1968a) investigated two-layer models of $K_{zz}(Z)$, consisting of two power functions, that were even more complicated.

According to the similarity theory of Chapt. 4 for the atmospheric surface layer, before selecting an analytical form for the function $K_{zz}(Z)$, it is expedient to transform to the dimensionless variables

$$\zeta = \frac{Z}{|L|}, \quad \tau = \frac{\varkappa u_* t}{|L|}, \quad \eta = \frac{H}{|L|}, \quad K(\zeta) = \frac{K_{zz}(Z)}{\varkappa u_* |L|}, \quad \theta = \frac{|L|\,\theta_{00}}{Q} \tag{10.96}$$

where L is a natural height scale in the surface layer defined by Eq. (7.12). If, as usual, we assume that the eddy diffusivity for heat equals that for any passive admixture, then Eq. (7.21) evidently implies that the quantity K in Eq. (10.96) differs from the Richardson number Ri, or in the case of unstable stratification, from its modulus, only by the factor $\alpha^2 = \left(\frac{K_\vartheta}{K}\right)^2$ which, apparently, in many cases is not very far from unity. However, the cases of extreme instability or stability are the exceptions. The function $\Theta(\zeta, \tau) = \Theta(\zeta, \tau | \eta)$ will now be defined by the conditions

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial \zeta} K(\zeta) \frac{\partial \theta}{\partial \zeta}, \quad K(\zeta) \frac{\partial \theta}{\partial \zeta}\bigg|_{\zeta=0} = 0,$$
$$\theta \to 0 \quad \text{for} \quad \zeta \to \infty, \quad \theta(\zeta, 0) = \delta(\zeta - \eta), \tag{10.97}$$

which are universal, that is, do not contain any meteorological parameters. Theoretically speaking, here we must consider only the two cases of unstable and stable stratification corresponding to two different functions $K(\zeta)$, and we obtain formulas which are applicable in all real cases. From a practical viewpoint, it is convenient to also add a third case of neutral stratification, although in principle it may be obtained from cither of the two preceding ones as $|L| \to \infty$. In this third case, as we have already noted, it is expedient to assume that $K(\zeta) = \zeta$, or, more precisely, $K(\zeta) = \alpha\zeta$, $\alpha \approx 1.1$, and to apply Eqs. (10.93) and (10.93′). As for the stable and unstable stratification, if the corresponding universal functions $K(\zeta)$ were known with sufficient accuracy, it would be possible to solve the two problems (10.97) numerically once and for all without any use of analytical approximations for $K(\zeta)$, setting up tables of corresponding functions $\Theta(\zeta, \tau | \eta)$ of three variables. However, we know from Sect. 8 that the function $K(\zeta)$ is known only in general features for unstable and especially for stable stratification. Moreover, the use of similarity theory under real conditions is always accompanied by considerable scatter of experimental data; consequently, we are almost limited to using approximate expressions for $K(\zeta)$, which can be selected to permit us to obtain explicit formulas for $\Theta(\zeta, \tau | \eta)$. In particular, for weak stable stratification or when L is large and only relatively small values of ζ are of interest, it is possible following Laykhtman (1944) and Deacon (1949), to approximate $K(\zeta)$ by a power function of the form $K_1\zeta^n$ with an index n somewhat less than one. For strong stability, or when we are interested in a broad range of values of ζ, we may use the fact that

$$K(\zeta) = \alpha_0\zeta \text{ for } \zeta \to 0 \text{ and } K(\zeta) \to \frac{\alpha_\infty}{R} = \text{const for } \zeta \to \infty$$

(see Sect. 7) and use a function $K(\zeta)$ of the form (10.95). Similarly, for weak unstable stratification it is possible to assume approximately that $K(\zeta) = K_1\zeta^n$, where n is somewhat greater than unity. For stronger instability a two-layer formula of the following form

$$K(\zeta)=\begin{cases}\alpha_0|\zeta| & \text{for} \quad 0\leqslant-\zeta\leqslant\zeta_1^*=\left(\dfrac{C_2\alpha_0}{3\alpha_{-\infty}}\right)^3, \\ \dfrac{3\alpha_{-\infty}}{C_2}|\zeta|^{4/3} & \text{for} \quad -\zeta>\zeta_1^*,\end{cases} \tag{10.98}$$

and which is derived from Eq. (7.51) is more suitable: this case was examined by Yordanov (1966; 1968a). Finally, for especially strong instability, when values of $|\zeta|$ which exceed 0.05 are of special interest, one may assume that $K(\zeta) = K_1\zeta^{4/3}$ and use Eqs. (10.94)–(10.94′) with $n = 4/3$.

The above-obtained results permit quite accurate semiempirical description of vertical diffusion in the atmospheric surface layer for any thermal stratification. However, for horizontal diffusion, the simplest (but very rough) method here is to replace approximately the wind velocity $\bar{u}(Z)$ and the coefficients $K_{xx}(Z)$ and $K_{yy}(Z)$ by the constants U_{av}, $(K_{xx})_{av}$ and $(K_{yy})_{av}$, equal to the average values of the corresponding functions of Z within the layer between the upper and lower bounds of the admixture cloud. (This may be calculated in advance from the function $\theta_{00}(Z,t)$.) Then the concentration $\bar{\vartheta}(X, Y, Z, t)$ from an instantaneous point source of output Q at the point $(0, 0, H)$ occurring at the time $t = t_0$, will be represented in first approximation as

$$\bar{\vartheta}(X, Y, Z, t) \approx \frac{1}{4\pi[(K_{xx})_{av}(K_{yy})_{av}]^{1/2}(t-t_0)} \times$$
$$\times \exp\left\{-\frac{[X-U_{av}(t-t_0)]^2}{4(K_{xx})_{av}(t-t_0)}-\frac{Y^2}{4(K_{yy})_{av}(t-t_0)}\right\}\theta_{00}(Z,t), \tag{10.99}$$

where $\theta_{00}(Z,t)$ is the solution of the corresponding equation (10.76) under the conditions (10.92). Since the coefficients $K_{xx}(Z)$ and $K_{yy}(Z)$ usually are rather poorly known, the practical application of Eq. (10.99) frequently turns out to be quite difficult and may lead, for various values of these coefficients, to quite different results. Even more significant, however, is the fact that when using equations of the form (10.99) for any selection of K_{xx} and K_{yy} with a constant average wind velocity, we completely lose the important effect of interaction between the vertical mixing and the wind shear. We have already seen above that in turbulent shear flows in tubes and channels and in an unbounded space, this cffect with sufficiently large diffusion time, plays a main part in the horizontal admixture dispersion. Thus it is clear that Eq. (10.99) may be applicable only for comparatively small values of $t = t_0$ in the atmospheric surface layer too.

The Effect of Wind Shear on Horizontal Atmospheric Dispersion

Saffman (1962b) used Aris' method for an approximate accounting of the influence of the wind shear on the horizontal dispersion. Here, the admixture distribution in the planes $Z=$ const is described by several first moments $\theta_{nm}(Z, t)$ given by Eq. (10.75). We obtain the famihar equations (10.76) for the moments θ_{00}, θ_{10}, θ_{20}, etc., from the semiempirical diffusion equation in which, however, the boundary conditions used in Sect. 10.4, $\left.\dfrac{\partial\theta_{nm}}{\partial Z}\right|_{Z=H} = 0$, on the upper bound of the channel $Z=H$ must now be replaced by the condition $\theta_{nm}(Z, t) \to 0$ when $Z \to \infty$. Since we are interested primarily in the asymptotic behavior of the admixture cloud from an instantaneous source at time t_0 when $t - t_0 \to \infty$, we can limit ourselves to a surface source at the coordinate origin, i.e., the point (0, 0, 0), since it is natural to suppose that the source height H will influence the dispersion process only for a limited time. To examine the basic qualitative peculiarities resulting from the interaction between the wind shear and the vertical mixing, first let us consider the simplest case where $\bar{u}(Z)=\Gamma Z$ and $K_{zz}(Z)=K=$ const. (The case of $\bar{u}(Z) = U_0+\Gamma Z$, $K_{zz}=$ const which is of interest for a source at finite height H, is easy to obtain from this with the help of a transformation to a new inertial system of coordinates.) Consequently, the desired solution of Eqs. (10.76) for $\theta_{00}(Z, t)$ will be

$$\theta_{00}(Z, t) = \theta_{00}(Z, t_0+\tau) = \frac{Q}{(\pi K\tau)^{1/2}} e^{-\frac{Z^2}{4K\tau}}.$$

(The symbol τ here and henceforth will designate the difference $t - t_0$.) The solution of Eq. (10.76′) for the function $\theta_{10}(Z, t)$ and thus $\theta_{00}(Z, t)$ when $K_{zz}=K$, $\bar{u}(Z)=\Gamma Z$ is more complex; however, it is clear that it must be proportional to $Q\Gamma$ and, consequently, on the basis of dimensional considerations must have the form

$$\theta_{10}(Z, t) = \theta_{10}(Z, t_0+\tau) = Q\Gamma\tau \cdot \varphi_{10}\left(\frac{Z}{(K\tau)^{1/2}}\right). \tag{10.100}$$

Here $\varphi_{10}(\xi)$ is a universal function of one variable such that $\varphi_{10}(\xi) \to 0$ when $\xi \to \infty$; with the help of Eq. (10.76′) this function is expressed simply in terms of $\exp(-\xi^2)$ and the so-called parabolic cylinder functions of mathematical physics [see Saffman (1962b)]. This holds also for the most important function $\theta_{20}(Z, t)$, which describes the variance of the admixture distribution in the plane Z = const in the direction of the OX axis. According to Eqs. (10.76″), this function depends also on the horizontal eddy diffusivity K_{xx}; more exactly, it is the sum of two terms, the first of which depends only on $\bar{u}(Z)$ and $K_{zz}=K$ but not on K_{xx}, and the second, on $K_{xx}(Z)$ and $K_{zz}=K$, but not on $\bar{u}(Z)$. The first of these terms is obviously proportional to $Q\Gamma^2$ and, therefore, on the basis of dimensional considerations must be written in the form of the product

$$Q\Gamma^2K^{-2}(K\tau)^{5/2}\varphi_{20}(Z/(K\tau)^{1/2});$$

the general form of the second term may also be written down easily if we assume that K_{xx} is a power function of Z. In particular, if $K_{xx}=K_1=$ const or $K_{xx}=K_1^*Z$ where $K_1^*=$const, then the solution of Eqs. (10.76) which we need can be written as

$$\theta_{20}(Z,\ t_0+\tau)=\frac{Q\Gamma^2}{K^2}(K\tau)^{5/2}\varphi_{20}\left(\frac{Z}{\sqrt{K\tau}}\right)+$$
$$+\begin{cases} QK_1\sqrt{\frac{\tau}{K}}\,\varphi_{20}^{(1)}\left(\frac{Z}{\sqrt{K\tau}}\right) & \text{for} \quad K_{xx}=K_1, \\ QK_1^*\tau\,\varphi_{20}^{(2)}\left(\frac{Z}{\sqrt{K\tau}}\right) & \text{for} \quad K_{xx}=K_1^*Z. \end{cases} \tag{10.101}$$

Here $\varphi_{20}(\xi)$, $\varphi_{20}^{(1)}(\xi)$ and $\varphi_{20}^{(2)}(\xi)$ are new functions which are damped at infinity and which, it turns out, may also be expressed in terms of the function $\exp(-\xi^2)$ and the parabolic cylinder functions. Similar but more complex expressions may be obtained also for the higher-order moments $\theta_{nm}(Z,\ t)$ [with $n+m>2$].

According to Eqs. (10.100), (10.101), etc., the moments of the admixture distribution in the plane $Z=$ const at time $t=t_0+\tau$ are power functions of τ multiplied by functions of $\xi=\frac{Z}{\sqrt{K\tau}}$ which, as is easily proved, assume finite values when $\xi=0$. Therefore, in any layer $0<Z<H$ of fixed thickness H, the moments of the admixture distribution in all the planes $Z=$ const have asymptotically identical form when $\tau\to\infty$. The total admixture distribution between the various planes $Z=$ const is described by the function

$$(Z,\ t_0+\tau)=Q\Gamma\tau\cdot\varphi_{10}\left(\frac{Z}{(K\tau)^{1/2}}\right)\cdot$$

This function does not depend on $\bar{u}(Z)$ and $K_{xx}(Z)$; it shows that for a fixed value of τ the thickness of the admixture cloud is on the order of $\sqrt{K\tau}$. Moreover, if $H\ll\sqrt{K\tau}$, then the admixture is distributed vertically almost uniformly within the layer of thickness H, with a constant vertical density $\theta_{00}=\frac{Q}{\sqrt{\pi K\tau}}$. According to Saffman, the equations for the moments $\theta_{10}(Z,\ t_0+\tau)$ and $\theta_{20}(Z,\ t_0+\tau)$ at $Z\ll\sqrt{K\tau}$, obtained after substitution into Eqs. (10.100) and (10.101) of the explicit expressions for the corresponding transcendental functions $\varphi_{10}(\xi)$ $\varphi_{20}(\xi)$, $\varphi_{20}^{(1)}(\xi)$ and $\varphi_{20}^{(2)}(\xi)$ and passage to the limit $\xi\to 0$, have the following form:

$$\theta_{10}=\frac{Q\Gamma\tau}{4},\quad \theta_{20}=\frac{7}{30\sqrt{\pi}}\frac{Q\Gamma^2}{K^2}(K\tau)^{5/2}+$$
$$+\begin{cases}\frac{2QK_1}{\sqrt{\pi}}\left(\frac{\tau}{K}\right)^{1/2} & \text{for} \quad K_{xx}=K_1,\\ \frac{QK_1^*\tau}{2} & \text{for} \quad K_{xx}=K_1^*Z.\end{cases} \tag{10.102}$$

Thus it follows that the coordinate X of the center of gravity of the admixture distribution in the lower layer, of thickness $H \ll \sqrt{K\tau}$), and the variance $\sigma^2_{\vartheta_x}$ of this distribution in the OX direction, have the following form:

$$\langle X\rangle=\frac{\theta_{10}}{\theta_{00}}=\frac{\Gamma}{4}\sqrt{\pi K\tau^3};\quad \sigma^2_{\vartheta_x}=\frac{\theta_{20}}{\theta_{00}}-\left(\frac{\theta_{10}}{\theta_{00}}\right)^2=$$
$$=\left(\frac{7}{30}-\frac{\pi}{16}\right)\Gamma^2K\tau^3+\begin{cases}2K_1\tau & \text{for} \quad K_{xx}=K_1,\\ \frac{K_1^*}{2}\sqrt{\pi K\tau^3} & \text{for} \quad K_{xx}=K_1^*Z.\end{cases} \tag{10.103}$$

We see that the center of gravity of the admixture cloud in the lower layer is shifted by the mean flow along the OX axis with an ever-increasing velocity $\frac{3\sqrt{\pi}}{8}\Gamma\sqrt{K\tau}$ which is proportional to the velocity gradient Γ and the square root of the vertical eddy diffusivity. The variance of this distribution is made up of two components. The first is proportional to $\Gamma^2K\tau^3$, or distinguished only by a constant factor from the horizontal variance of an admixture cloud diffusing in an unbounded shear flow; obviously it is produced by the interaction of the vertical turbulent mixing with the velocity gradient. However, the numerical coefficient of this component $7/30-\pi/16 \approx 0.036$ is equal to only about one-fifth of the corresponding coefficient and one-sixth for the case of an unbounded flow region; see the previous subsection. As a result, the presence of the wall greatly suppresses the horizontal dispersion. The second component in the expression for $\sigma^2_{\vartheta_x}$ describes the ordinary horizontal dispersion due to turbulent fluctuations of horizontal velocity, that is, turbulent diffusion with coefficient K_{xx}. This second component has the ordinary form $2K_1\tau$ when $K_{xx} = K_1$, but when $K_{xx}=K_1^*Z$, it also depends on $K_{zz}= K$, that is, it expresses the interaction of horizontal and vertical diffusion, and is proportional to $\tau^{3/2}$. However, in any case, it is essential that for sufficiently large values of τ, it turns out to be negligibly small compared with the first component of order τ^3. Let us also note that the equation for $\theta_{02}(Z, t)$ will differ from Eq. (10.76″) by the absence of a term with $\bar{u}(Z)$, under the assumption that the mean velocity for all Z is directed along the OX axis. Therefore, the variance $\sigma^2_{\vartheta_y}$ will be described only by the second component of Eq. (10.103) for $\sigma^2_{\vartheta_x}$, with the obvious replacement of K_{xx} by K_{yy}; that is, for large values of τ it will be much smaller than $\sigma^2_{\vartheta_x}$. The maximum surface concentration $\bar{\vartheta}_m = \bar{\vartheta}_m(\tau)$, achieved obviously at the point $\langle X \rangle$, 0, will decrease asymptotically in proportion to $\tau^{-5/2}$ when $\tau \to \infty$ if $K_{yy} = \text{const}$ and in proportion to $\tau^{-11/4}$ if $K_{yy} \sim Z$. Clearly, all these facts in no way agree with the simplified equation (10.99). In addition, this equation turns out to be incorrect also in the sense that the admixture distribution in the plane $Z =$ const, for sufficiently large values of $\tau = t - t_0$, does not resemble a Gaussian distribution in form. Indeed, using expressions that are similar to Eqs. (10.100)–(10.102) for the moment θ_{30}, it is possible to show that

$$\theta_{30} \approx \frac{21}{256} Q\Gamma^3 K\tau^4$$

for sufficiently large τ. Consequently, it follows that the asymmetry of the admixture distribution in the OX direction, for a fixed Z and $\tau \to \infty$, does not approach zero, but unity.

These results refer to the model in which $\bar{u}(Z) = \Gamma Z$, $K_{zz} = K =$ const. However, it is natural that in other cases also, if the wind velocity is height-dependent, a similar situation will exist. It would be of considerable interest to investigate the case

$$\bar{u}(Z) = \frac{u_*}{\varkappa} \ln \frac{Z}{Z_0}, \quad K_{zz} = \alpha \varkappa u_* Z ,$$

that is, diffusion in the logarithmic layer, and also the cases in which $u(Z)$ and $K_{zz}(Z)$ are defined by the equations derived in Chapt. 4 for a thermally stratified boundary layer. However, such an investigation, based on the semiempirical equation of turbulent diffusion, has serious analytical difficulties. Therefore, we shall limit ourselves briefly to considering diffusion in the neutrally stratified boundary layer [following Chatwin (1968)].

Let us write $\bar{\vartheta}(X, Z, t) = \int_{-\infty}^{\infty} \bar{\vartheta}(X, Y, Z, t)\, dY$, where $\bar{\vartheta}(X, Y, Z, t)$ is the mean concentration from an instantaneous point source of output Q at the point $(0, 0, H)$ at the time t_0; $\bar{\vartheta}(X, Z, t)$ is also the mean concentration from an instantaneous linear source along the line $X = 0, Z = H$ at the time t_0 of output Q per unit length. Then the semiempirical diffusion equation, in which the longitudinal turbulent diffusion term will be neglected as being small compared with the convection term, implies the following equation for $\bar{\vartheta}(X, Z, t)$:

$$\frac{\partial \bar{\vartheta}}{\partial t} + \frac{u_*}{\varkappa} \log \frac{Z}{Z_0} \frac{\partial \bar{\vartheta}}{\partial X} = \frac{\partial}{\partial Z}\left[\varkappa' u_* Z \frac{\partial \bar{\vartheta}}{\partial Z}\right], \varkappa' = \alpha\varkappa. \quad (10.104)$$

It is clear that the function $\theta_{00}(Z, t) = \int_{-\infty}^{\infty} \bar{\vartheta}(X, Z, t)\, dX$ must have the form (10.93), with $K_1 = \kappa' u$. Thus

$$\langle Z \rangle = \int_0^\infty Z\theta_{00}(Z, t)\, dZ \Big/ \int_0^\infty \theta_{00}(Z, t)\, dZ = \varkappa' u_* \left(\tau + \frac{H}{\varkappa' u_*}\right), \tau = t - t_0 \,. \quad (10.104')$$

The simplest derivation of Eq. (10.104′) is based on the equation $\frac{d\langle Z \rangle}{dt} = \varkappa' u_*$ which follows from Eq. (10.104) and on the initial condition $\langle Z \rangle = H$ when $\tau = 0$. When $H = 0$, the formula for $\theta_{00}(Z, t)$ is considerably simplified [see Eq. 10.93′)], and the result (10.104′) may be quite easily and immediately obtained.

Henceforth, for simplicity, H will usually be taken as zero and Q as unity. Then Eq. (10.93′) easily leads to the result

$$\sigma_{\vartheta_z}^2 = \langle Z - \langle Z \rangle \rangle^2 = \int_0^\infty (Z - \varkappa' u_* \tau)^2 \theta_{00}(Z, t)\, dZ = \varkappa'^2 u_*^2 \tau^2. \quad (10.104'')$$

The center of gravity of the admixture cloud from an instantaneous source in the boundary layer rises with the constant velocity $\varkappa' u_*$ and the cloud thickness, defined as twice its vertical standard deviation, is equal to twice the distance from its center of gravity to the earth.

The examination of the horizontal dispersion of the admixture cloud is more difficult. Nevertheless, if we consider the quantity

$$\langle X \rangle = \int_{-\infty}^{\infty} \int_{0}^{\infty} X \bar{\vartheta}(X, Z, t)\, dX dZ,$$

then according to Eqs. (10.93′) and (10.104)

$$\frac{d\langle X \rangle}{dt} = \frac{u_*}{\varkappa} \ln \frac{\varkappa' u_* t}{Z_0 e^{-\gamma}}, \text{ i.e., } \langle X \rangle = \frac{u_* \tau}{\varkappa} \left[\ln \frac{\varkappa' u_* \tau}{Z_0 e^{\gamma}} - 1 \right] \tag{10.105}$$

where

$$\gamma = -\int_{0}^{\infty} e^{-x} \ln x dx = \lim_{n \to \infty} [1 + 1/2 + \cdots + 1/n - \ln n] \approx 0.58$$

is the so-called Euler constant. For $H \neq 0$ the computations are more complicated; however, for this case Chatwin also obtained the exact asymptotic result, when $t \to \infty$:

$$\langle X \rangle \approx \frac{u_* \tau}{\varkappa} \left[\ln \frac{\varkappa' u_*}{Z_0 e^{\gamma}} \left(\tau + \frac{H}{\varkappa' u_*} \right) - 1 \right]. \tag{10.105′}$$

The evaluation ot the quantity

$$\sigma_{\vartheta_x}^2 = \langle X - \langle X \rangle \rangle^2 = \int_{-\infty}^{\infty} \int_{0}^{\infty} (X - \langle X \rangle)^2 \bar{\vartheta}(X, Z, t)\, dX\, dZ$$

is considerably more complex and requires tedious algebra; nevertheless, Chatwin's final result for this problem is also quite simple:

$$\sigma_{\vartheta_x}^2 = \left(\frac{\pi^2}{6} - 1 \right) \left(\frac{u_* \tau}{\varkappa} \right)^2, \text{ i.e., } \sigma_{\vartheta_x} \approx 0.8 \frac{u_* \tau}{\varkappa} \approx 2 u_* \tau. \tag{10.105″}$$

All the above-mentioned results are in full agreement with the predictions of the Lagrangian characteristics in a nonstratified boundary layer established in Sect. 9.4 by similarity arguments and dimensional analysis. We see also that the approximate semiempirical diffusion theory permits one to estimate the dimensionless universal constants entering the general equations (9.60), (9.61), and so on; with the same accuracy as that to which the semiempirical diffusion equation is valid, we must take the estimates: $b = \varkappa' = \alpha\varkappa \approx 0.45$, $c = be^{-\gamma} \approx 0.56\,b$ and $D_{33}(\tau) = \varkappa'^2 u_*^2\tau^2$, that is, $d_{33} = \varkappa'^2 \approx 0.2$,

$$D_{11}(\tau) = \left(\frac{\pi^2}{6} - 1\right) u_*^2\tau^2/\varkappa^2,$$

that is,

$$d_{11} = \left(\frac{\pi^2}{6} - 1\right)\varkappa^{-2} \approx 4.$$

The degree of accuracy of these estimates will be discussed at the end of Sect. 10.6.

Let us also note that it is not justified to make a direct comparison of these results for diffusion in a boundary layer with Eqs. (10.103) because these equations deal with the characteristics of the admixture distribution in the bottom layer only. However, Eq. (10.104) may also be used for the computation of the Z-dependent characteristics of the admixture distribution at a given height Z:

$$\langle X(Z,t)\rangle = \int_0^\infty X\bar{\vartheta}\,dX \Big/ \int_0^\infty \bar{\vartheta}\,dX,$$

$$\sigma_{\vartheta_x}^{\;2}(Z,t) = \int_{-\infty}^{\infty} (X - \langle X(Z,t)\rangle)^2\,\bar{\vartheta}\,dX \Big/ \int_{-\infty}^{\infty} \bar{\vartheta}\,dX.$$

It is clear that as $t \to \infty$ any fixed height Z gets into the bottom layer and therefore the asymptotic results for large t at any Z are independent of Z. According to the computations of Chatwin, as

$t = \infty$ for any fixed Z

$$\langle X(Z,t)\rangle = \langle X\rangle - \frac{u_*\tau}{\varkappa}, \sigma_{\vartheta_x}^{\ 2}(Z,t) \approx 0.6\,\frac{u_*\tau}{\varkappa}, \qquad (10.105''')$$

where $\langle X\rangle$ is given by Eq. (10.105); these results can be compared with Eq. (10.103). Equations (10.105‴) show specifically that the center of gravity of the admixture at any given level lags behind the center of gravity of the whole cloud. This fact is quite expected since, for any Z, there is a time after which nearly all the admixture is at a height greater than Z, and, consequently, has a horizontal velocity exceeding the wind velocity at height Z.

General results of a similar type for diffusion in a thermally stratified boundary layer are unknown at present. However, when considering vertical diffusion under very unstable conditions we can use an eddy diffusivity of the form (7.37); therefore the exact solution of the corresponding equation for $\theta_{00}(Z,t)$ can be found from Eqs. (10.94) and (10.94′). It is easy to show that

$$\langle Z\rangle = \frac{32}{27\sqrt{\pi}}\left(\frac{3}{C_1}\right)^{3/2}\left(\frac{q}{c_p\rho}\frac{g}{T_0}\right)^{1/2}\tau^{3/2}$$

is the mean height of the admixture cloud described by the solution (10.94′) corresponding to the diffusivity K_{zz} of the form (7.37). Therefore the semiempirical diffusion theory implies the value

$$a = \frac{32}{3\sqrt{3\pi}\,C_1^{\ 3/2}} \approx 3.5C_1^{-3/2}$$

for the coefficient a in Yaglom's equation (9.73); in fact this value is apparently higher than the true one. Thus we shall briefly discuss Saffman's results (1962b) for a model of a boundary-layer flow satisfying the equations

$$\begin{aligned}&\bar{u}(Z) = u_1 Z^m, \quad K_{zz}(Z) = KZ^n,\\ &K_{xx}(Z) = K_1 Z^k, \quad K_{yy}(Z) = K_2 Z^l,\end{aligned} \qquad (10.106)$$

where u_1, K, K_1, K_2 are constants and $n < 2$. Here, the general form of the solutions of Eqs. (10.76), (10.76′), (10.76″), etc., for an instantaneous surface point source of output Q at the coordinate origin at the time $t = t_0$, may again be established on the basis of dimensional considerations. In particular, in Eq. (10.76) for $\theta_{00}(Z,\ t)$ we now have a unique dimensional parameter K; thus from K, Z, and $\tau = t - t_0$, it is possible to construct a unique dimensionless combination $\xi = Z\,(K\tau)^{-\frac{1}{2-n}}$. We obtain

$$\theta_{00}(Z,t) = Q\,(K\tau)^{-\frac{1}{2-n}}\,\varphi_{00}(Z\,(K\tau)^{-1/(2-n)}), \qquad (10.107)$$

where $\varphi_{00}(\xi)$ is a dimensionless universal function, the explicit form of which is given by Eq. (10.94′). According to Eqs. (10.76′) and (10.106), $\theta_{10}(Z, t)$ must be proportional to Qu_1, and in addition also depends only on K, Z, and τ; consequently,

$$\theta_{10}(Z, t) = Qu_1\tau\,(K\tau)^{\frac{m-1}{2-n}}\,\varphi_{10}(Z\,(K\tau)^{-1/(2-n)}), \tag{10.107′}$$

where $\varphi_{10}(\xi)$ again is a universal function. Finally, on the basis of Eqs. (10.76″) and (10.106), θ_{20} may be represented as the sum of a component which is proportional to Qu_1^2, and a component which is proportional to QK_1, while θ_{02} is proportional to QK_2; therefore,

$$\theta_{20}(Z, t) = Qu_1^2\tau^2\,(K\tau)^{\frac{2m-1}{2-n}}\,\varphi_{20}^{(1)}(Z\,(K\tau)^{-1/(2-n)}) +$$
$$+ QK_1\tau\,(K\tau)^{\frac{k-1}{2-n}}\,\varphi_{20}^{(2)}(Z\,(K\tau)^{-1/(2-n)}),$$
$$\theta_{02}(Z, t) = QK_2\tau\,(K\tau)^{\frac{l-1}{2-n}}\,\varphi_{02}(Z\,(K\tau)^{-1/(2-n)}). \tag{10.107″}$$

Since $\bar{\theta}(Z, t) \to 0$ when $Z \to \infty$, all the functions $\varphi_{00}(\xi)$, $\varphi_{10}(\xi)$, $\varphi_{20}^{(1)}(\xi)$, etc., must decrease when $\xi \to \infty$. Thus on the basis of Eq. (10.107) it follows that the thickness of the admixture cloud at the time $t_0 + \tau$ must have the order of $(K\tau)^{\frac{1}{2-n}}$. In a layer of thickness $H \ll (K\tau)^{\frac{1}{2-n}}$, the admixture distribution will be practically independent of the height and will have a density

$$\theta_{00} = \frac{\alpha_{00}Q}{(K\tau)^{1/(2-n)}},$$

where

$$\alpha_{00} = \varphi_{00}(0); \quad \text{that is, } \alpha_{00} = (2-n)^{-\frac{n}{2-n}}\left[\Gamma\left(\frac{1}{2-n}\right)\right]^{-1}$$

on the basis of Eq. (10.94′). Moreover we have made the assumption that the velocity $\bar{u}(Z)$ is parallel to the OX axis at all levels, which, incidentally, may easily be relaxed, if necessary, by introducing another component $\bar{v}(Z) = v_1 Z^{m_1}$); this implies that the center of gravity of the admixture distribution in the layer of thickness $H \ll (K\tau)^{\frac{1}{2-n}}$ will have coordinates

$$\langle X\rangle = bu_1\tau\,(K\tau)^{\frac{m}{2-n}}, \qquad \langle Y\rangle = 0, \qquad b = \frac{\varphi_{10}(0)}{\varphi_{00}(0)}. \tag{10.108}$$

With the same assumption, the variances of the admixture distribution in the plane Z = const where $Z \ll (K\tau)^{\frac{l}{2-n}}$ will have the form

$$\sigma_{\vartheta_x}^2 = b_1u_1^2\tau^2\,(K\tau)^{\frac{2m}{2-n}} + b_2K_1\tau\,(K\tau)^{\frac{k}{2-n}}, \qquad \sigma_{\vartheta_y}^2 = b_3K_2\tau\,(K\tau)^{\frac{l}{2-n}}, \tag{10.108′}$$

where b_1, b_2, and b_3 are new constants, and the joint moment

$$\sigma_{xy} = \langle (X - \langle X \rangle) Y \rangle = \frac{\theta_{11}}{\theta_{00}} - \frac{\theta_{01}\theta_{10}}{\theta_{00}^2}$$

will be equal to zero since $\theta_{01} = \theta_{11} = 0$. The first term of the first equation of (10.108′) again describes the horizontal dispersion caused by the interaction of the wind shear with the vertical turbulent mixing, and the second term of this equation, the ordinary horizontal turbulent diffusion. Clearly, the first of these effects will be predominant for large τ only if $2m + 2 - n > k$. In the atmospheric surface layer, the latter condition apparently may always be considered as fulfilled. (Since for stable or neutral stratification $n \leqslant 1$, it may probably be assumed that $n + k \leqslant 2$, while for strong instability n is close to 4/3, and the coefficient $K_{xx}(Z)$, if it can be used here at all, will show a barely perceptible increase with height.) Therefore, it may be concluded that the presence of a wind shear always plays a basic role in horizontal dispersion in the atmosphere over comparatively large time intervals. Since it is natural to suppose that $K_{yy}(Z)$ is approximately proportional to $K_{xx}(Z)$, then neglecting the turning of the wind with height, the dispersion in the direction of the mean wind for large τ will considerably exceed the dispersion in the perpendicular direction. In addition, the standard deviation $\sigma_{\vartheta_x}^2$ here is quite large relative to the average thickness of the cloud, which is of the order $(K\tau)^{1/(2-n)}$; consequently, the admixture cloud for large τ is extended largely in the direction of the mean wind.

The effect of wind shear on horizontal admixture dispersion in the atmosphere is discussed also in Högström (1964), Smith (1965), and Tyldesley and Wallington (1965). These investigators used numerical methods and Lagrangian analysis in comparing their results with experimental data. However, we shall not discuss them in detail here.

Sutton's Equations for Atmospheric Dispersion

The estimates presented above for horizontal dispersion in the atmospheric surface layer create new possibilities for the mathematical analysis of the process of admixture propagation from instantaneous sources. It is clear, however, that this analysis will unavoidably be quite complex; therefore, various rather rough but simple approximate procedures have found application in the practical description of atmospheric diffusion. In particular, in England and in the United States, the approximate equations proposed by Sutton (1932; 1949; 1953) are frequently used to calculate admixture diffusion in the atmosphere. According to these equations, the admixture distribution from an instantaneous point source is assumed to have a Gaussian form (10.12) in the coordinate system displaced with the mean wind with a constant velocity $\bar{u}$, but with variances $D_{ii}(\tau)$ which increase more rapidly than the first power of τ. (This is in full agreement with Eqs. (10.108′) and with the fact that the decrease in the surface concentration corresponding to the variances $D_{ii}(\tau) = 2K_{ii}\tau$ in practice is too slow.) To determine the functional form of the variances $D_{ii}(\tau)$, Sutton proposed that for diffusion in the atmospheric surface layer, it is possible to use approximately Taylor's classical formula (9.31) for $D_{ii}(\tau)$, strictly obtained only under the assumption of homogeneity of the turbulence, assuming in this case that the Lagrangian velocity correlation functions $R_{ii}^{(L)}(\tau)$ have the form

$$R_{ii}^{(L)}(\tau) = \left(1 + \frac{\tau}{\tau_i}\right)^{-n}, \qquad i = 1, 2, 3. \tag{10.109}$$

Here τ_i are the constant time scales, and n according to Sutton assumes values on the order of 0.1–0.3, depending on stratification. Since it is assumed that $n < 1$, the Lagrangian integral time scale T of Eq. (9.32) turns out to be infinitely large; therefore, Sutton's theory does not lead to an asymptotically linear increase in the variances for an increase in time τ. Therefore, it follows in particular that, strictly speaking, this theory is not a version of the ordinary semiempirical theory of turbulent diffusion which is applicable, as noted above, only when $\tau \gg T_i$. However, similar to the ordinary semiempirical theory, in Sutton's theory also only sufficiently large values of τ are investigated, that is, large not by comparison with T_i, here equal to infinity, but by comparison with τ_i. According to Eqs. (10.109) and (9.31) when $\tau \gg \tau_i$ the variance $D_{ii}(\tau)$ is asymptotically proportional to τ^{2-n}; Sutton's corresponding asymptotic formula is written as

$$D_{ii}(\tau) = \frac{C_i^2}{2}(\bar{u}\tau)^{2-n}, \tag{10.110}$$

where $\bar{u}$ is the mean wind velocity in the atmospheric layer under discussion, and C_i are constant coefficients; when $\overline{u_i'^2} \sim \bar{u}^2$, this indicates that $\tau_i \sim (\bar{u})^{-1}$. In the case of admixture diffusion from elevated sources, Sutton recommends on the basis of experiment that the coefficients C_i be considered dependent on the source height H and decrease with an increase in H. Let us also note that with variances $D_{ii}(\tau)$ defined by Eq. (10.110), the function (10.12) is a solution of a diffusion equation of the form (10.55), but with the diffusivities

$$K_{ii} = \frac{2-n}{4} C_i^2 (\bar{u})^{2-n} \tau^{1-n},$$

which depend not on the height Z, but on the diffusion time τ. However, in this case, Eq. (10.55) cannot be derived from the statistical assumptions of the semiempirical diffusion theory which require [see Eq. (10.53)] that the diffusivities be determined only by the velocity field $\boldsymbol{u}(\boldsymbol{X}, t)$, and not by when and where the admixture sources are introduced into this field. (As G. I. Taylor (1959) noted, the fact that the eddy diffusivities in the semiempirical theory cannot depend on τ follows from the superposition principle, corresponding to the assumption of linearity of the diffusion equation: if K_{ii} depends on τ, K_{ii} at the fixed point (X, t) would be multivalued in the presence of several instantaneous admixture sources which do not emit simultaneously.)

Knowing the concentration distribution from an instantaneous point source, it is easy to calculate the concentrations from stationary point and linear sources; this can be done exactly as above for Roberts' equations, which correspond to constant diffusivities. Let us consider, for example, diffusion from a stationary point source of output Q at the point (0, 0, H), under the condition of admixture reflection by the surface of the earth. In this case, Sutton's theory gives the following formula related to Eq. (10.90):

$$\bar{\vartheta}(X, Y, Z) = \frac{Q}{\pi C_y C_z \bar{u} X^{2-n}} e^{-\frac{Y^2}{C_y^2 X^{2-n}}} \left[e^{-\frac{(Z-H)^2}{C_z^2 X^{2-n}}} + e^{-\frac{(Z+H)^2}{C_z^2 X^{2-n}}} \right]. \tag{10.111}$$

For a linear source on the straight line $X = 0$, $Z = H$, it is necessary only to drop the factor which depends on Y from the right side and replace Q by

$$\sqrt{\pi}\, Q C_y X^{1-\frac{n}{2}}.$$

According to Eq. (10.111), the admixture distribution along the straight line X = const, Z =

const is Gaussian with a variance

$$\frac{C_y^2}{2} X^{1-\frac{n}{2}},$$

and the distribution of the surface concentration in the direction of the mean wind has the form

$$\overline{\vartheta}(X, 0, 0) = \frac{2Q}{\pi C_y C_z \overline{u} X^{2-n}} e^{-\frac{H^2}{C_z^2 X^{2-n}}}, \tag{10.112}$$

that is, it decreases asymptotically in proportion to X^{n-2}, but in the case of a linear source, in proportion to $X^{\frac{n}{2}-1}$. The maximum surface concentration from Eq. (10.112) equals $\overline{\vartheta}_{\max} = \frac{2Q}{\pi e \overline{u} H^2} \frac{C_z}{C_y}$. That is, it is inversely proportional to the wind velocity and the square of the source height, and is achieved at a distance from the source

$$X = \left(\frac{H}{C_z}\right)^{\frac{2}{2-n}}$$

which, in contrast to the case of Roberts' theory, does not depend on the wind velocity $\overline{u}$.

A comparison of Sutton's equations with the data from atmospheric diffusion experiments, may be found, for example, in Barad and Haugen (1959), Drimmel and Reuter (1960), Haugen, Barad and Antanaitis (1961) Brummage (1968), Csanady, Hilst and Bowne (1968) and many others; the comparison of Eq. (10.109) with data on the Lagrangian velocity correlation functions is made by Munn (1963). The material presented in the works cited indicates that Sutton's equations give a very inaccurate description of the actual Lagrangian velocity correlation functions. They can be made to correspond more or less satisfactorily to the existing diffusion data only if the range of allowable values of n is considerably expanded, and if n, as well as the parameters C_i, may assume different values for diffusion in different directions, that is, in Eq. (10.110) the parameter n should be replaced by n_i. If this is done, however, the number of adjustable parameters in the equations of Sutton's theory become so large that the good agreement of these equations with the results of field experiments is fully explained; moreover, the dependency of the parameters C_i and n_i on the meteorological conditions is quite complex and not very well studied. Therefore, it is natural that other means are sought for the approximate description of the atmospheric diffusion data, which would be sufficiently simple, but theoretically better founded than Sutton's equations.

Stationary Solutions of the Semiempirical Diffusion Equation

In the special case of transverse dispersion $D_{yy}(\tau)$, in the cross-wind direction, new approximate equations were proposed by Byutner and Laykhtman (1963); however, these equations are quite complex and require empirical testing. A number of practical equations for calculating atmospheric diffusion may be found also in the monograph by Pasquill (1962b). Here we shall discuss only a few of these equations derived from the semiempirical equation of turbulent diffusion (10.55). Above, we investigated only the solution of this equation corresponding to the case of an instantaneous admixture source; it is specifically to this problem the remark applies that, as yet, no exact solutions have been obtained for the diffusion equation iwth height-dependent wind velocity. However, in practice, the distribution of concentration from a continuous stationary admixture source frequently is of primary interest, and this distribution is considerably easier to obtain by mathematical analysis.

Let us begin with the concentration $\overline{\vartheta}(X, Z)$ from a stationary linear horizontal cross-wind source, producing Q units of admixture mass per unit time per unit length; obviously it is also equal to the quantity

$$\overline{\vartheta}(X, Z) = \int_{-\infty}^{\infty} \overline{\vartheta}(X, Y, Z)\, dY,$$

where $\overline{\vartheta}(X, Y, Z)$ is the concentration from a stationary point source of productivity Q. In this case, in Eq. (10.55), the terms containing $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial Y}$ can obviously be omitted. If, in addition, we neglect also the term $-K_{xx}\frac{\partial^2 \overline{\vartheta}}{\partial X^2} = \frac{\partial}{\partial X}\overline{u'\vartheta'}$ which describes longitudinal (streamwise) diffusion by comparison with the convection term $\overline{u}(Z)\frac{\partial \overline{\vartheta}}{\partial X}$ (for stationary sources this neglect hardly ever leads to appreciable error), then Eq. (10.55) becomes the so-called two-dimensional diffusion equation:

$$\overline{u}(Z)\frac{\partial \overline{\vartheta}}{\partial X} = \frac{\partial}{\partial Z}\left[K_{zz}(Z)\frac{\partial \overline{\vartheta}}{\partial Z}\right]. \tag{10.113}$$

In the case of admixture reflection by the surface of the earth, and of a source at a height H, the desired solution must satisfy the following conditions:

$$\overline{\vartheta} \to 0 \quad \text{for} \quad X \to \infty \quad \text{or} \quad Z \to \infty; \quad K_{zz}\left.\frac{\partial \overline{\vartheta}}{\partial Z}\right|_{Z=0} = 0;$$

$$\overline{\vartheta} = 0 \quad \text{for} \quad X = 0, \quad Z \neq H; \quad \int_0^{\infty} \overline{u}\,\overline{\vartheta}\, dZ = Q \quad \text{for} \quad X \geqslant 0 \tag{10.114}$$

the last two of these conditions may also be written as $\overline{u}(H)\overline{\vartheta}(0, Z) = Q\delta(Z - H)$. Thus, we obtain an ordinary boundary-value problem with initial conditions for the parabolic partial differential equation (10.113), in which the role of time is now played by the variable X. In the special case, when $\overline{u}(Z) = \overline{u} =$ const, $K_{zz}(Z) = K_{zz} =$ const, the solution of the problem (10.113)–(10.114) obviously will be given by Roberts' equation (10.91), which, as is known, does not agree with the data. The case of constant wind velocity u, but linearly increasing vertical diffusivity $K_{zz}(Z) = K_1 Z$, was investigated by Bosanquet and Pearson (1936). For the special case of a surface source with $H = 0$, they obtained the very simple formula

$$\overline{\vartheta}(X, Z) = \frac{Q}{K_1 X} e^{-\frac{\overline{u}Z}{K_1 X}}. \tag{10.115}$$

This formula is already in comparatively good agreement with the observational data from nearly neutral temperature stratification. Furthermore, several investigators also studied the more general case of an arbitrary power law height dependency of the wind, and the diffusivities

$$K_{zz}(Z) = K_1 Z^n, \qquad \bar{u}(Z) = u_1 Z^m. \tag{10.116}$$

In the special case of $n = 1 - m$, or for the so-called Schmidt conjugate power law, Eq. (10.113) was solved comparatively long ago by Sutton (1934) with these coefficients, but for boundary conditions which differ from Eq. (10.114), in connection with an investigation of the evaporation in the atmospheric surface layer; a problem similar to this was also examined by Laykhtman (1947b). The solution of Eq. (10.113) under the conditions (10.114) corresponding to a surface source, for the coefficients (10.116), was obtained, in particular, by O. F. T. Roberts in an unpublished paper [see Calder (1949)], Frost (1946) [under the assumption that $n = 1 - m$.] and Laykhtman (1961; 1963); it has the following form:

$$\bar{\vartheta}(X, Z) = \frac{(m-n+2)\,Q}{u_1 \Gamma\left(\dfrac{m+1}{m-n+2}\right)} \left[\frac{u_1}{(m-n+2)^2 K_1 X}\right]^{\frac{m+1}{m-n+2}} \times$$
$$\times \exp\left[-\frac{u_1 Z^{m-n+2}}{(m-n+2)^2 K_1 X}\right] \tag{10.117}$$

(under the assumption that $m - n + 2 > 0$). The more general solution of this equation which corresponds to an elevated source at any height H may be found, for example, in Laykhtman (1961; 1963), Rounds (1955), Monin (1956b) and Smith (1957); in this case

$$\bar{\vartheta}(X, Z) = \frac{Q\,(HZ)^{\frac{1-n}{2}}}{(m-n+2)\,K_1 X} \times$$
$$\times \exp\left[-\frac{u_1\left(Z^{m-n+2} + H^{m-n+2}\right)}{(m-n+2)^2 K_1 X}\right] I_p\left(\frac{2u_1\,(HZ)^{(m-n+2)/2}}{(m-n+2)^2 K_1 X}\right), \tag{10.118}$$

where I_p is a Bessel function of imaginary argument, of order $p = -\dfrac{1-n}{m-n+2}$. A considerably more awkward solution of Eq. (10.113) with the coefficients (10.116) under boundary condition

$$K_{zz}\frac{\partial\bar{\vartheta}}{\partial Z}\bigg|_{Z=0} = \beta\bar{\vartheta}\bigg|_{Z=0}$$

which corresponds to the general case of partial or complete absorption of the admixture, was published by Laykhtman (1961; 1963); he, and also Tseytin (1962), discussed the case in which the wind velocity $\bar{u}$ and the vertical diffusivity K_{zz} satisfy the power equations (10.116) only up to some fixed height $Z = H_1$, being constant at greater heights. More general and more complex two-layer power-law models of the coefficients $\bar{u}(Z)$ and $K_{zz}(Z)$ in Eq. (10.113) were studied, in particular, by Yordanov (1968 a, b, c).

In the case $\bar{u}(Z) = u_1 \ln \frac{z}{z_0}$, $K_{zz}(Z) = K_1 Z$ which corresponds to the boundary layer under neutral stratification, the problem (10.113)–(10.114) has not been solved analytically with success, but clearly it may be solved numerically. Berlyand, Genikhovich, et al. (1963; 1964) obtained the numerical solution for several values of H. However, they assumed that the linear increase in the diffusivity $K_{zz}(Z)$ is retained only up to some "virtual height of the surface layer" h, and that beyond this height, $K_{zz}(Z)$ remains constant, or even changes with height in an arbitrary manner so that the graph of the function $K_{zz}(Z)$ has several breaks. Other examples of numerical solution of the diffusion problem for point and linear steady sources, were described in Klug and Wippermann (1967) and in Berlyand and Onikul (1968). Tyldesley also obtained a numerical solution of the two-dimensional diffusion equation for a logarithmic wind profile and a linear diffusivity profile. He found good agreement with the data from experiments in neutral conditions [see the discussion of Pasquill's paper (1966)]. Finally, in Yamamoto and Shimanuki (1960), a numerical solution of Eq. (10.113) was obtained under the assumption that $\tau = \rho_0 u_*^2 =$ const, and that according to the Reynolds analogy, the coefficient K_{zz} everywhere equals the turbulent viscosity, that is, is defined by the equation $K_{zz} \frac{\partial \bar{u}}{\partial Z} = u_*^2$, which for a logarithmic wind profile leads to the result $K_{zz} = \varkappa u_* Z$. However, here instead of the logarithmic formula for $\bar{u}(Z)$, the general similarity equation (7.24) was used for the wind profile with a universal function $\varphi(\zeta) = \zeta f'(\zeta)$ which satisfies the fourth-degree equation (7.61). The parameter σ of Eq. (7.61) was excluded from the problem with the help of a simple transformation to a new length scale $L_* = L/\sigma$. Consequently, the problem of admixture diffusion from a linear surface source at a roughness height $Z = Z_0$, for the condition of reflection at the earth surface, was reduced to the solution of one universal equation for a function of two variables depending on the single parameter $\zeta_0 = \frac{Z_0}{L_*}$, which arises in connection with the dependency of $\bar{u}(Z)$ on Z_0. It is clear that a positive value of ζ_0 corresponds to stable stratification, and negative values to unstable stratification. Yamamoto and Shimanuki obtained numerical solutions of the corresponding equation for a series of values of ζ_0, and presented them in the form of several graphs. The reproduction of one of the graphs in Fig. 82 shows the dependency of the index p, in the law for the decrease in the surface concentration when $Z = Z_0$, with increase in X, on the stability parameter ζ_0 and the dimensionless distance from the source X/Z_0. The parameter p here was determined from the equation

$$\frac{\bar{\vartheta}(X_2, Z_0)}{\bar{\vartheta}(X_1, Z_0)} = \left(\frac{X_2}{X_1}\right)^{-p},$$

where both values of X_2 and X_1 were taken close to the fixed coordinate X. We can see that for neutral stratification, or for a value of ζ_0 close to 0, the parameter p up to very large values of X/Z_0 remains approximately constant and close to 0.9. This is in complete agreement with the empirical facts discussed after Eq. (10.91). However, nonneutral stratification, as a rule, depends essentially on the distance X.

Tyldesley, in solving the two-dimensional diffusion equation numerically [see the discussion of Pasquill's paper (1966)] used several forms of the universal wind profile, namely, logarithmic plus linear profiles with various values of β in stable conditions, profiles satisfying Eq. (7.61) with various values of σ in unstable conditions, and Swinbank's exponential wind profile. The results obtained were compared with field data on atmospheric diffusion and show some unexplained discrepancies.

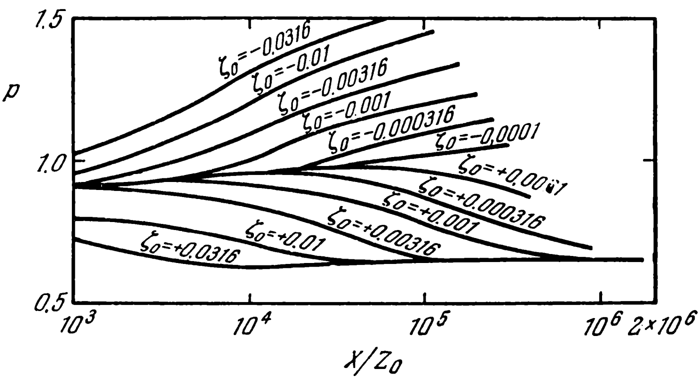


FIG. 82. The index p in the law for the decrease in admixture surface concentration as a function of distance from the source and of the stability parameter.

If, when using the power-law equations (10.116), we assume that $K_{zz}(Z)$ may differ from the turbulent viscosity at most by a constant factor α, that is, that $K_{zz}\frac{\partial \bar{u}}{\partial Z} = \text{const}$, then one must take $n = 1 - m$, or use Schmidt's conjugate power law. In the special case of neutral stratification the logarithmic profile of the wind velocity is approximated comparatively well over a significant height range by a power function with index $m = 1/7$ [see above, Sect. 5.6]; therefore, it is expedient here to assume that $m = 1/7$, $n = 6/7$. Actually, Eqs. (10.117)–(10.118) for these m and n, and the corresponding selection of the coefficients u_1 and K_1, correspond quite well to many existing data on diffusion under neutral stratification [see, for example, Calder (1949)]. It is obvious from Eqs. (10.117) and (10.118), in particular, that for such m and n the surface concentration decreases asymptotically in proportion to $X^{-8/9}$, which agrees well with experiment and with the data from Fig. 82 for very small ζ_0. With nonneutral stratification, many investigators have also attempted to apply Eqs. (10.117)–(10.118) in which the indices m and n are taken to depend on the meteorological conditions [see, for example, Deacon (1949), Vaughan (1961), and Gee (1966)], but here the results obtained depend quite strongly on the method of selecting the values of m and n, and therefore are less convincing.

Proceeding now to diffusion from a steady point source of productivity Q at the point $(0, 0, H)$, we again neglect diffusion in the direction OX. In comparison with the convection of the admixture by the mean flow, the semiempirical equation for the mean concentration, the so-called there-dimensional diffusion equations, will have the form

$$\bar{u}(Z)\frac{\partial \bar{\vartheta}}{\partial X} = \frac{\partial}{\partial Y}\left(K_{yy}\frac{\partial \bar{\vartheta}}{\partial Y}\right) + \frac{\partial}{\partial Z}\left(K_{zz}\frac{\partial \bar{\vartheta}}{\partial Z}\right). \tag{10.119}$$

The solution of Eq. (10.119) in which we are interested must satisfy the conditions

$$\begin{gathered}\bar{\vartheta} \to 0 \quad \text{for} \quad X^2 + Y^2 + Z^2 \to \infty, \quad K_{zz}\frac{\partial \bar{\vartheta}}{\partial Z}\bigg|_{Z=0} = 0,\\ \bar{\vartheta}(0, Y, Z) = \frac{Q}{\bar{u}(H)}\,\delta(Y)\,\delta(Z - H)\end{gathered} \tag{10.120}$$

[cf. Eq. (10.114)]. Increasing the number of independent variables from two to three in the transition from Eq. (10.13) to (10.119) significantly complicates the problem. Davies (1950) attempted to find an exact solution of the problem (10.119)–(10.120) for the case of a surface source, under the assumption that $\overline{u}(Z) \sim Z^m$, $K_{zz}(Z) \sim Z^{1-m}$ and $K_{yy}(Z) \sim Z^{\alpha}$; however, he succeeded only in the special case of $\alpha = m$, that is, when $K_{yy}(Z) = c\overline{u}(Z)$ where c = const. In the latter case it is easy to prove that

$$\overline{\vartheta}(X, Y, Z) = \frac{1}{2\sqrt{\pi c X}} \exp\left(-\frac{Y^2}{4cX}\right) \cdot \overline{\vartheta}(X, Z),$$

where $\overline{\vartheta}(X, Z)$ is the solution of problem (10.113)–(10.114). Thus, when $\alpha = m$, it is not difficult to consider the diffusion along the OY axis; therefore, the assumption that $K_{yy} \sim \overline{u}$, was made in a whole series of works on turbulent diffusion. The solution of the corresponding problem for an elevated point source under these assumptions is more complex but it also can be found in explicit form [see Walters (1965)]. Unfortunately, the equations obtained for concentration $\overline{\vartheta}(X, Y, Z)$ in many cases clearly contradict the data on admixture propagation from steady point sources in the atmospheric surface layer. In this connection, Smith (1957) reinvestigated the case of the general index α, but was unable to obtain an exact description of the function $\overline{\vartheta}(X, Y, Z)$; instead, he obtained the two first nonzero moments of the admixture distribution along the straight line X = const, Z = const:

$$\theta_0(X, Z) = \int_{-\infty}^{\infty} \overline{\vartheta}(X, Y, Z)\, dY, \quad \theta_2(X, Z) = \int_{-\infty}^{\infty} Y^2 \overline{\vartheta}(X, Y, Z)\, dY. \tag{10.121}$$

The function $\theta_0(X, Z)$ will obviously satisfy the same equation (10.113) [with the conditions (10.114)] which describes the concentration distribution for a steady linear source, that is, it may be determined from Eq. (10.118) with $n = 1 - m$. For the function $\theta_2(X, Z)$, a differential second-order equation, which also contains the coefficient $K_{yy}(Z)$, is obtained for it from Eq. (10.119). This equation may be solved explicitly for any H for some particular values of m and α; however, for arbitrary m and α its explicit solution, expressed in terms of complex transcendental functions, has been found only for the case of a surface source, that is, when $H = 0$. It is natural to suppose, however, that for not very large H and large X, the solution when $H > 0$ will differ little from the solution for a surface source. In addition, if only the surface distribution of the admixture is of interest, that is, values of $\overline{\vartheta}(X, Y, 0)$, then the solution for an elevated source may be derived from the general solution of a surface source with the help of an elegant duality theorem proved by Smith. This theorem expresses the statistical reversibility of the diffusion process described by the semiempirical equation (10.55). According to the theorem, *for turbulent diffusion with the reflection condition on the surface* $Z = 0$, *the value of* $\overline{\vartheta}(X, Y, 0)$ *of the function* $\overline{\vartheta}(X, Y, Z)$ *which corresponds to an elevated admixture source at the point* $X = Y = 0$, $Z = H$ *coincides with the value of* $\overline{\vartheta}(X, Y, H)$ *of the function* $\overline{\vartheta}(X, Y, Z)$ *which corresponds to a surface source at the point* $X = Y = Z = 0$ (so that for $\theta_2(X, Z)$ it is possible to write an explicit formula for any H). Knowing $\theta_0(X, Z)$ and $\theta_2(X, Z)$, it is possible to determine the function $\overline{\vartheta}(X, Y, Z)$ approximately by assuming, according to intuitive physical assumptions on transverse diffusion and existing data, that the admixture distribution along the straight line X = const, Z = const, for all X and Y, differs very little from a Gaussian distribution. Consequently, for any X, Y and Z

$$\overline{\vartheta}(X, Y, Z) \approx \frac{\exp\left[-Y^2/2\overline{Y^2}(X, Z)\right]}{\left[2\pi \overline{Y^2}(X, Z)\right]^{1/2}}\, \theta_0(X, Z), \tag{10.122}$$

where

$$\overline{Y^2(X, Z)} = \frac{\theta_2(X, Z)}{\theta_0(X, Z)}.$$

Using this formula, Smith, in particular, found the rate of decrease in the surface concentration in the OX direction on the axis of the admixture cloud, that is, when $Y = 0$, for $m = 1/7$ and $n = \alpha = 6/7$. In this case, $\bar{\vartheta} \sim X^{-5/3}$ asymptotically which more or less agrees with the data for steady point admixture sources in the nearly neutral atmosphere.

In the general case of arbitrary stratification, Yamamoto and Shimanuki (1964) attempted to evaluate the cross-wind (lateral) diffusivity by comparing the diffusion data from point sources with the results of numerical solution of the problem (10.119)–(10.120) for values of $K_{yy}(Z)$ containing an adjustable parameter. Here, as also in their preceding work, it was assumed that the wind is given by Eq. (7.24), where $\varphi(\zeta) = \zeta f'(\zeta)$ satisfies Eq. (7.61), and that $K_{zz} = u_* \varkappa Z/\varphi(\zeta)$. As a first approximation, it was assumed that the coefficient $K_{yy}(Z)$ depends linearly on height Z for any stratification, that is, it is given by a formula of the type

$$K_{yy}(Z) = u_* \varkappa Z \cdot \alpha(\zeta_0), \qquad \zeta_0 = \frac{Z_0}{L_*} = \frac{\sigma Z_0}{L},$$

where $\alpha(\zeta_0)$ is an unknown function of ζ_0, or, of stratification. The main advantage of this selection of K_{yy} is that it is easy to eliminate the parameter α with the help of a transformation to dimensionless coordinates

$$X_1 = X/Z_0, \; Y_1 = Y/\alpha^{1/2} Z_0, \; Z_1 = Z/Z_0,$$

which depend on α. With this transformation the problem of diffusion from a steady surface point source reduces to the solution of a universal equation for a function $\bar{\vartheta}(X_1, Y_1, Z_1)$ of three variables which depends on a single parameter ζ_0. This equation was solved numerically for a series of values of ζ_0, and the admixture distribution along the OY_1 axis, calculated for $X_1 = 32{,}000$ and $Z_1 = 240$, was compared with corresponding data from field diffusion experiments. The results of this comparison permitted approximate determination of the unknown values of $\alpha(\zeta_0)$; the values of Z_0 and L_* were estimated from the data of simultaneous wind and temperature profile measurements. With comparatively small scatter, the values of $\alpha(\zeta_0)$ obtained lay on a smooth curve monotonically decreasing with an increase in ζ_0; according to this curve, $\alpha \approx 13$ when $\zeta_0 = 0$, that is, for neutral stratification, $\alpha \approx 100$ when $\zeta_0 = -0.014$, and $\alpha \approx 3$ when $\zeta_0 = 0.01$. Furthermore, to check the validity of the assumption made concerning $K_{yy}(Z)$, the values of $\alpha(\zeta_0)$ found were used for theoretical calculation of the concentration distribution along the OY axis for some other values of X and Z, the results of which again were compared with the observations. Although there was in general satisfactory agreement between the theory and experiments, the scatter of the experimental points was so large that it was impossible to draw a reliable conclusion. For the decrease in surface concentration along the axis of the admixture cloud, with neutral stratifications, the calculation based on $K_{yy} = 13\,\varkappa u\, Z$ led to the expression $\bar{\vartheta} \sim X^{-1.78}$, which agrees well with the existing data.

The data also show that the horizontal diffusion along the OY direction becomes faster with an increase in the horizontal extension of the cloud, so that the coefficient K_{yy} increases with an increase in the diffusion time $\tau = t - t_0$ or the coordinate X. This is explained by the very important effect of *acceleration of the relative diffusion* of the admixture cloud or acceleration of the change in distance between the individual component particles of the cloud, with an increase in the size of the cloud. The effect is related to the fact that, as the size of the cloud increases, more and more large-scale velocity disturbances began to participate in the dispersion; we shall discuss this in Chapt. 8, Volume 2 of this book. However, as noted above, within the framework of the semiempirical theory of turbulent diffusion, it is impossible to assume that the diffusivities depend on the time τ. Nevertheless, Davies (1954a) tried to calculate this effect, formally assuming that in Eq. (10.119) the coefficients u and K_{zz} depend only on Z, specifically $\bar{u} \sim Z^m, K_{zz} \sim Z^{1-m}$, but $K_{yy} \sim Z^{\alpha} |Y|^{\beta}$. (That is, K_{yy} actually depends on the position of the admixture source since this is the only means by which the Y origin is identified.) In the special case $\alpha = m$, $\beta = 1 - 2m$, Davies found an exact solution of Eq. (10.119); however, it is clear that his basic assumption concerning the dependence of K_{yy} on Y sharply contradicts the intent of the semiempirical theory. From this viewpoint, the proposition of Laykhtman (1963) is more relevant. According to Laykhtman, the concentration $\bar{\vartheta}(X, Y, Z)$ in the case of a point source must be found in the form (10.122) where $\vartheta_0(X, Z)$ is given as the solution of Eq. (10.113); to determine $\overline{Y^2(X, Z)}$, it is recommended that the Lagrangian expressions be taken as the basis [e.g., the Taylor formula (9.31)], rather than the semiempirical theory.

Consideration of the Possible Deviation of the Principal Axes of the Eddy Diffusivity Tensor from the OX, OY and OZ Axes

The preceding conclusions were all based on the assumption that the semiempirical equation of atmospheric diffusion has the form (10.55). Let us now recall that even the most general semiempirical diffusion equation may only be obtained by using certain nonexact approximations; however, Eq. (10.55) contains also the additional assumption that the OZ, OX and OY axes directed vertically, along the mean wind and perpendicular to the wind, are the principal axes of the eddy diffusivity tensor K_{ij}. We have introduced this assumption since these directions are the preferred ones from the viewpoint of the geometric and kinematic conditions of the flow; however, it must be kept in mind that this argument is not rigorous. Therefore, we must not exclude the possibility that in fact the semiempirical equation of atmospheric diffusion should contain also some additional terms which are omitted in Eq. (10.55). Moreover, in Sect. 7.5 we have already noted that in the atmosphere above a homogeneous underlying surface, a horizontal turbulent heat flux must exist in the mean wind direction which is described by the joint moment $\overline{u'T'} = u_* T_* f_7(Z/L)$. This moment is positive for positive temperature gradients and negative for negative gradients, and rather great; at nearly neutral stratification it is about three times as great in absolute value as the moment $\overline{w'T'}$ and in stable conditions the ratio is even greater [see Sect. 8.5, especially Fig. C]. Within the framework of the semiempirical diffusion theory, the moment $\overline{u'T'}$ in plane-parallel turbulent flow must be represented in the form $\overline{u'T'} = -K_{xz}\dfrac{\partial \bar{T}}{\partial Z}$ (where $K_{xz} < 0$), while $\overline{w'T'} = -K_{zz}\dfrac{\partial \bar{\vartheta}}{\partial Z}$. The presence of a nonzero coefficient K_{xz} clearly indicates that the coordinate axes do not coincide with the principal axes of the tensor K_{ij}; moreover, we see that in a nonstratified boundary layer, the nondiagonal component K_{xz} of the tensor K_{ij} is about three times as great as the diagonal K_{zz}. Under the usual assumption that the eddy diffusivities for heat and for neutral mass admixture are equal, we write the semiempirical equation for the turbulent admixture flux in the mean wind direction as

$$\overline{u'\vartheta'} = -K_{xx}\frac{\partial \bar{\vartheta}}{\partial X} - K_{xz}\frac{\partial \bar{\vartheta}}{\partial Z}, \tag{10.123}$$

where K_{xz} is a negative coefficeint and in neutral conditions is approximately three times as great in absolute value as K_{zz}. (The third possible component of $\overline{u'\vartheta'}$, namely, $-K_{xy}\dfrac{\partial \bar{\vartheta}}{\partial Y}$, is equal to zero since $K_{xy} = K_{zy} = K_{yx} = K_{yz} = 0$ in a two-dimensional turbulent flow due to the symmetry of the flow relative to the mean velocity direction.) Of course the large value of K_{xz} compared to K_{zz} does not mean that the corresponding term $-\dfrac{\partial}{\partial X}K_{xz}(Z)\dfrac{\partial \bar{\vartheta}}{\partial Z}$ of the semi-empirical diffusion equation is always significant. In fact, this term is due to the turbulent admixture flux in the mean wind direction which is often small in comparison to the convection flux $\overline{u\vartheta}$; consequently, it may often be taken as small compared with $\bar{u}\dfrac{\partial \bar{\vartheta}}{\partial X}$. However, if $K_{xx}\dfrac{\partial^2 \bar{\vartheta}}{\partial X^2}$ is taken into account, then there is no reason to omit the term $K_{xz}\dfrac{\partial^2 \bar{\vartheta}}{\partial X \partial Z}$ in the diffusion equation. Moreover, if $K_{xz} \neq 0$, then also no reason exists for supposing that $K_{zx} = 0$. Qualitative physical reasoning on the mechanism of admixture transfer in a turbulent flow with $\bar{\vartheta} = \bar{\vartheta}(X)$, that is, independent of the coordinates Y and Z, at the initial time $t = t_0$ makes it natural to expect that K_{zx} will be negative in a two-dimensional flow; the approximate equality of the correlation coefficients r_{uT} and r_{wT} in absolute values in a nearly neutral atmospheric surface layer also is partly responsible for the supposition that $K_{zx} \approx -K_{xx}/3$ in such a layer [see Yaglom (1969)]. In every case the general form of the semiempirical diffusion equation in a plane-parallel turbulent flow must, strictly speaking, differ from Eq. (10.55) by two additional terms, and must be written as

$$\frac{\partial\bar{\vartheta}}{\partial t} + \bar{u}(Z)\frac{\partial\bar{\vartheta}}{\partial X} = K_{xx}(Z)\frac{\partial^2\bar{\vartheta}}{\partial X^2} + K_{xz}(Z)\frac{\partial^2\bar{\vartheta}}{\partial X\partial Z} + K_{yy}(Z)\frac{\partial^2\bar{\vartheta}}{\partial Y^2} +$$

$$+ \frac{\partial}{\partial Z}\left(K_{zx}(Z)\frac{\partial\bar{\vartheta}}{\partial X}\right) + \frac{\partial}{\partial Z}\left(K_{zz}(Z)\frac{\partial\bar{\vartheta}}{\partial Z}\right). \qquad (10.55')$$

Here the coefficients K_{xx}, K_{yy}, and K_{zz} are taken as positive, and K_{xz} and K_{zx}, as negative. The term containing K_{xz} is usually of the same importance as that containing K_{xx}, and often they can both be omitted without considerable error; similarly in many cases the term containing K_{zx} must be comparable with the usual vertical diffusion term containing K_{zz}. However, all the qualitative predictions of the semiempirical diffusion theory can hardly depend on the presence or absence of the terms with coefficients K_{xz} and K_{zx}; therefore all the general deductions obtained above using Eq. (10.55) are apparently correct. The last conclusion is confirmed also by the results of the few existing calculations based on the diffusion equation with nonzero coefficients K_{xz} and K_{zx}; we shall now consider these results.

Lettau (1952) [see also Priestley (1963)] was apparently the first to draw attention to the possibility that $K_{xz} \neq 0$ in the semiempirical equation of atmospheric diffusion. Later, Davies (1954b) tried to estimate the possible influence of the corresponding term of the diffusion equation on the diffusion from a steady point source. For his estimate he compared the solution of the equation

$$\bar{u}\frac{\partial\bar{\vartheta}}{\partial X} = \frac{\partial}{\partial X}\left(K_{xz}\frac{\partial\bar{\vartheta}}{\partial Z}\right) + \frac{\partial}{\partial Z}\left(K_{zz}\frac{\partial\bar{\vartheta}}{\partial Z}\right), \; K_{xz} < 0,$$

for some special assumptions on the functional form of the coefficients $\bar{u}(Z)$, $K_{xz}(Z)$ and $K_{zz}(Z)$, based partly on semiempirical reasoning, with the solution of the ordinary two-dimensional diffusion equation (10.113); as expected, the difference in solutions turned out to be quite small. Later, Gee and Davies (1963) evaluated the influence of the term $\frac{\partial}{\partial X}\left(K_{xz}\frac{\partial\bar{\vartheta}}{\partial Z}\right)$ on the values $\langle X\rangle$ and $\sigma^2_{\vartheta x}$ calculated by Saffman [see Eq. (10.103)], and found that for certain assumptions concerning $K_{xz}(Z)$ the corrections obtained may be of the order of 15–20%. At almost the same time, Matsuoka (1961; 1962) reported that apparently $K_{zx} \neq 0$ also, or that the diffusion equation must have the form (10.55′); he presented some nonrigorous semiempirical arguments in favor of the equality $K_{zx} = K_{xz}$ and supposed that $K_{zx} = K_{xz} = -(K_{xx}K_{zz})^{1/2}$. Matsuoka used his diffusion equation to compute the admixture distribution from a linear steady source. He found the corresponding values of the surface concentration, but not of the concentration at other heights, under his assumptions to agree exactly with the values given by the solution of the corresponding "classical" diffusion equation (10.55). Gee and Davies (1964) also used Eq. (10.55′) with $K_{xz} = K_{zx}$ and some special assumptions concerning all the diffusivities to compute horizontal dispersion from an instantaneous source; their results show that the influence of the two additional terms in Eq. (10.55′) may be quite significant on the diffusion from an instantaneous source. Also in Gee and Davies, a semiempirical argument was presented showing the equation $K_{zx} = K_{xz}/2$ to be more reasonable than Matsuoka's equation $K_{zx} = K_{xz}$. The relation $K_{zx} = K_{xz}/2$, together with some others of the same type, was used by Gee (1967) to compute diffusion from an instantaneous point source in a flow with $\bar{u}(Z) = U_0 + aZ$, $\bar{v}(Z) = V_0 + bZ$; in this case the tensor K_{ij} will have in general four nonzero off-diagonal components. The results of Gee's computations with $K_{ij} =$ const for all i and j show that the additional terms have httle effect on diffusion. This contradicts the results of Gee and Davies using other vertical profiles of the diffusivities, and illustrates the sensitivity of the above-mentioned effect to the forms chosen for these profiles. Since at present these profiles are still relatively unknown, it is pointless to go into further detail here.

The Use of the Lagrangian Similarity Hypothesis in Surface Layer Diffusion Theory

In the preceding, we always began with the semiempirical equation of turbulent diffusion. However, it is of interest to consider what conclusions may be drawn concerning diffusion in the atmospheric surface layer without using the semiempirical hypothesis of linear dependency of the turbulent admixture flux on the average concentration gradient. Thus, of significant help here may be the proposition, which appears quite natural, on the similarity of the Lagrangian turbulence characteristics in the surface layer which was introduced in Sect. 9.4. We shall now proceed to some corollaries of this proposition.

To begin with, let us recall that neglecting the effect of molecular diffusion, the mean admixture concentration from an instantaneous point source of unit output, at the point $\boldsymbol{x} = (x, y, z)$ at the time t_0, is equal to the probability density function of the Lagrangian coordinates (X, Y, Z) of a fluid particle which at the time t_0 is at the point $\boldsymbol{x}$ (see above, Sect. 10.3). It follows from this, in particular, that for an instantaneous surface source of output Q at the coordinate origin, the mean concentration at the time $t = t_0 + \tau$ for neutral thermal stratification and sufficiently large τ, will be determined by a formula of the type

$$\bar{\vartheta}(X, Y, Z, t_0 + \tau) = \frac{Q}{\bar{Z}^3} P_3\left(\frac{X - \bar{X}}{\bar{Z}}, \frac{Y}{\bar{Z}}, \frac{Z - \bar{Z}}{\bar{Z}}\right), \tag{10.124}$$

where P_3 is a universal function of three variables

$$\bar{Z} \approx b u_* \tau, \quad \bar{X} \approx \frac{u_* \tau}{\varkappa} \ln \frac{c u_* \tau}{e Z_0},$$

and b, c and $e = 2.718\ldots$ are universal constants [see Eqs. (9.65), (9.60′), and (9.61′)]. It is possible to use the same expression for a source at height H only if $\tau \gg \frac{H}{u_*}$. For not very large values of τ but which nevertheless appreciably exceed $\frac{H}{u_*}$, somewhat more accurate results are obtained if we assume that the statistical properties of the Lagrangian velocity of a fluid particle at time τ after its release at a given point at height H, are the same as those of a particle released at the ground ($H = 0$) at the instant $t_0 - \tau_1$ where τ_1 is of the order H/u_*, specifically, $\tau_1 = \beta H/u_*$, where β is a universal constant; in other words, if we assume that Eqs. (9.60′) and (9.61′) for $\bar{Z}(\tau)$ and $\bar{X}(\tau)$ are valid if the time τ is replaced by $\tau + \tau_1 = \tau + \beta H/u_*$ [Batchelor (1964)].[4] This correction is significant only in a comparatively small time interval. However, when processing data from real experiments, especially laboratory experiments, in which it is frequently necessary

[4] In this subsection we have already seen that the approximate semiempirical diffusion theory leads to the estimate $\beta = 1/\varkappa' = 1/\alpha\varkappa \approx 2.2$, and also to the values $b = \varkappa' \approx 0.45$, $c = e^{-\gamma} b \approx 0.56b$. A somewhat more accurate estimate of b will be given at the end of the next subsection.

to limit oneself to not very large values of τ, it sometimes turns out to be useful [see, for example, Cermak (1963)]. In the case of unstable or stable stratification, Eq. (10.124) is replaced by the more general expressions

$$\bar{\vartheta}(X, Y, Z, t_0+\tau)=\frac{Q}{\bar{Z}^3}P\left(\frac{X-\bar{X}}{\bar{Z}}, \frac{Y}{\bar{Z}}, \frac{Z-\bar{Z}}{\bar{Z}}; \frac{\bar{Z}}{L}\right), \qquad (10.125)$$

where P is a new universal function of four variables $\bar{Z}=\bar{Z}(\tau)$ and $\bar{X}=\bar{X}(\tau)$ are determined by Eqs. (9.69) and (9.70), and L is a natural length scale in the stratified boundary layer introduced in Chapt. 4. For τ which are not very large it is also possible here to introduce a correction for a source height H, but a discussion of this will not be presented here.

It follows from expression (10.124) in particular that the variances of the admixture cloud from an instantaneous point source, determined by the expressions

$$\begin{aligned}
\sigma^2_{\vartheta_x} &= Q^{-1}\int\int\int (X-\bar{X})^2\bar{\vartheta}(X, Y, Z)\,dX\,dY\,dZ,\\
\sigma^2_{\vartheta_y} &= Q^{-1}\int\int\int Y^2\bar{\vartheta}(X, Y, Z)\,dX\,dY\,dZ,\\
\sigma^2_{\vartheta_z} &= Q^{-1}\int\int\int (Z-\bar{Z})^2\bar{\vartheta}(X, Y, Z)\,dX\,dY\,dZ,
\end{aligned}$$

for neutral stratification, are asymptotically proportional to $\bar{Z}^2$. According to Eq. (9.61′), this indicates that the variances along all the three axes of the admixture cloud from the instantaneous point source with neutral stratification must be asymptotically proportional to $u_*^2\tau^2$, that is, to τ^2, and not to τ as in the case of homogeneous turbulence, and not to τ^3 as in the case of homogeneous shear flow. In the presence of thermal stratification, strictly speaking, it is impossible to ensure that these variances (even for very large τ) will be strictly proportional to $\bar{Z}^2$, since here the form of the distribution $\bar{\vartheta}(X, Y, Z, t_0+t)$ may depend on $\bar{Z}/L$, that is, may vary with variation in τ. It is natural to suppose, however, that the dependence of the distribution on $\bar{Z}/L$ in many cases will not be very strong; thus in the first approximation here also the variances may be considered proportional to $\bar{Z}^2$. In the case of strong instability the latter conclusion seems rather convincing and has additional support [see Eq. (9.75′)]; therefore, under these conditions, for values of τ

which are not very small, we must have the expressions

$$\sigma^2_{\vartheta_x} \sim \sigma^2_{\vartheta_y} \sim \sigma^2_{\vartheta_z} \sim \overline{Z}^2 \sim \frac{q}{c_p \rho} \frac{g}{T_0} \tau^3 .$$

(The most exact of the last-mentioned expressions probably will be that for $\sigma^2_{\vartheta_z}$.) In principle, all these conclusions may be compared with data from experiments with instantaneous point admixture sources. However, since there is still such little data, for experimental testing it is more convenient to use the results from Lagrangian similarity hypotheses concerning the concentration distribution from steady admixture sources. These results will be discussed below.

According to Eq. (10.125), the surface admixture concentration from a steady point source, at a sufficiently large distance X from this source in the direction of the mean wind, will be determined by the equation

$$\bar{\vartheta}(X, 0, 0) = Q \int_0^{\infty} P\left(\frac{X-\overline{X}}{\overline{Z}}, 0, -1; \frac{\overline{Z}}{L}\right) \frac{d\tau}{\overline{Z}^3}. \quad (10.126)$$

For a linear source this equation must be replaced by the following:

$$\bar{\vartheta}(X, 0) = Q \int_0^{\infty} \int_{-\infty}^{\infty} P\left(\frac{X-\overline{X}}{\overline{Z}}, \frac{Y}{\overline{Z}}, -1; \frac{\overline{Z}}{L}\right) \frac{dY\, d\tau}{\overline{Z}^3}. \quad (10.127)$$

The last two equations may obviously also be rewritten as

$$\bar{\vartheta}(X, 0, 0) = Q \int P\left(\frac{X-\overline{X}}{\overline{Z}}, 0, -1; \frac{\overline{Z}}{L}\right) \frac{d\left(\dfrac{X-\overline{X}}{\overline{Z}}\right)}{\overline{Z}^3 \dfrac{d\left(\dfrac{X-\overline{X}}{\overline{Z}}\right)}{d\tau}} =$$

$$= Q \int \frac{P\left(\dfrac{X-\overline{X}}{\overline{Z}}, 0, -1; \dfrac{\overline{Z}}{L}\right)}{\overline{Z}^2 \left[\dfrac{X-\overline{X}}{\overline{Z}} \dfrac{d\overline{Z}}{d\tau} + \dfrac{d\overline{X}}{d\tau}\right]} d\left(\frac{X-\overline{X}}{\overline{Z}}\right) \quad (10.126')$$

and analogously

$$\bar{\vartheta}(X, 0) = Q \int\int \frac{P\left(\frac{X-\bar{X}}{\bar{Z}}, \frac{Y}{\bar{Z}}, -1; \frac{\bar{Z}}{L}\right)}{\bar{Z}\left[\frac{\bar{X}-X}{\bar{Z}}\frac{d\bar{Z}}{d\tau} + \frac{d\bar{X}}{d\tau}\right]} d\left(\frac{X-\bar{X}}{\bar{Z}}\right) d\frac{Y}{\bar{Z}}. \quad (10.127')$$

For approximate calculation of the integrals on the right sides of Eqs. (10.126′) and (10.127′), following Batchelor (1959; 1964) and Ellison (1959), let us use the fact that for large X, the admixture cloud from the instantaneous point source at the coordinate origin must flow past the plane $X = \text{const}$ in a time much less than the time τ required for the cloud to reach this plane. This may be expressed as follows: since obviously $\bar{Z}/\bar{X} \to 0$ when $\tau \to \infty$, the range of values X for which

$$P\left(\frac{X-\bar{X}}{\bar{Z}}, 0, -1; \frac{\bar{Z}}{L}\right)\left[\text{or } P\left(\frac{X-\bar{X}}{\bar{Z}}, \frac{Y}{\bar{Z}}, -1; \frac{\bar{Z}}{L}\right)\right]$$

remains markedly different from zero, will make up a relatively small fraction of $\bar{X}$ itself for large X. Therefore, for sufficiently large $\bar{X}$ it is permissible to consider that

$$P\left(\frac{X-\bar{X}}{\bar{Z}}, \frac{Y}{\bar{Z}}, -1; \frac{\bar{Z}}{L}\right) \sim \delta\left(\frac{X-\bar{X}}{\bar{Z}}\right)$$

for any fixed $\frac{Y}{\bar{Z}}$. This assumption is quite similar to neglecting longitudinal diffusion in the mean wind direction, compared with the admixture convection by the mean flow, which is done in almost all applications of the semiempirical diffusion equation. It follows from this that Eqs. (10.126′) and (10.127′) may be replaced without any great error by the approximate expressions

$$\bar{\vartheta}(X, 0, 0) \approx \frac{Q}{\bar{Z}^2 \frac{d\bar{X}}{d\tau}} \int_{-\infty}^{\infty} P\left(\xi, 0, -1; \frac{\bar{Z}}{L}\right) d\xi \quad (10.126'')$$

and

$$\bar{\vartheta}(X, 0) \approx \frac{Q}{\bar{Z}\frac{d\bar{X}}{d\tau}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P\left(\xi, \eta, -1; \frac{\bar{Z}}{L}\right) d\xi\, d\eta, \quad (10.127'')$$

where $\bar{Z} = \bar{Z}(\tau)$ and $\bar{X} = \bar{X}(\tau)$ now are taken for a value of τ such that $\bar{X}(\tau) = X$. If we replace the height zero on the left sides of Eqs. (10.126″) and (10.127″) by an arbitrary height Z, then the argument -1 of the functions P must only be replaced by $Z/\bar{Z} - 1$.

First, let us investigate the simplest case of diffusion under neutral stratification. In this case the function P in Eqs. (10.126″) and (10.127″) does not depend on $\bar{Z}/L$, that is, for all $\bar{Z}$ it is the same, and $\bar{X}(\tau)$ and $\bar{Z}(\tau)$ satisfy Eqs. (9.60′)–(9.61′). Therefore,

$$\bar{\vartheta}(X, 0, 0) \sim \frac{Q}{u_*^3 \tau^2 \ln \frac{c u_* \tau}{z_0}} \approx \frac{Q \ln \frac{c\bar{Z}}{bz_0}}{u_* X^2} \tag{10.128}$$

and

$$\bar{\vartheta}(X, 0) \sim \frac{Q}{u_*^2 \tau \ln \frac{c u_* \tau}{z_0}} \approx \frac{Q}{u_* X}. \tag{10.129}$$

Thus it is seen that in neutral conditions, the admixture concentration from a steady linear source must decrease approximately proportional to X^{-1}, and from a steady point source, somewhat more slowly than X^{-2}; as we know, the data agree quite well with this conclusion [see above, following Eq. (10.91)].

The index p in the expression $\bar{\vartheta} \sim X^{-p}$ may be determined more accurately from the formula $p = -\frac{d \log \bar{\vartheta}}{d \log X}$; this was done by Cermak (1963), keeping in mind also the above-mentioned correction to $\bar{X}(\tau)$ and $\bar{Z}(\tau)$ related to the nonzero source height H. With this the theoretical values of $p = p(X)$ calculated from Eqs. (10.128) and (10.129) under the assumption that $b = c = 0.1$, were in even better agreement with the existing data of both laboratory experiments and field observations for neutral stratification than would have been expected. Later, however, Pasquill (1966) advocated the value $b = \kappa \approx 0.4$ which, as we know, follows from the semiempirical diffusion equation, under the assumption that $\alpha = 1$. With this in mind, he treated the data from field observations under neutral conditions of the vertical admixture distribution at a given distance X from a steady source. It was necessary to accept several nonrigorous assumptions in order to derive the relationship between the observed characteristics of the vertical distribution and the Lagrangian statistical characteristics X and Z; however, under these assumptions, Pasquill found that the field data agree with the estimate $b \approx 0.4$ with greater accuracy than that of the assumptions used in the theoretical derivation of this estimate.

The case for nonneutral stratification is even more complicated. First, the integrals on the right sides of Eqs. (10.126″) and (10.127″) may now depend on $\bar{Z}/L$, that is, on X. However, this dependency will be weaker than the dependency on X of the factors before the integrals. Thus, as a first approximation, it is possible also in this case to use the expressions

$$\bar{\vartheta}(X, 0, 0) \sim \frac{Q}{\bar{Z}^2 \left.\frac{d\bar{X}}{d\tau}\right|_{\bar{X}(\tau)=X}}, \qquad \bar{\vartheta}(X, 0) \sim \frac{Q}{\bar{Z} \left.\frac{d\bar{X}}{d\tau}\right|_{\bar{X}(\tau)=X}}.$$

It is more important that the functions $\bar{X}(\tau)$ and $\bar{Z}(\tau)$ here be defined by Eqs. (9.69) and (9.70) which contain the unknown universal functions $f(\zeta)$, $\varphi_1(\zeta)$ and $\varphi(\zeta)$; only their asymptotic behavior may be considered known with accuracy up to indeterminate numerical coefficients. Thus, for strong instability $\bar{Z}(\tau) \sim \tau^{3/2}$, $\bar{X}(\tau) \sim \tau$ and consequently

$$\vartheta(X, 0, 0) \sim X^{-3}, \quad \bar{\vartheta}(X, 0,) \sim X^{-3/2}.$$

Of course, we may hope (and the data of Sect. 8 give some justification for this) that only the asymptotic behavior of these functions will play a basic role in the calculation of concentration. If this is true, then, in first approximation, we can substitute for the unknown functions almost any functions that behave properly for small and for large values of the arguments. Such a possibility is partially confirmed by the calculated results of Gifford (1962). In accordance with Eq. (9.71), but in disagreement with the deductions of the semiempirical theory, Gifford proposed that $\varphi_1(\zeta) = f(\zeta)$ so that $\frac{d\bar{X}}{d\tau} = \bar{u}(\bar{Z})$. Then he selected as $f(\zeta)$ a function of the form

$$f(\zeta) = \begin{cases} \ln \zeta + 6\zeta & \text{for} \quad \zeta > 0, \\ \ln |\zeta| & \text{for} \quad 0 < \zeta < -0.03, \\ \zeta^{-\frac{1}{3}} + (0.03)^{-\frac{1}{3}} + \ln 0.03 & \text{for} \quad \zeta < -0.03 \end{cases} \tag{10.130}$$

and assumed [according to Kazanskiy and Monin (1957)] that

$$\varphi(\zeta) = \left[1 - \frac{1}{f'(\zeta)}\right]^{1/4}.$$

According to the dependency obtained from these assumptions, where the dimensionless concentration $\frac{u_* L^2}{Q \varkappa} \bar{\vartheta}$ for a steady point admixture source depends on the dimensionless distance $\varkappa b \frac{X}{L}$, Gifford calculated the index p in the expression $\bar{\vartheta} \sim X^{-p}$ for different thermal stratification and different distances X. The values of p he obtained turned out to agree comparatively well with the data gathered in the United States during Project Prairie Grass at O'Neill, Nebraska, and published by Barad (1958) et al. Later, Malhotra and Cermak (1963) compared the results of Gifford's calculations, again when $b = 0.1$, with the data from several diffusion experiments carried out in a special meteorological wind-tunnel with a heated lower wall, producing an unstable temperature stratification; the results obtained also were satisfactory. Cermak (1963) made an additional verification of Gifford's theory for the case of slightly nonneutral conditions; he used Swinbank's exponential function $f(\zeta)$ [cf. Eq. (7.73)] in his calculations and obtained satisfactory agreement between theoretical predictions and the field and laboratory data. Later Pasquill (1966) revised Gifford's calculations, using the value $b = 0.4$, of the dependence of $\bar{Z}$ on $\bar{X}$. He also analyzed the estimate of the same dependence by Panofsky and Prasad (1965); according to his conclusions, the similarity theory cannot explain the severe effect of thermal stratification on the vertical spread of admixture which is apparent in the published data of the O'Neill experiments. Still later, Klug (1968) repeated and expanded Gifford's calculations, using the same assumptions but with $b = 0.4$ and a slightly different form of universal wind profile. He also found that some features of the real concentration measurements are inconsistent with the theoretical predictions obtained. Consequently, no clear solution has yet been obtained and additional careful investigations, both experimental and theoretical, are greatly desirable.

10.6 Diffusion with a Finite Velocity

The fact that the semiempirical diffusion equation (10.55) is parabolic leads to the result that the admixture introduced into the fluid at the time t_0 spreads immediately throughout the entire space. Consequently, by the time $t_0+\tau$ where τ is as small as desired, the admixture, even in a very small amount, can already be detected at an arbitrarily large distance from the source [see, for example, Eq. (10.89)]. This clearly contradicts the actual boundedness of the velocities of the fluid particles. It is directly related to the fact that in the semiempirical theory, the instantaneous velocity of the fluid particles, as we have seen in Sect. 10.3, turns out to be infinite. In practice, however, the appearance of an infinite velocity in the theory usually does not play a large role, since the volume inside which the concentration is not negligible, is always bounded, and the distribution of the concentration inside this volume as a rule is described satisfactorily by the parabolic diffusion equation for diffusion times which are not too small. Nevertheless, close to the real boundaries of the admixture cloud, the use of the parabolic diffusion equation may lead to essential errors. For example, smoke coming from a stack at a height H indeed reaches the surface of the earth no closer than a distance from the stack equal to $\frac{u}{v}H$ where u is the minimum velocity of the wind, and v is the maximum velocity of the vertical motion of the smoke particles; at the same time, according to the semiempirical parabolic diffusion equation, smoke may be detected on the surface of the earth as close to the stack as desired. Therefore, in some cases it is desirable to have a more general semiempirical diffusion theory in which the boundedness of the fluid particle velocities would be taken into consideration. Such a generalization of the ordinary theory of diffusion, although not necessarily turbulent, was briefly outlined, with no details, by G. I. Taylor (1921) and later developed at different times by many investigators; in this subsection, we shall discuss it principally according to Monin (1955; 1956b).

We shall assume that the random function $\boldsymbol{X}(\boldsymbol{x}, t)$ which describes the fluid particle motion is differentiable with respect to t, and that its derivative, the velocity of the fluid particle $\boldsymbol{V}(\boldsymbol{x}, t)$, is bounded. In this case, the function $\boldsymbol{X}(\boldsymbol{x}, t)$ cannot be considered a Markov function of t. In fact, we have already seen that for a Markov function $\boldsymbol{X}(\boldsymbol{x}, t)$, the velocity $\frac{\partial \boldsymbol{X}(\boldsymbol{x}, t)}{\partial t}$ will always be infinite provided $K_{ij} \neq 0$. If we assume that $K_{ij}=0$, then the velocity will not be infinite, but, as is easily seen, the function $\boldsymbol{X}(\boldsymbol{x}, t)$ in this case will not be random. Thus, the condition of boundedness of the velocity of a fluid particle does not agree with the assumption that its coordinate $\boldsymbol{X}(\boldsymbol{x}, t)$ is a Markov random function of t.

A similar situation arises in the theory of molecular diffusion when attempting to take into account the inertia of a diffusing particle; due to inertia its trajectory $\boldsymbol{X}(\boldsymbol{x}, t)$ turns out to have a finite derivative $V(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{X}(\boldsymbol{x}, t)}{\partial t}$ everywhere. Thus, it follows that the function $\boldsymbol{X}(\boldsymbol{x}, t)$ cannot be a Markov function. It is known that in the theory of molecular diffusion, that is, Brownian motion, it is possible to consider inertia of the particles by assuming that, not the function $\boldsymbol{X}(\boldsymbol{x}, t)$, but the six-dimensional function $\{\boldsymbol{X}(\boldsymbol{x}, t), \boldsymbol{V}(\boldsymbol{x}, t)\}$ is Markovian [see, for example, Uhlenbeck and Ornstein (1930) or Chandrasekhar (1943)]. We shall proceed in a similar manner.

For simplicity, let us limit ourselves to investigating a one-dimensional diffusion problem in the direction of the OZ axis. According to what has been said above, let us try to construct a generalization of the semiempirical theory of turbulent diffusion on the basis of a new semiempirical assumption that is broader than the assumption of the Markovian nature of the function $\boldsymbol{X}(\boldsymbol{x}, t)$. In particular, let us assume that the two-dimensional random function $\{Z(z, t), W(z, t)\}$ is Markovian where $Z(z, t)$ is the coordinate of a fluid particle, and $W(Z, t) = \frac{\partial Z(z, t)}{\partial t}$ is its velocity. Since we are only interested in the effects of boundedness of the velocity W, we can limit ourselves to a very rough description of the possible values of W. Let $|W| < C$; let us divide the entire range of possible values of the velocity $-C < W < C$ into a finite number N of intervals Δ_n, and instead of the specific values of W we shall indicate only a number n of intervals Δ_n into which these values fall. In other words, instead of the random function $W(z, t)$, we shall consider the random function $n(t)$ which assumes discrete values of $1, 2, \ldots N$; also, we shall make the assumption that the random function $\{Z(z, t), n(t)\}$ is Markovian.

Let us now investigate the conditional probabilities

$$P\{Z(z, t) < Z;\ n(t) = i \,|\, Z(z, t_0) = z\} = \int_{-\infty}^{Z} p_i(Z \,|\, z, t)\, dZ. \qquad (10.131)$$

The probability density function for the coordinate of a fluid particle $Z(z, t)$ which plays a basic role in the theory of turbulent diffusion is expressed in terms of the functions $p_i(Z|z, t)$ by the equation

$$p(Z\,|\,z,\ t) = \sum_l p_l(Z\,|\,z,\ t). \tag{10.132}$$

The basic differential equations of Markov random processes, indicated by Kolmogorov (1931), reduce here to the following system of equations for the functions $p_i(Z|z,\ t)$, replacing the ordinary semiempirical diffusion equation:

$$\frac{\partial p_l}{\partial t} = \sum_j a_{jl} p_j - \frac{\partial \overline{W}_l p_l}{\partial Z}. \tag{10.133}$$

Here the following notations are used:

$$\overline{W}_i = \overline{W}_i(Z,\ t) = \left[\frac{\partial}{\partial\tau}\overline{Y_i(\tau\,|\,Z,\ t)}\right]_{\tau=0},$$

$$a_{ji} = a_{jl}(Z,\ t) = \left[\frac{\partial}{\partial\tau}P\,\{n(t+\tau) = i\,|\,Z(z,\ t) = Z,\ n(t) = j\}\right]_{\tau=0},$$

where

$$Y_i(\tau|Z,\ t) = Z(z,\ t+\tau) - Z(z,\ t)$$

is the particle displacement under the condition that the values of $Z(z,\ t) = Z$ and $n(t+\tau) = n(t) = i$ are fixed. Obviously, $\overline{W}_i$ is the mathematical expectation of the particle velocity W under the condition that W belongs to the interval Δ_i; thus, the value of $\overline{W}_i(Z,\ t)$ for all Z and t lies somewhere inside the interval Δ_i. The meaning of the variables a_{ji} may be clarified in the following way. Since the function $n(t)$ assumes only integral values, it may vary only in jumps, making the transition from the value $n(t-0) = j$ at the time which directly precedes t, to some other value $n(t+0) = l$ at a time which immediately follows t. The values of a_{ji} when $i \neq j$ are the average frequencies of transitions from $n(t-0) = j$ to $n(t+0) = l$, since $a_{ji}\Delta t$ is the probability of a jump from $n = j$ to $n = i$ during the time interval $(t,\ t+\Delta t)$ under the condition that $n(t) = j$.

Since

$$\sum_l P\{n(t+\tau) = l\,|\,Z(z,t) = Z,\ n(t) = j\} = 1,$$

it is clear then that $\sum_i a_{ji}=0$; consequently $a_{jj}=-\sum_{i\neq j} a_{ji}$ is equal to the average frequency of transitions from $n=j$ to some other value of n taken with the minus sign.

Equations (10.133) form a semiempirical system of diffusion equations corresponding to the propositions formulated above. These equations must be solved under the initial conditions

$$\begin{gathered} p_i(Z\,|\,z,\ t)\to\varepsilon_i(z)\,\delta(Z-z)\quad \text{for}\quad t\to t_0,\\ \varepsilon_i(z)=P\{n(t_0)=i\,|\,Z(z,\ t_0)=z\}. \end{gathered} \tag{10.134}$$

The variables ε_i give the probability distribution for $n(t_0)$, that is, actually, for values of the initial velocity of a fluid particle.

Let us investigate in more detail the simplest scheme in which the entire region of possible values of the fluid particle velocity $-C<W<C$ is divided into two intervals Δ_1 and Δ_2. We select as Δ_1 and Δ_2 the intervals $0<W<C$ and $-C<W<0$; in this case the random function $n(t)$ indicates the direction of fluid particle motion. For determinacy, assume that both directions are equivalent, so that, in particular, $\overline{W}=0$; in this case we may assume that $\overline{W}_1=-\overline{W}_2=W$, where W is the mean absolute velocity of the particle, and $a_{12}=a_{21}=-a_{11}=-a_{22}=a$, where a is the frequency of a change in the direction of motion. Then Eqs. (10.133) assume the following form:

$$\begin{aligned} \frac{\partial p_1}{\partial t}+\frac{\partial W p_1}{\partial Z}&=a\,(p_2-p_1),\\ \frac{\partial p_2}{\partial t}-\frac{\partial W p_2}{\partial Z}&=a\,(p_1-p_2). \end{aligned} \tag{10.135}$$

Equations of this type, or the telegraph equation which derives from them (see below), were first obtained to describe the one-dimensional diffusion of photons by Fock (1926). Afterwards they were used in describing molecular diffusion, taking into account the finiteness of the velocities of the diffusing particles, by Davidov (1935), Cattaneo (1948–1949; 1958), R. Davis (1954) Venotte (1958), and others. Such equations were proposed to describe turbulent diffusion by Lyapin (1948; 1950), Goldstein (1951; here the remarks of G. I. Taylor (1921) were developed into a mathematical theory), and Gupta (1959); similar equations for two- and three-dimensional turbulent diffusion were considered by Bourret (1961).

Assume that the diffusion takes place in a field of *stationary* turbulence. In this case the values of a and W do not depend on t and consequently may depend only on Z. Introducing new unknowns $p=p_1+p_2$, the probability density function for the coordinate of a diffusing particle, and $q=W(p_1-p_2)$, the flux density of the diffusing particles, we obtain

$$\begin{aligned} &\frac{\partial p}{\partial t}+\frac{\partial q}{\partial Z}=0; \\ &\frac{\partial q}{\partial t}+2aq=-W\frac{\partial Wp}{\partial Z}. \end{aligned} \tag{10.136}$$

The first of these equations, which is correct also for nonstationary turbulence, expresses the law of mass conservation for the diffusing admixture. According to this equation, one may assume

$$p=\frac{\partial \Psi}{\partial Z}, \quad q=-\frac{\partial \Psi}{\partial t}, \tag{10.137}$$

where

$$\Psi=\Psi(Z\,|\,z,\ t)=\int_{-\infty}^{Z} p(Z'\,|\,z,\ t)\,dZ'$$

denotes the probability distribution function for the coordinate $Z(z,\ t)$ of the diffusing particle. Substituting Eqs. (10.137) into the second of Eqs. (10.136), we obtain for the function Ψ the so-called *telegraph equation*[5]

$$\frac{\partial^2 \Psi}{\partial t^2}+2a\frac{\partial \Psi}{\partial t}=W\frac{\partial}{\partial Z}W\frac{\partial \Psi}{\partial Z}. \tag{10.138}$$

[5]This type of equation [Eq. 10.138)] is called a telegraph equation since it also describes the law of time variation of the current in a conductor, for example in a telegraphic cable, with resistance R, capacitance C and self-inductance L, if we assume that $a=\dfrac{R}{2L}$ and $W=(CL)^{-1/2}$. From this analogy, it may be concluded that the electrons somehow complete a random walk in the conductor with a mean velocity $(CL)^{-1/2}$, changing their direction of motion on the average with a frequency of $R/2L$; in this case, the value of $K=\dfrac{W^2}{2a}=\dfrac{1}{CR}$ is analogous to the diffusion coefficient.

This equation is of the hyperbolic type; it describes the propagation of a diffusing admixture having a finite velocity not exceeding W; in fact it is even supposed here that the velocity is always equal to W in absolute value. Also, this equation is only approximate since it is based on a nonrigorous hypothesis concerning the Markov nature of the function $\{Z(t),\, n(t)\}$; nevertheless, it is more accurate than the ordinary, one-dimensional, semiempirical equation of turbulent diffusion. Indeed, it is not difficult to show that our new diffusion equation is a generalization of the parabolic diffusion equation; the latter is obtained from Eq. (10.138) as a limiting case. This is easiest to demonstrate in an example of diffusion in a field of stationary homogeneous turbulence where the coefficients a and W depend on neither t nor Z, that is, are constant. In this case, p [and also p_1 and p_2] satisfies the same equation as Ψ, namely, the telegraph equation

$$\frac{\partial^2 p}{\partial t^2}+2a\frac{\partial p}{\partial t}=W^2\frac{\partial^2 p}{\partial Z^2}. \tag{10.139}$$

If the parameters a and W are allowed to increase without bound in such a way that the ratio $W^2/2a$ approaches a finite limit, which we shall designate as K, then the telegraph equation (10.139) will be converted into the parabolic diffusion equation $\frac{\partial p}{\partial t}=K\frac{\partial^2 p}{\partial Z^2}$, which corresponds to the case of homogeneous stationary turbulence.

The solution of Eq. (10.139), for a bounded space $-\infty<Z<\infty$ under the initial conditions (10.134), has the form

$$p(Z\,|\,z,\,t)=e^{-a\tau}\Bigg\{\varepsilon_1\delta(\zeta-W\tau)+\varepsilon_2\delta(\zeta+W\tau)+$$
$$+\frac{a}{2W}\left[I_0\left(a\tau\sqrt{1-\frac{\zeta^2}{W^2\tau^2}}\right)+\frac{1+(\varepsilon_1-\varepsilon_2)\frac{\zeta}{W\tau}}{\sqrt{1-\frac{\zeta^2}{W^2\tau^2}}}I_1\left(a\tau\sqrt{1-\frac{\zeta^2}{W^2\tau^2}}\right)\right]\Bigg\}, \tag{10.140}$$

where $\tau=t-t_0$, $\zeta=Z-z$, I_0 and I_1 are symbols of the Bessel functions of an imaginary argument, and the function p is assumed equal to zero when $|\zeta|>W\tau$. Equation (10.140) indicates that in this theory the admixture distribution from an instantaneous point source has a sharp edge, that is, a wave front $\zeta=\pm W\tau$ in which a finite fraction of the total amount of diffusing admixture is concentrated, where this fraction decreases exponentially with time. Behind this front there is a continuous wake formed by admixture particles which have experienced multiple scattering, or a change in direction of motion. In the center of this wake, namely, when $|\zeta| \ll W\tau$, it is not difficult to see that the right side of Eq. (10.140) differs little from the function

$$\sqrt{\frac{a}{2\pi W^2\tau}}\, e^{-\frac{a\zeta^2}{2W^2\tau}},$$

for $\tau \gg \frac{1}{a}$ and any ε_1 and ε_2; this is the solution of the parabolic equation $\frac{\partial p}{\partial \tau} = K\frac{\partial^2 p}{\partial \zeta^2}$, $K = \frac{W^2}{2a}$. Thus, the theory of diffusion with a finite velocity leads to a noticeable discrepancy in the conclusions from the parabolic diffusion equation only either for a very small diffusion time $\tau \lesssim \frac{1}{a}$, or close to the edge of the admixture cloud, where the concentration obtained from any theory is very small. As an illustration, graphs are presented in Fig. 83 for the continuous part of the distribution $p(Z|z, t)$ when $\varepsilon_1 = \varepsilon_2 = 1/2$ and $\varepsilon_1 = 1$, $\varepsilon_2 = 0$ for two values of $a\tau$, together with the graphs of the admixture distribution corresponding to the parabolic diffusion equation.

The mean value and variance of the distribution (10.140) equal

$$\begin{aligned} \bar{Z} &= z + \frac{W}{2a}(\varepsilon_1 - \varepsilon_2)(1 - e^{-2a\tau}); \\ \overline{(Z - \bar{Z})^2} &= \frac{W^2\tau}{a}\left[1 - \frac{1 - e^{-2a\tau}}{2a\tau} + (\varepsilon_1 - \varepsilon_2)^2 \frac{(1 - e^{-2a\tau})^2}{4a\tau}\right]; \end{aligned} \tag{10.141}$$

they do not differ essentially from the values of

$$\bar{Z} = z + \frac{W}{2a}(\varepsilon_1 - \varepsilon_2) \quad \text{and} \quad \overline{(Z - \bar{Z})^2} = 2K\tau = \frac{W^2\tau}{a}$$

when $a\tau \gg 1$. According to the model assumed here for the diffusion process, the probability of scattering in a small time interval τ equals $a\tau + o(\tau)$ and scattering in nonintersecting time intervals may be regarded as essentially independent random events. Therefore, the number $\nu(\tau)$ of scatterings in the time interval τ is a random variable with a Poisson distribution having the parameter $a\tau$:

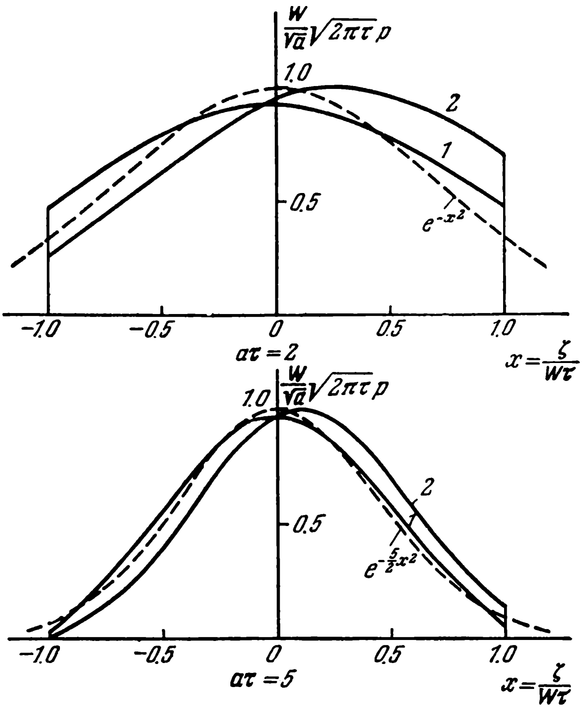


FIG. 83. A comparison of the continuous part of the distribution $p(Z \mid z, t)$ when $a\tau = 2$ and $a\tau = 5$ from diffusion theory with a finite velocity with the Gaussian distribution which follows from the parabolic diffusion equation. Curve 1 corresponds to the right side of Eq. (10.140), without the terms containing the δ-functions, when $\epsilon_1 = \epsilon_2 = 1/2$; Curve 2, the same, when $\epsilon_1 = 1, \epsilon_2 = 0$; the dashed curve is the Gaussian distribution.

$$P\{\nu(\tau) = m\} = \frac{(a\tau)^m}{m!}\, e^{-a\tau}.$$

If we designate as $p_m(\zeta, \tau)$ the conditional probability density function for $Z(z, t) = Z(z, t_0+\tau)$ under the conditions $n(t_0) = 1$, $\nu(\tau) = m$, then obviously

$$p(Z \mid z, t) = \sum_{m=0}^{\infty} \frac{(a\tau)^m}{m!}\, e^{-a\tau} \left[\varepsilon_1 p_m(\zeta, \tau) + \varepsilon_2 p_m(-\zeta, \tau)\right]. \qquad (10.142)$$

This equation gives the expansion of the distribution (10.140) with respect to the multiplicity of scatterings; using Eq. (10.140) it can be seen that $p_0(\zeta, \tau) = \delta(\zeta - W\tau)$, and when $m \geq 1$ the functions $p_m(\zeta, \tau)$ are equal to zero when $|\zeta| > W\tau$, and have the following form:

$$p_{2m+1}(\zeta, \tau) = \frac{\Gamma\left(m+\frac{3}{2}\right)}{m!\sqrt{\pi}} \frac{1}{W\tau}\left(1-\frac{\zeta^2}{W^2\tau^2}\right)^m, \tag{10.143}$$

$$p_{2m+2}(\zeta, \tau) = \frac{\Gamma\left(m+\frac{3}{2}\right)}{m!\sqrt{\pi}} \frac{1}{W\tau}\left(1+\frac{\zeta}{W\tau}\right)\left(1-\frac{\zeta^2}{W^2\tau^2}\right)^m,$$

when $|\zeta| \leqslant W\tau$. When $a \to \infty$, $W \to \infty$, $\frac{W^2}{2a} \to K$, the distribution (10.140) asymptotically approaches the normal distribution with mean value zero and variance $\frac{W^2\tau}{a} = 2K\tau$ which is the solution of the parabolic diffusion equation $\frac{\partial p}{\partial \tau} = K\frac{\partial^2 p}{\partial Z^2}$. The same limit, as we have seen, is reached if a and W, and also $\frac{W^2}{2a} = K$, are fixed and ζ is fixed, but τ tends to infinity. Thus, for sufficiently large τ at the points ζ far from the wave front $\zeta = \pm W\tau$, this procedure for the description of turbulent diffusion gives practically the same results as the ordinary semiempirical theory based on the use of the parabolic equation. This again indicates that this procedure is actually a generalization of the ordinary semiempirical theory, in the sense that it contains the latter as a limiting case.

As an example of the application of the theory of diffusion with a finite velocity to the case of inhomogeneous turbulence, consider the appearance of this theory as applied to vertical diffusion in the atmospheric surface layer. We begin with the simplest case of a nonstratified boundary layer. Within the limits of a logarithmic layer, from dimensional considerations we have: $W = \lambda u_*$, $\alpha = \mu u_*/Z$ where λ and μ are universal constants. It is easy to see that Eq. (10.138) with such a and W becomes the usual diffusion equation $\partial p/\partial t = \partial/\partial Z\; K\; \partial p/\partial z$ as $\alpha \to \infty$, or $\mu \to \infty$, $W \to \infty$, or $\lambda \to \infty$, $W^2/\alpha \to 2K$. Since $W^2/2a$ has the meaning of eddy diffusivity K, we must have the relation $\mu = \lambda^2/\varkappa$. (For simplicity, we shall suppose now that $\alpha = 1$; otherwise $\varkappa$ must be replaced by $\varkappa_1 = \alpha\varkappa$ in all the subsequent equations.) Transforming to dimensionless variables $\zeta = Z/L$, $\tau = \lambda u_*(t-t_0)/L$ where L is an arbitrary length scale, we may rewrite Eq. (10.138) in the form

$$\frac{\partial^2 \Psi}{\partial \tau^2} + \frac{2n}{\zeta} \frac{\partial \Psi}{\partial \tau} = \frac{\partial^2 \Psi}{\partial \zeta^2} \tag{10.144}$$

where $n = \lambda/2\kappa$ = const. Here, it is convenient to select the source height H as the scale L used to measure the height Z, so that the dimensionless variables (ζ, τ) are defined by $\zeta = \frac{Z}{H}$, and $\tau = \frac{\lambda u_* (t - t_0)}{H}$. Moreover, if $H = 0$, the case of a surface source, then it is most convenient to use the dimensional variables $\zeta = Z$, $\tau = \lambda u_*(t - t_0)$. The initial conditions which we wish to place on the solution $\Psi(\zeta, \tau)$ can be formulated first in terms of the conditional density functions p_1 and p_2 which satisfy Eqs. (10.135):

$$p_1 (Z \mid H, t) \to \varepsilon_1 \delta (Z - H), \quad p_2 (Z \mid H, t) \to \varepsilon_2 \delta (Z - H) \quad \text{for} \quad t \to t_0,$$

here ε_1 and ε_2 are the probabilities of the positive and negative initial velocity of the diffusion particle, determined from physical considerations. (For example, for smoke particles from a stack, it is natural to assume that $\varepsilon_1 = 1$, $\varepsilon_2 = 0$, and for gas particles being spread from the point of rupture of a shell, that $\varepsilon_1 = \varepsilon_2 = 1/2$.) However, since Eqs. (10.137) show that

$$\Psi = \int_0^Z (p_1 + p_2)\, dZ \text{ and } \frac{\partial \Psi}{\partial t} = - W (p_1 - p_2),$$

the initial conditions for the function Ψ may be written as

$$\Psi (\zeta, 0) = E (\zeta - 1), \quad \left. \frac{\partial \Psi (\zeta, \tau)}{\partial \tau} \right|_{\tau = 0} = (\varepsilon_2 - \varepsilon_1) \delta (\zeta - 1), \tag{10.145}$$

where $E(\zeta)$ is the probabihty distribution function concentrated at the point $\zeta = 0$, that is, a function equal to zero when $\zeta < 0$, and unity when $\zeta > 0$.

The solution of Eq. (10.144) in which we are interested and which satisfies the condition of reflection of the admixture on the earth surface $Z = 0$ will obviously be identical to the solution of the equation

$$\frac{\partial^2 \Psi}{\partial \tau^2} + \frac{2n}{|\zeta|} \frac{\partial \Psi}{\partial \tau} = \frac{\partial^2 \Psi}{\partial \zeta^2} \tag{10.146}$$

in the entire space $-\infty<\zeta<\infty$ corresponding to the presence of a symmetrical source at the point $-H$ in addition to the source at the point H. Therefore, we may assume

$$\Psi(\zeta, \tau)=\Psi_1(\zeta, \tau)+\Psi_1(-\zeta, \tau),$$

where $\Psi_1(\zeta, \tau)$ is the solution of Eq. (10.146) in the whole space which satisfies the initial conditions (10.145). In the special case of a surface source when $H=0$, it is obviously necessary to assume that $\varepsilon_1 = 1$, $\varepsilon_2 = 0$ in the conditions (10.145); the solution $\Psi_1(\zeta, \tau)$ here becomes the function $\Psi_0(\zeta, \tau)$ which may depend only on the ratio of its arguments $\xi=\frac{\zeta}{\tau}=\frac{Z}{\lambda u_*(t-t_0)}$, that is, has the form $\Psi_0(\zeta, \tau)=F(\xi)$. Here Eq. (10.146) becomes an ordianry second-order differential equation for $F(\xi)$, the solution of which, corresponding to the conditions of our problem, has the form

$$\Psi_0(\zeta, \tau)=F(\xi)=\begin{cases} 1-\left(\dfrac{1-\xi}{1+\xi}\right)^n & \text{for} \quad 0\leqslant\xi\leqslant 1, \\ 0 & \text{for} \quad \xi>1 \quad \text{or} \quad \xi<0 \end{cases} \tag{10.147}$$

[Monin (1956b)]. Consequently, the admixture concentration $\bar{\vartheta}=\frac{\partial\Psi_0}{\partial Z}$ is given here by

$$\bar{\vartheta}(Z, t\mid 0)=\frac{Q}{\varkappa u_*\tau}\,\frac{(1-Z/\lambda u_*\tau)^{n-1}}{(1+Z/\lambda u_*\tau)^{n+1}}, \quad 0\leqslant Z\leqslant\lambda u_*\tau. \tag{10.148}$$

Let us note that $F(1-0) = 1$ according to Eq. (10.147), that is, the function $F(\xi)$ here turns out to be continuous at the point $\xi=1$. In other words, the fraction of admixture which is found at the wave front $Z=\lambda u_*\tau$ will be equal to zero at all times. (This is explained by the fact that the frequency of scattering $a=\frac{\lambda^2}{2\varkappa}\frac{u_*}{Z}$ is infinitely large at the point $Z=0$, so that the particle which leaves the source at the point $Z=0$ must experience at least one scattering.) The behavior of the admixture concentration near the front depends strongly on the values of the numerical constant $n=\frac{\lambda}{2\varkappa}$. When $n<1$, or $\lambda<2\varkappa\approx 0.8$, the concentration tends to infinity on approaching the front, so that the latter has the nature of a strong discontinuity. On the other hand, when $n>1$, or $\lambda>2\varkappa\approx 0.8$, the concentration approaches zero on approaching the front, and when $n = 1$ it approaches some finite value. The experimental data of Kazanskiy and Monin (1957) [see also Monin (1959a)] correspond best to a value of n close to unity. These investigators measured the vertical profiles of the admixture concentration in a smoke jet from a stationary, linear, cross-wind source at a distance of 100 meters from the source. Afterwards, the profiles observed were compared with the theoretical curves $f(\xi) = (1-\xi)^{n-1}(1+\xi)^{-n-1}$ for different n; the curve for $n = 1$ gave the best fit. This deduction does not seem very convincing since it does not take into account the effect of wind shear on dispersion; however, it does agree well with other existing estimates. Thus in one of the experiments of Kazanskiy and Monin, the temporal rate of increase in the vertical diameter d of a single puff of smoke was studied with the aid of cinematography. The experimental curve $d(t)$ was found to be a good approximation of the straight line $d = 2W(t-t_0)$ where the value of $\lambda = W/u_*$ turned out to be equal to 0.75, which corresponds to a value of $n = \lambda/2K$ close to unity. On the other hand, it is natural to suppose that the root mean square velocity of diffusing particles, which is equal to $W = \lambda u_*$ according to the theory of this subsection, must be of the same order of magnitude as the typical Eulerian vertical velocity σ_W of the boundary-layer flow. Since $\sigma_W/u_* \approx 0.9$ to 1.2 [see Sects. 5.3 and 8.5], we again obtain the estimate $\lambda/2\varkappa \approx 1$.

To compare diffusion theory with finite velocity with the ordinary semiempirical theory of turbulent diffusion, the function (10.148) should be compared with the function (10.93′) which is the solution of a parabolic diffusion equation with eddy diffusivity of the form $K = \varkappa u_* Z = K_1 Z$; the difference between $\varkappa$ and $\varkappa' = \alpha\varkappa$ can evidently now be neglected. This comparison, under the assumption that $n = 1.25$, is presented in Fig. 85, taken from Pasquill (1962b). We can see that for each fixed $\tau = t - t_0$, the profile of the mean concentration $\bar{\vartheta}(Z)$ assumes very similar forms in the theory of diffusion with a finite velocity, and in the ordinary semiempirical theory within the range of values of Z corresponding to relatively large values of the concentration. The essential difference of the profiles is observed only near the front of the admixture cloud. Here, according to diffusion theory, the concentration suddenly drops to zero with a finite velocity, while according to the ordinary theory, it continues to drop smoothly, remaining different from zero for values of Z as large as desired.

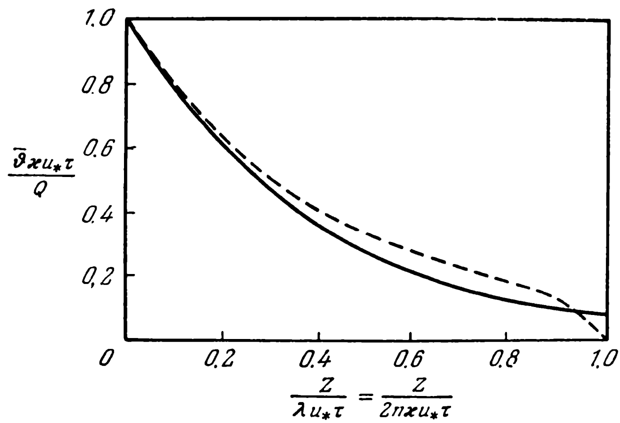


FIG. 85. A comparison of the vertical distribution of concentration according to diffusion theory with finite velocity (dashed line) with the distribution obtained from a parabolic diffusion equation with an eddy diffusivity which increases linearly with height (solid line).

It is interesting to compare the estimates of the Lagrangian statistical characteristics obtained in Sect. 10.5 using the ordinary semiempirical theory with the corresponding estimates from diffusion theory with a finite velocity; such a comparison may give an idea of the accuracy of the semiempirical theory. For simplicity, let us suppose that $n = 1$ and disregard the difference between $\varkappa$ and $\varkappa' = \alpha\varkappa$. According to Eq. (10.148),

$$\langle Z \rangle = \int_0^\infty Z\bar{\vartheta}(Z,t)dZ \Big/ \int_0^\infty \bar{\vartheta}(Z,t)dZ$$

$$= \frac{\lambda^2 u_*}{\varkappa}\tau \int_0^1 \frac{(1-\xi)^{n-1}}{(1+\xi)^{n+1}}\xi d\xi = \qquad (10.149)$$

$$= 4(\ln 2 - 0.5)\varkappa u_* \tau \approx 0.77\varkappa u_* \tau \quad \text{when } n = 1, \text{ that is, } \lambda = 2\varkappa.$$

We see that the theory of diffusion with a finite velocity leads to an estimate of the constant b in the Lagrangian equation $\overline{Z} = bu_*\tau$ [see Eq. (9.61′)] which is about 13% lower than the estimate from the ordinary diffusion theory when a reasonable value of vertical velocity W is chosen. On the other hand, if we estimate the value of $\sigma^2_{\vartheta_z} = \langle Z - \langle Z\rangle\rangle^2$ in a similar way, we obtain the following result:

$$\sigma^2_{\vartheta_z} = \int_0^1 Z^2 \bar{\vartheta}(Z, t) dZ \Big/ \int_0^1 \bar{\vartheta}(Z, t) dZ - \langle Z\rangle^2 =$$

$$= \frac{\lambda^3}{\varkappa} u_*^2 \tau^2 \int_0^1 \frac{(1-\xi)^{n-1}}{(1+\xi)^{n+1}} \xi^3 \, d\xi - \langle Z\rangle^2 =$$

$$= [8(1.5 - 2\ln 2) - 16(\ln 2 - 0.5)^2] \varkappa^2 u_*^2 \tau^2 \approx 0.32 \varkappa^2 u_*^2 \tau^2$$

$$\text{when } n = 1. \qquad (10.149')$$

We see that the estimate of the constant d_{33} in the Lagrangian equation $D_{33}(\tau) = d_{33} u_*^2 \tau^2$ is only about 30% of the corresponding estimate obtained with the aid of ordinary diffusion theory, and the estimate of the standard deviation σ_{ϑ_z} about 55%, when the same value of W is chosen. Therefore we must conclude that the ordinary semiempirical diffusion theory may be used for an approximate estimate of the position of the center of gravity of the admixture cloud in a boundary layer, but it will give a seriously incorrect estimate of the dispersion of the cloud.

When $\lambda \to \infty$ the function (10.54) naturally approaches the function (10.93′) which is a solution of the parabohc diffusion equation

$$\frac{\partial \bar{\vartheta}}{\partial t} = \frac{\partial}{\partial Z} \varkappa u_* Z \frac{\partial \bar{\vartheta}}{\partial Z}.$$

In the case of a surface point source which begins to produce admixture, at time $t = 0$, with constant (unit) rate, the concentration $\bar{\vartheta}_1(Z, t)$ is found by integration with respect to time of the solution of Eq. (10.148); when $t \to \infty$, it is easy to obtain from this the following expansion in powers of $1/t$:

$$\bar{\vartheta}_1(Z_2, t) - \bar{\vartheta}_1(Z_1, t) = \frac{1}{\varkappa u_*} \int_{Z_2/\lambda u_* t}^{Z_1/\lambda u_* t} \frac{(1-\xi)^{n-1}}{(1+\xi)^{n+1}} \frac{d\xi}{\xi} \approx$$

$$\approx \frac{1}{\varkappa u_*} \left(\ln \frac{Z_1}{Z_2} + \frac{Z_2 - Z_1}{\varkappa u_* t} + \dots \right). \qquad (10.150)$$

This equation permits one to evaluate the rate of approach of the concentration profile to the logarithmic profile which corresponds to a steady surface source.

In the case of an elevated source at an arbitrary height H, the solution of the corresponding initial value problem for Eq. (10.146) may be found using the general Riemann method for the solution of second-order hyperbolic differential equations. In general, this solution is quite awkward; it is in the form of an integral of a complex combination of hypergeometric functions; see Monin (1956b). However, in some cases it may be substantially simplified. Examples of this type may be found in Monin (1955; 1956b); however, we shall not consider them in detail in this book.

Let us consider briefly diffusion with a finite velocity in a thermally stratified boundary layer. Here, the concentration distribution $\bar{\vartheta}_1(Z, t)$ must be defined according to $\bar{\vartheta} = \frac{\partial \Psi}{\partial Z}$ where $\Psi = \Psi(Z, t)$ is the solution of the corresponding telegraph equation (10.138) for suitable boundary conditions; below we shall use only the condition of reflection of the admixture at $Z = 0$ and the initial condition which corresponds to the presence at the time $t - t_0$ of an instantaneous point admixture source of unit output at the point $(0, 0, H)$. On the basis of the similarity principles developed in Chapt. 4 we may expect that the coefficients W, with velocity dimensions, and a, with frequency dimensions, may be represented here in the form

$$W = \lambda u_* \varphi_1\left(\frac{Z}{L}\right), \qquad a = \frac{u_*}{Z}\varphi_2\left(\frac{Z}{L}\right), \tag{10.151}$$

where L is a length scale defined by Eq. (7.12), φ_1 and φ_2 are dimensionless universal functions, and λ is a numerical constant introduced for convenience, which allows one to impose the additional restriction $\varphi_1(0) = 1$ on the function $\varphi_1(\zeta)$. (In this case λ has the same meaning as above, that is, it denotes the limiting value of W/u_* as $\frac{Z}{L} \to 0$, or the ratio of W/u_* for neutral stratification.) Since our diffusion theory is already approximate, it is reasonable to refer to additional approximate expressions of a semiempirical nature to determine the functions $\varphi_1(\zeta)$ and $\varphi_2(\zeta)$. This may be done, for example, as follows. We take as a basis the semiempirical equation $K = u_* l(1 - \sigma \mathrm{Rf})^{1/4}$ [see Sect. 6]; if, in addition, we assume that $K \sim Wl$, then we obtain

$$W \sim u_*(1 - \sigma \mathrm{Rf})^{\frac{1}{4}}, \quad \text{that is,} \quad \varphi_1(\zeta) = \left[1 - \frac{\sigma}{f'(\zeta)}\right]^{\frac{1}{4}}, \tag{10.151'}$$

where $f(\zeta)$ is a universal function of the wind profile in the atmospheric surface layer. Now we use the fact that in the stationary case Eqs. (10.136) have the form

$$q = \text{const}, \quad 2aq = -W\frac{\partial W\bar{\vartheta}}{\partial Z}, \tag{10.152}$$

where q is a vertical admixture flux. For neutral stratification, $W = \lambda u_* = \text{const}$; therefore, the right side of the second of Eqs. (10.152) may be written here in the form $-W^2\frac{\partial \bar{\vartheta}}{\partial Z}$. Strictly speaking, in the general case this will not be so; however, it may be hoped that here also the corresponding correction to the term $-W^2\frac{\partial \bar{\vartheta}}{\partial Z}$ will be relatively small and insignificant. Thus, as a first approximation, the following may be assumed:

$$\frac{\partial\bar{\vartheta}}{\partial Z} = -\frac{2aq}{W^2} = -\frac{2q}{\lambda^2 u_* Z}\frac{\varphi_2(\zeta)}{[\varphi_1(\zeta)]^2}.$$

On the other hand, neglecting the possible differences between the eddy admixture diffusivity and the eddy viscosity, the gradient of a stationary concentration distribution, according to similarity theory, will equal

$$\frac{\partial\bar{\vartheta}}{\partial Z} = -\frac{q}{\varkappa u_* Z}\zeta f'(\zeta).$$

Comparing the two expressions for $\frac{\partial\bar{\vartheta}}{\partial Z}$ and using Eq. (10.151′), we obtain

$$\varphi_2(\zeta) = \frac{\lambda^2}{2\varkappa}\zeta f'(\zeta)\left[1 - \frac{\sigma}{f'(\zeta)}\right]^{\frac{1}{2}}. \tag{10.153}$$

Equations (10.150), (10.151′), and (10.153) with a known function $f(\zeta)$ and a known constant σ, which may be considered as assuming different values for stable and unstable stratification, completely determine the form of the coefficients of the telegraph equation (10.138) for diffusion in the atmospheric surfacc layer with arbitrary stratification. Here the limit $a \to \infty$, $W \to \infty$, $\frac{W^2}{2a} \to K = \text{const}$, which corresponds to replacing the telegraph equation by an ordinary parabohc diffusion equation, will obviously be equivalent to the limit when $\lambda \to \infty$.

Using the equations above and transforming to dimensionless variables $\zeta = \frac{Z}{L}$ and $\tau = \frac{\lambda u_* (t - t_0)}{L}$ we can reduce Eq. (10.138) to the form

$$\frac{\partial^2\Psi}{\partial\tau^2} + \frac{2n}{g(\xi)}\frac{\partial\Psi}{\partial\tau} = \frac{\partial^2\Psi}{\partial\xi^2}, \tag{10.154}$$

where $n = \frac{\lambda}{2\varkappa}$, and the variable ξ and the function $g(\xi)$ are determined by the following:

$$\xi = \int_0^{\zeta}\left[1 - \frac{\sigma}{f'(\zeta)}\right]^{-\frac{1}{4}} d\zeta, \quad g = \frac{1}{f'(\zeta)}\left[1 - \frac{\sigma}{f'(\zeta)}\right]^{-\frac{1}{2}}. \tag{10.155}$$

The functions $f(\zeta)$ and $g(\xi)$, and the variable $\xi = \xi(\zeta)$ have different forms for neutral, stable and unstable stratification. The case of *neutral stratification* was considered above; here in all the preceding equations the limit $L \to \infty$ or $\zeta = \frac{Z}{L} \to 0$ must be taken. But, $f'(\zeta) \approx \frac{1}{\zeta}$, $\xi \approx \zeta$ and $g \approx \zeta$ for small $|\zeta|$; therefore, Eq. (10.154) here assumes the form (10.144). Under *stable stratification*, we can use, for example, for $f(\zeta)$, the interpolation formula

$$f(\zeta) \approx \ln \zeta + C_2\zeta,$$

where $C_2 = 1/\mathrm{Rf}_{cr} = \sigma$. In the case of *unstable stratification* for large ζ we have $f'(\zeta) \sim \zeta^{-\frac{4}{3}}$. However, instead of dwelling here on the analysis of the situation for a stratified boundary layer, we remark only that the main difference from the neutrally stratified case consists in the appearance of the function $g(\xi)$ in the equation, which depends on stratification.

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