5

Pearson Correlation Coefficient

This chapter develops several forms of the Pearson correlation coefficient in the different domains. This coefficient can be used as an optimization criterion to derive different optimal noise reduction filters [14], but is even more useful for analyzing these optimal filters for their noise reduction performance.

5.1 Correlation Coefficient Between Two Random Variables

Let a and b be two zero-mean real-valued random variables. The Pearson correlation coefficient (PCC) is defined as[1] [41], [98], [105]

$$\rho\left(a, b\right) = \frac{E\left(ab\right)}{\sigma_a \sigma_b}, \tag{5.1}$$

where $E\left(ab\right)$ is the cross-correlation between a and b, and $\sigma_a^2 = E\left(a^2\right)$ and $\sigma_b^2 = E\left(b^2\right)$ are the variances of the signals a and b, respectively. In the rest, it will be more convenient to work with the squared Pearson correlation coefficient (SPCC):

$$\rho^2\left(a, b\right) = \frac{E^2\left(ab\right)}{\sigma_a^2 \sigma_b^2}. \tag{5.2}$$

One of the most important properties of the SPCC is that

$$0 \leq \rho^2\left(a, b\right) \leq 1. \tag{5.3}$$

The SPCC gives an indication on the strength of the linear relationship between the two random variables a and b. If $\rho^2\left(a, b\right) = 0$, then a and b are said

[1] This correlation coefficient is named after Karl Pearson who described many of its properties. Actually to be precise, the true definition of the PCC is when the expectation operator in (5.1) is replaced by a sum over the samples.

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to be uncorrelated. The closer the value of $\rho^2(a,b)$ is to 1, the stronger the correlation between the two variables. If the two variables are independent, then $\rho^2(a,b)=0$. But the converse is not true because the SPCC detects only *linear* dependencies between the two variables a and b. For a non-linear dependency, the SPCC may be equal to zero. However, in the special case when a and b are jointly normal, "independent" is equivalent to "uncorrelated."

5.2 Correlation Coefficient Between Two Random Vectors

We can generalize the idea of the SPCC between two random variables to two random vectors. Indeed, let

$$\mathbf{a} = \begin{bmatrix} a_1 \ a_2 \ \cdots \ a_L \end{bmatrix}^T,$$
$$\mathbf{b} = \begin{bmatrix} b_1 \ b_2 \ \cdots \ b_L \end{bmatrix}^T,$$

be two zero-mean real-valued random vectors of length L. We define the SPCC between $\mathbf{a}$ and $\mathbf{b}$ as

$$\rho^2(\mathbf{a},\mathbf{b}) = \frac{E^2\left(\mathbf{a}^T\mathbf{b}\right)}{E\left(\mathbf{a}^T\mathbf{a}\right)E\left(\mathbf{b}^T\mathbf{b}\right)}. \tag{5.4}$$

Let $\mathbf{\Pi}_a$ and $\mathbf{\Pi}_b$ be two permutation matrices. If $\mathbf{\Pi}_a = \mathbf{\Pi}_b$ then $\rho^2(\mathbf{\Pi}_a\mathbf{a},\mathbf{\Pi}_a\mathbf{b}) = \rho^2(\mathbf{a},\mathbf{b})$. In general, if $\mathbf{\Pi}_a \neq \mathbf{\Pi}_b$, we have $\rho^2(\mathbf{\Pi}_a\mathbf{a},\mathbf{\Pi}_b\mathbf{b}) \neq \rho^2(\mathbf{a},\mathbf{b})$.

Property 5.1. We always have

$$0 \leq \rho^2(\mathbf{a},\mathbf{b}) \leq 1. \tag{5.5}$$

Proof. From the definition (5.4), it is clear that $\rho^2(\mathbf{a},\mathbf{b}) \geq 0$. To show that $\rho^2(\mathbf{a},\mathbf{b}) \leq 1$, let us define the positive quantity:

$$E\left[(\mathbf{a}-c\mathbf{b})^T(\mathbf{a}-c\mathbf{b})\right] \geq 0, \tag{5.6}$$

where c is a real number. The development of the previous expression gives:

$$E\left[(\mathbf{a}-c\mathbf{b})^T(\mathbf{a}-c\mathbf{b})\right] = E\left(\mathbf{a}^T\mathbf{a}\right) - 2cE\left(\mathbf{a}^T\mathbf{b}\right) + c^2E\left(\mathbf{b}^T\mathbf{b}\right). \tag{5.7}$$

In particular, for

$$c = \frac{E\left(\mathbf{a}^T\mathbf{a}\right)}{E\left(\mathbf{a}^T\mathbf{b}\right)} \tag{5.8}$$

we get

$$E\left(\mathbf{a}^T\mathbf{a}\right) - 2E\left(\mathbf{a}^T\mathbf{a}\right) + \frac{E^2\left(\mathbf{a}^T\mathbf{a}\right)E\left(\mathbf{b}^T\mathbf{b}\right)}{E^2\left(\mathbf{a}^T\mathbf{b}\right)} \geq 0, \tag{5.9}$$

which implies that

$$\frac{E\left(\mathbf{a}^T\mathbf{a}\right)E\left(\mathbf{b}^T\mathbf{b}\right)}{E^2\left(\mathbf{a}^T\mathbf{b}\right)} \geq 1. \tag{5.10}$$

Therefore $\rho^2\left(\mathbf{a}, \mathbf{b}\right) \leq 1$.

5.3 Frequency-Domain Versions

Let $A(j\omega)$ and $B(j\omega)$ be the DTFTs of the two zero-mean real-valued random variables a and b. We define the subband SPCC, which is also known as the magnitude squared coherence function (MSCF), between $A(j\omega)$ and $B(j\omega)$ at frequency ω as

$$\begin{aligned} \left|\rho\left[A(j\omega), B(j\omega)\right]\right|^2 &= \frac{\left|E\left[A(j\omega)B^*(j\omega)\right]\right|^2}{E\left[\left|A(j\omega)\right|^2\right]E\left[\left|B(j\omega)\right|^2\right]} \\ &= \frac{\left|\phi_{ab}(j\omega)\right|^2}{\phi_a(\omega)\phi_b(\omega)}, \end{aligned} \tag{5.11}$$

where superscript * denotes complex conjugation. It is clear that the subband SPCC always takes its values between 0 and 1.

We can generalize this idea to infinite vectors (containing all frequencies). We will refer to this definition as the fullband SPCC, which also takes its values between 0 and 1. In this case, we have

$$\begin{aligned} \left|\rho\left(A, B\right)\right|^2 &= \frac{\left|E\left[\int_{-\pi}^{\pi} A(j\omega)B^*(j\omega)d\omega\right]\right|^2}{E\left[\int_{-\pi}^{\pi}\left|A(j\omega)\right|^2 d\omega\right]E\left[\int_{-\pi}^{\pi}\left|B(j\omega)\right|^2 d\omega\right]} \\ &= \frac{\left|\int_{-\pi}^{\pi}\phi_{ab}(j\omega)d\omega\right|^2}{\left[\int_{-\pi}^{\pi}\phi_a(\omega)d\omega\right]\left[\int_{-\pi}^{\pi}\phi_b(\omega)d\omega\right]} \\ &= \frac{E^2\left(ab\right)}{\sigma_a^2\sigma_b^2} \\ &= \rho^2\left(a, b\right). \end{aligned} \tag{5.12}$$

5.4 KLE-Domain Versions

Let a and b be two zero-mean real-valued random variables and $c_{a,l}$ and $c_{b,l}$ their respective representations in the KLE domain and in the subband l. We define the subband SPCC (or MSCF) between $c_{a,l}$ and $c_{b,l}$ in the subband l as

$$\rho^2 (c_{a,l}, c_{b,l}) = \frac{E^2 (c_{a,l} c_{b,l})}{E\left(c_{a,l}^2\right) E\left(c_{b,l}^2\right)}. \tag{5.13}$$

The vector form or fullband SPCC is

$$\rho^2 (\mathbf{c}_a, \mathbf{c}_b) = \frac{E^2 \left(\mathbf{c}_a^T \mathbf{c}_b\right)}{E\left(\mathbf{c}_a^T \mathbf{c}_a\right) E\left(\mathbf{c}_b^T \mathbf{c}_b\right)}, \tag{5.14}$$

where $\mathbf{c}_a$ and $\mathbf{c}_b$ are two vectors of length L containing all the elements $c_{a,l}$ and $c_{b,l}$, $l = 1, 2, \ldots, L$, respectively. It is clear that

$$0 \le \rho^2 (c_a, c_b) \le 1, \tag{5.15}$$

$$0 \le \rho^2 (\mathbf{c}_a, \mathbf{c}_b) \le 1. \tag{5.16}$$

5.5 Summary

This chapter developed different forms of the so-called Pearson correlation coefficient in the time, frequency, and KLE domains. Each of these forms has many interesting properties, which are very useful not only for deriving, but also for analyzing optimal filters in the context of noise reduction. We will elaborate on these properties in the next chapter.