# Shock Waves in Arbitrary Fluids

By HERMANN WEYL

## 1. *Introduction*

The following investigation tries to clear up the general hydrodynamical and thermodynamical foundations of the shock phenomenon.[1] The *first part,* Sections 2-5, answers the question: What are the conditions for the equation of state of a fluid under which shocks with their distinctive qualitative features may be produced. These conditions, enumerated in Section 3, are partly of differential, partly of global nature. The *second part,* Sections 6-7, investigates the physical structure of the shock layer whose "infinitesimal" width is of the order of magnitude $\epsilon$ provided heat conductivity and viscosity are small of the same order. Initial state and final state are singular points for the differential equations of the shock layer, and it is shown that they are of such a nature as to make one expect the problem to have a unique solution.

## Part I. Thermodynamics and the Shock Phenomenon

### 2. *Shocks*

At any place and time a fluid is in a definite *thermodynamical state* $Z$ and in a definite *state of motion,* the latter being characterized by the components $u, v, w$ of the velocity $\vec{u}$ in a Cartesian system of coordinates $x, y, z$. The possible thermodynamical states form a two-dimensional manifold $\mathfrak{Z}$; examples of quantities having definite values in a definite thermodynamical state are density $\rho$, pressure $p$, temperature $T$; volume $\tau = \rho^{-1}$, potential energy $e$, heat content or enthalpy $i = e + p\tau$, and entropy $S$ per unit mass. On $\mathfrak{Z}$ the fundamental thermodynamical equation

$$de = T\,dS - p\,d\tau \tag{1}$$

holds. It is once and for all assumed that we are dealing with a stationary state $(Z, \vec{u})$; i.e., that neither $Z$ nor $\vec{u}$ vary in time, but depend on $x, y, z$ only.

[1]This paper was originally submitted to the Applied Mathematics Panel in 1944 as a report. The same problem is treated in, *The theory of shock waves for an arbitrary equation of state,* H. A. Bethe, Division B, NDRC, OSRD, Report No. 545, to which the author had no access.

In the limit for vanishing viscosity and heat conductivity, a *discontinuity* is possible such that $(Z, \vec{u})$ has a constant value $(Z_0, \vec{u}_0)$ for $x < 0$ and another constant value $(Z_1, \vec{u}_1)$ for $x > 0$. The symbols [0], [1] refer to the two regions $x < 0$, $x > 0$ which are divided by the shock front $x = 0$. Because of conservation of mass, we have $\rho_0 u_0 = \rho_1 u_1$. Denoting by $M$ this common flow of mass, we may write

$$u_0 = M\tau_0, \qquad u_1 = M\tau_1 .$$

Then the laws of conservation of momentum and energy give the relations

$$Mu_0 + p_0 = Mu_1 + p_1 ,$$

$$Mv_0 = Mv_1 , \qquad Mw_0 = Mw_1 ,$$

$$M\{i_0 + \tfrac{1}{2}(u_0^2 + v_0^2 + w_0^2)\} = M\{i_1 + \tfrac{1}{2}(u_1^2 + v_1^2 + w_1^2)\}$$

while the law of increasing entropy ("More entropy flows in than out") requires

$$MS_1 \geq MS_0 .$$

The phenomenon that results for $M \neq 0$ is called a *shock*. The problem then splits into two parts, one involving only the normal velocity component $u$, the other referring to the tangential components and stating that they cross the shock front unchanged:

$$u_0 = M\tau_0 , \qquad u_1 = M\tau_1 , \tag{2.1}$$

$$Mu_0 + p_0 = Mu_1 + p_1 , \tag{2.2}$$

$$i_0 + \tfrac{1}{2}u_0^2 = i_1 + \tfrac{1}{2}u_1^2 , \tag{2.3}$$

$$v_0 = v_1 , \qquad w_0 = w_1 . \tag{3}$$

If, however, $M$ vanishes, we obtain a *vortex sheet* or *slip stream* characterized by the relations

$$u_0 = u_1 = 0, \qquad p_0 = p_1 .$$

The fluid in [1] glides tangentially past that in [0] and the pressure is the same on both sides of the discontinuity. Our object will largely be the study of shocks in an arbitrary fluid.

After introducing the values (2.1) for the velocities, (2.2) gives

$$M^2 = \frac{p_1 - p_0}{\tau_0 - \tau_1} , \tag{4}$$

and thereupon (2.3) yields a relation between the two thermodynamical states $Z_0$ and $Z_1$ ; namely,

$$i_1 - i_0 = \tfrac{1}{2}M^2(\tau_0 - \tau_1)(\tau_0 + \tau_1) = \tfrac{1}{2}(p_1 - p_0)(\tau_0 + \tau_1) \qquad \text{or}$$

$$e_1 - e_0 = \tfrac{1}{2}(p_1 + p_0)(\tau_0 - \tau_1).$$

Hence the problem of shocks is reduced to a study of this relation between two states $Z_0$, $Z_1$:

$$H \equiv H(Z_1, Z_0) \equiv (e_1 - e_0) - \tfrac{1}{2}(p_1 + p_0)(\tau_0 - \tau_1) = 0 \tag{5}$$

(Hugoniot equation).

We are only interested in shocks in one and two dimensions. If in the latter case the normal unit-vector of the shock front in the direction [0] → [1] is $(\alpha, \beta)$ instead of (1, 0) the equations (2.1) and (3) for the velocities read as follows:

$$\alpha u_0 + \beta v_0 = M\tau_0, \qquad \alpha u_1 + \beta v_1 = M\tau_1,$$

$$-\beta u_0 + \alpha v_0 = -\beta u_1 + \alpha v_1.$$

A simple analysis (which will presently occupy us in much greater detail) reveals that the value of $M$ resulting from (4) is of the order $\rho c$ where $c$ is the acoustic velocity. Thus we are dealing with a situation like this: A certain quantity might be zero, but if it is not, it is even $\geq 1$. Clearly quite different circumstances must be responsible for the two phenomena of shock and slip stream, and $M = 0$ is in no way to be considered here the limiting case of a non-vanishing $M$. The problem of the shock layer will be compared to that of the slip layer in Section 6 by taking viscosity and heat conductivity into account and then letting them tend to zero. In particular, this passage to the limit will explain why a shock is not conservative with respect to entropy although it conserves energy.

## 3. *Thermodynamical Assumptions*

Next we specify our assumptions concerning the thermodynamical behavior of our fluid. They will be enumerated, I-IV.

I. *Infinitesimal adiabatic increase of pressure produces compression:*

$$\left(\frac{d\tau}{dp}\right)_{ad} < 0.$$

II. *The rate of compression* $- d\tau/dp$ *diminishes in this process:*[2]

$$\left(\frac{d^2\tau}{dp^2}\right)_{ad} > 0.$$

These local hypotheses will be supplemented by two assumptions of "global" character.

[2]It can be made plausible that condition II is essential for the formation of a shock wave.

III. *In the continuous process of adiabatic compression one can raise pressure arbitrarily high.*

IV. *The state $Z$ is uniquely specified by pressure and specific volume, and the points $(p, \tau)$ representing the possible states $Z$ in a $(p, \tau)$—diagram form a convex region.*

[It would be more natural but less elementary to divide IV into a local and a global part, the local postulate asserting that in the neighborhood of a given state $Z_0$ the variables $p$, $\tau$ can be used as parameters for the specification of states; in other words that the projection $Z \to (p, \tau)$ of the manifold $\mathfrak{Z}$ of states $Z$ upon a $(p,\tau)$-plane is locally one-to-one. The global part would assert that the projection of $\mathfrak{Z}$ covers a convex region $\bar{\mathfrak{Z}}$ in the $(p,\tau)$-plane and that $Z_0$ being any given state and $(p_0,\tau_0)$ its projection, one never runs against an obstacle when, on starting at $Z_0$, one lets $Z$ vary so that its projection $(p, \tau)$ follows a given path in $\bar{\mathfrak{Z}}$ beginning at $(p_0,\tau_0)$ ("continuation"). All this means that $\mathfrak{Z}$ is a covering surface over the convex $\bar{\mathfrak{Z}}$ without singularities and relative boundaries. But the convex $\bar{\mathfrak{Z}}$ being simply connected, this covering surface necessarily consists of a single sheet, or $(p, \tau)$ determines $Z$ uniquely. We thus fall back on the more elementary formulation of IV. The assumption of convexity for the region $\bar{\mathfrak{Z}}$ will prove to be quite essential for our investigation. Henceforth we simply denote by $Z$ the projection of $Z$ in the $(p,\tau)$-plane.]

Because of IV, entropy $S$ is a (single-valued) function of $p$ and $\tau$, $S = S(p, \tau)$. The adiabatic process of compression is thus defined by

$$\frac{\partial S}{\partial p}\, dp + \frac{\partial S}{\partial \tau}\, d\tau = 0.$$

From the fact that (1) is a total differential we have

$$\frac{\partial}{\partial p}\left(T \frac{\partial S}{\partial \tau} - p\right) = \frac{\partial}{\partial \tau}\left(T \frac{\partial S}{\partial p}\right)$$

or

$$\frac{\partial T}{\partial p}\frac{\partial S}{\partial \tau} - \frac{\partial T}{\partial \tau}\frac{\partial S}{\partial p} = 1. \tag{6}$$

Hence $\partial S/\partial p$, $\partial S/\partial \tau$ cannot vanish simultaneously, and thus hypothesis I requires that $\partial S/\partial p$, $\partial S/\partial \tau$ are of the same sign:

$$\frac{\partial S}{\partial p} > 0, \quad \frac{\partial S}{\partial \tau} > 0 \quad \text{or} \quad \frac{\partial S}{\partial p} < 0, \quad \frac{\partial S}{\partial \tau} < 0.$$

As the region $\mathfrak{Z}$ is connected, the one set or the other will hold *everywhere*. We assume the first:

$$\frac{\partial S}{\partial p} > 0, \quad \frac{\partial S}{\partial \tau} > 0. \tag{I$'$}$$

(The other alternative would make little difference.) Infinitesimal adiabatic compression is now described by

$$dp = a^* \, dt, \qquad d\tau = b^* \, dt; \qquad a^* = \frac{\partial S}{\partial \tau}, \qquad b^* = -\frac{\partial S}{\partial p}$$

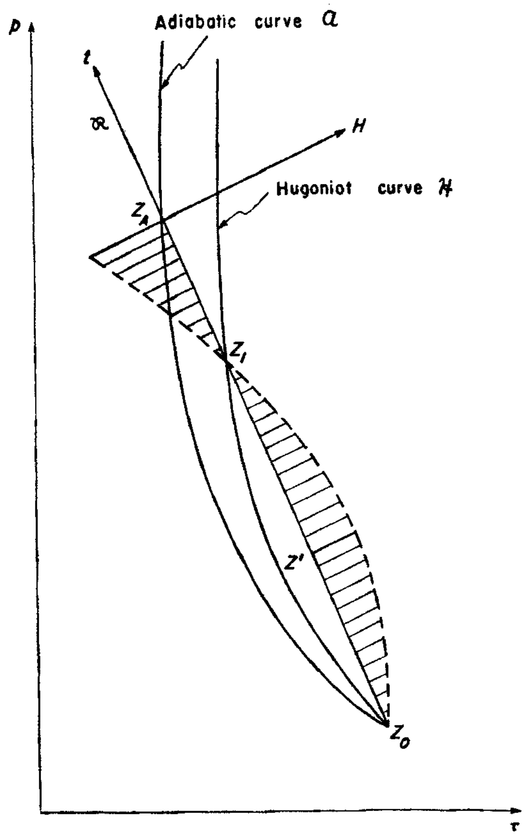


FIG. 1. This diagram shows the adiabatic through $Z_0$ and the Hugoniot line in a $p,\tau$-diagram; moreover, in the form of a shaded profile, $H$ as function of the distance $t$ along a ray: $H$ reaches its maximum at $Z'$, passes through zero at $Z_1$, and is negative at $Z_A$.

(a *positive* factor of proportionality $dt$ corresponding to an *increase* of pressure). Hypothesis II yields

$$\frac{d}{dt}\left(\frac{b^*}{a^*}\right) > 0 \qquad \text{or} \qquad b^* \frac{da^*}{dt} - a^* \frac{db^*}{dt} < 0,$$

i.e. $$b^*\left(\frac{\partial a^*}{\partial p} a^* + \frac{\partial a^*}{\partial \tau} b^*\right) - a^*\left(\frac{\partial b^*}{\partial p} a^* + \frac{\partial b^*}{\partial \tau} b^*\right)$$

$$= -a^{*2}\frac{\partial b^*}{\partial p} + a^*b^*\left(\frac{\partial a^*}{\partial p} - \frac{\partial b^*}{\partial \tau}\right) + b^{*2}\frac{\partial a^*}{\partial \tau} < 0$$

or

$$\text{(II}'\text{)} \qquad \frac{\partial^2 S}{\partial p^2}\left(\frac{\partial S}{\partial \tau}\right)^2 - 2\frac{\partial^2 S}{\partial p\, \partial \tau}\frac{\partial S}{\partial \tau}\frac{\partial S}{\partial p} + \frac{\partial^2 S}{\partial \tau^2}\left(\frac{\partial S}{\partial p}\right)^2 < 0.$$

In the future we shall use hypotheses I and II in their analytic forms (I′) and (II′).

Our next move consists in developing a number of consequences from the assumptions I-III. Let $(a, b) \neq (0, 0)$ be two given numbers which determine the direction of a straight line through $Z_0 = (p_0, \tau_0)$,

$$p = p_0 + at, \qquad \tau = \tau_0 + bt,$$

the half-line or ray being obtained by the restriction $t \geq 0$ on the parameter $t$. We follow the straight line or the ray as long as it stays within $\mathcal{Z}$, and form

$$\frac{dS}{dt} = \frac{\partial S}{\partial p} a + \frac{\partial S}{\partial \tau} b = S'$$

$$\frac{dS'}{dt} = \frac{\partial S'}{\partial p} a + \frac{\partial S'}{\partial \tau} b = \frac{\partial^2 S}{\partial p^2} a^2 + 2\frac{\partial^2 S}{\partial p\, \partial \tau} ab + \frac{\partial^2 S}{\partial \tau^2} b^2.$$

*Lemma* 1. *$S'$ is positive (negative) and hence $S$ increasing (decreasing) along the whole straight line, provided $a \geq 0$, $b \geq 0$ ($a \leq 0$, $b \leq 0$).*

This follows from assumption (I′).

*Lemma* 2. *If $S' = 0$ for a certain value of $t$, then $dS'/dt < 0$ for the same value.*

Proof. Substitute the values of $a$ and $b$ derived from $S' = 0$ into $dS'/dt$ and use the inequality (II′).

*Lemma* 3. *If $S' \leq 0$ for $t = 0$, then $S' < 0$* for *$t > 0$.*

Proof. First assume $S' < 0$ for $t = 0$. Should $S$ change sign as $Z$ travels along the ray, it would have to vanish somewhere. Suppose this occurs for the first time at $t = t_1$. As $S' < 0$ before $t$ reaches this value, $S'$ passes the zero level at $t_1$ ascending; hence $dS'/dt \geq 0$ for $t = t_1$. But these two relations $S' = 0$, $dS'/dt \geq 0$ for $t = t_1$, contradict Lemma 2.

If $S' = 0$ for $t = 0$ we have $dS'/dt < 0$ for $t = 0$; consequently $S'$ becomes negative immediately after the point $Z$ has started on its way from $Z_0$, and from then on, as we have seen, $S'$ must remain negative.

We turn to III. Starting at a given point $Z_0 = (p_0, \tau_0)$ and always raising $p$, we follow the adiabatic through $Z_0$: we thus obtain a continuous monotonely descending function $\tau(p)$, and according to III we can make $p$ travel over the entire infinite interval $p \geq p_0$. While this happens, the directional coefficient

$$s = \frac{p - p_0}{\tau_0 - \tau} \tag{7}$$

of the straight line joining $(p_0, \tau_0)$ with the point $(p, \tau)$ on the adiabatic $S = S_0$ increases monotonely from a certain positive value $m_0$ to $+\infty$. $m$ denotes the adiabatic derivative $-dp/d\tau$ and $m_0$ its value at the point $Z_0$. Consequently, there is exactly one point on the upper branch $p > p_0$ of the adiabatic that lies on the ray from $(p_0, \tau_0)$ in a given direction $s$, provided $s$ lies between $m_0$ and $\infty$.

The analytic proof for these intuitively evident statements runs as follows. Form the (non-total) differential

$$dr = (\tau_0 - \tau)\, dp + (p - p_0)\, d\tau,$$

which vanishes along any ray through $Z_0$. We want to show that along the adiabatic $dr$ is positive for a positive increment $dp$:

$$\frac{dr}{dp} = R = (\tau_0 - \tau) + (p - p_0)\frac{d\tau}{dp} > 0 \qquad \text{for} \qquad p > p_0 . \tag{8}$$

By hypothesis II

$$\frac{dR}{dp} = (p - p_0)\frac{d^2\tau}{dp^2} > 0 \qquad \text{for} \qquad p > p_0 , \tag{9}$$

and since $R$ vanishes for $p = p_0$ it must indeed be positive for $p > p_0$:

$$R = \int_{p_0}^{p} (p - p_0)\frac{d^2\tau}{dp^2}\, dp > 0. \tag{10}$$

Using (7) we may write

$$dr = (\tau_0 - \tau)^2\, ds \tag{11}$$

and thus the monotone behavior of $s$ is exhibited by the equation

$$\frac{ds}{dp} = R/(\tau_0 - \tau)^2.$$

It is clear that $s$ tends to $m_0$ for $p \to p_0$, and the combination of hypothesis III with the trivial inequality $s > (p - p_0)/\tau_0$ leads to $s \to \infty$ for $p \to \infty$.

### 4. *The Fundamental Inequality for the Direction* $\overrightarrow{Z_0 Z_1}$

For the differential of

$$H(Z, Z_0) = (e - e_0) - \tfrac{1}{2}(p + p_0)(\tau_0 - \tau) \tag{12}$$

we find by (1)

$$dH = de - \tfrac{1}{2}(\tau_0 - \tau)\, dp + \tfrac{1}{2}(p_0 + p)\, d\tau$$
$$= T\, dS - \tfrac{1}{2}\{(\tau_0 - \tau)\, dp + (p - p_0)\, d\tau\},$$

or

$$dH = T\, dS - \tfrac{1}{2}\, dr, \tag{13}$$

and hence *along any ray*

$$dH = T\, dS. \tag{14}$$

Let $Z_0$, $Z_1$ be two distinct points satisfying the Hugoniot equation $H(Z_1, Z_0) = 0$. Because of the convexity of the region $Z$ (hypothesis IV) we may join these two points by a straight segment lying in $Z$. Integrating (14) along it we obtain

$$H(Z_1, Z_0) = \int_{Z_0}^{Z_1} T\, dS = 0. \tag{15}$$

From this simple relation we are going to deduce

*Theorem* 1. *For any two distinct states $Z_0$, $Z_1$ linked by Hugoniot's equation $H = 0$ the following inequalities hold*

$$W = (p_1 - p_0)(\tau_0 - \tau_1) > 0, \tag{16}$$

$$(p_1 - p_0) + m_0(\tau_1 - \tau_0) > 0, \qquad (p_1 - p_0) + m_1(\tau_1 - \tau_0) < 0. \tag{17}$$

Before proving the theorem let us discuss its physical significance. We have seen that elimination of the velocities from the conditions for a shock lead to the *Hugoniot equation* and the inequality $W \geq 0$, cf. (4). We now realize that we may omit the supplementary relation $W \geq 0$ because it follows, in fact in the sharper form $W > 0$, from the Hugoniot equation.

The adiabatic derivative

$$\frac{dp}{d\rho} = -\tau^2 \frac{dp}{d\tau} = \tau^2 m$$

is the square of the "acoustic velocity" $c$. Assume $\tau_0 > \tau_1$. Then the two relations (17) and relation (4) give

$$m_0 < M^2 = \frac{p_1 - p_0}{\tau_0 - \tau_1} < m_1 \tag{18}$$

or $\rho_0 c_0 < |M| < \rho_1 c_1$, thus confirming a statement made in Section 2 on the magnitude of $M$. In terms of the velocities $u_0$, $u_1$ given by (2.1) or

$$u_0^2 = \frac{p_1 - p_0}{\tau_0 - \tau_1}\tau_0^2, \qquad u_1^2 = \frac{p_1 - p_0}{\tau_0 - \tau_1}\tau_1^2,$$

(18) may be written as

(19) $$|u_0| > c_0, \qquad |u_1| < c_1:$$

*Relative to the shock front the flow in* [0] *is supersonic, in* [1] *subsonic.*

Proceeding to the proof of our theorem, let us travel from $Z_0$ along that ray which passes through $Z_1$ and hence set in Lemma 1,

$$a = p_1 - p_0, \qquad b = \tau_1 - \tau_0.$$

Were $a \geq 0$, $b \geq 0$, then $S'$ would be positive and hence $S$ monotone increasing along the segment $Z_0Z_1$, which contradicts the relation (15). Consequently this combination (and in the same way the other, $a \leq 0$, $b \leq 0$) is ruled out, and therefore $ab < 0$, or (16) must hold.

Next consider

$$S' = a\frac{\partial S}{\partial p} + b\frac{\partial S}{\partial \tau}.$$

Were $S' \leq 0$ for $t = 0$, then $S'$ would always be negative by Lemma 3 and hence $S$ would monotonely decrease while $Z$ travels along the straight segment from $Z_0$ to $Z_1$, in contradiction to equation (15). Therefore

$$a\left(\frac{\partial S}{\partial p}\right)_0 + b\left(\frac{\partial S}{\partial \tau}\right)_0 > 0,$$

and this is equivalent to the first of the inequalities (17). The second follows from the first by interchanging $Z_0$ and $Z_1$.

But our argument shows much more. While $Z$ travels along the ray from $Z_0$ passing through $Z_1$, $S'$ starts with a positive value; because of (15) it must change sign before $Z$ reaches $Z_1$. But $S'$ remains negative from the moment it vanishes for the first time. Any rise and fall in $H = H(Z, Z_0)$ along the ray is coupled with the same in $S$ by the relation $dH = T\,dS$. Hence $H$ first rises monotonely to a positive maximum and then decreases, passing on the descent through 0 for $Z = Z_1$. This description shows that $H$ is positive from $Z_0$ to $Z_1$ and negative thereafter. In particular:

On the ray from $Z_0$ through $Z_1$ the point $Z = Z_1$ is the *only one* aside from $Z_0$ itself which satisfies the Hugoniot equation $H(Z, Z_0) = 0$.

According to (16) either the inequalities $p_1 > p_0$, $\tau_0 > \tau_1$ or $p_0 > p_1$, $\tau_1 > \tau_0$ hold for two distinct states $Z_0$, $Z_1$, satisfying the Hugoniot equation $H(Z_1, Z_0) = 0$. We shall indicate these alternatives by $Z_1 > Z_0$, $Z_1 < Z_0$ respectively. By the *Hugoniot contour* $\mathcal{H}$ (for a given $Z_0$) we understand the locus of all points $Z_1 = (p_1, \tau_1)$ for which

$$H(Z_1, Z_0) = 0 \quad \text{and} \quad Z_1 > Z_0.$$

It is quite essential for our argument to pick this *upper* branch $Z_1 > Z_0$. Our above result may then be stated thus: On the Hugoniot contour $s = (p_1 - p_0)/$

$(\tau_0 - \tau_1)$ lies between $m_0$ and $\infty$; $s$ is a uniformizing parameter inasmuch as the value of $s$ specifies uniquely the point $Z_1$. But it must be borne in mind that so far we have not yet proved that to every preassigned value of $s > m_0$ there actually corresponds a point on the Hugoniot contour; we only know that there cannot be more than one.

## 5. *Entropy and Parametrization of the Hugoniot Line*

For a moment let us return to the upper branch $\mathfrak{A}$ of the adiabatic through $Z_0$, on which $p$ increases monotonely from $p_0$ to $+\infty$ and $s = (p - p_0)/(\tau_0 - \tau)$ from $m_0$ to $\infty$. Moving along $\mathfrak{A}$ we have by (13)

$$dH = -\tfrac{1}{2}\, dr = -\tfrac{1}{2}R\, dp \tag{20}$$

and hence (10) implies

*Lemma* 4. *$H = H(Z, Z_0)$ is negative along $\mathfrak{A}$.*

As a matter of fact, we even know that $H$ is falling, $R > 0$, and by condition (9) falling with increasing rapidity, $dR/dp > 0$. The explicit formula giving $H(Z_A, Z_0)$ for any point $Z_A$ on the adiabatic $\mathfrak{A}$ is by (20), (8) and (9),

$$H(Z_A, Z_0) = -\frac{1}{2}\int_{\mathfrak{A}\,p_0}^{p_A} R\, dp = -\frac{1}{2}R(p - p_A)\Big|_{p_0}^{p_A} - \frac{1}{2}\int_{\mathfrak{A}\,p_0}^{p_A} (p_A - p)\frac{dR}{dp}\, dp$$

$$= -\frac{1}{2}\int_{\mathfrak{A}\,p_0}^{p_A} (p - p_0)(p_A - p)\frac{d^2\tau}{dp^2}\, dp < 0.$$

This lemma is instrumental in establishing the following propositions:

*Theorem* 2. *For any two distinct states $Z_0$, $Z_1$ satisfying the Hugoniot equation one has $S_1 > S_0$ or $S_1 < S_0$ according to whether $Z_1 > Z_0$ or $Z_1 < Z_0$.*

*Theorem* 3. *The Hugoniot contour is a simple line starting from $Z_0$ on which $s$ and $S$ are monotone increasing* ($s$ traveling from $m_0$ to $+\infty$, $S$ from $S_0$ to an unknown destination).

Proof. Assume the arrangement $Z_1 > Z_0$ for the two points $Z_0$, $Z_1$ linked by the Hugoniot equation. Draw the ray $\mathfrak{R}$ from $Z_0$ passing through $Z_1$; its directional coefficient $s = (p_1 - p_0)/(\tau_0 - \tau_1)$ lies between $m_0$ and $+\infty$, and hence $\mathfrak{R}$ meets the upper branch of the adiabatic line $\mathfrak{A}$ in a point $Z_A = (p_A, \tau_A)$ where, according to Lemma 4,

$$H = \int_{\mathfrak{R}\,Z_0}^{Z_A} T\, dS = -\frac{1}{2}\int_{\mathfrak{A}\,p_0}^{p_A} R\, dp < 0.$$

Hence $Z$ on its road along the ray must have passed the point $Z'$ where it reaches its maximum, and running downhill have passed a point $Z_1$ for which $H =$

$\int_{Z_0}^{Z_1} T\,dS = 0$ before coming to $Z_A$. Its value $S_1$ at the point $Z_1$ of the Hugoniot contour is therefore higher than its value $S_0$ at $Z_A$ (Theorem 2).

Again, let $s$ be any value between $m_0$ and $\infty$. The ray going out from $Z_0$ in the direction $s$ meets $\mathcal{A}$ in a point $Z_A$ where $H$ is negative. Hence $H$ must vanish at a certain point $Z_1$ between $Z'$ and $Z_A$ (where $S$ is already on the down grade), and thus a point on the Hugoniot contour is obtained corresponding to this preassigned value of $s = (p_1 - p_0)/(\tau_0 - \tau_1)$. We know it is the only one. The question raised at the end of Section 4 is answered in the affirmative, and we may now speak of the Hugoniot contour as a simple line $Z_1 = \Phi(s)$ with the uniformizing parameter $s$.

Because the point $Z_1 = \Phi(s)$ is thought of as varying along the Hugoniot line, let us drop the index 1. Along $\mathcal{H}$ we may use the differential relation $dH = 0$ or, according to (13) and (11),

$$T\,dS - \tfrac{1}{2}\,dr = T\,dS - \tfrac{1}{2}\,(\tau_0 - \tau)^2\,ds = 0.$$

Because

$$\frac{dS}{ds} = \frac{(\tau_0 - \tau)^2}{2T}$$

thus turns out to be positive, $S$ increases all the way along the Hugoniot line with $s$. This gives a new proof for the inequality $S_1 > S_0$ by integration along that line,

$$S_1 - S_0 = \int_{\mathcal{H}^{m_0}}^{s_1} \frac{(\tau_0 - \tau)^2}{2T}\,ds \qquad (s_1 > m_0)$$

Thus Theorem 3 is proved, and Theorem 2 by even two different methods.

## Part II. The Problem of the Shock Layer

### 6. *Formulation of the Problem*

For the sake of brevity we now denote partial derivatives with respect to $(p, \tau)$ by subscripts as in

$$\frac{\partial S}{\partial p} = S_p\,, \qquad \frac{\partial S}{\partial \tau} = S_\tau\,.$$

The rest of our argument will be based on the thermodynamical assumptions I-IV, Section 3, but in its course it will become clear that one more condition of highly plausible nature is to be added to our list:

Ia. *Heating a quantity of fluid at constant volume raises its pressure and temperature.*

It must be remembered that the analytic formulation (I′), (II′) of the hypotheses I and II depended on the fixation of a sign. For the previous investigation this did not matter greatly, but it is essential for the behavior of the shock layer. We therefore require explicitly by the statement concerning pressure in Ia, that $(dp/TdS)_{T=\text{const.}} > 0$ or $S_p > 0$. The other part of Ia then adds the inequality $T_p/S_p > 0$ or $T_p > 0$.

In a stationary field the state $(Z, \vec{u})$ of a fluid of given thermodynamical nature depends on the spatial coordinates $x, y, z$ only, and not on time. The five equations which, in the absence of an external force, express conservation of mass, momentum and energy, take heat conductivity and viscosity into account by the flow of heat $\vec{j} = -\lambda \operatorname{grad} T$ and a stress tensor $S$ of the form

$$S_{ik} = p\delta_{ik} - S^*_{ik}\,, \qquad S^*_{ik} = \mu' \operatorname{div} \vec{u}\cdot\delta_{ik} + \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

(in the writing down of which we have for the moment adopted the subscript notation for the coordinates $x_i$ and velocity components $u_i$). $\lambda = \lambda[Z]$, $\mu$, $\mu'$ are given functions of the thermodynamical state $Z$; $\lambda > 0$, $\mu > 0$, $\mu' + \frac{2}{3}\mu > 0$. As they are small we intend to let them pass to zero. In two ways such a passage to the limit $\lambda, \mu, \mu' \to 0$ can give rise to a discontinuity in the plane $x = 0$:

(A) Write $\epsilon x$ for $x$ and $\epsilon\lambda$, $\epsilon\mu$, $\epsilon\mu'$ for $\lambda, \mu, \mu'$, leaving all other quantities untouched, and then let the positive constant factor $\epsilon$ tend to zero (*shock layer*).

(B) Let $\epsilon$ tend to zero after writing $\epsilon x$, $\epsilon u$ for $x$, $u$ and $\epsilon^2\lambda$, $\epsilon^2\mu$, $\epsilon^2\mu'$ for $\lambda, \mu, \mu'$ (*slip layer*).

The different effect is clearly exemplified by the continuity equation

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \tag{21}$$

Substitution (A) changes it into

$$\frac{\partial}{\partial x}(\rho u) + \epsilon\frac{\partial}{\partial y}(\rho v) + \epsilon\frac{\partial}{\partial z}(\rho w) = 0$$

which for $\epsilon \to 0$ reduces to

$$\frac{\partial}{\partial x}(\rho u) = 0,$$

whereas process (B) leaves the three-term equation (21) unchanged. This is typical. None but the derivatives $\partial/\partial x$ survive the limiting process (A), the others $\partial/\partial y$, $\partial/\partial z$ disappear from the equations. The shock layer is thus seen to be a one-dimensional problem, no interaction taking place between the one-dimensional fibers of space $y$ = constant, $z$ = constant. The slip layer on the other hand constitutes a far more complicated three-dimensional problem, identical with that of Prandtl's boundary layer. We are concerned here only with the shock layer.

With the abbreviation $\mu^* = \mu' + 2\mu$, $\mu^* > 0$, the limiting process (A) leads to the following equations:

$$\rho u = \text{constant} = M, \tag{22.1}$$

$$Mu + p - \mu^* \frac{du}{dx} = \text{constant}, \tag{22.2}$$

$$Mv - \mu \frac{dv}{dx} = \text{constant}, \qquad Mw - \mu \frac{dw}{dx} = \text{constant}, \tag{22.3}$$

$$\left(p - \mu^* \frac{du}{dx} + \rho e^*\right)u - \mu\left(v \frac{dv}{dx} + w \frac{dw}{dx}\right) - \lambda \frac{dT}{dx} = \text{constant}, \tag{22.4}$$

where $e^*$ denotes the total energy $e + \frac{1}{2}(u^2 + v^2 + w^2)$. Here the quantities $\lambda$, $\mu$, $\mu'$ are no longer small. We assume that $(Z, \vec{u})$ approaches definite constant values $(Z_0, \vec{u}_0)$, $(Z_1, \vec{u}_1)$ for $x \to -\infty$ and $x \to +\infty$ respectively, whereas the derivatives $d/dx$ tend to zero.

The energy equation is equivalent to the entropy equation

$$T \frac{d}{dx}\left(\rho u S - \frac{\lambda}{T}\frac{dT}{dx}\right) = \mu^*\left(\frac{du}{dx}\right)^2 + \mu\left\{\left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right\} + \frac{\lambda}{T}\left(\frac{dT}{dx}\right)^2, \tag{23}$$

from which one learns that the flow of entropy

$$\rho u S - \frac{\lambda}{T}\frac{dT}{dx} = MS - \frac{\lambda}{T}\frac{dT}{dx}$$

is a monotone increasing function of $x$ and therefore $MS_1 \geq MS_0$. The equality sign could prevail only if $u$, $v$, $w$, $T$ and hence by (22.2) also $p$ were constant. Excluding this trivial case of a constant state[3] $(Z, \vec{u})$ we thus obtain the *strict inequality*

$$MS_1 > MS_0 \tag{24}$$

implying $M \neq 0$. The latter remark shows how misleading it is to link the alternative $M \neq 0$, $M = 0$ to the discrimination of shock and slip layer; rather do these two possibilities correspond to the two ways $(A)$ and $(B)$ of passing to the limit of vanishing viscosity and heat conductivity. Being sure now that $M$ does not vanish, we fall back upon our old relations connecting $(Z_0, \vec{u}_0)$ and $(Z_1, \vec{u}_1)$, including the Hugoniot relation. Let us assign the labels 0, 1

[3] Indeed, in general, $p$ = constant, $T$ = constant, imply $\tau$ = constant. There is, however, one extremely exceptional case in which the conclusion does not stand, namely if $M = 0$ and if for some constant value of $p$ the temperature $T$ as function of $\tau$ stays constant in an entire $\tau$-interval. Maybe this is a warning against the danger of a singularity in the shock layer problem which one steers clear of with certainty only by assuming $T_\tau \neq 0$ or, what is the same, if the hypotheses I–IV, Ia are preceded by another to the effect that $p$, $T$ may serve as local parameters on $\mathcal{Z}$ everywhere.

to the two states $Z_0$, $Z_1$, and accordingly orient the $x$-axis, so that $Z_0 < Z_1$. Then Theorem 2 combined with the law of increasing entropy (24) gives $M > 0$. The meaning of this inequality is that the shock front moves in the direction $[1] \rightarrow [0]$, or that $Z_0$ and not $Z_1$ is the initial state of the fluid not yet reached by the propagating shock.

## 7. *Character of the Two Singular Points $Z_0$, $Z_1$: Node and Saddle*[4]

It is easy to dispose of the equations (22.3) for $v$ and $w$. Single out the one for $v$,

$$\mu \frac{dv}{dx} = M(v - b), \qquad b = \text{constant},$$

and introduce the variable

$$\xi = M \int_0^x \frac{dx}{\mu(x)} \qquad \text{where } \mu(x) = \mu[Z(x)].$$

We find

$$v(x) - b = Ce^{\xi}, \qquad C = \text{constant}.$$

$\mu(x)$ tends to positive values $\mu_0 = \mu[Z_0]$, $\mu_1 = \mu[Z_1]$ for $x \rightarrow -\infty$, $x \rightarrow +\infty$ respectively; therefore

$$\xi = \frac{M}{\mu_0} x + o(x) \qquad \text{for } x \rightarrow -\infty,$$

$$\xi = \frac{M}{\mu_1} x + o(x) \qquad \text{for } x \rightarrow +\infty.$$

Considering that $M > 0$ one realizes that $v$ cannot tend to a finite limit $v_1$ for $x \rightarrow +\infty$ unless $C = 0$. Thus $v$ and $w$ must be constant throughout the shock layer.

(22.1) is satisfied by $u = M\tau$. Denoting by $M^2a$ the constant at the right of (22.2) and substituting from (22.2) the value of $p - \mu^* du/dx$ into (22.4) we are left, as is readily seen, with the following two ordinary differential equations of first order for the state $Z = Z(x)$:

$$\mu^* M \frac{d\tau}{dx} = M^2(\tau - a) + p$$

(25)

$$\frac{\lambda}{M}\frac{dT}{dx} = e - \frac{1}{2}M^2(\tau - a)^2 - c.$$

[4]For ideal gases of constant specific heat with certain special values of the adiabatic exponent $\gamma$, cf. R. Becker, Zeitschrift für Physik 8 (1922), 321–362; in particular, pp. 339–347.

The constants $M^2$, $a$, $c$ are determined by

$$M^2\tau_0 + p_0 = M^2\tau_1 + p_1 = M^2 a,$$

$$e_0 - \tfrac{1}{2}M^2(\tau_0 - a)^2 = e_1 - \tfrac{1}{2}M^2(\tau_1 - a)^2 = c.$$

As the right members of (25) vanish for $Z = Z_0$ and $Z = Z_1$, the two points $Z_0$, $Z_1$ turn out to be *singular points* for the differential system (25). If we care only for the trajectory in the $(p,\tau)$-diagram along which $Z(x)$ moves while $x$ runs from $-\infty$ to $+\infty$ we are faced with one ordinary differential equation of first order in the $(p,\tau)$-plane,

$$\frac{\lambda\, dT}{e - \tfrac{1}{2}M^2(\tau - a)^2 - c} = \frac{\mu^*\, d\tau}{(\tau - a) + (p/M^2)}.$$

But contrary to the customary viewpoint taken in the classical investigations on singular points, which were started by Briot and Bouquet almost one hundred years ago and are reproduced in all treatises on differential equations, our interest is in the *parametrized* trajectory $Z = Z(x)$.

Let us briefly recount the well-known facts about singular points relevant for our purposes.[5] Placing the singular point at the origin 0 we study two simultaneous differential equations for the "vector" $\varphi = (\varphi_1(x), \varphi_2(x))$ in a $\varphi_1$, $\varphi_2$-plane,

$$\frac{d\varphi_1}{dx} = F_1(\varphi_1, \varphi_2), \qquad \frac{d\varphi_2}{dx} = F_2(\varphi_1, \varphi_2)$$

where $F_1$, $F_2$ are given functions of $\varphi_1$, $\varphi_2$ vanishing at the origin. From the outset it is evident that any translation $x \to x + h$ carries a solution $\varphi = (\varphi_1(x), \varphi_2(x))$ into a solution ("equivalent solutions"). Suppose the Taylor expansions of $F_1$, $F_2$ begin with the terms

$$F_1(\varphi_1, \varphi_2) \sim R_{11}\varphi_1 + R_{12}\varphi_2, \qquad F_2(\varphi_1, \varphi_2) \sim R_{21}\varphi_1 + R_{22}\varphi_2.$$

The equations

$$\varphi_1' = R_{11}\varphi_1 + R_{12}\varphi_2, \qquad \varphi_2' = R_{21}\varphi_1 + R_{22}\varphi_2$$

define a linear mapping $\varphi \to \varphi' = R\varphi$. We determine its two eigen-values $k$ and corresponding eigen-vectors $\varphi = \alpha$ (or rather eigen-directions) by $R\alpha = k\alpha$, explicitly

$$(k - R_{11})\alpha_1 - R_{12}\alpha_2 = 0,$$

$$-R_{21}\alpha_1 + (k - R_{22})\alpha_2 = 0,$$

[5]See for direct constructive proofs: H. Weyl, *Concerning a classical problem in the theory of singular points of ordinary differential equations.* Actas de la Academia Nacional de Ciencias de Lima, Volume 7, 1944, pp. 21–60.

and assume that the two roots $k_1$ , $k_2$ of the secular equation

$$\begin{vmatrix} k - R_{11} , & - R_{12} \\ - R_{21} , & k - R_{22} \end{vmatrix} = 0$$

are real, distinct and $\neq 0$. In a suitable affine coordinate system in the $\varphi_1$ , $\varphi_2$-plane our mapping is then described by $\varphi_1' = k_1\varphi_1$ , $\varphi_2' = k_2\varphi_2$ , so that $F_1$ , $F_2$ are of the form

$$F_1(\varphi_1 , \varphi_2) = k_1\varphi_1 + \cdots ,$$

$$F_2(\varphi_1 , \varphi_2) = k_2\varphi_2 + \cdots .$$

We distinguish three cases according to the signs of the eigen-values $k_1$ , $k_2$ . In the case (i) of two negative eigen-values, $k_2 < k_1 < 0$, every solution $\varphi$ which comes sufficiently near to the origin plunges into it with $x$ tending to $+\infty$; its asymptotic behavior is described by

$$\varphi_1(x) \sim a\cdot e^{k_1 x}, \qquad \varphi_2(x) \sim 0\cdot e^{k_1 x} \qquad (x \to \infty)$$

in the sense that

$$e^{-k_1 x}\varphi_1(x) = a + O(e^{-\epsilon x}),$$

$$e^{-k_1 x}\varphi_2(x) = O(e^{-\epsilon x})$$

where $\epsilon$ is a positive constant, the same for all solutions, while the constant $a$ is specific (*node*, see Figure 2). In the case (ii) of one negative, one positive

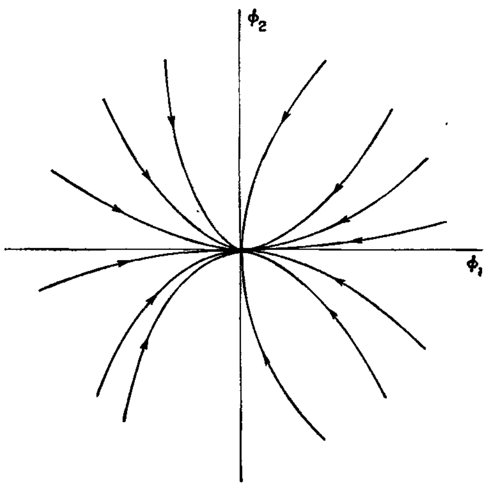


FIG. 2. Node.

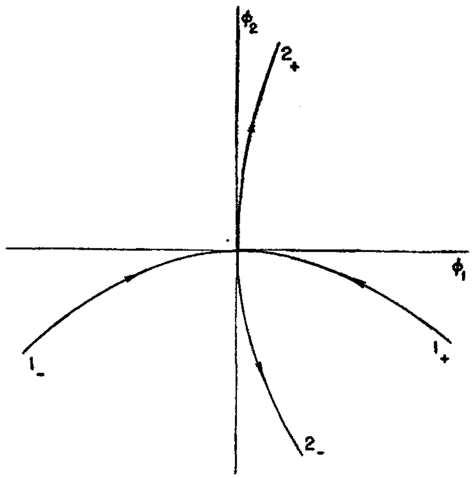


FIG. 3. Saddle.

eigen-value, $k_1 < 0$, $k_2 > 0$, there exist exactly two solutions ($1_+$, $1_-$ in Figure 3) approaching 0 with $x \to +\infty$; their asymptotic behavior is described by

$$\varphi_1(x) \sim \pm e^{k_1 x}, \qquad \varphi_2(x) \sim 0 \cdot e^{k_1 x} \qquad (x \to +\infty)$$

(*saddle*). Of course this statement is to be interpreted so that equivalent solutions are considered as one and the same. The term saddle is better understood by observing that in this case (ii) two other solutions, $2_+$ and $2_-$, plunge into 0 in the direction of the $\varphi_2$-axis, while $x$ goes to $-\infty$. In case (iii) of two positive eigen-values there is no solution whatsoever which approaches the origin as $x$ tends to $+\infty$, except the trivial one $\varphi_1 = \varphi_2 = 0$ (whereas, of course, every solution that comes sufficiently near 0 takes the plunge when $x$ moves toward $-\infty$).

With these facts in mind we now advance our decisive

*Theorem* 6. *For the differential equations of the shock layer determining the transition from a state $Z_0$ to a state $Z_1 > Z_0$, the initial state $Z_0$ is a node, the end state $Z_1$ a saddle.*

Clearly this behavior in the neighborhood of the two singular points $Z_0$, $Z_1$ is such as to favor the existence of a unique solution of the shock layer problem. All other combinations would make one expect either no solution or an infinity of solutions

Again, the proof depends, above all, on the fundamental inequalities (18) for $M^2 = (p_1 - p_0)/(\tau_0 - \tau_1)$. Since $S_\tau/S_p$ is the adiabatic derivative $m = -dp/d\tau$ we may write them as

$$M^2 S_p^{(0)} - S_\tau^{(0)} > 0, \qquad M^2 S_p^{(1)} - S_\tau^{(1)} < 0. \tag{26}$$

They must be combined with

(27) $$M > 0; \qquad S_p > 0, \qquad T_p > 0.$$

First investigate the neighborhood of $Z_0$ and therefore write

$$M^2(\tau - a) + p = M^2(\tau - \tau_0) + (p - p_0).$$

For infinitesimal $\delta p = p - p_0$, $\delta\tau = \tau - \tau_0$

$$\delta e = T_0\,\delta S - p_0\,\delta\tau, \qquad \delta\tfrac{1}{2}(\tau - a)^2 = (\tau_0 - a)\,\delta\tau, \qquad \text{hence}$$

$$\delta\{e - \tfrac{1}{2}M^2(\tau - a)^2\} = T_0\,\delta S - \{p_0 + M^2(\tau_0 - a)\}\,\delta\tau = T_0\,\delta S,$$

therefore in neglecting terms of higher order

$$e - \tfrac{1}{2}M^2(\tau - a)^2 - c \sim T_0(S - S_0).$$

The following approximate linear system for $\delta\tau$, $\delta p$ results:

$$\frac{d}{dx}(\delta\tau) = g(M^2\,\delta\tau + \delta p),$$

$$\frac{d}{dx}(\delta T) = g_0'\,\delta S$$

where

$$g = 1/M\mu^* > 0, \qquad g' = TM/\lambda > 0$$

and $\delta S$, $\delta T$ stand for

$$\delta S = S_p^{(0)}\,\delta p + S_\tau^{(0)}\,\delta\tau, \qquad \delta T = T_p^{(0)}\,\delta p + T_\tau^{(0)}\,\delta\tau.$$

In order to avoid unnecessary encumbrances, we drop the index 0. Then the equations determining the eigen-values $k$ and corresponding eigen-directions $(\delta\tau, \delta p)$ at $Z_0$ become

$$k\cdot\delta\tau = g(M^2\delta\tau + \delta p), \qquad k\cdot\delta T = g'\delta S$$

giving rise to the following secular equation for $k$,

$$\begin{vmatrix} k - gM^2, & -g \\ kT_\tau - g'S_\tau, & kT_p - g'S_p \end{vmatrix} = 0 \qquad \text{or}$$

(28) $$T_p\cdot k^2 - \{g(M^2T_p - T_\tau) + g'S_p\}\cdot k + gg'(M^2S_p - S_\tau) = 0.$$

Compute the discriminant

$$\Delta = \{g(M^2T_p - T_\tau) + g'S_p\}^2 - 4gg'T_p(M^2S_p - S_\tau)$$

$$= \{g(M^2T_p - T_\tau) - g'S_p\}^2 + 4gg'(T_pS_\tau - T_\tau S_p)$$

and thus verify by (6) that $\Delta$ is positive,

$$\Delta = \{g(M^2T_p - T_\tau) - g'S_p\}^2 + 4gg' \geq 4gg' = \frac{4T}{\lambda\mu^2} > 0.$$

Consequently the eigen-values are real and distinct, at the singular point $Z_0$ as well as at $Z_1$.

Their product equals

$$gg'\cdot(M^2S_p - S_\tau)/T_p,$$

and because of (27) and the fundamental inequalities (26) that product is positive at $Z_0$, negative at $Z_1$. Hence the two eigen values are of opposite sign at $Z_1$, of equal sign at $Z_0$. Whether the latter sign is positive or negative can be decided by the sum of the two $k$'s for which (28) gives the value

$$\{g(M^2T_p - T_\tau) + g_0'S_p\}/T_p.$$

All constituents of this expression are positive, including

$$M^2T_p^{(0)} - T_\tau^{(0)} > (S_\tau^{(0)}/S_p^{(0)})T_p^{(0)} - T_\tau^{(0)} = 1/S_p^{(0)} > 0.$$

Thus we arrive at the result that the two roots $k$ at $Z_0$ are *positive*. Our theorem has now been proved: the fundamental inequalities (17) not only show that by a shock the flow changes from supersonic to subsonic but also that one of the two points $Z_0$, $Z_1$ is a node, the other a saddle.

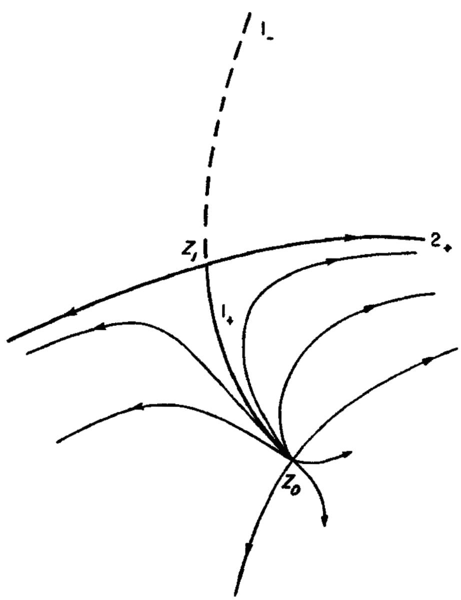


FIG. 4. The topological picture of the trajectories in the $(p, \tau)$-plane.

In order actually to prove the existence of a unique solution of the shock layer problem, one has to bridge the gap between the neighborhoods of the two singular points $Z_0$, $Z_1$ by some sort of topological argument. In that respect one fact is of decisive importance:

*Theorem* 7. *The differential equations of the shock layer have no singular point besides* $Z_0$, $Z_1$.

Indeed if there existed three distinct singular points $Z_0$, $Z_1$, $Z_2$ they would satisfy the equations

$$M^2\tau_0 + p_0 = M^2\tau_1 + p_1 = M^2\tau_2 + p_2 = M^2a,$$

$$e_0 - \tfrac{1}{2}M^2(\tau_0 - a)^2 = e_1 - \tfrac{1}{2}M^2(\tau_1 - a)^2 = e_2 - \tfrac{1}{2}M^2(\tau_2 - a)^2.$$

The first set states that $Z_0$, $Z_1$, $Z_2$ lie on a straight line of negative inclination $-M^2$, and we may assume the arrangement $Z_0 < Z_1 < Z_2$. Then the second set requires that both $Z = Z_1$ and $Z = Z_2$ satisfy the Hugoniot equation $H(Z, Z_0) = 0$, in contradiction to a statement previously proved by which there cannot be more than one point $Z$ on any ray from $Z_0$ satisfying that equation. The observation is urged upon us that whereas the Hugoniot relation is not transitive, the relation between $Z_0$, $Z_1$ defined by the *simultaneous* equations

$$M^2\tau_0 + p_0 = M^2\tau_1 + p_1, \qquad H(Z_1, Z_0) = 0$$

is ($M^2$ being a given constant).

The topological picture of the trajectories which one expects is indicated by Figure 4. If this is correct, the region covered by all trajectories running into $Z_0$ for $x \to -\infty$ would be bounded by the saddle line through $Z_1$ consisting of the two branches $2_+$ and $2_-$ (Figure 3). The outlook is encouraging.