The original total variation inpainting can be simply stated by

$$\min_{f \in F(\Omega)} \int_{\Omega \setminus \Omega_0} |f-g|^2 \ dx\ dy + \lambda \int_{\Omega} |\nabla f| \ dx\ dy \tag{1}$$

Using the indicator function δ, we can rewrite the problem as

$$\min_{f \in F(\Omega)} \int_{\Omega} |f-g|^2 \delta + \lambda |\nabla f| \ dx\ dy \tag{2}$$

Where

$$\delta(x,y) = \begin{cases} 0 & \text{if } (x,y) \in \Omega_0 \\ 1 & \text{otherwise} \end{cases} \tag{3}$$

Since $|\cdot|$ is not differentiable at the origin, we will use the relaxation

$$\min_{f \in F(\Omega)} \int_{\Omega} |f-g|^2 \delta + \lambda \sqrt{\epsilon^2 + f_x^2 + f_y^2}\ dx\ dy \tag{4}$$

Using the Lagrange-Euler equation to solve the functional $\int_\Omega \mathcal{L}\ dx\ dy = \int_\Omega |f-g|^2 \delta + \lambda \sqrt{\epsilon^2 + f_x^2 + f_y^2}\ dx\ dy$, we can derivate the first optimality condition

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta f} &= \frac{\partial \mathcal{L}}{\partial f} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial f_x} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial f_y} \\ &= 2(f-g)\delta - \frac{\partial}{\partial x}\lambda \frac{f_x}{\sqrt{\epsilon^2 + f_x^2 + f_y^2}} - \frac{\partial}{\partial y}\lambda \frac{f_y}{\sqrt{\epsilon^2 + f_x^2 + f_y^2}} \\ &= 2(f-g)\delta - \lambda \mathrm{div}(\frac{\nabla f}{\sqrt{\epsilon^2 + f_x^2 + f_y^2}}) \end{aligned} \tag{5}$$

From which the curve evolution equation can be stated

$$f_t = -\frac{\delta \mathcal{L}}{\delta f} = -2(f-g)\delta + \lambda\ \mathrm{div}\left(\frac{\nabla f}{\sqrt{\epsilon^2 + f_x^2 + f_y^2}}\right) \tag{6}$$

If we focus on the evolution of the equation in Ω_0, δ vanishes. Thus,

$$f_{t \restriction \Omega_0} = \lambda\ \mathrm{div}\left(\frac{\nabla f}{\sqrt{\epsilon^2 + f_x^2 + f_y^2}}\right) \tag{7}$$

Using equations 8, 9 (where $R := \epsilon^2 + f_x^2 + f_y^2$)

$$\frac{\partial}{\partial x}\frac{f_x}{\sqrt{\epsilon^2 + f_x^2 + f_y^2}} = R^{-3/2}(f_{xx}\epsilon^2 + f_{xx}f_y^2 - f_x f_y f_{yx}) \tag{8}$$

$$\frac{\partial}{\partial x}\frac{f_y}{\sqrt{\epsilon^2 + f_x^2 + f_y^2}} = R^{-3/2}(f_{yy}\epsilon^2 + f_{yy}f_x^2 - f_y f_x f_{yx}) \tag{9}$$

and the fact that $f_{yx} = f_{xy}$, we can rewrite equation 7 as

$$\begin{aligned} f_{t\restriction\Omega_0} &= \lambda \operatorname{div}\left(\frac{\nabla f}{\sqrt{\epsilon^2 + f_x^2 + f_y^2}}\right) \\ &= \lambda\, R^{-3/2}(\epsilon^2 f_{xx} + \epsilon^2 f_{yy} + f_{yy}f_x^2 + f_{xx}f_y^2 - 2f_x f_y f_{xy}) \end{aligned} \tag{10}$$

To get a explicit scheme, we use central differences for the right hand side (RHS) and forward differences for the left hand side (LHS).

For the LHS, the discretization is simply

$$f_t \approx \frac{f_{t+1} - f_t}{\Delta t} \tag{11}$$

For the RHS,

$$\begin{aligned} f_{xx}(\epsilon^2 + f_y^2) + f_{yy}(\epsilon^2 + f_x^2) - 2f_x f_y f_{xy} \approx \\ (f_{i+1,j} + f_{i-1,j} - 2f_{i,j})(\epsilon^2 + \frac{1}{4}(f_{i,j+1} - f_{i,j-1})) \\ + (f_{i,j+1} + f_{i,j-1} - 2f_{i,j})(\epsilon^2 + \frac{1}{4}(f_{i+1,j} - f_{i-1,j})) \\ - 2(f_{i,j+1} - f_{i,j-1})(f_{i+1,j} - f_{i=1,j})(f_{i+1,j+1} + f_{i-1,j-1} - f_{i-1,j+1} - f_{i+1,j-1}) \end{aligned} \tag{12}$$

To simplify the following steps, we use the following naming scheme

$$\begin{bmatrix} f_{i-1,j+1} & f_{i,j+1} & f_{i+1,j+1} \\ f_{i-1,j} & f_{i,j} & f_{i+1,j} \\ f_{i-1,j-1} & f_{i,j-1} & f_{i+1,j-1} \end{bmatrix} = \begin{bmatrix} a & b & q \\ c & d & e \\ h & g & p \end{bmatrix}$$

and continue the derivation of equation 12

$$
\begin{aligned}
&= (e + c - 2d)(\epsilon^2 + \frac{1}{4}(b - g)^2) \\
&\qquad\qquad + (b + f - 2d)(\epsilon^2 + \frac{1}{4}(e - c)^2) \\
&\qquad\qquad\qquad\qquad - \frac{2}{16}(b - g)(e - c)(q + h - a - p)
\end{aligned} \tag{13}
$$

Defining the following coefficients

$$c_1 = (\epsilon^2 + \frac{1}{4}(b - g)^2) \tag{14}$$

$$c_2 = (\epsilon^2 + \frac{1}{4}(e - c)^2) \tag{15}$$

$$c_3 = -\frac{2}{16}(b - g)(e - c) \tag{16}$$

We arrive to the following approximation of the RHS

$$
\begin{aligned}
&\lambda\ R^{-3/2}(\epsilon^2 f_{xx} + \epsilon^2 f_{yy} + f_{yy} f_x^2 + f_{xx} f_y^2 - 2 f_x f_y f_{xy}) \approx \\
&\qquad \lambda \hat{R}^{-3/2}(ec_1 + cc_1 - 2d(c_1 + c_2) + bc_2 + gc_2 + qc_3 + hc_3 - ac_3 - pc_3)
\end{aligned} \tag{17}
$$

where

$$
\begin{aligned}
R &= \epsilon^2 + f_x^2 + f_y^2 \\
&\approx \epsilon^2 + \frac{(f_{i+1,j} - f_{i-1,j})^2 + (f_{i,j+1} - f_{i,j-1})^2}{4} \\
&= \epsilon^2 + \frac{(e - c)^2 + (b - g)^2}{4} := \hat{R}
\end{aligned} \tag{18}
$$

Joining the finite elements approximation for the LHS and RHS, we get the following explicit scheme

$$(f_{t+1})_{i,j} = (f_{t+1})_{i,j} + \Delta t \lambda \hat{R}^{-3/2}(ec_1 + cc_1 - 2d(c_1 + c_2) + bc_2 + gc_2 + qc_3 + hc_3 - ac_3 - pc_3) \tag{19}$$

Furthermore, since $(f_{t+1})_{i,j} = f_{i,j} = d$,

$$(f_{t+1})_{i,j} = d(1-\Delta t \lambda \hat{R}^{-3/2} 2(c_1+c_2)) + \Delta t \lambda \hat{R}^{-3/2}(ec_1+cc_1+bc_2+gc_2+qc_3+hc_3-ac_3-pc_3) \tag{20}$$

Equation 20 can be rewritten in matrix form as

$$f^{t+1} = Qf^t \tag{21}$$

Here f^t is the row-wise image vector at time t. For instance,

$$\text{Image} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \implies f^t = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \tag{22}$$

and Q is a sparse matrix with 9 diagonals[1]

$$Q = \begin{bmatrix} \ddots & & & & & & & & & & & & \\ \cdots 0 \cdots & c_{a_i} & c_{b_i} & c_{q_i} & \cdots 0 \cdots & c_{c_i} & c_{d_i} & c_{e_i} & \cdots 0 \cdots & c_{h_i} & c_{g_i} & c_{p_i} & \cdots 0 \cdots \\ & & & & & & & & & & & & \ddots \end{bmatrix} \tag{23}$$

where

[1]If the dimensions of the image are NxM, then the dimensions of Q are (N*M)x(N*M)

$$c_{a_i} = \begin{cases} -\lambda \Delta t \hat{R}^{-3/2} c_3 & \text{if } i \text{ has a north west neighbor} \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

$$c_{b_i} = \begin{cases} \lambda \Delta t \hat{R}^{-3/2} c_2 & \text{if } i \text{ has a north neighbor} \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

$$c_{q_i} = \begin{cases} \lambda \Delta t \hat{R}^{-3/2} c_3 & \text{if } i \text{ has a norht east neighbor} \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

$$c_{c_i} = \begin{cases} \lambda \Delta t \hat{R}^{-3/2} c_1 & \text{if } i \text{ has a west neighbor} \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

$$c_{d_i} = 1 - 2\lambda \Delta t \hat{R}^{-3/2}(c_1 + c_2) \tag{28}$$

$$c_{e_i} = \begin{cases} \lambda \Delta t \hat{R}^{-3/2} c_1 & \text{if } i \text{ has a east neighbor} \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

$$c_{h_i} = \begin{cases} \lambda \Delta t \hat{R}^{-3/2} c_3 & \text{if } i \text{ has a south west neighbor} \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

$$c_{g_i} = \begin{cases} \lambda \Delta t \hat{R}^{-3/2} c_2 & \text{if } i \text{ has a south neighbor} \\ 0 & \text{otherwise} \end{cases} \tag{31}$$

$$c_{p_i} = \begin{cases} -\lambda \Delta t \hat{R}^{-3/2} c_3 & \text{if } i \text{ has a south east neighbor} \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

In order to guarantee convergence, we need to assure the stability of the explicit scheme 21. For this, the requirement is that the diagonal of the Q matrix must be positive, that is, $\forall_i c_{d_i} > 0$. To guarantee this, we ask

$$\begin{aligned} 1 - 2\lambda \Delta t \hat{R}^{-3/2}(c_1 + c_2) &> 0 \\ 2\lambda \Delta t \hat{R}^{-3/2}(c_1 + c_2) &< 1 \\ \lambda \Delta t &< \frac{\hat{R}^{3/2}}{2(c_1 + c_2)} \\ \lambda \Delta t &< \frac{\min_i(\hat{R}^{3/2})}{\max_i(2(c_1 + c_2))} \\ \lambda \Delta t &< \frac{\epsilon^3}{4(\epsilon^2 + \frac{1}{4})} = \frac{\epsilon^3}{4\epsilon^2 + 1} \end{aligned} \tag{33}$$

where in step 4 we assumed an uncorrelated numerator and denominator. Thus, we got a unrealistic upper bound, in the sense that it is more conservative that the true upper bound. For instance, using an unitary ϵ, gives the upper bound $\lambda \Delta t < 0.2$.

The implementations can be found in the file `sol.m`[2]. Figure 1 shows the output of the explicit scheme using 500 iterations, unitary ϵ and $\lambda\Delta t = 0.19$.

Figure 1: Output of the explicit scheme for the total variation inpainting problem. Parameters: 500 iterations, unitary ϵ and $\lambda\Delta t = 0.19$.

[2]Please note that in the implementation there is no λ parameter. The Δt parameter in the file is equivalent to the $\lambda\Delta t$ mentioned in this report.