Signed Permutation Statistics

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January 4, 2020

1 Project Description

A permutation statistic is a function which takes a permutation in the symmetric group (i.e., the Weyl group of type A) to a non-negative integer. The peak number, valley number, double ascent number, and double descent number are permutation statistics which have been very well-studied in the literature. For example, there are recurrence formulas and generating function formulas for the distributions of these permutation statistics, and the limiting distributions of these statistics are known—big surprise, they all converge to the normal distribution! There are also cyclic analogues of these statistics, and they have been similarly well-studied. All of these statistics—the peak number, valley number, double ascent number, double descent number, and their cyclic analogues—can also be defined for signed permutations, which are elements of the hyperoctahedral group (the Weyl group of type B). Perhaps sur- prisingly, not much seems to be known about the distribution of these statistics in the type B setting. The goal of this honors thesis project is to study these statistics for signed per- mutations, and to prove enumeration formulas and asymptotic normality results in this setting. For a more in-depth introduction to the research topics in this document, refer to [1], [2], [3], [4], and [5].

2 Eulerian Numbers

Let $\mathfrak{S}_n$ denote the set of permutations of $[n] = \{1, 2, ..., n\}$. The notation S_n is more common, but $\mathfrak{S}_n$ is standard in enumerative combinatorics. Given a permutation π, let $\operatorname{des}(\pi)$ denote the number of descents of π as defined in [2]. Define $A_{n,k}$ to be the number of permutations in $\mathfrak{S}_n$ with $\operatorname{des}(\pi) = k$; these are called *Eulerian numbers*. Note that Bóna defines Eulerian numbers differently in [2]. Then we have the recurrence

$$A_{n,k} = (n-k)A_{n-1,k-1} + (k+1)A_{n-1,k} \tag{1}$$

for $k \geq n$. Formula 1 with the initial condition $A_{0,0} = 1$ can be used to recursively generate all Eulerian numbers. This is equivalent to Bóna's Theorem 1.7 [2]:

Theorem 2.1. *For all positive integers k and n satisfying $k \leq n$, we have*

$$A(n, k+1) = (k+1)A(n-1, k+1) + (n-k)A(n-1, k).$$

The proof of Theorem 2.1 from [2] is as follows:

Proof. There are two ways we can get an n-permutation p with k descents from an $(n-1)$-permutation p' by inserting the entry n into p'. Either p' has k descents, and the insertion of n does not form a new descent, or p' has $k-1$ descents, and the insertion of n does form a new descent.

In the first case, we have to put the entry n at the end of p', or we have to insert n between two entries that form one of the k descents of p'. This means we have $k+1$ choices for the position of n. As we have $A(n-1, k+1)$ choices for p', the first term of the right-hand side is explained.

In the second case, we have to put the entry n at the front of p', or we have to insert n between two entries that form one of the $(n-2)-(k-1)$ ascents of p'. This means that we have $n-k$ choices for the position of n. As we have $A(n-1, k)$ choices for p', the second part of the right-hand side is explained and the theorem is proved. $\square$

3 Eulerian Polynomials

Define the nth *Eulerian polynomial* $A_n(t)$ by

$$A_n(t) := \sum_{k=0}^{n-1} A_{n,k} t^k = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)}$$

for $n \geq 1$ and $A_0(t) = 1$. Then, $A_n(t)$ encodes the distribution of the descent number statistic over $\mathfrak{S}_n$. For example, $A_3(t) = 1 + 4t + t^2$, which means that among all permutations in $\mathfrak{S}_3$, there is one with no descents, four with one descent, and one with two descents.

4 Recursive-Differential Equation for the Eulerian Numbers

We can use Formula 1 to prove the *recursive-differential equation*

$$A_n(t) = (1 + (n-1)t)A_{n-1}(t) + t(1-t)A'_{n-1}(t) \tag{2}$$

for $n \geq 1$. Here, $A'_n(t)$ is the derivative of $A_n(t)$. This is shown below, beginning with Formula 2 and ending with Formula 1:

Proof.

$$
\begin{aligned}
A_n(t) &= (1+(n-1)t)A_{n-1}(t) + t(1-t)A'_{n-1}(t)\\
\sum_{k=0}^{n-1} A_{n,k}t^k &= (1+(n-1)t)\sum_{k=0}^{n-2} A_{n-1,k}t^k + t(1-t)\sum_{k=0}^{n-2} kA_{n-1,k}t^{k-1}\\
\sum_{k=0}^{n-1} A_{n,k}t^k &= \sum_{k=0}^{n-2} A_{n-1,k}t^k + (n-1)\sum_{k=0}^{n-2} A_{n-1,k}t^{k+1} + (1-t)\sum_{k=0}^{n-2} kA_{n-1,k}t^k\\
\sum_{k=0}^{n-1} A_{n,k}t^k &= \sum_{k=0}^{n-2} A_{n-1,k}t^k + (n-1)\sum_{k=0}^{n-2} A_{n-1,k}t^{k+1} + \sum_{k=0}^{n-2} kA_{n-1,k}t^k - \sum_{k=0}^{n-2} kA_{n-1,k}t^{k+1}\\
\sum_{k=0}^{n-1} A_{n,k}t^k &= \sum_{k=0}^{n-2}(k+1)A_{n-1,k}t^k + \sum_{k=0}^{n-2}(n+k-1)A_{n-1,k}t^{k+1}\\
1+\sum_{k=1}^{n-1} A_{n,k}t^k &= \sum_{k=1}^{n-1}(k+1)A_{n-1,k}t^k + 1 + \sum_{k=1}^{n-1}(n-k)A_{n-1,k-1}t^k\\
A_{n,k}\sum_{k=1}^{n-1} t^k &= (k+1)A_{n-1,k-1}\sum_{k=1}^{n-1} t^k + (n-k)A_{n-1,k-1}\sum_{k=1}^{n-1} t^k\\
A_{n,k} &= (k+1)A_{n-1,k-1} + (n-k)A_{n-1,k-1}.
\end{aligned}
$$

□

5 Partial Differential Equation for the Eulerian Numbers

Let us consider the exponential generating function for the sequence of Eulerian polynomials:

$$A(t;x) := \sum_{n=0}^{\infty} A_n(t)\frac{x^n}{n!}.$$

We can Formula 2 to prove the *partial differential equation*

$$A(t;x) = (1-xt)\frac{\partial}{\partial x}A(t;x) + t(t-1)\frac{\partial}{\partial t}A(t;x). \tag{3}$$

This is shown below, beginning with Formula 2 and ending with Formula 3:

Proof.

$$\begin{aligned}
A_n(t) &= (1+(n-1)t)A_{n-1}(t) + t(1-t)A'_{n-1}(t)\\
\sum_{n=1}^{\infty} A_n(t)\frac{x^{n-1}}{(n-1)!} &= \sum_{n=1}^{\infty}(1+(n-1)t)A_{n-1}(t)\frac{x^{n-1}}{(n-1)!} + \sum_{n=1}^{\infty} t(1-t)A'_{n-1}(t)\frac{x^{n-1}}{(n-1)!}\\
\sum_{n=1}^{\infty} A_n(t)\frac{x^{n-1}}{(n-1)!} &= \sum_{n=1}^{\infty} A_{n-1}(t)\frac{x^{n-1}}{(n-1)!} + \sum_{n=1}^{\infty}(n-1)tA_{n-1}(t)\frac{x^{n-1}}{(n-1)!} + \sum_{n=1}^{\infty} t(1-t)A'_{n-1}(t)\frac{x^{n-1}}{(n-1)!}\\
\sum_{n=1}^{\infty} A_n(t)\frac{x^{n-1}}{(n-1)!} &= \sum_{n=0}^{\infty} A_n(t)\frac{x^n}{n!} + \sum_{n=1}^{\infty}(n-1)tA_{n-1}(t)\frac{x^{n-1}}{(n-1)!} + \sum_{n=1}^{\infty} t(1-t)A'_{n-1}(t)\frac{x^{n-1}}{(n-1)!}\\
\sum_{n=0}^{\infty} A_n(t)\frac{x^n}{n!} &= \sum_{n=1}^{\infty} A_n(t)\frac{x^{n-1}}{(n-1)!} - \sum_{n=1}^{\infty}(n-1)tA_{n-1}(t)\frac{x^{n-1}}{(n-1)!!} + \sum_{n=1}^{\infty} t(t-1)A'_{n-1}(t)\frac{x^{n-1}}{(n-1)!}\\
A(t;x) &= \sum_{n=0}^{\infty} A_{n+1}(t)\frac{x^n}{n!} - \sum_{n=1}^{\infty}(n-1)tA_{n-1}(t)\frac{x^{n-1}}{(n-1)!} + \sum_{n=1}^{\infty} t(t-1)A'_{n-1}(t)\frac{x^{n-1}}{(n-1)!}.
\end{aligned}$$

Here, we want to show that $\sum_{n=1}^{\infty} t(n-1)A_{n-1}(t)\frac{x^{n-1}}{(n-1)!} = xt\frac{\partial}{\partial x}A(t;x)$. Simplifying, we want to show $\sum_{n=1}^{\infty} A_{n-1}(t)\frac{x^{n-1}}{(n-2)!} = x\frac{\partial}{\partial x}A(t;x)$.

$$\begin{aligned}
\sum_{n=1}^{\infty} A_{n-1}(t)\frac{x^{n-1}}{(n-2)!} &= \sum_{n=0}^{\infty} A_{n+1}(t)\frac{x^{n+1}}{n!}\\
\sum_{n=1}^{\infty} A_n(t)\frac{x^n}{(n-1)!} &= \sum_{n=0}^{\infty} A_{n+1}(t)\frac{x^{n+1}}{n!}\\
\sum_{n=0}^{\infty} A_{n+1}(t)\frac{x^{n+1}}{n!} &= \sum_{n=0}^{\infty} A_{n+1}(t)\frac{x^{n+1}}{n!}.
\end{aligned}$$

Continuing,

$$\begin{aligned}
A(t;x) &= \sum_{n=0}^{\infty} A_{n+1}(t)\frac{x^n}{n!} - xt\frac{\partial}{\partial x}A(t;x) + \sum_{n=1}^{\infty} t(t-1)A'_{n-1}(t)\frac{x^{n-1}}{(n-1)!}\\
A(t;x) &= \sum_{n=1}^{\infty} A_n(t)\frac{x^{n-1}}{(n-1)!} - xt\frac{\partial}{\partial x}A(t;x) + \sum_{n=1}^{\infty} t(t-1)A'_{n-1}(t)\frac{x^{n-1}}{(n-1)!}\\
A(t;x) &= \sum_{n=1}^{\infty} A_n(t)n\cdot\frac{x^n}{n!} - xt\frac{\partial}{\partial x}A(t;x) + \sum_{n=1}^{\infty} t(t-1)A'_{n-1}(t)\frac{x^{n-1}}{(n-1)!}\\
A(t;x) &= \frac{\partial}{\partial x}\sum_{n=0}^{\infty} A_n(t)\frac{x^n}{n!} - xt\frac{\partial}{\partial x}A(t;x) + \sum_{n=1}^{\infty} t(t-1)A'_{n-1}(t)\frac{x^{n-1}}{(n-1)!}\\
A(t;x) &= (1-xt)\frac{\partial}{\partial x}A(t;x) + \sum_{n=1}^{\infty} t(t-1)A'_{n-1}(t)\frac{x^{n-1}}{(n-1)!}
\end{aligned}$$

Here, we want to show that $\sum_{n=1}^{\infty} t(t-1)A'_{n-1}(t)\frac{x^{n-1}}{(n-1)!} = t(t-1)\frac{\partial}{\partial t}A(t;x)$.

$$\begin{aligned}
\sum_{n=1}^{\infty} t(t-1)A'_{n-1}(t)\frac{x^{n-1}}{(n-1)!} &= t(t-1)\frac{\partial}{\partial t}\sum_{n=0}^{\infty} A_n(t)\frac{x^n}{n!} \\
t(t-1)\sum_{n=1}^{\infty} A'_{n-1}(t)\frac{x^{n-1}}{(n-1)!} &= t(t-1)\frac{\partial}{\partial t}\sum_{n=0}^{\infty}\sum_{k=0}^{n-1} A_{n,k}t^k\frac{x^n}{n!} \\
t(t-1)\sum_{n=1}^{\infty} A'_{n-1}(t)\frac{x^{n-1}}{(n-1)!} &= t(t-1)\sum_{n=0}^{\infty}\sum_{k=0}^{n-1} kA_{n,k}t^{k-1}\frac{x^n}{n!} \\
t(t-1)\sum_{n=1}^{\infty} A'_{n-1}(t)\frac{x^{n-1}}{(n-1)!} &= t(t-1)\sum_{n=1}^{\infty}\sum_{k=0}^{n-2} kA_{n-1,k}t^{k-1}\frac{x^{n-1}}{(n-1)!} \\
t(t-1)\sum_{n=1}^{\infty} A'_{n-1}(t)\frac{x^{n-1}}{(n-1)!} &= t(t-1)\sum_{n=1}^{\infty} A'_{n-1}(t)\frac{x^{n-1}}{(n-1)!}.
\end{aligned}$$

Thus, we have shown that

$$A(t;x) = (1-xt)\frac{\partial}{\partial x}A(t;x) + t(t-1)\frac{\partial}{\partial t}A(t;x),$$

as desired. □

Solving this PDE with the boundary condition $A(t;0) = 1$ yields

$$A(t;x) = \frac{1-t}{e^{-x(1-t)} - t},$$

which is equivalent to Bóna's Theorem 1.23 [2]:

Theorem 5.1. *Let*

$$r(t,u) = \sum_{n\geq 0}\sum_{k\geq 0} A(n,k)t^k\frac{u^n}{n!}.$$

Then we have

$$r(t,u) = \frac{1-t}{1-te^{u(1-t)}}.$$

Every Eulerian number is encoded inside this function.

6 Peak Numbers

Given a permutation $\pi \in \mathfrak{S}_n$, we say that i (where $2 \leq i \leq n-1$) is a *peak* of π if $\pi_{k-1} < \pi_k > \pi_{k+1}$. Let $\mathrm{pk}(\pi)$ be the number of peaks of π; this is the *peak number* statistic. Also, let $P_{n,k}$ denote the number of permutations in $\mathfrak{S}_n$ with $\mathrm{pk}(\pi) = k$. We will refer to these as *peak numbers*.

The value $P_{n,k}$ for $k > \lfloor\frac{n-1}{2}\rfloor$ is 0. The maximum number of peaks is $k = \lfloor\frac{n-1}{2}\rfloor$. These peaks cannot be at positions $i = 1$ or $i = n$. This leaves $n-2$ possible peak positions. There cannot be consecutive peak indices. Thus, the permutation must alternate between peaks and valleys (i.e. *peak-valley-peak-valley-...*). Thus, $\lceil \frac{n-2}{2} \rceil$ is the maximum number of peaks, which is equivalent to $\lfloor \frac{n-1}{2} \rfloor$.

The value of $P_{n,0}$ represents the number of permutations in $\mathfrak{S}_n$ with no peaks. The permutation π can be split into two sub-permutations, π_1 and π_2. For there to be no peaks, the complete permutation must be arranged as $\pi = \pi_1 1 \pi_2$. The permutation π_1 is a decreasing permutation containing a subset of $[2, n]$. There are 2^{n-1} ways to arrange π_1. Then, π_2 must be an increasing permutation that contains the remaining values of $[2, n]$ not in π_1, and there is only one way to arrange this after π_1 is chosen. This means that $P_{n,0} = 2^{n-1}$.

7 Recursive Formula for the Peak Numbers

The following equation provides a recursive formula, analogous to 1 for the peak numbers $P_{n,k}$:

$$P_{n,k} = 2(k+1)P_{n-1,k} + (n-2k)P_{n-1,k-1} \tag{4}$$

for $n \geq 1$ and $1 \leq k \leq n$. The proof of Formula 4 follows:

Proof. There are two ways we can get an n-permutation p with k peaks from an $(n-1)$-permutation p' by inserting the entry n into p'. Either p' has k peaks and the insertion of n does not form a new peak, or p' has $k-1$ peaks and the insertion of n forms a new peak.

In the first case, we can put the entry n at the beginning or end of p'. We can also insert n between the entry on either side of a peak and the peak index itself. There are two choices for each of the k peaks in addition to the placement at the beginning and end. There are $P_{n-1,k}$ choices for p'. So, there are $(2k+2)P_{n-1,k}$ possibilities, and the first term of the right hand side of 4 is explained.

In the second case, we cannot put the entry n between the entry on either side of a peak and the peak index itself. There are two choices for each of the k peaks, and we are able to insert n in any of the remaining positions. Thus, there are $n-2k$ possible positions. There are $P_{n-1,k-1}$ choices for p'. So, there are $(n-2k)P_{n-1,k-1}$ possibilities, and the second term of the right hand side of 4 is explained and the recurrence is proved. $\square$

8 Recursive-Differential Equation and Partial Differential Equation for the Peak Numbers

Now, define

$$P_n(t) := \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} P_{n,k} t^k = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{pk}(\pi)}$$

for $n \geq 1$ with $P_0(t) = 1$. Next, define

$$P(t;x) := \sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!}.$$

The following equation provides a recursive-differential equation, analogous to 2, for the polynomials $P_n(t)$:

$$P_n(t) = 2t(1-t)P'_{n-1}(t) + (2 + t(n-2))P_{n-1}(t). \tag{5}$$

The proof of Formula 5 follows:

Proof. We will begin with Formula 4.

$$\begin{aligned}
P_{n,k} &= (2k+2)P_{n-1,k} + (n-2k)P_{n-1,k-1} \\
\sum_{k=0}^{n} P_{n,k}t^k &= \sum_{k=0}^{n}(2k+2)P_{n-1,k}t^k + \sum_{k=1}^{n}(n-2k)P_{n-1,k-1}t^k \\
\sum_{k=0}^{n} P_{n,k}t^k + 2^{n-1} &= \sum_{k=1}^{n}(2k+2)P_{n-1,k}t^k + 2^{n-1} + t\sum_{k=1}^{n}(n-2k)P_{n-1,k-1}t^{k-1} \\
P_n(t) &= \sum_{k=1}^{n}(2k+2)P_{n-1,k}t^k + 2^{n-1} + t\sum_{k=1}^{n}(n-2k)P_{n-1,k-1}t^{k-1} \\
P_n(t) &= \sum_{k=1}^{n}(2k+2)P_{n-1,k}t^k + 2^{n-1} + t\sum_{k=0}^{n}(n-2(k+1))P_{n-1,k}t^k \\
P_n(t) &= 2\sum_{k=1}^{n} kP_{n-1,k}t^k + 2\sum_{k=1}^{n} P_{n-1,k}t^k + 2^{n-1} + tn\sum_{k=0}^{n} P_{n-1,k}t^k - 2t\sum_{k=0}^{n} kP_{n-1,k}t^k \\
&\quad - 2t\sum_{k=0}^{n} P_{n-1,k}t^k \\
P_n(t) &= 2\sum_{k=0}^{n} kP_{n-1,k}t^k + 2\sum_{k=1}^{n} P_{n-1,k}t^k + 2^{n-2} + tn\sum_{k=0}^{n} P_{n-1,k}t^k - 2t\sum_{k=0}^{n} kP_{n-1,k}t^k \\
&\quad - 2t\sum_{k=0}^{n} P_{n-1,k}t^k \\
P_n(t) &= 2\sum_{k=0}^{n} kP_{n-1,k}t^k + 2P_{n-1}(t) + tn\sum_{k=0}^{n} P_{n-1,k}t^k - 2t\sum_{k=0}^{n} kP_{n-1,k}t^k - 2t\sum_{k=0}^{n} P_{n-1,k}t^k \\
P_n(t) &= 2\sum_{k=0}^{n} kP_{n-1,k}t^k + P_{n-1}(t) + t(n-2)\sum_{k=0}^{n} P_{n-1,k}t^k - 2t\sum_{k=0}^{n} kP_{n-1,k}t^k \\
P_n(t) &= 2(1-t)\sum_{k=0}^{n} kP_{n-1,k}t^k + P_{n-1}(t) + t(n-2)\sum_{k=0}^{n} P_{n-1,k}t^k \\
P_n(t) &= 2t(1-t)\sum_{k=0}^{n} kP_{n-1,k}t^{k-1} + P_{n-1}(t) + t(n-2)P_{n-1}(t) \\
P_n(t) &= 2t(1-t)P'_{n-1}(t) + (2 + t(n-2))P_{n-1}(t).
\end{aligned}$$

□

Similarly, a partial differential equation be defined for $P(t;x)$, analogous to Formula 3.

9 Type B n-Permutations and Type B Eulerian numbers

From Section 2.3 of Zhuang [3]:

> Let $\mathfrak{B}_n$ be the set of permutations $\pi = \pi_{-n} \cdots \pi_{-1}\pi_0\pi_1 \cdots \pi_n$ of $\{-n, \ldots, -1, 0, 1, \ldots, n\}$ satisfying $\pi_{-i} = -\pi_i$ for all $-n \leq i \leq n$; we call these *signed n-permutations* (or *type B n-permutations*). Let $\mathfrak{B} := \bigcup_{n=0}^{\infty} \mathfrak{B}_n$. For any signed n-permutation π, we must have $\pi_0 = 0$ and π is completely determined by $\{\pi_1, \ldots, \pi_n\}$, so we can write π as $\pi = \pi_1 \cdots \pi_n$ with the understanding that $\pi_0 = 0$ and $\pi_{-i} = -\pi_i$ for all i. In this way, we can think of $\mathfrak{S}_n$ as the subset of signed permutations in $\mathfrak{B}_n$ with no negative letters among $\{\pi_1, \ldots, \pi_n\}$.
> For cleaner notation, let us write $\bar{i}$ rather than $-i$ when writing out the letters of a signed permutation. For example, if $\pi = \pi_1\pi_2\pi_3$ with $\pi_1 = 3$, $\pi_2 = -2$, and $\pi_3 = -1$, then we write $\pi = 3\bar{2}\bar{1}$.
> We say that $i \in \{0\} \cup [n-1]$ is a *descent* (or *type B descent*) of $\pi \in \mathfrak{B}_n$ if $\pi_i > \pi_{i+1}$. Note that we allow 0 to be descent, which happens precisely when π_1 is negative. There are two notions of descent number for signed permutations that we consider. The *descent number* (or *type B descent number*) $\mathrm{des}_B(\pi)$ is simply the number of descents of $\pi \in \mathfrak{B}_n$... For example, let $\pi = \bar{4}72\bar{6}\bar{3}51$. Then the descents of π are 0, 2, 3, and 6, so $\mathrm{des}_B(\pi) = 4$...
> We define $B_n(t) := \sum_{\pi \in \mathfrak{B}_n} t^{\mathrm{des}_B(\pi)}$... which is the type B analogues of Eulerian polynomials using the descent number. We call $B_n(t)$ the nth *type B Eulerian polynomial.*

The set of permutations $\mathfrak{B}_n$ contains $2^n n!$ elements. We define $B_{n,k}$ to be the number of permutations in $\mathfrak{B}_n$ with $\mathrm{des}_B(\pi) = k$; these are called *type B Eulerian numbers.* We also define

$$B_n(t) := \sum_{k=0}^{n} B_{n,k} t^k = \sum_{\pi \in \mathfrak{B}_n} t^{\mathrm{des}_B(\pi)}$$

for $n \geq 1$ with $B_0(t) = 1$, and define

$$B(t; x) := \sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!}.$$

The polynomials $B_n(t)$ are called *type B Eulerian polynomials*, and $B(t;x)$ is their exponential generating function.

10 Recursive Formula for the Type B Eulerian Numbers

The following equation provides a recursive formula for the type B Eulerian numbers $B_{n,k}$:

$$B_{n,k} = (2n - 2k + 1)B_{n-1,k-1} + (2k + 1)B_{n-1,k}. \tag{6}$$

The proof of Formula 6 follows:

Proof. There are two ways we can get an n-signed permutation p with k descents from an $(n-1)-$permutation p' by inserting the entries n and $\bar{n}$ into p'. Either p' has k descents and the insertion of n and $\bar{n}$ does not form a new descent, or p' has $k-1$ descents, and the insertion of n and $\bar{n}$ forms a new descent.

In the first case, p' has k descents and inserting n and $\bar{n}$ does not add a new descent. For each of the k descents $\pi_i > \pi_{i+1}$, n can be inserted between π_i and π_{i+1}. There are k ways to do this. Similarly, $\bar{n}$ can be inserted between each of the k descents $\pi_i > \pi_{i+1}$. There are k ways to do this. Also, $\bar{n}$ can be placed at the end of the permutation. Thus, since there are $B_{n-1,k}$ choices for p', there are $(2k+1)B_{n-1,k}$ possibilities, and the second term of 6 is explained.

In the second case, p' has $k-1$ descents and inserting n and $\bar{n}$ adds a new descent. There are $k-1$ descents, and n cannot be placed between any of the $k-1$ descents $\pi_i > \pi_{i+1}$. The entry n also cannot be placed at the end of p'. So n cannot be placed in k places, leaving $n-k$ possible places to insert n. The entry $\bar{n}$ also cannot be placed between any of the $k-1$ descents $\pi_i > \pi_{i+1}$. The entry $\bar{n}$ can be placed at the end of p'. This leaves $n-(k-1) = n-k+1$ possible locations. Since there are $B_{n-1,k-1}$ choices for p', there are $(2n-2k+1)B_{n-1,k-1}$ possibilities, and the first term of 6 is explained and the equation is proved. □

11 Recursive-Differential Equation and Partial Differential Equation for the Type B Eulerian Polynomials

We can use Formula 6 to prove the recursive-differential equation

$$B_n(t) = (1 + t(2n-1))B_{n-1}(t) + 2t(1-t)B'_{n-1}(t) \tag{7}$$

for the type B Eulerian polynomials $B_n(t)$. The proof of Formula 7 follows:

Proof. We will begin with Formula 6.

$$\begin{aligned}
B_{n,k} &= (2n-2k+1)B_{n-1,k-1} + (2k+1)B_{n-1,k} \\
1 + \sum_{k=1}^{n} B_{n,k}t^k &= t\sum_{k=1}^{n}(2n-2k+1)B_{n-1,k-1}t^{k-1} + \sum_{k=1}^{n}(2k+1)B_{n-1,k}t^k + 1 \\
B_n(t) &= t\sum_{k=0}^{n}(2n-2k-1)B_{n-1,k}t^k + \sum_{k=0}^{n}(2k+1)B_{n-1,k}t^k \\
B_n(t) &= 2tn\sum_{k=0}^{n}B_{n-1,k}t^k - 2t^2\sum_{k=0}^{n}kB_{n-1,k}t^{k-1} - t\sum_{k=0}^{n}B_{n-1,k}t^k + \sum_{k=0}^{n}(2k+1)B_{n-1,k}t^k \\
B_n(t) &= (2tn - t)B_{n-1}(t) - 2t^2\sum_{k=0}^{n}kB_{n-1,k}t^{k-1} + 2t\sum_{k=0}^{n}kB_{n-1,k}t^{k-1} + \sum_{k=0}^{n}B_{n-1,k}t^k \\
B_n(t) &= (1 + t(2n-1))B_{n-1}(t) + 2t(1-t)B'_{n-1}(t).
\end{aligned}$$

□

Similarly, a partial differential equation be defined for $B(t;x)$.

12 Python Code for Generating Type B Eulerian Numbers and Polynomials

In our research, we wrote Python code to generating the Type B Eulerian numbers and polynomials for various values of n using a code structure based in memoization. The GitHub repository can be found at: https://github.com/lkbbad/typeBeulerian.

13 Recurrence Formulas for $P^+_{n,k}$ and $P^-_{n,k}$

In our research, we reproduced results discovered by Chak-On Chow and Shi-Mei Ma in 2014 [5]. Recurrence formulas for the numbers $P^+_{n,k}$ and $P^-_{n,k}$, where $P^+_{n,k}$ is the number of permutations $\pi \in \mathfrak{B}_n$ with $\pi(1) > 0$ and $\mathrm{pk}_B(\pi) = k$, and $P^-_{n,k}$ is defined analogously but with $\pi(1) < 0$. According to Lemma 6 in [5]:

$$P^+_{n,k} = (4k+1)P^+_{n-1,k} + (2n-4k+2)P^+_{n-1,k-1} + P^-_{n-1,k-1} \tag{8}$$

$$P^-_{n,k} = (4k+3)P^-_{n-1,k} + (2n-4k)P^-_{n-1,k-1} + P^+_{n-1,k} \tag{9}$$

The proof of these two recurrences can also be found in [5].

14 Joint Distribution of the Type B Peak Number and Type B Descent Number over $\mathfrak{B}_n$

We extended the work of Chak-On Chow and Shi-Mei Ma by investigating the joint distribution of the type B peak number and type B descent number over $\mathfrak{B}_n$. Let $P^+_{n,j,k}$ be the number of permutations $\pi \in \mathfrak{B}_n$ with $\pi(1) > 0$, $\mathrm{pk}_B(\pi) = j$, and $\mathrm{des}_B(\pi) = k$. Also, let $P^-_{n,j,k}$ be define analogously with $\pi(1) < 0$. These refine $P^+_{n,k}$ and $P^-_{n,k}$ by the number of type B descents so that

$$P^+n, k = \sum_{m=0}^{n} P^+_{n,k,m}$$

and

$$P^-_{n,k} = \sum_{m=0}^{n} P^-_{n,k,m}.$$

We can also define the polynomials

$$P^+_n(s,t) := \sum_{k=0}^{n}\sum_{j=0}^{k} P^+_{n,j,k} s^j t^k = \sum_{\substack{\pi\in\mathfrak{B}_n \\ \pi(1)>0}} s^{\mathrm{pk}_B(\pi)} t^{\mathrm{des}_B(\pi)}$$

and

$$P^-_n(s,t) := \sum_{k=0}^{n}\sum_{j=0}^{k} P^-_{n,j,k} s^j t^k = \sum_{\substack{\pi\in\mathfrak{B}_n \\ \pi(1)<0}} s^{\mathrm{pk}_B(\pi)} t^{\mathrm{des}_B(\pi)},$$

as well as the exponential generating functions

$$P^+(s,t;x) := \sum_{n=1}^{\infty} P_n^+(s,t)\frac{x^n}{n!}$$

and

$$P^-(s,t;x) := \sum_{n=1}^{\infty} P_n^-(s,t)\frac{x^n}{n!}.$$

15 System of Recursive Equations for $P^+_{n,j,k}$ and $P^-_{n,j,k}$

We found and proved a system of two recursive equations for the numbers $P^+_{n,j,k}$ and $P^-_{n,j,k}$, as follows:

$$\begin{aligned}P^+_{n,j,k} = (2j+1)P^+_{n-1,j,k} + 2jP^+_{n-1,j,k-1} + (2k-2j+2)P^+_{n-1,j-1,k} \\ + (2n-2j-2k+2)P^+_{n-1,j-1,k-1} + P^-_{n-1,j-1,k}\end{aligned}$$

$$\begin{aligned}P^-_{n,j,k} = (2j+2)P^-_{n-1,j,k} + (2j+1)P^-_{n-1,j,k-1} + (2k-2j)P^-_{n-1,j-1,k} \\ + (2n-2j-2k+2)P^-_{n-1,j-1,k-1} + P^+_{n-1,j,k-1}\end{aligned}$$

The proof for the recursive equation of $P^+_{n,j,k}$ follows:

Proof. There are 5 ways we can get an n-signed permutation p that starts with an ascent with k descents and j from an $(n-1)$-permutation p' that starts with an ascent or descent by inserting the entries n and $\bar{n}$ into p'.

1. The permutation p' can start with an ascent and have k descents and j peaks, and adding n or $\bar{n}$ does not add a new descent or a new peak. In this case, we can insert n on the right side of a peak and insert $\bar{n}$ on the left side of a valley. There are j peaks in p', and there is a bijection between the number of valleys and peaks because each peak will be followed by a valley in the permutation. So, this gives $2j$ places to insert n or $\bar{n}$. Additionally, we can insert n at the end of the permutation. Thus, since there are $P^+_{n-1,j,k}$ choices for p', there are $(2j+1)P^+_{n-1,j,k}$ possibilities, and the first term of the right hand side is explained.

2. The permutation p' can start with an ascent and have $k-1$ descents and j peaks, and adding n or $\bar{n}$ adds a new descent, but not a new peak. In this case, we can insert n on the left side of a peak and insert $\bar{n}$ on the right side of a valley. Since there is a bijection between the number of valleys and peaks, this gives $2j$ places to insert n or $\bar{n}$. Thus, since there are $P^+_{n-1,j,k-1}$ choices for p', there are $2jP^+_{n-1,j,k-1}$ possibilities, and the second term of the right hand side is explained.

3. The permutation p' can start with an ascent and have k descents and $j-1$ peaks, and adding n or $\bar{n}$ adds a new peak, but not a new descent. In this case, we cannot insert n on the right side of a peak, and we cannot insert $\bar{n}$ on the left side of a valley. So, there are k descents and $2k$ places to put the entries, but we must eliminate the position to the

right of every peak and the position to the left of every valley. Since there are $j-1$ peaks and there is a bijection between peaks and valleys, this eliminates $2(j-1)$ positions. This is equivalent to inserting n after every double descent and $\bar{n}$ before every double descent. Thus, since there are $P^+_{n-1,j-1,k}$ choices for p', there are $(2k-2j+2)P^+_{n-1,j-1,k}$ possibilities, and the third term of the right hand side is explained.

4. The permutation p' can start with an ascent and have $k-1$ descents and $j-1$ peaks, and adding n or $\bar{n}$ adds a new peak and a new descent. In this case, there are $n-1-(k-1) = n-k$ ascents in p'. We can insert n before and $\bar{n}$ after any ascent that is not a peak. There are $(n-k)-(j-1)$ of these, and $2((n-k)-(j-1)) = 2n-2k-2j+2$ possible places for the entries. This is equivalent to inserting n before every double ascent and $\bar{n}$ after every double ascent. Thus, since there are $P^+_{n-1,j-1,k-1}$ choices for p', there are $(2n-2k-2j+2)P^+_{n-1,j-1,k-1}$ possibilities, and the fourth term of the right hand side is explained.

5. The permutation p' can also start with a descent and have $j-1$ peaks and k descents. You can insert the entry n at the beginning of p', creating a new peak and making the permutation start with an ascent. There are $P^-_{n-1,j-1,k}$ choices for p', and thus the fifth term of the right hand side is explained.

Thus, this recursive formula is proved. □

The proof for the recursive equation of $P^-_{n,j,k}$ follows:

Proof. There are 5 ways we can get an n-signed permutation p that starts with a descent with k descents and j from an $(n-1)$-permutation p' that starts with an ascent or descent by inserting the entries n and $\bar{n}$ into p'.

1. The permutation p' can start with an descent and have k descents and j peaks, and adding n or $\bar{n}$ does not add a new descent or a new peak. In this case, we can insert n on the right side of a peak and insert $\bar{n}$ on the left side of a valley. There are j peaks in p', and there is a bijection between the number of valleys and peaks. So, this gives $2j$ places to insert n or $\bar{n}$. Additionally, we can insert n at the end of the permutation. We will consider the first descent of p' to be a valley, but this valley does not fall in bijection with the peaks of the permutation. So, we must account for this valley because $\bar{n}$ can always be placed to the left this valley. Thus, there are $2j+2$ places to insert n or $\bar{n}$. Since there are $P^-_{n-1,j,k}$ choices for p', there are $(2j+2)P^-_{n-1,j,k}$ possibilities, and the first term of the right hand side is explained.

2. The permutation p' can start with an descent and have $k-1$ descents and j peaks, and adding n or $\bar{n}$ adds a new descent, but not a new peak. In this case, we can insert n on the left side of all peaks and $\bar{n}$ on the right side of all valleys. Since there is a bijection between the number of valleys and peaks, this gives $2j$ places to insert n or $\bar{n}$. We must also consider the first descent as a valley because $\bar{n}$ can always be inserted on the right side of this valley. Thus, there are $2j+1$ places to insert n or $\bar{n}$. Since there are $P^-_{n-1,j,k-1}$ choices for p', there are $(2j+1)P^-_{n-1,j,k-1}$ possibilities, and the second term of the right hand side is explained.

3. The permutation p' can start with an descent and have k descents and $j-1$ peaks, and adding n or $\bar{n}$ adds a new peak, but not a new descent. In this case, there are

$2k$ possible places for n and $\bar{n}$ to be inserted before or after all k descents. We cannot insert n after a peak, and we cannot insert $\bar{n}$ before a valley. So, we must eliminate the position to the right of every peak and the position to the left of every valley. Since there are $j-1$ peaks and there is a bijection between peaks and valleys, this eliminates $2(j-1)$ positions. This is equivalent to inserting n after every double descent and $\bar{n}$ before every double descent. Additionally, n or $\bar{n}$ cannot go at the beginning of the permutation. Thus, there are $2k-(2j-2)-2 = 2k-2j$ places to insert n or $\bar{n}$. Since there are $P^-_{n-1,j-1,k}$ choices for p', there are $(2k-2j)P^-_{n-1,j-1,k}$ possibilities, and the third term of the right hand side is explained.

4. The permutation p' can start with an descent and have $k-1$ descents and $j-1$ peaks, and adding n or $\bar{n}$ adds a new peak and a new descent. In this case, there are $n-1-(k-1) = n-k$ ascents in p'. We can insert n before and $\bar{n}$ after any ascent that is not a peak. There are $(n-k)-(j-1)$ of these, and $2((n-k)-(j-1)) = 2n-2k-2j+2$ possible places for the entries. This is equivalent to inserting n before every double ascent and $\bar{n}$ after every double ascent. Thus, there are $2n-2k-2j+2$ places to insert n or $\bar{n}$. Since there are $P^-_{n-1,j-1,k-1}$ choices for p', there are $(2n-2k-2j+2)P^-_{n-1,j-1,k-1}$ possibilities, and the fourth term of the right hand side is explained.

5. The permutation p' can also start with a ascent and have j peaks and $k-1$ descents. You can insert the entry $\bar{n}$ at the beginning of p', creating a new descent and making the permutation start with an descent. There are $P^+_{n-1,j,k-1}$ choices for p', and thus the fifth term of the right hand side is explained.

Thus, this recursive formula is proved. □

16 System of Recursive-Differential Equations for $P^+_n(s,t$ and $P^-_n(s,t)$

We began work on discovering a system of two recursive-differential equations for the polynomials $P^+_n(s,t)$ and $P^-_n(s,t)$, but were unable to solidify a result in the time allotted.

17 Asymptotic Normality for Type B Peaks

18 Acknowledgements

We wish to acknowledge the support and direction of Dr. Yan Zhuang. This research would not have been possible without his knowledgeable guidance, input, and direction.

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