Numerical Analysis Homework 1

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Abstract

Solutions to various numerical analysis problems.

I

For $f \in \mathcal{C}^2[x_0, x_1]$ and $x \in (x_0, x_1)$, linear interpolation of f at x_0 and x_1 yields

$$f(x) - p_1(f; x) = \frac{f''(\xi(x))}{2}(x - x_0)(x - x_1).$$

Consider the case $f(x) = \frac{1}{x}$, $x_0 = 1$, $x_1 = 2$.

1. Determine $\xi(x)$ explicitly.
2. Extend the domain of ξ continuously from (x_0, x_1) to $[x_0, x_1]$. Find $\max \xi(x)$, $\min \xi(x)$, and $\max f''(\xi(x))$.

II

Let P_m^+ be the set of all polynomials of degree $\leq m$ that are non-negative on the real line,

$$P_m^+ = \{p : p \in P_m, \forall x \in \mathbb{R}, p(x) \geq 0\}.$$

Find $p \in P_{2n}^+$ such that $p(x_i) = f_i$ for $i = 0, 1, \ldots, n$ where $f_i \geq 0$ and x_i are distinct points on $\mathbb{R}$.

III

Consider $f(x) = e^x$.

1. Prove by induction that

$$\forall t \in \mathbb{R}, \quad f[t, t+1, \ldots, t+n] = \frac{(e-1)^n}{n!} e^t.$$

2. From Corollary 2.22 we know

$$\exists \xi \in (0, n) \text{ s.t. } f[0, 1, \ldots, n] = \frac{1}{n!} f^{(n)}(\xi).$$

   Determine ξ from the above two equations. Is ξ located to the left or to the right of the midpoint $n/2$?

IV

Consider $f(0) = 5$, $f(1) = 3$, $f(3) = 5$, $f(4) = 12$.

- Use the Newton formula to obtain $p_3(f; x)$;
- The data suggest that f has a minimum in $x \in (1, 3)$. Find an approximate value for the location $x_{\min}$ of the minimum.

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V

Consider $f(x) = x^7$.

- Compute $f[0, 1, 1, 1, 2, 2]$.
- We know that this divided difference is expressible in terms of the 5th derivative of f evaluated at some $\xi \in (0, 2)$. Determine ξ.

VI

f is a function on $[0, 3]$ for which one knows that

$$f(0) = 1, \quad f(1) = 2, \quad f'(1) = -1, \quad f(3) = f'(3) = 0.$$

- Estimate $f(2)$ using Hermite interpolation.
- Estimate the maximum possible error of the above answer if one knows, in addition, that $f \in \mathcal{C}^5[0, 3]$ and $|f^{(5)}(x)| \le M$ on $[0, 3]$. Express the answer in terms of M.

VII

Given

$$\begin{aligned} \Delta f(x) &= f(x+h) - f(x), \\ \Delta^{k+1} f(x) &= \Delta\Delta^k f(x) = \Delta^k f(x+h) - \Delta^k f(x) \end{aligned}$$

and backward difference by

$$\begin{aligned} \nabla f(x) &= f(x) - f(x-h), \\ \nabla^{k+1} f(x) &= \nabla\nabla^k f(x) = \nabla^k f(x) - \nabla^k f(x-h). \end{aligned}$$

For $x_j = x + jh$, prove

$$\begin{aligned} \Delta^k f(x) &= k! h^k f\left[x_0, x_1, \ldots, x_k\right], \\ \nabla^k f(x) &= k! h^k f\left[x_0, x_{-1}, \ldots, x_{-k}\right]. \end{aligned}$$

VIII

Assume f is differentiable at x_0. Prove

$$\frac{\partial}{\partial x_0} f[x_0, x_1, \ldots, x_n] = f[x_0, x_0, x_1, \ldots, x_n].$$

What about the partial derivative with respect to one of the other variables?

IX

A min-max problem. For $n \in \mathbb{N}^+$ and a fixed $a_0 \neq 0$, determine

$$\min \max_{x \in [a,b]} \left| a_0 x^n + a_1 x^{n-1} + \cdots + a_n \right|$$

where the minimum is taken over all $a_i \in \mathbb{R}, i = 1, 2, \ldots, n$.

X

Imitate the proof of Chebyshev Theorem.

Express the Chebyshev polynomial of degree $n \in \mathbb{N}$ as a polynomial T_n and change its domain from $[-1, 1]$ to $\mathbb{R}$. For a fixed $a > 1$, define $P_n^a := \{p \in P_n : p(a) = 1\}$ and a polynomial $\hat{p}_n(x) \in P_n^a$,

$$\hat{p}_n(x) := \frac{T_n(x)}{T_n(a)}.$$

For $\|f\|_\infty = \max_{x \in [-1,1]} |f(x)|$, prove $\forall p \in P_n^a, \quad \|\hat{p}_n\|_\infty \le \|p\|_\infty$.

XI

Give a detailed proof of Lemma 2.53.

XII

Give a detailed proof of Lemma 2.55.

Acknowledgement

Use GPT-4 for quick template transformation, and use Kimi AI to correct English grammar.