Numerical Analysis Homework 6

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Due time: 2024 年 12 月 23 日

摘要

Solutions to various numerical analysis problems.

I

(a). Using Newton's interpolation formula, we can derive the interpolation formula

$$p(x) = \frac{f(1) - f(-1) - 2 * f'(0)}{2}(x+1)x^2 + (f'(0) - f(0) + f(-1))(x+1)x + (f(0) - f(-1))(x+1) + f(-1)$$

Substituting $\int_{-1}^{1} p(x)dx = \frac{2}{6}(f(-1) + f(1) + 4f(0))$
This gives us Simpson's rule

$$\int_{-1}^{1} y(x)dx = \int_{-1}^{1} p(x) + E_s(y)$$

(b). According to the linear interpolation remainder theorem, we have

$$f(x) - P(x) = \frac{f^{(4)}(\xi)}{4!}(x^2 - 1)x^2, \quad -1 \le \xi \le 1$$

Integrating both sides from -1 to 1

$$\int_{-1}^{1} f(x)dx - \int_{-1}^{1} P(x)dx = \frac{1}{4!}\int_{-1}^{1} f^{(4)}(\xi)x^2(x^2 - 1)dx$$

$$\int_{-1}^{1} p(x)dx = \frac{2}{6}(f(-1) + f(1) + 4f(0))$$

So

$$E_S(y) = \frac{1}{4!}\int_{-1}^{1} f^{(4)}(\xi)x^2(x^2 - 1)dx$$

When $x \in [-1, 1]$, $x^2(x^2 - 1) \le 0$,

$$\int_{-1}^{1} f^{(4)}(\xi)x^2(x^2 - 1)dx \in [min(f^{(4)})\int_{-1}^{1} x^2(x^2 - 1)dx, max(f^{(4)})\int_{-1}^{1} x^2(x^2 - 1)dx]$$

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$f^{(4)}(x)$ is continuous on the interval, by the mean value theorem, there exists a point η on $[a, b]$, such that

$$\int_{-1}^{1} f^{(4)}(\xi)x^2(x^2-1)dx = f^{(4)}(\eta)\int_{-1}^{1} x^2(x^2-1)dx$$

$$\int_{-1}^{1} x^2(x^2-1)dx = -\frac{4}{15}$$

Therefore, the error estimate is

$$E_S(y) = -\frac{1}{90}f^{(4)}(\eta)dx$$

(c) Divide the interval into 2m equal parts, and apply Simpson's rule on each $[x_{2k-2}, x_{2k}]$, we have

$$\int_{x_{2k-2}}^{x_{2k}} y(x)dx$$

$$= \int_{-1}^{1} y(t * h + x_{2k-1})hdt$$

$$= \frac{2h}{6}(y(x_{2k-2} + x_{2k} + 4x_{2k-1})) - \frac{h^5}{90}y^{(4)}(\eta_k)$$

Summing up
The original expression equals $= \Sigma_{k=1}^{m} \frac{2h}{6}(y(x_{2k-2} + x_{2k} + 4x_{2k-1}))$
The error is $= \Sigma_{k=1}^{m} - \frac{h^5}{90}y^{(4)}(\eta_k) = -\frac{h^4}{90}y^{(4)}(\eta)$

II

(a) It is easy to know $\int_0^1 e^{-x^2} \in C^2$
By Theorem 6.18

$$E_n^T(f) = -\frac{1}{12}(\frac{1}{n})^2 f''(\xi)$$

$$|f''(\xi)| = |(4x^2-2)|e^{-x^2}$$

$$f'''(x) = (8x - 8x^3 + 4x)e^{-x^2} > 0$$

$$|f''(\xi)| \leq max(f''(0), f''(1)) = 2$$

To make the error less than $0.5 * 10^{-6}$
We only need to make $n^2 \geq 10^6 * 4/12$

$$n \geq 578$$

(b) By the Chinese lecture notes Theorem 6.3.3

$$R_{f,S_n} = -\frac{(b-a)}{2880}(\frac{2(b-a)}{n})^4 f^{(4)}(\eta)$$

Substituting we get

$$R_{f,S_n} = -\frac{1}{2880}(\frac{16}{n^4})f^{(4)}(\eta)$$

$$f^{(4)}(x) = (16x^4 - 48x^2 + 12)e^{-x^2}$$

$$f^{(5)}(x) = 8e^{-x^2}x(-4x^4 + 20x^2 - 15)$$

$$|f^{(4)}(x)| \leq max(f^{(4)}(0), f^{(4)}(1), f^{(4)}(\frac{5-\sqrt{10}}{2})) = 12$$

$$n > 19.1088...$$

$$n \geq 20$$

III

(a)
Find π_2, such that

$$\forall p \in \mathcal{P}_1, \quad \int_0^{+\infty} p(t)\pi_2(t)\rho(t)\,dt = 0.$$

Let $p(t) = kt + c, \pi_2(t) = t^2 + at + b$

$$\int_0^{+\infty} p(t)\pi_2(t)\rho(t)\,dt = \int_0^{+\infty} e^{-t}(kt^3 + (ak+c)t^2 + (ac+bk)t + bc)dt$$

$$= k3! + (ak + c)2! + (ac + bk)1! + bc0! = 0$$

for any k, c

Calculate to get $a = -4, b = 2$

(b)

Find the roots to get $t^2 - 4x + 2 = 0$

$$x = 2 - \sqrt{2}, 2 + \sqrt{2}$$

There is $w_1 + w_2 = \int_0^{+\infty} e^{-t} = 1$

$$(2 - \sqrt{2})w_1 + (2 + \sqrt{2})w_2 = 1$$

$$w_1 = \frac{2 + \sqrt{2}}{4}, w_2 = \frac{2 - \sqrt{2}}{4}$$

Therefore,

$$\int_0^{\infty} f(x)e^{-x}dx = \frac{2 + \sqrt{2}}{4} f(2 - \sqrt{2}) + \frac{2 - \sqrt{2}}{4} f(2 + \sqrt{2}) + E_2(f)$$

Refer to the proof of problem (4), using Hermite interpolation

$$f(x) = \Sigma_{m=1}^{n}(h_m(x)f_m + q_m(x)f'_m) + \frac{f^{2n}(\xi)}{(2n)!}\Pi(x - x_i)^2$$

Let $I(\Pi(x - x_i)\Pi_{i \neq m}(x - x_i)) = 0$, solve for x_i to get the same Gauss-Laguerre formula

The error is

$$\int_0^{+\infty} \frac{f^4(\xi)}{(4)!} e^{-t}(x - (2 - \sqrt{2}))^2(x - (2 + \sqrt{2}))^2 dx$$

By the mean value theorem, there exists τ, such that

$$E_2(f) = \frac{f^4(\tau)}{24} \int_0^{+\infty} e^{-t}(x - (2 - \sqrt{2}))^2(x - (2 + \sqrt{2}))^2$$

$$= \frac{f^4(\tau)}{6}$$

(c)

The predicted result is

$$\frac{2 + \sqrt{2}}{4} * \frac{1}{3 - \sqrt{2}} + \frac{2 - \sqrt{2}}{4} * \frac{1}{3 + \sqrt{2}} = 0.5714285714...$$

The true value $I = 0.596347361$

The error value 0.0249187896

$f^{(4)}(x) = \frac{24}{(1+t)^5}$

Corresponding $\tau = 1.7612556$

(IV) (a) $l_m(x) = \Pi_{i \neq m} \frac{x - x_i}{x_m - x_i} \in P_{n-1}, l(x_m) = 1$

For $n \neq m$, $l_n(x_m) = 0$

Therefore, substituting $x = x_m$

We get $(a_m + b_m x_m)f(x_m) + (c_m + d_m x_m)f'(x_m) = f(x_m)$

Differentiate

Since $m \neq n$,

$l_n^2(x_m) = 0, l_n(x_m)l_n'(x_m) = 0$

The first derivative is still determined by a single formula

$f(x_m)(b_m + 2(a_m + b_m x_m)(l_m'(x_m))) + f'(x_m)(d_m + 2(c_m + d_m x_m)((l_m'(x_m)))) = f'(x_m)$ Construct the solution, let

$a_m + b_m x_m = 1, c_m + d_m x_m = 0, b_m + 2(a_m + b_m x_m)(l_m'(x_m)) = 0, d_m + 2(c_m + d_m x_m)((l_m'(x_m))) = 1$

Solve to get $a_m = 1 + 2x_m l_m'(x_m), b_m = -2l_m'(x_m)$ $c_m = -x_m, d_m = 1$

By the uniqueness of Hermit interpolation, this is also the only solution

(b) $\forall p \in P_{2n-1}, x_1 ... x_n$

$p(x) = \Sigma_{m=1}^n (h_m(x)f_m + q_m(x)f_m')$ $I_n(f) = \Sigma_{m=1}^n (I(h_m(x))f_m + I(q_m(x))f_m')$ Where $I(h_m(x))$ is the corresponding integral result of $h_m(x)$, and $I(q_m(x))$ is the same

From (a) we know

$w_m = I(h_m(x)) = I((1+2x_m l_m'(x_m) - 2l_m'(x_m)t)(l_m^2(t)))$ $u_m = I(q_m(x)) = I((t - x_m)(l_m^2(t)))$ (c) $q_m(x) = x - x_m$

We get $I(\Pi(x - x_i)\Pi_{i\neq m}(x - x_i)) = 0$ Let $v(x) = \Pi(x - x_i)$, which is orthogonal to $l_k(x)$

If $l_k(x)$ is linearly dependent, there exist coefficients $\Sigma_k \lambda_k l_k(t) = 0$ Regard λ_k as f_k, it can be known that the corresponding interpolation polynomial is always 0. Thus, it can be concluded that $f_k = \lambda_k = 0$ Therefore, $l_k(x)$ is linearly independent

n-1 linearly independent polynomials in P_{n-1} can span the entire P_{n-1} space.

So in order to make $u_k = 0$ We need to make $v(x) = \Pi(x - x_i)$ orthogonal to the P_{n-1} space

Acknowledgement

Use GPT-4 for quick template transformation, and use Kimi AI to correct English grammar.