Numerical Analysis Homework 3

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摘要

Solutions to various numerical analysis problems.

I

$s'(x) = -3(2-x)^2 x \in [1,2]$
$s''(x) = 6(2-x)x \in [1,2]$
Substitute to get
$s''(1) = 6, s'(1) = -3$
$p \in P_3, p(0) = 0, p(1) = 1, p'(1) = -3, p''(1) = 6$
Construct divided difference
x——f
0 0
1 1 1
1 1 -3 -4
1 1 -3 3 7
$p(x) = 0 + x - 4x(x-1) + 7x(x-1)^2 = 7x^3 - 18x^2 + 12x$
Substitute s(x)
s"(2)=0
s"(0)=-36
Therefore, it is not a natural cubic spline

II

(a)
By Theorem 3.14, the spline space is of dimension $n+1$, but currently there are only n equations.
There are many polynomials that satisfy the existing conditions.
One construction is given here. Let $s'(x_1) = m_1$
In $[x_1, x_2]$ have $s(x) = f(x_2) + (x - x_2) * \frac{f(x_1)-f(x_2)}{x_1-x_2} + (x-x_1)(x-x_2)\frac{(x_1-x_2)m_1-(f(x_1)-f(x_2))}{(x_1-x_2)^2}$
Get $s'(x_2) = \frac{2(f(x_1)-f(x_2))-(x_1-x_2)*m_1}{x_1-x_2}$
By $s(x) = f(x_{i+1}) + (x - x_{i+1}) * \frac{f(x_i)-f(x_{i+1})}{x_i-x_{i+1}} + (x-x_i)(x-x_{i+1})\frac{(x_i-x_{i+1})m_i-(f(x_i)-f(x_{i+1}))}{(x_i-x_{i+1})^2} x \in [x_i.x_{i+1}]$
Construct recursively. And for different m_1 there are different results.

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(b) Construct divided difference

$x||f$

x_{i+1}, f_{i+1}

$x_i, f_i, \frac{f_i - f_{i+1}}{x_i - x_{i+1}}$

$x_i, f_i, m_i, \frac{(x_i - x_{i+1})m_i - (f(x_i) - f(x_{i+1}))}{(x_i - x_{i+1})^2}$

Construct to get

$p_i(x) = f(x_{i+1}) + (x - x_{i+1}) * \frac{f(x_i) - f(x_{i+1})}{x_i - x_{i+1}} + (x - x_i)(x - x_{i+1})\frac{(x_i - x_{i+1})m_i - (f(x_i) - f(x_{i+1}))}{(x_i - x_{i+1})^2} x \in [x_i.x_{i+1}]$

(c)

In $[x_1, x_2]$ have $s(x) = f(x_2) + (x - x_2) * \frac{f(x_1) - f(x_2)}{x_1 - x_2} + (x - x_1)(x - x_2)\frac{(x_1 - x_2)m_1 - (f(x_1) - f(x_2))}{(x_1 - x_2)^2}$

Get $s'(x_2) = \frac{2(f(x_1) - f(x_2) - (x_1 - x_2) * m_1)}{x_1 - x_2}$

Known m_i, substitute into $[x_i, x_{i+1}]$ to get the function on the interval.

$s(x) = f(x_{i+1}) + (x - x_{i+1}) * \frac{f(x_i) - f(x_{i+1})}{x_i - x_{i+1}} + (x - x_i)(x - x_{i+1})\frac{(x_i - x_{i+1})m_i - (f(x_i) - f(x_{i+1}))}{(x_i - x_{i+1})^2} x \in [x_i.x_{i+1}]$

Can calculate $m_{i+1} = \frac{2(f(x_i) - f(x_{i+1})) - (x_i - x_{i+1}) * m_i}{x_i - x_{i+1}}$

So it can be calculated recursively

III

$s \in S_3^2$

Need to make $s''(-1) = 0, s''(1) = 0, s(1) = -1$

$s'(x) = 3c(x+1)^2, x \in [-1, 0]$

$s''(x) = 6c(x+1)$

$s''(0) = 6c, s'(0) = 3c, s(0) = 1 + c, s(1) = -1$

Construct divided difference

x——f

1 -1

0 1+c -2-c

0 1+c 3c -2-4c

0 1+c 3c 3c -2-7c

$s_2(x) = -1 + (-2 - c)(x - 1) + (-2 - 4c)x(x - 1) + (-2 - 7c)x^2(x - 1)$

Calculate the second derivative

$s_2''(1) = -12 - 36c = 0$

$c = -\frac{1}{3}$

IV

$f(x) = cos(\frac{\pi}{2}x)$

(a)

$f(-1) = 0, f(0) = 1, f(1) = 0$

$f''(-1) = f''(1) = 0$

By Lemma 3.4

$M_0 + 2M_1 + M_2 = 6f[-1, 0, 1] = -6$

$M_1 = -3$

Let $f = 1 + a_1x + a_2x^2 + a_3x^3$ on [-1,0].

$f''(0) = 2a_2 = M_1, f(-1) = 1 - a_1 + a_2 - a_3 = 0, f''(-1) = 2a_2 - 6a_3 = 0$

Solve the system of linear equations to get $f = 1 - \frac{3}{2}x^2 - \frac{1}{2}x^3$

Similarly on [0,1], let $f = 1 + a_1x + a_2x^2 + a_3x^3$

$f''(0) = 2a_2 = M_1, f(1) = 1 + a_1 + a_2 + a_3 = 0,$

$f''(1) = 2a_2 + 6a_3 = 0$

$f = 1 - \frac{3}{2}x^2 + \frac{1}{2}x^3$

In summary, on $x \in [-1, 0]$

$f = 1 - \frac{3}{2}x^2 - \frac{1}{2}x^3$

On $x \in [0, 1]$

$f = 1 - \frac{3}{2}x^2 + \frac{1}{2}x^3$

After verification, it meets the criteria.

(b)

On $x \in [-1, 0]$

$s''(x) = -3 - 3x$

On $x \in [0, 1]$

$s''(x) = -3 + 3x$

$\int_{-1}^{1}[s''(x)]^2\, dx = \int_{-1}^{0}[-3 - 3x]^2\, dx + \int_{0}^{1}[-3 + 3x]^2\, dx = 6$

(i)

By the Lagrange interpolation formula, get g(x)

$g(x) = 0 * \frac{(x-0)(x-1)}{(-1-0)(-1-1)} + 1 * \frac{(x-(-1))(x-1)}{(0-(-1))(0-1)} + 0 * \frac{(x+1)(x-0)}{(1-(-1))(1-0)}$

$g(x) = 1 - x^2, g''(x) = -2$

$\int_{-1}^{1}[g''(x)]^2\, dx = 8 > \int_{-1}^{1}[s''(x)]^2\, dx = 6$ so it meets the condition

(ii)

Let $g(x) = f(x)$

$\int_{-1}^{1}[g''(x)]^2\, dx = \frac{\pi^4}{16} > \frac{97}{16} > \int_{-1}^{1}[s''(x)]^2\, dx = 6$ so it meets the condition

V

(a)

$B_i^{n+1}(x) = \frac{x-t_{i-1}}{t_{i+n}-t_{i-1}} B_i^n(x) + \frac{t_{i+n+1}-x}{t_{i+n+1}-t_i} B_{i+1}^n(x).$

The hat function at t_i is

$$\hat{B}_i(x) = \begin{cases} \frac{x-t_{i-1}}{t_i-t_{i-1}} & x \in (t_{i-1}, t_i], \\ \frac{t_{i+1}-x}{t_{i+1}-t_i} & x \in (t_i, t_{i+1}], \\ 0 & \text{otherwise.} \end{cases}$$

Substitute to get

$B_i^2(x) = \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} \hat{B}_i^n(x) + \frac{t_{i+2}-x}{t_{i+2}-t_i} \hat{B}_{i+1}^n(x).$

For $x \in (t_{i-1}, t_i]$

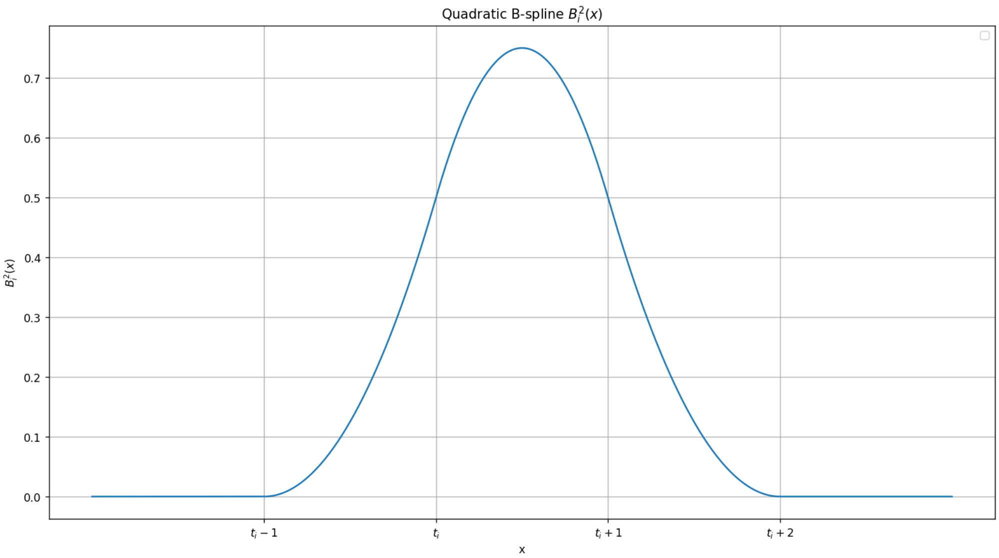
$B_i^2(x) = \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} \frac{x-t_{i-1}}{t_i-t_{i-1}} + \frac{t_{i+n+1}-x}{t_{i+n+1}-t_i} * 0 = \frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})}$

For $x \in (t_i, t_{i+1}]$

$B_i^2(x) = \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} \frac{t_{i+1}-x}{t_{i+1}-t_i} + \frac{t_{i+2}-x}{t_{i+2}-t_i} \frac{x-t_i}{t_{i+1}-t_i} = \frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(t_{i+2}-x)(x-t_i)}{(t_{i+2}-t_i)(t_{i+1}-t_i)}$.
For $x \in (t_{i+1}, t_{i+2}]$
$B_i^2(x) = \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} * 0 + \frac{t_{i+2}-x}{t_{i+2}-t_i} \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}} = \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}$.
Consistent with the conclusion in the book.
(b)
On $x \in (t_{i-1}, t_i]$,
$\frac{d}{dx} B_i^2(x) = \frac{2(x-t_{i-1})}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})}$
The left derivative at t_i is $\frac{2}{(t_{i+1}-t_{i-1})}$, and the left derivative is continuous.
On $x \in (t_i, t_{i+1}]$,
$\frac{d}{dx} B_i^2(x) = \frac{t_{i+1}+t_{i-1}-2x}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{t_{i+2}+t_i-2x}{(t_{i+2}-t_i)(t_{i+1}-t_i)}$.
The right derivative at t_i is $lim_{\epsilon->0^+} \frac{B_i^2(t_i+\epsilon)-B_i^2(t_i)}{\epsilon} = lim_{\epsilon->0^+} \frac{\frac{(t_i+\epsilon-t_{i-1})(t_{i+1}-t_i+\epsilon)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(t_{i+2}-t_i+\epsilon)(t_i+\epsilon-t_i)}{(t_{i+2}-t_i)(t_{i+1}-t_i)} - \frac{(t_i-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})}}{\epsilon} = \frac{2}{(t_{i+1}-t_{i-1})}$
And the derivative is continuous on $(t_{i-1}, t_i]$, so the left derivative equals the right derivative at t_i, and both sides are continuous.
Also, the left derivative at t_{i+1} is $\frac{t_{i-1}-t_{i+1}}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{t_{i+2}+t_i-2t_{i+1}}{(t_{i+2}-t_i)(t_{i+1}-t_i)} = \frac{-2}{t_{i+2}-t_i}$
On $x \in (t_{i+1}, t_{i+2}]$,
$\frac{d}{dx} B_i^2(x) = \frac{2(x-t_{i+2})}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}$
The right derivative at t_{i+1} is $lim_{\epsilon->0^+} \frac{B_i^2(t_{i+1}+\epsilon)-B_i^2(t_{i+1})}{\epsilon} = lim_{\epsilon->0^+} \frac{\frac{(t_{i+2}-t_{i+1}-\epsilon)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} - \frac{(t_{i+2}-t_{i+1})}{(t_{i+2}-t_i)}}{\epsilon} = \frac{-2}{(t_{i+2}-t_i)}$
So at t_{i+1}, the left derivative equals the right derivative, and both sides are continuous, proven.
(c) From the derivative values calculated in (b), the derivatives at t_i, t_{i+1} are not 0.
On $x \in (t_{i-1}, t_i)$,
If $\frac{d}{dx} B_i^2(x) = \frac{2(x-t_{i-1})}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} = 0, x = t_{i-1}$, does not meet the condition
On $x \in (t_i, t_{i+1}]$,
Let $\frac{d}{dx} B_i^2(x) = \frac{t_{i+1}+t_{i-1}-2x}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{t_{i+2}+t_i-2x}{(t_{i+2}-t_i)(t_{i+1}-t_i)} = 0$.
There is a unique solution $t = \frac{t_{i+1}t_{i+2}-t_it_{i-1}}{t_{i+1}+t_{i+2}-t_i-t_{i-1}}$
And $t - t_i = \frac{(t_{i+1}-t_i)(t_{i+2}-t_i)}{t_{i+1}+t_{i+2}-t_i-t_{i-1}} > 0$
$t_{i+1} - t = \frac{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})}{t_{i+1}+t_{i+2}-t_i-t_{i-1}} > 0$
So the solution is on $x \in (t_i, t_{i+1}]$ meets the condition, the conclusion is valid.
(d) When $x = t_{i-1}$, $B_i^2(x) = 0$
When $x \in (t_{i-1}, t_i]$, $\frac{d}{dx} B_i^2(x) = \frac{2(x-t_{i-1})}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} > 0$
$B_i^2(x) \leq B_i^2(t_i) = \frac{t_i-t_{i-1}}{t_{i+1}-t_{i-1}} < 1$
And $B_i^2(x) = \frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} > 0$ holds.
When $x \in [t_i, t_{i+1}]$, $B_i^2(x)$ is continuous on this closed interval and has an extreme point. If the maximum point x satisfies $B_i^2(x) \geq 1$
Considering $B_i^2(t_i) = \frac{t_i-t_{i-1}}{t_{i+1}-t_{i-1}} < 1$, $B_i^2(t_{i+1}) = \frac{(t_{i+2}-t_{i+1})}{(t_{i+2}-t_i)} < 1$
Thus $x \in (t_i, t_{i+1})$, the derivative exists, so it must be 0. There is only one point with a derivative of 0 in (t_i, t_{i+1}).
From (c) $x = \frac{t_{i+1}t_{i+2}-t_it_{i-1}}{t_{i+1}+t_{i+2}-t_i-t_{i-1}}$
Corresponding $B_i^2(x) = \frac{\frac{(t_{i+1}-t_i)(t_{i+2}-t_i)}{t_{i+1}+t_{i+2}-t_i-t_{i-1}}}{t_{i+1}-t_{i-1}} \frac{\frac{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})}{t_{i+1}+t_{i+2}-t_i-t_{i-1}}}{t_{i+1}-t_i} + \frac{t_{i+2}-t}{t_{i+2}-t_i} \frac{\frac{(t_{i+1}-t_i)(t_{i+2}-t_i)}{t_{i+1}+t_{i+2}-t_i-t_{i-1}}}{t_{i+1}-t_i}$
$= \frac{(t_{i+2}-t_i)^2}{((t_{i+2}-t_i)+(t_{i+1}-t_{i-1}))^2} < 1$ contradiction. Therefore, the maximum point function value is less than 1.
The minimum point is similar. If the minimum point x satisfies $B_i^2(x) < 0$
Considering $B_i^2(t_i) = \frac{t_i-t_{i-1}}{t_{i+1}-t_{i-1}} > 0$, $B_i^2(t_{i+1}) = \frac{(t_{i+2}-t_{i+1})}{(t_{i+2}-t_i)} > 0$

Thus $x \in (t_i, t_{i+1})$, the derivative exists, so it must be 0. There is only one point with a derivative of 0 in (t_i, t_{i+1}).
From (c) $x = \frac{t_{i+1}t_{i+2}-t_i t_{i-1}}{t_{i+1}+t_{i+2}-t_i-t_{i-1}}$
Corresponding $B_i^2(x) = \frac{\frac{(t_{i+1}-t_i)(t_{i+2}-t_i)}{t_{i+1}+t_{i+2}-t_i-t_{i-1}}}{t_{i+1}-t_{i-1}} \frac{\frac{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})}{t_{i+1}+t_{i+2}-t_i-t_{i-1}}}{t_{i+1}-t_i} + \frac{t_{i+2}-t}{t_{i+2}-t_i}\frac{\frac{(t_{i+1}-t_i)(t_{i+2}-t_i)}{t_{i+1}+t_{i+2}-t_i-t_{i-1}}}{t_{i+1}-t_i}$
$= \frac{(t_{i+2}-t_i)^2}{((t_{i+2}-t_i)+(t_{i+1}-t_{i-1}))^2} > 0$ contradiction. Therefore, the minimum point function value is greater than or equal to 0.
When $x \in (t_{i+1}, t_{i+2}]$ the derivative is
$\frac{d}{dx}B_i^2(x) = \frac{2(x-t_{i+2})}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} \le 0$
Thus $B_i^2(x) \ge B_i^2(t_{i+2}) = 0$
And $B_i^2(x) = \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} < \frac{(t_{i+2}-t_{i+1})^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} < 1$
In summary, the conclusion holds.

(e)



VI

Prove $(t_{i+2}-t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 = B_i^2$.
Construct divided difference
$t_{i-1}(t_{i-1}-x)_+^2$
$t_i, (t_i-x)_+^2, \frac{(t_i-x)_+^2-(t_{i-1}-x)_+^2}{t_i-t_{i-1}}$
$t_{i+1}, (t_{i+1}-x)_+^2, \frac{(t_{i+1}-x)_+^2-(t_i-x)_+^2}{t_{i+1}-t_i}, \frac{\frac{(t_{i+1}-x)_+^2-(t_i-x)_+^2}{t_{i+1}-t_i} - \frac{(t_i-x)_+^2-(t_{i-1}-x)_+^2}{t_i-t_{i-1}}}{t_{i+1}-t_{i-1}}$
$t_{i+2}, (t_{i+2}-x)_+^2, \frac{(t_{i+2}-x)_+^2-(t_{i+1}-x)_+^2}{t_{i+2}-t_{i+1}}, \frac{\frac{(t_{i+2}-x)_+^2-(t_{i+1}-x)_+^2}{t_{i+2}-t_{i+1}} - \frac{(t_{i+1}-x)_+^2-(t_i-x)_+^2}{t_{i+1}-t_i}}{t_{i+2}-t_i}$
$[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 = \frac{\frac{\frac{(t_{i+2}-x)_+^2-(t_{i+1}-x)_+^2}{t_{i+2}-t_{i+1}} - \frac{(t_{i+1}-x)_+^2-(t_i-x)_+^2}{t_{i+1}-t_i}}{t_{i+2}-t_i} - \frac{\frac{(t_{i+1}-x)_+^2-(t_i-x)_+^2}{t_{i+1}-t_i} - \frac{(t_i-x)_+^2-(t_{i-1}-x)_+^2}{t_i-t_{i-1}}}{t_{i+1}-t_{i-1}}}{t_{i+2}-t_{i-1}}$
$(t_{i+2}-t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 = \frac{\frac{(t_{i+2}-x)_+^2-(t_{i+1}-x)_+^2}{t_{i+2}-t_{i+1}} - \frac{(t_{i+1}-x)_+^2-(t_i-x)_+^2}{t_{i+1}-t_i}}{t_{i+2}-t_i} - \frac{\frac{(t_{i+1}-x)_+^2-(t_i-x)_+^2}{t_{i+1}-t_i} - \frac{(t_i-x)_+^2-(t_{i-1}-x)_+^2}{t_i-t_{i-1}}}{t_{i+1}-t_{i-1}}$
For $x \in (t_{i-1}, t_i]$, $(t_{i+2}-x)_+^2 = (t_{i+2}-x)^2, (t_{i+1}-x)_+^2 = (t_{i+1}-x)^2$
$, (t_i-x)_+^2 = (t_i-x)^2, (t_{i-1}-x)_+^2 = 0$ substitute into the original formula to get
$LHS = \frac{(t_{i+2}-x)^2-(t_{i+1}-x)^2}{(t_{i+2}-t_{i+1})(t_{i+2}-t_i)} - \frac{(t_{i+1}-x)^2-(t_i-x)^2}{(t_{i+1}-t_i)(t_{i+2}-t_i)} - \frac{(t_{i+1}-x)^2-(t_i-x)^2}{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})} + \frac{(t_i-x)^2}{(t_i-t_{i-1})(t_{i+1}-t_{i-1})}$

$= 1 - \frac{t_{i+1}+t_i-2x}{t_{i+1}-t_{i-1}} + \frac{(t_i-x)^2}{(t_i-t_{i-1})(t_{i+1}-t_{i-1})}$
$= \frac{x^2+t_i^2-2xt_{i-1}-t_i^2+t_{i-1}^2}{(t_i-t_{i-1})(t_{i+1}-t_{i-1})} = \frac{(t_{i-1}-x)^2}{(t_i-t_{i-1})(t_{i+1}-t_{i-1})} = B_i^2$

For $x \in (t_i, t_{i+1}]$, $(t_{i+2}-x)_+^2 = (t_{i+2}-x)^2, (t_{i+1}-x)_+^2 = (t_{i+1}-x)^2, (t_i-x)_+^2 = 0, (t_{i-1}-x)_+^2 = 0$ substitute into the original formula to get

$LHS = \frac{(t_{i+2}-x)^2-(t_{i+1}-x)^2}{(t_{i+2}-t_{i+1})(t_{i+2}-t_i)} - \frac{(t_{i+1}-x)^2}{(t_{i+1}-t_i)(t_{i+2}-t_i)} - \frac{(t_{i+1}-x)^2}{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})}$
$= \frac{(t_{i+2}-t_{i+1})(t_{i+2}+t_{i+1}-2x)}{(t_{i+2}-t_{i+1})(t_{i+2}-t_i)} - \frac{(t_{i+1}-x)^2}{(t_{i+1}-t_i)(t_{i+2}-t_i)} - \frac{(t_{i+1}-x)^2}{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})}$
$= \frac{(t_{i+2}-x)(x-t_i)+(t_i-t_{i+2})x-t_it_{i+1}+t_{i+1}t_{i+2}}{(t_{i+1}-t_i)(t_{i+2}-t_i)} + \frac{(t_{i+1}-x)(x-t_{i-1})+(t_{i+1}-t_{i-1})x+t_{i-1}t_{i+1}-t_{i+1}^2}{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})}$
$= B_i^2(x) + \frac{-t_it_{i+1}+t_{i+1}t_{i+2}}{(t_{i+1}-t_i)(t_{i+2}-t_i)} + \frac{t_{i-1}t_{i+1}-t_{i+1}^2}{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})}$
$= B_i^2(x)$ is valid

For $x \in (t_{i+1}, t_{i+2}]$, $(t_{i+2}-x)_+^2 = (t_{i+2}-x)^2, (t_{i+1}-x)_+^2 = 0, (t_i-x)_+^2 = 0, (t_{i-1}-x)_+^2 = 0$ substitute into the original formula to get

$LHS = \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_{i+1})(t_{i+2}-t_i)} = B_i^2$

All are equal, so $(t_{i+2}-t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 = B_i^2$. is valid.

VII

If $n = 0$, the integral $\int_{t_{i-1}}^{t_i} \frac{B_i^n(x)}{t_i-t_{i-1}}\,dx = 1$ always holds true.

For $n+1 \geq 2$,

$\frac{d}{dx}B_i^{n+1}(x) = \frac{(n+1)B_i^n(x)}{t_{i+n}-t_{i-1}} - \frac{(n+1)B_{i+1}^n}{t_{i+n+1}-t_i}$
$\int_{t_{i-1}}^{t_{i+n+1}} \frac{d}{dx}B_i^{n+1}(x)\,dx = \int_{t_{i-1}}^{t_{i+n+1}} \left(\frac{(n+1)B_i^n(x)}{t_{i+n}-t_{i-1}} - \frac{(n+1)B_{i+1}^n}{t_{i+n+1}-t_i}\right) dx$
$= B_i^{n+1}(x)\Big|_{t_{i-1}}^{t_{i+n+1}} = 0$
$0 = \int_{t_{i-1}}^{t_{i+n+1}} \left(\frac{(n+1)B_i^n(x)}{t_{i+n}-t_{i-1}} - \frac{(n+1)B_{i+1}^n}{t_{i+n+1}-t_i}\right) dx = \int_{t_{i-1}}^{t_{i+n+1}} \frac{(n+1)B_i^n(x)}{t_{i+n}-t_{i-1}}dx - \int_{t_{i-1}}^{t_{i+n+1}} \frac{(n+1)B_{i+1}^n}{t_{i+n+1}-t_i}dx$
$= (n+1)\left(\int_{t_{i-1}}^{t_{i+n}} \frac{B_i^n(x)}{t_{i+n}-t_{i-1}}dx - \int_{t_i}^{t_{i+n+1}} \frac{B_{i+1}^n}{t_{i+n+1}-t_i}dx\right)$ (The function value is 0 outside the support set)

Thus,

$0 = \int_{t_{i-1}}^{t_{i+n}} \frac{B_i^n(x)}{t_{i+n}-t_{i-1}}dx - \int_{t_i}^{t_{i+n+1}} \frac{B_{i+1}^n}{t_{i+n+1}-t_i}dx$
$\int_{t_{i-1}}^{t_{i+n}} \frac{B_i^n(x)}{t_{i+n}-t_{i-1}}dx = \int_{t_i}^{t_{i+n+1}} \frac{B_{i+1}^n}{t_{i+n+1}-t_i}dx$

The scaled integral of a B-spline over its support is independent of its index i even if the spacing of the knots is not uniform.

VIII

(a) Prove for m=4,n=2.

$$\forall m \in \mathbb{N}^+, \forall i \in \mathbb{N}, \forall n = 0, 1, \ldots, m, \tau_{m-n}(x_i, \ldots, x_{i+n}) = [x_i, \ldots, x_{i+n}]x^m.$$

$\tau_2(x_i, \ldots, x_{i+2}) = x_i^2 + x_{i+1}^2 + x_{i+2}^2 + x_ix_{i+1} + x_ix_{i+2} + x_{i+1}x_{i+2}$

Construct divided difference

x_i, x_i^4
$x_{i+1}, x_{i+1}^4, \frac{x_{i+1}^4-x_i^4}{x_{i+1}-x_i}$

$x_{i+2}, x_{i+2}^4, \frac{x_{i+2}^4 - x_{i+1}^4}{x_{i+2} - x_{i+1}}, \frac{\frac{x_{i+2}^4 - x_{i+1}^4}{x_{i+2} - x_{i+1}} - \frac{x_{i+1}^4 - x_i^4}{x_{i+1} - x_i}}{x_{i+2} - x_i}$

At this time $[x_i, \ldots, x_{i+n}]x^m = \frac{\frac{x_{i+2}^4 - x_{i+1}^4}{x_{i+2} - x_{i+1}} - \frac{x_{i+1}^4 - x_i^4}{x_{i+1} - x_i}}{x_{i+2} - x_i} = \frac{(x_{i+2} - x_i)(x_{i+1}(x_{i+2} + x_i) + x_{i+1}^2 + x_{i+2}^2 + x_{i+2}x_i + x_i^2)}{x_{i+2} - x_i}$

$= \tau_2(x_i, \ldots, x_{i+2})$ is valid.

(b)

For any m.

For n=0,

$\tau_m(x_i) = x_i^m = [x_i]x^m$ is valid.

Assume it holds for n=$k < m$

For n=k+1, by the recursive formula

$(x_{i+k+1})\tau_{m-k-1}(x_i, \ldots, x_{i+k+1}) = (\tau_{m-k}(x_i, \ldots, x_{i+k+1}) - \tau_{m-k}(x_i, \ldots, x_{i+k}))$

$(x_i)\tau_{m-k-1}(x_i, \ldots, x_{i+k+1}) = (\tau_{m-k}(x_i, \ldots, x_{i+k+1}) - \tau_{m-k}(x_{i+1}, \ldots, x_{i+k+1}))$ Subtract.

$(x_{i+k+1} - x_i)\tau_{m-k-1}(x_i, \ldots, x_{i+k+1}) = -\tau_{m-k}(x_i, \ldots, x_{i+k}) + \tau_{m-k}(x_{i+1}, \ldots, x_{i+k+1})$

By the induction hypothesis.

$\tau_{m-k}(x_i, \ldots, x_{i+k}) = [x_i \ldots x_{i+k}]x^m, \tau_{m-k}(x_{i+1}, \ldots, x_{i+k+1}) = [x_{i+1} \ldots x_{i+k+1}]x^m$

Substitute $\tau_{m-k-1}(x_i, \ldots, x_{i+k+1}) = \frac{[x_{i+1} \ldots x_{i+k+1}]x^m - [x_i \ldots x_{i+k}]x^m}{(x_{i+k+1} - x_i)} = [x_i \ldots x_{i+k+1}]x^m$ is valid.

So it holds.

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Use GPT-4 for quick template transformation, and use Kimi AI to correct English grammar.