Numerical Analysis Homework 1

罗开诚 Luo Kaicheng 3220103383 *

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摘要

Solutions to various numerical analysis problems.

I

1.We have $f(x_0) = 1, f(x_1) = 1/2$,using interpolation method we got $p_1(f;x) = -\frac{1}{2}x + \frac{3}{2}$
so $f(x) - p_1(f;x) = \frac{1}{x} + \frac{x}{2} - \frac{3}{2} = \frac{f''(\xi(x))}{2}(x-1)(x-2)$
$\xi(x) = (2x)^{\frac{1}{3}}$

2.After extending the domain of ξ continuously from $(x_0, x_1) to [x_0, x_1]$
$\xi(x_0) = lim_{x->x_0}\xi(x) = (2)^{\frac{1}{3}}, \xi(x_1) = lim_{x->x_1}\xi(x) = (4)^{\frac{1}{3}}$
$\xi^{'}(x) = \frac{2}{3} * (2x)^{-\frac{2}{3}} > 0$ in $[x_0, x_1]$
so $\max\xi(x) = (4)^{\frac{1}{3}}$, $\min\xi(x) = (2)^{\frac{1}{3}}$, and $\max f''(\xi(x)) = \max\frac{2}{\xi^3(x)} = 1$

II

Let P_m^+ be the set of all polynomials of degree $\leq m$ that are non-negative on the real line,

$$P_m^+ = \{p : p \in P_m, \forall x \in \mathbb{R}, p(x) \geq 0\}.$$

Find $p \in P_{2n}^+$ such that $p(x_i) = f_i$ for $i = 0, 1, \ldots, n$ where $f_i \geq 0$ and x_i are distinct points on $\mathbb{R}$.
Let

$$l_k(x) = \prod_{i=0;i\neq k}(\frac{x - x_k}{x_k - x_i})^2$$

we have $l_k \in P_{2n}^+$,make $p(x) = \Sigma_{i=0}^{k} f_i * l_k(x)$ witch satisfies $p_(x_i) = f_i, p(x) \in P_{2n}^+$
Note that they are not unique.

III

1.$\forall t \in \mathbb{R}$,有$f[t] = f(t) = e^t$，即n=0时符合条件
若对n=k符合条件，
n=k+1时
$f[t, t+1, ..., t+k+1] = \frac{f[t+1,...,t+k+1] - f[t,t+1,...,t+k]}{t+k+1-t}$

*Electronic address: 3220103383@zju.edu.com

由归纳假设可知$f[t+1,...,t+k+1]=\frac{(e-1)^k}{k!}e^{t+1}$
$f[t,...,t+k]=\frac{(e-1)^k}{k!}e^t$
得$f[t,t+1,...,t+k+1]=\frac{(e-1)^k*(e^{t+1}-e^t)}{k!*(k+1)}=\frac{(e-1)^{k+1}}{(k+1)!}e^t$成立。
由归纳法知结论成立.

2.由(1)可知， $\forall t\in\mathbb{R},\quad f[t,t+1,\ldots,t+n]=\frac{(e-1)^n}{n!}e^t.$
即 $\quad f[0,1,\ldots,n]=\frac{(e-1)^n}{n!}.$
带入得$\frac{(e-1)^n}{n!}=\frac{1}{n!}f^{(n)}(\xi)$
$\xi=n*ln(e-1)$
ln(e-1)>0.5,在$\frac{n}{2}$右侧

IV

1.构建divided diffrence如下：
x f
0 5
1 3 -2
3 5 1 1
4 12 7 2 0.25
得到结果 $p_3(f;x)=5-2x+x(x-1)+\frac{1}{4}x(x-1)(x-3)$
$p_3(f;x)=\frac{1}{4}x^3-\frac{9}{4}x+5$

2.用插值多项式的最小值点估计f最小值点。
$p_3(f;x)'=\frac{3}{4}x^2-\frac{9}{4}$
可知$p_3(f;x)$在$[-\sqrt{3},\sqrt{3}]$上递减，在$[\sqrt{3},\infty)$上递增，最小值点估计为$\sqrt{3}$

V

1.
同样构建divided diffrence x
0: 0
1: 1 1
1: 1 7 6
1: 1 7 21 15
2: 128 127 120 99 42
2: 128 448 321 201 102 30
$f[0,1,1,1,2,2]=30$
2.
$f^{(5)}(x)=2520*x^2$
带入$2520*x^2=30$,得$x=\sqrt{\frac{1}{84}}$

VI

f is a function on $[0,3]$ for which one knows that

$$f(0)=1, \quad f(1)=2, \quad f'(1)=-1, \quad f(3)=f'(3)=0.$$

- Estimate $f(2)$ using Hermite interpolation.
- Estimate the maximum possible error of the above answer if one knows, in addition, that $f \in \mathcal{C}^5[0,3]$ and $|f^{(5)}(x)| \le M$ on $[0,3]$. Express the answer in terms of M.

1.
构建divided diffrence: x
0: 1
1: 2 1
1: 2 -1 -2
3: 0 -1 0 0.666667
3: 0 0 0.5 0.25 -0.138889
$p_4(f;x) = \frac{-5}{36}(x-3)(x-1)^2x + \frac{2}{3}x(x-1)^2 - 2x(x-1) + x + 1$
带入有$f(2) \approx \frac{11}{18}$
2.
由Theorem2.37
For the Hermite interpolation problem, denote $N = k + m_i$. Denote by $p(f;x)$ the unique element of P_n 满足条件. Suppose $f^{(N+1)}(r)$ exists in (a,b). Then there exists some $\xi \in (a,b)$ such that
$f(x) - p_N(f;x) = \frac{f^{(N+1)}(\xi)}{(N+1)!}\prod_{i=0}^{k}(x-x_i)^{m_i+1}$
在这个题目中，N取4，$f^{(5)}(r)$ exists in $(0,3)$
$|f(x) - p_4(f;x)| = |\frac{f^{(5)}(\xi)}{5!}x(x-1)^2(x-3)^2| <= \frac{M}{5!}x(x-1)^2(x-3)^2$
令$g(x) = x(x-1)^2(x-3)^2$
$g'(x) = 5(x - \frac{6-\sqrt{21}}{5})(x-1)(x - \frac{6+\sqrt{21}}{5})(x-3)$
$g(x) <= max\{g(\frac{6-\sqrt{21}}{5}), g(\frac{6+\sqrt{21}}{5})\} x \in [0,3]$
得误差$|f(x) - p_4(f;x)| <= M\frac{4896+336\sqrt{21}}{37500}$
取$x = \frac{6+\sqrt{21}}{5}$且取此x时Theorem2.37对应得$\xi$有$f^{(5)}(\xi) = M$得到该maximum possible error

VII

用归纳法，对于式子1，
有对任意x,if k=1：
$\Delta f(x) = f(x+h) - f(x) = h\frac{f(x+h)-f(x)}{h} = 1!hf[x_0, x_1]$成立，
假设上式对任意x,k=n成立，k=n+1时：
$\Delta^{n+1}f(x) = \Delta\Delta^n f(x) = \Delta^n f(x+h) - \Delta^n f(x)$
由归纳假设，取k=n，有
$\Delta^n f(x) = n!h^n f[x_0, x_1, \ldots, x_n]$
因为按照归纳假设，k=n时结论对任意x成立，用x+h代替x，可得
$\Delta^n f(x+h) = n!h^n f[x_1, x_2, \ldots, x_{n+1}]$
带入

$\Delta^{n+1}f(x) = \Delta\Delta^n f(x) = \Delta^n f(x+h) - \Delta^n f(x) =$
$n!h^n(f[x_1, x_2, \dots, x_{n+1}] - f[x_0, x_1, \dots, x_n]) = (n+1)!h^{n+1}\frac{f[x_1,x_2,\dots,x_{n+1}]-f[x_0,x_1,\dots,x_n]}{(n+1)h}$
$= (n+1)!h^{n+1}f[x_0, x_1, \dots, x_{n+1}]$成立
由归纳法知结论成立

类似地证明第二个式子
有对任意x,if k=1：
$\nabla f(x) = f(x) - f(x-h) = h\frac{f(x)-f(x-h)}{h} = 1!hf[x_0, x_{-1}]$成立，
假设上式对任意x,k=n成立，k=n+1时：
$\nabla^{n+1}f(x) = \nabla\nabla^n f(x) = \nabla^n f(x) - \nabla^n f(x-h)$
由归纳假设，取k=n，有
$\nabla^n f(x) = n!h^n f[x_0, x_{-1}, \dots, x_{-n}]$
因为按照归纳假设，k=n时结论对任意x成立，用x-h代替x，可得
$\nabla^n f(x-h) = n!h^n f[x_{-1}, x_{-2}, \dots, x_{-(n+1)}]$
带入
$\nabla^{n+1}f(x) = \nabla\nabla^n f(x) = \nabla^n f(x) - \nabla^n f(x-h) =$
$n!h^n(f[x_0, x_{-1}, \dots, x_{-n}] - f[x_{-1}, x_{-2}, \dots, x_{-(n+1)}]) = (n+1)!h^{n+1}\frac{f[x_0,x_{-1},\dots,x_{-n}]-f[x_{-1},x_{-2},\dots,x_{-(n+1)}]}{(n+1)h}$
$= (n+1)!h^{n+1}f[x_0, x_{-1}, \dots, x_{-n+1}]$成立

VIII

Assume f is differentiable at x_0. Prove

$$\frac{\partial}{\partial x_0}f[x_0, x_1, \dots, x_n] = f[x_0, x_0, x_1, \dots, x_n].$$

What about the partial derivative with respect to one of the other variables? 从偏导定义出发，考虑
$lim_{\epsilon->0}\frac{f[x_0+\epsilon,x_1,\dots,x_n]-f[x_0,x_1,\dots,x_n]}{\epsilon}$
由最小度数插值多项式的唯一性，可以更改顺序
原式$=lim_{\epsilon->0}\frac{f[x_1,\dots,x_n,x_0+\epsilon]-f[x_0,x_1,\dots,x_n]}{\epsilon}$
$=lim_{\epsilon->0}\frac{f[x_1,\dots,x_n,x_0+\epsilon]-f[x_0,x_1,\dots,x_n]}{x_0+\epsilon-x_0} = f[x_0, x_1, \dots, x_n, x_0 + epsilon]$
$=lim_{\epsilon->0}f[x_0 + \epsilon, x_0, x_1, \dots, x_n]$
即证$lim_{\epsilon->0}f[x_0 + \epsilon, x_0, x_1, \dots, x_n] = f[x_0, x_0, x_1, \dots, x_n]$
可知
$lim_{\epsilon->0}f[x_0 + \epsilon, x_0] = lim_{\epsilon->0}\frac{f(x_0+\epsilon)-f(x_0)}{\epsilon} = f'(x_0) = f[x_0, x_0]$
$lim_{\epsilon->0}f[x_0 + \epsilon, x_0, x_1] = lim_{\epsilon->0}\frac{f[x_0+\epsilon,x_0]-f[x_0,x_1]}{\epsilon+x_0-x_1} = \frac{lim_{\epsilon->0}(f[x_0+\epsilon,x_0]-f[x_0,x_1])}{lim_{\epsilon->0}(\epsilon+x_0-x_1)} = f[x_0, x_0, x_1]$(分子分母，关于$\epsilon->0$极限均存在，且$x_1 \neq x_0$,否则应该要求在$x_0$处二阶可微)
递推$lim_{\epsilon->0}f[x_0 + \epsilon, x_0, x_1, \dots, x_k] = f[x_0, x_0, x_1, \dots, x_k]$对k=m成立，k=m+1时
$lim_{\epsilon->0}f[x_0 + \epsilon, x_0, x_1, ...x_{m+1}] = lim_{\epsilon->0}\frac{f[x_0+\epsilon,x_0\dots,x_m]-f[x_0,x_1,\dots x_{m+1}]}{\epsilon+x_0-x_{m+1}}$(由k=m时得结论，$f[x_0 + \epsilon, x_0..., x_m]$关于$lim_{\epsilon->0}$极限存在,且$x_{m+1} \neq x_0$,否则应该要求在$x_0$处二阶可微，故有)
$= \frac{lim_{\epsilon->0}(f[x_0+\epsilon,x_0\dots,x_m]-f[x_0,x_1\dots,x_{m+1}])}{lim_{\epsilon->0}(\epsilon+x_0-x_{m+1})} = f[x_0, x_0, x_1, ...x_{m+1}]$成立
递推至m=n即可
故上述结论成立
若考虑对其他变量，若f is differentiable at x_i.
$\frac{\partial}{\partial x_i}f[x_0, x_1, ...x_i, ..., x_n]$

$=\frac{\partial}{\partial x_i}f[x_i,x_1,...x_0,...,x_n]$
$=lim_{\epsilon->0}f[x_i+\epsilon,x_i,x_1,\ldots,x_n]$用同样方法可证明
$=f[x_i,x_i,x_1,\ldots,x_n]=f[x_1,x_2,...x_i,x_i,\ldots,x_n]$

IX

令$t=\frac{x-\frac{a+b}{2}}{\frac{b-a}{2}}$
$x=\frac{b-a}{2}t+\frac{b+a}{2}$
$\min\max_{x\in[a,b]}|a_0x^n+a_1x^{n-1}+\cdots+a_n|$
$=\min\max_{t\in[-1,1]}|a_0(\frac{b-a}{2})^nt^n+a_1't^{n-1}+\cdots+a_n'|$
$=\min\max_{t\in[-1,1]}a_0(\frac{b-a}{2})^n|t^n+\frac{a_1'}{a_0(\frac{b-a}{2})^n}*t^{n-1}+\cdots+\frac{a_n'}{a_0(\frac{b-a}{2})^n}|$
由Theorem 2.47:
Denote by $\tilde{P}_n$ the class of all polynomials of degree $n\in\mathbb{N}^+$ with leading coefficient 1. Then

$$\forall p\in\tilde{P}_n,\quad \max_{x\in[-1,1]}\left|\frac{T_n(x)}{2^{n-1}}\right|\le\max_{x\in[-1,1]}|p(x)|. \qquad (2.45)$$

简略重述Proof. By Theorem 2.45, $T_n(x)$ assumes its extrema $n+1$ times at the points x_k'. Suppose 上述结论 does not hold. Then

$$\exists p\in\tilde{P}_n\quad \text{s.t.}\quad \max_{x\in[-1,1]}|p(x)|<\frac{1}{2^{n-1}}.$$

Then the polynomial $Q(x)=\frac{1}{2^{n-1}}T_n(x)-p(x)$满足：

$$Q(x_k')=\frac{(-1)^k}{2^{n-1}}-p(x_k'),\quad k=0,1,\ldots,n.$$

$Q(x)$ 在 $n+1$ points依次变换符号，至少有n个零点. 但是 $\frac{1}{2^{n-1}}T_n(x)$和p(x)的n次项系数相同, the degree of $Q(x)$ is at most $n-1$. Therefore, $Q(x)\equiv 0$ and $p(x)=\frac{1}{2^{n-1}}T_n(x)$, which implies $\max|p(x)|=\frac{1}{2^{n-1}}$，矛盾.
故
$\min\max_{t\in[-1,1]}|t^n+\frac{a_1'}{a_0(\frac{b-a}{2})^n}*t^{n-1}+\cdots+\frac{a_n'}{a_0(\frac{b-a}{2})^n}|=\frac{1}{2^{n-1}}$
$\min\max_{x\in[a,b]}|a_0x^n+a_1x^{n-1}+\cdots+a_n|=a_0(\frac{b-a}{2})^n*\frac{1}{2^{n-1}}$
当$a_0x^n+a_1x^{n-1}+\cdots+a_n=a_0(\frac{b-a}{2})^n*\frac{1}{2^{n-1}}T(\frac{x-\frac{a+b}{2}}{\frac{b-a}{2}})$取等

X

证明$\forall p\in P_n^a\quad \|\hat{p}_n\|_\infty\le\|p\|_\infty$
即证$\forall p\in P_n^a\ max_{x\in[-1,1]}|T_n(x)|\le max_{x\in[-1,1]}|T(a)||p(x)|$
$\forall p\in P_n^a$
已知$max_{x\in[-1,1]}|T_n(x)|=1$ 由于$T(n)\in P_n$,且T(n)在[-1,1]上已经有n个零点,a¿1，故$T(a)\neq 0$
若结论不成立
$\exists p\in P_n^a\quad \text{s.t.}\quad \max_{x\in[-1,1]}|p(x)|<\frac{1}{T(a)}$. 取这个p,令：
$Q(x)=T_n(x)-T(a)p(x)$ Q(a)=0.
By Theorem 2.45, $T_n(x)$ assumes its extrema $n+1$ times at the points x_k',这些点上的函数值依次为1或-1交替
故$Q(x)$ 在 $n+1$ points依次变换符号，至少有n个零点，考虑Q(a)=0,$a\notin[-1,1]$，故Q(x)有至少n+1个0点，但$Q(x)\in P_n$,故Q(x)=0恒成立。
$p(x)=\frac{T_n(x)}{T(a)}$,$\max_{x\in[-1,1]}|p(x)|=\frac{1}{T(a)}$,矛盾

故结论成立

XI

由定义

$b_{n,k}(t) := \binom{n}{k} t^k (1-t)^{n-k}$

$b_{n,k+1}(t) := \binom{n}{k+1} t^{k+1} (1-t)^{n-k-1}$

带入得

$\frac{n-k}{n} b_{n,k}(t) + \frac{k+1}{n} b_{n,k+1}(t) = \binom{n}{k} \frac{n-k}{n} t^k (1-t)^{n-k} + \frac{k+1}{n} \binom{n}{k+1} t^{k+1} (1-t)^{n-k-1}$

$= t^k (1-t)^{n-k-1} ((1-t) \frac{n-k}{n} \binom{n}{k} + t \frac{k+1}{n} \binom{n}{k+1})$

又： $(1-t) \frac{n-k}{n} \binom{n}{k} + t \frac{k+1}{n} \binom{n}{k+1} = (1-t) \frac{n-k}{n} \frac{n!}{k!(n-k)!} + t \frac{k+1}{n} \frac{n!}{(k+1)!(n-k-1)!} = (1-t+t) \frac{(n-1)!}{k!(n-1-k)!} = \binom{n-1}{k}$

综上：

$\frac{n-k}{n} b_{n,k}(t) + \frac{k+1}{n} b_{n,k+1}(t) = \binom{n-1}{k} t^k (1-t)^{n-k-1} = b_{n-1,k}(t)$

XII

需证明：

$\forall k = 0, 1, \ldots, n, \quad \int_0^1 b_{n,k}(t)\, dt = \frac{1}{n+1}.\ b_{n,k}(t) := \binom{n}{k} t^k (1-t)^{n-k}$

$\int_0^1 b_{n,k}(t)\, dt = \binom{n}{k} \quad \int_0^1 t^k (1-t)^{n-k}\, dt$

若k=0

原式=$\int_0^1 t^n\, dt = \frac{1}{n+1}$成立

若k=n

原式=$\int_0^1 (1-t)^n\, dt = \frac{1}{n+1}$也成立

$k \geq 1$且$k \leq n-1$时

$\int_0^1 t^k (1-t)^{n-k}\, dt = \frac{1}{k+1} t^{k+1} (1-t)^{n-k} |_0^1 - (-1) \int_0^1 \frac{n-k}{k+1} t^{k+1} (1-t)^{n-k-1}\, dt$

=$\frac{n-k}{k+1} \int_0^1 t^{k+1} (1-t)^{n-k-1}\, dt$

重复，直到(1-t)的指数为1

=$\frac{(n-k)(n-k-1)\ldots 2}{(k+1)(k+2)\ldots(n-1)} \int_0^1 t^{n-1} (1-t)\, dt$

=$\frac{(n-k)(n-k-1)\ldots 2}{(k+1)(k+2)\ldots(n-1)} * (\frac{1}{n} - \frac{1}{n+1})$

=$1/\binom{n}{k} \frac{1}{n+1}$

故 $\int_0^1 b_{n,k}(t)\, dt = \binom{n}{k} \quad \int_0^1 t^k (1-t)^{n-k}\, dt = \frac{1}{n+1}$

结论成立

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