Time Series Exam 1

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1.

a. If $\{W_t\}$=$\nabla^d Y_t$ is a stationary ARMA(p,q) process, then

$$W_t = \phi_1 W_{t-1} + \phi_2 W_{t-2} + \ldots + \phi_p w_{t-p} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q}$$

and $\{Y_t\}$ is an ARIMA(p,d,q) process with

$$Y_t = (Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2}) - \ldots - (Y_{t-p} - Y_{t-p-d})$$

We assume there are no common factors between the AR and MA characteristic polynomials of $\{W_t\}$.

b. For an ARIMA(p,d,q) process,

$$\phi(B)(1-B)^d Y_t = \theta(B)e_t$$

so

$$Y_t = \frac{\theta(B)e_t}{\phi(B)(1-B)^d}$$

assuming there are no common factors between $\phi(B)$ and $\theta(B)$.

c. On Invertibility: Assuming an ARIMA(p,d,q) with q>0, th process is invertible if and only if all roots of the MA characteristic equation exceed one in absolute value (lie outside the unit circle).
On Stationarity: For p,d=0, the ARIMA process is an MA process, which is automatically stationary. For d=0 and p,q>0, an ARIMA process is simply an ARMA process and it is stationary if all roots of the AR characteristic polynomial lie outside the unit circle. For cases in which d>0, an ARIMA process, $\{Y_t\}$ is by definition nonstationary, however its d^{th} difference, $\nabla^d Y_t$, is a stationary ARMA(p,q) process.

2.

$$Y_t = 1.6Y_{t-1} - .68Y_{t-2} + .08Y_{t-3} + e_t - .8e_{t-1} - .12e_{t-2}$$

a. Since the AR and MA characteristic polynomials have a common factor, $(1 - .2B)$, in their reduced forms:

$$\phi(B) = (1-B)(1-.4B)$$

and

$$\theta(B) = (1-.6B)$$

so $\{Y_t\}$ is an ARIMA(1,1,1) process.

b. The ARIMA(1,1,1) process in (a) is by definition nonstationary, since it has a root=1 which lies on the unit circle. The remaining AR root=2.5 is the root of the AR characteristic function of $\nabla Y_t = W_t = .4W_{t-1} + e_t - .6e_{t-1}$ which is a stationary ARMA(1,1) process. The MA root=1.67 lies outside the unit circle, indicating that the process is invertible.

3.

a. In terms of Y_t the process is nonstationary, so there is no MA representation of it. The MA representation of the ARMA(1,1) process, W_t is:

$$W_t = \psi(B)e_t$$

$$= \sum_{j=0}^{\infty} \psi_j e_{t-j}$$

$$= \sum_{j=0}^{1} \psi_j e_{t-j}$$

with $\theta_1 = .6$ and$\phi_1 = .4$

$$\psi_0 = 1$$

$$\psi_1 = -\theta_1 + \phi_1 = -.2$$

$\implies W_t = e_t - .2e_{t-1}$

b. The series is invertible both in terms of Y_t and W_t . The AR representation of the ARMA(1,1) process, W_t is:

$$\pi(B)W_t = e_t$$

$$= \sum_{i=0}^{\infty} \pi_j W_{t-j}$$

with $\theta_1 = .6$ and $\phi_1 = .4$

$$\pi_0 = 1$$

$$\pi_1 = -\theta_1 + \phi_1 = -.2$$

$\implies e_t = W_t + .2W_{t-1}$

4.

$$Y_t = e_t - .6e_{t-1} + .05e_{t-2}$$

a. This is an MA(2) process and is as such automatically stationary.

b.

$$\gamma_k = Cov(Y_t, Y_{t-1})$$

$$= Cov(e_t - .6e_{t-1} + .05e_{t-2}, e_{t-k} - .6e_{t-k-1} + .05e_{t-k-2})$$

$$= Cov(e_t, e_{t-k}) - .6Cov(e_{t-1}, e_{t-k}) + .36Cov(e_{t-1}, e_{t-k-1}) + .05Cov(e_{t-2}, e_{t-k}) - .03Cov(e_{t-2}, e_{t-k-1}) + .0025Cov(e_{t-2}, e_{t-k-2}$$

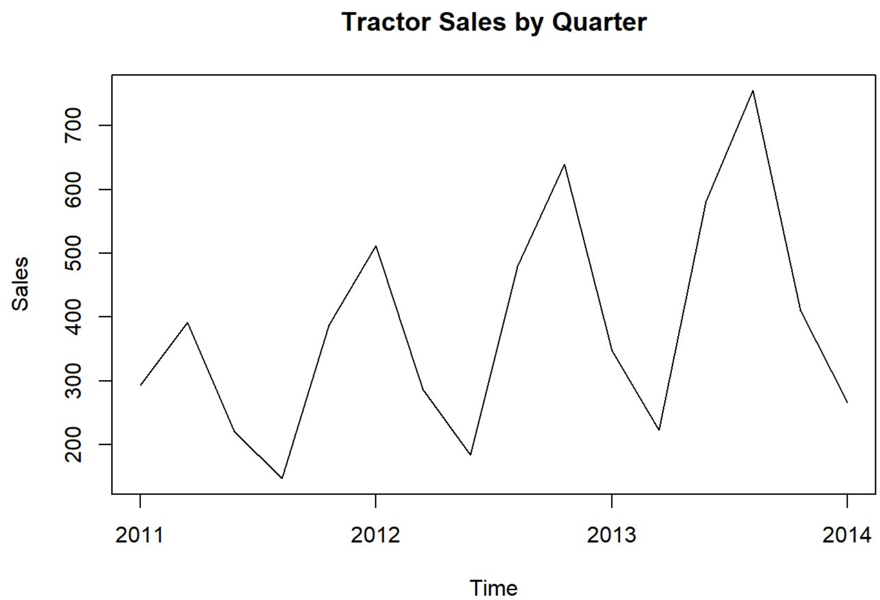
$\gamma_0 = \sigma_e^2 + .36\sigma_e^2 + .0025\sigma_e^2 = 1.3625\sigma_e^2 = 6.8125$
$\gamma_1 = -.6\sigma_e^2 - .03\sigma_e^2 = -.63\sigma_e^2 = -3.15$
$\gamma_2 = .05\sigma_e^2 = .25$
$\gamma_k = 0, k > 2$

$\rho_0 = \frac{\gamma_0}{\gamma_0} = 1$
$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{-3.15}{6.8125} = -.462385$
$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{.25}{6.8125} = .036697$

c. for $j \geq 3, \gamma_j = 0$ so $\rho_j = \frac{\gamma_j}{\gamma_0} = 0$

5. (See Excel spreadsheet on final page of appendix)

a. The plot exhibits nonconstant mean and variance. The series is nonstationary.



b. The seasonal components for quarters 1-4 are: $sn_1 = 1.191412$
   $sn_2 = 1.521231$
   $sn_3 = 0.803606$
   $sn_4 = 0.483751$
   There does appear to be a seasonal effect with the highest sales tending to occur in Q2 and the lowest in Q4.

c.

$$\hat{Y}_t = 415.1094 + 13.80869t$$

d. Sales are increasing over time in both mean and variance while continuing to follow a seasonal trend.

$$\hat{Y}_{17} = 666.818$$

with 95% $PI = (652.7854, 680.8505)$

$$\hat{Y}_{18} = 881.7619$$

with 95% $PI = (867.441, 896.0828)$

$$\hat{Y}_{19} = 481.8318$$

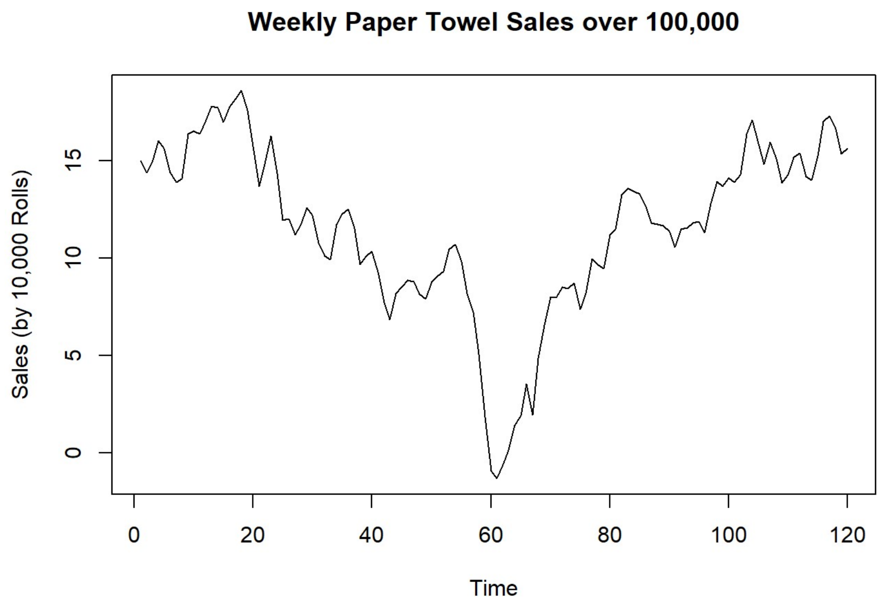
with 95% $PI = (467.1971, 496.4665)$

$$\hat{Y}_{20} = 299.7015$$

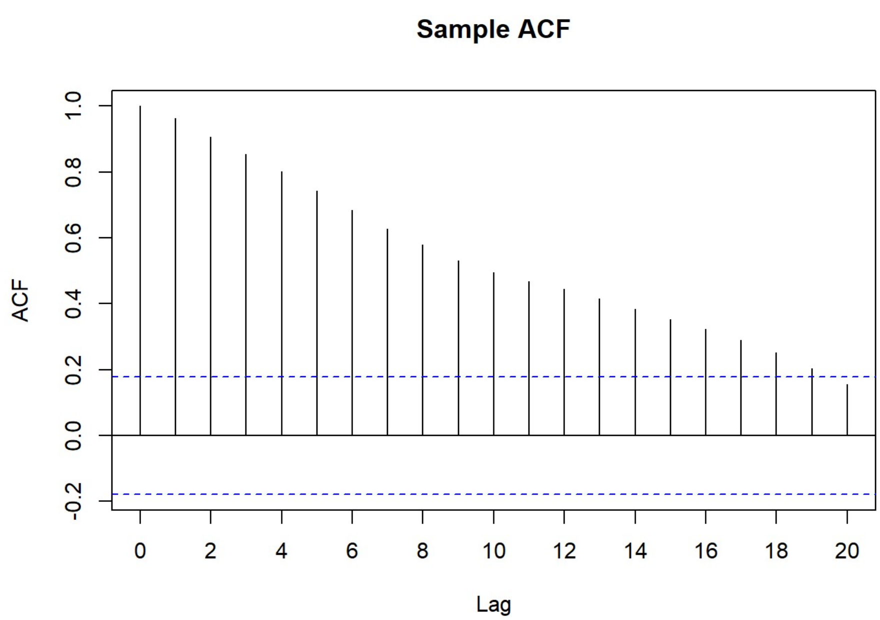
with 95% $PI = (284.7293, 314.6737)$

6.

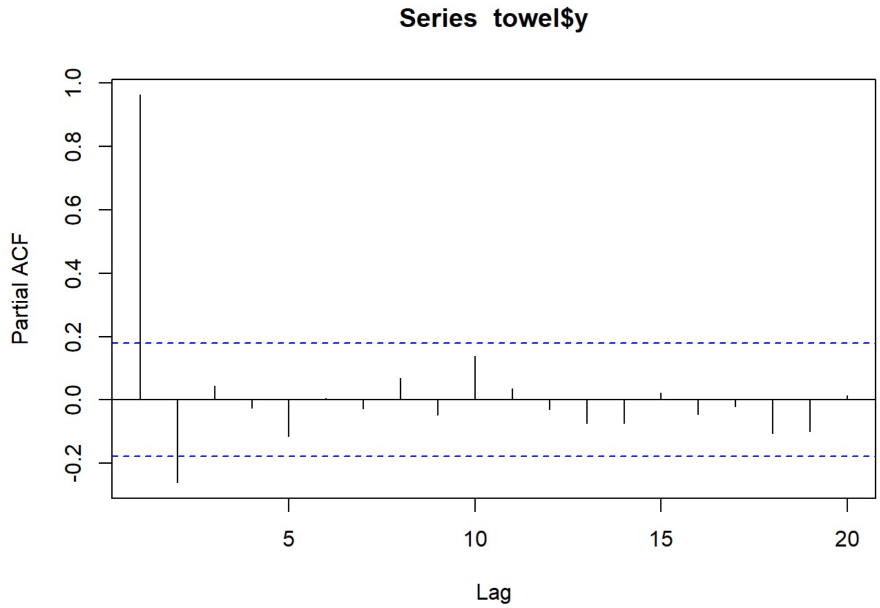
a. The time series plot shows that the series has a nonconstant mean and nonconstant variance. The sample ACF plot does not appear to die out. Both plots indicate that the series is nonstationary.



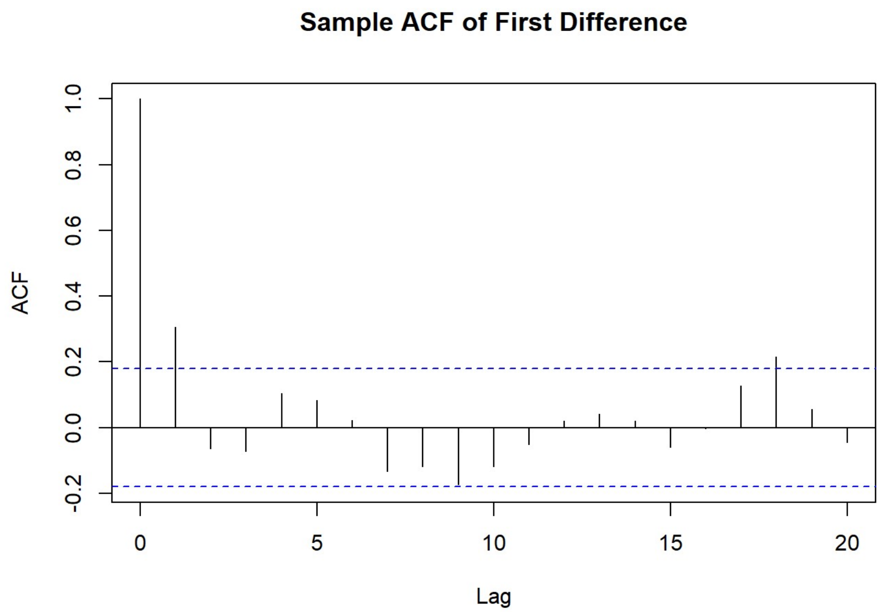
Weekly Paper Towel Sales over 100,000
Sales (by 10,000 Rolls)
0
5
10
15
0
20
40
60
80
100
120
Time

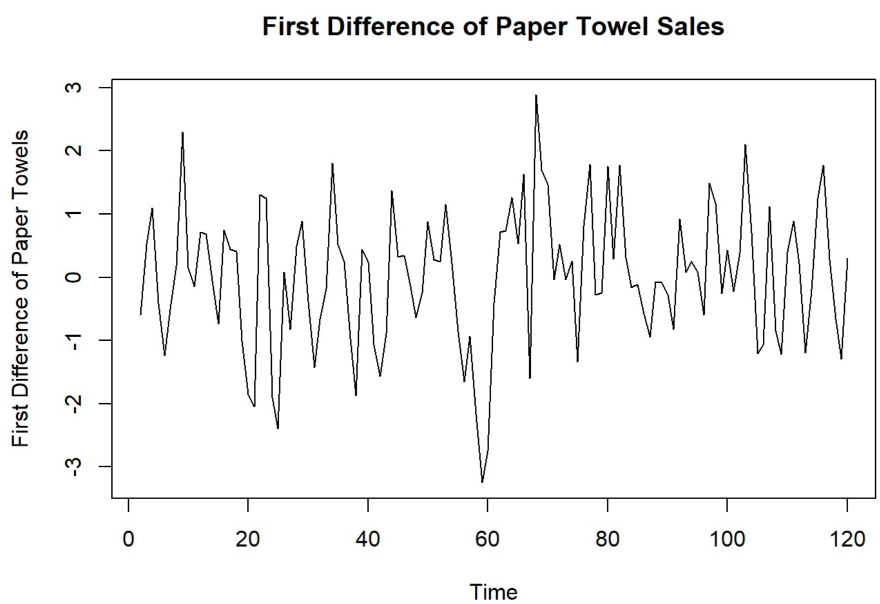


Sample ACF
ACF
-0.2
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
14
16
18
20
Lag

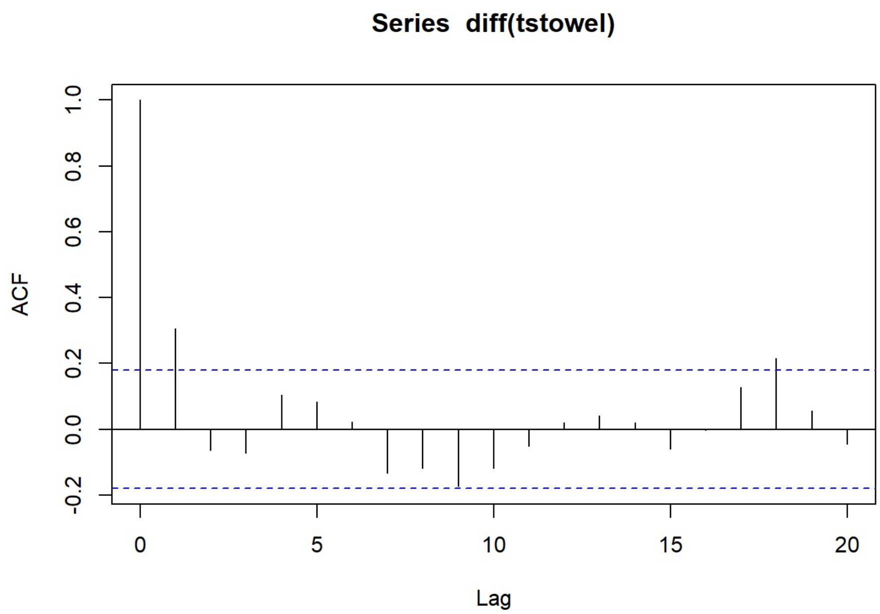

Series towel$y
Partial ACF
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
5
10
15
20
Lag

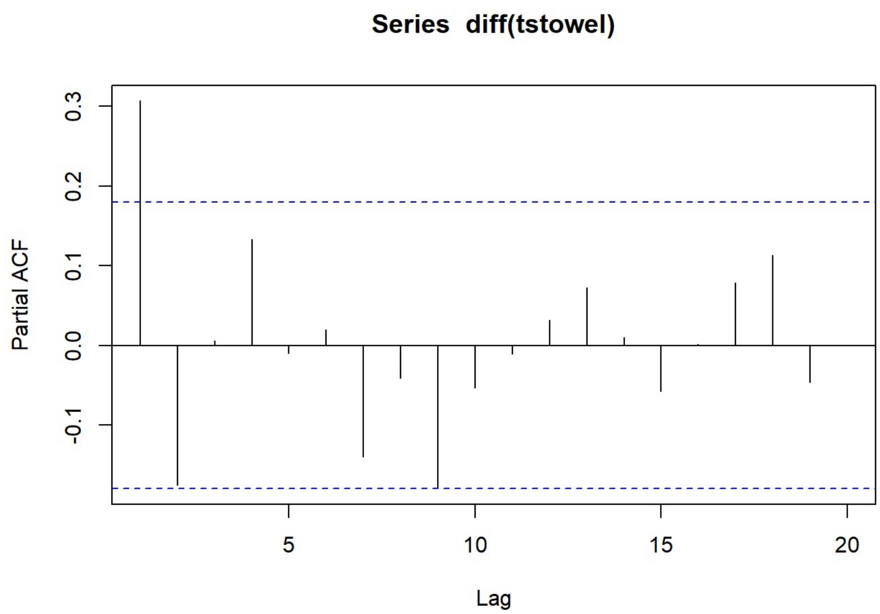
b. The first difference appears to be stationary. The ACF dies off quicly and the time series plot exhibits white noise behavior.





c. The ACF of the sample first difference does die off and implies stationarity. Since both the ACF and PACF die off, an appropriate model is not immediately apparent. An ARI(1,1) model seems appropriate since the original sample PACF did die of and the sample ACF did not, which is indicative of an AR process.





d. The best model (by lowest AIC) is an IMA(1,1) with estimated model:

$$Y_t = Y_{t-1} + e_t + 0.3518e_{t-1}$$

The residuals are approximately normally distributed and exhibit white noise behavior. Additionally, a Ljung-Box test fails to reject the null hypothesis that the model does not show lack of fit.

```
## Registered S3 method overwritten by 'xts':
##   method     from
##   as.zoo.xts zoo
```

```
## Registered S3 method overwritten by 'quantmod':
##   method            from
##   as.zoo.data.frame zoo
```

```
## Registered S3 methods overwritten by 'forecast':
##   method             from
##   fitted.fracdiff    fracdiff
##   residuals.fracdiff fracdiff
```

```
## Loading required package: zoo
```

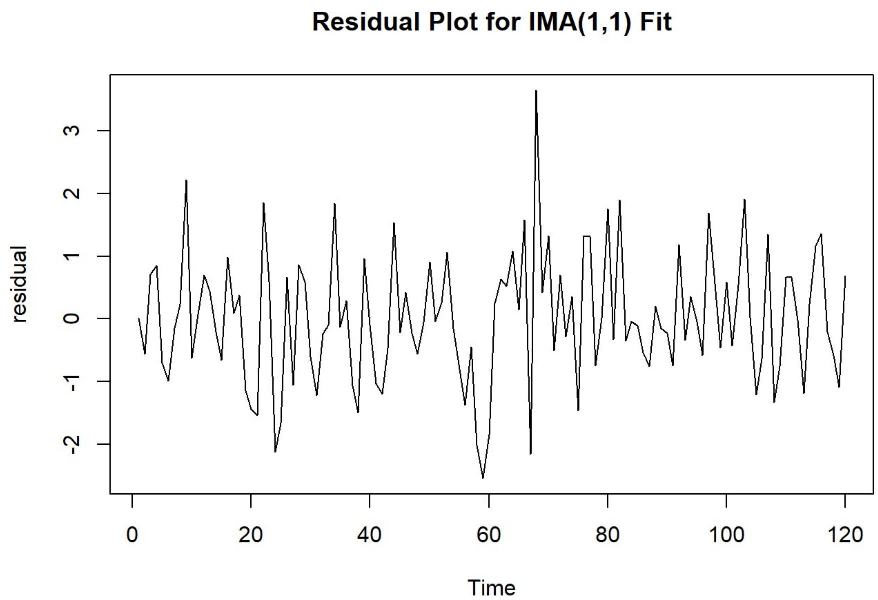
```
##
## Attaching package: 'zoo'
```

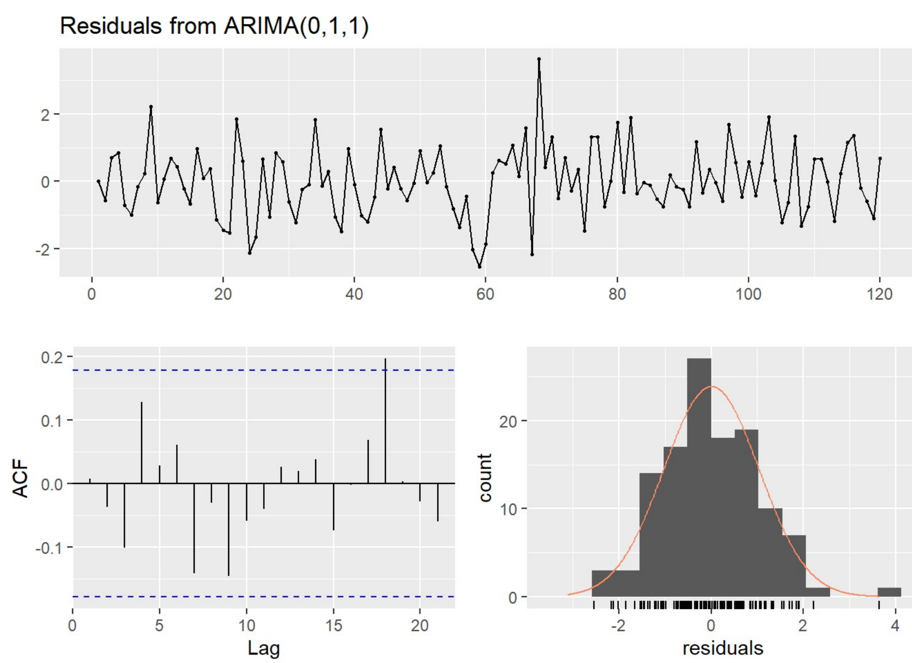
```
## The following objects are masked from 'package:base':
##
##     as.Date, as.Date.numeric
```

```
##
##  ARIMA(2,1,2) with drift         : 356.3176
##  ARIMA(0,1,0) with drift         : 364.3682
##  ARIMA(1,1,0) with drift         : 354.7813
##  ARIMA(0,1,1) with drift         : 352.1823
##  ARIMA(0,1,0)                    : 362.3019
##  ARIMA(1,1,1) with drift         : 354.3083
##  ARIMA(0,1,2) with drift         : 354.2944
##  ARIMA(1,1,2) with drift         : 353.8354
##  ARIMA(0,1,1)                    : 350.0789
##  ARIMA(1,1,1)                    : 352.1679
##  ARIMA(0,1,2)                    : 352.154
##  ARIMA(1,1,0)                    : 352.677
##  ARIMA(1,1,2)                    : 351.6571
##
##  Best model: ARIMA(0,1,1)
```

```
## Series: tstowel
## ARIMA(0,1,1)
##
## Coefficients:
##          ma1
##       0.3518
## s.e.  0.0800
##
## sigma^2 estimated as 1.08:  log likelihood=-172.99
## AIC=349.98   AICc=350.08   BIC=355.53
```

```
##
## z test of coefficients:
##
##     Estimate Std. Error z value  Pr(>|z|)
## ma1 0.351842   0.080047  4.3954 1.106e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```






```
##
##  Ljung-Box test
##
## data:  Residuals from ARIMA(0,1,1)
## Q* = 10.13, df = 9, p-value = 0.3401
##
## Model df: 1.   Total lags used: 10
```

7. ACF for the AR(2) Process (pg. 72 in textbook) $\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}$

a. $Y_t - .4Y_{t-1} - .45Y_{t-2} = e_t$

$\gamma_k = E(Y_tY_{t-k}) = E[(.4Y_{t-1} + .45Y_{t-2} + e_t)Y_{t-k}]$
$\gamma_k = .4\gamma_{k-1} + .45\gamma_{k-2}$, for k>0
dividing through by γ_0 we have: $\rho_k = .4\rho_{k-1} + .45\rho_{k-2}$

From this, we solve for roots of the polynomial equation: $(1 - .4B - .45\ ) = 0$ $B_2\ \rho_k$

$R_1 = B_1^{-1} = .9$ and $R_2 = B_2^{-1} = -.5$ with initial conditions $\rho_0 = 1$ and $\rho_{-1} = \rho_1 = \frac{\phi_1}{1-\phi_2} = 0.7273$ and the difference equations given by $\rho_k = \sum_{i=1}^{n} b_i R_i^k = .9^k b_1 + (-.5)^k b_2$, we solve for b_1 and b_2 .

$$\rho_0 = 1 = b_1 + b_2$$

$$\rho_1 = 0.7273 = .9b_1 - .5b_2$$

$$b_1 = 0.8766$$

$$b_2 = 0.1234$$

Since ρ_0 and ρ_1 are given in the initial conditions, we use b_1 and b_2 to solve for ρ_2 :

$$\rho_2 = .9^2(0.8766) + (-.5)^2(0.1234) = 0.7409$$

```
##         0         1         2
## 1.0000000 0.7272727 0.7409091
```

b. $Y_t - 1.2Y_{t-1} + .85Y_{t-2} = e_t$

$\gamma_k = E(Y_t Y_{t-k}) = E[(1.2Y_{t-1} - .85Y_{t-2} + e_t)Y_{t-k}]$
$\gamma_k = 1.2\gamma_{k-1} - .85\gamma_{k-2}$, for k>0
dividing through by γ_0 we have: $\rho_k = 1.2\rho_{k-1} - .85\rho_{k-2}$
From this, we solve for roots of the polynomial equation: $(1 - 1.2B + .85B^2)\rho_k = 0$
$R_1 = B_1^{-1} = \frac{1}{.7059+.8235i} = \frac{.7059-.8235i}{1.17645} = .6 - .7i$ and $R_2 = B_2^{-1} = .6 + .7i$

$$\alpha = (c^2 + d^2)^{1/2} = (.6^2 + .7^2)^{1/2} = .92195$$

$$\phi = tan^{-1}(d/c) = tan^{-1}(.7/.6) = .86217$$

$$\rho_k = b_1 + b_2(.92195)^k cos(.86217k)$$

with initial conditions $\rho_0 = 1$ and $\rho_{-1} = \rho_1 = \frac{\phi_1}{1-\phi_2} = 0.6486$ and difference equations:

$$\rho_0 = 1 = b_1 + b_2$$

$$\rho_1 = .6486 = b_1 + b_2(.92195)cos(.86217)$$

solving for b_1 and b_2 gives us:

$$b_1 = .1215$$

and

$$b_2 = .8785$$

which gives

$$\rho_2 = .1215 + .8785(.92195^2)cos(2 * .86217) = .0073$$

which means there is an error somewhere in my work because $\rho_2 = -.0716$

```
##          0           1           2
##  1.00000000  0.64864865 -0.07162162
```