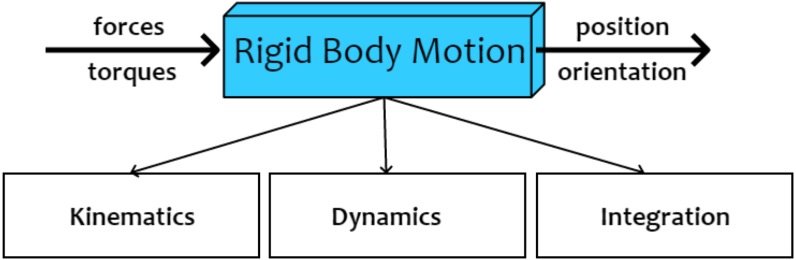
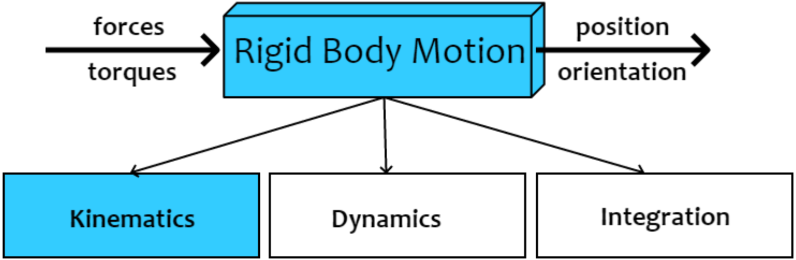
Kinematics


forces
torques
Rigid Body Motion
position
orientation
Kinematics
Dynamics
Integration

Kinematics


forces
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Agenda
Kinematics

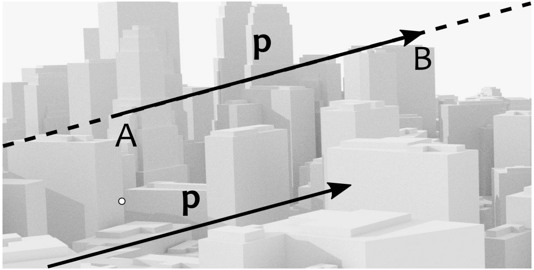
- Vectors and Coordinate Systems
- Translational Motion
- Rotational Motion
- Translation + Rotation

3D Vectors

Kinematics

- most quantities of relevant to motion (forces, torques, linear and angular velocities, linear and angular accelerations) are 3D vectors
- wind velocity, gravity force
- a vector is an object that has magnitude and direction

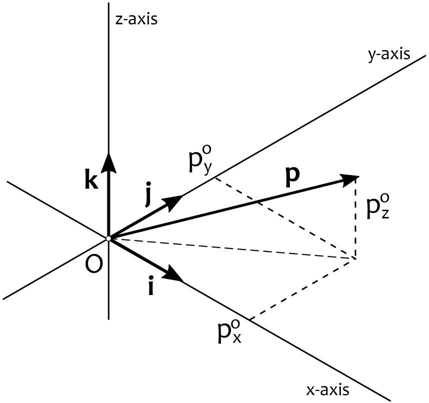
3D Vectors
Kinematics



- a line segment with an initial point and a terminal point
- magnitude is length
- direction is given by the oriented line of the segment or any other parallel to it
- $\boldsymbol{p}$ is a geometric vector

Cartesian Coordinate System

Kinematics



- 3 axes perpendicular to each other, and intersecting at the origin point (RH) with basis vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$
- geometrical vector description

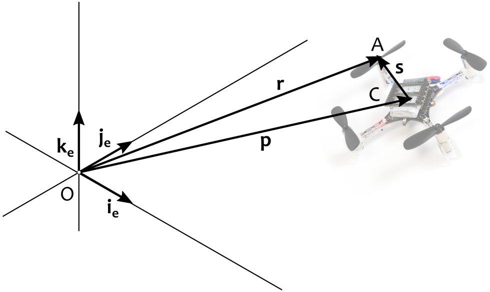
$$\boldsymbol{p} = p_x^0 \boldsymbol{i} + p_y^0 \boldsymbol{j} + p_z^0 \boldsymbol{k}, \tag{1}$$

- algebraic vector description

$$\boldsymbol{p}^0 = \begin{bmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{bmatrix} \boldsymbol{i}^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{j}^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{k}^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2}$$

Translation

Kinematics



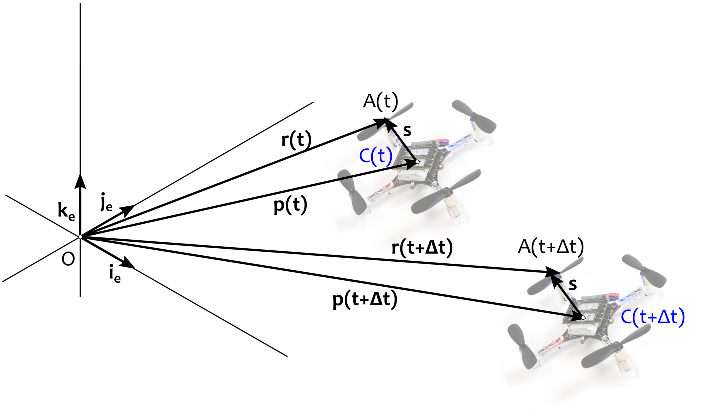
- $\boldsymbol{p}$, position vector of C relative to O
- $\boldsymbol{r}$, position vector of A relative to O
- $\boldsymbol{s}$, position vector of A relative to C

$$\boxed{\boldsymbol{r} = \boldsymbol{p} + \boldsymbol{s}} \tag{3}$$

- Under a translation motion, vector $\boldsymbol{s}$ will be constant.

Translation

Kinematics



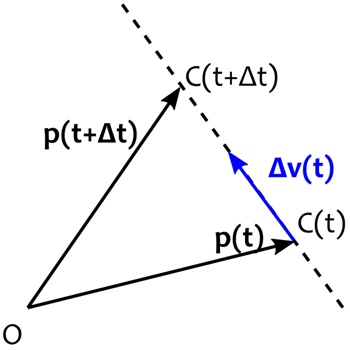
Translation Velocity & Acceleration

Kinematics

- the velocity and acceleration of the quadrotor are the time derivative of vector $\boldsymbol{p}$:
  $\boldsymbol{v}_p(t) \triangleq \dot{\boldsymbol{p}}(t) = \dfrac{d\boldsymbol{p}(t)}{dt} = \lim_{\Delta t \to 0} \dfrac{\boldsymbol{p}(t+\Delta t) - \boldsymbol{p}(t)}{\Delta t}$, [m/s] - linear velocity
  $\boldsymbol{a}_p(t) \triangleq \dot{\boldsymbol{v}}_p(t) = \ddot{\boldsymbol{p}}(t) = \dfrac{d\boldsymbol{v}(t)}{dt} = \dfrac{d^2\boldsymbol{p}(t)}{dt^2}$, [m/s^2] - linear acceleration

Translation Velocity

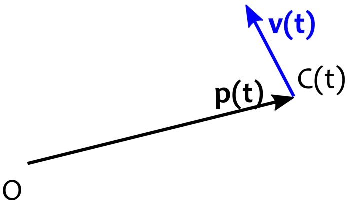
Kinematics



$$\boldsymbol{v}_p(t) \triangleq \lim_{\Delta t \to 0} \frac{p(t+\Delta t) - p(t)}{\Delta t}$$

Translation Velocity

Kinematics



$$\boldsymbol{v}_p(t) \triangleq \lim_{\Delta t \to 0} \frac{p(t+\Delta t) - p(t)}{\Delta t}$$

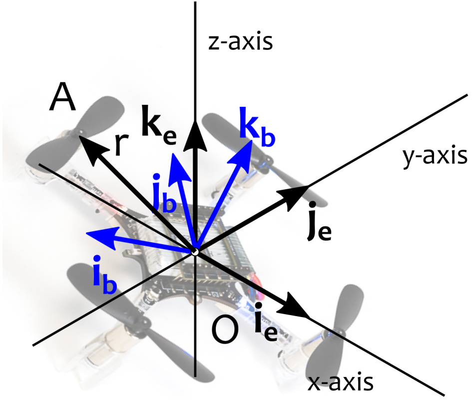
TranslationVelocity & Acceleration

Kinematics

- How does the vector $\boldsymbol{r}$ (position vector of point A) change in time ? Remember that $\boxed{\boldsymbol{r} = \boldsymbol{p} + \boldsymbol{s}}$ and $\boxed{\boldsymbol{s} \text{ is constant}}$.
- $\boldsymbol{v}_r = \dot{\boldsymbol{r}} = \dot{\boldsymbol{p}} + \underbrace{\dot{\boldsymbol{s}}}_{=0} = \dot{\boldsymbol{p}} = \boldsymbol{v}_p, \boxed{\boldsymbol{v}_r = \boldsymbol{v}_p}$
- $\boldsymbol{a}_r = \ddot{\boldsymbol{r}} = \dot{\boldsymbol{v}}_r = \dot{\boldsymbol{v}}_p = \boldsymbol{a}_p, \boxed{\boldsymbol{a}_r = \boldsymbol{a}_p}$
- Under a translation movement, all points of the moving object have the same velocity and acceleration

Rotation

Kinematics



- Let $\boldsymbol{p}(t) = 0$, no translation wrt to the e- frame
- A body CS, with axes "glued" to the quadrotors' fw/bw, left/right and up/down. The b-frame is rotating together with the quadrotor around the origin, its center of mass.
- Position vector $\boldsymbol{r}$ defined by OA, which points to a fixed point on the body-frame
- Velocity and acceleration of vector $\boldsymbol{r}$?

Rotation matrix

Kinematics

- the b-frame basis vectors in terms of the e-frame basis vectors

$$\boldsymbol{i}_b = i^e_{b,x}\boldsymbol{i}_e + i^e_{b,y}\boldsymbol{j}_e + i^e_{b,z}\boldsymbol{k}_e \qquad (4a)$$

$$\boldsymbol{j}_b = j^e_{b,x}\boldsymbol{i}_e + j^e_{b,y}\boldsymbol{j}_e + j^e_{b,z}\boldsymbol{k}_e \qquad (4b)$$

$$\boldsymbol{k}_b = k^e_{b,x}\boldsymbol{i}_e + k^e_{b,y}\boldsymbol{j}_e + k^e_{b,z}\boldsymbol{k}_e \qquad (4c)$$

$$\boldsymbol{i}^e_b = \begin{bmatrix} i^e_{b,x} & i^e_{b,y} & i^e_{b,z} \end{bmatrix}^T \qquad (5a)$$

$$\boldsymbol{j}^e_b = \begin{bmatrix} j^e_{b,x} & j^e_{b,y} & z^e_{b,z} \end{bmatrix}^T \qquad (5b)$$

$$\boldsymbol{k}^e_b = \begin{bmatrix} k^e_{b,x} & k^e_{b,y} & k^e_{b,z} \end{bmatrix}^T \qquad (5c)$$

- Notice that

$$\boldsymbol{i}_e = 1\boldsymbol{i}_e + 0\boldsymbol{j}_e + 0\boldsymbol{k}_e \qquad (6a)$$

$$\boldsymbol{j}_e = 0\boldsymbol{i}_e + 1\boldsymbol{j}_e + 0\boldsymbol{k}_e \qquad (6b)$$

$$\boldsymbol{k}_e = 0\boldsymbol{i}_e + 0\boldsymbol{j}_e + 1\boldsymbol{k}_e \qquad (6c)$$

$$\boldsymbol{i}^e_e = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \qquad (7a)$$

$$\boldsymbol{j}^e_e = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \qquad (7b)$$

$$\boldsymbol{k}^e_e = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \qquad (7c)$$

Rotation matrix
Kinematics

- Let's then express vector $\boldsymbol{r}$ in the b-frame,

$$\boldsymbol{r} = r_x^b \boldsymbol{i}_b + r_y^b \boldsymbol{j}_b + r_z^b \boldsymbol{k}_b \qquad (8a) \qquad\qquad \boldsymbol{r}^b = \begin{bmatrix} r_x^b & r_y^b & r_z^b \end{bmatrix}^T \qquad (8b)$$

- and resolve this equation in the e-frame, $\boldsymbol{r}^e = r_x^b \boldsymbol{i}_b^e + r_y^b \boldsymbol{j}_b^e + r_z^b \boldsymbol{k}_b^e$,

$$\begin{bmatrix} r_x^e \\ r_y^e \\ r_z^e \end{bmatrix} = r_x^b \begin{bmatrix} i_{b,x}^e \\ i_{b,y}^e \\ i_{b,z}^e \end{bmatrix} + r_y^b \begin{bmatrix} j_{b,x}^e \\ j_{b,y}^e \\ z_{b,z}^e \end{bmatrix} + r_z^b \begin{bmatrix} k_{b,x}^e \\ k_{b,y}^e \\ k_{b,z}^e \end{bmatrix} = \underbrace{\begin{bmatrix} i_{b,x}^e & j_{b,x}^e & k_{b,x}^e \\ i_{b,y}^e & j_{b,y}^e & k_{b,y}^e \\ i_{b,z}^e & j_{b,z}^e & k_{b,z}^e \end{bmatrix}}_{\mathbf{R}_b^e} \begin{bmatrix} r_x^b \\ r_y^b \\ r_z^b \end{bmatrix}$$

RotationRotation Matrix
Kinematics

- resolving a vector $\boldsymbol{r}$ in one of the CS when we know the component form in the other CS is done by multiplication with a rotation matrix

$$\boxed{\boldsymbol{r}^e = \mathbf{R}_b^e \boldsymbol{r}^b} \text{ where } \boldsymbol{r}^e = \begin{bmatrix} r_x^e \\ r_y^e \\ r_z^e \end{bmatrix}, \boldsymbol{r}^b = \begin{bmatrix} r_x^b \\ r_y^b \\ r_z^b \end{bmatrix} \mathbf{R}_b^e = \begin{bmatrix} i_{b,x}^e & j_{b,x}^e & k_{b,x}^e \\ i_{b,y}^e & j_{b,y}^e & k_{b,y}^e \\ i_{b,z}^e & j_{b,z}^e & k_{b,z}^e \end{bmatrix} \tag{9}$$

- The inverse operation is given by the transpose matrix,

$$\boxed{\boldsymbol{r}^b = \mathbf{R}_e^b \boldsymbol{r}^e} \text{ with } \boxed{\mathbf{R}_e^b = (\mathbf{R}_b^e)^T} \text{ meaning } \boxed{(\mathbf{R}_e^b)^{-1} = (\mathbf{R}_b^e)^T}$$

- An identity rotation matrix $\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ => e-frame = b-frame

RotationVelocity

Kinematics

- We have seen that matrix $\mathbf{R}_b^e$ transforms coordinate vectors resolved in b-frame to coordinate vectors resolved in e-frame (equivalently, it rotates frame e-frame over to b-frame)
- If the quadrotor and the "glued" b-frame are rotating, vector $\boldsymbol{r}^b$ is constant. However, $\boldsymbol{r}^e$ is changing with time.

$$\frac{d\boldsymbol{r}^b(t)}{dt} = \dot{\boldsymbol{r}}^b = 0 \text{ and } \frac{d\boldsymbol{r}^e(t)}{dt} = \dot{\boldsymbol{r}}^e \neq 0 = ?$$

RotationVelocity

Kinematics

$$\boldsymbol{r}^e(t) = \mathbf{R}_b^e(t)\boldsymbol{r}^b(t)$$

$$\dot{\boldsymbol{r}}^e(t) = \frac{d}{dt}\left(\mathbf{R}_b^e(t)\boldsymbol{r}^b(t)\right)$$

- Product rule of differentiation: $(\dot{ab}) = \dot{a}b + a\dot{b}$

$$\boldsymbol{v}^e(t) = \dot{\mathbf{R}}_b^e(t)\boldsymbol{r}^b(t) + \mathbf{R}_b^e(t)\underbrace{\dot{\boldsymbol{r}}^b(t)}_{=0}$$

$$\boxed{\boldsymbol{v}^e(t) = \dot{\mathbf{R}}_b^e(t)\boldsymbol{r}^b(t)} \tag{10}$$

- What is $\dot{\mathbf{R}}_b^e(t)$?

RotationAngular Velocity

Kinematics

- It turns out that $\dot{\mathbf{R}}_b^e(t)$ can be written in the following way,

$$\dot{\mathbf{R}}_b^e(t) = \Omega^e \mathbf{R}_b^e(t) = \mathbf{R}_b^e(t)\Omega^b, \text{ where } \Omega^e, \Omega^b \in \mathbb{R}^{3\times 3}$$

- Matrices Ω are skew-symmetric, i.e. have the following form

$$\text{Let } \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \text{then } \Omega = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = [\boldsymbol{\omega}]_\times \tag{11}$$

- Multiplication with the skew-symmetric matrix is equivalent to the vector cross-product, $\boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a}]_\times \boldsymbol{b}$

RotationAngular Velocity

Kinematics

- Vector $\boldsymbol{\omega}$ is called the angular velocity vector, and it can be expressed like any vector in a chosen coordinate frame.
- We have now a relation for $\dot{\mathbf{R}}_b^e(t)$, namely

$$\boxed{\dot{\mathbf{R}}_b^e(t) = [\boldsymbol{\omega}^e]_\times \mathbf{R}_b^e(t) = \mathbf{R}_b^e(t) \left[\boldsymbol{\omega}^b\right]_\times} \text{ with } \boxed{\boldsymbol{\omega}^e = \mathbf{R}_b^e \boldsymbol{\omega}^b} \tag{12}$$

- The angular velocity vector in the rotating b-frame, $\boldsymbol{\omega}^b = \begin{bmatrix} \omega_x^b & \omega_y^b & \omega_z^b \end{bmatrix}^T$, corresponds to the measurements we would receive from a MEMS gyroscope sensor placed on the b-frame
- $\boldsymbol{\omega}_{\mathbf{R}_e^b} = -\boldsymbol{\omega}_{\mathbf{R}_b^e} = -\boldsymbol{\omega}$

RotationVelocity

Kinematics

- Let us continue the equation for the velocity vector $\boldsymbol{r}$,

$$\begin{aligned}\boldsymbol{v}^e(t) = \dot{\mathbf{R}}_b^e(t)\boldsymbol{r}^b(t) = [\omega^e(t)]_\times \mathbf{R}_b^e(t)\boldsymbol{r}^b(t) = [\omega^e(t)]_\times \boldsymbol{r}^e(t) = \omega^e(t) \times \boldsymbol{r}^e(t) \\ = \mathbf{R}_b^e(t)\omega^b(t)\boldsymbol{r}^b(t)\end{aligned}$$

- Thus

$$\boxed{\boldsymbol{v}^e = [\omega^e]_\times \boldsymbol{r}^e} \tag{13}$$

RotationAngular Acceleration
Kinematics

- The angular acceleration is defined as

$$\alpha(t) = \dot{\omega}(t) = \frac{d\omega(t)}{dt}$$

- Unit of angular velocity ω is [rad/s]
- Unit of angular acceleration α is [rad/s^2]

RotationVelocity and Accelerations

Kinematics

- The position and linear velocity relations

$$\boxed{\boldsymbol{r}^e = \mathbf{R}_b^e \boldsymbol{r}^b} \qquad \boxed{\boldsymbol{v}^e = [\omega^e]_\times \boldsymbol{r}^e}$$

- And the linear acceleration:

$$\boldsymbol{a}^e = \left[\dot{\omega}^e\right]_\times \boldsymbol{r}^e + [\omega^e]_\times \boldsymbol{v}^e_{re} = [\alpha^e]_\times \boldsymbol{r}^e + [\omega^e]_\times [\omega^e]_\times \boldsymbol{r}^e$$

$$\boxed{\boldsymbol{a}^e = [\alpha^e]_\times \boldsymbol{r}^e + [\omega^e]_\times [\omega^e]_\times \boldsymbol{r}^e} \tag{14}$$

- $\boldsymbol{a}^b = \mathbf{R}_e^b \boldsymbol{a}^e$ is the output of the quadrotor's accelerometer sensors

Notice

Kinematics

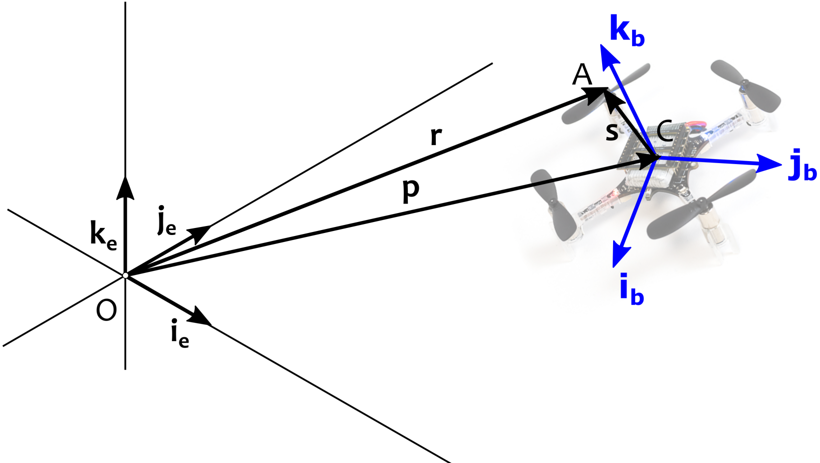
- There is a danger with the derivative notation of coordinate vectors, because of two possible meanings.
- Let $\boldsymbol{y} = \frac{d\boldsymbol{x}}{dt}$, where $\boldsymbol{x}$ can be $\boldsymbol{r}$, $\boldsymbol{v}$ or ω, and $\boldsymbol{y}$ can be $\boldsymbol{v}$, $\boldsymbol{a}$ or α. And e-frame be fixed while b-frame is moving
- Then there are two possible meaning for $\boldsymbol{y}^b$:

$$\boxed{\boldsymbol{y}^b = \mathbf{R}_e^b \boldsymbol{y}^e = \mathbf{R}_b^e \frac{d\boldsymbol{x}^e}{dt}} \quad \text{OR} \quad \boxed{\boldsymbol{y}^b = \frac{d\boldsymbol{x}^b}{dt}}.$$

- The first notation is implied in this work. That is the derivative takes place in the fixed coordinate frame, and derivative quantities can then be expressed in the moving frame.

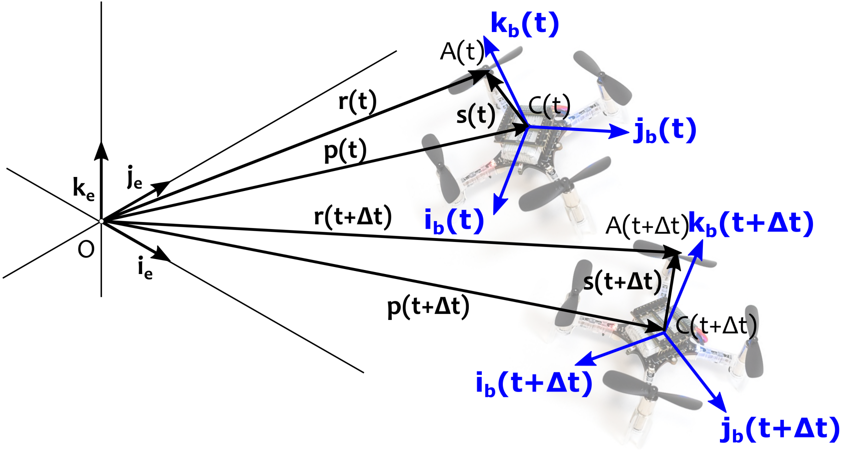
Translation + Rotation

Kinematics



Translation + Rotation

Kinematics



Translation + Rotation

Kinematics

- The b-frame is moving with respect to the e-frame, both translating and rotating, so the vectors $\boldsymbol{p}^e$ and $\boldsymbol{r}^e$ and $\boldsymbol{s}^e$ are time dependent, $\boldsymbol{p}^e = \begin{bmatrix} p_x^e(t) & p_y^e(t) & p_z^e(t) \end{bmatrix}^T$, $\boldsymbol{r}^e = \begin{bmatrix} r_x^e(t) & r_y^e(t) & r_z^e(t) \end{bmatrix}^T$,
- Vector $\boldsymbol{s}$ is constant when expressed in the b-frame, $\boldsymbol{s}^b = \begin{bmatrix} s_x^b & s_y^b & s_z^b \end{bmatrix}^T = \text{ct. } \boldsymbol{s}^e = \begin{bmatrix} s_x^e(t) & s_y^e(t) & s_z^e(t) \end{bmatrix}^T$.
- How will position, velocity and acceleration of vector $\boldsymbol{r}$ look like from the e-frame ?

Translation + Rotation

Kinematics

- We combine the translation and rotation motion remember that

$$\boldsymbol{r}^e = \boldsymbol{p}^e + \boldsymbol{s}^e = \boldsymbol{p}^e + \mathbf{R}_b^e \boldsymbol{s}^b \tag{15}$$

- The velocity relation

$$\boldsymbol{v}_r^e = \dot{\boldsymbol{r}}^e = \dot{\boldsymbol{p}}^e + \frac{d\left(\mathbf{R}_b^e \boldsymbol{s}^b\right)}{dt} = \boldsymbol{v}_p^e + \dot{\mathbf{R}}_b^e \boldsymbol{s}^b + \mathbf{R}_b^e \overbrace{\dot{\boldsymbol{s}}^b}^{=0}$$

$$\boxed{\boldsymbol{v}_r^e = \boldsymbol{v}_p^e + \mathbf{R}_b^e \left[\omega^b\right]_\times \boldsymbol{s}^b} \qquad \boxed{\boldsymbol{v}_r^e = \boldsymbol{v}_p^e + \left[\omega^e\right]_\times \underbrace{\mathbf{R}_b^e \boldsymbol{s}^b}_{\boldsymbol{s}^e}}$$

Translation + Rotation

Kinematics

$$\boxed{\boldsymbol{v}_r^e = \boldsymbol{v}_p^e + [\omega^e]_\times \boldsymbol{s}^e} \tag{16}$$

$$\begin{aligned}
\boldsymbol{a}_r^e = \dot{\boldsymbol{v}}_r^e &= \dot{\boldsymbol{v}}_p^e + [\dot{\omega}^e]_\times \boldsymbol{s}^e + [\omega^e]_\times \dot{\boldsymbol{s}}^e \\
&= \boldsymbol{a}_p^e + [\alpha^e]_\times \boldsymbol{s}^e + [\omega^e]_\times \frac{d\left(\mathbf{R}_b^e \boldsymbol{s}^b\right)}{dt} \\
&= \boldsymbol{a}_p^e + [\alpha^e]_\times \boldsymbol{s}^e + [\omega^e]_\times \dot{\mathbf{R}}_b^e \boldsymbol{s}^b \\
&= \boldsymbol{a}_p^e + [\alpha^e]_\times \boldsymbol{s}^e + [\omega^e]_\times [\omega^e]_\times \mathbf{R}_b^e \boldsymbol{s}^b
\end{aligned}$$

$$\boxed{\boldsymbol{a}_r^e = \boldsymbol{a}_p^e + [\alpha^e]_\times \boldsymbol{s}^e + [\omega^e]_\times [\omega^e]_\times \boldsymbol{s}^e} \tag{17}$$