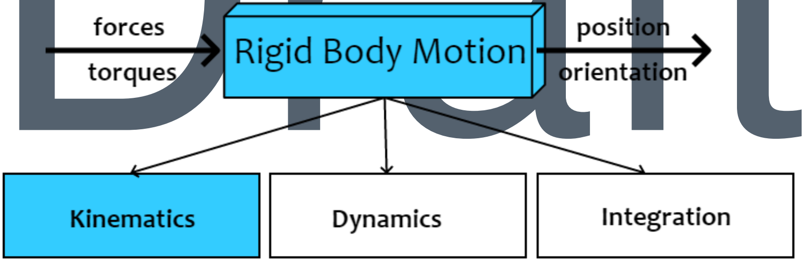
Quaternions and Euler Angles

Lecture 5



Agenda

Lecture 5 | Quaternions and Euler Angles

- Unit quaternions (aka versors, or Euler parameters)
- Euler angles

Motivation

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- The rotation matrix consists of 9 elements, although only [illegible] are required to specify a rotation orientation in 3D space. Can it be done better ?
- Unit Quaternions are a set of 4 numbers (a 4D vector). They are equivalent to rotation matrices, and are the most efficient and nume[illegible] tool for expressing rotations
- Euler angles are a set of 3 numbers, that are equivalent to rotation matrices but have the disadvantage that they have singularities in operation (divisions by zero in expressions) that need to be treated as special cases

Rotation Matrix

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- Main relations:

$$\boldsymbol{r}^e = \mathbf{R}_b^e \boldsymbol{r}^b \tag{1}$$

$$\dot{\mathbf{R}}_b^e = [\omega^e]_\times \mathbf{R}_b^e = \mathbf{R}_b^e [\omega^b]_\times \tag{2}$$

Unit Quaternions

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- Quaternions are a 4-dimensional vector $\in \mathbb{R}^4$ and can be seen as combination of a scalar and a 3-dimensional vector i.e
$$\boldsymbol{q} = \begin{bmatrix} s \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} s & v_1 & v_2 & v_3 \end{bmatrix}^T.$$
- unit quaternion has norm 1, that is $s^2 + v_1^2 + v_2^2 + v_3^2 = 1$
- A rotation matrix can be expressed as a function of a quaternion in the following way:

$$\mathbf{r}^e = \begin{bmatrix} -\boldsymbol{v} & s\mathbf{I}_3 + [\mathbf{v}]_\times \end{bmatrix} \begin{bmatrix} -\boldsymbol{v}^T \\ s\mathbf{I}_3 + [\mathbf{v}]_\times \end{bmatrix} \boldsymbol{r}^b = \mathbf{R}_b^e(\boldsymbol{q})\boldsymbol{r}^b \qquad (3)$$

Unit Quaternions

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$$\mathbf{R}_b^e(\mathbf{q}) = \begin{bmatrix} s^2 + v_1^2 - v_2^2 - v_3^2 & 2v_1v_2 - 2sv_3 & 2v_1v_3 + 2sv_2 \\ 2v_1v_2 + 2v_3s & s^2 - v_1^2 + v_2^2 - v_3^2 & -2sv_1 + 2v_2v_3 \\ 2v_1v_3 - 2sv_2 & 2v_2v_3 + 2v_1s & s^2 - v_1^2 - v_2^2 + v_3^2 \end{bmatrix} \quad (4)$$

$$= 2\begin{bmatrix} s^2 + v_1^2 - 0.5 & v_1v_2 - sv_3 & v_1v_3 + sv_2 \\ v_1v_2 + v_3s & s^2 + v_2^2 - 0.5 & -sv_1 + v_2v_3 \\ v_1v_3 - sv_2 & v_2v_3 + v_1s & s^2 + v_3^2 - 0.5 \end{bmatrix} \quad (5)$$

Unit Quaternions

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- The identical rotation is represented by unit quaternion $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$
- The time derivative of the quaternion is the following

$$\dot{\boldsymbol{q}} = \frac{1}{2} \begin{bmatrix} -\boldsymbol{v}^T \\ s\mathbf{I}_3 + [\mathbf{v}]_\times \end{bmatrix} \boldsymbol{\omega}^b = \frac{1}{2} \begin{bmatrix} -v_1\omega_x^b - v_2\omega_y^b - v_3\omega_z^b \\ s\omega_x^b - v_3\omega_y^b + v_2\omega_z^b \\ v_3\omega_x^b + s\omega_y^b - v_1\omega_z^b \\ -v_2\omega_x^b + v_1\omega_y^b + s\omega_z^b \end{bmatrix} \tag{6}$$

Euler Angles

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The two interpretations of the rotation matrix $\mathbf{R}^e_b$:

- The passive interpretation: multiplication with a vector expressed in the b-frame results in the same vector now expressed in the e-frame, $\boldsymbol{r}^e = \mathbf{R}^e_b \boldsymbol{r}^b$.
- The active interpretation is that it takes the e-frame basis vectors and rotates them over to the b-frame basis vector, in effect rotating the e-frame over to the b-frame

$$\text{Since } \mathbf{R}^e_b = \begin{bmatrix} i^e_{b,x} & j^e_{b,x} & k^e_{b,x} \\ i^e_{b,y} & j^e_{b,y} & k^e_{b,y} \\ i^e_{b,z} & j^e_{b,z} & k^e_{b,z} \end{bmatrix}, \; \boldsymbol{i}^e_b = \mathbf{R}^e_b \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; \boldsymbol{j}^e_b = \mathbf{R}^e_b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \; \boldsymbol{k}^e_b = \mathbf{R}^e_b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Euler Angles

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- The rotation $\mathbf{R}_b^e$ of the e-frame $\boldsymbol{i}_e, \boldsymbol{j}_e, \boldsymbol{k}_e$ such that it becomes identical/overlaps with the b-frame $\boldsymbol{i}_b, \boldsymbol{j}_b, \boldsymbol{k}_b$ can be split in a series of three individual rotations around the coordinate system axes. This allows to write the rotation matrix $\mathbf{R}_b^e$ as a product of three 2D rotations.
- For example

$$\mathbf{R}_b^e = \mathbf{R}_z(\phi)\mathbf{R}_y(\theta)\mathbf{R}_x(\psi), \text{ where}$$

$$\mathbf{R}_z(\phi) = \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{R}_y(\theta) = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}, \mathbf{R}_x(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\psi & -s\psi \\ 0 & s\psi & c\psi \end{bmatrix}$$

Euler Angles

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$$\mathbf{R}_b^e = \begin{bmatrix} c\theta c\phi & s\psi s\theta c\phi - c\psi s\phi & c\psi s\theta c\phi + s\psi s\phi \\ c\theta s\phi & s\psi s\theta s\phi + c\psi c\phi & c\psi s\theta s\phi - s\psi c\phi \\ -s\theta & s\psi c\theta & c\psi c\theta \end{bmatrix} \quad (7)$$

- here ψ is roll (rotation around the x-axis), θ is pitch (rotation around the new y-axis), ϕ is yaw (rotation around the new z-axis)
- Notice the order of rotation:

$$i_b^e = \mathbf{R}_z\left(\mathbf{R}_y\left(\mathbf{R}_x i_e^e\right)\right) \quad (8)$$

- Rotations (matrix products in general) are not commutative:

$$\mathbf{R}_1\mathbf{R}_2 \neq \mathbf{R}_2\mathbf{R}_1$$

Euler Angles

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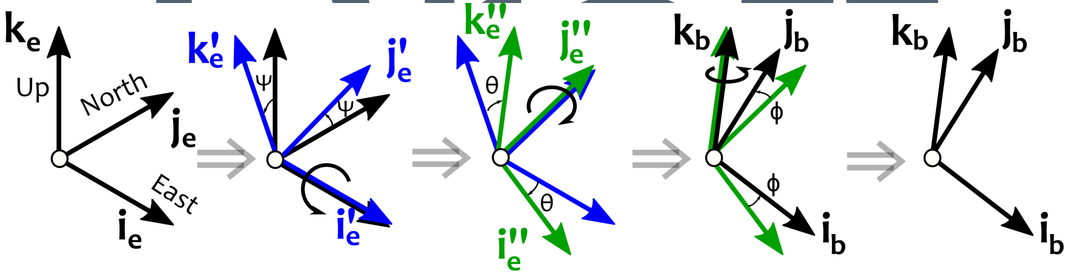

An x-y-z intrinsic rotation taking the e-frame to the b-frame

Euler Angles

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- We can write other combinations for the 2[illegible] axes rotation, there exist in all [illegible] intrinsic combinations and [illegible] extrinsic combinations. We are going to look only at the one described above - the x,y,z extrinsic sequence (called also Tait-Bryan angles or Cardan angles).
- We are going to use the Euler angles to make plots of the rotation, we are interested in transforming the rotation matrix to Euler angles.

Euler Angles

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$$\mathbf{R}_b^e = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} c\theta c\phi & s\psi s\theta c\phi - c\psi s\phi & c\psi s\theta c\phi + s\psi s\phi \\ c\theta s\phi & s\psi s\theta s\phi + c\psi c\phi & c\psi s\theta s\phi - s\psi c\phi \\ -s\theta & s\psi c\theta & c\psi c\theta \end{bmatrix}$$

The solutions are, see note by Gregory G. Slabaugh in the lecture resources, if $R_{31} \neq \pm 1$

$$R_{31} = -\sin(\theta) \rightarrow \theta_1 = -\arcsin(R_{31}), \theta_2 = \pi + \arcsin(R_{31}) \tag{9}$$

$$\frac{R_{32}}{R_{33}} = \tan(\psi) \rightarrow \psi_1 = \text{atan2}\left(\frac{R_{32}}{\cos\theta_1}, \frac{R_{33}}{\cos\theta_1}\right), \psi_2 = \text{atan2}\left(\frac{R_{32}}{\cos\theta_2}, \frac{R_{33}}{\cos\theta_2}\right) \tag{10}$$

$$\frac{R_{21}}{R_{11}} = \tan(\phi) \rightarrow \phi_1 = \text{atan2}\left(\frac{R_{21}}{\cos\theta_1}, \frac{R_{11}}{\cos\theta_1}\right), \phi_2 = \text{atan2}\left(\frac{R_{21}}{\cos\theta_2}, \frac{R_{11}}{\cos\theta_2}\right) \tag{11}$$

and if $R_{31} = \pm 1$,

$$\theta = \pm\frac{\pi}{2}, \psi = \pm\phi + \text{atan2}\left(\pm R_{12}, \pm R_13\right), \phi \in \mathbb{R} \tag{12}$$

Derivates of Euler Angles

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Looking at the relation $\dot{\mathbf{R}}_b^e = \mathbf{R}_b^e \left[\omega^b\right]_\times$, we can derive the expressions of the Euler angles derivatives. First, looking at term R_{31}

$$-\frac{ds\theta}{dt} = [illegible]\, c\theta\omega_z [illegible]\, c\psi c\theta\omega_y \Rightarrow \boxed{\dot{\theta} = -[illegible]\,\omega_z + [illegible]\,\omega_y} \tag{13}$$

Then looking at the term $R_{3[illegible]}$

$$\frac{[illegible]\psi c\theta}{dt} = [illegible]\,\omega_z + c\psi c\theta [illegible] \Rightarrow \boxed{\dot{\psi} = [illegible]\tan\theta\,\omega_y + [illegible]\psi\tan\theta\,\omega_{[illegible]}} \tag{14}$$

And finally, looking at R_{21},

$$\frac{ds\theta s\phi}{dt} = (s\psi s\theta s\phi + c\psi c\phi)\omega_z - (c\psi s\theta s\phi - s\psi c\psi)\omega_y \Rightarrow \boxed{\dot{\phi} = \frac{c\psi}{c\theta}\omega_z + \frac{s\psi}{c\theta}\omega_y} \tag{15}$$

Derivates of Euler Angles

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This can also be written in a matrix form:

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 1 & s\psi \tan\theta & c\psi \tan\theta \\ 0 & c\psi & -s\psi \\ 0 & \dfrac{s\psi}{c\theta} & \dfrac{c\psi}{c\theta} \end{bmatrix} \omega^b \tag{16}$$

Onto the quiz