State Estimation ...



... of Non-Linea Dynamic: Sy: +ms:

- Observabil
- ► Extended | Iman F er (E F)
- ► Unscented Kalman Filter (UKF)
- ▶ UKF with state constraints

Non-Linear Discrete-Time Dynamical Systems Lecture 17



and initial state \mathbf{x}_0 . We keep the noise description the same as in the linear context, $E[\mathbf{w}_k] = \mathbf{0}_n$, $E[\mathbf{v}_k] = \mathbf{0}_m$, $E[\mathbf{w}_k \mathbf{w}_k^T] = \mathbf{Q} \in \mathbb{R}^{n \times n}$, and $E[\mathbf{v}_k \mathbf{v}_k^T] = \mathbf{R} \in \mathbb{R}^{m \times m}$.

Observability Lecture 17



- ► Generally is put dependent system can be observable under a set of inputs and pt observable inder of configuration.
- A much less atraight (ward conce for the linear systems, but the question remains very valid. The estimation problem might be ill-posed for particular combinations of dynamics and measurements.

Extended Kalman Filter



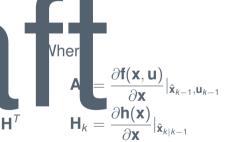
Extended Kalman Filter Predict Step:

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{1} \mathbf{x}_{k-1}, \mathbf{u}_{k-1}$$

$$\mathbf{P}_{k|k-1} = \mathbf{I} \mathbf{P}_{k-1} \mathbf{A}_k^T - \mathbf{GQ}$$

► Kalman Filter U late St

$$\begin{split} \hat{\mathbf{y}}_k &= \mathbf{h}(\hat{\mathbf{x}}_{k|k-1}); \ \mathbf{P}_y = \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^T; \ \mathbf{P}_{xy} = \mathbf{P}_{k|k-1} \mathbf{H}^T \\ \mathbf{K}_k &= \mathbf{P}_{xy} (\mathbf{P}_y + \mathbf{R})^{-1} \\ \hat{\mathbf{x}}_k &= \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k (\mathbf{y}_k - \hat{\mathbf{y}}_k) \\ \mathbf{P}_k &= (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_{k|k-1} \end{split}$$



EKF Example 1 Lecture 17



A 1D rocke model, we state of the state of

$$\dot{\mathbf{x}} = \begin{bmatrix} (u - 0.2v^2)m \\ -0.01^2u \end{bmatrix} \qquad \mathbf{f}(\mathbf{x}, u) = \begin{bmatrix} 3 + v\Delta t \\ v + (u - 0.2v^2)m\Delta t \\ m - 0.01^2u\Delta t \end{bmatrix}$$

The measurement equation is $h(\mathbf{x}) = s$.

EKF Example 2 Lecture 17



An autonor pus pend tum model $\mathbf{x} = \begin{bmatrix} \theta & \omega & L & \alpha \end{bmatrix}^T \in \mathbb{R}^4$, where θ is the angle, ω is the angular procity also length of L pendulum wire, and α is a coefficient efficient.

Continuous-time odel:

$$\dot{\mathbf{x}} = \begin{bmatrix} \omega \\ -\frac{g}{L}\sin\theta - \alpha\omega \\ 0 \\ 0 \end{bmatrix}$$

discre -tim appr kimation:

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \theta + \omega \Delta t \\ \omega - (\frac{g}{L}\sin\theta + \alpha\omega)\Delta t \\ L \\ \alpha \end{bmatrix}$$

The measurement equation is $h(\mathbf{x}) = \theta$.

EKF vs UKF



- For the EK at each time step, the Jacobian of linear function e nd with curr is evaluate s can be then The be used in manter ed atior e "standa orocess essentially hearizes h-linear Ind til current n ard estimate.
- Notice also, may the nonlinear function of and he could bused in their original (non-linearized) form in the dynamics prediction step and for the measurement prediction. We only needed a linearized approximation for covariance estimation/calculation.
- ► The UKF comes with another solution to the problem of calculating these covariance terms in the non-linear context.

The Unscented Transform Lecture 17



- Consider a andom variable $a \mathbb{R}^n$ real [a] and (auto-)covariance $E[(1-\bar{a},a-\bar{a},b]=0]$
- Next consider a non-linear function $\mathbb{R}^n \to \mathbb{R}^n$ and posider the characteristics of the lander variable $\mathbf{b} \to \mathbb{R}^n$
- ➤ The UT is a method for approximating the mean and auto-covariance of the random variable **b**, when we know the function **g**, but only the mean and auto-covariance of **a** (we would need to know the entire pdf of **a** to attempt an exact computation).

The Unscented Transform



- ► It is a type heurist
- ▶ The main i a is to dr cally cho en amples point determinis he ra dom variable *a*. from apply the n n-linear f ctic g to nple and se a type of sample me hand varia se as mple statistics of the random variable b = g(a).
- ► A main idea with the UT, is to carefully select the sample points and keep their number reduced (not doing Monte Carlo here).
- ► The sample points are called **sigma-points**. The name **Sigma-Points Kalman Filter** is also used.

Lecture 17

The Scaled Unscented Transform



- Scalar parameters α , β , κ , e.g. $\alpha = 10^{-3}$, $\kappa = 0$, $\beta = 2$ Derived scalar
- quantity $\lambda = (n + h) \cdot n$; Let $(n + \lambda) \cdot C$ = LL^T, the Choose, decomposition to the sched covariance matrix \mathbf{C}_a we re L is low ratio gular; The sigma part of the sigma $\mathbf{c}_{i+1+n} = \bar{a}$ of the sigma \mathbf{c}_{i+1+
- ▶ Two sets (set *m* and set *c*) of weights are defined as well $W_0^m = \lambda/(n+\lambda)$, $W_0^c = \lambda/(n+\lambda) + (1-\alpha^2+\beta)$, and $W_i^m = W_i^c = 1/(2(n+\lambda))$, j = 1...2n;
- ► Then $\bar{\boldsymbol{b}} = E[\mathbf{g}(\boldsymbol{a})] \approx \tilde{\boldsymbol{b}} = \sum_{i=0}^{2n} W_i^m \mathbf{g}(\mathbf{s}_i)$, and $\mathbf{C}_b = E[(\mathbf{g}(a) - \bar{\boldsymbol{b}})(\mathbf{g}(a) - \bar{\boldsymbol{b}})^T] \approx \tilde{\mathbf{C}}_b = \sum_{i=0}^{2n} W_i^c(\mathbf{g}(\mathbf{s}_i) - \tilde{\boldsymbol{b}})(\mathbf{g}(\mathbf{s}_i) - \tilde{\boldsymbol{b}})^T.$

Exercise Lecture 17



► Generate the sigma-points according to the uncoded ascented transform for the rand in variable $\mathcal{L} \in \mathbb{R}^4$ ith the following statistics:

$$\bar{a} = 0$$
 1 -2

$$\mathbf{C}_a = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

Unscented Kalman Filter



► Unscented January "ter Predict Step:

$$\mathbf{x}_{j} = \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{P}_{k-1}); \quad \mathbf{x}_{j-1} = \sum_{j=0}^{2n} W_{j}^{m} \mathbf{x}_{j}; \quad \mathbf{P}_{j-1} = \sum_{j=0}^{2n} W_{j}^{m} \mathbf{x}_{j}; \quad \mathbf{X}_{j} = \mathbf{x}_{j-1} \mathbf{x}_{j} - \mathbf{x}_{j-1})^{T} + \mathbf{GQG}^{T}$$

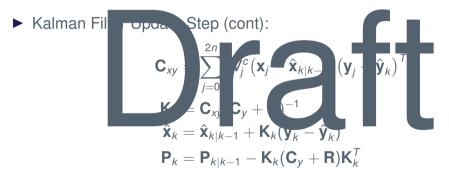
► Kalman Filter Update Step:

$$\mathbf{x}_j = \mathbf{s}_j(\hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1}), \ \mathbf{y}_j = \mathbf{h}(\mathbf{x}_j), \ j = 0, \dots, 2n$$

$$\hat{\mathbf{y}}_k = \sum_{i=0}^{2n} W_j^m \mathbf{y}_j, \ \mathbf{C}_y = \sum_{i=0}^{2n} W_j^c (\mathbf{y}_j - \hat{\mathbf{y}}_k) (\mathbf{y}_j - \hat{\mathbf{y}}_k)^T$$

Unscented Kalman Filter





ightharpoonup with some given initial conditions $\hat{\mathbf{x}}_0$, \mathbf{P}_0 .

There is an alternative form where the sigma-points are extracted from an augmentated state including the noise.

Joseph Form and square Root Filter Lecture 17



- ▶ Joseph for impleme is directly for EKE
 ▶ Joseph for for UKF lake use (the fo wing a
- ► Joseph for the for UKF lake use the following approximation $\mathbf{H} = \mathbf{C}_{xy}^T \mathbf{P}_{k|_{K-1}}^T$
- ► Square Root implementations [ToDo]

UKF Exercise Lecture 17



Implement the pendulum mod

EKF and UKF



Occasional progens with livergence. Some potent Lfixe

- ► [Perea et a 2007] "A fici increases in covarance e.g. B-EKF1: $\mathbf{C}_y = \alpha \mathbf{H}_k \mathbf{F}_{k-1} \mathbf{H}_k^T + \alpha \in \mathbb{R}^+$, e.g. see also EKF2, B-EKF3, B-EKF4 in is paper
- ▶ UKFz [Perea et all. 2007]: neplacing $\hat{\mathbf{y}}_k = \sum_{j=0}^{j} W_j^m \mathbf{y}_j$ to $\mathbf{h}(\hat{\mathbf{x}}^{k|k-1})$. Affects the calculation of \mathbf{C}_y and \mathbf{C}_{sy} and $\hat{\mathbf{x}}_k$;
- ► IUKF [Perea et all. 2008]: Use $\hat{\mathbf{y}}_k$ from $\hat{\mathbf{y}}_k = \sum_{j=0}^{2n} W_j^m \mathbf{y}_j$ to $\mathbf{h}(\hat{\mathbf{x}}^{k|k-1})$ but only in $\hat{\mathbf{x}}_k$;

Equality Constrained EKF/UKF Lecture 17



dure as i ılman ilter roje the state onto Same prod the

the constra

Interval-Constrained UKF



- all 20101 ► [Teixeira et a verv useful nbinauon wh case). See specially sed in the le d and at the end f the uncation predict ste ep a perform d (th TIUK procedure
- ► ICUT is an interval constraint sigma-point generation procedure;
- ▶ The PDF truncation procedure modifies both the $\hat{\mathbf{x}}_k$ and \mathbf{P}_k using an iterative procedure to enforce each of the interval constraints one by one.

Well done!

DIall

