

# An intro to multilinear maps and their attacks using the example of CLT13

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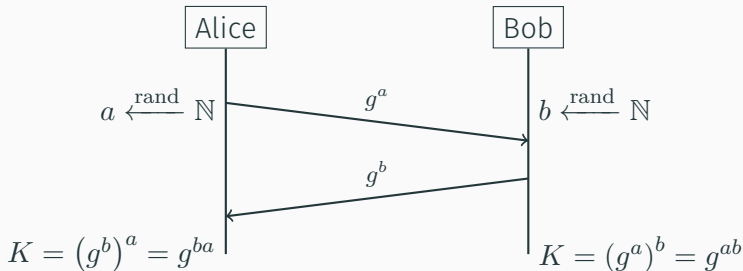
Lukas Kempf

2021-12-10

- Multilinear maps
- CLT13
- DH with CLT13
- Attacking CLT13
- State of multilinear maps

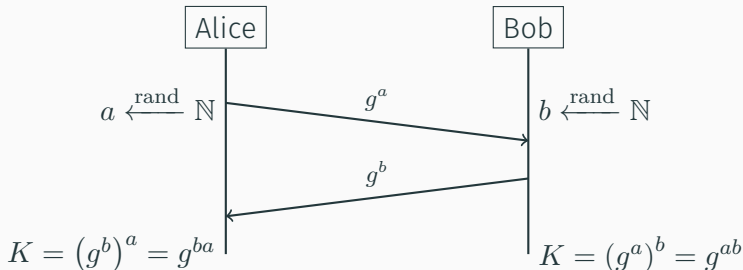
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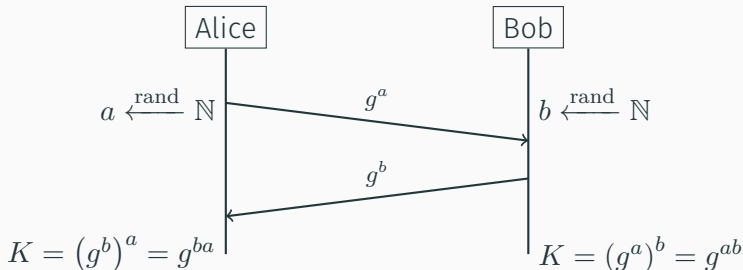
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$\Rightarrow$  Obtaining  $g^{ab}$  from  $g^a$  and  $g^b$  this must be hard (computational assumption).

Alternative stronger assumption: Given  $g^a$  and  $g^b$  it must be hard to differentiate  $g^{ab}$  from random (decisional assumption).

## Definition (Multilinear Map Boneh et al. 2003)

A map  $e : G_1^\kappa \rightarrow G_2$  is a  $\kappa$ -multilinear map if it satisfies the following properties:

1.  $G_1$  and  $G_2$  are groups of the same prime order
2. if  $a_1, \dots, a_\kappa \in \mathbb{Z}$  and  $x_1, \dots, x_\kappa \in G_1$  then

$$e(x_1^{a_1}, \dots, x_\kappa^{a_\kappa}) = e(x_1, \dots, x_\kappa)^{a_1 \dots a_\kappa}$$

3. if  $g \in G_1$  is a generator of  $G_1$  then  $e(g, \dots, g)$  is a generator of  $G_2$ .

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Let  $e : G_1^\kappa \rightarrow G_2$  be a  $\kappa$ -multilinear map and  $g$  a generator of  $G_1$ .

Each of the  $\kappa + 1$  parties chooses  $a_i \xleftarrow{\text{rand}} \mathbb{N}$  and broadcasts  $f_i = g^{a_i}$ .

$$\begin{aligned} e(f_2, \dots, f_{\kappa+1})^{a_1} &= e(f_1, f_3, \dots, f_{\kappa+1})^{a_2} = \dots \\ &= e(f_1, \dots, f_\kappa)^{a_{\kappa+1}} = e(g, \dots, g)^{a_1 \cdots a_{\kappa+1}} \end{aligned}$$



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Security assumption: Given  $e, g, g^{a_1}, \dots, g^{a_{\kappa+1}} \in G_1$  it is hard to compute  $e(g, \dots, g)^{a_1 \cdots a_{\kappa+1}} \in G_2$ .

## How to construct such maps?

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“We also give evidence that such maps might have to either come from outside the realm of algebraic geometry, or occur as ‘unnatural’ computable maps arising from geometry.” (Boneh et al. 2003)

# Graded encoding schemes

Somewhat similar to homomorphic encodings.

For every  $r \in R$  and level  $t \in \mathbb{N}$  we have a set  $S_t^{(r)}$  of possible encodings.

Addition is possible: For  $v_1 \in S_t^{(r_1)}$  and  $v_2 \in S_t^{(r_2)}$  we have  $v_1 + v_2 \in S_t^{(r_1+r_2)}$ .

Multiplication is possible: For  $v_1 \in S_{t_1}^{(r_1)}$  and  $v_2 \in S_{t_2}^{(r_2)}$  we have  $v_1 \cdot v_2 \in S_{t_1+t_2}^{(r_1 \cdot r_2)}$  if  $t_1 + t_2 \leq \kappa$ , the maximum level.

## Interpretation of graded multilinear maps

Let  $v \in S_t^{(r)}$  be a level  $t$  encoding of  $r \in \mathbb{Z}$ . Let  $G_1$  be a group with generator  $g$ .

For  $t = 0$  interpret  $v$  as a scalar  $r$ .

For  $t = 1$  interpret  $v$  as group element  $g^r \in G_1$ .

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Let  $v_1$  and  $v_2$  be two level-1 encodings of  $r_1$  and  $r_2$ . Interpret  $v_1 + v_2$  as  $g^{r_1+r_2}$ . Interpret  $v_1 \cdot v_2$  as evaluation of some multilinear pairing  $e(g^{r_1}, g^{r_2}) = e(g, g)^{r_1 \cdot r_2}$ .

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$\Rightarrow$  We can see  $e(g, g)$  as the generator of some group  $G_2$  of level-2 encodings.

# Chinese remainder theorem

## Theorem (Chinese remainder theorem over the integers)

Let  $\{p_1, \dots, p_n\}$  be pairwise coprime integers and  $M = \prod_{i=1}^n p_i$ . Then there exists a unique ring isomorphism

$$\mathbb{Z}_M \cong \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_n}.$$

More concretely we have maps

$$f : x \mapsto (x \bmod p_1, \dots, x \bmod p_n)$$

and

$$g : (x_1, \dots, x_n) \mapsto \sum_{i=1}^n 1_{p_i} x_i \bmod M$$

where  $1_{p_i}$  is chosen so that  $f(1_{p_i})$  is 1 in exactly the  $i$ -th component and  $f \circ g = \text{id}$ .

Notation:  $x = \text{CRT}_{(p_i)}([x_i])$



Section based on the paper of Coron, Lepoint, and Tibouchi 2013.

- Use CRT to operate in small secret rings while only one large value is public
- Generate  $n$  primes  $p_1, \dots, p_n$  for CRT and compute  $M := \prod_i^n p_i$
- Generate “small” primes  $g_1, \dots, g_n$
- Let  $r_1, \dots, r_n$  be “small” integers (noise)
- Let  $d$  be random integer
- Let  $\kappa$  be max encoding level

Encoding of vector  $m \in \mathbb{Z}^n$  in level  $t$ :

$$c \equiv \frac{r_i \cdot g_i + m_i}{d^t} \pmod{p_i}$$

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Adding encodings:

$$\frac{r_i \cdot g_i + m_i}{d^t} + \frac{r'_i \cdot g_i + m'_i}{d^t} \equiv \frac{(r_i + r'_i) \cdot g_i + m_i + m'_i}{d^t} \pmod{p_i}$$

Multiplying encodings:

$$\frac{r_i \cdot g_i + m_i}{d^t} \cdot \frac{r'_i \cdot g_i + m'_i}{d^{t'}} \equiv \frac{r_i^\dagger \cdot g_i + m_i \cdot m'_i}{d^{t+t'}} \pmod{p_i}$$

$$c \equiv \frac{r_i \cdot g_i + m_i}{d^t} \pmod{p_i}$$

Zero-test parameter:

$$p_{zt} = \sum_{i=1}^n h_i (d^\kappa \cdot g_i^{-1} \pmod{p_i}) \cdot \prod_{i' \neq i} p_{i'} \pmod{M}$$

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Applying the zero-test to a  $\kappa$ -level encoding  $c$ :

$$p_{zt} \cdot c = \sum_{i=1}^n h_i (r_i + m_i \cdot (g_i^{-1} \pmod{p_i})) \cdot \prod_{i' \neq i} p_{i'} \pmod{M}$$

Security parameter  $\lambda$  influences multiple internal parameters (omitted).

Notation: `rand` means random number of appropriate size.

Public key `pubKey` (simplified):

- $M := \prod_i^n p_i$
- $p_{zt}$
- Level-1 encodings of zero  $\{z_j\}$  meaning
$$z_j = \text{CRT}_{(p_i)} \left( \left[ \frac{\text{rand} \cdot g_i}{d} \right] \right)$$
- Level-0 encodings of random values  $\{x'_j\}$  meaning
$$x'_j = \text{CRT}_{(p_i)} ([\text{rand} \cdot g_i + \text{rand}])$$
- Level-1 encoding of 1  $y = \text{CRT}_{(p_i)} \left( \left[ \frac{\text{rand} \cdot g_i + 1}{d} \right] \right)$

**samp**(pubKey): To get random level-0 encoding pick random subset of  $\{x'_j\}$  and add together.

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**ext**(pubKey,  $c$ ): Collect most significant bits of  $c \cdot p_{zt}$  (simplified version).



Key exchange between  $\kappa + 1$  parties.

Each party  $i$ : Let  $a_i = \mathbf{samp}(\text{pubKey})$  be secret random value and broadcast

$$f_i = \mathbf{enc}(\text{pubKey}, a_i, 1)$$

Shared encoding:

$$\mathbf{ext}(a_1 f_2 \cdots f_{\kappa+1}) = \mathbf{ext}(f_1 a_2 f_3 \cdots f_{\kappa+1}) = \cdots = \mathbf{ext}(f_1 \cdots f_{\kappa} a_{\kappa+1})$$

## DH with CLT13 — First idea

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Problem:  $f_i \cdot y^{-1} \equiv a_i \cdot y \cdot y^{-1} \equiv a_i \pmod{M!}$

Solution: Add noise to the encoding.

**reRand**(pubKey,  $c$ ): Re-randomize level-1 encoding  $c$  by adding sum of random subset of  $\{z_j\}$  to  $c$  (simplified).

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Shared encoding doesn't change:

$$\mathbf{ext}(a_1 f_2 \cdots f_{\kappa+1}) = \mathbf{ext}(f_1 a_2 f_3 \cdots f_{\kappa+1}) = \cdots = \mathbf{ext}(f_1 \cdots f_{\kappa} a_{\kappa+1})$$

Given a public key `pubKey` with security parameter  $\lambda$  generate:

Real encoding:

1. Choose random level-0 encoding  $a_j$  for all  $1 \leq j \leq \kappa + 1$ .
2. Set  $u_j = \mathbf{reRand}(\text{pubKey}, \mathbf{enc}(\text{pubKey}, 1, a_j))$  to obtain level-1 encoding for all  $1 \leq j \leq \kappa + 1$ .
3. Set  $v = \mathbf{reRand}(\text{pubKey}, \mathbf{enc}(\text{pubKey}, \kappa, \prod_{i=1}^{\kappa+1} a_i))$ .

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Random encoding:

1. Choose random level-0 encoding  $b$ .
2. Set  $w = \mathbf{reRand}(\text{pubKey}, \mathbf{enc}(\text{pubKey}, \kappa, b))$ .

The GDDH assumption states that a polynomial attacker has only negligible chance to differentiate  $v$  and  $w$  given the  $u_j$  and  $\text{pubKey}$ .

Section based on the paper of Cheon et al. 2014.

Before attacking CLT13 we look at a simpler but similar problem:

## Definition (CRT-ACD Problem)

Let  $n, \eta, \varepsilon \in \mathbb{N}$ . For given  $\eta$ -bit primes  $p_1, \dots, p_n$  define the distribution

$$D_{\varepsilon, \eta, n}(p_1, \dots, p_n) = \left\{ \text{CRT}_{(p_i)}([r_i]) \mid r_i \xleftarrow{\text{rand}} (-2^\varepsilon, 2^\varepsilon) \cap \mathbb{Z} \right\}.$$

CRT-ACD Problem: Given many samples from  $D_{\varepsilon, \eta, n}(p_1, \dots, p_n)$  and  $M = \prod_{i=1}^n p_i$  find all  $p_i$ .



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Let  $\hat{p}_i = M/p_i$ .  $\hat{P} = \text{CRT}_{(p_i)}([\hat{p}_i])$  is called auxillary input. This can be seen as simpler version of  $p_{zt}$ .

## Lemma

Given  $a = \text{CRT}_{(p_i)} ([r_i]) \xleftarrow{\text{rand}} D_{\varepsilon, \eta, n}(p_1, \dots, p_n)$  and  $\hat{P} = \text{CRT}_{(p_i)} ([\hat{p}_i])$  it holds that

$$\hat{P} \cdot a \bmod M = \text{CRT}_{(p_i)} ([\hat{p}_i \cdot r_i]) = \sum_{i=1}^n \hat{p}_i \cdot r_i$$

if  $\varepsilon + \log n + 1 < \eta$ .

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⇒ Reduces modular arithmetics to linear equation.

Sketch of proof: Consider second equation modulo  $p_i$ . Ensure that left side is smaller than  $M$ . Result follows from uniqueness of CRT.

We now cleverly apply the reduction to linear equations to recover the secret  $p_i$ .

Let  $a = \text{CRT}_{(p_i)}([a_i])$ ,  $b = \text{CRT}_{(p_i)}([b_i])$ . Assume lemma is applicable:

$$ab\hat{P} \bmod M = \sum_{i=1}^n a_i b_i \hat{p}_i$$

## Attacking CLT13 — CRT-ACD

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As matrix equation:

$$ab\hat{P} \bmod M = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} \hat{p}_1 & 0 & \cdots & 0 \\ 0 & \hat{p}_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{p}_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Collecting more samples ( $1 \leq i, j \leq n$ ):

$$a_i = \text{CRT}_{(p_k)}([a_{k,i}]), b = \text{CRT}_{(p_k)}([b_k]), c_j = \text{CRT}_{(p_k)}([c_{k,j}])$$

Using the samples to state matrix equations:

$$w_{i,j} = a_i \cdot \hat{P} b \cdot c_j \bmod M = \begin{pmatrix} a_{1,i} & \cdots & a_{n,i} \end{pmatrix} \begin{pmatrix} b_1 \hat{p}_1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & b_n \hat{p}_n \end{pmatrix} \begin{pmatrix} c_{1,j} \\ \vdots \\ c_{n,j} \end{pmatrix}$$

$$w'_{i,j} = a_i \cdot \hat{P} \cdot c_j \bmod M = \begin{pmatrix} a_{1,i} & \cdots & a_{n,i} \end{pmatrix} \begin{pmatrix} \hat{p}_1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \hat{p}_n \end{pmatrix} \begin{pmatrix} c_{1,j} \\ \vdots \\ c_{n,j} \end{pmatrix}$$

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Collecting  $w_{i,j}$  and  $w'_{i,j}$  into matrices  $\mathbf{W}$  and  $\mathbf{W}'$ :

$$\mathbf{W} = \mathbf{A}^T \cdot \text{diag}(b_1 \hat{p}_1, \dots, b_n \hat{p}_n) \cdot \mathbf{C}$$

$$\mathbf{W}' = \mathbf{A}^T \cdot \text{diag}(\hat{p}_1, \dots, \hat{p}_n) \cdot \mathbf{C}$$

with  $\mathbf{A}^T = (a_{k,i})$  and  $\mathbf{C} = (c_{k,j})$ .



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with  $\mathbf{A}^T = (a_{k,i})$  and  $\mathbf{C} = (c_{k,j})$ .

Assume  $\mathbf{A}$  and  $\mathbf{C}$  are invertible:

$$\mathbf{W} \cdot \mathbf{W}'^{-1} = \mathbf{A}^T \cdot \text{diag}(b_1, \dots, b_n) \cdot \mathbf{A}^{T^{-1}}$$

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Assume are  $b_i$  pairwise distinct:

$$\gcd(b - b_i, M) = p_i$$

## Attacking CLT13 — Adapting the attack

Recall:

$$p_{zt} = \sum_{i=1}^n h_i (d^\kappa \cdot g_i^{-1} \bmod p_i) \cdot \prod_{i' \neq i} p_{i'} \bmod M$$

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Let  $a = \text{CRT}_{(p_i)}([r_i g_i / d^\kappa])$  be top-level encoding of 0.

$$p_{zt} \cdot a \bmod M = \text{CRT}_{(p_i)}([\hat{p}_i h_i r_i]) = \sum_{i=1}^n \hat{p}_i h_i r_i$$

Lemma applies because for encodings of 0  $p_{zt} \cdot a$  is short enough.

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We get similar attack by spanning

$$x'_j \cdot x'_1 \cdot z_k \cdot y^{\kappa-1} \cdot p_{zt} \bmod M \text{ and } x'_j \cdot z_k \cdot y^{\kappa-1} \cdot p_{zt} \bmod M$$

for  $1 \leq j, k \leq n$ .

## Why multilinear maps matter

- DH key-exchange between multiple parties in one round.
- Efficient message broadcast to subset of recipients.
- Existence of cryptographic multilinear maps linked to the existence of indistinguishability obfuscation.
- Building block for time-lock encryption.

## Current state of multilinear maps

CLT13 and improvement CLT15 broken in regards to GDDH. iO based on CLT13 has been broken in multiple cases.

MZ17 (based on CLT13) is still standing regarding the GDDH assumption.

Lattice based approaches have been successfully attacked regarding GDDH and iO.

More (partially outdated) info: <https://malb.io/are-graded-encoding-schemes-broken-yet.html>



- Multilinear maps are building block for a lot of new and interesting constructions.
- Current known instantiations are quite complex.
- Many have been broken.
- Existence of practical and secure multilinear maps still unknown.
- $\Rightarrow$  Lots of ongoing research.

## Why this topic?

- Topic was chosen as basis for CTF-Challenge originally.
- Plan: Implement DH with CLT13 and let players implement the attack.

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- Topic was chosen as basis for CTF-Challenge originally.
- Plan: Implement DH with CLT13 and let players implement the attack.
- However: Calculating eigenvalues of larger matrix with large integers is a bit slow.
- Solution: Use weak re-randomization instead.
- If you like applying cryptography it might be fun to look at CTFs.