EE2012/ST2334 Discussion 3*

Liu Zichen

March 14, 2019

1. **[2-D random variable]** Let E be an experiment and S a sample space associated with E. Let X and Y be two functions each assigning a real number to each $s \in S$. We call (X, Y) a two-dimensional random variable (or random vector). Its **range** is $R_{X,Y} = \{(x, y) | x = X(s), y = Y(s), s \in S\}$. The definition can be extended to higher dimension, and they can be defined for both discrete and continuous random variable.

2. **[Joint probability function]**

   Discrete:

$$\begin{aligned} f_{X,Y}(x_i, y_j) &= \Pr(X = x_i, Y = y_j) \\ f_{X,Y}(x_i, y_j) &\geq 0 \text{ for all } (x_i, y_j) \in R_{X,Y} \\ \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{X,Y}(x_i, y_j) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \Pr(X = x_i, Y = y_j) = 1 \end{aligned} \tag{1}$$

   Continuous: change $\sum$ to $\int$.

3. **[Marginal distribution]** Recall the law of total probability (discrete case):

$$\begin{aligned} \Pr(A) &= \sum_n \Pr(A \cap B_n) \\ &= \sum_n \Pr(A | B_n) \Pr(B_n) \end{aligned} \tag{2}$$

   If (X, Y) is a 2-D discrete random variable, and its joint probability is $f_{X,Y}(x, y)$, the marginal distributions are:

$$\begin{aligned} f_X(x) &= \sum_y f_{X,Y}(x, y) \\ f_Y(y) &= \sum_x f_{X,Y}(x, y) \end{aligned} \tag{3}$$

4. **[Conditional distribution]**

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}, \text{ if } f_X(x) > 0 \tag{4}$$

5. **[Independence]**

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B) \Longleftrightarrow \mathrm{P}(A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \mathrm{P}(A|B) \tag{5}$$

6. **[Expectation]**

$$E[g(X, Y)] = \begin{cases} \sum_x \sum_y g(x, y) f_{X,Y}(x, y), & \text{for Discrete RV's} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dx dy, & \text{for Continuous RV's} \end{cases} \tag{6}$$

   (a) A special case is that when $g(X, Y) = (X - \mu_X)(Y - \mu_Y)$, the expectation is the **covariance** of (X, Y). $\mathrm{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - \mu_X \mu_Y$.

   (b) If X and Y are independent, $\mathrm{cov}(X, Y) = 0$. But $\mathrm{cov}(X, Y) = 0$ does **NOT** imply independence.

   (c) $\mathrm{Cov}(aX + b, cY + d) = ac\,\mathrm{Cov}(X, Y)$, $V(aX + bY) = a^2 V(X) + b^2 V(Y) + 2ab\,\mathrm{Cov}(X, Y)$

7. **[Correlation coefficient]**

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X, Y)}{\sqrt{V(X)}\sqrt{V(Y)}} \tag{7}$$

   (a) $-1 \leq \rho_{X,Y} \leq 1$

   (b) $\rho_{X,Y}$ measures the degree of **linear relationship** between X and Y.

   (c) If X and Y are independent, $\rho_{X,Y} = 0$. But $\rho_{X,Y} = 0$ does **NOT** imply independence.

*Some examples from bookmarked pages of the lecture notes