Multiple-source shortest paths with unit weights in embedded graphs

Abstract

We describe a new algorithm that computes multiple-source shortest paths from vertices in a given boundary face to all other vertices in an embedded graph with unit weight edges.

1 Introduction

Recently, Eisanstat and Klein [?] introduced a new algorithm for computing multiple source shortest path problem for a planar graphs in linear time. Our paper attempts to generalize this idea to embedded graphs. In their paper, they maintain so-called leafmost shortest tree to get around an issue of ambiguous shortest paths between a pair of vertices. There is no direct way to generalize leafmost tree computing process to an embedded graphs, largely because there is no way to define leafmost tree of a data structure in an embedded graphs. In the following section we introduce terms that alleviate the process of ambiguous pivoting.

We can formally define multiple-source shortest paths problem as follows:

Introduce notations, homology, and dual graphs.

Given. *Let G be a directed graph $(V, \vec{E})$, embedded on a surface with genus g. All edge weights are unit and $|V| = n$. Using Euler's theorem, we can derive $|E| = n + 2g - 1$.*

Find. *Consider boundary face f of G. $\forall v \in f$, find a shortest path to $\forall u \in V$.*

2 Holy Tree

Let T be a breadth first search(BFS) tree of G, and C^* be a BFS co-tree in G. Then there is exactly $2g$ leftover edges $L = \{e_1, e_2, \ldots, e_{2g}\}$.

There exists a unique cycle λ_i in $C^* \cup e_i$, and $(\lambda_1, \lambda_2, \ldots, \lambda_{2g}) = \Lambda$ defining homology basis.
We define homological signature of an edge as follows:

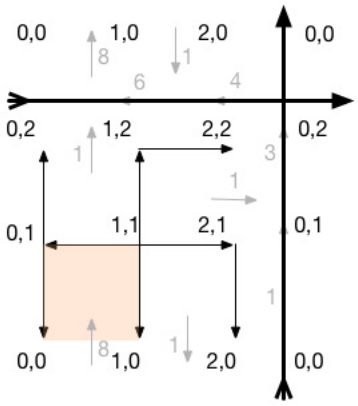
$$[e]_i = \begin{cases} 1 & \text{, if } e \in \lambda_i \\ -1 & \text{, if } rev(e) \in \lambda_i \\ 0 & \text{, o/w} \end{cases}$$

Furthermore, we define leafmost term α recursively as follows:

$$\alpha(e^*) = \begin{cases} 1 & \text{, if } rev(e^*) \text{ is a leaf dart in } C^* \\ \sum_{\text{tail}(e'^*)=\text{head}(e^*)} \alpha(e') & \text{, o/w} \end{cases}$$

We can extend above definition with $\alpha(e) = \alpha(e^*)$ and $\alpha(e)^* = -\alpha(\text{rev}(e^*))$.

Let $\tilde{w}(e) = (1, [\vec{e}], \alpha(e))$ be new weight vector for each edge in G. We refer to each component this weight vector as length, homology, and leafmost terms respectively.



Definition. *An edge e_1 is holier than e_2, if $\tilde{w}(e_1) < \tilde{w}(e_2)$ in lexicographic comparison. Therefore, we can define holiness of any $S \subset G$ as follows:*

$$H(S) = \sum_{e \in S} \tilde{w}(e)$$

3 Need of holiness

We provide a simple explanation on why it is necessary to introduce holiness to cope with ambiguity.

1. Describe why homology and leafmost term resolves the issue in higher genus graphs.

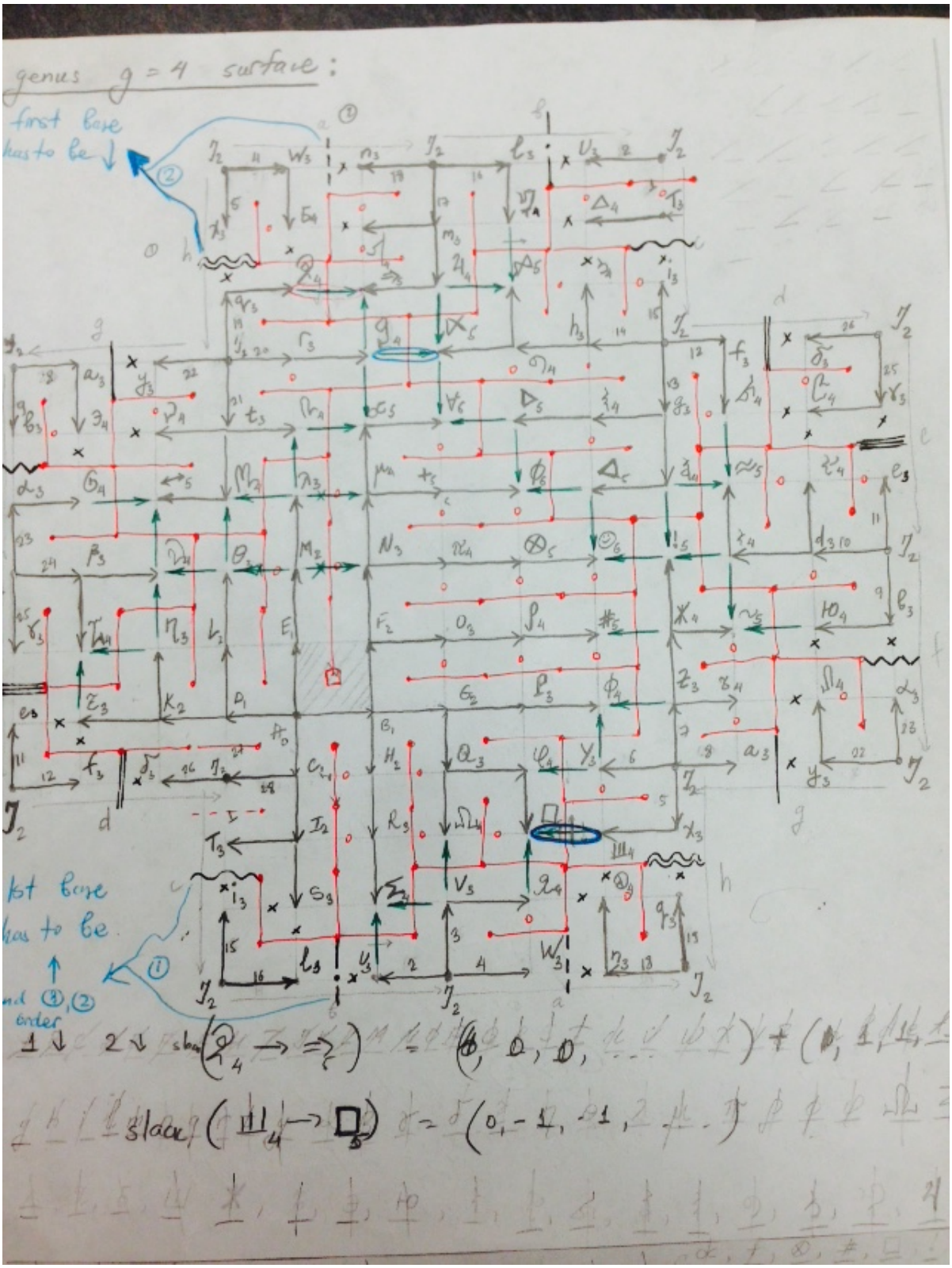
4 Leftmost tree vs Holy tree

1. Explain our initial attempts on leafmost tree maintaining to show it is sufficient for dealing with non-unique shortest paths.
2. Show counterexample.

Difference of holiest Tree and leftmost tree:
Our initial hope was that by computing only leftmost tree, we might be able to compute holy tree. However, on genus $g = 4$ surface, we show that no configuration of homology and leafmost terms can result in equal holy and leftmost trees. Consider below picture,

- Leftmost tree is rooted at A and noted by solid black darts.
- There are exactly 8 homology cycles, each specifically denoted with special darts crossing the boundaries.
- Leafmost term building tree is denoted with red darts.

The observation on the figure is 2 darts highlighted in blue will be tense, regardless of the homology and leafmost choice configurations.



5 Initial holy tree

Holiest tree is a spanning tree with minimal holiness. We build holiest tree rooted at r, using slight tweak in the Bellman-Ford algorithm for finding shortest path tree rooted at r.

BuildHoliestTree$(G, \tilde{w}, r)$:
  Set $dist[r] \leftarrow (0, [\vec{0}], 0)$
    pred$(r) \leftarrow$ NULL
  for all $v : v \neq r$
    $dist[r] \leftarrow (\infty, [\vec{\infty}], \infty)$
    pred$(r) \leftarrow$ NULL
  put r into queue
  while queue is not empty:
    Let $u \leftarrow$ dequeue item
    for all $u \rightarrow v$
      if v is not marked
        mark v and put in the queue
      if isTense$(u \rightarrow v)$
        relax$(u \rightarrow v)$

isTense$(u \rightarrow v)$:
  return $dist[u] + \tilde{w}(u \rightarrow v) < dist[v]$

relax$(u \rightarrow v)$:
  $dist[v] \leftarrow dist[u] + \tilde{w}(u \rightarrow v)$
  pred[v] $\leftarrow u$

Observation. *Each vertex will be added once to the queue.*

Corollary. *Each edge will be relaxed at most once.*

Lemma 2.1. *If there is no tense edge in G, then for each $v : r \rightarrow \ldots \rightarrow pred(pred(v)) \rightarrow pred(v) \rightarrow v$ is the holiest path from r to v.*

Proof: Let's prove it by induction on $dist[v].length$ distance from the root r.

Base. *$dist[v].length = 0$, then v = r, so the claim holds trivially.*

Induction Step. *Suppose the claim is true for all vertex $v \in V$ such that $dist[v].length < d$ for some d. Consider vertex v such that $dist[v].length = d$. By induction hypothesis, all vertices with $dist[u].length = d-1$ have holiest path correctly updated. By definition, $dist[v] = \min_{u \rightarrow v}\{dist[u] + \tilde{w}(u \rightarrow v)\}$, here $dist[u].length = d - 1$. By Induction hypothesis, $dist[u]$ is not tense and can construct holiest path to u, so if there is no tense edge in G then $dist[v] = min\{dist[u] + \tilde{w}(u \rightarrow v)\}$ holds.*

□

Corollary. *The algorithm will produce holiest tree rooted at r in $O(n + g)$ time.*

6 Moving Along an Edge

Let T_u be a current holiest tree, $u \rightarrow v$ be an edge that we are trying to move along. So at the end of the process we would like to obtain T_v, the holiest tree rooted at v.

We follow Caballo, Chambers, and Erickson's [?] method to move across the edge, by bisecting the given edge and inserting new source s which is connected to both u and v. At the start of the process, $s == u$ and continuously move to v, and when $s == v$, we would have our $T_s = T_v$.

Initial attempt:
Let $(1, \vec{h}, \alpha) = w(u \rightarrow v)$. We treat λ as a parameter with satisfying following equations:

$$w_u(s \to u) = (0, [\vec{0}], 0), \quad w_u(s \to v) = (1, \vec{h}, \alpha) \tag{1}$$
$$w_v(s \to u) = (1, -[\vec{h}], -\alpha), \quad w_v(s \to v) = (0, \vec{0}, 0) \tag{2}$$

The natural definition would be as follows, but it does not satisfy our constraints (1) − (2).

$$w_\lambda(s \to u) = (\lambda, [\vec{0}], 0) \tag{3}$$
$$w_\lambda(s \to v) = (1 - \lambda, \vec{h}, \alpha) \tag{4}$$

Therefore, we modify our parametric definition by decreasing distance to u by $w(u \to v) = (0, -[\vec{h}], -\alpha)$. This change allows us to define

$$w_\lambda(s \to u) = (0, -[\vec{h}], -\alpha) + \lambda * (1, [\vec{h}], \alpha) \tag{5}$$
$$w_\lambda(s \to v) = (1, [\vec{h}], \alpha) + \lambda * (1, [\vec{h}], \alpha) \tag{6}$$

Since we decrease distance to u at the start, this could potentially introduce pivots. Suppose, for instance, x be descendant of u and y be of v in T_s, then:

$$slack(x \to y) = dist(x) + w(x \to y) - dist(y) = \tag{7}$$
$$= dist_0(x) + (0, -[\vec{h}], -\alpha) + w(x \to y) - dist_0(y) = \tag{8}$$
$$= slack_0(x \to y) + (0, -[\vec{h}], -\alpha) \Rightarrow \tag{9}$$
$$slack(x \to y) < 0, \text{ if } slack_0(x \to y) < (0, [\vec{h}], \alpha) \tag{10}$$

Therefore, this could be potentially problamatic. Furthermore, **What if $u \to v$ has non-trivial homology or leafmost term?**

We need to take into account that $u \to v$ has non-trivial homology and leafmost terms.

1. First attempt of not maintain holy tree at all times.
2. Second attempt of continuously moving by changing homology and leafmost terms, also reducing destination distance initially.
3. Necessity of resolving homology and leafmost values for an edge that source is moving.

$$w_0(s \to u) = (0, [\vec{0}], 0) \tag{11}$$
$$w_0(s \to v) = (1, [\vec{h}], \alpha) = w(u \to v) \tag{12}$$

Observe that this condition implies $s = u$, therefore $T_s = T_u$. We reduce distances to u and v as follows:

$$w_\epsilon(s \to u) = (0, -[\vec{h}], -\alpha) \tag{13}$$
$$w_\epsilon(s \to v) = (1, [\vec{0}], 0) \tag{14}$$

Since we reduced distance to all vertices in the graph equally, the process does not introduce any pivots. Then we define a parametric weights as follows:

$$w_\lambda(s \to u) = (0, -[\vec{h}], -\alpha) - \lambda \tag{15}$$
$$w_\lambda(s \to v) = (1, [\vec{0}], 0) + \lambda \tag{16}$$

Every other dart $x \to y$ has constant parametric weight $w_\lambda(x \to y) = w(x \to y)$. We then maintain the holy tree T_λ rooted at s, with respect to the weight function w_λ, as λ increases continuously from 0 to $(1, [\vec{0}], 0)$. When $\lambda = w(u \to v)$, $T_\lambda = T_v$.

In the following algorithm, **pred** defines holy tree rooted at u, and **dist** is corresponding distance to each vertex in the graph.

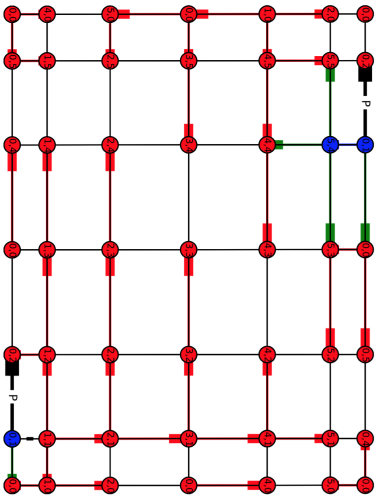
MoveAlongEdge$(G, u \to v, dist, pred)$:
  Add new vertex s
  $pred[u], pred[v] \leftarrow s$
  $\lambda \leftarrow 0$

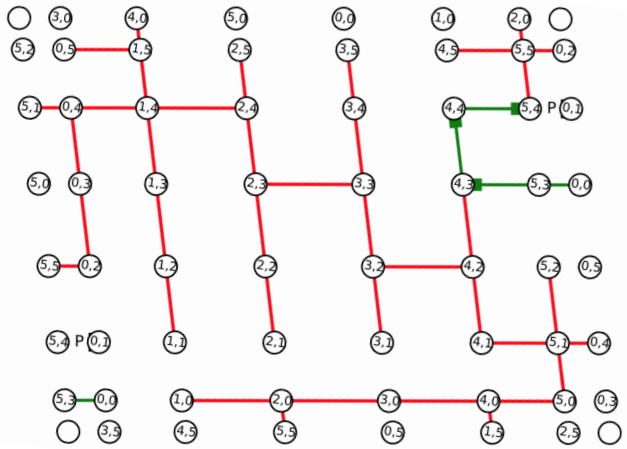
  $w(s \to u) \leftarrow (0, -[w(s \to u)], -\alpha(w(s \to u)))$
  AddSubtree$((0, -[w(s \to u)], -\alpha(w(s \to u))), u)$

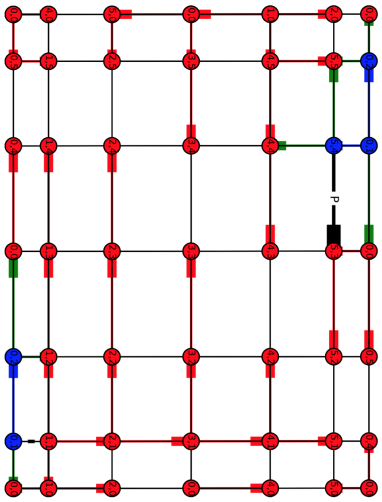
  $w(s \to v) \leftarrow (1, [\vec{0}], 0)$
  AddSubtree$((0, -[w(s \to u)], -\alpha(w(s \to u))), v)$

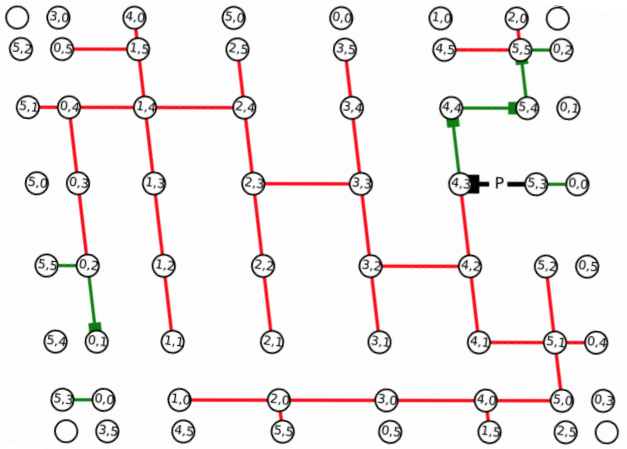
  while $\lambda < (1, [\vec{0}], 0)$:
    pivot $\leftarrow$ FindNextPivot
    If **pivot** is non NULL **AND** $(\lambda + slack(\textbf{pivot})/2) < (1, [\vec{0}], 0)$
      Pivot(**pivot**)
      $\lambda \leftarrow \lambda + slack(\textbf{pivot})/2$
    else
      $\delta = (1, [\vec{0}], 0) - \lambda$
      AddSubtree(δ, u)
      AddSubtree$(-\delta, v)$
      $\lambda \leftarrow \lambda + \delta$

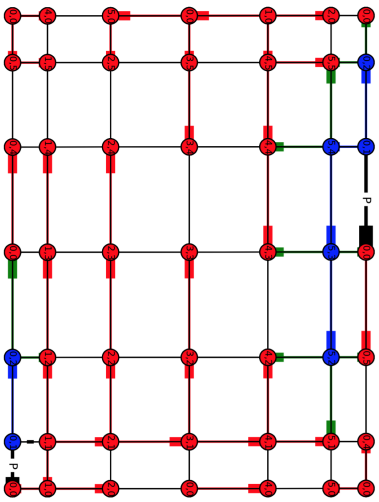
Below, we show an example of moving along an edge process on genus $g = 2$ grid.
On a left side, we see primal holy tree H with nodes increasing in distance are red, decreasing are blue, and next pivot is denoted as P.
On right side, we see the dual graph $(G/T)^*$ and active darts are noted with green color.

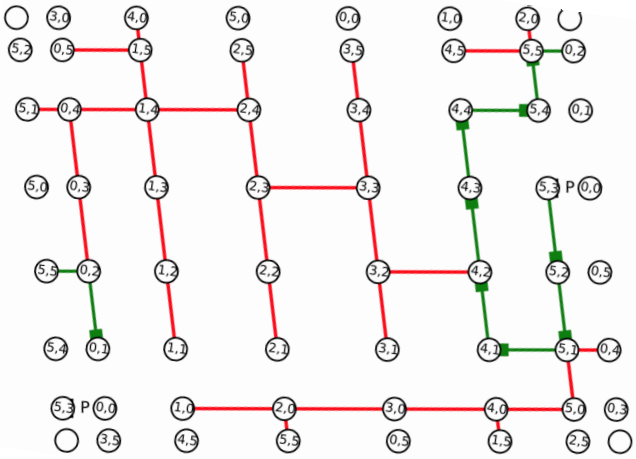












7 Bounding number of pivots

1. Describe how Eisenstat Klein clock lemma works and why it is not easily generalizable in higher genus graphs.
2. Describe P_i^v, P_j^v non-crossing.

We introce a clocking lemma to prove that each edge is involved in pivoting process at most $O(g)$ times.

Lemma 4.1. *Let v_0 be a first vertex in our given face f, and v_i be the source right after i^{th} pivot. Consider a vertex y. We denote holiest path from v_i to y as P_i^y. Then P_i^y and P_j^y are non-crossing for all i, j.*

Proof: This is immediate from our definition of holiest path, as no two paths can have same value. [???] □

Lemma 4.2. *As source vertex s moves around the given face f, any dart d has exactly $2g$ continuous clock state:*

$$d \in T \tag{17}$$
$$d^* \in (G/T)^* \tag{18}$$
$$rev(d) \in T \tag{19}$$
$$rev(d^*) \in (G/T)^* \tag{20}$$

Proof: Here is the proof. □

Theorem 4.1. *Total running time of MSSP is $O(gn)$.*

Proof: Building initial holy tree takes $O(n+g)$ time. The process of moving arond the face and pivoting takes $O(gn)$ as each dart enters and replaces $O(g)$ times. □

8 Finding pivot quickly

- What data structure do we maintain in the G*? Finding shortest path in network can also be understood as a Linear Programming problem as follows:
- How do we find next pivot quickly using above structure?

9 Analysis

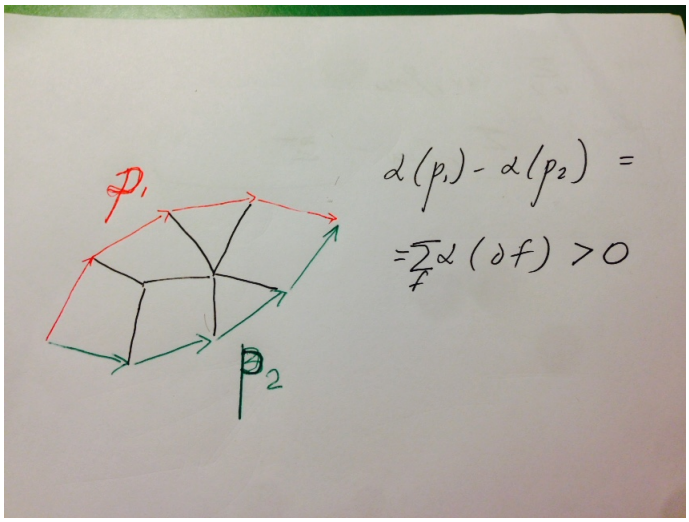
- Building initial tree
- Pivoting
- Number of times each edge is pivoted
- Overall running time

Couple questions regarding the slack and flow:

- If the fixed tree T is arbitrary tree (not necessarily the holiest Tree, then the flow in the answer does not have to be optimal) but solution to the linear program $min < f, \text{slack}_T >$ is equal to the answer from fixed tree T, not necessarily the optimal solution. That is because for each vertex, the demand satisfies the constraint and if we consider the tree T, then the value of $min < f, \text{slack}_T >$ would be 0, implying it is the optimal solution. (Sum cannot be negative since otherwise there is no optimal solution)
- The reason we picked the slack as the way we defined is due to the fact that slack is not negative, ensuring that the nothing bad happens.
- What makes the non-planar case special with 2g extra constraints?
- How does the slack in dual representation help us to find the pivots quickly?

10 Additional

NOTE: Necessity of the α definition on the edges for holiness.
Consider the following picture:



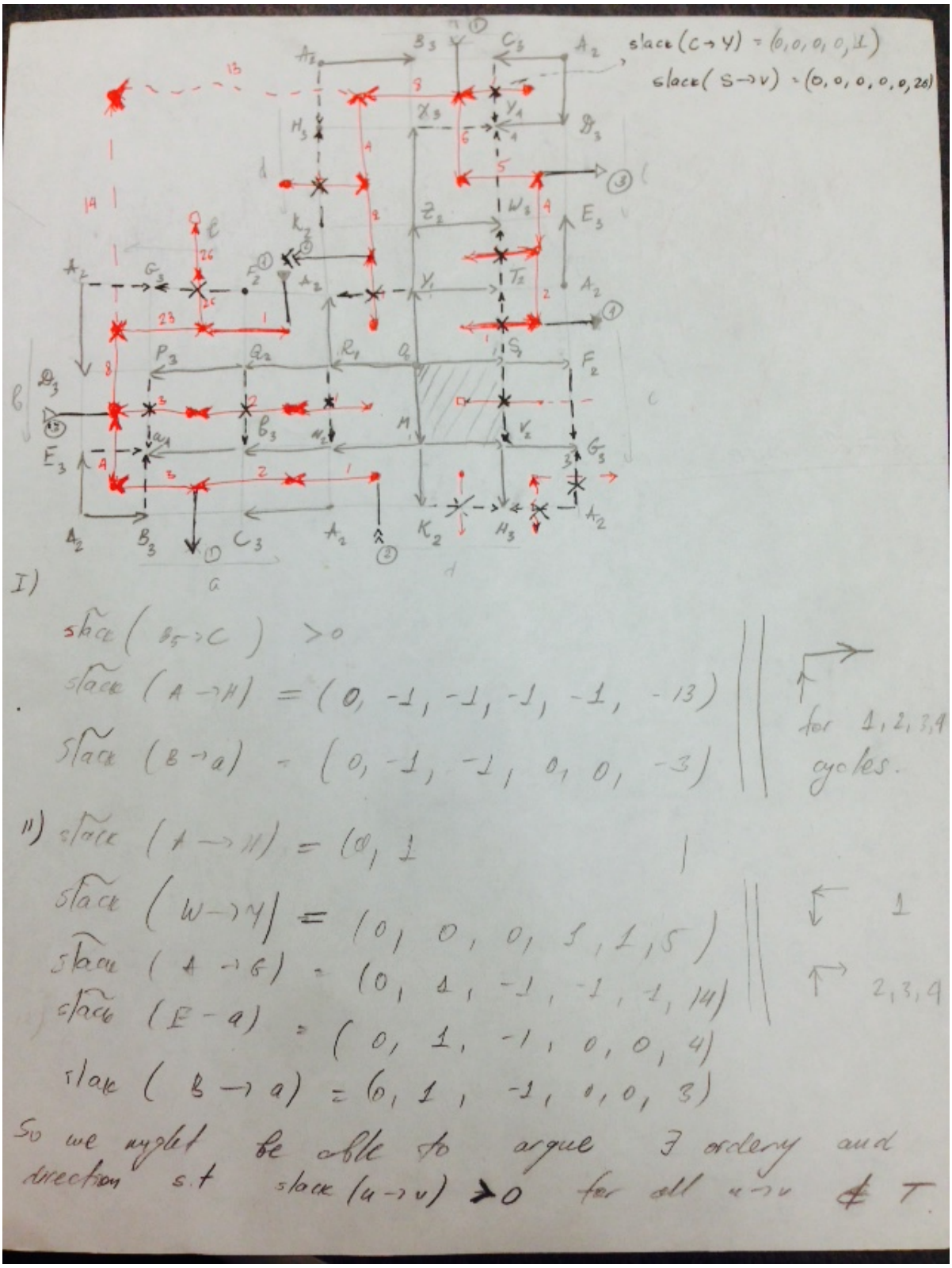
By the definition of *alpha*:

$$\alpha(p_1) - \alpha(p_2) = \sum_f \alpha(\partial f) > 0$$

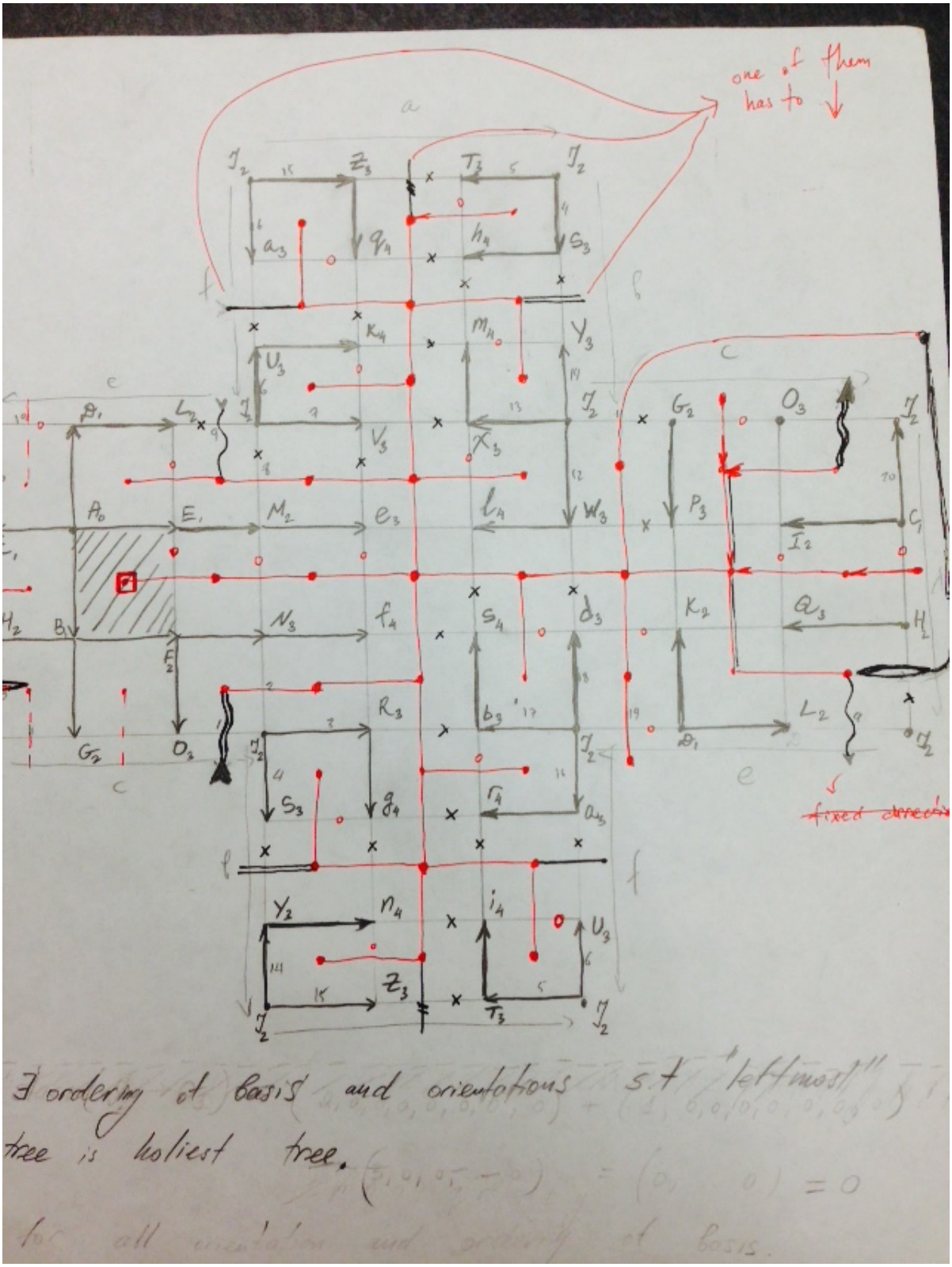
This will ensure that any two paths p_1, p_2, whose $w(p_1) = w(p_2)$ and $[p_1]_\Lambda = [p_2]_\Lambda$, has $\alpha(p_1) \neq \alpha(p_2)$

11 Working on examples:

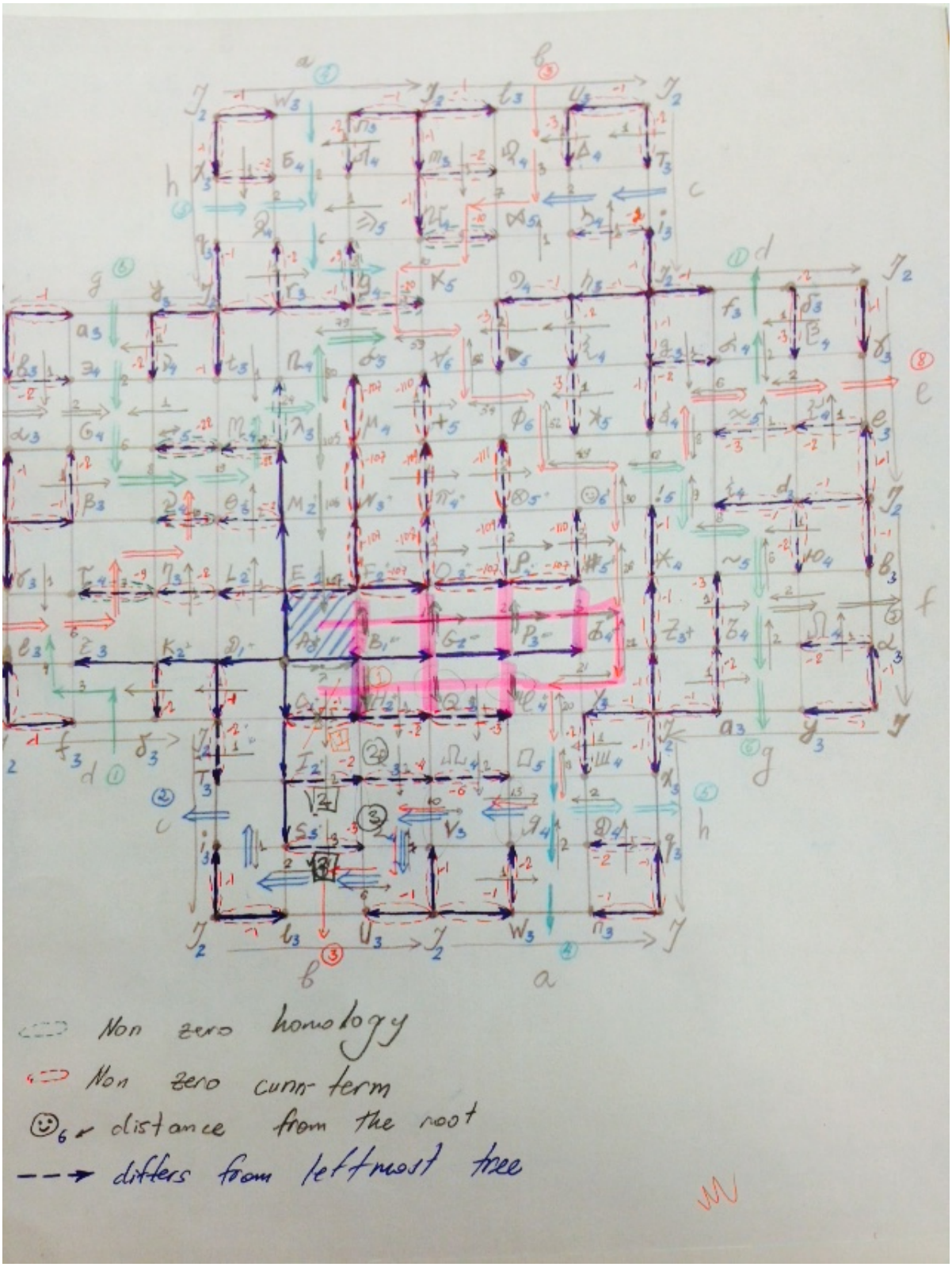
On genus $g = 2$ surface:



On genus $g = 3$ surface:



On genus $g = 4$ surface with initial pivot:



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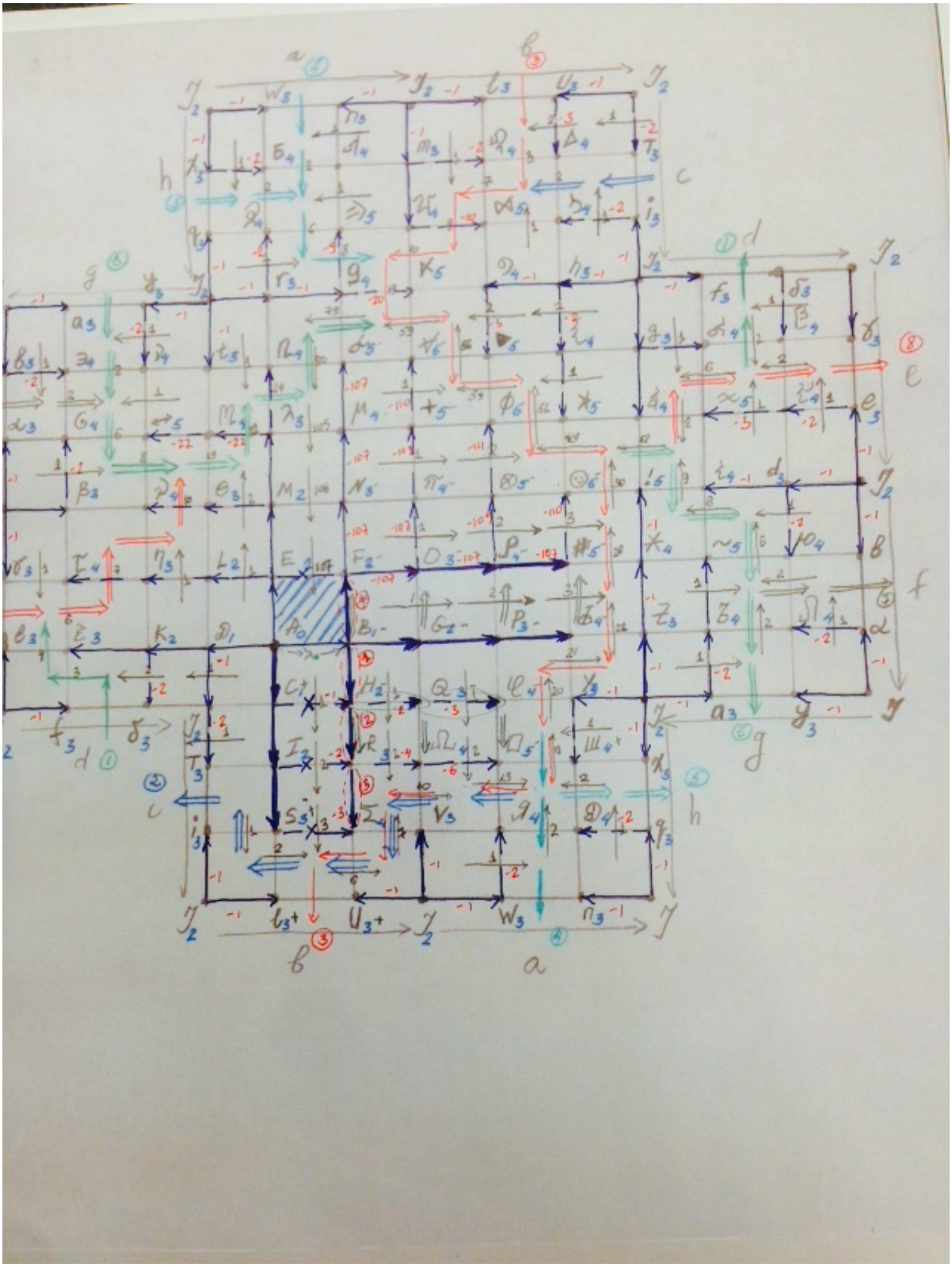
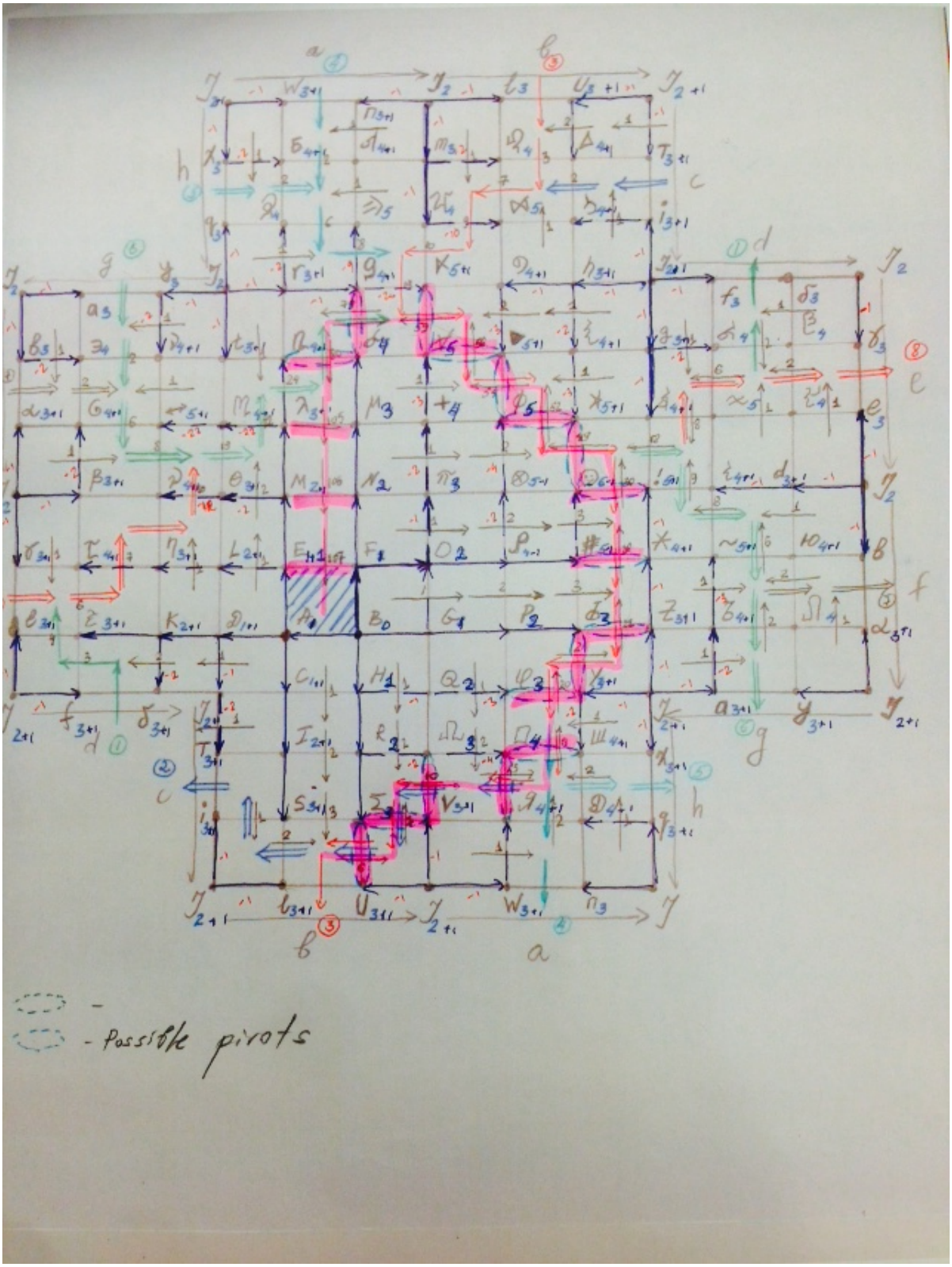


On genus $g = 5$ surface with initial pivot:



12 References:

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