Multiple Source Shortest Path with unit weights

1 Introduction

Given: Let G be a directed graph $(V, \vec{E})$, embedded on a surface with genus g. All edge weights are unit.
Find: Consider boundary f of G. $\forall v \in f$, find a shortest path to $\forall u \in V$.

Let T be the BFS (Breadth first search) tree of G, and C be the BFS co-tree in G. Then there is exactly $2g$ leftover edges $L = \{e_1, e_2, \ldots, e_{2g}\}$.

There exists a unique cycle λ_i in $C \cup e_i$, and $(\lambda_1, \lambda_2, \ldots, \lambda_{2g}) = \Lambda$ defining homology basis.
We define homological signature of an edge as follows:

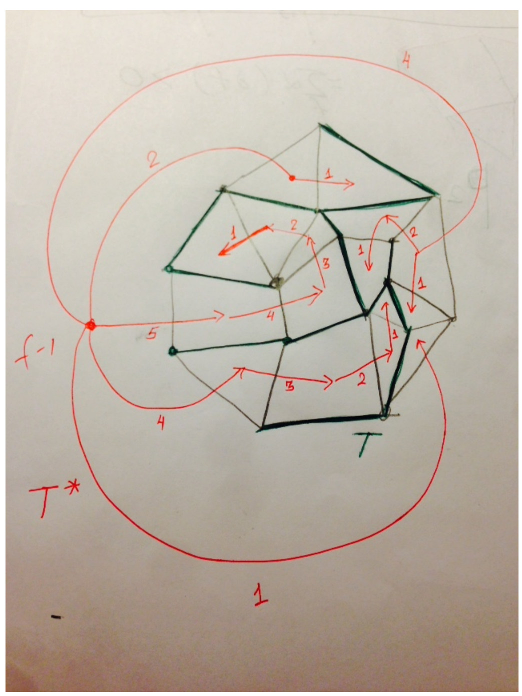
$$[e]_i = \begin{cases} 1 & \text{, if } e \in \lambda_i \\ -1 & \text{, if } rev(r) \in \lambda_i \\ 0 & \text{, otherwise} \end{cases}$$

Furthermore, we define leafmost term α recursively as follows:

$$\alpha(\vec{e}^*) = \begin{cases} 1 & \text{, if } e^* \text{ is a leaf edge of } C \\ \displaystyle\sum_{\text{tail}(\vec{e}'^*)=\text{head}(\vec{e}^*)} \alpha(\vec{e}') & \text{, otherwise} \end{cases}$$

We can extend above definition with $\alpha(\vec{e}) = \alpha(\vec{e}^*)$ and $\alpha(e)^* = -\alpha(\text{rev}(\vec{e}^*))$.

Let $\tilde{w}(\vec{e}) = \langle 1, [\vec{e}], \alpha(\vec{e}) \rangle$ be new weight vector for each edge in G.



Def: An edge $\vec{e}$ is "holier" than $\vec{e}'$, if $\tilde{w}(\vec{e}) < \tilde{w}(\vec{e}')$ in lexicographic comparison. Therefore, we can define "holiness" of any $S \subset G$ as follows:

$$\text{Ho}(S) = \sum_{\vec{e} \in S} \tilde{w}(\vec{e})$$

Holiest tree is a spanning tree with minimal "holiness". We build Holiest tree rooted at r, using slight tweak in the Bellman-Ford algorithm for finding shortest path tree rooted at r.

```
BuildHoliestTree(G, w̃, r):
  Set d[r] ← ⟨0, [0], 0⟩
    pred(r) ← NULL
  for all v : v ≠ r
    d[r] ← ⟨∞, [∞], ∞⟩
    pred(r) ← NULL
  put r into queue
  while queue is not empty:
    Let u ← dequeue item
    for all u → v
      if v is not marked
        mark v and put in the queue
      if isTense(u → v)
        relax(u → v)
```

```
isTense(u → v):
  return d[u] + w̃(u → v) < d[v]
```

```
relax(u → v):
  d[v] ← d[u] + w̃(u → v)
  pred[v] ← u
```

Observation: Each vertex will be added once to the queue.

Corollary: Each edge will be relaxed at most once.

Lemma-1: If there is no tense edge in G, then for each $v : r \to \ldots \to \mathrm{pred}(\mathrm{pred}(v)) \to \mathrm{pred}(v) \to v$ is the holiest path from r to v.

Proof: Let's prove it by induction on $d[v][0]$ distance from the root r.
Base: $d[v][0] = 0$, then v = r, so the claim holds trivially.
Induction Step: Suppose the claim is true for all vertex $v \in V$ such that $d[v][0] < d$ for some d. Consider vertex v such that $d[v][0] = d$. By induction hypothesis, all vertices with $d[u][0] = d - 1$ have "holiest" path correctly updated. By definition, $d[v] = \min_{u \to v} d[u] + \tilde{w}(u \to v)$, here $d[u][0] = d - 1$. By Induction hypothesis, $d[u]$ is not tense and can construct "holiest" path to u, so if there is no tense edge in G then $d[v] = \min_{u \to v} d[u] + \tilde{w}(u \to v)$ holds. □

Corollary: The algorithm will produce "holiest" tree rooted at r in linear time.
We now have produced our initial "Holiest" tree.

2 Moving Along an Edge

Consider a single edge uv in G. Suppose we already computed the shortest-path tree T_u rooted at u. We transform T_u into the shortest-path tree T_v as follows. First, we insert a new vertex s in the inerior of the uv, bisecting it into two edges su and sv with weights:

$$w_\lambda(s \to u) = \langle 0, -2 * [w(u \to v)], -2 * \alpha(w(u \to v)) \rangle + \lambda$$

$$w_\lambda(s \to v) = \langle 1, [w(u \to v)], \alpha(w(u \to v)) \rangle - \lambda = w(u \to v) - \lambda$$

Every other dart $x \to y$ has constant parametric weight $w_\lambda(x \to y) = w(x \to y)$ We then maintain the shortest-path tree T_λ rooted at s, with respect to the weight function w_λ, as λ increases continuously from 0 to $w(u \to v)$. When $\lambda = w(u \to v)$, $T_\lambda = T_v$.

In the following algorithm, **pred** defines SSSP rooted at u, and **dist** is corresponding distance to each vertex in the graph.

```
MoveAlongAnEdge(G, u → v, dist, pred):
  Add new vertex s
  λ ← 0
  w(s → u) ← ⟨0, −2 ∗ [w(s → u)], −2 ∗ α(w(s → u))⟩
  w(s → v) ← w(u → v)
  pred[u], pred[v] ← s
  AddSubtree(w(s → u), u)
  while λ < w(u → v):
    pivot ← FindNextPivot
    If pivot is non NULL
      Pivot(pivot)
    else
      δ = w(u → v) − λ
      AddSubtree(δ, u)
      AddSubtree(−δ, v)
      λ ← λ + δ
```

3 Minimum Cost Flow Problem and methods to solve them

There are several ways to define minimum cost flow and with different types of flows(non-negative and skew-symmetric), capacity or upper bound, edge demand or lower bound, edge cost, and with flow balance. The standard way to define:

$$\begin{array}{l} \underline{\mathbf{min} < \mathbf{cent}, \mathbf{flow} >} \\ \text{s.t} \sum_{u \to v} \text{flow}(u \to v) - \sum_{v \to w} \text{flow}(v \to w) = 0 \\ \qquad \text{flow}(u \to v) = b(u \to v) \\ \qquad flow \geq 0 \end{array}$$

The easiest solution we can find is using augmenting cycle:
We can run any maximum flow algorithm to find feasible solution to the problem(neglecting the cost). Let the augmenting cycle be a cycle with negative cost. Then sending a flow through this cycle would reduce the total cost, while maintaining the feasible property.
We can also define reduced cost as follows:
Let $\phi(v)$ be a any potential function on a vertex. Then $\bar{\text{cost}}(u \to v) = \phi(u) - \phi(v) + \bar{\text{cost}}(u \to v)$ satisfies the condition that cost(C) = $\bar{\text{cost}}(C)$ for any cycle C.

Lemma: A feasible flow f is optimal $\leftrightarrow$ there is no augmenting cycle in the residual graph.

Proof: $\to$ Suppose the flow is optimal. If there is an augmenting cycle C, then we can send flow through C and reduce the cost of current flow, contradicting the f is optimal.
$\leftarrow$ Suppose there is no augmenting cycle. Let $\phi(v)$ = [Shortest path from s to v with respect to the cost function]. Then $\bar{\text{cost}}(u \to v) = \phi(u) - \phi(v) + \text{cost}(u \to v) \geq 0$. For the new cost function, sending more flow in a new residual graph would increase the cost of any flow, therefore f is optimal. □

We can find the augmenting cycle in a graph systematically as follows:
Let T be any fixed spanning tree of G. Define new potential function for each vertex

$$\text{slack}_T(u \to v) = \begin{cases} 0 & \text{, if } u \to v \in T \\ \sum_{e \in \text{cycle in } T \cup \{u \to v\}} \text{cost}(u \to v) & \text{, Otherwise} \end{cases}$$

Above definition preserves the property that $\text{slack}_T(C) = \text{cycle}(C)$ for any cycle C in G. Therefore, we can essentially find negative reduced cost edge in residual graph and push flow through it and do pivoting.

Spanning tree T will be updated as follows:

Update T:
Find bottleneck capacity in a cycle
Push the flow amount equal to bottleneck capacity through the cycle
Pivot out the bottleneck capacity edge, and pivot in the dart with negative reduced cost
Recompute vertex potentials

4 Finding pivot quickly

- What data structure do we maintain in the G*? Finding shortest path in network can also be understood as a Linear Programming problem as follows:
- How do we find next pivot quickly using above structure?

There are two ways to represent the flow in the graph:

- f(u → v) = - f(v → u), for all edges
- f(u → v) ≥ 0 and f(u → v) = 0 if f(u → v) > 0

Transhipment problem

$$\begin{array}{|l|}\hline \mathbf{min} < f, \$ > \\ \text{s.t } \partial f = b \\ \quad\quad f \geq 0 \\ \hline\end{array}$$

Under generic assumption on $f, \$$:

- Basis spanning tree T, there is a unique flow$_T$ that satisfies the condition $\partial\text{flow}_T = b, flow_T(e)$ is nonzero only for edges in T.
- There exist a unique spanning tree T_{OPT} with $flow_{T_{\text{OPT}}}$ is optimal.

Subsequently, we can define slack as follows:
For fixed spanning tree T, there is unique cycle $C = T \cup \{e\}$ for edge e not in T. $slack(e) = \sum_{l \in C} \$(l)$, and 0 otherwise. **Observe that slack is not negative, since otherwise there is no optimal solution to our LP**
The main LP is:

$$\begin{array}{|l|}\hline \mathbf{min} < f, \mathbf{slack}_T > \\ \text{s.t } \partial f = \partial \text{ flow}_T \\ \quad\quad f \geq 0 \\ \hline\end{array} \qquad \begin{array}{|l|}\hline \mathbf{min} < s, \mathbf{flow}_T > \\ \text{s.t } \partial s = \partial \text{ slack}_T \\ \quad\quad s \geq 0 \\ \hline\end{array}$$

And in the case of non-planar embedded graph with genus g:

$$\begin{array}{|l|}\hline \mathbf{min} < f, \mathbf{slack}_T > \\ \text{s.t } \partial f = \partial \text{ flow}_T \\ \quad\quad f \geq 0 \\ \hline\end{array} \qquad \begin{array}{|l|}\hline \mathbf{min} < s, \mathbf{flow}_T > \\ \text{s.t } \partial s = \partial \text{ slack}_T \\ \quad\quad [s] = [\text{slack}_T] \\ \quad\quad s \geq 0 \\ \hline\end{array}$$

Consider the primal LP, We can rewrite the constraints as follows:

$$\partial f = \partial \text{flow}_T \iff \sum_u f(u \to v) - f(v \to u) \iff$$

$$\gamma(u) \sum_u (f(u \to v) - f(v \to u)) = \sum_{u \to v} f(u \to v)(\gamma(u) - \gamma(v)) = \sum_{u \to v} f(u \to v)\gamma(u \to v)$$

Observe here that $\gamma(u \to v) = -\gamma(v \to u)$ We can define s in terms of γ by setting the negative values to be 0, and we will get the exact dual program defined above.

5 Analysis

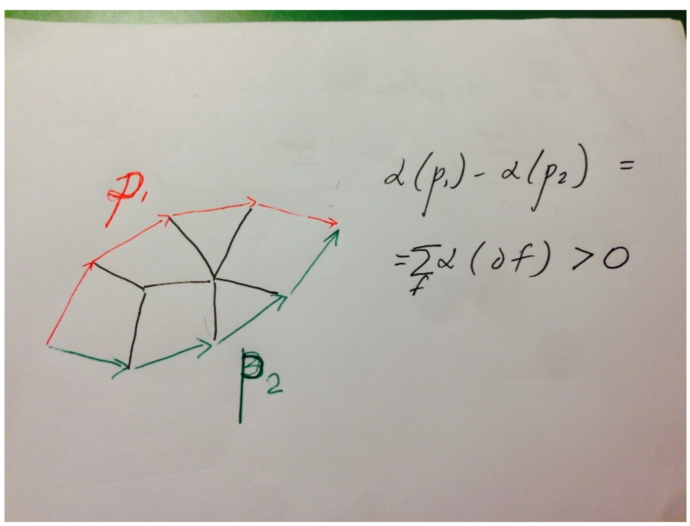
- Building initial tree
- Pivoting
- Number of times each edge is pivoted
- Overall running time

Couple questions regarding the slack and flow:

- If the fixed tree T is arbitrary tree (not necessarily the Holiest Tree, then the flow in the answer does not have to be optimal) but solution to the linear program $min < f, \text{slack}_T >$ is equal to the answer from fixed tree T, not necessarily the optimal solution. That is because for each vertex, the demand satisfies the constraint and if we consider the tree T, then the value of $min < f, \text{slack}_T >$ would be 0, implying it is the optimal solution. (Sum cannot be negative since otherwise there is no optimal solution)
- The reason we picked the slack as the way we defined is due to the fact that slack is not negative, ensuring that the nothing bad happens.
- What makes the non-planar case special with 2g extra constraints?
- How does the slack in dual representation help us to find the pivots quickly?

6 Additional

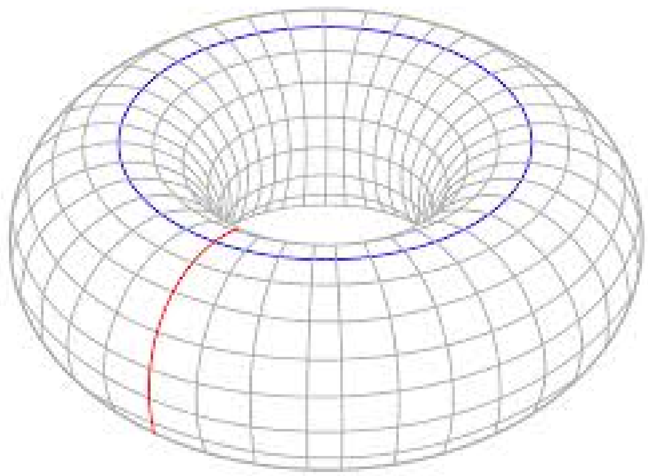
NOTE: Necessity of the α definition on the edges for "Holiness"
Consider the following picture:



By the definition of *alpha*:

$$\alpha(p_1) - \alpha(p_2) = \sum_f \alpha(\partial f) > 0$$

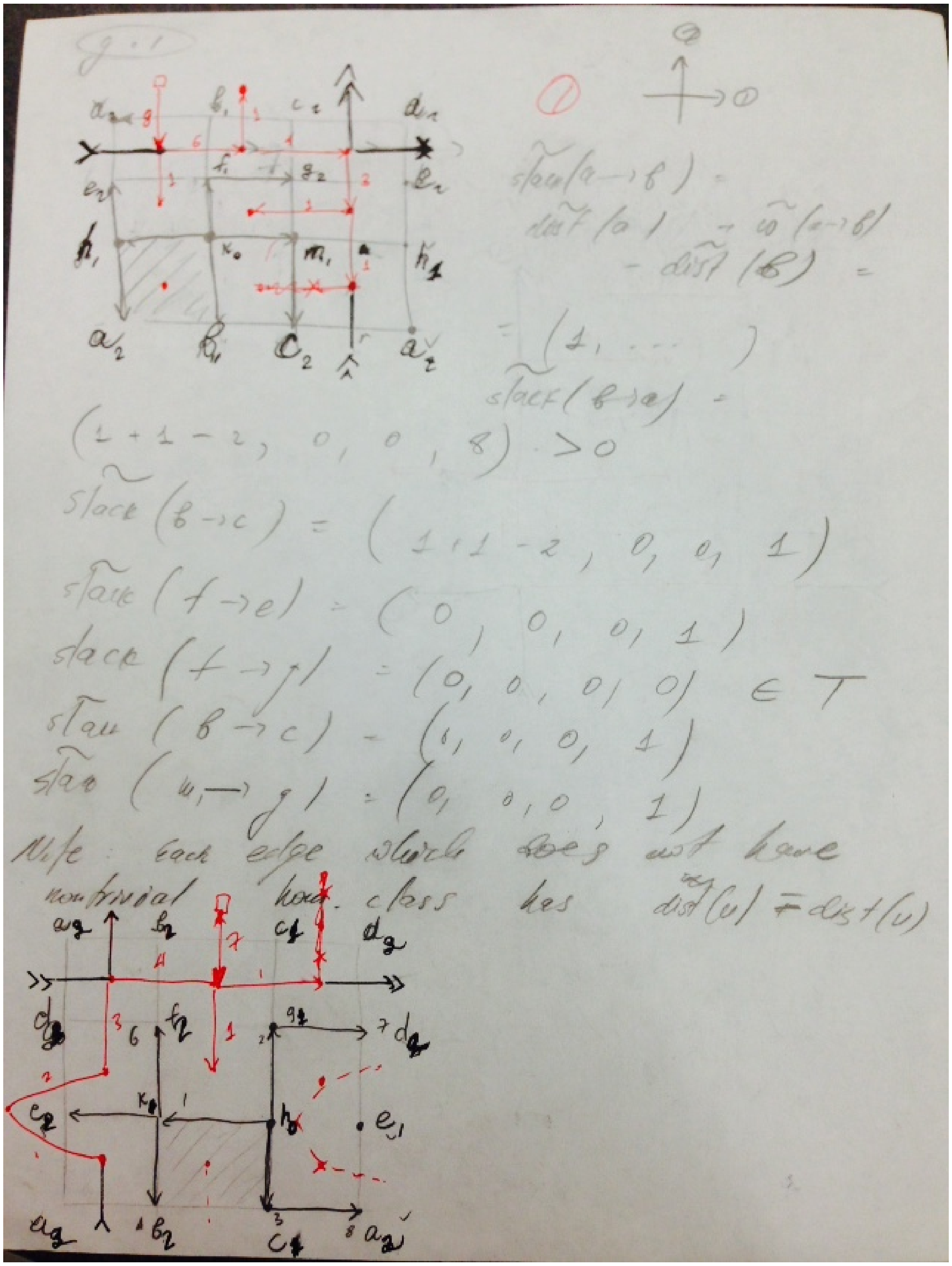
This will ensure that any two paths p_1, p_2, whose $w(p_1) = w(p_2)$ and $[p_1]_\Lambda = [p_2]_\Lambda$, has $\alpha(p_1) \neq \alpha(p_2)$



$http : //en.wikipedia.org/wiki/Homology_(mathematics)$

7 Working on examples:

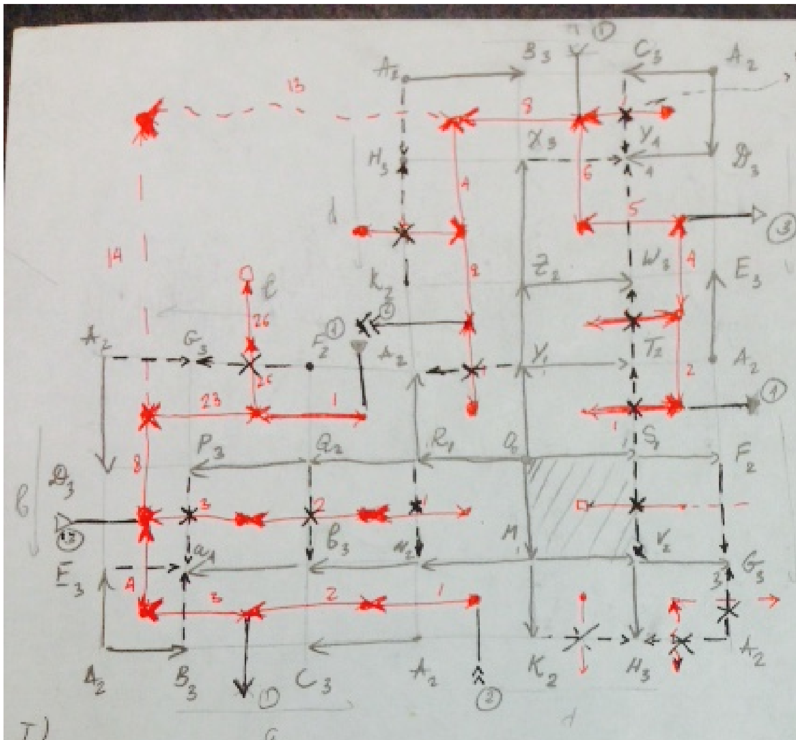
Difference of Holiest Tree and leftmost tree:
On genus $g = 1$ surface:



On genus $g = 2$ surface:

$\mathrm{slack}(C \to Y) = (0,0,0,0,1)$

$\mathrm{slack}(S \to V) = (0,0,0,0,0,26)$



I)

$\widetilde{\mathrm{slack}}(B_5 \to C) > 0$

$\widetilde{\mathrm{slack}}(A \to H) = (0, -1, -1, -1, -1, -13)$

$\widetilde{\mathrm{slack}}(B \to a) = (0, -1, -1, 0, 0, -3)$

for 1, 2, 3, 4 cycles.

II) $\widetilde{\mathrm{slack}}(A \to H) = (0, 1$

$\widetilde{\mathrm{slack}}(W \to Y) = (0, 0, 0, 1, 1, 5)$

$\widetilde{\mathrm{slack}}(A \to G) = (0, 1, -1, -1, -1, 14)$

$\widetilde{\mathrm{slack}}(E - a) = (0, 1, -1, 0, 0, 4)$

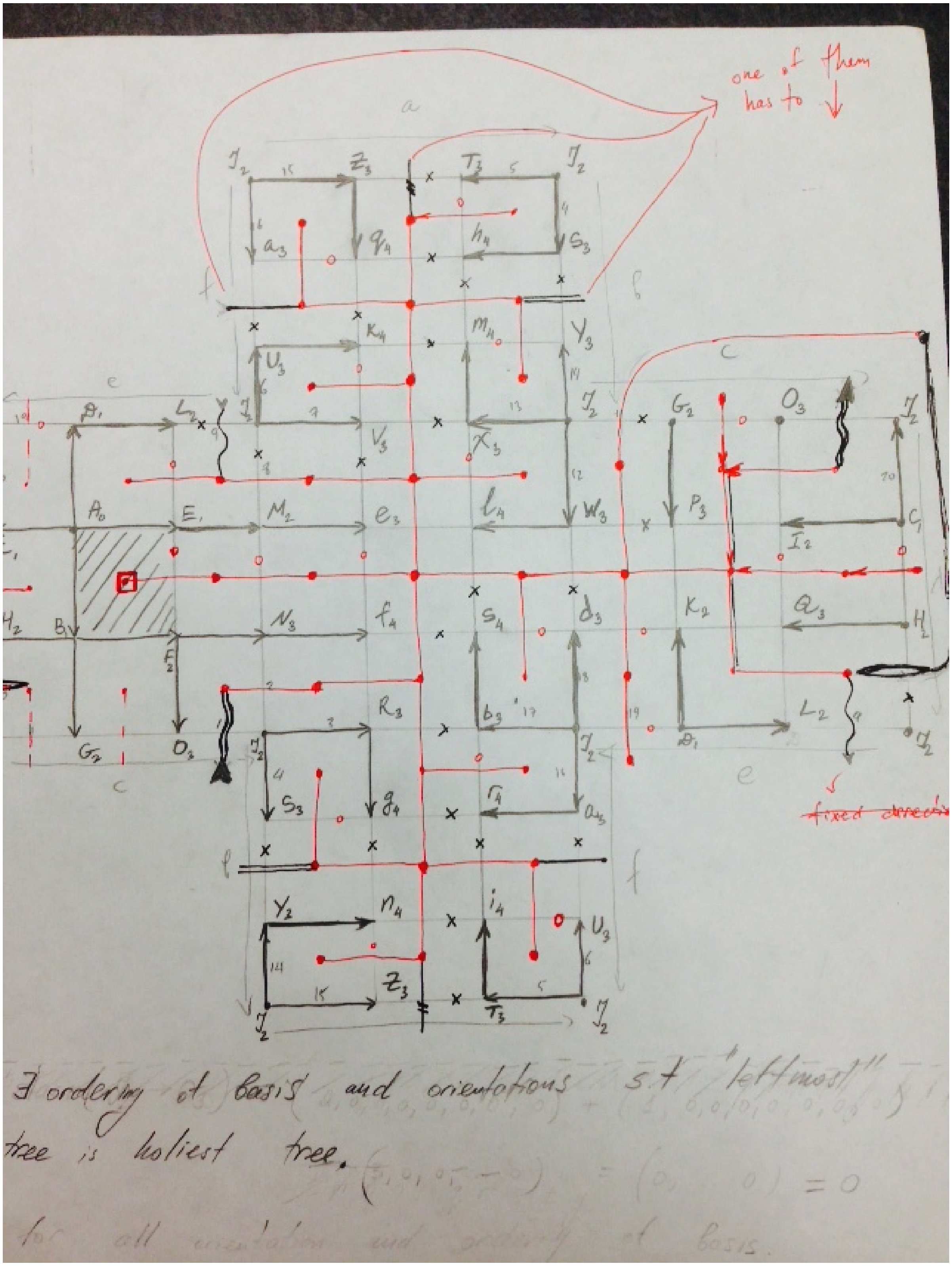
$\mathrm{slack}(B \to a) = (0, 1, -1, 0, 0, 3)$

1

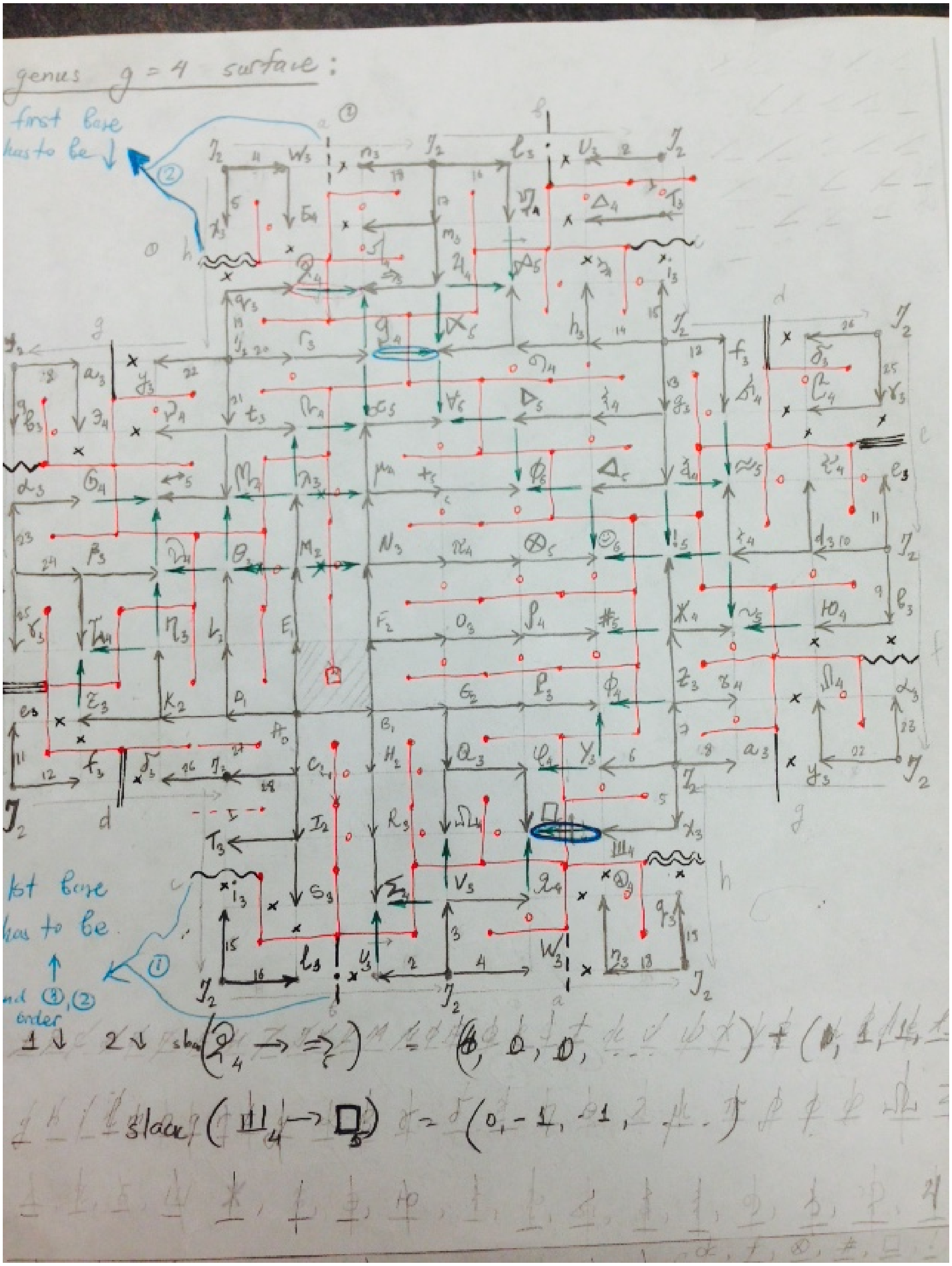
2, 3, 4

So we might be able to argue $\exists$ ordering and direction s.t $\mathrm{slack}(u \to v) > 0$ for all $u \to v \notin T$.

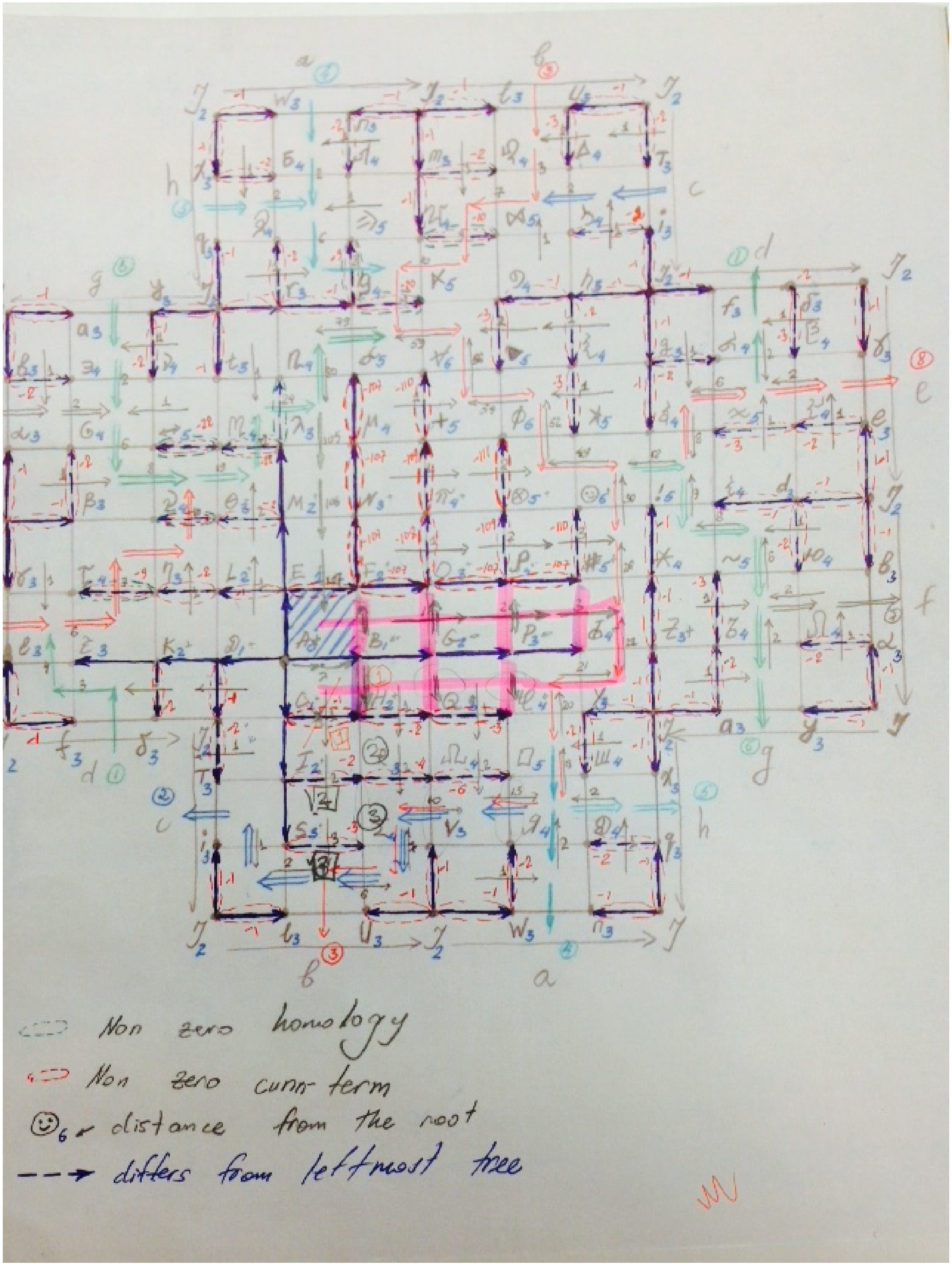
On genus $g = 3$ surface:



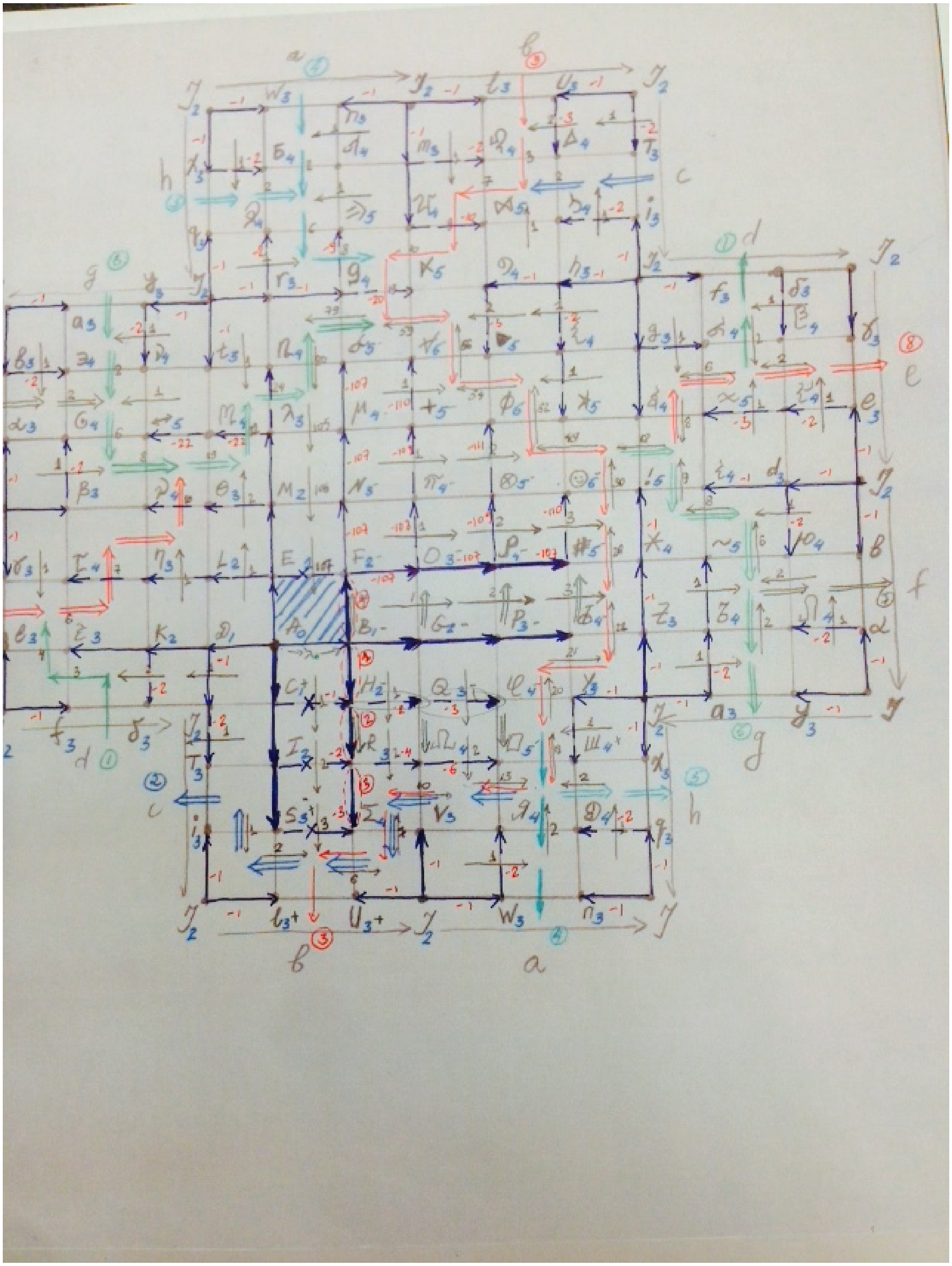
On genus $g = 4$ surface with initial Holy Tree build:



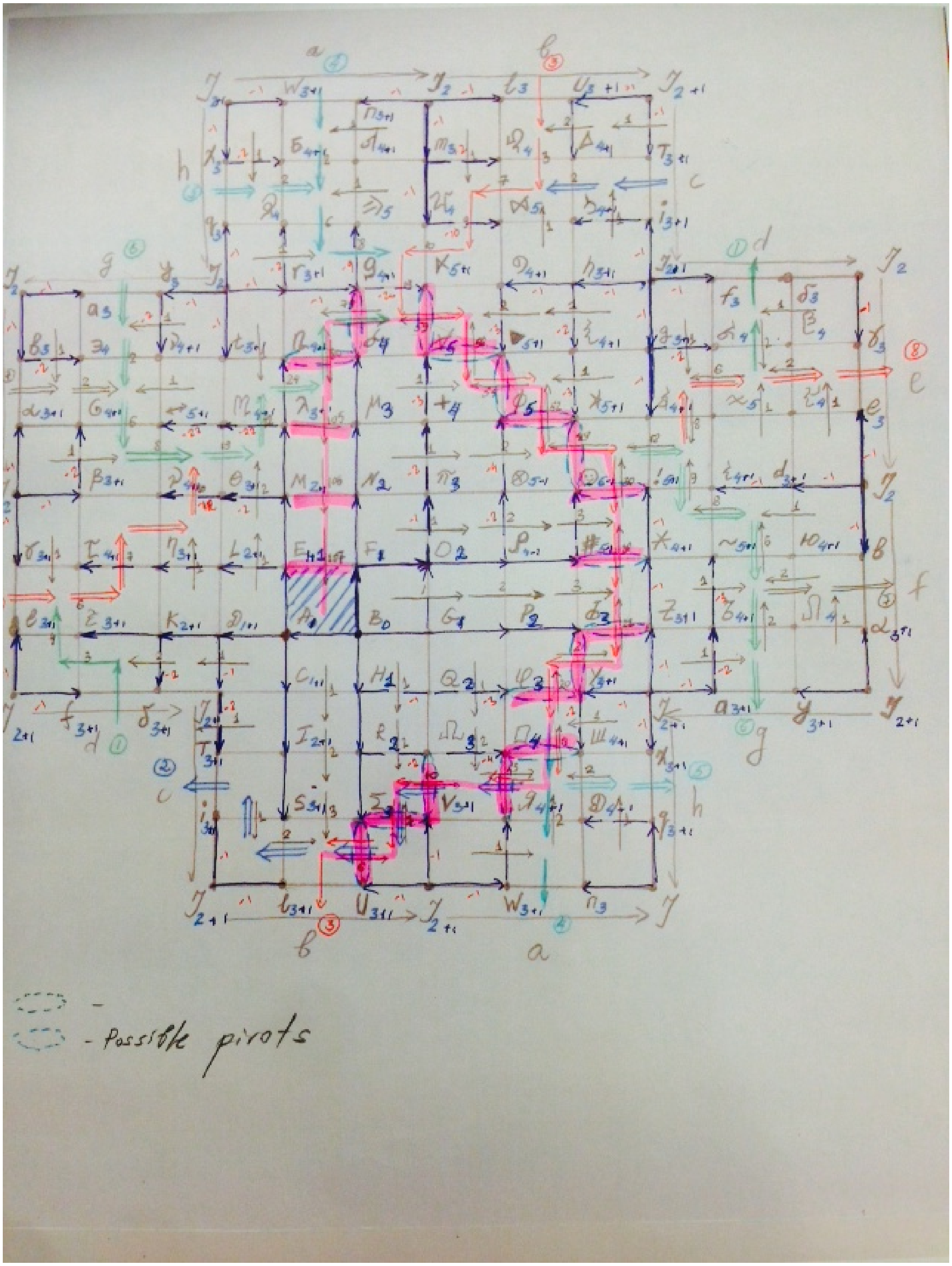
On genus $g = 4$ surface with initial pivot:



On genus $g = 4$ surface with initial pivot:



On genus $g = 5$ surface with initial pivot:



8 References:

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