

Multiple Source Shortest Path with unit weights

1 Introduction

Given: Let G be a directed graph (V, \vec{E}) , embedded on a surface with genus g . All edge weights are unit.

Find: Consider boundary f of G . $\forall v \in f$, find a shortest path to $\forall u \in V$.

Let T be the BFS (Breadth first search) tree of G , and C be the BFS co-tree in G . Then there is exactly $2g$ leftover edges $L = e_1, e_2, \dots, e_{2g}$.

There exists a unique cycle λ_i in $C \cup e_i$, and $(\lambda_1, \lambda_2, \dots, \lambda_{2g}) = \Lambda$ defines homology basis. We define homological signature of an edge as follows:

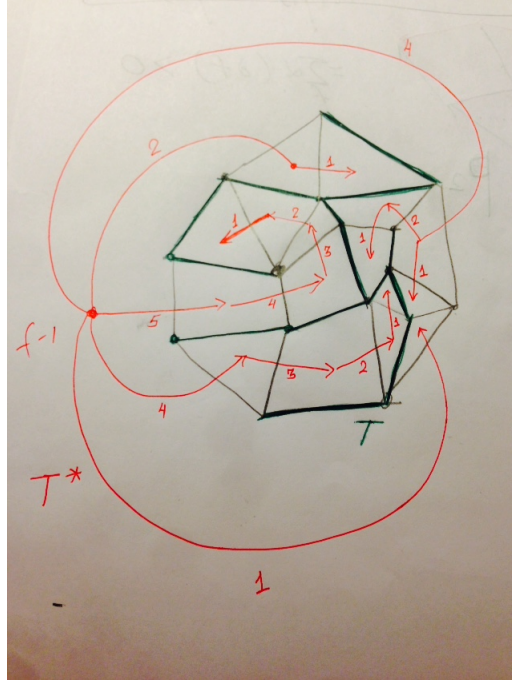
$$[e]_i = \begin{cases} 1 & , \text{if } e \in \lambda_i \\ -1 & , \text{if } rev(r) \in \lambda_i \\ 0 & , \text{otherwise} \end{cases}$$

Furthermore, we define α recursively as follows:

$$\alpha(\vec{e}^*) = \begin{cases} 1 & , \text{if } e^* \text{ is a leaf edge of } C \\ \sum_{tail(\vec{e}^*)=head(\vec{e}')} \alpha(\vec{e}') & , \text{otherwise} \end{cases}$$

We can extend above definition with $\alpha(\vec{e}) = \alpha(\vec{e}^*)$ and $\alpha(e)^* = -\alpha(rev(\vec{e}^*))$.

Let $\tilde{w}(\vec{e}) = \langle 1, [\vec{e}], \alpha(\vec{e}) \rangle$ be new weight vector for each edge in G .



Def: An edge \vec{e} is "holier" than \vec{e}' , if $\tilde{w}(\vec{e}) < \tilde{w}(\vec{e}')$ in lexicographic comparison. Therefore, we can define "holiness" of any $S \subset G$ as follows:

$$Ho(S) = \sum_{\vec{e} \in S} \tilde{w}(\vec{e})$$

Holiest tree is a spanning tree with minimal "holiness". We build Holiest tree rooted at r , using slight tweak in the Bellman-Ford algorithm for finding shortest path tree rooted at r .

BuildHoliestTree(G, \tilde{w}, r):

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Set  $d[r] \leftarrow \langle 0, [0], 0 \rangle$ 
 $\text{pred}(r) \leftarrow \text{NULL}$ 
for all  $v : v \neq r$ 
   $d[v] \leftarrow \langle \infty, [\infty], \infty \rangle$ 
   $\text{pred}(v) \leftarrow \text{NULL}$ 
put  $r$  into queue
while queue is not Empty:
  Let  $s \leftarrow$  dequeue item
  for all  $u : s \rightarrow u$ 
    if  $u$  is not marked
      mark  $u$  and put in the queue
    if  $\text{isTense}(s \rightarrow u)$ 
       $\text{relax}(s \rightarrow u)$ 

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isTense($s \rightarrow u$):

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return  $d[s] + \tilde{w}(s \rightarrow u) < d[u]$ 

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relax($s \rightarrow u$):

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 $d[u] \leftarrow d[s] + \tilde{w}(s \rightarrow u)$ 
 $\text{pred}[u] \leftarrow s$ 

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Observation: Each edge will be added once to the queue.

Corollary: Each edge will be relaxed at most once.

Lemma-1: If there is no tense edge in G , then for each $v : r \rightarrow \dots \rightarrow \text{pred}(\text{pred}(v)) \rightarrow \text{pred}(v) \rightarrow v$ is the holiest path from r to v .

Proof: Let's prove it by induction on $d[v]$ distance from the root r .

Base: $d[v] = 0$, then $v = r$, so the claim is definitely true.

Induction Step: Suppose the claim is true for all vertex $v \in V$ such that $d[v] < d$ for some d . Consider vertex v such that $d[v] = d$. By induction hypothesis, all vertices with $d[u] = d - 1$ have "holiest" path correctly updated. By definition, $d[v] = \min_{s \rightarrow v} d[s] + \tilde{w}(s \rightarrow v)$, here $d[s] = d - 1$. By Induction hypothesis, $d[s]$ is "holiest" path, so if there is no tense edge in G then $d[v] = \min_{s \rightarrow v} d[s] + \tilde{w}(s \rightarrow v)$ holds. \square

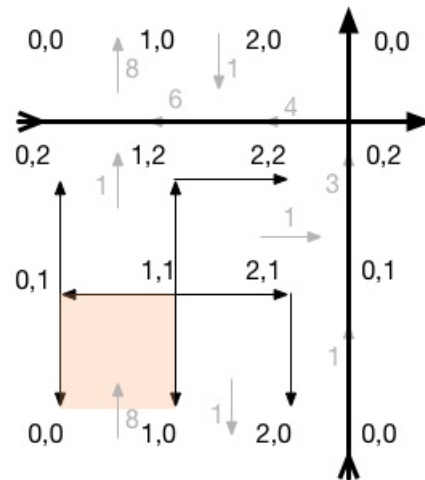
Corollary: The algorithm will produce "holiest" tree rooted at r in linear time.

We now have produced our initial "Holiest" tree.

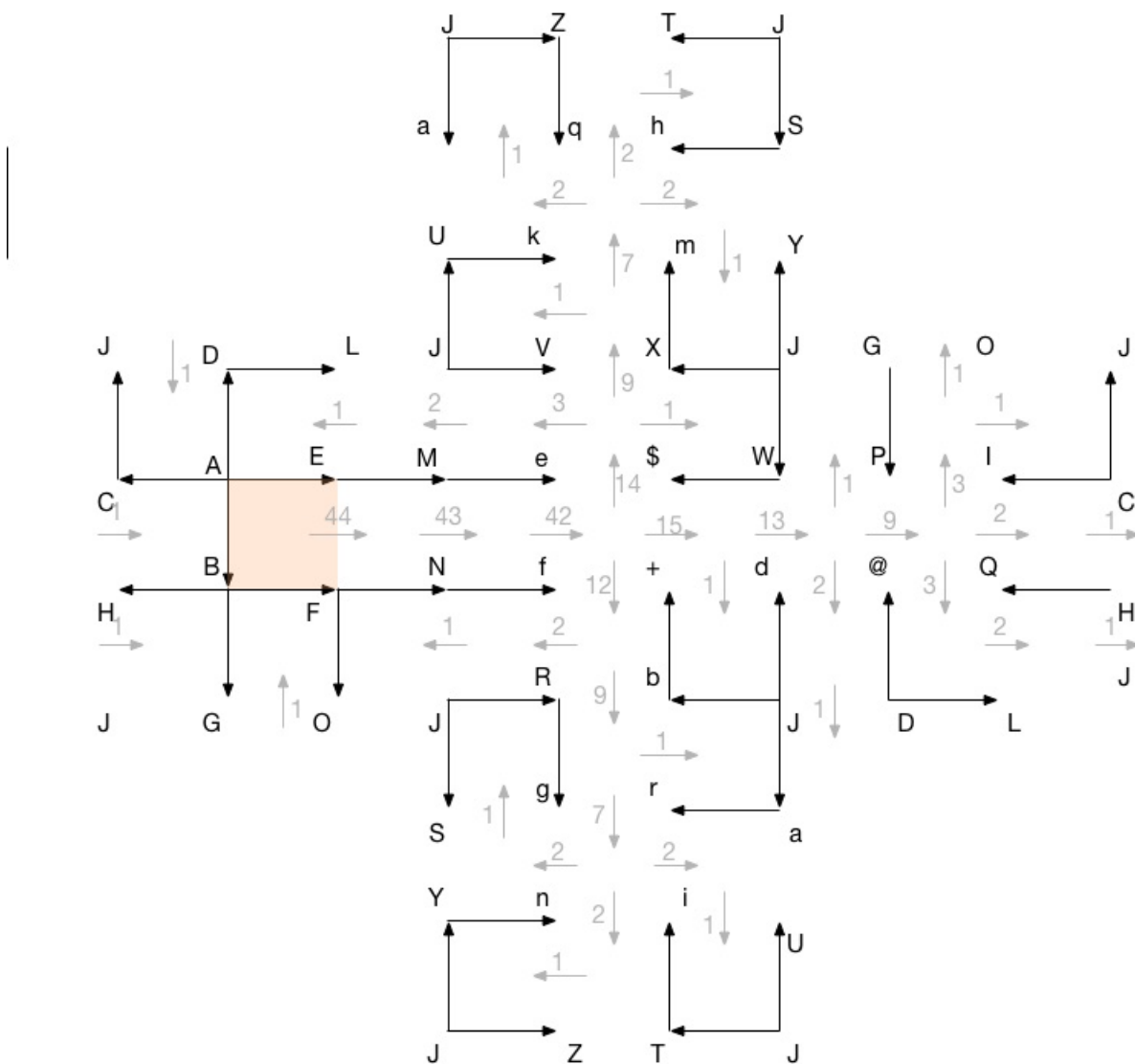
2 Working on examples:

Holiest Tree:

On genus $g = 1$ surface:



On genus $g = 3$ surface:



3 References:

- Cabello, Sergio, Erin W. Chambers, and Jeff Erickson. "Multiple-source shortest paths in embedded graphs." *SIAM Journal on Computing* 42.4 (2013): 1542-1571.
- Eisenstat, David, and Philip N. Klein. "Linear-time algorithms for max flow and multiple-source shortest paths in unit-weight planar graphs." *Proceedings of the forty-fifth annual ACM symposium on Theory of computing*. ACM, 2013.
- Erickson, Jeff. "Maximum flows and parametric shortest paths in planar graphs." *Proceedings of the twenty-first annual ACM-SIAM symposium on Discrete Algorithms*. Society for Industrial and Applied Mathematics, 2010.