J. Geomag. Geoelectr., **47**, 191–212, 1995

Ionospheric Electrodynamics Using Magnetic Apex Coordinates

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(Received September 19, 1994; Revised November 23, 1994; Accepted November 24, 1994)

The equations of ionospheric electrodynamics are developed for a geomagnetic field of general configuration, with specific application to coordinate systems based on Magnetic Apex Coordinates. Two related coordinate systems are proposed: Modified Apex Coordinates, appropriate for calculations involving electric fields and magnetic-field-aligned currents; and Quasi-Dipole Coordinates, appropriate for calculations involving height-integrated ionospheric currents. Distortions of the geomagnetic field from a dipole cause modifications to the equations of electrodynamics, with distortion factors exceeding 50% at some geographical locations. Under the assumption of equipotential geomagnetic-field lines, it is shown how the field-line-integrated electrodynamic equations can be expressed in two dimensions in magnetic latitude and longitude, and how the height-integrated and field-aligned current densities can be calculated. Expressions are derived for the simplified calculation of magnetic perturbations above and below the ionosphere associated with the three-dimensional current system. It is shown how the base vectors for the Modified Apex coordinate system can be applied to map electric fields, plasma-drift velocities, magnetic perturbations, and Poynting fluxes along the geomagnetic field to other altitudes, automatically taking into account changes in magnitude and direction of these vector quantities along the field line. Similarly, it is shown how Quasi-Dipole coordinates are useful for expressing horizontal ionospheric currents, equivalent currents, and ground-level magnetic perturbations. A computer code is made available for efficient calculation of the various coordinates, base vectors, and related quantities described in this article.

1. Introduction

The strong influence of the geomagnetic field on charged-particle motion causes many ionospheric phenomena to be naturally organized with respect to the geomagnetic field. Ionospheric conductivity is highly anisotropic, and the conductivity parallel to the geomagnetic field is so large that field lines are very nearly equipotential in the ionosphere for quasi-static large-scale electrodynamic conditions. Even conjugate points in the northern and southern magnetic hemispheres lying on the same field line have nearly the same electric potential for field lines that do not extend more than several Earth radii away from the Earth, that is, interior to the region where acceleration of auroral particles occurs. Near the magnetic equator, where the geomagnetic field becomes horizontal, electrodynamic phenomena like the equatorial electrojet are closely tied to the local geomagnetic-field configuration. For these reasons it is often convenient to organize data and to construct models in a coordinate system aligned with the geomagnetic field.

To a first approximation, the geomagnetic field is dipolar, and dipole coordinates have frequently been used in ionospheric studies like those modeling the ionospheric wind dynamo (e.g. Stening, 1968; Schieldge *et al.*, 1973; Takeda, 1982; Singh and Cole, 1987; Richmond and Roble, 1987). However, the non-dipolar component of the geomagnetic field produces significant distortions that must be taken into account when observational data are analyzed. The distortions are also found to have a significant influence on the results of simulation models (e.g. Walton and Bowhill, 1979). Three magnetic coordinate systems using realistic models of the Earth's main field of internal origin that have been proposed for ionospheric work are: Magnetic Apex Coordinates (VanZandt *et al.*, 1972); *Coordinées de l'Anneau Équatorial* or Corrected Geomagnetic Coordinates (Mayaud, 1960; Hakura, 1965; Gustafsson *et al.*, 1992), upon which are based the PACE Geomagnetic Coordinates (Baker and Wing, 1989); and Constant B-Minimum Coordinates (Gustafsson *et al.*, 1992). These systems have constant values of magnetic latitude and longitude along field lines (apart from a possible sign switch in defining magnetic latitudes for the two magnetic hemispheres). Another commonly used magnetic-latitude coordinate is invariant latitude (see McIlwain, 1966), which is nearly, though not strictly, constant along geomagnetic-field lines. More recently, Papitashvili *et al.* (1992) have proposed the "Momentary" Constant B-Minimum Coordinate system that takes into account distortions of the field by magnetospheric currents, and thus varies with Universal Time and with magnetic activity. Given a well-defined magnetic-latitude/magnetic-longitude coordinate system, it is relatively easy to locate the magnetic coordinates of any geographic point, but it is considerably more difficult to know how the non-dipolar geomagnetic-field distortions should be taken into account when vector quantities like the electric field or electric current density are considered. Realistic magnetic-field-oriented coordinate systems are generally non-orthogonal, i.e., the gradients of the magnetic coordinates are not at mutual right angles, which introduces complications in dealing with vector quantities.

The purpose of this article is to describe how electrodynamic calculations can be carried out in the ionosphere using a generalized coordinate system that is oriented with respect to the geomagnetic field. Consideration is given to modeling the ionospheric wind dynamo; to organizing scalar and vector quantities with respect to magnetic latitude and longitude; to mapping some of those quantities along the geomagnetic field to a common reference height; and to calculating magnetic perturbations produced by ionospheric and geomagnetic-field-aligned currents. Two different coordinate systems based on Magnetic Apex Coordinates are defined in Sections 3 and 6, one of which is useful for phenomena that are primarily organized along geomagnetic-field lines, and the other of which is useful for phenomena related to horizontally stratified ionospheric currents. In order to facilitate the application of the formulas presented, a computer code is made available through the Coupling, Energetics and Dynamics of Atmospheric Regions (CEDAR) Data Base that provides the needed transformation parameters in numerical form.

2. Steady-State Electrodynamic Equations in Generalized Magnetic-Field-Oriented Coordinates

For time scales longer than about one minute, it is valid to treat global ionospheric electrodynamics as being in a steady state, with the electric field $\mathbf{E}$ being electrostatic and the current density $\mathbf{J}$ being divergence-free. It can usually be assumed, for simplicity, that the electric-field component along the geomagnetic field $\mathbf{B}$ vanishes, so that geomagnetic field lines are equipotentials. The field $\mathbf{B}$ can be expressed as the sum of the main field $\mathbf{B}_0$ and an additional component $\Delta\mathbf{B}$ produced by the currents external to the Earth, where $\Delta\mathbf{B}$ is only of perturbation magnitude in the vicinity of the Earth. The current density $\mathbf{J}$ essentially consists of an ohmic component transverse to the magnetic field in the ionosphere, a non-ohmic transverse magnetospheric component $\mathbf{J}_M$, and a field-aligned component $\mathbf{J}_\parallel$:

$$\mathbf{J} = \sigma_P(\mathbf{E} + \mathbf{u} \times \mathbf{B}) + \sigma_H \mathbf{b} \times (\mathbf{E} + \mathbf{u} \times \mathbf{B}) + \mathbf{J}_M + \mathbf{J}_\parallel, \tag{2.1}$$

where σ_P and σ_H are the Pedersen and Hall components of the conductivity tensor, $\mathbf{u}$ is the neutral-wind velocity, and $\mathbf{b}$ is a unit vector parallel to $\mathbf{B}$.

Let (x_1, x_2, x_3) be a generalized coordinate system, with x_1 and x_2 defined to be constant on a geomagnetic-field line and x_3 varying monotonically along the field line. The coordinates need not be orthogonal, but the volume parameter W, defined as

$$W \equiv (\nabla x_1 \times \nabla x_2 \cdot \nabla x_3)^{-1}, \tag{2.2}$$

is assumed to be positive, i.e., the system is right-handed. Area vectors $\mathbf{a}_i$ are defined as

$$\mathbf{a}_i = W\nabla x_i. \tag{2.3}$$

The appropriate expressions for gradient and divergence in generalized coordinates can be used to give

$$\mathbf{E} = -\nabla\Phi = -\frac{1}{W}\sum_{i=1}^{2}\mathbf{a}_i\frac{\partial\Phi}{\partial x_i}, \tag{2.4}$$

$$\nabla\cdot\mathbf{J} = \frac{1}{W}\sum_{i=1}^{3}\frac{\partial(\mathbf{a}_i\cdot\mathbf{J})}{\partial x_i} = 0, \tag{2.5}$$

where Φ is the electrostatic potential. [Note that the $i = 3$ component does not appear in the summation of (2.4) because the equipotentiality of magnetic-field lines implies that $\partial\Phi/\partial x_3 = 0$.] Combining (2.1), (2.4), and (2.5), and integrating $W\nabla\cdot\mathbf{J}$ with respect to x_3 from the base of the conducting ionosphere in one hemisphere ($x_3 = x_a$) to the other hemisphere ($x_3 = x_b$) on closed geomagnetic field lines yields the following partial differential equation for Φ:

$$\frac{\partial}{\partial x_1}\left(S_{11}\frac{\partial\Phi}{\partial x_1} + (S_{12}+S_H)\frac{\partial\Phi}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left((S_{21}-S_H)\frac{\partial\Phi}{\partial x_1} + S_{22}\frac{\partial\Phi}{\partial x_2}\right) = \frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2}, \tag{2.6}$$

$$S_{ij} = \int_{x_a}^{x_b}\left(\frac{\mathbf{a}_i\cdot\mathbf{a}_j}{W}\right)\sigma_P\,dx_3 = \int_{s_a}^{s_b}\left(\frac{\nabla x_i\cdot\nabla x_j}{|\nabla x_1\times\nabla x_2|}\right)\sigma_P\,ds, \tag{2.7}$$

$$S_H = \int_{x_a}^{x_b}\left(\frac{-\mathbf{b}\cdot\mathbf{a}_i\times\mathbf{a}_j}{W}\right)\sigma_H\,dx_3 = \int_{s_a}^{s_b}(-\mathbf{b}\cdot\nabla s)\,\sigma_H\,ds, \tag{2.8}$$

$$Q_i = \int_{x_a}^{x_b}\mathbf{a}_i\cdot(\sigma_P\mathbf{u}\times\mathbf{B}+\sigma_H B\mathbf{u}+\mathbf{J}_M)\,dx_3 = \int_{s_a}^{s_b}\left(\frac{\nabla x_i}{|\nabla x_1\times\nabla x_2|}\right)\cdot(\sigma_P\mathbf{u}\times\mathbf{B}+\sigma_H B\mathbf{u}+\mathbf{J}_M)\,ds, \tag{2.9}$$

where s is geometric distance along the field line, increasing with x_3, and the pairs (x_a, x_b) or (s_a, s_b) are the integration limits corresponding to the base of the conducting region at the ends of the field line. In (2.8) the quantity $(-\mathbf{b}\cdot\nabla s)$ is either -1 or 1, depending on whether x_3 and s increase in the direction parallel or antiparallel to $\mathbf{B}$, respectively.

If the conductivities, neutral wind velocity, and magnetospheric current density perpendicular to $\mathbf{B}$ are given, and if suitable boundary conditions for Φ are given for the domain of interest, then (2.6) can in principle be solved for Φ within the domain. A suitable boundary condition at the edge of the domain corresponding to geomagnetic field lines that just graze the bottom of the conducting region at the magnetic equator is that the vertical component of current vanish, or

$$(\sigma_P\mathbf{k}+\sigma_H\mathbf{k}\times\mathbf{b})\cdot\sum_{i=1}^{2}\frac{\mathbf{a}_i}{W}\frac{\partial\Phi}{\partial x_i} = (\sigma_P\mathbf{k}+\sigma_H\mathbf{k}\times\mathbf{b})\cdot\mathbf{u}\times\mathbf{B}, \tag{2.10}$$

where $\mathbf{k}$ is a unit vector in the upward direction. A suitable boundary condition at high latitudes is more difficult, as it depends on properties of the magnetosphere that are not yet adequately understood. Since (2.6) is applicable only in regions where magnetic field lines have both ends connected to the ionosphere, it does not apply directly to open-field regions in the polar caps. The high-latitude boundary for (2.6) must therefore lie at or below the boundary between open and closed field lines, yet strong electromagnetic coupling across this boundary is expected. Further discussion of this problem goes beyond the scope of this article.

Once the electric potential is determined, the electric field $\mathbf{E}$ and the current component $\mathbf{J}_\perp$ perpendicular to $\mathbf{B}$ in the ionosphere are readily calculated from (2.4) and (2.1). Calculation of $\mathbf{J}_\parallel$ is somewhat more complicated, though straightforward. Since both $(\mathbf{J}_\parallel + \mathbf{J}_\perp)$ and $\mathbf{B}$ are divergenceless,

$$\nabla \cdot \mathbf{J}_\parallel = \nabla \cdot (J_\parallel \mathbf{B}/B) = \mathbf{B} \cdot \nabla(J_\parallel/B) = \frac{\mathbf{B}\cdot\mathbf{a}_3}{W}\frac{\partial}{\partial x_3}\left(\frac{J_\parallel}{B}\right) = -\nabla\cdot\mathbf{J}_\perp = -\frac{1}{W}\sum_{i=1}^{3}\frac{\partial(\mathbf{a}_i\cdot\mathbf{J}_\perp)}{\partial x_i}. \tag{2.11}$$

Since $\mathbf{a}_1 \cdot \mathbf{B} = \mathbf{a}_2 \cdot \mathbf{B} = 0$,

$$\nabla \cdot \mathbf{B} = \frac{1}{W}\frac{\partial(\mathbf{a}_3\cdot\mathbf{B})}{\partial x_3} = 0, \tag{2.12}$$

so that $\mathbf{a}_3 \cdot \mathbf{B}$ is constant along a field line. Multiplying (2.11) by $W/(\mathbf{a}_3\cdot\mathbf{B})$, integrating with respect to x_3, and multiplying the result by B gives

$$J_\parallel = \frac{-B}{\mathbf{a}_3\cdot\mathbf{B}}\int_{x_a}^{x_3}\sum_{i=1}^{3}\frac{\partial(\mathbf{a}_i\cdot\mathbf{J}_\perp)}{\partial x_i}\,dx_3' = \frac{B}{\mathbf{a}_3\cdot\mathbf{B}}\int_{x_3}^{x_b}\sum_{i=1}^{3}\frac{\partial(\mathbf{a}_i\cdot\mathbf{J}_\perp)}{\partial x_i}\,dx_3'. \tag{2.13}$$

The magnetic perturbation field $\Delta\mathbf{B}$ associated with the three-dimensional current system can be calculated without directly calculating $J_\parallel$. It is possible to write $\Delta\mathbf{B}$ generally as

$$\Delta\mathbf{B} = \Delta\mathbf{B}^{(1)} + \Delta\mathbf{B}^{(2)}, \tag{2.14}$$

$$\Delta\mathbf{B}^{(1)} = \frac{\beta_1\mathbf{a}_1 + \beta_2\mathbf{a}_2}{W} = \beta_1\nabla x_1 + \beta_2\nabla x_2, \tag{2.15}$$

$$\Delta\mathbf{B}^{(2)} = -\nabla\eta. \tag{2.16}$$

Then

$$\begin{aligned}\mu_0\mathbf{J} &= \nabla\times(\Delta\mathbf{B}) = \nabla\beta_1\times\nabla x_1 + \nabla\beta_2\times\nabla x_2 = \frac{\nabla\beta_1\times\mathbf{a}_1 + \nabla\beta_2\times\mathbf{a}_2}{W}\\ &= \frac{1}{W^2}\left(\mathbf{a}_2\times\mathbf{a}_1\frac{\partial\beta_1}{\partial x_2} + \mathbf{a}_3\times\mathbf{a}_1\frac{\partial\beta_1}{\partial x_3} + \mathbf{a}_1\times\mathbf{a}_2\frac{\partial\beta_2}{\partial x_1} + \mathbf{a}_3\times\mathbf{a}_2\frac{\partial\beta_2}{\partial x_3}\right),\end{aligned} \tag{2.17}$$

so that

$$\mu_0\mathbf{a}_1\cdot\mathbf{J} = -\frac{\partial\beta_2}{\partial x_3}, \tag{2.18}$$

$$\mu_0\mathbf{a}_2\cdot\mathbf{J} = \frac{\partial\beta_1}{\partial x_3}, \tag{2.19}$$

$$\mu_0\mathbf{a}_3\cdot\mathbf{J} = \frac{\partial\beta_2}{\partial x_1} - \frac{\partial\beta_1}{\partial x_2}, \tag{2.20}$$

where use has been made of the fact that

$$\mathbf{a}_1\cdot\mathbf{a}_2\times\mathbf{a}_3 = W^2. \tag{2.21}$$

The following formulas for β_1 and β_2 are found to satisfy (2.18)–(2.20):

$$\beta_1 = \mu_0\int_{x_a}^{x_3}\mathbf{a}_2\cdot\mathbf{J}\,dx_3' + \frac{\partial\zeta(x_1,x_2)}{\partial x_1}, \tag{2.22}$$

$$\beta_2 = -\mu_0\int_{x_a}^{x_3}\mathbf{a}_1\cdot\mathbf{J}\,dx_3' + \frac{\partial\zeta(x_1,x_2)}{\partial x_2}, \tag{2.23}$$

where ζ is an arbitrary function of x_1 and x_2. The quantity η is a function of β_1 and β_2, and is found by requiring the divergence of $\Delta\mathbf{B}$ to vanish, which gives the equation

$$\nabla^2\eta = \nabla \cdot (\beta_1 \nabla x_1 + \beta_2 \nabla x_2). \tag{2.24}$$

The reason for including the arbitrary function ζ in (2.22) and (2.23) is to allow possible simplification of (2.24) under certain conditions. For example, ζ can be chosen so as to make the right-hand side of (2.24) small in certain regions of space. Once β_1, β_2, and η have been obtained, the perturbation magnetic field is computed from (2.14)–(2.16). Since this procedure involves one-dimensional integrations only through the thickness of the conducting ionosphere, together with the solution of Poisson's Equation for the scalar magnetic potential η, it can represent a simplification over the three-dimensional vector integrations implied by the Biot-Savart Law.

It is worth making two comments regarding coordinate systems for the calculation of the magnetic perturbations. First, all of the equations (2.14)–(2.24) are valid in any coordinate system, not just in a geomagnetic-field-aligned system. For example, x_1 and x_2 could be defined to be approximately constant in altitude, and x_3 could be defined to be altitude h (see Section 6), so that the integrals in (2.22) and (2.23) would be carried out vertically rather than along a field line. That would be useful if all the current were confined to a relatively thin layer in the ionosphere, with no field-aligned current above, because then ζ could be chosen to make β_1 and β_2 vanish everywhere above the ionosphere. Second, even if β_1 and β_2 are calculated in a geomagnetic-field-aligned coordinate system, as might be convenient if field-aligned current does exist above the ionosphere, the left-hand side of (2.24) can be expressed in a simpler coordinate system in order to solve for η, thereby avoiding the complicated form of the Laplacian operator in generalized coordinates.

3. Modified Apex Coordinates

In the previous section, a specific definition of the coordinates (x_1, x_2, x_3) was not given. For further development of the electrodynamic equations, let us now focus on Magnetic Apex Coordinates (VanZandt *et al.*, 1972). Their calculation uses a model of the geomagnetic main field to trace a field line to its apex, defined as its highest point above the Earth's surface, taking into account the slightly spheroidal shape of the Earth. The geomagnetic-dipole longitude of the apex defines one of the coordinates, the apex longitude ϕ_A, which will be associated with x_1 in this article. The apex altitude h_A or some function thereof defines the second coordinate, which will be associated with x_2. VanZandt *et al.* (1972) defined an apex radius A as

$$A = 1 + \frac{h_A}{R_{eq}}, \tag{3.1}$$

where R_{eq} is the equatorial radius of the geoid, 6.378160×10^6 m. They also defined an apex latitude λ_A as

$$\lambda_A = \pm \cos^{-1} A^{-1/2}, \tag{3.2}$$

where the plus sign is for locations north of the magnetic equator and the minus sign for locations south of the magnetic equator. λ_A would equal geomagnetic-dipole latitude at the Earth's surface if the true field were dipolar and if the Earth were a true sphere.

Magnetic Apex Coordinates have the convenient property that all field lines with a given apex height are defined to have the same value of apex latitude, so that the equatorial boundary condition (2.10) can be applied at a constant value of magnetic latitude. For geomagnetic field lines that intersect the ionosphere at high latitudes, Magnetic Apex Coordinates are very similar to

Corrected Geomagnetic Coordinates and Constant B-Minimum Coordinates. Invariant latitude is similar to the magnetic latitudes of all three of these systems at high latitudes. Conjugate points in the northern and southern hemispheres have the same values of Apex coordinates for real geomagnetic field lines that have little distortion due to magnetospheric currents, or for field lines whose distortion is essentially symmetric about the geomagnetic equator. When field-line distortion is large and not symmetric, or when field lines are open to the solar wind, high-latitude ionospheric locations in the northern and southern hemispheres with the same values of ϕ_A and equal-but-opposite values of λ_A no longer lie on the same field line. Papitashvili *et al.* (1992) have discussed this effect for Constant B-Minimum Coordinates.

For many purposes, it is convenient to define the magnetic latitude by a modified form of (3.1)–(3.2):

$$\lambda_m = \pm \cos^{-1}\left(\frac{R}{R_E + h_A}\right)^{1/2}, \tag{3.3}$$

$$R = R_E + h_R, \tag{3.4}$$

where R_E is the mean radius of the Earth, 6.3712×10^6 m, h_R is some constant reference altitude, and where again the upper (lower) sign is for the northern (southern) magnetic hemisphere. The subscript "m" stands for "Modified-Apex." Although the value of λ_m is a real number only for $h_A \geq h_R$, the value of $\cos\lambda_m = \sqrt{R/(R_E + h_A)}$ is real for any positive values of h_A and h_R. Like Apex latitude, Modified-Apex latitude is defined to be constant along a geomagnetic field line except for a sign change at the magnetic equator. Figure 1 shows, on the right side of the slice through the geomagnetic torus, the relation between λ_A and λ_m for a reference height of 4000 km; this value of h_R is chosen only for illustrative and not practical purposes. If the Earth were truly spherical and the geomagnetic field were a dipole, a field line traced out from dipole latitude 56.3° at the Earth's surface would pass through the altitude 4000 km at dipole latitude 45°, and a field line traced out from dipole latitude 45° at the Earth's surface would pass through the altitude 4000 km at dipole latitude 25.6°. Since the equation for a dipolar field line is used in the definitions of λ_A and λ_m, $\lambda_A = 56.3°$ corresponds to $\lambda_m = 45°$ and $\lambda_A = 45°$ corresponds to $\lambda_m = 25.6°$, even for a geomagnetic field that is not perfectly dipolar. A dipolar field line traced out from dipole latitude 30° at the Earth's surface would not reach 4000 km altitude, and thus the field line with $\lambda_A = 30°$ does not have a real value of λ_m for this choice of h_R.

If h_R is chosen to be 0, λ_m nearly equals λ_A, but if h_R is chosen to have any value greater than $R_{eq} - R_E$ (6.96 km), $|\lambda_m|$ is always less than $|\lambda_A|$ when λ_m is real, with the greatest differences occurring close to the magnetic equator. If one is calculating ionospheric electrodynamics in a model whose lower boundary is the base of the conducting ionosphere, h_R can be set equal to the assumed altitude of this base, and one could have a latitude coordinate that is continuous across the magnetic equator, unlike what one would have if λ_A were the latitudinal coordinate. Another option, which can be useful at middle and high magnetic latitudes, is to set h_R equal to the mean height of the toroidal component of horizontal ionospheric currents, so that when ground-level magnetic perturbations are related to the currents, the horizontal ionospheric currents can be treated as though they flow in a thin shell at altitude h_R. At high latitudes, toroidal ionospheric currents are associated mainly with Hall currents, whose mean altitude is around 110 km. Since the definition of magnetic latitude in the modified-apex coordinate system depends on the value of h_R, it is important to specify what value is being used when referring to these coordinates. Let us label Modified Apex Coordinates as M(h_R), where h_R is in units of kilometers: thus "M(110) coordinates" refer to Modified Apex Coordinates with $h_R = 110$ km.

The modified apex longitude ϕ_m is defined to equal ϕ_A. The reason for introducing it as a separate coordinate is for consistency in subscripting notation in the modified apex coordinate system. In lieu of magnetic longitude, magnetic local time is often used in ionospheric studies. It is defined as the difference between the magnetic longitude of the point in question and the

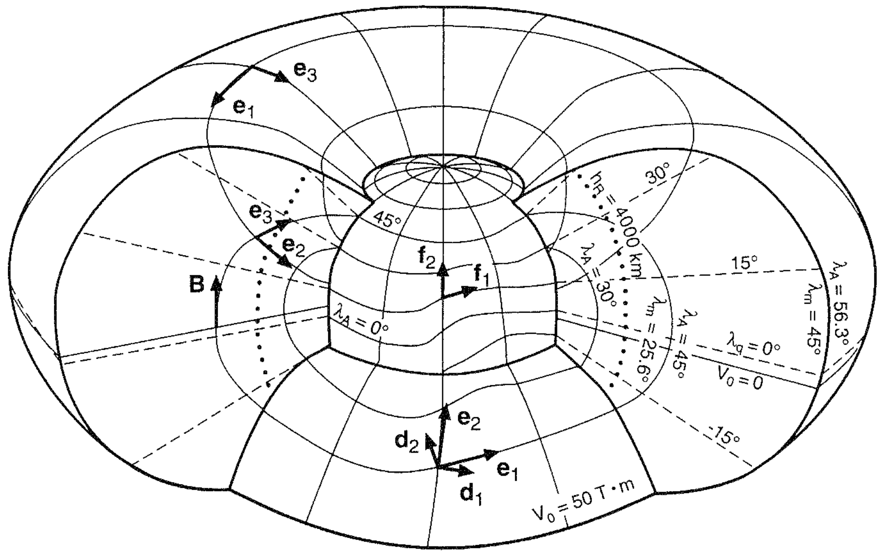

Fig. 1. Schematic diagram of Magnetic Apex Coordinates, Modified Apex Coordinates, Quasi-Dipole Coordinates, and associated base vectors. Five surfaces are visible: (1) Earth's surface, with contours of constant Apex latitude λ_A and longitude ϕ_A; (2) a partial torus composed of geomagnetic field lines with a fixed value of λ_A (56.3°), with contours of constant ϕ_A and geomagnetic potential V_0, as well as the magnetic equator [dashed line]; (3 and 4) two slices through the torus along surfaces of fixed Apex longitude ϕ_A, with contours of constant λ_A, V_0, quasi-dipole latitude λ_q [dashed lines], and reference altitude h_R (4000 km [dotted lines]); and (5) a surface of fixed geomagnetic potential V_0 (50 T·m), with contours of constant λ_A and ϕ_A. On the right surface of fixed ϕ_A, values of λ_m are given in M(4000) coordinates for two field lines: the field line with $\lambda_A = 56.3°$ passes through h_R at $\lambda_q = 45°$, and thus has a M(4000) latitude of $\lambda_m = 45°$; and the field line with $\lambda_A = 45°$ passes through h_R at $\lambda_q = 25.6°$, and thus has a M(4000) latitude of $\lambda_m = 25.6°$. The field line with $\lambda_A = 30°$ peaks below h_R, and thus does not have a real value of λ_m. On the surface of fixed V_0, the base vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are normal to the contours of constant ϕ_A and λ_A, respectively, while $\mathbf{e}_2$ and $\mathbf{e}_1$ are tangent to those surfaces. On the left surface of fixed ϕ_A, the base vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ are tangent to the contours of constant V_0 and λ_A, respectively. $\mathbf{B}$ is the geomagnetic field. On the torus of fixed λ_A, the base vectors $\mathbf{e}_1$ and $\mathbf{e}_3$ are tangent to the contours of constant V_0 and ϕ_A, respectively. On the surface of the Earth, the base vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ are tangent to the contours of constant λ_A and ϕ_A, respectively.

geomagnetic dipole longitude of a line extending away from the center of the Earth in the anti-solar direction, expressed as an hour angle.

The third coordinate, x_3, can be chosen to be any quantity that varies along the field line. VanZandt *et al.* (1972) suggested altitude h as the third coordinate. Another choice can be some function of the magnetic potential V_0 of the main field $\mathbf{B}_0$. Since

$$\mathbf{B}_0 = -\nabla V_0, \tag{3.5}$$

choosing x_3 to be a function of V_0 causes $\mathbf{a}_3$ to be orthogonal to $\mathbf{a}_1$ and $\mathbf{a}_2$, and to lie essentially along $\mathbf{B}$ in the ionosphere, where the geomagnetic field is adequately represented by the main field alone. Some contours of constant V_0 and a surface of $V_0 = 50$ T·m are illustrated in Fig. 1. These are normal to geomagnetic field lines.

Figure 2 shows a map of modified apex latitude and longitude for M(110) coordinates at an altitude of 110 km (i.e., with $h = h_R$) for the epoch 1995.0.

Let us define the quantity I_m by the following relations:

$$\cos I_m = \cos \lambda_m (4 - 3\cos^2 \lambda_m)^{-1/2}, \tag{3.6}$$

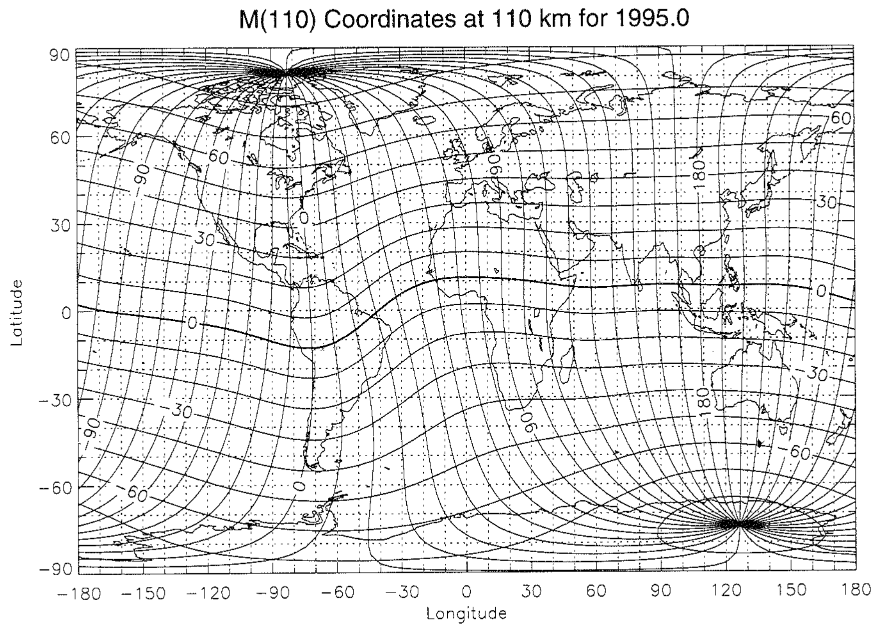

Fig. 2. M(110) latitude and longitude at 10° intervals, at an altitude of 110 km for epoch 1995.0.

$$\sin I_m = 2\sin\lambda_m(4 - 3\cos^2\lambda_m)^{-1/2}. \tag{3.7}$$

I_m, which by definition is constant along a magnetic-field line in each hemisphere (changing sign at the magnetic equator), would equal the angle of inclination of the geomagnetic field below the horizontal for a dipolar geomagnetic field on a spherical Earth at $h = h_R$, but not at other altitudes. Note that $\cos I_m$ is real for all values of $\cos\lambda_m < 4/3$, i.e., for $h_A > \frac{3}{4}h_R - \frac{1}{4}R_E$, even if I_m itself is not real. Let us also define the following sets of base vectors $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$ and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$:

$$\mathbf{d}_1 = R\cos\lambda_m \nabla\phi_m, \tag{3.8}$$

$$\mathbf{d}_2 = -\cos^2\lambda_m \cos I_m \nabla h_A = -R\sin I_m \nabla\lambda_m, \tag{3.9}$$

$$\mathbf{d}_3 = \frac{-\nabla V_0}{B_0 D} = \frac{\mathbf{b}_0}{D}, \tag{3.10}$$

$$\mathbf{e}_1 = \mathbf{d}_2 \times \mathbf{d}_3, \tag{3.11}$$

$$\mathbf{e}_2 = \mathbf{d}_3 \times \mathbf{d}_1, \tag{3.12}$$

$$\mathbf{e}_3 = \mathbf{d}_1 \times \mathbf{d}_2 = D\mathbf{b}_0, \tag{3.13}$$

$$\mathbf{b}_0 \equiv \mathbf{B}_0/B_0, \tag{3.14}$$

$$D \equiv |\mathbf{d}_1 \times \mathbf{d}_2| = R\cos^3\lambda_m \cos I_m |\nabla\phi_m \times \nabla h_a| = R^2\cos\lambda_m |\sin I_m \nabla\phi_m \times \nabla\lambda_m|. \tag{3.15}$$

The base vectors are real quantities everywhere that $\cos\lambda_m < 4/3$, even for many field lines that peak below h_R. They satisfy the relations

$$\mathbf{d}_1 = \mathbf{e}_2 \times \mathbf{e}_3, \qquad \mathbf{d}_2 = \mathbf{e}_3 \times \mathbf{e}_1, \qquad \mathbf{d}_3 = \mathbf{e}_1 \times \mathbf{e}_2, \tag{3.16}$$

$$\mathbf{d}_1 \times \mathbf{d}_2 \cdot \mathbf{d}_3 = \mathbf{e}_1 \times \mathbf{e}_2 \cdot \mathbf{e}_3 = 1, \tag{3.17}$$

$$\mathbf{d}_i \cdot \mathbf{e}_j = \delta_{ij}, \tag{3.18}$$

where δ_{ij} is the Kronecker delta.

Figure 1 illustrates these base vectors: $\mathbf{d}_1$ and $\mathbf{e}_1$ are more-or-less in the magnetic eastward direction; $\mathbf{d}_2$ and $\mathbf{e}_2$ are generally downward and/or equatorward; while $\mathbf{d}_3$ (not shown) and $\mathbf{e}_3$ are along $\mathbf{B}_0$. In general, the base vectors of either set ($\mathbf{d}_i$ or $\mathbf{e}_i$) are neither mutually orthogonal nor of unit length, though they would be orthogonal for a dipolar field on a spherical Earth, where they would be of unit length at $h = h_R$. The magnitudes of $\mathbf{d}_1, \mathbf{d}_2$, and $\mathbf{e}_3$ decrease with increasing altitude along a field line, while those of $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{d}_3$ increase with increasing altitude.

The quantity D varies along geomagnetic-field lines in proportion to the magnetic-field strength B_0. For a dipolar field and a spherical Earth D would be unity at $h = h_R$, and thus its departures from unity at this level are basically a measure of the distortion of the field from a dipolar configuration. Figure 3a shows D at $h = h_R = 110$ km for M(110) coordinates. The maximum distortion it displays occurs in the ocean south of Africa, where D becomes as small as 0.60.

Depending on the sign choice in (3.3), i.e., depending on the magnetic hemisphere upon which one is focusing attention, the appropriate right-handed modified apex coordinate system can be either (ϕ_m, λ_m, V_0; northern hemisphere, where λ_m is positive) or ($\phi_m, \lambda_m, -V_0$; southern hemisphere, where λ_m is negative). The associated volume parameter W_m and the area vectors

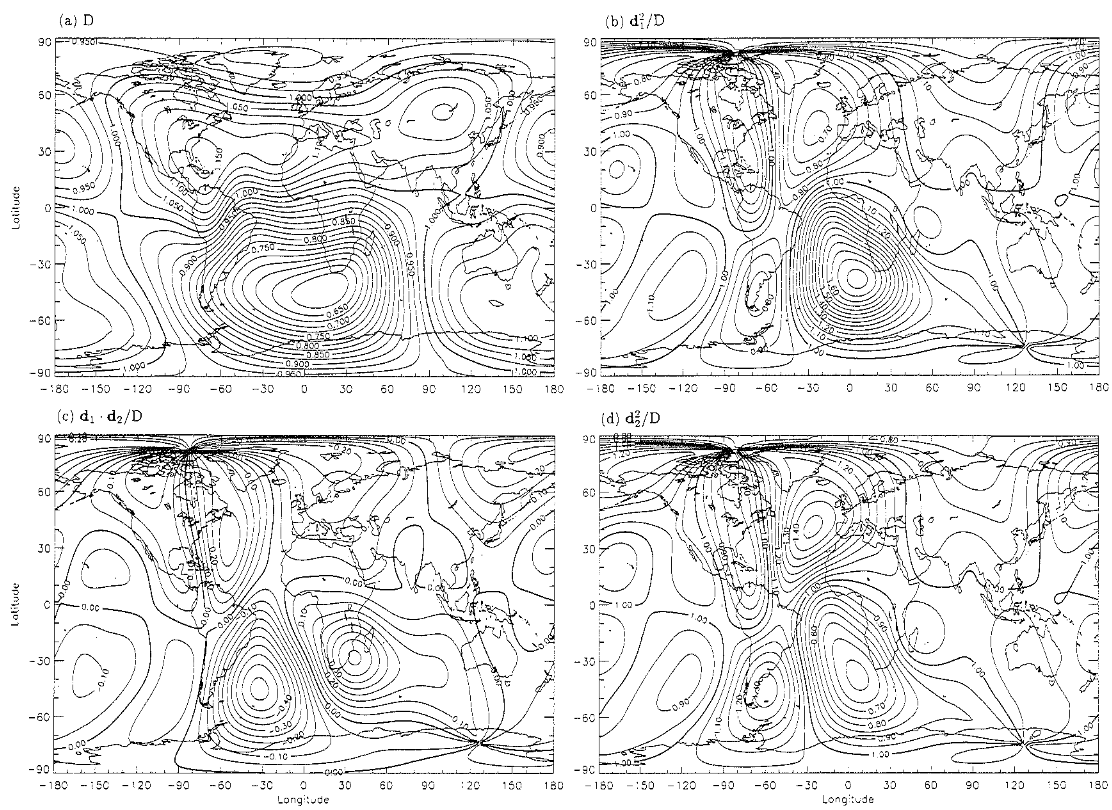

Fig. 3. Various M(110) parameters at an altitude of 110 km for epoch 1995.0 (see text): (a) D, contour interval 0.025; (b) d_1^2/D, contour interval 0.05; (c) $\mathbf{d}_1 \cdot \mathbf{d}_2/D$, contour interval 0.05; (d) d_2^2/D, contour interval 0.05.

$\mathbf{a}_{1m}$, $\mathbf{a}_{2m}$, and $\mathbf{a}_{3m}$ are

$$W_m = |\nabla\phi_m \times \nabla\lambda_m \cdot \mathbf{B}_0|^{-1} = \frac{R^2 \cos\lambda_m |\sin I_m|}{B_0 D}, \tag{3.19}$$

$$\mathbf{a}_{1m} = W_m \nabla\phi_m = \frac{W_m \mathbf{d}_1}{R\cos\lambda_m} = \frac{R|\sin I_m|\mathbf{d}_1}{B_0 D}, \tag{3.20}$$

$$\mathbf{a}_{2m} = W_m \nabla\lambda_m = \frac{-W_m \mathbf{d}_2}{R\sin I_m} = \frac{\mp R\cos\lambda_m \mathbf{d}_2}{B_0 D}, \tag{3.21}$$

$$\mathbf{a}_{3m} = \pm W_m \nabla V_0 = \mp W_m \mathbf{B}_0 = -R^2 \cos\lambda_m \sin I_m \mathbf{d}_3, \tag{3.22}$$

where the upper (lower) sign applies to the northern (southern) magnetic hemisphere.

4. Vector Decomposition and Mapping

Any vector quantity can be decomposed into components multiplying either the $\mathbf{d}_i$ or $\mathbf{e}_i$ set of base vectors, for example

$$\mathbf{A} = \sum_{i=1}^{3} A_{di}\mathbf{d}_i = \sum_{i=1}^{3} A_{ei}\mathbf{e}_i, \tag{4.1}$$

$$A_{di} = \mathbf{e}_i \cdot \mathbf{A}, \tag{4.2}$$

$$A_{ei} = \mathbf{d}_i \cdot \mathbf{A}. \tag{4.3}$$

Relations (3.18) are used to obtain (4.2)–(4.3). For a dipolar geomagnetic field on a spherical Earth A_{di} and A_{ei} equal the actual vector components in the magnetic-eastward, magnetic-downward/equatorward, and field-parallel directions for components 1, 2, and 3, respectively, at the altitude h_R, but only at that altitude. For the curvilinear coordinate system defined here $(\phi_m, \lambda_m, \pm V_0)$, the contravariant components of vector $\mathbf{A}$ are $(A_{e1}/R\cos\lambda_m,\ -A_{e2}/R\sin I_m,\ \mp A_{d3}B_0/D)$, and the covariant components are $(A_{d1}R\cos\lambda_m,\ -A_{d2}R\sin I_m,\ \mp A_{e3}D/B_0)$, where the multipliers of the A_{di}'s and A_{ei}'s in these expressions are constant along geomagnetic fields lines.

Depending on the quantity represented, it usually turns out that only one or the other decomposition in (4.1) is convenient. The fact that the vectors $\mathbf{a}_i$ are parallel to the respective vectors $\mathbf{d}_i$, together with the manner in which the $\mathbf{a}_i$'s enter into the electrodynamic equations (2.4), (2.5), (2.12), and (2.15), indicate that it is convenient to expand $\mathbf{E}$ and $\Delta\mathbf{B}$ in terms of the base vectors $\mathbf{d}_i$ and to expand $\mathbf{J}$ and $\mathbf{B}_0$ in terms of the base vectors $\mathbf{e}_i$:

$$\mathbf{B}_0 = B_{e3}\mathbf{e}_3, \tag{4.4}$$

$$\mathbf{E} = E_{d1}\mathbf{d}_1 + E_{d2}\mathbf{d}_2, \tag{4.5}$$

$$\mathbf{J} = \sum_{i=1}^{3} J_{ei}\mathbf{e}_i, \tag{4.6}$$

$$\Delta\mathbf{B} = \sum_{i=1}^{3} \Delta B_{di}\mathbf{d}_i. \tag{4.7}$$

The components of $\mathbf{E}$ and $\Delta\mathbf{B}$ are:

$$E_{d1} = \frac{-1}{R\cos\lambda_m}\frac{\partial\Phi}{\partial\phi_m}, \tag{4.8}$$

$$E_{d2} = \frac{1}{R \sin I_m} \frac{\partial \Phi}{\partial \lambda_m}, \tag{4.9}$$

$$\Delta B_{d1} = \frac{1}{R \cos \lambda_m} \left(\beta_1 - \frac{\partial \eta}{\partial \phi_m} \right), \tag{4.10}$$

$$\Delta B_{d2} = \frac{-1}{R \sin I_m} \left(\beta_2 - \frac{\partial \eta}{\partial \lambda_m} \right), \tag{4.11}$$

$$\Delta B_{d3} = -D \mathbf{b}_0 \cdot \nabla \eta, \tag{4.12}$$

while B_{e3} and J_{e3} are

$$B_{e3} = B_0 / D, \tag{4.13}$$

$$J_{e3} = J_{\|} / D. \tag{4.14}$$

B_0 and B_{e3} are mapped over the Earth at $h = h_R = 110$ km in Fig. 4 for M(110) coordinates. The magnetic flux per unit intervals of ϕ_m and λ_m, which is a quantity of importance for dynamic modeling of the ionosphere in M(h_R) coordinates, is

$$|\mathbf{a}_{3m} \cdot \mathbf{B}_0| = R^2 \cos \lambda_m |\sin I_m| B_{e3}. \tag{4.15}$$

It can be seen that E_{d1}, E_{d2}, and B_{e3} are constant along magnetic field lines, and that J_{e3}, $\Delta B_{d1}^{(1)}$, and $\Delta B_{d2}^{(1)}$ are constant along field lines at altitudes where $\nabla \cdot \mathbf{J}_\perp$ is insignificant (the superscript (1) referring, as in Section 2, to the component of $\Delta \mathbf{B}$ involving the β_1 and β_2 terms only, not η). Thus the decompositions can be useful for mapping those field components along geomagnetic-field lines from one altitude to another. Other quantities can also be mapped. The electrodynamic drift velocity,

$$\mathbf{v}_E \equiv \mathbf{E} \times \mathbf{B}_0 / B_0^2, \tag{4.16}$$

which is essentially the ion velocity perpendicular to $\mathbf{B}_0$ in the upper ionosphere, maps along the geomagnetic field with different scaling relations than does $\mathbf{E}$. It is conveniently expressed as

$$\mathbf{v}_E = v_{e1} \mathbf{e}_1 + v_{e2} \mathbf{e}_2, \tag{4.17}$$

in which v_{e1} and v_{e2} are constant along geomagnetic field lines, and are related to E_{d1}, E_{d2}, and B_{e3} according to

$$v_{e1} = E_{d2} / B_{e3}, \tag{4.18}$$

$$v_{e2} = -E_{d1} / B_{e3}. \tag{4.19}$$

The Poynting flux,

$$\mathbf{P} \equiv \mathbf{E} \times \Delta \mathbf{B}, \tag{4.20}$$

can be expressed as the sum of two components, corresponding to the components of $\Delta \mathbf{B}$ superscripted (1) and (2). It is easy to show that $\mathbf{P}^{(2)}$ is divergence-free, and thus does not contribute to the production or dissipation of electromagnetic energy. $\mathbf{P}^{(1)}$ is aligned along the magnetic field, and is given by

$$\mathbf{P}^{(1)} = (E_{d1} \Delta B_{d2}^{(1)} - E_{d2} \Delta B_{d1}^{(1)}) \mathbf{e}_3 = P_{e3}^{(1)} \mathbf{e}_3. \tag{4.21}$$

$P_{e3}^{(1)}$ is constant along geomagnetic-field lines at altitudes where $\mathbf{J}_\perp$ vanishes, so that the divergence of the Poynting flux also vanishes at those altitudes.

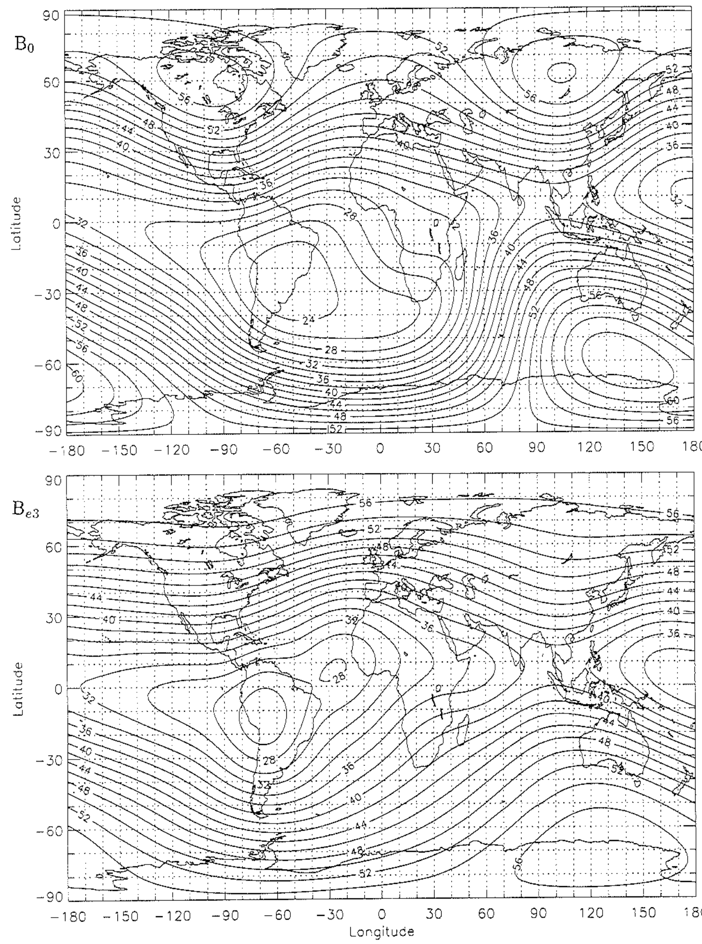

Fig. 4. (top) Geomagnetic-field strength B_0 at an altitude of 110 km for epoch 1995.0. Units are μT. (bottom) B_{e3} for M(110) coordinates at 110 km.

5. Electrodynamic Equations Mapped to the Shell at $h = h_R$

In Modified Apex Coordinates, let x_l and x_u be the values of x_3 at the lower and upper boundaries, respectively, of the conducting ionosphere. Let s be distance along a field line, increasing upward as does x_3, and let s_l and s_u be the values of s corresponding to x_l and x_u. For low-latitude field lines that do not reach to the top of the conducting ionosphere, x_u and s_u are defined to be the values of x_3 and s at the field-line apex. Let us also define the following quantities:

$$K_{m\phi} = \int_{x_l}^{x_u} \frac{\mathbf{a}_{1m} \cdot \mathbf{J}}{R} dx_3 = |\sin I_m| \int_{s_l}^{s_u} \frac{J_{e1}}{D} ds, \tag{5.1}$$

$$K_{m\lambda} = \int_{x_l}^{x_u} \frac{\mathbf{a}_{2m} \cdot \mathbf{J}}{R \cos \lambda_m} dx_3 = \mp \int_{s_l}^{s_u} \frac{J_{e2}}{D} ds, \tag{5.2}$$

$$J_{mr} = \frac{-1}{R \cos \lambda_m} \left(\frac{\partial K_{m\phi}}{\partial \phi_m} + \frac{\partial (K_{m\lambda} \cos \lambda_m)}{\partial \lambda_m} \right). \tag{5.3}$$

From (2.13), (4.14), and (5.1)–(5.3) the following relations can be found for field lines that reach the top of the conducting ionosphere [for which the order of integration and differentiation in (2.13) can be interchanged]:

$$J_{mr} = \frac{\mathbf{a}_{3m} \cdot \mathbf{B}_0 J_{\|}}{B_0 R^2 \cos\lambda_m}\bigg|_{x_3=x_u} = \frac{-\sin I_m J_{\|}}{D}\bigg|_{s=s_u} = -\sin I_m J_{e3}\bigg|_{s=s_u}. \tag{5.4}$$

Equation (5.3) has the appearance of a thin-shell current-continuity equation on a sphere of radius R, in which the upward current density at the top of the shell equals the convergence of horizontal height-integrated sheet-current density. Indeed, J_{mr} has the dimensions of current density (A/m^2), while $K_{m\phi}$ and $K_{m\lambda}$ have the dimensions of height-integrated current density (A/m). However, J_{mr}, $K_{m\phi}$, and $K_{m\lambda}$ do not in general equal the vertical current density and height-integrated horizontal ionospheric current density components. Their relation to those quantities is discussed in Section 7. Nonetheless, the form of (5.3) does help to introduce the concept of mapping the electrodynamic equations to a sphere of radius R.

Ohm's Law (2.1) can be written in terms of the base vectors as

$$\mathbf{J} = (\sigma_P \mathbf{d}_1 + \sigma_H D\mathbf{e}_2)(E_{d1} + u_{e2}B_{e3}) + (\sigma_P \mathbf{d}_2 - \sigma_H D\mathbf{e}_1)(E_{d2} - u_{e1}B_{e3}) + \mathbf{J}_M + J_{e3}\mathbf{e}_3, \tag{5.5}$$

$$u_{ei} = \mathbf{d}_i \cdot \mathbf{u}. \tag{5.6}$$

The components of $\mathbf{J}$ perpendicular to $\mathbf{B}_0$ in the ionosphere (where $\mathbf{J}_M = 0$) are found by taking the dot product of (5.5) with $\mathbf{d}_1$ and $\mathbf{d}_2$:

$$J_{e1} = \sigma_P d_1^2(E_{d1} + u_{e2}B_{e3}) + (\sigma_P \mathbf{d}_1 \cdot \mathbf{d}_2 - \sigma_H D)(E_{d2} - u_{e1}B_{e3}), \tag{5.7}$$

$$J_{e2} = (\sigma_P \mathbf{d}_1 \cdot \mathbf{d}_2 + \sigma_H D)(E_{d1} + u_{e2}B_{e3}) + \sigma_P d_2^2(E_{d2} - u_{e1}B_{e3}). \tag{5.8}$$

Let us define

$$E_{m\phi} = E_{d1} = \frac{-1}{R\cos\lambda_m}\frac{\partial\Phi}{\partial\phi_m}, \tag{5.9}$$

$$E_{m\lambda} = -E_{d2}\sin I_m = \frac{-1}{R}\frac{\partial\Phi}{\partial\lambda_m}. \tag{5.10}$$

Then (5.1) and (5.2) give

$$K_{m\phi} = \Sigma_{\phi\phi}E_{m\phi} + \Sigma_{\phi\lambda}E_{m\lambda} + K^D_{m\phi}, \tag{5.11}$$

$$K_{m\lambda} = \Sigma_{\lambda\phi}E_{m\phi} + \Sigma_{\lambda\lambda}E_{m\lambda} + K^D_{m\lambda}, \tag{5.12}$$

$$\Sigma_{\phi\phi} = |\sin I_m| \int_{s_l}^{s_u} \frac{\sigma_P d_1^2}{D} ds, \tag{5.13}$$

$$\Sigma_{\lambda\lambda} = \frac{1}{|\sin I_m|} \int_{s_l}^{s_u} \frac{\sigma_P d_2^2}{D} ds, \tag{5.14}$$

$$\Sigma_{\phi\lambda} = \pm(\Sigma_H - \Sigma_c), \tag{5.15}$$

$$\Sigma_{\lambda\phi} = \mp(\Sigma_H + \Sigma_c), \tag{5.16}$$

$$\Sigma_H = \int_{s_l}^{s_u} \sigma_H ds, \tag{5.17}$$

$$\Sigma_c = \int_{s_l}^{s_u} \frac{\sigma_P \mathbf{d}_1 \cdot \mathbf{d}_2}{D} ds, \tag{5.18}$$

$$K^D_{m\phi} = B_{e3}|\sin I_m| \int_{s_l}^{s_u} \left[\frac{\sigma_P d_1^2}{D} u_{e2} + \left(\sigma_H - \frac{\sigma_P \mathbf{d}_1 \cdot \mathbf{d}_2}{D}\right) u_{e1}\right] ds, \tag{5.19}$$

$$K_{m\lambda}^{D} = \mp B_{e3} \int_{s_l}^{s_u} \left[\left(\sigma_H + \frac{\sigma_P \mathbf{d}_1 \cdot \mathbf{d}_2}{D} \right) u_{e2} - \frac{\sigma_P d_2^2}{D} u_{e1} \right] ds. \tag{5.20}$$

Equations (5.11) and (5.12) represent a two-dimensional form of Ohm's Law, relating the sheet-current components to the horizontal components of the electric field and to the dynamo terms $K_{m\phi}^{D}$ and $K_{m\lambda}^{D}$. The quantities $\mathbf{d}_i \cdot \mathbf{d}_j / D$ that enter into the integrals in (5.13), (5.14), (5.19), and (5.20) vary but little along geomagnetic-field lines over the thickness of the conducting ionosphere, and for practical applications can be taken outside the integrals by setting them equal to their values where the field line passes through the surface $h = h_R$. The variations of these quantities over the Earth at $h = h_R = 110$ km are shown in Fig. 3b–3d for M(110) coordinates. For a dipole field on a spherical Earth, d_1^2/D and d_2^2/D would be unity at $h = h_R$, while $\mathbf{d}_1 \cdot \mathbf{d}_2 / D$ would be zero. The field distortion represented by $\mathbf{d}_1 \cdot \mathbf{d}_2 / D$ in Fig. 3c maximizes in the western Atlantic Ocean, reaching a maximum magnitude of about 50%.

Note that besides the non-dipole modifications to the electrodynamic equations represented by the factors $\mathbf{d}_i \cdot \mathbf{d}_j / D$, there are additional non-dipole influences on the electrodynamics due to modulations of the conductivities and of the dynamo electric field $\mathbf{u} \times \mathbf{B}$ by the varying magnetic-field strength and direction (Stening, 1971; Walton and Bowhill, 1979; Wallis and Budzinski, 1981). The daytime height-integrated Pedersen conductivity is found to scale roughly as $B^{-1.6}$, and the height-integrated Hall conductivity roughly as $B^{-1.3}$ (Richmond, 1995). Around 50° S Apex latitude the geomagnetic field strength varies by a factor of about 2.2, corresponding to height-integrated Pedersen and Hall conductivity variations by factors of roughly 3.5 and 2.8, respectively, for a given daytime solar zenith angle.

Let us now obtain a differential equation for Φ for regions of closed geomagnetic-field lines that are reasonably conjugate, that is, for which the values of ϕ_m and $|\lambda_m|$ are essentially the same in the northern and southern hemispheres. Let us define the magnetospheric source of current, associated with the convergence of transverse magnetospheric currents along the field line, as

$$J_{Mr} = J_{mr}^{N} + J_{mr}^{S}, \tag{5.21}$$

where J_{mr}^{N} and J_{mr}^{S} are the values of J_{mr} in the northern and southern hemispheres, respectively. When (5.3) is used to replace J_{mr}^{N} and J_{mr}^{S} in (5.21), and use is made of the fact that

$$\left(\frac{\partial}{\partial \lambda_m} \right)^{S} = -\frac{\partial}{\partial |\lambda_m|} = -\left(\frac{\partial}{\partial \lambda_m} \right)^{N}, \tag{5.22}$$

the following partial differential equation for Φ is obtained:

$$\begin{aligned} &\frac{\partial}{\partial \phi_m} \left(\frac{\Sigma_{\phi\phi}^{T}}{\cos \lambda_m} \frac{\partial \Phi}{\partial \phi_m} + \Sigma_{\phi\lambda}^{T} \frac{\partial \Phi}{\partial |\lambda_m|} \right) + \frac{\partial}{\partial |\lambda_m|} \left(\Sigma_{\lambda\phi}^{T} \frac{\partial \Phi}{\partial \phi_m} + \Sigma_{\lambda\lambda}^{T} \cos \lambda_m \frac{\partial \Phi}{\partial |\lambda_m|} \right) \\ &= R \frac{\partial K_{m\phi}^{DT}}{\partial \phi_m} + R \frac{\partial (K_{m\lambda}^{DT} \cos \lambda_m)}{\partial |\lambda_m|} + R^2 \cos \lambda_m J_{Mr}, \end{aligned} \tag{5.23}$$

$$\Sigma_{\phi\phi}^{T} = \Sigma_{\phi\phi}^{N} + \Sigma_{\phi\phi}^{S}, \tag{5.24}$$

$$\Sigma_{\phi\lambda}^{T} = \Sigma_{\phi\lambda}^{N} - \Sigma_{\phi\lambda}^{S}, \tag{5.25}$$

$$\Sigma_{\lambda\phi}^{T} = \Sigma_{\lambda\phi}^{N} - \Sigma_{\lambda\phi}^{S}, \tag{5.26}$$

$$\Sigma_{\lambda\lambda}^{T} = \Sigma_{\lambda\lambda}^{N} + \Sigma_{\lambda\lambda}^{S}, \tag{5.27}$$

$$K_{m\phi}^{DT} = K_{m\phi}^{DN} + K_{m\phi}^{DS}, \tag{5.28}$$

$$K_{m\lambda}^{DT} = K_{m\lambda}^{DN} - K_{m\lambda}^{DS}. \tag{5.29}$$

Equation (5.23) is a specialized form, for Modified Apex Coordinates, of the more general equation (2.6). If h_R is placed at or below the base of the conducting ionosphere, (5.23) is applicable at all altitudes within the equatorial regions as well as at middle latitudes. In that case, the equatorial boundary condition is

$$K_{m\lambda}^T = 0 \tag{5.30}$$

or

$$\frac{\Sigma_{\lambda\phi}^T}{\cos\lambda_m}\frac{\partial\Phi}{\partial\phi_m} + \Sigma_{\lambda\lambda}^T\frac{\partial\Phi}{\partial\lambda_m} = RK_{m\lambda}^{DT}. \tag{5.31}$$

Because $\Sigma_{\lambda\lambda}$ goes to infinity at $\lambda_m = 0$ unless σ_P is zero there, the differential equation (5.23) and the boundary condition (5.31) must be carefully handled. One way to avoid difficulties is to define h_R to lie below the region of solution, so that the boundary lies at some value of $|\lambda_m|$ that is greater than zero, where $\Sigma_{\lambda\lambda}$ is well-behaved. Another way to avoid coefficients in the equations that blow up at the equator is to transform the variable $|\lambda_m|$ to something whose gradient (in the vertical direction) does not become infinite at the equator. For example, the choices $x_2 = \lambda_m^2$ or $x_2 = (h_A - h_R)/(R_E + h_A)$ would make the equations well-behaved at the equator.

6. Quasi-Dipole Coordinates

Although the geomagnetic field strongly organizes many of the electrodynamic phenomena, ionospheric currents tend to be predominantly horizontal by virtue of the fact that the ionospheric Pedersen and Hall conductivities are confined to a relatively thin layer surrounding the Earth. When analyzing ionospheric currents and their associated magnetic perturbations, it can be useful to use a magnetically oriented coordinate system in which x_3 is altitude and in which the longitude and latitude variables x_1 and x_2 vary only little with altitude, though still being linked to the geomagnetic field. In this case x_1 and x_2 will vary along $\mathbf{B}_0$. For a dipolar field on a spherical Earth, x_1 and x_2 could be simply dipole longitude and latitude, which are independent of altitude. For a more general field Quasi-Dipole Coordinates can be defined as follows. Quasi-dipole latitude λ_q is

$$\lambda_q \equiv \pm\cos^{-1}\left(\frac{R_E + h}{R_E + h_A}\right)^{\frac{1}{2}}, \tag{6.1}$$

so that

$$\cos\lambda_q = \left(\frac{R_E + h}{R}\right)^{\frac{1}{2}}\cos\lambda_m. \tag{6.2}$$

Quasi-dipole longitude ϕ_q equals ϕ_A and ϕ_m. The third quasi-dipole coordinate is altitude h. Quasi-Dipole Coordinates do not have any reference height associated with them. Lines of constant quasi-dipole longitude and latitude are approximately vertical, as illustrated in Fig. 1 by the dashed lines on the two surfaces of constant magnetic longitude. Unlike λ_m, λ_q varies along magnetic-field lines, and goes to zero at the magnetic equator all altitudes. Although $\phi_q = \phi_m$, partial derivatives with respect to these two longitude coordinates will not be the same, since a partial derivative with respect to ϕ_m will imply that λ_m and V_0 are held constant, while a partial derivative with respect to ϕ_q will imply that λ_q and h are held constant.

Base vectors can be defined in a manner similar to that for the base vectors $\mathbf{d}_1 - \mathbf{d}_3$ and $\mathbf{e}_1 - \mathbf{e}_3$ defined in Section 3 for Modified Apex Coordinates. Let us define

$$\mathbf{g}_1 = \frac{(R_E + h)\cos\lambda_q}{F}\nabla\phi_q = \left(\frac{R_E + h}{R}\right)^{\frac{3}{2}}\frac{\mathbf{d}_1}{F}, \tag{6.3}$$

$$\mathbf{g}_2 = \frac{(R_E + h)}{F} \nabla \lambda_q = -\frac{\cot \lambda_q}{2F} \left[\mathbf{k} + \left(\frac{R_E + h}{R} \right) \frac{\mathbf{d}_2}{\cos I_m} \right], \tag{6.4}$$

$$\mathbf{g}_3 = F\mathbf{k}, \tag{6.5}$$

$$\mathbf{f}_1 = \mathbf{g}_2 \times \mathbf{g}_3 = -(R_E + h)\mathbf{k} \times \nabla \lambda_q = \left(\frac{R_E + h}{R} \right) \frac{\cot \lambda_q \mathbf{k} \times \mathbf{d}_2}{2 \cos I_m}, \tag{6.6}$$

$$\mathbf{f}_2 = \mathbf{g}_3 \times \mathbf{g}_1 = (R_E + h) \cos \lambda_q \mathbf{k} \times \nabla \phi_q = \left(\frac{R_E + h}{R} \right)^{\frac{3}{2}} \mathbf{k} \times \mathbf{d}_1, \tag{6.7}$$

$$\mathbf{f}_3 = \mathbf{g}_1 \times \mathbf{g}_2, \tag{6.8}$$

$$F = \mathbf{f}_1 \times \mathbf{f}_2 \cdot \mathbf{k} = (R_E + h)^2 \cos \lambda_q \mathbf{k} \cdot \nabla \phi_q \times \nabla \lambda_q = \frac{\sin \lambda_m \sin I}{\sin \lambda_q \sin I_m} \left(\frac{R_E + h}{R} \right)^3 D, \tag{6.9}$$

where I is the real angle of inclination of the geomagnetic field below the horizontal. The vectors

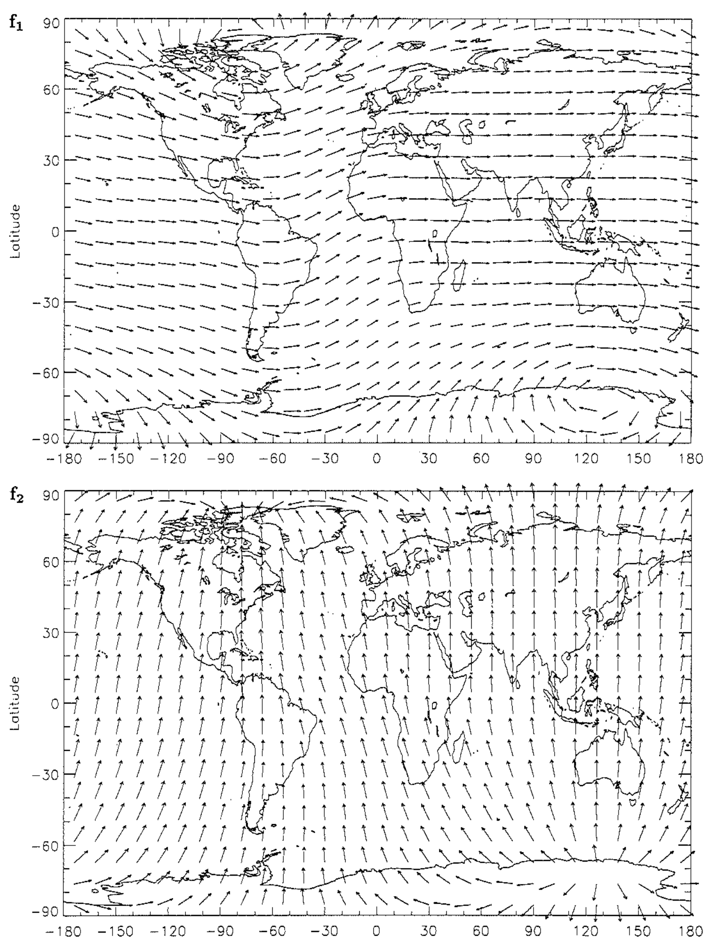

Fig. 5. (top) Base vectors $\mathbf{f}_1$ at an altitude of 110 km for epoch 1995.0. (bottom) Base vectors $\mathbf{f}_2$ at 110 km. 10° of longitude corresponds to unit length.

$\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ and $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$ are interrelated in a manner analogous to $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$ and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ [see relations (3.11)–(3.13) and (3.16)–(3.17)]. Figure 1 illustrates $\mathbf{f}_1$ and $\mathbf{f}_2$ at the surface of the Earth, while Fig. 5 shows maps of $\mathbf{f}_1$ and $\mathbf{f}_2$ at an altitude of 110 km. At $h = h_R$, F equals $D \sin I / \sin I_m$, and thus its variations over the Earth at $h = h_R$ have similar features to those of D, which is shown in Fig. 3a. For a dipolar field on a spherical Earth F would be unity and the sets of vectors $\mathbf{g}_i$ and $\mathbf{f}_i$ would be orthogonal unit vectors everywhere.

The volume parameter and area vectors for Quasi-Dipole Coordinates are

$$W_q = \frac{(R_E + h)^2 \cos \lambda_q}{F}, \tag{6.10}$$

$$\mathbf{a}_{1q} = (R_E + h)\mathbf{g}_1 = \left(\frac{R_E + h}{R}\right)^{\frac{5}{2}} \frac{R\mathbf{d}_1}{F}, \tag{6.11}$$

$$\mathbf{a}_{2q} = (R_E + h) \cos \lambda_q \mathbf{g}_2, \tag{6.12}$$

$$\mathbf{a}_{3q} = \frac{(R_E + h)^2 \cos \lambda_q \mathbf{g}_3}{F^2} = \frac{(R_E + h)^2 \cos \lambda_q \mathbf{k}}{F}. \tag{6.13}$$

7. Shell Currents

To see the relation of $K_{m\phi}$ and $K_{m\lambda}$ [defined by (5.1)–(5.2)] to actual densities of horizontal sheet currents, note that the following three-dimensional current distribution can be shown to satisfy (5.1)–(5.3):

$$\mathbf{J} = \mathbf{K}\delta(h - h_R) - \frac{J_{mr}}{\sin I_m} \mathbf{e}_3 S(h - h_R), \tag{7.1}$$

$$\mathbf{K} = K_{m\phi}\mathbf{f}_1 + K_{m\lambda}\mathbf{f}_2, \tag{7.2}$$

where δ is the delta function, S is a step function (0 if $h \leq h_R$, 1 if $h > h_R$), and where the base vectors $\mathbf{e}_3$, $\mathbf{f}_1$ and $\mathbf{f}_2$ are defined by (3.13), (6.6), and (6.7). The upward current density at the top of the shell is

$$J_r = -J_\parallel \sin I = \frac{D \sin I}{\sin I_m} J_{mr} = F J_{mr}, \tag{7.3}$$

where F is the scaling factor given by (6.9), and where all quantities are evaluated at $h = h_R$.

It is tempting to think of $\mathbf{K}$ and J_r as representative of the height-integrated horizontal ionospheric current and the vertical current at the top of the ionosphere, respectively, at the latitude and longitude where the field line defined by (ϕ_m, λ_m) crosses the surface $h = h_R$. Provided that h_R is not too far removed from the mean altitude of the horizontal ionospheric currents, that is in fact a reasonable approximation at middle and high latitudes, where the horizontal position of a field line does not vary greatly in its traversal of the conducting ionosphere. Near the magnetic equator the situation is quite different. First of all, unless h_R is chosen to lie at the base of the conducting region, some field lines peak below h_R, and do not traverse the surface $h = h_R$. The current flowing in the region of these field lines cannot be represented by (7.2). Secondly, the distribution of $\mathbf{K}$ in latitude, as given by (7.2), does not represent the actual height-integrated current density close to the equator, because the integrations in (5.1) and (5.2) are carried out along the curving and nearly horizontal field lines, not vertically. Thirdly, in the equatorial region the vertical current density varies strongly in altitude, even in the upper ionosphere where current flow is nearly parallel to $\mathbf{B}_0$. J_r, as given by (7.3), does not represent the actual vertical current density at any altitude near the magnetic equator.

For purposes of calculating magnetic perturbations above or below the ionosphere, it is useful to have an expression for the sheet-current density that more nearly represents the height-integrated horizontal current at all latitudes. Let us define new quantities $K_{q\phi}$ and $K_{q\lambda}$ that are functions of ϕ_q and λ_q as

$$K_{q\phi} = \int_{h_l}^{h_u} \frac{\mathbf{a}_{1q} \cdot \mathbf{J}}{R} dh = \int_{h_l}^{h_u} \left(\frac{R_E + h}{R} \right)^{\frac{5}{2}} \frac{J_{e1}}{F} dh, \tag{7.4}$$

$$K_{q\lambda} = -\frac{1}{\cos \lambda_q} \int_{-\frac{\pi}{2}}^{\lambda_q} \left[J_{mr}(\phi_q, \lambda'_q) R \cos \lambda'_q + \frac{\partial K_{q\phi}(\phi_q, \lambda'_q)}{\partial \phi_q} \right] d\lambda'_q, \tag{7.5}$$

where h_l and h_u are the heights of the lower and upper boundaries of the conducting ionosphere and where the integration in (7.4) is carried out with λ_q held constant rather than λ_m, i.e., approximately in the vertical direction rather than along a magnetic-field line. Equation (7.5) ensures that

$$J_{mr} = \frac{-1}{R \cos \lambda_q} \left[\frac{\partial K_{q\phi}}{\partial \phi_q} + \frac{\partial (K_{q\lambda} \cos \lambda_q)}{\partial \lambda_q} \right], \tag{7.6}$$

while the form of (7.4) can be shown to result in the equality of the following expressions for total eastward current when $R \leq R_E + h_l$:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K_{q\phi} d\lambda_q = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K_{m\phi} d\lambda_m. \tag{7.7}$$

At the southern magnetic pole ($\lambda_q = \lambda_m = -\frac{\pi}{2}$), (7.5) gives $K_{q\lambda} = -\partial K_{q\phi}/\partial \phi_q$, while at the northern pole ($\lambda_q = \lambda_m = \frac{\pi}{2}$), $K_{q\lambda}$ converges to $\partial K_{q\phi}/\partial \phi_q$ by virtue of the conditions (5.3), (7.6), and (7.7). However, numerical evaluation of the integral in (7.5) is likely not to be precisely zero at $\lambda_q = \frac{\pi}{2}$, so that the division by $\cos\lambda_q$ as the north magnetic pole is approached may cause problems unless corrections for the numerical inaccuracies are made.

Note that any modification to J_{mr} that is antisymmetric about the magnetic equator will preserve the requirements for global current conservation and for convergence of (7.5) at the north pole. Such a modification could be effected by raising h_R above h_l when computing J_{mr}. Since J_{mr} near the magnetic equator depends strongly on the choice of h_R, $K_{q\lambda}$ will also be sensitive to that choice. Raising h_R would shift the accounting of the essentially horizontal cross-equatorial current from magnetic-field-aligned flow (as represented by J_{mr}) to ionospheric-sheet current (i.e., $K_{q\lambda}$), while having very little influence on the calculation of ground magnetic perturbations. Thus J_{mr} could be calculated by using a value of, say, $h_R = 300$ km in order to have J_{mr} represent only field-aligned current flow that is totally above the region of significant transverse conductivities, and in order to have $K_{m\lambda}$ include all cross-equatorial current below that height.

Magnetic perturbations can be calculated from (2.14)–(2.16) if (2.22)–(2.24) are first solved, or the perturbations can be calculated by other means, such as carrying out the integrations of the Biot-Savart Law. Below or above the ionosphere, a reasonable approximation would be to represent the current distribution as a horizontal sheet current connected to geomagnetic-field-aligned currents above the ionosphere, i.e., in the form (7.1) but with $\mathbf{K}$ given by the following expression rather than (7.2):

$$\mathbf{K} = K_{q\phi} \mathbf{f}_1 + K_{q\lambda} \mathbf{f}_2. \tag{7.8}$$

For representing ground-level magnetic perturbations, it can be useful to introduce the concept of "equivalent current," which is that (fictitious) overhead horizontal sheet current that would produce the same magnetic perturbations on the ground as does the true three-dimensional current system. It must be divergence-free, and thus can be written as

$$\mathbf{K}_{\text{equiv}} = \mathbf{k} \times \nabla \psi, \tag{7.9}$$

where ψ is the equivalent current function. It depends on the altitude at which the equivalent current sheet is defined to flow, which may be taken to be $h = h_R$. The chain rule for differentiation allows us to write (7.9) as

$$\mathbf{K}_{\text{equiv}} = \mathbf{k} \times \left(\nabla\phi_q \frac{\partial\psi}{\partial\phi_q} + \nabla\lambda_q \frac{\partial\psi}{\partial\lambda_q} + \mathbf{k}\frac{\partial\psi}{\partial h} \right) = \frac{-\mathbf{f}_1}{R}\frac{\partial\psi}{\partial\lambda_q} + \frac{\mathbf{f}_2}{R\cos\lambda_q}\frac{\partial\psi}{\partial\phi_q}, \tag{7.10}$$

where the terms on the right-hand side are evaluated at $h = h_R$.

The horizontal component of magnetic perturbation on the ground is approximately, though not exactly, proportional to the overhead equivalent current rotated counterclockwise by 90° as viewed from above. Since the ground magnetic perturbation can be expressed as the negative gradient of a perturbation magnetic potential ΔV [which is not necessarily the same as η in (2.16)], its horizontal component is

$$(\Delta\mathbf{B})_{\text{hor}} = \mathbf{k} \times [\mathbf{k}\times\nabla(\Delta V)] = \Delta B_{q\phi}\mathbf{f}_2 \times \mathbf{k} + \Delta B_{q\lambda}\mathbf{k} \times \mathbf{f}_1, \tag{7.11}$$

$$\Delta B_{q\phi} = \frac{-1}{(R_E + h)\cos\lambda_q}\frac{\partial\Delta V}{\partial\phi_q} = \frac{\mathbf{f}_1 \cdot \Delta\mathbf{B}}{F}, \tag{7.12}$$

$$\Delta B_{q\lambda} = \frac{-1}{R_E + h}\frac{\partial\Delta V}{\partial\lambda_q} = \frac{\mathbf{f}_2 \cdot \Delta\mathbf{B}}{F}. \tag{7.13}$$

8. Data Organization with Respect to the Shell at $h = h_R$

For comparison with theoretical models, or for construction of empirical models, it can be convenient to represent observed electrodynamic parameters in terms of quantities mapped to the shell at $h = h_R$ like $E_{m\phi}$ and $E_{m\lambda}$, defined by (5.9) and (5.10). For a general geomagnetic field $E_{m\phi}$ and $E_{m\lambda}$ are not actual electric field components, but they enter into the electrodynamic equations like the two-dimensional form of Ohm's Law (5.11)–(5.12) in a manner analogous to true electric-field components. They are constant along geomagnetic-field lines. By relating observations to quantities like $E_{m\phi}$ and $E_{m\lambda}$, account can be taken of the influence of coordinate-system distortion on quantities that tend to be organized largely with respect to magnetic latitude and magnetic local time.

Electric fields can be measured in the upper ionosphere by *in-situ* instruments (e.g. Heppner and Maynard, 1987), while plasma drift velocities can be measured either *in situ* or by radar (e.g. Holt *et al.*, 1987; Senior *et al.*, 1990; de la Beaujardière *et al.*, 1991; Rich and Hairston, 1994; Ruohoniemi *et al.*, 1994). The components of the electric-field or ion-drift vector are generally measured in arbitrary directions, e.g. along a satellite trajectory or along the line of sight of a radar. Let us use the unit vector $\mathbf{l}$ to denote some such direction. An observation of the electric field along $\mathbf{l}$ is related to $E_{m\phi}$ and $E_{m\lambda}$ by

$$\mathbf{l} \cdot \mathbf{E} = \mathbf{l} \cdot \mathbf{d}_1 E_{m\phi} - \frac{\mathbf{l} \cdot \mathbf{d}_2}{\sin I_m} E_{m\lambda}. \tag{8.1}$$

An observation of a single component of the electromagnetic drift velocity $\mathbf{v}_E$ [see (4.16)] along $\mathbf{l}$ at any altitude is related to $E_{m\phi}$ and $E_{m\lambda}$ by

$$\mathbf{l} \cdot \mathbf{v}_E = -\frac{\mathbf{l} \cdot \mathbf{e}_2}{B_{e3}} E_{m\phi} - \frac{\mathbf{l} \cdot \mathbf{e}_1}{B_{e3}\sin I_m} E_{m\lambda}. \tag{8.2}$$

In general, magnetic perturbations cannot be simply characterized by a two-dimensional mapped field. However, certain features of the magnetic perturbations can sometimes be approximately characterized in two dimensions. If ζ is chosen so as to make the right-hand side of

(2.24) small above the ionosphere, and if one observes magnetic perturbations in space at a large distance from regions where $\mathbf{J}_\perp$ is significant, then $\Delta\mathbf{B}^{(2)}$ will vary smoothly in space, and may be relatively small. Spatially structured features of $\Delta\mathbf{B}$ can then be attributed largely to $\Delta\mathbf{B}^{(1)}$, which maps readily along the geomagnetic field. If $\Delta\mathbf{B}^{(1)}$ is written as

$$\Delta\mathbf{B}^{(1)} = \Delta B_{m\phi}^{(1)}\mathbf{d}_1 - \frac{\Delta B_{m\lambda}^{(1)}}{\sin I_m}\mathbf{d}_2, \tag{8.3}$$

$$\Delta B_{m\phi}^{(1)} = \Delta B_{d1}^{(1)} = \frac{\beta_1}{R\cos\lambda_m} = \mu_0 K_{m\lambda} + \frac{1}{R\cos\lambda_m}\frac{\partial\zeta}{\partial\phi_m}, \tag{8.4}$$

$$\Delta B_{m\lambda}^{(1)} = -\Delta B_{d2}^{(1)}\sin I_m = \frac{\beta_2}{R} = -\mu_0 K_{m\lambda} + \frac{1}{R}\frac{\partial\zeta}{\partial\lambda_m} \tag{8.5}$$

then an observation of a single component of $\Delta\mathbf{B}^{(1)}$ along the arbitrary unit vector $\mathbf{l}$ is

$$\mathbf{l}\cdot\Delta\mathbf{B}^{(1)} = \mathbf{l}\cdot\mathbf{d}_1\Delta B_{m\phi}^{(1)} - \frac{\mathbf{l}\cdot\mathbf{d}_2}{\sin I_m}\Delta B_{m\lambda}^{(1)}. \tag{8.6}$$

It is straightforward to show that

$$\mu_0 J_{mr} = \frac{1}{R\cos\lambda_m}\left[\frac{\partial(\Delta B_{m\lambda}^{(1)})}{\partial\phi_m} - \frac{\partial(\Delta B_{m\phi}^{(1)}\cos\lambda_m)}{\partial\lambda_m}\right]. \tag{8.7}$$

[Note that the superscript "(1)" could be dropped in (8.7), since $\Delta\mathbf{B}^{(2)}$, being derived from a potential, would give no net contribution to the right-hand side.]

At the ground, (7.11)–(7.13) can be used to express an observation of a horizontal component of $\Delta\mathbf{B}$ along the arbitrary horizontal unit vector $\mathbf{h}$ as

$$\mathbf{h}\cdot\Delta\mathbf{B} = \mathbf{h}\cdot\mathbf{f}_2\times\mathbf{k}\Delta B_{q\phi} + \mathbf{h}\cdot\mathbf{k}\times\mathbf{f}_1\Delta B_{q\lambda}. \tag{8.8}$$

Any particular magnetic-component observation (e.g. ΔH, ΔD, ΔX, or ΔY in conventional geomagnetic notation) will be related to both $\Delta B_{q\phi}$ and $\Delta B_{q\lambda}$ unless the direction of $\mathbf{h}$ associated with that observation happens to lie parallel to $\mathbf{f}_1$ or $\mathbf{f}_2$. The horizontal magnetic perturbations can be approximately related to the equivalent current (7.10) by treating $(\phi_q, \lambda_q, R_E + h)$ as though they were true spherical coordinates, treating $\Delta B_{q\phi}$ and $\Delta B_{q\lambda}$ as though they were true magnetic components, and applying standard analysis techniques like spherical harmonic analysis.

The downward magnetic perturbation at the ground ΔZ does not have such a straightforward approximate relation to the equivalent current. However, for a current system that is stretched or contracted uniformly in all directions, ΔZ might be expected to scale inversely with the linear dimension of the current system in the same fashion as the horizontal magnetic perturbation components, that is, to scale as $|\mathbf{f}_1|$, $|\mathbf{f}_2|$, or $F^{\frac{1}{2}}$. Thus a scaled downward magnetic perturbation component ΔB_{qz} may be defined such that

$$\Delta Z = F^{\frac{1}{2}}\Delta B_{qz}. \tag{8.9}$$

ΔB_{qz} can then be incorporated into the approximate analysis procedure mentioned above to relate the magnetic perturbations to the equivalent current.

9. Concluding Remarks

Departures of the geomagnetic field from a dipolar configuration can be substantial, and the distortions can significantly influence the parameters used in models of ionospheric electrodynamics. The present state of model development, for both empirical and first-principles models, has reached a stage where it is becoming necessary to take realistic geomagnetic-field models into account for analysis of data and for model-data comparisons. The geomagnetic coordinate systems described in this article are useful for carrying out calculations of electrodynamic quantities and for organizing observations of these quantities. These coordinate systems have already been incorporated into the National Center for Atmospheric Research thermosphere-ionosphere-electrodynamics general circulation model (NCAR TIE-GCM (Richmond *et al.*, 1992)) and thermosphere-ionosphere-mesosphere-electrodynamics general circulation model (TIME-GCM (Roble and Ridley, 1994)), and are being incorporated into the empirical Assimilative Mapping of Ionospheric Electrodynamics procedure (AMIE (Richmond, 1992)).

In order to utilize apex coordinates and related quantities, it is essential to be able to determine them conveniently. Computer codes have been developed which determine Modified Apex Coordinates, Quasi-Dipole Coordinates, and associated base vectors and scaling factors. These codes are available through the CEDAR Data Base or by request to the author at richmond@ncar.ucar.edu.

I am grateful to E. C. Ridley for helpful discussions, to G. Lu for pointing out errors in an earlier draft, and to W. Golesorkhi for assistance in testing the computer routines and in producing the figures. The computer routines incorporate code provided by W. Clark, V. Wickwar, H. Passi, and R. Barnes. This study made use of the CEDAR Data Base, which is supported by the National Science Foundation, and was also supported by the NASA Space Plasma Theory Program and by NASA grants W-17,384 and W-18,536.

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