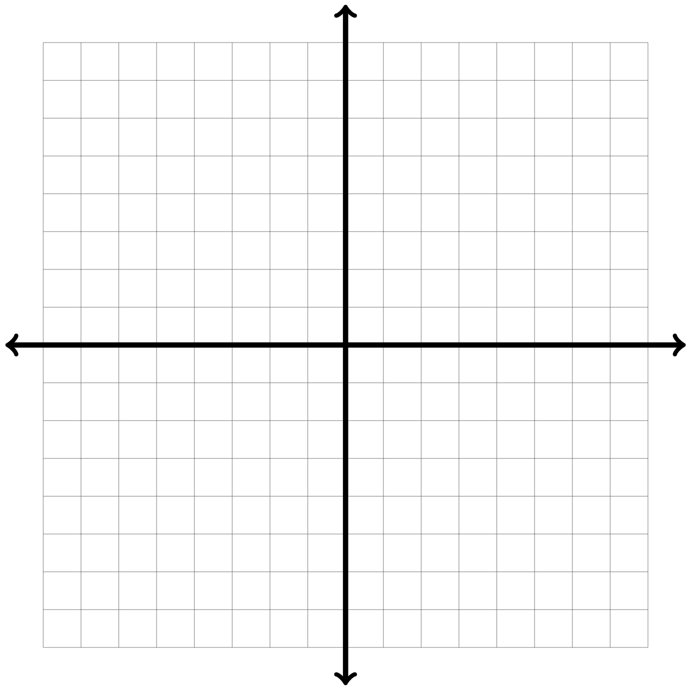
Algebra 2

1. Graphing Linear Equations

   (a) Graph the line $y = 2x + 4$ and, hence, solve for x if $y = 0$

   (b) Explain the signficance of the x-intercept

   (c) Use a graphical approach to solve $2x + 4 = 2$



2. Solving Linear Equations

   Solve $\frac{y+2}{3} - \frac{1}{6}(5y+2) = 1$

3. Solving Simultaneous Linear Equations

   Solve $\begin{array}{c} 2x - 4 = y \\ 2x + y = 2 \end{array}$ first by graphing and then by algebra

4. Solving Simultaneous Linear Equations with Fractions

Solve $\begin{array}{l} \frac{2}{3}x + \frac{2}{5}y = \frac{8}{5} \\ \frac{1}{2}x + \frac{2}{9}y = \frac{17}{18} \end{array}$

5. Solving Simultaneous Equations in 3 Variables

> Suggested Method:
>
> Step 1: Pick two equations, and eliminate a variable from them.
>
> Step 2: Pick a different pair of equations, and from them, eliminate the same variable as before.
>
> Step 3: Now we have two equations, each with two unknowns. Solve them simulataneously to find the values of them.
>
> Step 4: Substitute the values into one of three original equations.

Solve $\begin{array}{l} x + y + z = 6 \\ 5x + 3y - 2z = 5 \\ 3x - 7y + z = -8 \end{array}$

6. Harder Version

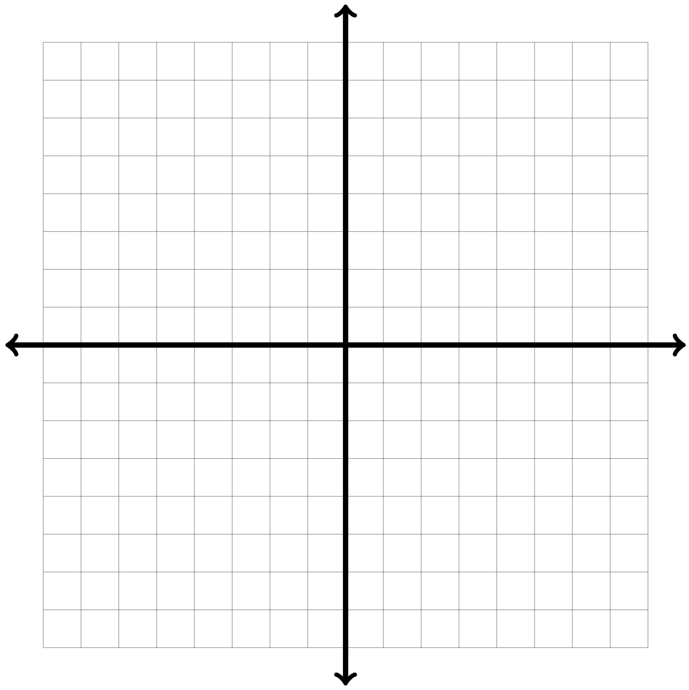
(a) Solve
$$\begin{aligned} 3a + b &= 8 \\ -3b + c &= -3 \\ a - 3c &= -7 \end{aligned}$$

(b) Solve

$$\begin{aligned} 2x + 3y - z &= -4 \\ 3x + 2y + 2z &= 14 \\ x - 3z &= -13 \end{aligned}$$

7. Solving Quadratic Equations

(a) Estimate the roots of $5x^2 - 2x = 25$ using a graph (with x varying from -3 to 3)



(b) Estimate the roots of $5x^2 - 2x = 25$ using algebra
Hint: Use '-b' formula as it doesn't factorise

8. Solving Quadratic Equations
   Solve $3x^2 + 5x - 12 = 0$

9. Solving Quadratic Equations
   Solve $x^2 + 7x - 12 = 0$,
   and hence solve $(y^2 + 4y)^2 + 7(y^2 + 4y) + 12 = 0$

10. Solving Using Fractions

Suggested Method: Multiply each term by the lowest common denominator.

(a) Solve $\frac{10}{x+3} - \frac{2}{x} = 1$ for $x \neq -3, 0$

(b) Solve $\frac{8}{y+2} - \frac{2}{y+3} = \frac{8}{5}$ for $y \neq -3, -2$

(c) Solve $\frac{3}{x^2} = \frac{11}{x} - \frac{10}{1}$ for $x \neq 0$

11. Roots of a Quadratic Equation

> Note:
>
> If $x = 2$ is a root, then $(x - 2)$ is a factor.
>
> If $x = 1$ is a root, then $(x + 1)$ is a factor.
>
> If $x = \frac{2}{3}$ is a root, then $(x - \frac{2}{3})$ or $(3x - 2)$ is a factor.

(a) Form an equation whose roots are 2 and -5.
Method 1: $x = -2$ and $x = 5$
Method 2: $x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0$

(b) Form an equation whose roots are $\frac{1}{2}$ and $\frac{-3}{4}$.

(c) Form an equation whose roots are $3 + \sqrt{5}$ and $3 - \sqrt{5}$.

12. Simultaneous Equations (Linear and Non-Linear)

> Suggested Method:
>
> Start with the linear equation, and isolate one variable.
>
> Substitute that into the non-linear equation.

(a) Solve $\begin{array}{l} x + y = 5 \\ x^2 + y^2 = 13 \end{array}$

(b) Solve $\begin{array}{l} x + y = 9 \\ xy = 10 \end{array}$

(c) Solve $\begin{array}{l} 2x + 3y = -1 \\ x^2 + xy + 2y^2 = 4 \end{array}$

13. Factor Theorem

A polynomial $f(x)$ has a factor $(x - a)$ if and only if $f(a) = 0$

(a) Show $(2x - 3)$ is a factor of $2x^3 - 5x^2 + 5x - 3$

(b) Verify $(x - 1)$ is a factor of $x^3 + 2x^2 - x - 2$, and find the other two factors

(c) If $(x-1)$ is a factor of x^3-2x^2-5x+b, find the value of b.

(d) Write in the form ax^3+bx^2+cx+d an equation with roots -1, 2, 5.

(e) Solve $x^3 - 2x^2 - 5x + 6$

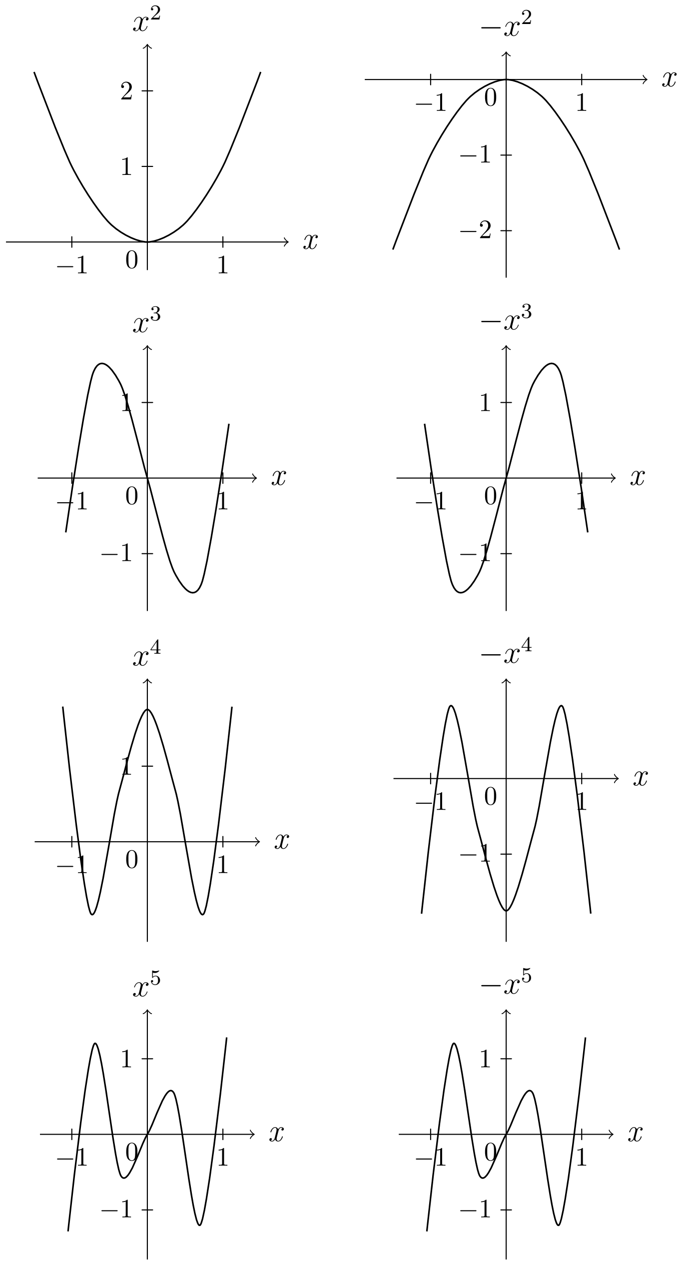
Suggested Method:
Use trial and error to find the first factor.
Start with $x = 1$, then $x = -1$, $x = 2$, and so on.
When you find the root, write the factor, and use long division to find the other factors, and hence the roots.

14. Graphs/Polynomials

Theory:

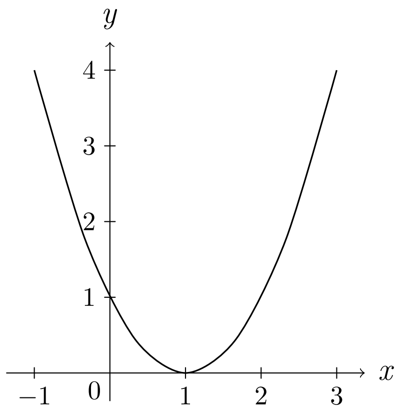
- The values of x for which $f(x) = 0$ are called **roots**.
- The **degree** of the polynomial is the highest power.
- The maximum number of real roots a polynomial can have is the same as its degree.
- The **leading coefficient** is the coefficient of the term with the highest power.

Take, for example, $x^3 - 5x^2 + 7x - 11$.
The degree is 3.
The leading coefficient is 1.
The max number of roots is 3.

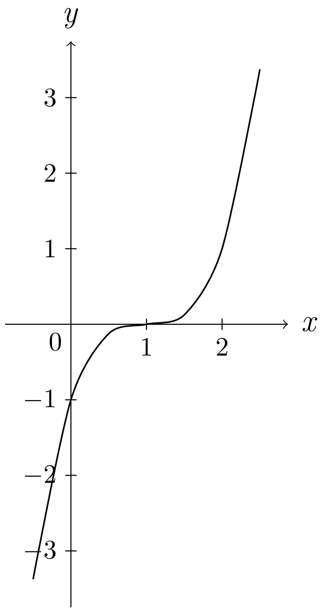


Notice that if the polynomial is of **even degree**, then the both arms point in **same** direction: up if positive, and down if negative. In contrast, if the polynomial is of **odd degree**, then the the arms point in **different** directions.

A polynomial may have a factor or root that occurs multiple times. This is called **multiplicity**. For example, the polynomial $x^2 - 2x + 1$, or $(x-1)^2$, has the root $(x = 1)$ twice. Notice, when graphed, the polynomial does not cross the x-axis at $x = 1$ but only touches it. Because it's squared, the root is said to have multiplicity 2.



On the other hand, the polynomial $x^3 - 3x^2 + 3x - 1$, or $(x-1)^3$, has the root $(x = 1)$ three times, and thus, the graph appears to flutter out at either side of $x = 1$.



(a) Sketch a graph of the polynomial $y = x(x+3)(x-2)^2(x-4)$

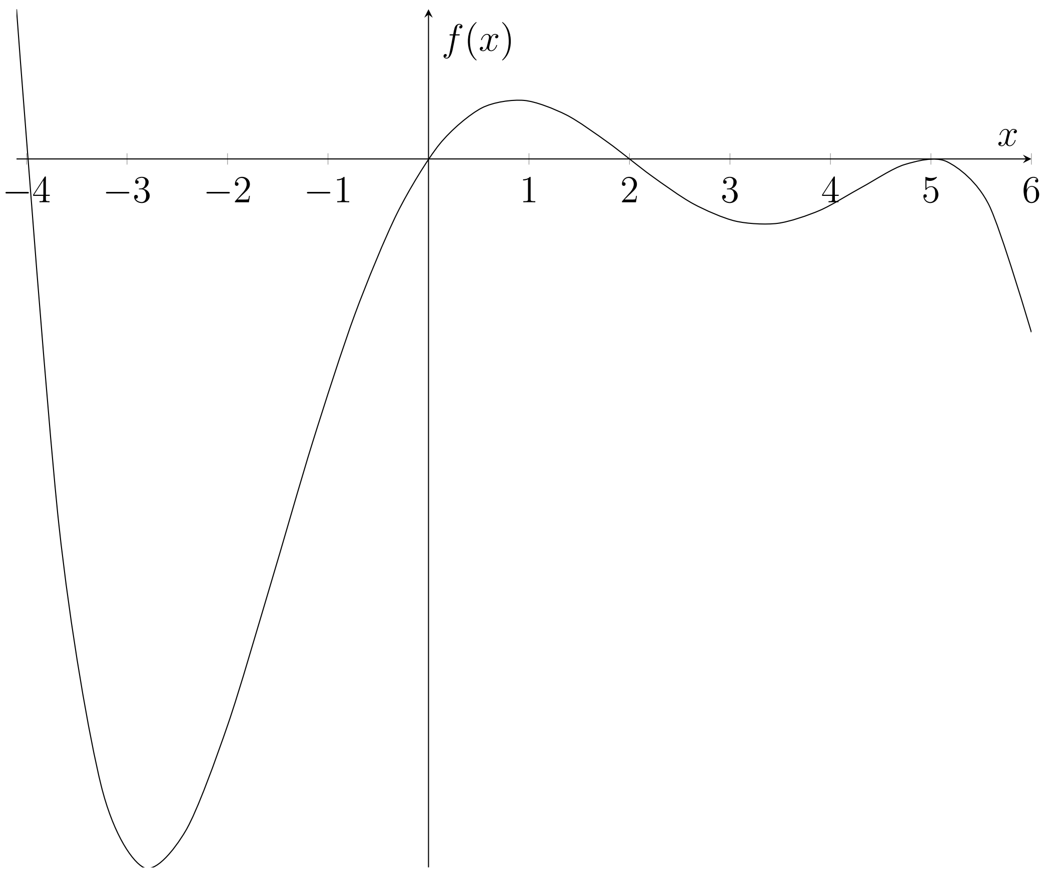
Considerations:

- What is the degree? Is it positive or negative? (This affects the shape.)
- Write down factors and roots.
- Are there any roots that appear multiple times? If so, are they even or odd? If even, the polynomial touches the x-axis; otherwise, it crosses it.

(b) Sketch a graph of the polynomial $y = (x+2)(x-1)^2(x-4)$

(c) Sketch a graph of the polynomial $y = -x(x+4)(x-2)^2(x-3)$

(d) Find a polynomial of degree 5 whose graph is shown below.



(e) If $(x-1)$ and $(x+2)$ are factors of $x^3 + ax^2 - bx - 2$, find the value of a and b.

(f) Given $(x+3)$ and $(x-1)$ are factors of ax^3+bx^2-9, find the value of a and b.

15. Manipulation of Formulae

    (a) Express y in terms of the other variables.
    $x = 3yr - k$

(b) Express c in terms of the other variables.
$p = \frac{a+b+c}{2}$

(c) Express b in terms of the other variables.
$\sqrt{\frac{a}{b}} = c$

(d) Express q in terms of the other variables.
$a = \sqrt{\frac{p}{1-q}}$

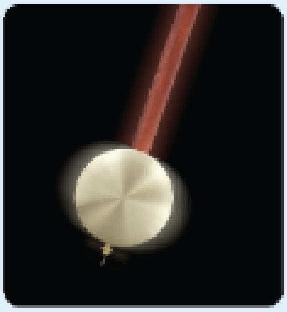
(e) Express v in terms of the other variable.
$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$

(f) Express r in terms of the other variables.
$p = \frac{1}{2}q + \sqrt{r}$

(g) Q20. on Pg 49 of Active Maths

20. Galileo discovered that the time it takes a pendulum to complete one full swing back and forth (one period) depends only on the length of the pendulum and the acceleration due to gravity.

This relationship only applies when it is a simple pendulum swinging back and forth over a small distance.



The period (T) of a simple pendulum is given by the formula

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where:

T = period, time taken for one complete swing back and forth, measured in seconds

l = length of the pendulum, measured in metres

g = acceleration due to gravity, which is 9.81 m s^{-2} at sea level.

Assume that $\pi = 3.14$.

(i) What is the period, to the nearest second, of a pendulum of 25 m in length?

(ii) Express l in terms of the other variables.

(iii) What is the length, to the nearest metre, of a pendulum with a period of 35 seconds?

(iv) Express g in terms of the other variables.

(v) An astronaut is standing on Mars with a pendulum. The period of the pendulum is 2.8 seconds and the length of the pendulum is 75 cm. What is the acceleration due to gravity, to one decimal place, on Mars?

16. Identities / Unknown Coefficients

(a) If $(x+t)^2 = x^2 + 6x + k$ for all values of x, find t and k.

Suggested Method:

- Multiply out the brackets first.
- Equate the left side to the right side.
- Line up x^2 on left with x^2 on the right. Line up x on left with x on the right, and do the same for the constants.

(b) If $c(x-a)^2 + b = 3x^2 - 6x + 5$, find the values of a, b, c (where $a, b, c \in R$).

(c) If $(x-3)$ and $(x-4)$ are factors of $x^3 + ax^2 + bx + c$, express b in terms of a and c in terms of b.

(d) If $x^2 - ax + b$ is a factor of $x^3 + cx + d$, prove (i) $b = a^2 + c$ and (ii) $d = a^3 + ac$

(e) If $x^2 - px + q$ is a factor of $x^3 + 3px^2 + 3qx + r$, show that $q = -2p^2$ and $r = -8p^3$

(f) $x^2 - ax + 2$ is a factor of $x^3 - x^2 + 5ax - b$

i. Express a in terms of b.

ii. Find the value of a. Express your answer in surd form.

(g) From 2011 Exam Paper: A cubic function is defined for x as $f(x) = x^3 + (1 - k^2)x + k$, where k is a constant.

i. Show -k is a root.

ii. Find in terms of k the other 2 roots.

17. Problem Solving

Problem Solving Rules:

- Read the question all the way through.
- Identify an unknown value, and represent it with a letter (say x).
- Most problems will consist of several unknown values, so let your letter (say x) represent the simplest of these.
- Express all other unknown values in terms of this simplest value.
- Convert the word equation into a mathematical equation.

(a) Example p52

A manufacturer produces a calculator that sells for €12. It costs €x to manufacture each calculator.

(i) Write an expression to show the profit per calculator.

The manufacturer must sell $50 - x$ of these calculators per day to achieve a daily profit of €215.

(ii) Find the total cost of producing the required number of calculators.

Solution

(i) Profit per calculator = Selling price − cost price

= €$(12 - x)$

(ii) Daily profit of €215 = Profit per calculator × number of calculators sold

$= (12 - x)(50 - x)$

$\therefore (12 - x)(50 - x) = 215$

$600 - 12x - 50x + x^2 = 215$

$x^2 - 62x + 385 = 0$

$(x - 7)(x - 55) = 0$

$x - 7 = 0$ OR $x - 55 = 0$

$x = 7$ OR $x = 55$

Double check your work in context questions, as you may need to reject an answer due to the nature of the question.

Check both solutions:

- If $x = 55$, then $50 - x = -5$. It is not possible to sell −5 calculators. Therefore, reject $x = 55$.
- $x = 7$ is acceptable, as $50 - x = 43$ and it is possible to sell 43 calculators.

$\therefore$ Cost per calculator = €7

Find the total cost of producing the required number of calculators.

= Cost per calculator × number of calculators sold

$= 7 \times 43$

= €301

(b) Q11 p55

11. A cinema contains 315 seats. It has x rows with an equal number of seats in each row. Six rows of seats are removed to create a fire escape. To ensure the same capacity, the number of seats per row must be increased by six.

How many rows of seats did the cinema originally have?

(c) Q19 p56

19. A landlord rents out 30 apartments.
The monthly rent of each apartment is €750.
For each €75 increase in rent, the landlord expects to lose one tenant.

Let x represent each €75 increase in the rent.

Calculate the monthly rent needed to ensure that the landlord makes €30,000 in rent per month.

(d) Q22 p57

22. Two taps running together take 6 minutes to fill a bath. The hot tap, working by itself, takes 100 seconds longer to fill the bath than the cold tap working by itself. To the nearest second, how long would it take each tap to fill the bath by itself?