MathStats

Moments

kth order moment

$$m_k = \mathbb{E}[X^k] = \int_{-\infty}^{\infty} x^k f_X(x) = \sum_{x:f_X(x)>0} x^k f_X(x)$$

kth order central moment

$$\sigma_k = \mathbb{E}[(X - m_1)^k]$$

Expectation

$$\mathbb{E}[g(X,Y)] = \sum_y \sum_x g(x,y) f_{X,Y}(x,y)$$

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} \mathbb{E}[Y|X=x] f_X(x)\, dx$$

If $X \geq 0$ (continuous or discrete)

$$\mathbb{E}[X] = \int_0^{\infty} (1 - F_X(x))\ dx$$

Conditional expectation

$$\mathbb{E}[Y|X=x] = \int y f_{y|x}(y|x) dy$$

$$\mathbb{E}[g(Y)|X=x] = \int g(y) f_{y|x}(y|x) dy$$

Law of iterated expectation

$$\mathbb{E}_X[E_{Y|X}\{g(Y)|X\}] = \mathbb{E}[g(Y)] = \sum_x \mathbb{E}[g(Y)|X=x] f_X(x)$$

$$\mathbb{E}[h(X)g(Y)] = \mathbb{E}[h(X)\mathbb{E}[g(Y)|X]]$$

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \sum_x \mathbb{E}[Y|X=x] f_X(x)$$

If $X \perp\!\!\!\perp Y$, then

$$\mathbb{E}[g(Y)|X=x] = E[g(Y)]$$

$$\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \text{Cov}(X,X)$$

Special Case: Let $Y^* = I(Y=y)$

$$\mathbb{E}[Y^*] = \mathbb{E}[I(Y=y)] = \mathbb{P}(Y=y)$$

Special case cont

$$\mathbb{E}[Y^*] = \sum_x \mathbb{E}[Y^*|X=x] f_X(x) = \sum_x \mathbb{P}(Y=y|X=x) f_X(x) = \mathbb{P}(Y=y)$$

Basics

$$\text{Cov}[X,Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$\rho(X,Y) = \frac{\text{Cov}[X,Y]}{\sqrt{\text{Var}[X] \cdot \text{Var}[Y]}} = \frac{\text{Cov}[X,Y]}{\sqrt{\sigma_X \sigma_Y}}$$

Expectation Algebra

$$\mathbb{E}[X] = \mathbb{E}[X_+] - \mathbb{E}[X_-] \text{ if } X_+ \geq 0$$

$$\mathbb{E}[|X|] = \mathbb{E}[X_+] + \mathbb{E}[X_-] \text{ if } X_- \geq 0$$

$$\mathbb{E}[a + bX] = a + b\mathbb{E}[X]$$

$$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$$

If $X \perp\!\!\!\perp Y$, then

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

Variance Algebra

$$\text{Var}[X+Y] = \text{Var}[X] + 2\,\text{Cov}[X,Y] + \text{Var}[Y]$$

$$\text{Var}[X-Y] = \text{Var}[X] - 2\,\text{Cov}[X,Y] + \text{Var}[Y]$$

If $X \perp\!\!\!\perp Y$, then

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$$

If $X \perp\!\!\!\perp Y$, then

$$\text{Var}[X-Y] = \text{Var}[X] - \text{Var}[Y]$$

$$\text{Var}[XY] = \mathbb{E}[X^2] \cdot \mathbb{E}[Y^2] - (\mathbb{E}[X] \cdot \mathbb{E}[Y])^2$$

$$\text{Var}[X/Y] = \text{Var}[X \cdot (1/Y)] = \text{Var}[(1/Y) \cdot X]$$

$$\text{Var}[aX + bY] = a^2\,\text{Var}[X] + b^2 Var[Y] + 2ab\,\text{Cov}[X,Y]$$

Correlation

$$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)}\sqrt{Var(Y)}}$$

Calc

Chain rule

$$f(g(x))' = f'(g(x))g'(x)$$

Product rule

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

Quotient rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Indeterminate Forms
$\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$, $\infty - \infty$, $0 * \infty$, 0^0, 1^∞, ∞^0
Determinate Forms
$\infty + \infty = \infty$, $-\infty - \infty = -\infty$, $0^\infty = 0$, $0^{-\infty} = \infty$, $\infty * \infty = \infty$

Series

Exponential

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

Geometric

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \text{ where } |r| < 1$$

$$\sum_{k=1}^{\infty} kr^{k-1} = \frac{1}{(1-r)^2} \text{ where } |r| < 1$$

$$\sum_{k=2}^{\infty} k(k-1)r^{k-2} = \frac{-2}{(1-r)^3} \text{ where } |r| < 1$$

Arithmetico-geometric

$$\sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{1}{p^2} = p^{-2}$$

$$\sum_{k=2}^{\infty} k(k-1)(1-p)^{k-2} = \frac{2}{p^3}$$

Binomial

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$n(x+y)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1} y^{n-k}$$

$$n(n-1)(x+y)^{n-2} = \sum_{k=2}^{n} k(k-1)\binom{n}{k} x^{k-2} y^{n-k}$$

Binomial Coefficient

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{k} = \binom{n-1}{k-1}\left(\frac{n}{k}\right) \text{ if } k > 0$$

$$(n+1)\binom{r}{n+1} = r\binom{r-1}{n}$$

$$\binom{n}{r+a}\binom{r+a}{r} = \binom{n}{r}\binom{n-r}{a}$$

$$\frac{1}{k+1}\binom{n}{k} = \binom{n+1}{k+1}\frac{1}{n+1}$$

Distribution Functions

$$\lim_{x \to -\infty} F(x) = 0 \quad \lim_{x \to \infty} F(x) = 1$$

$$F(x) \leq F(y) \text{ if } x < y$$

$$\lim_{h \downarrow 0} F(x+h) = F(x)$$

$$F(x) = \int_{-\infty}^{x} f(t)\ dt \text{ where } f(t) \geq 0 \text{ and} \int_{-\infty}^{\infty} f(t)\ dt = 1$$

$$F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$$

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = \mathbb{P}(\boldsymbol{X} \leq \boldsymbol{x}) = \mathbb{P}(X_1 \leq x_1, \ldots, X_k \leq x_k)$$

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_k} f(t_1, \cdots, t_k)\ dt_1 \cdots dt_k$$

$$\mathbb{P}(a < X \leq b, c < Y \leq d) = F(b,d) - F(a,d) - F(b,c) + F(a,c)$$

Joint Density

$$F(x,y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x} \left\{ \int_{-\infty}^{y} f(u,v)\ dv \right\}\ du$$

Random Variables

$$X : \Omega \to \mathbb{R}$$

$$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$$

$$F(x) = \mathbb{P}(X \leq x) = \mathbb{P}(\{\omega : X(\omega) \leq x\})$$

$$\mathbb{P}(X > x) = 1 - F(x)$$

$$\mathbb{P}(x < X \leq y) = F(y) - F(x)$$

$$\mathbb{P}(X < x) = \lim_{n \to \infty} F(x - \frac{1}{n})$$

$$\mathbb{P}(X = x) = \mathbb{P}(X \leq x) - \mathbb{P}(X < x) = F(x) - \lim_{n \to \infty} F(x - \frac{1}{n})$$

PDF

Conditional PDF

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)\, dy} = \int_{-\infty}^{y} \frac{f(x,v)}{f_X(x)}\, dv$$

Joint PDF

$$f(x,y) = \frac{\partial^2}{\partial y \partial x} F(x,y)$$

Marginal PDF

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,v)\ dv,\ f_Y(y) = \int_{-\infty}^{+\infty} f(u,y)\ du$$

Convolution

Let $Z = X + Y$

$$f_Z(z) = \sum_x f_X(x) f_{Y|X}(z - x|x) = \sum_y f_Y(y) f_{X|Y}(z - y|y)$$

If $X \perp\!\!\!\perp Y$, then

$$f_Z(z) = \sum_x f_X(x) f_Y(z - x) = \sum_y f_Y(y) f_X(z - y)$$

Definitions

Ω sample space
$\mathcal{F}$ collection of subsets of Ω

$$\text{if } A_1, A_2, \ldots, A_n \in \mathcal{F}, \text{then } \bigcup_{i=1}^{n} A_i \in \mathcal{F}$$

$$\text{if } A \in \mathcal{F}, \text{then } A^c \in \mathcal{F}$$

ω is a member of Ω, it is an elementary event, outcome.
$\mathbb{P}$ prob meas on $(\Omega, \mathcal{F})$ $\mathbb{P} : f \to [0, 1]$ $\mathbb{P}(\Omega) = 1, \mathbb{P}(\emptyset) = 0$
$\mathbb{P}$ is finitely additive and has countable additivity if events disjoint
$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$
$(\Omega, \mathcal{F}, \mathbb{P})$ prob space
RV
$X : \Omega \to \mathbb{R}$
$\mathcal{F}$ measurable
$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$ for each $x \in \mathbb{R}$
$F_X(x) : \mathbb{R} \to [0, 1]$
right cont, monotone inc, lims 0, 1