MathStats

Moments

kth order moment

$$m_k = \mathbb{E}[X^k] = \int_{-\infty}^{\infty} x^k f_X(x) = \sum_{x:f_X(x)>0} x^k f_X(x)$$

kth order central moment

$$\sigma_k = \mathbb{E}[(X - m_1)^k]$$

Expectation

$$\mathbb{E}[g(X,Y)] = \sum_y \sum_x g(x,y) f_{X,Y}(x,y)$$

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} \mathbb{E}[Y|X=x] f_X(x)\, dx$$

If $X \geq 0$ (continuous or discrete)

$$\mathbb{E}[X] = \int_0^{\infty} (1 - F_X(x))\ dx$$

Conditional expectation

$$\mathbb{E}[Y|X=x] = \int y f_{y|x}(y|x) dy$$

$$\mathbb{E}[g(Y)|X=x] = \int g(y) f_{y|x}(y|x) dy$$

Law of iterated expectation

$$\mathbb{E}_X[E_{Y|X}\{g(Y)|X\}] = \mathbb{E}[g(Y)] = \sum_x \mathbb{E}[g(Y)|X=x] f_X(x)$$

$$\mathbb{E}[h(X)g(Y)] = \mathbb{E}[h(X)\mathbb{E}[g(Y)|X]]$$

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \sum_x \mathbb{E}[Y|X=x] f_X(x)$$

If $X \perp\!\!\!\perp Y$, then

$$\mathbb{E}[g(Y)|X=x] = E[g(Y)]$$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathrm{Cov}(X,X)$$

Special Case: Let $Y^* = I(Y=y)$

$$\mathbb{E}[Y^*] = \mathbb{E}[I(Y=y)] = \mathbb{P}(Y=y)$$

Special case cont

$$\mathbb{E}[Y^*] = \sum_x \mathbb{E}[Y^*|X=x] f_X(x) = \sum_x \mathbb{P}(Y=y|X=x) f_X(x) = \mathbb{P}(Y=y)$$

Basics

$$\mathrm{Cov}[X,Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$\rho(X,Y) = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X] \cdot \mathrm{Var}[Y]}} = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\sigma_X \sigma_Y}}$$

Expectation Algebra

$$\mathbb{E}[X] = \mathbb{E}[X_+] - \mathbb{E}[X_-] \text{ if } X_+ \geq 0$$

$$\mathbb{E}[|X|] = \mathbb{E}[X_+] + \mathbb{E}[X_-] \text{ if } X_- \geq 0$$

$$\mathbb{E}[a + bX] = a + b\mathbb{E}[X]$$

$$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$$

If $X \perp\!\!\!\perp Y$, then

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

Variance Algebra

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + 2\,\mathrm{Cov}[X,Y] + \mathrm{Var}[Y]$$

$$\mathrm{Var}[X-Y] = \mathrm{Var}[X] - 2\,\mathrm{Cov}[X,Y] + \mathrm{Var}[Y]$$

If $X \perp\!\!\!\perp Y$, then

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$$

If $X \perp\!\!\!\perp Y$, then

$$\mathrm{Var}[X-Y] = \mathrm{Var}[X] - \mathrm{Var}[Y]$$

$$\mathrm{Var}[XY] = \mathbb{E}[X^2] \cdot \mathbb{E}[Y^2] - (\mathbb{E}[X] \cdot \mathbb{E}[Y])^2$$

$$\mathrm{Var}[X/Y] = \mathrm{Var}[X \cdot (1/Y)] = \mathrm{Var}[(1/Y) \cdot X]$$

$$\mathrm{Var}[aX + bY] = a^2\,\mathrm{Var}[X] + b^2 Var[Y] + 2ab\,\mathrm{Cov}[X,Y]$$

Correlation

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{Var(Y)}}$$

Calc

Chain rule

$$f(g(x))' = f'(g(x))g'(x)$$

Product rule

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

Quotient rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Indeterminate Forms
$\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$, $\infty - \infty$, $0 * \infty$, 0^0, 1^∞, ∞^0
Determinate Forms
$\infty + \infty = \infty$, $-\infty - \infty = -\infty$, $0^\infty = 0$, $0^{-\infty} = \infty$, $\infty * \infty = \infty$

Series

Exponential

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

Geometric

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \text{ where } |r| < 1$$

$$\sum_{k=1}^{\infty} kr^{k-1} = \frac{1}{(1-r)^2} \text{ where } |r| < 1$$

$$\sum_{k=2}^{\infty} k(k-1)r^{k-2} = \frac{-2}{(1-r)^3} \text{ where } |r| < 1$$

Arithmetico-geometric

$$\sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{1}{p^2} = p^{-2}$$

$$\sum_{k=2}^{\infty} k(k-1)(1-p)^{k-2} = \frac{2}{p^3}$$

Binomial

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$n(x+y)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1} y^{n-k}$$

$$n(n-1)(x+y)^{n-2} = \sum_{k=2}^{n} k(k-1)\binom{n}{k} x^{k-2} y^{n-k}$$

Binomial Coefficient

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{k} = \binom{n-1}{k-1}\binom{n}{k} \text{ if } k > 0$$

$$(n+1)\binom{r}{n+1} = r\binom{r-1}{n}$$

$$\binom{n}{r+a}\binom{r+a}{r} = \binom{n}{r}\binom{n-r}{a}$$

$$\frac{1}{k+1}\binom{n}{k} = \binom{n+1}{k+1}\frac{1}{n+1}$$

Distribution Functions

$$\lim_{x \to -\infty} F(x) = 0 \quad \lim_{x \to \infty} F(x) = 1$$

$$F(x) \leq F(y) \text{ if } x < y$$

$$\lim_{h \downarrow 0} F(x+h) = F(x)$$

$$F(x) = \int_{-\infty}^{x} f(t)\ dt \text{ where } f(t) \geq 0 \text{ and } \int_{-\infty}^{\infty} f(t)\ dt = 1$$

$$F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$$

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = \mathbb{P}(\boldsymbol{X} \leq \boldsymbol{x}) = \mathbb{P}(X_1 \leq x_1, \ldots, X_k \leq x_k)$$

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_k} f(t_1, \cdots, t_k)\ dt_1 \cdots dt_k$$

$$\mathbb{P}(a < X \leq b, c < Y \leq d) = F(b,d) - F(a,d) - F(b,c) + F(a,c)$$

Joint Density

$$F(x,y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x} \left\{ \int_{-\infty}^{y} f(u,v)\ dv \right\}\ du$$

Random Variables

$$X : \Omega \to \mathbb{R}$$

$$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$$

$$F(x) = \mathbb{P}(X \leq x) = \mathbb{P}(\{\omega : X(\omega) \leq x\})$$

$$\mathbb{P}(X > x) = 1 - F(x)$$

$$\mathbb{P}(x < X \leq y) = F(y) - F(x)$$

$$\mathbb{P}(X < x) = \lim_{n \to \infty} F(x - \frac{1}{n})$$

$$\mathbb{P}(X = x) = \mathbb{P}(X \leq x) - \mathbb{P}(X < x) = F(x) - \lim_{n \to \infty} F(x - \frac{1}{n})$$

PDF

Conditional PDF

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)\, dy} = \int_{-\infty}^{y} \frac{f(x,v)}{f_X(x)}\, dv$$

Joint PDF

$$f(x,y) = \frac{\partial^2}{\partial y \partial x} F(x,y)$$

Marginal PDF

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,v)\ dv,\ f_Y(y) = \int_{-\infty}^{+\infty} f(u,y)\ du$$

Convolution

Let $Z = X + Y$

$$f_Z(z) = \sum_x f_X(x) f_{Y|X}(z-x|x) = \sum_y f_Y(y) f_{X|Y}(z-y|y)$$

If $X \perp\!\!\!\perp Y$, then

$$f_Z(z) = \sum_x f_X(x) f_Y(z-x) = \sum_y f_Y(y) f_X(z-y)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\ dx = \int_{-\infty}^{\infty} f_{X,Y}(z-y, y)\ dy$$

If $X \perp\!\!\!\perp Y$, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\ dx = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\ dy$$

Bivariate Normal

$$\Sigma = \mathrm{Cov}(\boldsymbol{X}, \boldsymbol{X}) = \begin{pmatrix} \sigma_1^2 & \mathrm{Cov}(X_1, X_2) \\ \mathrm{Cov}(X_1, X_2) & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

$$\mathrm{Cov}(X_1, X_2) = 0 \iff X_1 \perp\!\!\!\perp X_2$$

$$\boldsymbol{U} = \boldsymbol{v} + C\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{v} + C\boldsymbol{\mu}, C\Sigma C)$$

$$X_2 | X_1 = x_1 \sim \mathcal{N}(\mu_2 + \rho\frac{\sigma_2}{\sigma 1}(x_1 - \mu_1), \sigma_2^2(1-\rho^2))$$

$$\mathbb{E}[X_2|X_1] = \mu_2 + \rho\frac{\sigma_2}{\sigma 1}(X_1 - \mu_1)$$

Distributions arising from $\mathcal{N}$

Gamma distribution

$$\Gamma\left(\frac{1}{2}, \frac{k}{2}\right) \to \chi_k^2 \to t_k \overset{D}{=} \frac{\mathcal{N}(0,1)}{\sqrt{\chi_k^2/k}} \to F_{m,k} \overset{D}{=} \frac{\chi_m^2/m}{\chi_k^2/k}.(F_{1,k} \overset{D}{=} t_k^2)$$

$$X \sim \Gamma(\lambda, p) \iff \lambda X \sim \Gamma(1, p) \overset{c>0}{\implies} cX \sim \Gamma(\frac{\lambda}{c}, p)$$

$$X_j \sim \Gamma(\lambda, p_j), j = 1, \ldots, k, X_1, \ldots, X_k \perp\!\!\!\perp \implies \sum_{j=1}^k X_j \sim \Gamma(\lambda, \sum_{j=1}^k p_j)$$

$$\begin{cases} X_1 & \sim \Gamma(\lambda, p) \\ X_2 & \sim \Gamma(\lambda, q) \\ X_1 & \perp\!\!\!\perp X_2 \end{cases} \implies \begin{cases} Y_1 = X_1 + X_2 & \sim \Gamma(\lambda, p+q) \\ Y_2 = \frac{X_1}{X_1 + X_2} & \sim \mathrm{Beta}(p, q) \\ Y_1 & \perp\!\!\!\perp Y_2 \end{cases}$$

χ_k^2 distribution (k degrees of freedom)

$$Z_1, \ldots, Z_k \overset{\text{iid}}{\sim} \mathcal{N}(0,1) \implies V = Z_1^2 + \cdots + Z_k^2 \sim \chi_k^2$$

$$f_V(v) = \frac{v^{(k-2)/2} e^{-v/2}}{2^{k/2}\Gamma(k/2)} \sim \Gamma\left(\frac{1}{2}, \frac{k}{2}\right), v \geq 0$$

$$\mathcal{N}(0,1)^2 \sim \chi_1^2 \overset{D}{=} \Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\chi_2^2 = \Gamma\left(\frac{1}{2}, 1\right) = \mathrm{Exp}\left(\frac{1}{2}\right)$$

t_k distribution (k degrees of freedom)

if

$$Z \sim \mathcal{N}(0,1),\ V \sim \chi_k^2,\ Z \perp\!\!\!\perp V$$

then

$$Q = \frac{Z}{\sqrt{V/k}} \sim t_k$$

$$f_Q(q) = c_1 (1 + \frac{q^2}{k})^{-(k+1)/2}, q \in \mathbb{R}$$

where $c_1 > 0$ is a constant

$$\frac{1}{1+q^2} \sim \mathrm{Cauchy}(0,1) \overset{D}{=} t_1$$

$$t_k \to \mathcal{N}(0,1) \text{ as } k \to \infty$$

F Distribution

if

$$V \sim \chi_m^2,\ W \sim \chi_k^2,\ V \perp\!\!\!\perp W$$

then

$$S = \frac{V/m}{W/k} \sim F_{m,k}$$

$$f_S(s) = c_2 s^{(m-2)/2}(1 + \frac{m}{k}s)^{-(k+m)/2}, s > 0$$

where $c_2 > 0$ is a constant

Special case: if $m = 1$, then

$$F_{m,k} = F1, k \overset{D}{=} t_k^2$$

Estimators

Estimate μ

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$$

Estimate σ^2

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2$$

Suppose $\{X_1, \ldots, X_n\} \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$, then

$$\sqrt{n}\frac{\overline{X}_n - \mu}{\sigma} \sim \mathcal{N}(0,1),\ \overline{X}_n \perp\!\!\!\perp S_n^2,\ \mathbb{E}[S_n^2] = \sigma^2,\ \frac{(n-1)S_n^2}{\sigma^2} \sim \chi_{n-1}^2$$

And

$$\sqrt{n}\frac{\overline{X}_n - \mu}{\sigma} \sim \mathcal{N}(0,1),\ \sqrt{n}\frac{\overline{X}_n - \mu}{S_n} \sim t_{n-1}$$

Sampling

Inverse Transform

Let $U \sim \mathrm{Unif}(0,1)$, F be some continuous distribution. Assume $F(x)$ is strictly $\uparrow$ in x, then: $X = F^{-1}(U) \sim F$

Discrete

Let $U \sim \mathrm{Unif}(0,1)$, suppose $X \in \{0, 1, 2, \ldots\} \sim F(x)$. We wish to generate X. For $k = 0, 1, 2, \ldots$, define $Y = k \iff F(k-1) < U \leq F(k)$, then $Y \sim F$

Rejection

Suppose $X \sim F_X$ with a known PDF: f_X. Let $U \sim \mathrm{Unif}(0,1)$, $Z \perp\!\!\!\perp U$, $Z \sim f_Z(z)$, where the pdf is known. $\exists$ a known constant $a > 0$, such that $f_X(z) \leq a f_Z(z) \forall z$

Event $E = \{aUf_Z(Z) \leq f_X(Z)\}$. $P(Z \leq x|E) = F_X)(x)$

Generating Functions – PGF

(non-negative integer-valued RVs) – No finiteness guarantee

$$G_X(s) = \mathbb{E}(s^X) = \sum_{j=0}^{\infty} s^j \mathbb{P}(X = j) = \sum_{j=0}^{\infty} s^j f_X(j)$$

Factorial moments

$$\mathbb{E}[X(X-1)\cdots(X-k+1)] = G^{(k)}(1)$$

$$X \perp\!\!\!\perp Y \implies G_{X+Y}(s) = G_X(s) G_Y(s)$$

If $X_1, X_2, \ldots$ iid with PGF G_X and $N (\geq 0)$ is a rv w/ PGF G_N, then the sum $S_N = X_1 + \cdots + X_N$ has the PGF:

$$G_{S_N}(s) = G_N(G_X(t))$$

$$G_{X_1, X_2}(s_1, s_2) = \mathbb{E}[s_1^{X_1} s_2^{X_2}]$$

If $X_1 \perp\!\!\!\perp X_2$:

$$G_{X_1, X_2}(s_1, s_2) = G_{X_1}(s_1) G_{X_2}(s_2)$$

Generating Functions – MGF

(General RVs)

$$M_X(t) = \mathbb{E}(e^{tX}) = G_X(e^t) = \sum_{j=0}^{\infty} e^{tj} f_X(j) = \int e^{tx} f_X(x)\ dx$$

$$M_{X_1, X_2}(t_1, t_2) = \mathbb{E}[e^{t_1 X_1} e^{t_2 X_2}] = G_{X_1, X_2}(e^{t_1}, e^{t_2})$$

If $X_1 \perp\!\!\!\perp X_2$:

$$M_{X_1, X_2}(t_1, t_2) = M_{X_1}(t_1) M_{X_2}(t_2)$$

For $S_n = X_1 + \cdots + X_n$, if $\{X_i\}_{i=1}^n$ are independent, then:

$$M_{S_n}(t) = \prod_{k=1}^n M_{X_k}(t)$$

In addition, if $\{X_i\}_{i=1}^n$ have the same distribution as X, then:

$$M_{S_n}(t) = (M_X(t))^n$$

For $Y = a + bX$, if M_X exists:

$$M_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{ta + tbX}] = e^{ta}\mathbb{E}[e^{tbX}] = e^{ta} M_X(tb)$$

If $|M_X(t)| < \infty$ for $t \in (-a, a)$, where $a > 0$ is some constant, then:

$$M_X^{(k)}(0) = \mathbb{E}[X^k]$$

Stein's Lemma

If $g(z)$ is differentiable and $\lim_{z \to -\infty} g(z)\phi(z) = 0$ and $\lim_{z \to +\infty} g(z)\phi(z) = 0$

Then for $Z \sim \mathcal{N}(0,1)$: $\mathbb{E}[Zg(Z)] = \mathrm{Cov}(Z, g(Z)) = \mathbb{E}[g'(Z)]$

Applications

For $Z \sim \mathcal{N}(0,1)$: $\mathbb{E}[Z^{2k}] = (2k-1)!!$ if $\lim_{z \to \pm\infty} m(\mu + \sigma z)\phi(z) = 0$, then: $Cov(X, m(X)) = \sigma^2 \mathbb{E}[m'(X)]$

CLT

Suppose $\{X_1, \ldots, X_n\} \overset{\text{iid}}{\sim} X$, with $\mu = \mathbb{E}[X]$ and $\sigma^2 = \mathrm{Var}(X) \in (0, \infty)$. Define $Z_n = \sqrt{n}\frac{\overline{X}_n - \mu}{\sigma}$. Then

$$Z_n \xrightarrow{D} Z \sim \mathcal{N}(0,1)$$

LLN

Suppose $\{X_1, \ldots, X_n\} \overset{\text{iid}}{\sim} X$, with $\mu = \mathbb{E}[X]$. Then

$$\overline{X}_n \xrightarrow{D} \mu$$

MGFs

Gamma: $\Gamma(\lambda, \alpha) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha, t < \lambda$

Binomial: $(q + pe^t)^n$

Poisson: $\exp\{\lambda(e^t - 1)\}$

Geometric: $\frac{pe^t}{1 - (1-p)e^t}, t < -\ln(1-p)$