Problem 1. (20 points total). This problem has six parts in total. Consider the following linear system

$$4x_1 + x_2 + 6x_3 = 2$$
$$2x_1 + 2x_2 + 3x_3 = 1$$

(i) (1 point). Write down the associated augmented matrix for the above system. **ANS:** $\begin{bmatrix} 4 & 1 & 6 & 2 \\ 2 & 2 & 3 & 1 \end{bmatrix}$

(ii) (7 points). Use the reduction algorithm to solve the above system. Note: *no other method will receive credit, you must use the reduction algorithm.* **ANS:** The system reduces to $\begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 \end{bmatrix}$. Then we have $x_1 = \frac{1}{2} - \frac{3}{2}x_3$ and $x_2 = 0$, $x_3 = free$. Can also express in parametric vector form.

(iii) (2 points). Is the original system consistent or inconsistent? **ANS:** Consistent.

(iv) (3 points). How many variables are basic and how many variables are free? **ANS:** 2 are basic, one is free.

(v) (5 points). Set $\mathbf{a}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Is $\mathbf{b}$ a linear combination of $\mathbf{a}_1, \mathbf{a_2}$ and $\mathbf{a}_3$? If not, explain. If yes, write $\mathbf{b}$ as a linear combination of $\mathbf{a}_1, \mathbf{a_2}$ and $\mathbf{a}_3$. **ANS:** Yes. Can use $x_3 = 0$, say, to get $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 \\ 2 \end{bmatrix} + 0\mathbf{a_2} + 0\mathbf{a_3}$. There are infinitely many other choices.

(vi) (2 points). Is $\mathbf{b} \in \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$? Why or why not? **ANS:** Yes. This is because $\mathbf{b}$ is a linear combination of the $\mathbf{a}_i$'s.

Problem 2. (10 points). Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(i) (5 points). Does $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$ have a unique solution? Justify your answer. **ANS:** Yes. Notice that $A = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so the homogeneous problem has only the trivial solution.

(ii) (5 points). For which $h, k \in \mathbb{R}$ is $\begin{bmatrix} h \\ k \end{bmatrix} \in \operatorname{span}\{\mathbf{u}, \mathbf{v}\}$? Justify your answer. **ANS:** Since $\begin{bmatrix} 2 & 2 & h \\ -1 & 1 & k \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -k \\ 0 & 1 & \frac{h+2k}{4} \end{bmatrix}$ we have pivots in both rows so that h, k arbitrary will work, i.e. always solvable for $h, k \in \mathbb{R}$

Problem 3. (12 points). Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Prove that $||\mathbf{v}+\mathbf{w}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2$ if and only if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal. **ANS:** $||\mathbf{v}+\mathbf{w}||^2 = (\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w}) = ||\mathbf{v}||^2 + ||\mathbf{w}||^2 + 2\mathbf{v}\cdot\mathbf{w} = ||\mathbf{v}||^2 + ||\mathbf{w}||^2 \iff 2\mathbf{v}\cdot\mathbf{w} = 0$

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Problem 5. (21 points total, 3 points each) For the following questions, answer using the word "True" or the word "False". You **don't need to justify your answer** to receive full credit. There's no partial credit.

(i) True/False: If A is of size $n \times (n+1)$ and $\mathbf{x} \in \mathbb{R}^{n+1}$ then $A\mathbf{x} = \mathbf{0}$ is solvable. **ANS:** True, homogeneous implies consistent.

(ii) True/False: A linear system with two basic variables cannot be consistent. **ANS:** false, consider coefficient matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(iii) True/False: Pivot columns cannot correspond to free variables. **ANS:** True by definition

(iv) True/False: If $\mathbf{u} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ then $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\{\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2\}$. **ANS:** True.

(v) True/False: If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ then $|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}||^2 \, ||\mathbf{v}||^2$. **ANS:** False. Let $u = \begin{bmatrix} 1 \end{bmatrix}, v = \begin{bmatrix} \frac{1}{2} \end{bmatrix}$ this would imply that $\frac{1}{2} < \frac{1}{4}$.

(vi) True/False: If $\mathbf{v}, \mathbf{w}$ are nonzero vectors in $\mathbb{R}^n$ then $\mathbf{proj}_{\mathbf{v}}\mathbf{w} = \mathbf{proj}_{\mathbf{w}}\mathbf{v}$. **ANS:** False. One is parallel to $\mathbf{v}$ while the other is parallel to $\mathbf{w}$

(vii) True/False: If A is an $m \times n$ matrix whose columns do not span $\mathbb{R}^m$ then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b} \in \mathbb{R}^m$. **ANS:** True. This is given in a theorem in the text.