Final for Quantum Mechanics 137A

December 13, 7:00-9:00pm

Closed notes, closed book, no electronics of any kind

Your Student Number: 303753649 6

Scores:

Problem 1 (60): ________________

Problem 2 (20): ________________

Problem 3 (25): ________________

Problem 4 (25): ________________

Problem 5 (25): ________________

Problem 6 (25): ________________

Total (180): ________________

1. *(30 pts) Survey of quantum mechanical ideas: It is not expected that you will need to do any substantial calculations to answer these questions.*

a) A Hilbert space has $d \geq 2$ single-particle states. How many two-fermion states can be formed? How many two-boson states can be formed?

$|s_1 m_{s_1}, s_2 m_{s_2}\rangle$ two-fermion $\frac{d(d-1)}{2}$

two-boson 2^d

b) I have a Hermitian operator $\hat{Q}$. Why are Hermitian operators important in quantum mechanics? Define a Hermitian operator in terms of its properties when taking expectation values in a state $|\alpha\rangle$. If the operator acts in a finite Hilbert space, then $\hat{Q}$ can be represented as a square matrix. What condition must this matrix satisfy to be Hermitian?

$\hat{Q}^\dagger = \hat{Q}$ Because $\langle x \rangle = \langle \phi \rangle$ $\langle \phi | Q | \phi \rangle = \langle \phi | Q\phi \rangle = \langle Q^\dagger \phi | \phi \rangle = \langle Q\phi | \phi$

matrix $\hat{Q}^T = \hat{Q}^*$

用 importance

because Hermitian operator commute

c) A system has two spatial states a and b and two magnetic spin states $m = \frac{1}{2}$ and $m = -\frac{1}{2}$. Write down the antisymmetric two-particle states as products of spatial and spin states.

~~antisym~~

spatial spin

sym $\begin{cases} |aa\rangle \\ \frac{1}{\sqrt{2}}|ab + ba\rangle + \text{antisym } \frac{1}{\sqrt{2}}|\uparrow\downarrow - \downarrow\uparrow\rangle \\ |bb\rangle \end{cases}$ $\Rightarrow$ antisym

antisym $\frac{1}{\sqrt{2}}|ab - ba\rangle$ + sym $\begin{cases} |\uparrow\uparrow\rangle \\ \frac{1}{\sqrt{2}}|\uparrow\downarrow + \downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \end{cases}$

d) How many bound states does the hydrogen atom have? The binding energy $|E|$ of the ground state is 13.6 eV. In terms of this binding energy, what are the energies of the other bound states? Which transitions among these states generate the Lyman and Balmer series? What part of the electromagnetic spectrum – ultraviolet, visible, infrared – dominates each series?

Infinite. $|E_n| = \frac{|E_1|}{n^2}$

ultraviolet → Lyman $\frac{1}{\lambda} = \frac{1}{911.5Å}(1 - \frac{1}{n_i^2}$

visible → Balmer $\frac{1}{\lambda} = \frac{1}{911.5Å}(\frac{1}{4} - \frac{1}{n_i^2}$

infrared → Pachen $\frac{1}{\lambda} = \frac{1}{911.5Å}(\frac{1}{9} - \frac{1}{n_i^2}$

$h\nu = |E_1|(\frac{1}{n_f^2} - \frac{1}{n_i^2})$

$\Rightarrow \frac{2\pi\hbar c}{\lambda} = 13.6\,eV(\frac{1}{n_f^2} - \frac{1}{n_i^2}) \Rightarrow \frac{1}{\lambda} = \frac{13.6\,eV}{2\pi\hbar c}(\frac{1}{n_f^2} - \frac{1}{n_i^2})$

$= \frac{1}{911.5Å}(\frac{1}{n_f^2} - \frac{1}{n_i^2})$

e) When we solve problems with time-independent potentials, we use stationary states. What is a stationary state? In what sense is a stationary state's wavefunction stationary? A normalized wave packet has been expanded in stationary states, each having a probability $|c_i|^2$. If we now evolve the state in time, how will the probabilities $|c_i|^2$ change? How does the energy of the wave packet evolve with time and how is the energy connected to the $|c_i|^2$?

$\langle E \rangle = \sum |c_i|^2 E_i$

It is a state will not evolve with time.

~~It ha~~ Each of ~~them~~ have an exact energy, for example: $\frac{E}{n}$

$|c_i|^2$ don't change, energy don't evolve with time

Connected ~~energy is a constan~~ $\boxed{\psi(t) = \sum c_i \phi_i e^{-i\frac{E_i t}{\hbar}}}$

f) One is given a 3D Schrödinger equation single-particle problem for which the potential is central, that is, $V(r)$ depends only on the radial coordinate. Name two additional operators that commute with the Hamiltonian $\hat{H}$. Do state energies depend on the quantum numbers of these two operators? E.g., note any issues like degeneracies.

L^2 and L_z commute with $\hat{H}$

$L^2|\psi\rangle_{l,m} = \hbar^2 l(l+1)|\psi\rangle_{l,m}$

$L_z|\psi\rangle_{l,m} = m\hbar|\psi\rangle_{l,m}$

Yes.

g) Use Clebsch-Gordan coefficients to express an uncoupled state $|\ell_1 m_1 \ell_2 m_2\rangle$ in terms of a sum over coupled states $|(\ell_1\ell_2)LM\rangle$.

$$|\ell_1 m_1 \ell_2 m_2\rangle = \sum_{L=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \sum_{M=-L}^{L} |(\ell_1\ell_2)LM\rangle\langle(\ell_1\ell_2)LM|\ell_1 m_1 \ell_2 m_2\rangle$$

h) An electron state is prepared so that its spin points along the positive $\hat{z}$ axis. A measurement is then done to determine whether the spin is pointing along the positive $\hat{x}$ axis. What is the probably of this result being yes? Please provide a qualitative reason for your answer.

$Z_+ \Rightarrow |\chi_+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad Z_- \rightarrow |\chi_-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad P = \langle\chi_+^x|\chi_+\rangle^2 = \left(\frac{1}{\sqrt{2}}\langle\chi_+ + \chi_-|\chi_+\rangle\right)^2$

$\chi_+ \Rightarrow |\chi_+^x\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|\chi_+\rangle + \frac{1}{\sqrt{2}}|\chi_-\rangle$ $\quad \boxed{= \frac{1}{2}}$

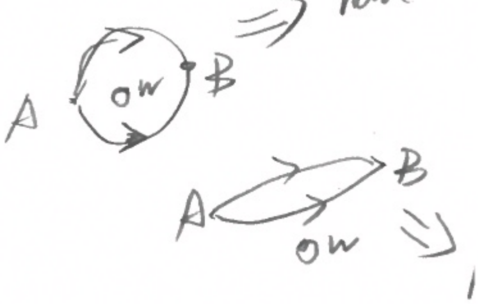
i) The Aharonov-Bohm effect arises in the electron diffraction two-slit experiment illustrated in the figure. The magnetic field is confined to the interior of an infinite solenoid perpendicular to the plane (the circle in the figure), but the vector potential is nonzero throughout the plane. i) An electron can travel from source → slit S1 → screen via path C1, and another from source → slit S1 → screen via path C1′. Will these two paths generate an Aharonov-Bohm interference? ii) Alternatively an electron can travel travel from source → slit S1 → screen via path C1, and another from source → slit S2 → screen via path C2. Will these two paths generate an Aharonov-Bohm interference? iii) Explain your answers, given that both cases involve electrons moving through regions of nonzero vector potential, but zero magnetic field.

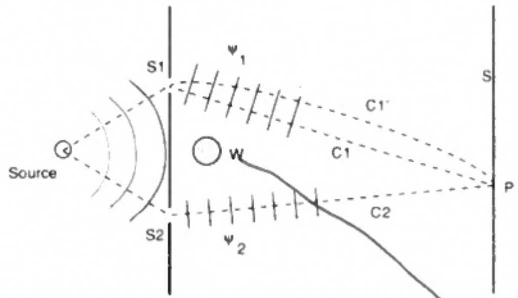
Yes ~~have~~

(i) No

(ii) Yes

(iii)





$$\iint \vec{B}\cdot d\vec{S} = \iint \nabla\times\vec{A}\, d\vec{S} = \oint \vec{A}\cdot d\vec{l}$$

Stokes formula

only the loop contain W inside

there will have a phase difference

It is topological properties for A-B effect.

2. *He atom specroscopy (20 pts):*

As you did in problem set 12, form all possible two-electron antisymmetric He basis states of the form $|\alpha(LS)JM_J\rangle$ where 1) only the $n=1$ and $n=2$ levels are occupied and 2) the $1s$ level contains at least one electron. Here α will be the occupation of the $n=1$ level, and thus by assumption must be 1 or 2. List your results in the table below, the first line of which has been completed. Please write out the radial wave function in the manner of $(1s2p-2p1s)/\sqrt{2}$, etc., to make things clear. "Symmetry" below refers to exchange symmetry.

	α	*radial wave function*	L	S	*radial symmetry*	*spin symmetry*	J	M_J *values*	
$^1S_0$	2	1s1s	0	0	even	odd	0	0	1
$^1S_0$	1	$\frac{1}{\sqrt{2}}\lvert 1s2s+2s1s\rangle$	0	0	even	odd	0	0	1
$^3S_1$	1	$\frac{1}{\sqrt{2}}\lvert 1s2s-2s1s\rangle$	0	1	odd	even	1	$\begin{cases}1\\0\\-1\end{cases}$	3
$^1P_1$	1	$\frac{1}{\sqrt{2}}\lvert 1s2p+2p1s\rangle$	1	0	even	odd	1	$\begin{cases}1\\0\\-1\end{cases}$	3
$^3P_{0,1,2}$	1	$\frac{1}{\sqrt{2}}\lvert 1s2p-2p1s\rangle$	1	1	odd	even	$\begin{cases}0\\1\\2\end{cases}$	0; $1\to\begin{cases}1\\0\\-1\end{cases}$; $2\to\begin{cases}2\\1\\0\\-1\\-2\end{cases}$	9

9+6+2=17 states

$S\begin{cases}0 & \text{odd}\\ 1 & \text{even}\end{cases}$

$\left(\frac{A}{4}\right)^2 (\langle 4_3| e^{i\frac{E_3 t}{\hbar}} + 3\langle 4_1| e^{i\frac{E_1 t}{\hbar}})\ |4_1\rangle\langle 4_1|\ |4_3 e^{-i\frac{E_3 t}{\hbar}} + 3|4_1\rangle e^{-i\frac{E_1 t}{\hbar}}\rangle$

3. *1D review problem* (25 pts)

$\frac{2a}{\pi} A^2 \int_0^{\frac{\pi}{2}} \cos^6\theta\, d\theta$

In class we derived the stationary states $\phi_n(x)$ and their energies E_n for a particle of mass m confined to an infinite well of width a centered on the origin, $-\frac{a}{2} < x < \frac{a}{2}$. This yielded plane-wave even- and odd-parity states that vanish at the boundaries of the well

$= \frac{2a}{\pi} A^2 \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 1$

$$\phi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \cos \frac{\pi n x}{a} & n = 1, 3, 5, ... \\ \sqrt{\frac{2}{a}} \sin \frac{\pi n x}{a} & n = 2, 4, 6, ... \end{cases} \qquad \text{where} \quad E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$

$\Rightarrow \frac{15a}{48} A^2 = 1 \Rightarrow A = \sqrt{\frac{48}{15a}}$

$\cos^3\theta + 3\cos^3\theta - 3\cos\theta = \cos 3\theta$

a) You are given an initial wave packet $\psi(x, t=0) = A\cos^3 \frac{\pi x}{a}$, $-\frac{a}{2} < x < \frac{a}{2}$. By using

$\int_{-\frac{a}{2}}^{\frac{a}{2}} A^2 \cos^6 \frac{\pi x}{a}\, dx = 1$

$$\cos 3\theta = \text{Re}[e^{3i\theta}] = \text{Re}[(\cos\theta + i\sin\theta)^3]$$

$(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$

$= \frac{a}{\pi} A^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^6\theta\, d\theta$

find an expression for $\cos^3\theta$ in terms of $\cos\theta$ and $\cos 3\theta$, and use this to expand the wave packet in terms of square-well stationary states, including determining A.

$\cos 3\theta = \text{Re}[(\cos^2\theta - \sin^2\theta + i2\sin\theta\cos\theta)(\cos\theta + i\sin\theta)] = \text{Re}[\cos^3\theta + i\sin\theta\cos^2\theta - \cos\theta\sin^2\theta - i\sin^3\theta + i2\sin\theta\cos^2\theta - 2\sin^2\theta\cos\theta]$

$= \cos^3\theta - 3\sin^2\theta\cos\theta = \cos^3\theta - 3(1-\cos^2\theta)\cos\theta$

$\Rightarrow \cos 3\theta = 4\cos^3\theta - 3\cos\theta \Rightarrow \cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$

$\Rightarrow \psi(x, t=0) = A\left(\frac{1}{4}\cos\frac{3\pi x}{a} + \frac{3}{4}\cos\frac{\pi x}{a}\right) = \frac{4}{\sqrt{5a}}\left(\frac{1}{4}\cos\frac{3\pi x}{a} + \frac{3}{4}\cos\frac{\pi x}{a}\right)$

$\Rightarrow A = \sqrt{\frac{48}{15a}} = \frac{4}{\sqrt{5a}}$

b) Following the prime directive, write down $\psi(x, t)$.

$E_3 = \frac{9\hbar^2\pi^2}{2ma^2}$

$\psi(x,t) = \sum c_i \phi_i e^{-i\frac{E_i t}{\hbar}}$

$E_1 = \frac{\hbar^2\pi^2}{2ma^2}$

$= \frac{A}{4}(|\psi_3\rangle e^{-i\frac{E_3 t}{\hbar}} + 3|\psi_1\rangle e^{-i\frac{E_1 t}{\hbar}}) = \frac{4}{\sqrt{5a}}\left(\frac{1}{4}\cos\frac{3\pi x}{a} e^{-i\frac{E_3 t}{\hbar}} + \frac{3}{4}\cos\frac{\pi x}{a} e^{-i\frac{E_1 t}{\hbar}}\right)$

c) Calculate the probability that a measurement at time t will find the particle in state n.

for $n \neq 1, 3 \Rightarrow P = 0$

for $n = 1 \quad P = \langle\psi(x,t)|\phi_1(x)\rangle\langle\phi_1(x)|\psi(x,t)\rangle = 0.9$

for $n = 3 \quad P = \langle\psi(x,t)|\phi_3(x)\rangle\langle\phi_3(x)|\psi(x,t)\rangle = 0.1$

d) Find the expectation $\langle H\rangle$ for the wave packet $\psi(x,t)$.

$\langle H\rangle = \int \psi^*(x,t)\, \hat{H}\, \psi(x,t)\, dx$

or $\langle H\rangle = \sum_{i=1,3} P_i E_i = 0.9 \times \frac{\hbar^2\pi^2}{2ma^2} + 0.1 \times \frac{9\hbar^2\pi^2}{2ma^2}$

$= \frac{0.9\hbar^2\pi^2}{ma^2}$

4. *Hermitian operators, Dirac notation* (25 pts)

a) What is meant by a Hilbert space in quantum mechanics. Give an example of a useful basis in quantum mechanics where the basis states are not part of the Hilbert space.

$1 = \sum |\phi_n\rangle\langle\phi_n|$ ↙ HO solution

b) $\hat{A}$, $\hat{B}$, and $\hat{C}$ are Hermitian operators. Determine which of the following operator combinations are Hermitian. (Please show a proof in each case.)

$\hat{A}\hat{B}\hat{C} + \hat{C}\hat{B}\hat{A}$ $\quad (\hat{A}\hat{B}\hat{C} + \hat{C}\hat{B}\hat{A})^{+} = \hat{C}^{+}\hat{B}^{+}\hat{A}^{+} + \hat{A}^{+}\hat{B}^{+}\hat{C}^{+} = \hat{C}\hat{B}\hat{A} + \hat{A}\hat{B}\hat{C}$ Yes

$\hat{A}\hat{B}\hat{C} - \hat{C}\hat{B}\hat{A}$: $\quad (\hat{A}\hat{B}\hat{C} - \hat{C}\hat{B}\hat{A})^{+} = \hat{C}^{+}\hat{B}^{+}\hat{A}^{+} - \hat{A}^{+}\hat{B}^{+}\hat{C}^{+} = \hat{C}\hat{B}\hat{A} - \hat{A}\hat{B}\hat{C}$ No

$i(\hat{A}\hat{B}\hat{C} - \hat{C}\hat{B}\hat{A})$: $\quad [i(\hat{A}\hat{B}\hat{C} - \hat{C}\hat{B}\hat{A})]^{+} = -i(\hat{C}^{+}\hat{B}^{+}\hat{A}^{+} - \hat{A}^{+}\hat{B}^{+}\hat{C}^{+}) = i(\hat{A}\hat{B}\hat{C} - \hat{C}\hat{B}\hat{A})$ Y

$\hat{A}\hat{B}\hat{C} + \hat{B}\hat{C}\hat{A}$: $\quad (\hat{A}\hat{B}\hat{C} + \hat{B}\hat{C}\hat{A})^{+} = \hat{C}^{+}\hat{B}^{+}\hat{A} + \hat{A}^{+}\hat{C}^{+}\hat{B}^{+} = \hat{C}\hat{B}\hat{A} + \hat{A}\hat{C}\hat{B}$ No

c) Show that if $\hat{P}$ and $\hat{Q}$ have a common, complete set of normalized eigenfunctions $|p_i q_i\rangle$, so that $\hat{P}|p_i q_i\rangle = p_i|p_i q_i\rangle$ and $\hat{Q}|p_i q_i\rangle = q_i|p_i q_i\rangle$ for all $|p_i q_i\rangle$ in the Hilbert space, then $[\hat{P}, \hat{Q}] = 0$.

$[\hat{P}, \hat{Q}]\,|p_i q_i\rangle = (\hat{P}\hat{Q} - \hat{Q}\hat{P})\,|p_i q_i\rangle$

$= p_i q_i\,|p_i q_i\rangle - q_i p_i\,|p_i q_i\rangle = 0$

$\Rightarrow [\hat{P}, \hat{Q}] = 0$ Commute

$\frac{4}{3}|1A\rangle = \frac{4}{5}|1B\rangle + \frac{16}{15}|2B\rangle$

$\frac{4}{3}|1A\rangle - |2A\rangle = \frac{5}{3}|2B\rangle$

$|1A\rangle + \frac{4}{3}|2A\rangle = \frac{5}{3}|1B\rangle \quad = |1B\rangle = \frac{3}{5}|1A\rangle + \frac{4}{5}|2A\rangle$

$\frac{4}{3}|2A\rangle = \frac{16}{15}|1B\rangle - \frac{4}{5}|2B\rangle$

d) *Sequential measurements:* The Hermitian operator $\hat{A}$ has normalized eigenstates $|1_A\rangle$ and $|2_A\rangle$ with eigenvalues a_1 and a_2, respectively. The Hermitian operator $\hat{B}$ has normalized eigenstates $|1_B\rangle$ and $|2_B\rangle$ with eigenvalues b_1 and b_2, respectively. The eigenstates are related by $|1_A\rangle = \frac{3}{5}|1_B\rangle + \frac{4}{5}|2_B\rangle$ and $|2_A\rangle = \frac{4}{5}|1_B\rangle - \frac{3}{5}|2_B\rangle$. You measure observable $\hat{A}$, finding outcome a_1; then immediately after, your lab partner measures observable $\hat{B}$, but fails to communicate the outcome to you; then promptly after this, you measure observable $\hat{A}$ again. What is the probability that you find outcome a_1 in the second measurement?

$|2B\rangle = \frac{4}{5}|1A\rangle - \frac{3}{5}|2A\rangle$

$\frac{16}{15} - 1B \quad \frac{25}{15} \quad \frac{5}{3}$

$\hat{A}|1A\rangle = a_1|1A\rangle$

$\hat{A}|2A\rangle = a_2|2A\rangle$

$\hat{B}|1B\rangle = b_1|1B\rangle$

$\hat{B}|2B\rangle = b_2|2B\rangle$

$|1A\rangle = \frac{3}{5}|1B\rangle + \frac{4}{5}|2B\rangle$

$|2A\rangle = \frac{4}{5}|1B\rangle - \frac{3}{5}|2B\rangle$

$\Rightarrow \{$

① $\hat{A} \Rightarrow$ you got $a_1 \Rightarrow$ the initial state is $|1A\rangle = \frac{3}{5}|1B\rangle + \frac{4}{5}|2B\rangle$

② then measure $\hat{B} \Rightarrow$ you have $\begin{cases} P = \frac{9}{25} \text{ to get } |1B\rangle \\ P = \frac{16}{25} \text{ to get } |2B\rangle \end{cases}$

$|\langle 1B|1A\rangle|^2 = \frac{9}{25}$

$|\langle 2B|1A\rangle|^2 = \frac{16}{25}$

case 1 if you got $|1B\rangle$

③ $\hat{A}$ again $\Rightarrow$ $|1B\rangle = \frac{3}{5}|1A\rangle + \frac{4}{5}|2A\rangle$ $\begin{cases} P = \frac{9}{25} \text{ get } |1A\rangle \\ P = \frac{16}{25} \text{ get } |2A\rangle \end{cases}$ $\Rightarrow \frac{9}{25} \times \frac{9}{25}$

$\boxed{\frac{81+256}{625}}$

add together

case 2 if you got $|2B\rangle$

find out a_1 again

$P = \frac{337}{625}$

$|2B\rangle = \frac{4}{5}|1A\rangle - \frac{3}{5}|2A\rangle$ $\begin{cases} P = \frac{16}{25} \text{ get } |1A\rangle \\ P = \frac{9}{25} \text{ get } |2A\rangle \end{cases}$ $\Rightarrow \frac{16}{25} \times \frac{16}{25}$

and $R_1(a) = R_2(a) \Rightarrow A\frac{\sin ka}{a} = D\frac{e^{-ka}}{a}$

$R_1'(a) = R_2'(a) \Rightarrow A\frac{ka\cos ka - \sin ka}{a^2} = D\frac{-kae^{-ka} - e^{-ka}}{a^2}$

5. *3D boundary value problems:* (25 pts) $\Rightarrow A = De^{-ka}\sin ka$

$\Rightarrow R(r) = \begin{cases} De^{-ka}\sin ka\frac{\sin kr}{r} & r<a \\ D\frac{e^{-kr}}{r} & r>a \end{cases}$

normalization

The Schrödinger equation in 3D for a central potential can be written

$R_1'(r) = A\frac{kr\cos kr - \sin kr}{r^2}$

$$\left[-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\right] + V(r) + \frac{1}{2mr^2}\mathbf{L}^2\right]\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

On making the substitution $\psi(\mathbf{r}) = R_\ell(r)Y_{\ell m}(\Omega) = \frac{u_\ell(r)}{r}Y_{\ell m}(\Omega)$ one finds ~~the radial equation~~

$R_2'(r) = D\frac{-kre^{-kr} - e^{-kr}}{r^2}$

$$-\frac{\hbar^2}{2m}\frac{d^2u_\ell(r)}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u_\ell(r) = Eu_\ell(r)$$

$\int_0^a De^{-ka}\sin ka\frac{\sin kr}{r}r^2dr + \int_a^\infty D\frac{e^{-kr}}{r}r^2dr = 1$

a) Find the bound state $\ell = 0$ interior and exterior radial solutions of the ~~finite~~ spherical

$$V(r) = \begin{cases} -V_0 & r<a \\ 0 & r>a \end{cases}$$

use this, you can solve $\boxed{D}$

You should require $R(0)$ to be finite. By matching the solutions at $r = a$ determine the radial wave function up to an overall normalization.

$r>a$ $V=0$

$l=0 \Rightarrow \frac{-\hbar^2}{2m}\frac{d^2u(r)}{dr^2} + V(r)u_\ell(r) = Eu_\ell(r)$

$\frac{-\hbar^2}{2m}u''(r) = -|E|u(r)$

$k = \frac{\sqrt{2m|E|}}{\hbar}$

$r<a$ $V=-V_0$

$\Rightarrow u''(r) = \frac{2m|E|}{\hbar^2}u(r) = k^2u(r)$

$\frac{-\hbar^2}{2m}\frac{d^2u(r)}{dr^2} = (V_0+E)u(r)$

$\Rightarrow u_2(r) = Ce^{kr} + De^{-kr}$

$k = \frac{\sqrt{2m(V_0-|E|)}}{\hbar}$

boundary condition

$u''(r) = -\frac{2m(V_0-|E|)}{\hbar^2}u(r)$

$R_1(r) = \frac{u_1(r)}{r}$ $R_2(r) = \frac{u_2(r)}{r}$

$R_1(0) = \text{finite} \Rightarrow B=0$ $R_2(\infty) = \text{finite} \Rightarrow C=0$

$u''(r) = -k^2u(r)$

$\Rightarrow u_1(r) = A\sin kr + B\cos kr$

$\Rightarrow R_1(r) = A\frac{\sin kr}{r}$ $R_2(r) = D\frac{e^{-kr}}{r}$

b) By requiring the derivative to be continuous, determine the eigenvalue equation.

$R_1(a) = R_2(a) \Rightarrow A\sin ka = De^{-ka}$ ①

$R_1'(a) = R_2'(a) \Rightarrow A(ka\cos ka - \sin ka) = -D(ka+1)e^{-ka}$ ②

① / ② $\Rightarrow \frac{\sin ka}{ka\cos ka - \sin ka} = -\frac{1}{ka+1} \Rightarrow \frac{\tan ka}{ka - \tan ka} = -\frac{1}{ka+1}$

set $ka = z$ $\quad$ $Ka = \sqrt{z_0^2 - z^2}$

$\Rightarrow \boxed{\frac{\tan z}{z - \tan z} = -\frac{1}{\sqrt{z_0^2-z^2}+1}}$ ②

c) What is the minimum depth V_0 required to support a bound state?

$z_0 = \sqrt{\frac{2mV_0}{\hbar^2}}$

$E \to 0 \Rightarrow K = \frac{\sqrt{2m|E|}}{\hbar} \to 0 \Rightarrow \sqrt{z_0^2 - z^2} \to 0$

plug into ② $\Rightarrow \frac{\tan z}{z - \tan z} = -1 \Rightarrow z = \frac{\pi}{2}$

E small

$\Rightarrow \sqrt{\frac{2m(V_0-|E|)}{\hbar^2}} = \frac{\pi}{2} \Rightarrow \sqrt{\frac{2mV_0}{\hbar^2}} \approx \frac{\pi}{2} \Rightarrow \frac{2mV_0}{\hbar^2} = \frac{\pi^2}{4} \Rightarrow \boxed{V_0 = \frac{\pi^2}{8\ldots}}$

d) The interior solution $u(r)$ found above corresponds to $R_0(r) \sim j_0(kr)$, the $\ell = 0$ spherical Bessel function. If we had solved the **infinite** 3D spherical well for arbitrary ℓ, what would then be the requirement for an eigenvalue?

$R_{n\ell}(r) = A_{n\ell}\, j_\ell(kr) + B_{n,\ell}\, n_\ell(kr)$ Neumann is diverge in ∞

finite $R(\infty) \Rightarrow B_{n\ell} = 0$

and we find $Y_{\ell m}(\theta\phi) = \Theta(\theta) \cdot \Phi(\phi)$

$\Theta(\theta) = C e^{im\phi}$

$\Phi(\phi) = A_{n\ell} P_\ell^m(\cos\phi)$

$\psi = R_{n\ell}(r) \cdot Y_{\ell m}(\theta, \phi)$

then $\hat{H}\psi \Rightarrow E_{n\ell}\psi$, we get eigenvalue.

6. *Larmor precession, prime directive* (25 pts): Consider a spin-1/2 system quantized as usual using $\hat{\mathbf{S}}^2$ and $\hat{S}_z$. The initial wave function at $t = 0$ is

$$|\psi(0)\rangle = |\chi_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so pointed along the $+\hat{z}$ axis.

a) The system evolves for a time t under influence of a magnetic field of strength B pointed along the positive x axis, so

$$\hat{H} = -\gamma \mathbf{B} \cdot \hat{\mathbf{S}} = -\gamma B \hat{S}_x = -\frac{\gamma B \hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Find the stationary states and eigenvalues for this Hamiltonian, and express $|\psi(0)\rangle$ in terms of the stationary states.

$$\hat{H}\chi_+^x = -\frac{\gamma B\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = -\frac{\gamma B\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow$$

stationary state $\chi_+^x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

eigen value : $-\frac{\gamma B\hbar}{2}$

$$\hat{H}\chi_-^x = -\frac{\gamma B\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{\gamma B\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow$$

$\chi_-^x \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

eigenvalue $\frac{\gamma B\hbar}{2}$

$$|\psi(0)\rangle = |\chi_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\chi_+^x + \frac{1}{\sqrt{2}}\chi_-^x$$

b) Using the prime directive, find an expression for $|\psi(t)\rangle$.

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\chi_+^x e^{i\frac{\gamma Bt}{2}} + \frac{1}{\sqrt{2}}\chi_-^x e^{-i\frac{\gamma Bt}{2}} = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(\chi_+ + \chi_-)e^{i\frac{\gamma Bt}{2}} + \frac{1}{\sqrt{2}}(\chi_+ - \chi_-)e^{-i}\right.$$

$$= \frac{1}{2}\left[\chi_+ 2\cos\frac{\gamma Bt}{2} + \chi_- 2i\sin\right.$$

$$= \chi_+\cos\frac{\gamma Bt}{2} + i\chi_-\sin\frac{\gamma Bt}{2}$$

c) Calculate the probability of finding the system in the states $|\chi_+\rangle$ and $|\chi_-\rangle$ as a function of time.

$$P_1 = \langle\psi(t)|\chi_+\rangle\langle\chi_+|\psi(t)\rangle$$

$$P_{\chi_+} = \langle\chi_+\cos\frac{\gamma Bt}{2} - i\chi_-\sin\frac{\gamma Bt}{2}|\chi_+\rangle\langle\chi_+|\chi_+\cos\frac{\gamma Bt}{2} + i\chi_-\sin\frac{\gamma Bt}{2}\rangle$$

$$= \cos^2\frac{\gamma Bt}{2}$$

$$\boxed{P_{\chi_+} + P_{\chi_-} = 1}$$

$$P_{\chi_-} = \langle\psi(t)|\chi_-\rangle\langle\chi_-|\psi(t)\rangle$$

$$= \langle\chi_+\cos\frac{\gamma Bt}{2} - i\chi_-\sin\frac{\gamma Bt}{2}|\chi_-\rangle\langle\chi_-|\chi_+\cos\frac{\gamma Bt}{2} + i\chi_-\sin\frac{\gamma Bt}{2}\rangle$$

$$= \sin^2\frac{\gamma Bt}{2}$$

d) Calculate the probabilities of finding the system in the states $|\chi^y_+\rangle$, and $|\chi^y_-\rangle$ as a function of time.

$$\chi_+^y = \begin{pmatrix} \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} \end{pmatrix} = \frac{1}{\sqrt2}\chi_+ + \frac{i}{\sqrt2}\chi_- \qquad \chi_-^y = \begin{pmatrix} \frac{1}{\sqrt2} \\ -\frac{i}{\sqrt2} \end{pmatrix} = \frac{1}{\sqrt2}\chi_+ - \frac{i}{\sqrt2}\chi_-$$

$$P_{y+} = \langle \psi(t) | \chi_+^y \rangle \langle \chi_+^y | \psi(t) \rangle$$

$$= \langle \chi_+ \cos\frac{\gamma Bt}{2} - i\chi_- \sin\frac{\gamma Bt}{2} | \frac{1}{\sqrt2}\chi_+ + \frac{i}{\sqrt2}\chi_- \rangle \langle \frac{1}{\sqrt2}\chi_+ - \frac{i}{\sqrt2}\chi_- | \chi_+ \cos\frac{\gamma Bt}{2} + i\chi_- \sin\frac{\gamma Bt}{2} \rangle$$

$$= \left(\frac{1}{\sqrt2}\cos\frac{\gamma Bt}{2} + \frac{1}{\sqrt2}\sin\frac{\gamma Bt}{2}\right)^2 = \frac{1}{2} + \cos\frac{\gamma Bt}{2}\sin\frac{\gamma Bt}{2}$$

$$P_{y-} = \langle \psi(t) | \chi_-^y \rangle \langle \chi_-^y | \psi(t) \rangle$$

$$= \langle \chi_+ \cos\frac{\gamma Bt}{2} - i\chi_- \sin\frac{\gamma Bt}{2} | \frac{1}{\sqrt2}\chi_+ - \frac{i}{\sqrt2}\chi_- \rangle \langle \frac{1}{\sqrt2}\chi_+ + \frac{i}{\sqrt2}\chi_- | \chi_+ \cos\frac{\gamma Bt}{2} + i\chi_- \sin\frac{\gamma Bt}{2} \rangle$$

$$= \left(\frac{1}{\sqrt2}\cos\frac{\gamma Bt}{2} - \frac{1}{\sqrt2}\sin\frac{\gamma Bt}{2}\right)^2 = \frac{1}{2} - \cos\frac{\gamma Bt}{2}\sin\frac{\gamma Bt}{2}$$

e) Describe in words how the spin precesses in time. Around what does it precess, in what plane does it reside, and how long does it take the spin to return to its initial position? At what times are your guaranteed that if you make a measurement of $\hat{S}_z$ or $\hat{S}_y$, you will get only one possible answer?

at first, the spin ~~at f~~ point z_+

and rotate in Y-z ~~plain~~ plane

$\frac{\gamma Bt}{2} = 2\pi \Rightarrow t = \frac{4\pi}{\gamma B}$ return to its initial position

$\cancel{\frac{\gamma Bt}{2} = n\pi \Rightarrow} t =$

$$\left.\begin{array}{l} \sin\frac{\gamma Bt}{2} = 0 \\ \text{or } \cos\frac{\gamma Bt}{2} = 0 \end{array}\right\} \Rightarrow \text{one possible answer}$$

$$\Rightarrow \frac{\gamma Bt}{2} = \frac{n\pi}{2}$$

$$\Rightarrow t = \boxed{\frac{n\pi}{\gamma B}}$$ one possible answer

Useful items:

The problems are constructed so you should have everything you need. But just in case:

1. Useful Clebsch Gordan coefficients $\langle \ell_1 m_1 \ell_2 m_2 | (\ell_1 \ell_2) LM \rangle$:

$$\langle \tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}|(\tfrac{1}{2}\tfrac{1}{2})11\rangle = 1 \quad \langle \tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}|(\tfrac{1}{2}\tfrac{1}{2})10\rangle = \tfrac{1}{\sqrt{2}} \quad \langle \tfrac{1}{2}-\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}|(\tfrac{1}{2}\tfrac{1}{2})10\rangle = \tfrac{1}{\sqrt{2}} \quad \langle \tfrac{1}{2}-\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}|(\tfrac{1}{2}\tfrac{1}{2})1-1\rangle =$$

$$\langle \tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}|(\tfrac{1}{2}\tfrac{1}{2})00\rangle = \tfrac{1}{\sqrt{2}} \quad \langle \tfrac{1}{2}-\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}|(\tfrac{1}{2}\tfrac{1}{2})00\rangle = -\tfrac{1}{\sqrt{2}}$$

2. Triangle condition for coupling $\vec{\ell}_1$ and $\vec{\ell}_2$ to $\vec{L}$:

$$|\ell_1 - \ell_2| \leq L \leq \ell_1 + \ell_2$$

3. Spin vectors

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \chi_+^x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \chi_-^x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \quad \chi_+^y = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \quad \chi_-^y = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$$