

PHYSICS 137A-LEC-001 Final Exam

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TOTAL POINTS

160 / 180

QUESTION 1

60 pts

1.1. a 3 / 6

- 0 pts Correct
- 6 pts fermion and boson counting both wrong
- 6 pts answer blank
- ✓ - 3 pts **boson counting wrong**
 - 2 pts Boson counting wrong, but you did have the qualitative result that there are more boson states

1.2 1. b 4 / 6

- 0 pts Correct
- 6 pts Answer blank
- ✓ - 2 pts **No connection to observables**
 - 2 pts Lack of a definition
 - 1 pts Transpose error or other minor error
 - 2 pts No definition of adjoint matrix

1.3 1. c 8 / 8

- ✓ - 0 pts **Correct**
 - 8 pts Missing all states
 - 6 pts Some indication that they know what an antisymmetric spin or space wf is
 - 4 pts Three of six states
 - 3 pts Did not put pieces together

1.4 1. d 6 / 6

- ✓ - 0 pts **Correct**
 - 2 pts not correct $1/n^2$
 - 2 pts Not proper series identification
 - 6 pts Blank
 - 1 pts Small error or omission

1.5 1. e 3 / 8

- 0 pts Correct

- ✓ - 2 pts why stationary
- ✓ - 2 pts Solution of the T-independent SE
- ✓ - 1 pts Energy constant, but not connected to $lcil^2$
- 2 pts No coherent energy explanation

1.6 1. f 4 / 6

- 0 pts Correct
- 1 pts No observation that energies depend on L^2
- ✓ - 2 pts **No observation of Lz degeneracy**
 - 2 pts two wrong operators
 - 1 pts One wrong operator
 - 6 pts Blank or nothing correct stated

1.7 1. g 6 / 6

- ✓ - 0 pts **Correct**
 - 6 pts No CG coefficient appears or no state appears
 - 4 pts Sums over unsummed variables
 - 2 pts Sum limits wrong or absent
 - 4 pts Sums and ket absent

1.8 1. h 6 / 6

- ✓ - 0 pts **Correct**
 - 6 pts Not answered; or answered as zero or some other answer with no reasonable explanation

1.9 1 i 8 / 8

- ✓ - 0 pts **Correct**
 - 2 pts Lacks a comment about line integral of the A field related by Stokes to the magnetic flux through the cylinder
 - 4 pts No mention of magnetic flux, the line integral over A, or the relation to B dS via stokes
 - 8 pts No answers provided, or the answers are wrong and thus fail to connect the phase to the enclosed magnetic flux

- 5 pts One answer wrong but some understanding that the effect is nonclassical and generated by A field

QUESTION 2

2 Problem 2 20 / 20

✓ - 0 pts Correct

- 3 pts m values not written out, or are wrong

- 4 pts Radial wave functions not written out as requested

- 4 pts S=1 claimed as odd, or S=0 as even, or otherwise the overall exchange symmetry is wrong

- 2 pts State count moderately wrong, e.g. one or two too many or too few

- 8 pts State count greatly off

- 2 pts S confused with MS or J with MJ or S not provided

- 4 pts Single particle labels wrong

- 20 pts Blank or effectively so

- 3 pts Missing J couplings, or incorrect J coupling or other vector ranks

- 8 pts Radial symmetry systematically wrong

+ 1 pts (a) Incorrect coefficient

✓ + 5 pts (b) Full $\psi(x, t) = \frac{1}{\sqrt{10}} (\phi_3(x) e^{-i E_1 t / \hbar} + \frac{3}{\sqrt{10}} e^{-i E_1 t / \hbar})$, $E_1 = \frac{\hbar^2}{2m} \pi^2$

+ 3 pts (b) correct time evolution for each energy eigenstates

✓ + 5 pts (c) Full

$\langle \psi | \psi \rangle = \sum_n |\psi_n|^2 = \frac{9}{10}$

$\langle \psi | \psi \rangle = \sum_n |\psi_n|^2 = \frac{1}{10}$

+ 3 pts (c) coefficient square

✓ + 5 pts (d) Full

$\langle H | \psi \rangle = \sum_n \langle \psi_n | H | \psi_n \rangle = \sum_n E_n |\psi_n|^2 = \frac{9}{10} E_1 + \frac{1}{10} E_2$

$E_1 = \frac{9\hbar^2}{2m} \pi^2$, $E_2 = \frac{\hbar^2}{2m} \pi^2$

+ 3 pts (d) $\langle H | \psi \rangle = \sum_n \langle \psi_n | H | \psi_n \rangle = \sum_n E_n |\psi_n|^2 = \frac{9}{10} E_1 + \frac{1}{10} E_2$

$H = \sum_n \langle \psi_n | H | \psi_n \rangle E_n = \frac{9}{10} E_1 + \frac{1}{10} E_2$

+ 3 pts (d) Integral expressions for expectation value with a proper representation of \hat{H}

- 1 pts Minor algebraic error

+ 0 pts Wrong

QUESTION 3

Problem 3 25 pts

3.1 3 25 / 25

✓ + 10 pts (a) Full

$\cos^3 \theta = \frac{1}{4} (3 \cos \theta + i \sin \theta)$

$\psi(x, 0) = A \left(\frac{1}{4} \cos \theta + \frac{3}{4} \cos(\theta + \pi/3) \right)$

$= \frac{1}{4} \sqrt{10} (\phi_3(x) + \frac{3}{\sqrt{10}} \phi_1(x))$

$\phi_1(x)$

+ 4 pts (a) $\cos^3 \theta = \frac{1}{4} (3 \cos \theta + i \sin \theta)$

$\cos \theta + \frac{3}{4} \cos(\theta + \pi/3)$

+ 2 pts (a) $\cos^3 \theta$ in terms of

$\cos \theta$ and $\cos 3\theta$

+ 3 pts (a) Correct $A = \frac{1}{\sqrt{5}}$

+ 2 pts (a) Incorrect A

+ 3 pts (a) Correct coefficient (dimension of ψ)

ψ

QUESTION 4

Problem 4 25 pts

4.1 a 1.5 / 5

- 0 pts Correct, eg "A Hilbert space in quantum mechanics is a complex vector space with an inner product. For wave functions (states represented in position space) the existence of the inner product corresponds to

wave functions being square integrable, and thus normalizable. Such wave functions can represent physical states. An example of a useful basis in quantum mechanics where the basis states are not part of the Hilbert space are the free-particle wave functions of definite momentum p, e^{ipx} , which are not normalizable."

✓ - 0.5 pts No mention of inner product, or normalizability

✓ - 2 pts Does not give an example of a basis which

is not a part of hilbert space

- **1 pts** The basis provided is part of hilbert space, or is not a basis, or is lacking some information

✓ - **1 pts** Does not mention that a Hilbert space is a vector space.

- **0.5 pts** Terminology misused in some way, so as to be not technically correct, even though it might be the right idea

- **5 pts** No relevant work

4.2 b 5 / 5

✓ - **0 pts** Correct

- **1 pts** minus sign or similar mistake: one wrong answer

- **1 pts** another minus sign or similar mistake: another wrong answer

- **2 pts** a wrong answer with no work shown

- **2 pts** Another wrong answer with no work shown

- **5 pts** no relevant work

4.3 C 4 / 5

+ **5 pts** Correct. Proof is rigorous (at the level of a physics class) and logical steps are shown and explained.

✓ + **4 pts** Proof is "morally" correct, but has some loose ends.

+ **3 pts** Work is on-track, but the explanation falls short

+ **2 pts** Work is on-track, but with little to no explanation, and/or ambiguous/mis-used notation

+ **1 pts** Little relevant work

+ **0 pts** No relevant work

4.4 d 10 / 10

✓ - **0 pts** Correct. sufficient work is shown

- **4 pts** Gets halfway through the problem with no errors

- **0.5 pts** Problem completed. Small computational mistake (eg algebra, arithmetic)

- **1 pts** Problem completed. Medium computational mistake

- **2 pts** Trouble with conditional probability

- **4 pts** Work shows slight misunderstanding of how measurements work in quantum mechanics

- **8 pts** Work betrays serious a lack of understanding of how measurements work in quantum mechanics

- **0 pts** Little relevant work (-7 to -9, depending on quality and quantity of what is written)

- **10 pts** No relevant work

QUESTION 5

Problem 5 25 pts

5.1 a 10 / 10

+ **10 pts** Correct. (Shortcut instead of using other rubric items)

✓ + **3 pts** Writes the $l=0$ radial equation (or solves without writing it. This rubric item is mostly for giving people partial credit).

✓ + **2 pts** Solved the $l=0$ radial equation for $r < a$

✓ + **1 pts** Used boundary condition at $r=0$

✓ + **2 pts** Solved the $l=0$ radial equation for $r > a$

✓ + **1 pts** Used boundary condition at $r=\infty$

✓ + **1 pts** Made wavefunction continuous at $r=a$

+ **0 pts** No relevant work

5.2 b 4.5 / 5

✓ - **0 pts** Correct.

✓ - **0.5 pts** Small algebra error

- **1 pts** Medium algebra error

- **2 pts** Unable to eliminate constants A and C (or whatever you called them) from equation

- **2.5 pts** Did not get past setting derivatives equal to each other

- **4 pts** Little relevant work

- **5 pts** No relevant work

5.3 C 5 / 5

✓ - **0 pts** Correct

- **1 pts** Answer stated with no work or justification

- **0.5 pts** Small algebra mistake

- **1 pts** Medium algebra mistake

- **2.5 pts** Some setup, but root of the equation is not identified

- **3 pts** The limit in which the equation is solved is not the correct limit (the correct limit is: kappa goes to zero, or the equivalent)

- **4 pts** Little relevant work

- **5 pts** No relevant work

• answer is cut off, but looks right

5.4 d 4 / 5

- **0 pts** Correct

- **2 pts** Does not correctly describe the boundary conditions imposed by the infinite 3D well

- **2 pts** No mention of the spherical bessel functions

✓ - **1 pts** No relevant comment about the energy eigenvalues in the infinite case

- **5 pts** No relevant work

$\cos \frac{\gamma B t - \frac{\pi}{2}}{2}$
 $\left\langle \chi_{-y} | \psi(t) \right\rangle = \cos \frac{\gamma B t + \frac{\pi}{2}}{2}$

+ **4 pts** d) Minor error in algebra, correct concept

+ **2 pts** d) Significant error in calculation or incomplete

✓ + **1 pts** e) rotating x-axis

✓ + **1 pts** e) y-z plane

+ **1 pts** e) $T = \frac{2\pi}{\gamma B}$

+ **2 pts** e) $t_y = \frac{\pi}{2\gamma B}, t_z = 0, t_x = \frac{\pi}{\gamma B}$

✓ + **1 pts** e) Minor error in calculating period

+ **0 pts** Wrong

QUESTION 6

6 Problem 6 23 / 25

✓ + **3 pts** a) Eigenvalues

$$E = -\frac{\gamma B \hbar^2}{2}, \quad +\frac{\gamma B \hbar^2}{2}$$

✓ + **3 pts** a) correct eigenvectors

✓ + **1 pts** a) correct expansion

$$\psi(0) = \frac{1}{\sqrt{2}} (\chi_+ + \chi_-)$$

✓ + **3 pts** b)

$$\psi(t) = \frac{1}{\sqrt{2}} e^{i(\gamma B t / 2)} (\chi_+ + e^{-i(\gamma B t / 2)})$$

✓ + **5 pts** c) correct inner product and probability

$$\langle \psi(t) | \chi_+ | \psi(t) \rangle = \cos \frac{\gamma B t}{2}$$

$$\langle \psi(t) | \chi_- | \psi(t) \rangle = i \sin \frac{\gamma B t}{2}$$

+ **4 pts** c) Minor error in algebra or not fully simplified

+ **2 pts** c) Significant error in calculation or incomplete

✓ + **5 pts** d) correct inner product and probability

$$\langle \chi_+ | \psi(t) \rangle = \cos \frac{\gamma B t}{2}$$

Final for Quantum Mechanics 137A

December 13, 7:00-9:00pm

Closed notes, closed book, no electronics of any kind

Your Student Number: 3037536496

Scores:

Problem 1 (60): _____

Problem 2 (20): _____

Problem 3 (25): _____

Problem 4 (25): _____

Problem 5 (25): _____

Problem 6 (25): _____

Total (180): _____

1. (30 pts) Survey of quantum mechanical ideas: It is not expected that you will need to do any substantial calculations to answer these questions.

a) A Hilbert space has $d \geq 2$ single-particle states. How many two-fermion states can be formed? How many two-boson states can be formed?

$$\begin{array}{lll} |s_1 m_1, s_2 m_2\rangle & \text{two-fermion} & \frac{d(d-1)}{2} \\ & & \\ & \text{two-boson} & 2^d \end{array}$$

b) I have a Hermitian operator \hat{Q} . Why are Hermitian operators important in quantum mechanics? Define a Hermitian operator in terms of its properties when taking expectation values in a state $|\alpha\rangle$. If the operator acts in a finite Hilbert space, then \hat{Q} can be represented as a square matrix. What condition must this matrix satisfy to be Hermitian?

$$\hat{Q}^\dagger = \hat{Q} \quad \text{Because } \langle \phi | \hat{Q} | \phi \rangle = \langle \phi | \hat{Q}^\dagger | \phi \rangle = \langle Q^\dagger \phi | \phi \rangle = \langle Q \phi | \phi \rangle$$

importance

because Hermitian operator commute

$$\hat{Q}^\dagger = \hat{Q}^*$$

c) A system has two spatial states a and b and two magnetic spin states $m = \frac{1}{2}$ and $m = -\frac{1}{2}$. Write down the antisymmetric two-particle states as products of spatial and spin states.

antisym

Spatial	Spin
---------	------

Sym $\left\{ \begin{array}{l} |aa\rangle \\ |ab+ba\rangle + \text{Antisym} \frac{1}{\sqrt{2}} |N-J\rangle \\ |bb\rangle \end{array} \right\} \Rightarrow \text{antisym}$

Antisym $\frac{1}{\sqrt{2}} |ab-ba\rangle + \text{Sym} \left\{ \begin{array}{l} |N\rangle \\ \frac{1}{\sqrt{2}} |NJ+JN\rangle \\ |NN\rangle \end{array} \right\}$

- d) How many bound states does the hydrogen atom have? The binding energy $|E|$ of the ground state is 13.6 eV. In terms of this binding energy, what are the energies of the other bound states? Which transitions among these states generate the Lyman and Balmer series? What part of the electromagnetic spectrum – ultraviolet, visible, infrared – dominates each series?

$$\text{Infinite } |E_n| = \frac{|E_1|}{n^2} \xrightarrow{\text{ultraviolet}} \text{Lyman } \frac{1}{\lambda} = \frac{1}{911.5 \text{ Å}} (1 - \frac{1}{n})$$

$$h\nu = E(\frac{1}{n_f^2} - \frac{1}{n_i^2})$$

$$\Rightarrow \frac{h\nu}{\lambda} = 13.6 \text{ eV} (\frac{1}{n_f^2} - \frac{1}{n_i^2}) \Rightarrow \frac{1}{\lambda} = \frac{13.6 \text{ eV}}{2\pi hc} (\frac{1}{n_f^2} - \frac{1}{n_i^2}) = \frac{1}{911.5 \text{ Å}} (\frac{1}{n_f^2} - \frac{1}{n_i^2})$$

$$\xrightarrow{\text{visible}} \text{Balmer } \frac{1}{\lambda} = \frac{1}{911.5 \text{ Å}} (\frac{1}{4} - \frac{1}{n})$$

$$\xrightarrow{\text{infrared}} \text{Pachsen } \frac{1}{\lambda} = \frac{1}{911.5 \text{ Å}} (\frac{1}{9} - \frac{1}{n})$$

- e) When we solve problems with time-independent potentials, we use stationary states. What is a stationary state? In what sense is a stationary state's wavefunction stationary? A normalized wave packet has been expanded in stationary states, each having a probability $|c_i|^2$. If we now evolve the state in time, how will the probabilities $|c_i|^2$ change? How does the energy of the wave packet evolve with time and how is the energy connected to the $|c_i|^2$?

$\langle E \rangle = \sum |c_i|^2 E_i$ It is a state will not evolve with time.

It ~~means~~ each of them have an exact energy, for example: $\frac{E}{n}$

Connected $|c_i|^2$ don't change, energy don't evolve with time
energy is a constan $|\psi(t)\rangle = \sum c_i \phi_i e^{-iEt/\hbar}$

- f) One is given a 3D Schrödinger equation single-particle problem for which the potential is central, that is, $V(r)$ depends only on the radial coordinate. Name two additional operators that commute with the Hamiltonian \hat{H} . Do state energies depend on the quantum numbers of these two operators? E.g., note any issues like degeneracies.

$$\begin{array}{l} \downarrow \\ L^2 |\psi_{\ell m}\rangle = \ell(\ell+1) |\psi_{\ell m}\rangle \\ L_z |\psi_{\ell m}\rangle = m\hbar |\psi_{\ell m}\rangle \end{array}$$

L^2 and L_z Commute with \hat{H}

Yes.

g) Use Clebsch-Gordan coefficients to express an uncoupled state $|\ell_1 m_1 \ell_2 m_2\rangle$ in terms of a sum over coupled states $|(l_1 l_2) LM\rangle$.

$$|\ell_1 m_1 \ell_2 m_2\rangle = \sum_{L=l_1+l_2}^{\ell_1+\ell_2} \sum_{M=-L}^L |LM\rangle \langle \ell_1 m_1 \ell_2 m_2|$$

h) An electron state is prepared so that its spin points along the positive \hat{z} axis. A measurement is then done to determine whether the spin is pointing along the positive \hat{x} axis. What is the probability of this result being yes? Please provide a qualitative reason for your answer.

$$\begin{aligned} \chi_+ &\rightarrow |\chi_+\rangle = \left| \begin{array}{c} 1 \\ 0 \end{array} \right\rangle \quad \chi_- \rightarrow |\chi_-\rangle = \left| \begin{array}{c} 0 \\ 1 \end{array} \right\rangle \quad P = \langle \chi_+ | \chi_+ \rangle^2 = \left(\frac{1}{\sqrt{2}} \langle \chi_+ | \chi_+ \rangle \right)^2 \\ \chi_+ &\rightarrow |\chi_+\rangle = \left| \begin{array}{c} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{array} \right\rangle = \frac{1}{\sqrt{2}} |\chi_+\rangle + \frac{1}{\sqrt{2}} |\chi_-\rangle \end{aligned}$$

$$\boxed{\frac{1}{2}}$$

i) The Aharonov-Bohm effect arises in the electron diffraction two-slit experiment illustrated in the figure. The magnetic field is confined to the interior of an infinite solenoid perpendicular to the plane (the circle in the figure), but the vector potential is nonzero throughout the plane. i) An electron can travel from source \rightarrow slit S1 \rightarrow screen via path C1, and another from source \rightarrow slit S1 \rightarrow screen via path C1'. Will these two paths generate an Aharonov-Bohm interference? ii) Alternatively an electron can travel from source \rightarrow slit S2 \rightarrow screen via path C2. Will these two paths generate an Aharonov-Bohm interference? iii) Explain your answers, given that both cases involve electrons moving through regions of nonzero vector potential, but zero magnetic field. Yes

(i) No

(ii) Yes

(iii)

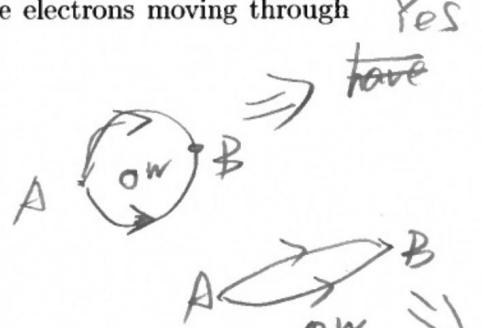
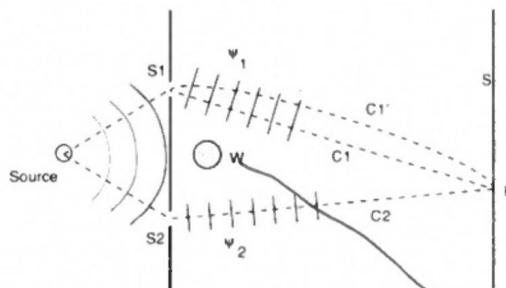
$$\iint \vec{B} \cdot d\vec{s}$$

$$= \iint \nabla \times \vec{A} \cdot d\vec{s}$$

$$= \oint \vec{A} \cdot d\vec{l}$$

Σ_l is topological

properties for A-B effect there will have a phase difference



Stokes formula

only the loop contains W inside

2. He atom spectroscopy (20 pts):

As you did in problem set 12, form all possible two-electron antisymmetric He basis states of the form $|\alpha(LS)JM_J\rangle$ where 1) only the $n = 1$ and $n = 2$ levels are occupied and 2) the $1s$ level contains at least one electron. Here α will be the occupation of the $n = 1$ level, and thus by assumption must be 1 or 2. List your results in the table below, the first line of which has been completed. Please write out the radial wave function in the manner of $(1s2p - 2p1s)/\sqrt{2}$, etc., to make things clear. "Symmetry" below refers to exchange symmetry.

α	radial wave function	L	S	radial symmetry	spin symmetry	J	M_J values
1S_0	2 $1s1s$	0	0	even	odd	0	0
1S_0	1 $\frac{1}{\sqrt{2}}(1s^2 + 2s^2)$	0	0	even	odd	0	0
3S_1	1 $\frac{1}{\sqrt{2}}(1s^2 - 2s^2)$	0	1	odd	even	1	$\begin{cases} 1 \\ 0 \\ -1 \end{cases}$
1P_1	1 $\frac{1}{\sqrt{2}}(1s^2 p + 2p^2)$	1	0	even	odd	1	$\begin{cases} 1 \\ 0 \\ -1 \end{cases}$
$^3P_{0,1,2}$	1 $\frac{1}{\sqrt{2}}(1s^2 p - 2p^2)$	1	1	odd	even	$\begin{cases} 0 \\ 1 \\ 2 \end{cases}$	$\begin{cases} 0 \\ 1 \\ 2 \end{cases}$

$$9 + 6 + 2 = 17 \text{ states}$$

$$\begin{cases} 0 & \text{odd} \\ 1 & \text{even} \end{cases}$$



$$\begin{cases} 0 & 0 \\ 1 & \rightarrow \begin{cases} 1 \\ 0 \\ -1 \end{cases} \\ 2 & \rightarrow \begin{cases} 2 \\ 1 \\ 0 \\ -1 \\ -2 \end{cases} \end{cases}$$

$$\left(\frac{A}{4}\right) \left(-4|e^{i\frac{\pi x}{a}} + 3|e^{i\frac{3\pi x}{a}} \right) |4\rangle \langle 4| \left(4|e^{i\frac{2\pi x}{a}} + 3|e^{i\frac{6\pi x}{a}} \right)$$

3. 1D review problem (25 pts)

In class we derived the stationary states $\phi_n(x)$ and their energies E_n for a particle of mass m confined to an infinite well of width a centered on the origin, $-\frac{a}{2} < x < \frac{a}{2}$. This yielded plane-wave even- and odd-parity states that vanish at the boundaries of the well

$$= \frac{6a}{\pi} A^2 \cdot \frac{5}{6} \frac{3}{4} \frac{1}{2} = 1$$

$$\Rightarrow \frac{15a}{48} A^2 = 1 \Rightarrow A = \sqrt{\frac{48}{15a}}$$

a) You are given an initial wave packet $\psi(x, t=0) = A \cos^3 \frac{\pi x}{a}$, $-\frac{a}{2} < x < \frac{a}{2}$. By using

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} A^2 \cos^6 \frac{\pi x}{a} dx = 1$$

$$= \frac{a}{\pi} A^2 \int_{-\frac{a}{2}}^{\frac{a}{2}} \cos^6 \theta d\theta$$

find an expression for $\cos^3 \theta$ in terms of $\cos \theta$ and $\cos 3\theta$, and use this to expand the wave packet

in terms of square-well stationary states, including determining A .

$$\cos^3 \theta = \operatorname{Re}[e^{3i\theta}] = \operatorname{Re}[(\cos \theta + i \sin \theta)^3]$$

$$= \cos^3 \theta - 3\sin^2 \theta \cos \theta = \cos^3 \theta - 3(1 - \cos^2 \theta) \cos \theta$$

$$\Rightarrow \cos^3 \theta = 4\cos^3 \theta - 3\cos \theta \Rightarrow \cos^3 \theta = \frac{1}{4} \cos 3\theta + \frac{3}{4} \cos \theta$$

$$\Rightarrow \psi(x, t=0) = A \left(\frac{1}{4} \cos \frac{3\pi x}{a} + \frac{3}{4} \cos \frac{\pi x}{a} \right) = \frac{4}{\sqrt{15a}} \left(\frac{1}{4} \cos \frac{3\pi x}{a} + \frac{3}{4} \cos \frac{\pi x}{a} \right)$$

$$\Rightarrow A = \sqrt{\frac{48}{15a}} = \frac{4}{\sqrt{15a}}$$

b) Following the prime directive, write down $\psi(x, t)$.

$$\psi(x, t) = \sum c_i \phi_i e^{-i\frac{E_i t}{\hbar}}$$

$$E_3 = \frac{9\hbar^2 \pi^2}{2ma^2}$$

$$= \frac{4}{\sqrt{15a}} \left(\frac{1}{4} \cos \frac{3\pi x}{a} e^{-i\frac{E_3 t}{\hbar}} + \frac{3}{4} \cos \frac{\pi x}{a} e^{-i\frac{E_1 t}{\hbar}} \right) \quad E_1 = \frac{\hbar^2 \pi^2}{2ma^2}$$

c) Calculate the probability that a measurement at time t will find the particle in state n .

$$\text{for } n \neq 1, 3 \Rightarrow P=0$$

$$\text{for } n=1 \quad P = \langle \psi(x, t) | \phi_1(x) \rangle \langle \phi_1(x) | \psi(x, t) \rangle = 0.9$$

$$\text{for } n=3 \quad P = \langle \psi(x, t) | \phi_3(x) \rangle \langle \phi_3(x) | \psi(x, t) \rangle = 0.1$$

d) Find the expectation $\langle H \rangle$ for the wave packet $\psi(x, t)$.

$$\langle H \rangle = \int \psi(x, t) \hat{H} \psi(x, t) dx$$

$$\text{or } \langle H \rangle = \sum_{i=1,3} p_i E_i = 0.9 \times \frac{\hbar^2 \pi^2}{2ma^2} + 0.1 \times \frac{9\hbar^2 \pi^2}{2ma^2}$$

$$= \frac{0.9\hbar^2 \pi^2}{ma^2}$$

4. Hermitian operators, Dirac notation (25 pts)

- a) What is meant by a Hilbert space in quantum mechanics. Give an example of a useful basis in quantum mechanics where the basis states are not part of the Hilbert space.

$$| = \sum |\phi_n\rangle \langle \phi_n| \quad \text{H0 solution}$$

- b) \hat{A} , \hat{B} , and \hat{C} are Hermitian operators. Determine which of the following operator combinations are Hermitian. (Please show a proof in each case.)

$$\begin{aligned} \hat{A}\hat{B}\hat{C} + \hat{C}\hat{B}\hat{A} & (\hat{A}\hat{B}\hat{C} + \hat{C}\hat{B}\hat{A})^+ = \hat{C}^\dagger \hat{B}^\dagger \hat{A}^\dagger + \hat{A}^\dagger \hat{B}^\dagger \hat{C}^\dagger = \hat{C}\hat{B}\hat{A} - \hat{A}\hat{B}\hat{C} \quad \text{Yes} \\ \hat{A}\hat{B}\hat{C} - \hat{C}\hat{B}\hat{A} & (\hat{A}\hat{B}\hat{C} - \hat{C}\hat{B}\hat{A})^+ = \hat{C}^\dagger \hat{B}^\dagger \hat{A}^\dagger - \hat{A}^\dagger \hat{B}^\dagger \hat{C}^\dagger = \hat{C}\hat{B}\hat{A} - \hat{A}\hat{B}\hat{C} \quad \text{No} \\ i(\hat{A}\hat{B}\hat{C} - \hat{C}\hat{B}\hat{A}) & [i(\hat{A}\hat{B}\hat{C} - \hat{C}\hat{B}\hat{A})]^+ = -i(\hat{C}^\dagger \hat{B}^\dagger \hat{A}^\dagger - \hat{A}^\dagger \hat{B}^\dagger \hat{C}^\dagger) = i(\hat{A}\hat{B}\hat{C} - \hat{C}\hat{B}\hat{A}) \\ \hat{A}\hat{B}\hat{C} + \hat{B}\hat{C}\hat{A} & (\hat{A}\hat{B}\hat{C} + \hat{B}\hat{C}\hat{A})^+ = \hat{C}^\dagger \hat{B}^\dagger \hat{A}^\dagger + \hat{A}^\dagger \hat{B}^\dagger \hat{C}^\dagger = \hat{C}\hat{B}\hat{A} + \hat{A}\hat{C}\hat{B} \quad \text{No} \end{aligned}$$

- c) Show that if \hat{P} and \hat{Q} have a common, complete set of normalized eigenfunctions $|p_i q_i\rangle$, so that $\hat{P}|p_i q_i\rangle = p_i |p_i q_i\rangle$ and $\hat{Q}|p_i q_i\rangle = q_i |p_i q_i\rangle$ for all $|p_i q_i\rangle$ in the Hilbert space, then $[\hat{P}, \hat{Q}] = 0$.

$$[\hat{P}, \hat{Q}] |p_i q_i\rangle = (\hat{P}\hat{Q} - \hat{Q}\hat{P}) |p_i q_i\rangle$$

$$= p_i q_i |p_i q_i\rangle - q_i p_i |p_i q_i\rangle = 0$$

$$\Rightarrow [\hat{P}, \hat{Q}] = 0 \quad \text{Commute}$$

$$\frac{4}{3}|1A\rangle = \frac{4}{5}|1B\rangle + \frac{1}{5}|2B\rangle$$

$$|1A\rangle - |1B\rangle = \frac{1}{3}|2B\rangle$$

$$|1A\rangle + \frac{4}{3}|2B\rangle = \frac{5}{3}|1B\rangle$$

$$\frac{4}{3}|1B\rangle = \frac{16}{15}|1B\rangle - \frac{4}{5}|2B\rangle$$

d) Sequential measurements: The Hermitian operator \hat{A} has normalized eigenstates $|1_A\rangle$ and $|2_A\rangle$ with eigenvalues a_1 and a_2 , respectively. The Hermitian operator \hat{B} has normalized eigenstates $|1_B\rangle$ and $|2_B\rangle$ with eigenvalues b_1 and b_2 , respectively. The eigenstates are related by $|1_A\rangle = \frac{3}{5}|1_B\rangle + \frac{4}{5}|2_B\rangle$ and $|2_A\rangle = \frac{4}{5}|1_B\rangle - \frac{3}{5}|2_B\rangle$. You measure observable \hat{A} , finding outcome a_1 ; then immediately after, your lab partner measures observable \hat{B} , but fails to communicate the outcome to you; then promptly after this, you measure observable \hat{A} again. What is the probability that you find outcome a_1 in the second measurement?

$$\hat{A}|1A\rangle = a_1|1A\rangle$$

$$|1A\rangle = \frac{3}{5}|1B\rangle + \frac{4}{5}|2B\rangle$$

$$\hat{A}|2A\rangle = a_2|2A\rangle$$

$$|2A\rangle = \frac{4}{5}|1B\rangle - \frac{3}{5}|2B\rangle$$

$$\hat{B}|1B\rangle = b_1|1B\rangle$$

$$\Rightarrow \left\{ \begin{array}{l} \\ \end{array} \right.$$

$$\hat{B}|2B\rangle = b_2|2B\rangle$$

① $\hat{A} \Rightarrow$ you got $a_1 \Rightarrow$ the initial state is $|1A\rangle = \frac{3}{5}|1B\rangle + \frac{4}{5}|2B\rangle$

② then measure $\hat{B} \Rightarrow$ you have $\begin{cases} P = \frac{9}{25} \text{ to get } |1B\rangle \\ P = \frac{16}{25} \text{ to get } |2B\rangle \end{cases}$

$$|\langle 1B|1A\rangle|^2 = \frac{9}{25}$$

$$|\langle 2B|1A\rangle|^2 = \frac{16}{25}$$

case 1 if you got $|1B\rangle$

$$|1B\rangle = \frac{3}{5}|1A\rangle + \frac{4}{5}|2A\rangle$$

$$\begin{cases} P = \frac{9}{25} \text{ get } |1A\rangle \\ P = \frac{16}{25} \text{ get } |2A\rangle \end{cases}$$

$$\frac{9}{25} \times \frac{1}{2}$$

$\frac{81+256}{625}$

case 2 if you got $|2B\rangle$

add figs.

find out again

$$P = \frac{337}{625}$$

$$\begin{aligned} a. \quad \langle PB \rangle &= \frac{4}{5}|1A\rangle - \frac{3}{5}|2A\rangle & \begin{cases} P = \frac{16}{25} \text{ get } |1A\rangle \Rightarrow \frac{16}{25} \times \frac{16}{25} \\ P = \frac{9}{25} \text{ get } |2A\rangle \end{cases} \\ & \end{aligned}$$

$$\text{And } R_1(a) = R_2(a) \Rightarrow A \frac{\sin ka}{a} = D \frac{e^{-ka}}{a}$$

$$R_1'(a) = R_2'(a) \Rightarrow A \frac{ka \cos ka - \sin ka}{a^2} = D \frac{-ka e^{-ka} - e^{-ka}}{a^2}$$

5. 3D boundary value problems: (25 pts)

The Schrödinger equation in 3D for a central potential can be written

$$R(r) = A \frac{k r \cos kr - \sin kr}{r^2}$$

$$\left[-\frac{\hbar^2}{2m} \left[\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} \right] + V(r) + \frac{1}{2mr^2} L^2 \right] \psi(\mathbf{r}) = E \psi(\mathbf{r})$$

On making the substitution $\psi(\mathbf{r}) = R_\ell(r) Y_{\ell m}(\Omega) = \frac{u_\ell(r)}{r} Y_{\ell m}(\Omega)$ one finds the radial equation

$$R_\ell'(r) = D \frac{-k r e^{-kr} - e^{-kr}}{r^2} - \frac{\hbar^2}{2m} \frac{d^2 u_\ell(r)}{dr^2} + \left[V(r) + \frac{\hbar^2 \ell(\ell+1)}{2m r^2} \right] u_\ell(r) = E u_\ell(r)$$

normalization $\oint_a^\infty \frac{De^{-kr} \sin kr}{r} \frac{1}{r^2} dr = 1$

$$+ \int_a^\infty \frac{De^{-kr}}{r} r^2 dr = 1$$

a) Find the bound state $\ell = 0$ interior and exterior radial solutions of the finite spherical

$$V(r) = \begin{cases} -V_0 & r < a \\ 0 & r > a \end{cases}$$

use this - you can solve \boxed{D}

You should require $R(0)$ to be finite. By matching the solutions at $r = a$ determine the radial wave function up to an overall normalization.

$$\ell=0 \Rightarrow \frac{-\hbar^2}{2m} \frac{d^2 u(r)}{dr^2} + V(r) u(r) = E u(r) \quad \frac{-\hbar^2}{2m} u''(r) = -E u(r)$$

$$r < a \quad V = -V_0$$

$$\frac{-\hbar^2}{2m} \frac{d^2 u(r)}{dr^2} = (V_0 - E) u(r)$$

$$u''(r) = -\frac{2m(V_0 - E)}{\hbar^2} u(r)$$

$$u''(r) = -k^2 u(r)$$

$$\Rightarrow u_1(r) = A \sin kr + B \cos kr$$

$$k = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

$$r > a \quad V = 0$$

$$\Rightarrow u''(r) = \frac{2m(E)}{\hbar^2} u(r) = k^2 u(r)$$

$$\Rightarrow u_2(r) = C e^{kr} + D e^{-kr}$$

boundary condition

$$R_1(r) = \frac{u_1(r)}{r} \quad R_2(r) = \frac{u_2(r)}{r}$$

$$R_1(0) = \text{finite} \Rightarrow B = 0$$

$$R_2(0) = \text{finite} \Rightarrow C = 0$$

$$\Rightarrow R_1(r) = A \frac{\sin kr}{r} \quad R_2(r) = D \frac{e^{-kr}}{r}$$

b) By requiring the derivative to be continuous, determine the eigenvalue equation.

$$R_1(a) = R_2(a) \Rightarrow A \sin ka = D e^{-ka} \quad \textcircled{1}$$

$$R_1'(a) = R_2'(a) \Rightarrow A(k \cos ka - \sin ka) = -D(k a + 1) e^{-ka} \quad \textcircled{2}$$

$$\textcircled{1}/\textcircled{2} \Rightarrow \frac{\sin ka}{k \cos ka - \sin ka} = -\frac{1}{ka + 1} \Rightarrow \frac{\tan ka}{ka - \tan ka} = -\frac{1}{ka + 1}$$

$$\text{Set } ka = z$$

$$ka = \sqrt{2m(E)/\hbar^2} \Rightarrow \frac{\tan z}{z - \tan z} = -\frac{1}{z + 1}$$

c) What is the minimum depth V_0 required to support a bound state?

$$z = \sqrt{\frac{2mV_0}{\hbar^2}}$$

$$E \rightarrow 0 \Rightarrow k = \frac{\sqrt{2m|E|}}{\hbar} \rightarrow 0 \Rightarrow \sqrt{z^2 - z^2} \rightarrow 0$$

$$\text{plug into } \textcircled{2} \Rightarrow \frac{\tan z}{z - \tan z} = -1 \Rightarrow z = \frac{\pi}{2} \stackrel{z \text{ small}}{=} \frac{\pi}{2}$$

$$\Rightarrow \sqrt{\frac{2m(V_0 + E)}{\hbar^2}} = \frac{\pi}{2} \Rightarrow \sqrt{\frac{2mV_0}{\hbar^2}} \approx \frac{\pi}{2} \Rightarrow \frac{2mV_0}{\hbar^2} = \frac{\pi^2}{4}$$

$$V_0 = \frac{\pi^2}{8}$$

d) The interior solution $u(r)$ found above corresponds to $R_0(r) \sim j_0(kr)$, the $\ell = 0$ spherical Bessel function. If we had solved the **infinite** 3D spherical well for arbitrary ℓ , what would then be the requirement for an eigenvalue?

$$R_{nl}(r) = A_n r j_\ell(kr) + B_n r n_\ell(kr)$$

↓ Neumann is diverge in ∞

finite $R(\infty) \Rightarrow B_n = 0$

and we find $Y_{lm}(\theta, \phi) = \Theta(\theta) \cdot \Phi(\phi)$

$$\Theta(\theta) = C e^{im\theta}$$

$$\Phi(\phi) = A_n r P_\ell^m(\cos\phi)$$

$$\Psi = R_{nl}(r) \cdot Y_{lm}(\theta, \phi)$$

then $\hat{H}\Psi \Rightarrow E_{nlm}$, we get eigenvalue.

6. Larmor precession, prime directive (25 pts): Consider a spin-1/2 system quantized as usual using $\hat{\mathbf{S}}^2$ and \hat{S}_z . The initial wave function at $t = 0$ is

$$|\psi(0)\rangle = |\chi_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so pointed along the $+\hat{z}$ axis.

- a) The system evolves for a time t under influence of a magnetic field of strength B pointed along the positive x axis, so

$$\hat{H} = -\gamma \mathbf{B} \cdot \hat{\mathbf{S}} = -\gamma B \hat{S}_x = -\frac{\gamma B \hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Find the stationary states and eigenvalues for this Hamiltonian, and express $|\psi(0)\rangle$ in terms of the stationary states.

$$\hat{H} |\chi_+\rangle = -\frac{\gamma B \hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{\gamma B \hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} \text{stationary state} \\ |\chi_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{matrix}$$

$$\hat{H} |\chi_-\rangle = -\frac{\gamma B \hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\gamma B \hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} \text{eigenvalue: } -\frac{\gamma B \hbar}{2} \\ |\chi_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{matrix}$$

$$|\psi(0)\rangle = |\chi_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} |\chi_+\rangle + \frac{1}{\sqrt{2}} |\chi_-\rangle \quad \begin{matrix} \text{eigenvalue } \frac{\gamma B \hbar}{2} \end{matrix}$$

- b) Using the prime directive, find an expression for $|\psi(t)\rangle$.

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} |\chi_+\rangle e^{i \frac{\gamma B t}{2}} + \frac{1}{\sqrt{2}} |\chi_-\rangle e^{-i \frac{\gamma B t}{2}} = \frac{1}{\sqrt{2}} \left[\frac{1}{\sqrt{2}} (|\chi_+ + \chi_-\rangle) e^{i \frac{\gamma B t}{2}} + \frac{1}{\sqrt{2}} (|\chi_+ - \chi_-\rangle) e^{-i \frac{\gamma B t}{2}} \right]$$

- c) Calculate the probability of finding the system in the states $|\chi_+\rangle$ and $|\chi_-\rangle$ as a function of time.

$$P_+ = \langle \psi(t) | \chi_+ \rangle \langle \chi_+ | \psi(t) \rangle = \frac{1}{2} [X_+ \cos \frac{\gamma B t}{2} + X_- \sin \frac{\gamma B t}{2}]$$

$$P_+ = \langle \chi_+ | \cos \frac{\gamma B t}{2} - i \chi_- \sin \frac{\gamma B t}{2} | \chi_+ \rangle \langle \chi_+ | \chi_+ | \chi_+ | \cos \frac{\gamma B t}{2} + i \chi_- \sin \frac{\gamma B t}{2} \rangle = X_+ \cos \frac{\gamma B t}{2} + i X_- \sin \frac{\gamma B t}{2}$$

$$= \cos \frac{\gamma B t}{2}$$

$$\boxed{P_+ + P_- = 1}$$

$$P_- = \langle \psi(t) | \chi_- \rangle \langle \chi_- | \psi(t) \rangle$$

$$= \langle \chi_+ | \cos \frac{\gamma B t}{2} - i \chi_- \sin \frac{\gamma B t}{2} | \chi_- \rangle \langle \chi_- | \chi_- | \chi_+ | \cos \frac{\gamma B t}{2} + i \chi_- \sin \frac{\gamma B t}{2} \rangle$$

$$= \sin \frac{\gamma B t}{2}$$

d) Calculate the probabilities of finding the system in the states $|x_+\rangle$, and $|x_-\rangle$ as a function of time.

$$|x_+\rangle^y = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}|x_+\rangle + \frac{i}{\sqrt{2}}|x_-\rangle \quad |x_-\rangle^y = \begin{pmatrix} \frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{i}{\sqrt{2}}|x_+\rangle - \frac{i}{\sqrt{2}}|x_-\rangle$$

$$P_{x+} = \langle 4|t\rangle |x_+\rangle^y \langle x_+^y | 4|t\rangle = \langle x_+ |\cos \frac{\partial B t}{2} - i x_- \sin \frac{\partial B t}{2} | \frac{1}{\sqrt{2}}|x_+\rangle + \frac{i}{\sqrt{2}}|x_-\rangle \rangle \langle \frac{1}{\sqrt{2}}|x_+\rangle - \frac{i}{\sqrt{2}}|x_-\rangle | x_+ |\cos \frac{\partial B t}{2} + i x_- \sin \frac{\partial B t}{2} |$$

$$= \left(\frac{1}{\sqrt{2}} \cos \frac{\partial B t}{2} + \frac{1}{\sqrt{2}} \sin \frac{\partial B t}{2} \right)^2 = \frac{1}{2} + \cos \frac{\partial B t}{2} \sin \frac{\partial B t}{2}$$

$$P_{x-} = \langle 4|t\rangle |x_-\rangle^y \langle x_-^y | 4|t\rangle =$$

$$= \langle x_+ |\cos \frac{\partial B t}{2} - i x_- \sin \frac{\partial B t}{2} | \frac{1}{\sqrt{2}}|x_+\rangle - \frac{i}{\sqrt{2}}|x_-\rangle \rangle \langle \frac{1}{\sqrt{2}}|x_+\rangle + \frac{i}{\sqrt{2}}|x_-\rangle | x_+ |\cos \frac{\partial B t}{2} - i x_- \sin \frac{\partial B t}{2} |$$

$$= \left(\frac{1}{\sqrt{2}} \cos \frac{\partial B t}{2} - \frac{1}{\sqrt{2}} \sin \frac{\partial B t}{2} \right)^2 = \frac{1}{2} - \cos \frac{\partial B t}{2} \sin \frac{\partial B t}{2}$$

e) Describe in words how the spin precesses in time. Around what does it precess, in what plane does it reside, and how long does it take the spin to return to its initial position? At what times are you guaranteed that if you make a measurement of \hat{S}_z or \hat{S}_y , you will get only one possible answer?



at first,

the spin ~~at~~ point Z_t

and rotate in $Y-Z$ ~~plane~~ plane

$$\frac{\partial B t}{2} = 2\pi \Rightarrow t = \frac{4\pi}{\partial B} \quad \text{return to its initial position}$$

$$\underline{\frac{\partial B t}{2} = n\pi} \Rightarrow t =$$

$$\left. \begin{array}{l} \sin \frac{\partial B t}{2} = 0 \\ \text{or } \cos \frac{\partial B t}{2} = 0 \end{array} \right\} \Rightarrow \text{one possible answer}$$

$$\Rightarrow \frac{\partial B t}{2} = \frac{n\pi}{2}$$

$$\Rightarrow t = \boxed{\frac{n\pi}{\partial B}}$$

one possible answer

Useful items:

The problems are constructed so you should have everything you need. But just in case:

1. Useful Clebsch Gordan coefficients $\langle \ell_1 m_1 \ell_2 m_2 | (\ell_1 \ell_2) LM \rangle$:

$$\begin{aligned} \langle \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} | (\frac{1}{2} \frac{1}{2}) 11 \rangle &= 1 & \langle \frac{1}{2} \frac{1}{2} \frac{1}{2} - \frac{1}{2} | (\frac{1}{2} \frac{1}{2}) 10 \rangle &= \frac{1}{\sqrt{2}} & \langle \frac{1}{2} - \frac{1}{2} \frac{1}{2} \frac{1}{2} | (\frac{1}{2} \frac{1}{2}) 10 \rangle &= \frac{1}{\sqrt{2}} & \langle \frac{1}{2} - \frac{1}{2} \frac{1}{2} - \frac{1}{2} | (\frac{1}{2} \frac{1}{2}) 1 - 1 \rangle &= \\ \langle \frac{1}{2} \frac{1}{2} \frac{1}{2} - \frac{1}{2} | (\frac{1}{2} \frac{1}{2}) 00 \rangle &= \frac{1}{\sqrt{2}} & \langle \frac{1}{2} - \frac{1}{2} \frac{1}{2} \frac{1}{2} | (\frac{1}{2} \frac{1}{2}) 00 \rangle &= -\frac{1}{\sqrt{2}} \end{aligned}$$

2. Triangle condition for coupling $\vec{\ell}_1$ and $\vec{\ell}_2$ to \vec{L} :

$$|\ell_1 - \ell_2| \leq L \leq \ell_1 + \ell_2$$

3. Spin vectors

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \chi_+^x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \chi_-^x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \quad \chi_+^y = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \quad \chi_-^y = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$$