$$\vec{a}_+ = \frac{1}{\sqrt{2}}(-i\vec{p}_\xi + \vec{\xi}) \qquad [\hat{x}, \vec{p}] = i\hbar$$

$$\vec{a}_- = \frac{1}{\sqrt{2}}(+i\vec{p}_\xi + \vec{\xi})$$

or

$$\hat{a}_+ = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}\vec{p}\right) \qquad \Rightarrow \begin{cases} \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_- + \hat{a}_+) \\ \vec{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}_+ - \hat{a}_-) \end{cases}$$

$$\hat{a}_- = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}\vec{p}\right)$$

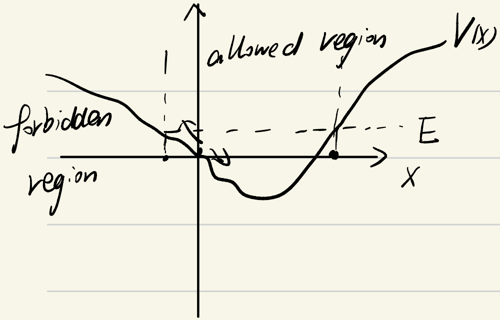
$$[\hat{a}_-, \hat{a}_+] = \frac{m\omega}{2\hbar}\left[\hat{x} - \frac{i}{m\omega}\vec{p}, \hat{x} - \frac{i}{m\omega}\vec{p}\right]$$

$$= \frac{m\omega}{2\hbar}\left[[\hat{x}, \hat{x}] - \frac{i}{m\omega}[x, p] - \cdots\right] = 1$$

bound system

characterized by allowed & forbidden system

$V(x)$, $F = -\nabla V$, solve $F = ma$



Quantum Bond System

V(x), solve TISE

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]\psi_E(x)=E\psi(x)$$

for $(\psi_E(x), E)$

$E(x,0) = c_0 \phi_0(x) + c_1 \phi_1(x)$

(a) $E(x,t) = e^{-i\omega_0 t} c_0 \phi_0(x) + e^{-i\omega_1 t} c_1 \phi_1(x)$

$\omega_n = \frac{E_n}{\hbar}$

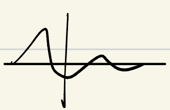
(b) $\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$

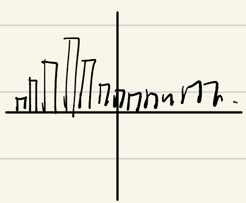
$\phi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \sqrt{\frac{2m\omega}{\hbar}}\, x\, e^{-\frac{m\omega x^2}{2\hbar}}$

$\langle x \rangle$

measure
x

$\rightarrow \boxed{} \rightarrow \psi(x)$





sai star times sai

$\frac{d\langle x \rangle}{dt} = 0$

$$|\Psi|^2 = (c_0^* \phi_0 e^{i\omega_0 t} + c_1^* \phi_1 e^{i\omega_1 t})(c_0 \phi_0 e^{-i\omega_0 t} + c_1 \phi_1 e^{-i\omega_1 t})$$

$$= |c_0|^2 |\phi_0|^2 + |c_1|^2 |\phi_1|^2 + \phi_1 \phi_0 \left(c_0^* c_1 e^{-i(\omega_1-\omega_0)t} + c_0 c_1^* e^{i(\omega_1-\omega_0)t} \right)$$

$$\int_{-\infty}^{+\infty} dx\, \phi_0(x)\, \phi_1(x)\, x = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \sqrt{\frac{2m\omega}{\hbar}} \int_{-\infty}^{+\infty} dx\, x^2 e^{-m\omega x^2/\hbar}$$

$$u = \sqrt{\frac{m\omega}{\hbar}}\, x \qquad du = \sqrt{\frac{m\omega}{\hbar}}\, dx$$

$$= \frac{m\omega}{\hbar} \frac{\sqrt{2}}{\sqrt{\pi}} \left(\frac{\hbar}{m\omega}\right)^{\frac{3}{2}} \int_{-\infty}^{+\infty} du\, u^2 e^{-u^2} \leftarrow \frac{\sqrt{\pi}}{2}$$

$$= \sqrt{\frac{\hbar}{m\omega}} \cdot \frac{1}{\sqrt{2}}$$

$$\langle x \rangle = \sqrt{\frac{2\hbar}{m\omega}} \cdot \mathrm{Re}\left[c_0^* c_1 e^{-i(\omega_1-\omega_0)t} \right]$$

$$\langle x \rangle = \langle \psi(t) | \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_+ + \hat{a}_-) | \psi(t) \rangle$$

$$= \langle \psi(t) | \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_+ + \hat{a}_-) (c_0 |\psi_0\rangle e^{-i\omega_0 t} + c_1 |\phi_1\rangle e^{-i\omega_1 t})$$

$$(c_0 |\psi_1\rangle e^{-i\omega_0 t} + 0 + c_1 |\phi_0\rangle e^{-i\omega_1 t} + c_1 |\phi_2\rangle e^{-i\omega_1 t})$$

$$\sqrt{\frac{\hbar}{2m\omega}} (c_0^* \langle \psi_0 | e^{i\omega_0 t} + c_1^* \langle \psi_1 | e^{i\omega_1 t}) (Ans)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} (c_0^* c_1 e^{-i(\omega_1 - \omega_0)t} + c_0 c_1^* e^{i(\omega_1 - \omega_0)t})$$

$$\langle \psi_i | \psi_j \rangle = \delta_{ij}$$

hw 4

9.20 1. Hint

$H_n(x) = (\quad) H_{n-1}(x)$

$H_0(x) = 1$

$H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$

$H_3(x) = 8x^3 - 12x$

$\phi_n(x) = \frac{1}{\sqrt{n}} \frac{1}{\sqrt{2}} \left[-\frac{d}{d\xi} + \xi \right] \phi_{n-1}(\xi)$

$\hat{a}_+ |\phi_n\rangle = \sqrt{n+1}\, |\phi_{n+1}\rangle \qquad \frac{1}{\sqrt{n}} \hat{a}_+ |\phi_{n-1}\rangle = |\phi_n\rangle$

$\hat{a}_+ \xrightarrow{x} \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega} \frac{d}{dx} \right)$

$\phi_n(x) = \frac{1}{\sqrt{n}} \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{\hbar}{m\omega} \frac{d}{dx} \right) \phi_{n-1}(x)$

$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$

$\uparrow$

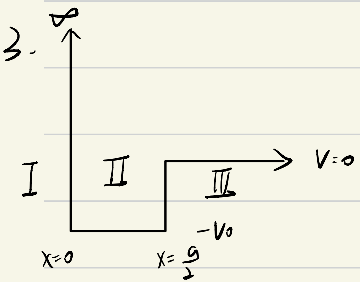
$\frac{1}{b}$

2. Hint

$$\int_{-\infty}^{+\infty} dx\, e^{-ax^2+bx+c} = \frac{\sqrt{\pi}}{\sqrt{a}}\, e^{\frac{b^2}{4a}+c}$$

$$\cos(kx) = \frac{e^{ikx}+e^{-ikx}}{2}$$

3.



$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]\psi(x) = E\psi(x)$$

$$\downarrow$$

$$V \to \begin{cases} e^{kx} \\ e^{ikx} \end{cases}$$

$$\psi'' = -\frac{2m(E-V)}{\hbar^2}\psi$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi \qquad k = \sqrt{\frac{2m(E-V)}{\hbar^2}} \qquad e^{\pm ikx} \text{ for } E>V$$

2 linear independent

$$\frac{d^2\psi}{dx^2} = +k\psi \qquad k = \sqrt{\frac{2m(V-E)}{\hbar^2}} \qquad e^{\pm kx} \text{ for } E<V$$

$$\psi(x) = \begin{cases} 0 & \text{in I} \quad E \ll V \\ A\sin(kx) + B\cos(kx) & E > V \quad \text{in II} \\ Ce^{-kx} & E < V \quad \text{in III} \end{cases}$$

$x \to \infty \quad \psi \to 0$

so e^{kx} vanish

(b) continous at $x=0 \Rightarrow B = 0$ ①

continous at $x=\frac{a}{2} \Rightarrow A \cdot \sin(\frac{ka}{2}) = Ce^{-k\frac{a}{2}}$ ②

ψ' conti at $x = \frac{a}{2} \Rightarrow Ak\cos(\frac{ka}{2}) = -kCe^{-\frac{ka}{2}}$ ③

momentum

$Z = \frac{ka}{2} \quad Z_0 = \frac{a}{2\hbar}\sqrt{2mV_0} \quad Z\cot Z = -\sqrt{Z_0^2 - Z^2}$

$A \cdot C E$

③/②

$$k\cot\left(\frac{ka}{2}\right) = -K = -\sqrt{\frac{2mV_0}{\hbar^2} - k^2} \Rightarrow z\cot z = -\sqrt{\frac{mV_0a^2}{2\hbar^2} - z^2} = -\sqrt{z_0^2 - z^2}$$

$$k(E) = \sqrt{\frac{2m(V_0 - |E|)}{\hbar^2}}$$

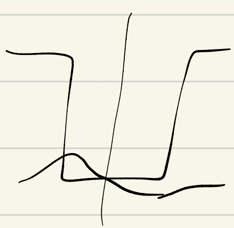
$$K(E) = \sqrt{\frac{2m|E|}{\hbar^2}}$$

$$k^2 + K^2 = \frac{2mV_0}{\hbar^2} - \frac{2m|E|}{\hbar} + \frac{2m|E|}{\hbar^2} = \frac{2mV_0}{\hbar^2}$$

$$K = \sqrt{\frac{2mV_0}{\hbar^2} - k^2}$$

$z_0 = \frac{\pi}{2}$ first solution

$$V_0 = \frac{4\pi^2\hbar^2}{2ma^2} \quad E_2$$



$$\frac{a}{2\hbar}\sqrt{2mV_0} = \frac{\pi}{2}$$

$$\frac{a^2}{4\hbar^2}2mV = \frac{\pi^2}{4}$$

HW 5

9.27

Today : Linear Algebra in Bra-ket notation (Part 1)

Quantum Mechanics		Linear Algebra
wave function	$\longleftrightarrow$	vector
linear operator	$\longleftrightarrow$	matrix
eigenstate	$\longleftrightarrow$	eigenvector
physical system	$\longleftrightarrow$	Hilbert space
physical observable	$\longleftrightarrow$	Hermitian matrix

Def A vector space is a set V of vectors together with rules for

(1) Adding two vectors $v + u$

(2) Multiplying by a scalar αv

Such that:

(i) Addition is associative and commutative

$$u + v = v + u \qquad (u + v) + w = u + (v + w)$$

(ii) The set V is closed under addition

$u + v$ is in V

(iii) Scalars distribute, and scalar multiplication is closed

$$\alpha(u + v) = \alpha u + \alpha v, \qquad \alpha u \text{ is in } V$$

(iv) There is a <u>zero vector</u> 0 such that $0+v = v+0 = v$ for all $v \in V$.

(v) Every vector v has an "inverse vector" $-v$ such that $v + (-v) = 0$

Warm-Up

Confirm / Disprove if the following are vector spaces:

① The set of all continuous functions $f\ \mathbb{R} \to \mathbb{R}$ with

$$(f+g)(x) = f(x) + g(x), \qquad (\alpha f)(x) = \alpha f(x)$$ ✓

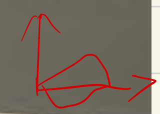
② The set of polynomials of degree 1 to 10, with the same rules as above ✓

③ The set of complex functions on the interval $[0,1]$, $f\ [0,1] \to \mathbb{C}$, which satisfy

complex

$$f(0) = 0, \quad f(1) = 0$$ ✓

(Addition + sclar mult. as before)



④ Same as ③, but now $f(0) = 0$, $f(1) = 1$. ✗

⑤ Same as ②, but only degree 3 to 10. ✗ zero polynomial 多项式的

scalar 标量

Nielsen & Chuang Quantum Computation (Chap 2)

$|\psi\rangle$ $|\phi\rangle$

sai fai

[1] $\mathbb{C}^n$ the set on n-tuples of complex numbers.

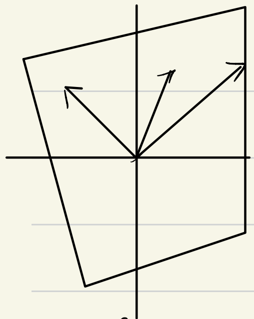
$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + \begin{pmatrix} c_1' \\ \vdots \\ c_n' \end{pmatrix} = \begin{pmatrix} c_1+c_1' \\ \vdots \\ c_n+c_n' \end{pmatrix} \qquad \alpha \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \alpha c_1 \\ \vdots \\ \alpha c_n \end{pmatrix}$$

[2] The set of continuous normalizable functions $f: \mathbb{R} \to \mathbb{C}$

$$\int_{-\infty}^{+\infty} dx\, |f(x)|^2 < \infty$$

$\boxed{\text{Basis}}$ $\vec{v} = v_x \hat{e}_x + v_y \hat{e}_y + v_z \hat{e}_z$

Def Linear Independance, $\{|v_1\rangle, |v_2\rangle, |v_3\rangle, \dots, |v_n\rangle\}$



$\sum_{j=1}^{n} c_i |v_i\rangle = 0 \not\Rightarrow c_i = 0$ for all i

dependent

Def Span

$\text{span}\{|v_1\rangle, \dots, |v_n\rangle\} = \{|w\rangle \mid |w\rangle = \sum_{i=1}^{n} c_i |v_i\rangle\}$

Def Dimension Maximum number of vectors which can be linearly independant.

Claim $\{0\}$ is finite dim w/ dim //

~~Assume~~ $\cancel{\sum_{i=1}^{n} c_i |v_i\rangle = 0}$ ~~for~~ $\cancel{c_i \neq 0}$

Basis Let $\text{Dim}(V) = n$. Then any set of n linearly independant vectors is a basis

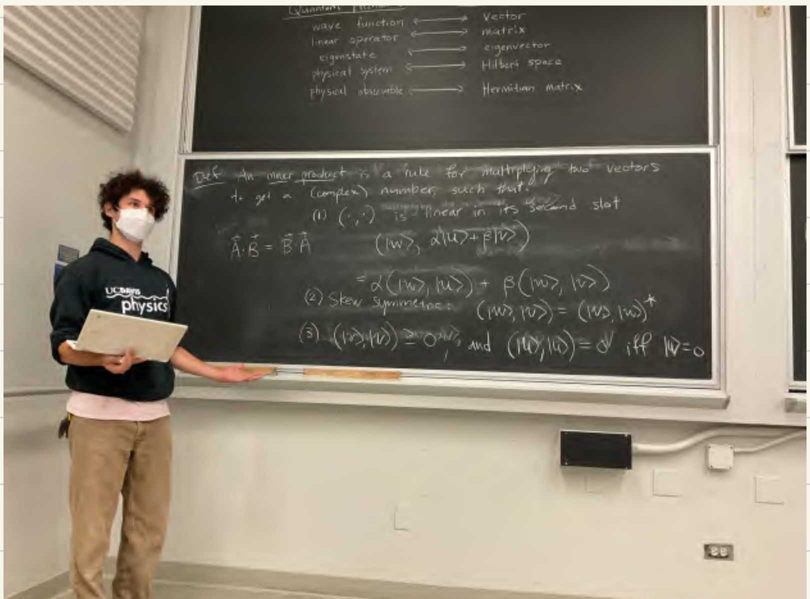
Let $\{|b_1\rangle, \dots, |b_n\rangle\}$, Then $|v\rangle = \sum_{i=1}^{n} v_i |b_i\rangle \rightarrow \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

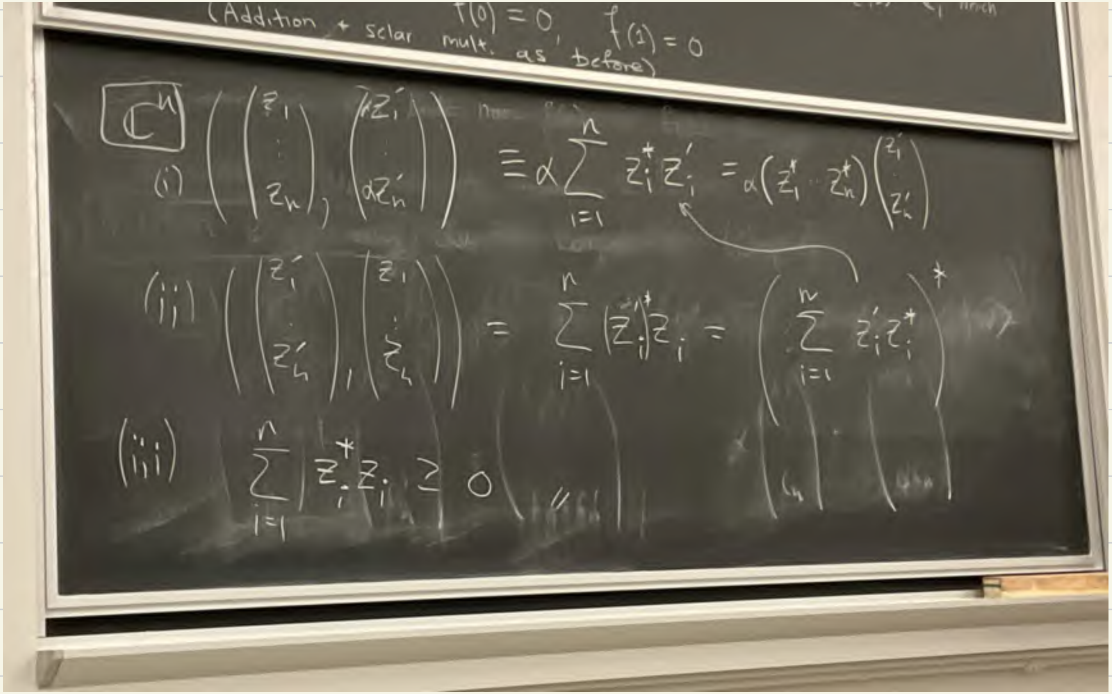
$(|v\rangle, |w\rangle) = \langle v|w\rangle$

$\langle v|(\hat{A}|w\rangle)$ $\quad \langle v|\hat{A}|w\rangle$

Def An inner product is a rule for multiplying two vectors to get a (complex) number, such that:

(1) $(\cdot,\cdot)$ is linear in its second slots




$f(0) = 0, \quad f(1) = 0$
(Addition + sclar mult. as before)
$\boxed{\mathbb{C}^n}$
(i) $\left(\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}, \begin{pmatrix} \alpha z_1' \\ \vdots \\ \alpha z_n' \end{pmatrix} \right) = \alpha \sum_{i=1}^{n} z_i^* z_i' = \alpha (z_1^* \cdots z_n^*) \begin{pmatrix} z_1' \\ \vdots \\ z_n' \end{pmatrix}$
(ii) $\left(\begin{pmatrix} z_1' \\ \vdots \\ z_n' \end{pmatrix}, \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \right) = \sum_{i=1}^{n} (z_i')^* z_i = \left(\sum_{i=1}^{n} z_i' z_i^* \right)^*$
(iii) $\sum_{i=1}^{n} z_i^* z_i \geq 0$

10.4 HW 6.

orthogonal: $\langle u|v\rangle = 0$

$(|u\rangle, |v\rangle) = 0$

模 norm: $\| |u\rangle \| = \sqrt{\langle u|u\rangle}$

$|u\rangle / \| |u\rangle \|$

orthornormal $\{|e_i\rangle\}$ $i = 1, 2, 3$ $\quad \langle e_i|e_j\rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

$|v\rangle = \sum_{j=1}^{n} v_i |b_i\rangle$

column

$\xrightarrow{b} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ $\quad (b_1 \ b_2 \ \cdots \ b_n) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

$|u\rangle = \sum_{i=1}^{n} u_i|e_i\rangle \qquad |v\rangle = \sum_{j=1}^{n} v_j|e_j\rangle$

$\langle u|v\rangle = (|u\rangle, |v\rangle) = \sum_{i=1}^{n}\sum_{j=1}^{n} u_i^* v_j \langle e_i|e_j\rangle = \sum_{i=1}^{n} u_i^* v_i$

$\langle u, v\rangle = \left(\sum_{i=1}^{n} u_i |e_i\rangle\right)^* \sum_{j=1}^{n} v_j |e_j\rangle$

$= \sum_{i=1}^{n}\sum_{j=1}^{n} u_i^* v_j \langle e_i|e_j\rangle = \sum_{i=1}^{n} u_i^* v_i$

$\xrightarrow{e} (u_1^* \ u_2^* \ \cdots \ u_n^*) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

Linear Operators

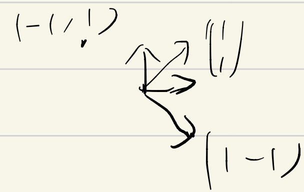
http: shad.io/MatVis

(trans format in)

A li op is a function from a vector space V to another vector space W such that

$$\hat{A}(\alpha|v_1\rangle+\beta|v_2\rangle)=\alpha\hat{A}(|v_1\rangle)+\beta\hat{A}(|v_2\rangle)$$

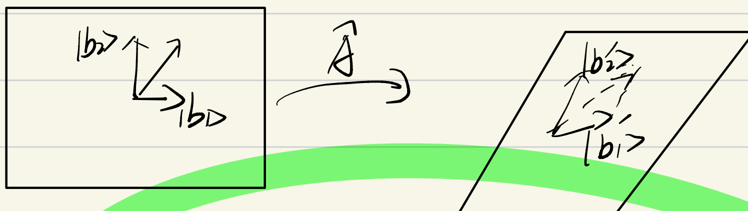
$$\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$



- $n\times n$ matrices
- $\frac{d}{dx}$ on functions
- $\hat{S}[f(x)]=f^2(x)$
- Dual vectors $\langle v|$

从一个视角到另一个视角.

$|v_0\rangle\in V$ $\quad |v_0\rangle: V\to\mathbb{C}$



$$\langle v_0|(|w\rangle)=(|v_0\rangle,|w\rangle).$$

$\hat{A}|b_i$,

$$\hat{A}|v\rangle=\hat{A}\Big(\sum_i v_i|b_i\rangle\Big)=\sum_i v_i|b_i'\rangle=|v'\rangle$$

v_i is the same

$\hat{A},\hat{B}: V\to V$

$(\hat{A}\hat{B})|v\rangle=\hat{A}(\hat{B}|v\rangle)$

$[\hat{A},\hat{B}]=\hat{A}\hat{B}-\hat{B}\hat{A}$

$\hat{A}|b_i\rangle=|b_i'\rangle$

$$\hat{A}|v\rangle=\hat{A}\sum_{i=1}^{n}v_i|b_i\rangle=\sum_{i=1}^{n}v_i|b_i'\rangle=|v'\rangle$$

$\hat{A}|b_i\rangle = |b_i'\rangle$

$$\begin{bmatrix} b_1' \\ \vdots \\ b_n' \end{bmatrix} = \begin{bmatrix} A_{ij} \\ A_{ij} \\ \vdots \\ A_{ij} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

b_j'

$|b_j'\rangle = \sum_i A_{ij} |b_i\rangle$

看以 $A|b\rangle$ 定则

$\vec{b}$ A j $\Rightarrow$ b'

$\hat{A}|v\rangle = \hat{A}\left(\sum_j v_j |b_j\rangle\right)$

$= \sum_j v_j \sum_i A_{ij} |b_i\rangle$

$= \sum_{ij} A_{ij} v_j |b_i\rangle \rightarrow [A_{ij}][v_j]$

$\hat{A}|v\rangle = \hat{A}\sum_j v_j |b_j\rangle$

$= \sum_j v_j |b_j'\rangle$

$= \sum_j v_j \sum_i A_{ij}|b_i\rangle$

$= \sum_{ij} A_{ij} v_j |b_i\rangle = (A_{ij})(v_j)$

ex $\quad \text{span}\{1, x, x^2, x^3\} = \text{span}\{|e_0\rangle, |e_1\rangle, |e_2\rangle, |e_3\rangle\}$

$\frac{1}{dx}$, $\quad \frac{d}{dx}x = 1 \quad \frac{d}{dx}x^2 = 2x \quad \frac{dx^3}{dx} = 3x^2$

$\hat{A}_i$

$|e_0\rangle \to 0$

$|e_1\rangle \to |e_0\rangle$

$|e_2\rangle \to 2|e_1\rangle$

$|e_3\rangle \to 3|e_2\rangle$

$\hat{A} \xrightarrow{\{e\}}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

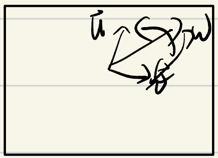
$$(e_0 \; e_1 \; e_2 \; e_3) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = (0 \; e_0 \; 2e_1 \; 3e_2)$$

outer product

$|u\rangle\langle v| : V \to V$

$(|u\rangle\langle v|)(|w\rangle) = |u\rangle\langle v|w\rangle$

$= |u\rangle (|v\rangle, |w\rangle)$



$\begin{pmatrix} a & b & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \to a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdots\cdots$

$\to \sum_{ij} A_{ij} |e_i\rangle\langle e_j| = \hat{A}$

$\langle e_l | A | e_k\rangle = A_{lk}$

$A \cdot \langle e_l | \sum (e_i A_{ij} \langle e_j|) | e_k\rangle$

$\Downarrow$

A_{lk}

$|b_j\rangle = \sum A_{ij} |b_i\rangle$

$e_1, e_2 \cdots e_n, \quad [A] = [e_1 \; 0 \; 0 \; 0 \; 0 \;]$

$|e_1\rangle\langle e_1| \xrightarrow[\text{act on}]{\{e\}} \begin{pmatrix} 1 & 0 & 0 & \cdot \\ 0 & 0 & 0 & \\ 0 & 0 & & \vdots \\ 0 & \vdots & \vdots & \vdots \\ 0 & \vdots & & \end{pmatrix}$

A

$|e\rangle\langle e_1| \xrightarrow{\{e\}}$

$A = \begin{pmatrix} 1 & 0 & & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix}$

$|e_1\rangle\langle e_2| \xrightarrow{\{e\}} \begin{pmatrix} 0 & 1 & 0 & \cdot \\ 0 & 0 & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$

$|e_i\rangle\langle e_j| \longrightarrow \delta_{ij}$

$|e \cdots e_j| = e_j =$

$j \quad j$

$\begin{array}{cc} & i \\ & \begin{bmatrix} & & \\ & A_{ji} = 1 & \\ & & \end{bmatrix} \end{array} \quad \begin{array}{c} j \\ (\; \cdots \; e_i \; \cdots \;) \end{array}$

e_j

$A[j][i] = 1$

Completeness relation

$$\mathbb{I} = \mathbb{1} = \sum_{i \in I} |e_i\rangle\langle e_i|$$

$I = \{1,2,3\}$

$I = \{1,2,3,\ldots\}$

$I = \{\mathbb{Z}\}$

$\{|e_i\rangle\} \Rightarrow$ Ortho Normal Bases

$$|v\rangle = \sum_i v_i |e_i\rangle$$

where

$$\langle e_j|v\rangle = v_j$$

$$\mathbb{I}\,|v\rangle = \sum_i |e_i\rangle \underbrace{\langle e_i|v\rangle}_{v_i} = |v\rangle$$

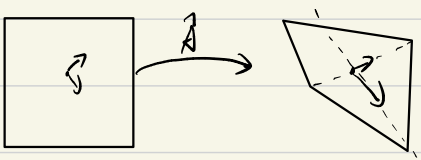
$$\mathbb{I}\,|v\rangle = \sum_i |e_i\rangle \underbrace{\langle e_i|v\rangle}_{v_j} = |v\rangle$$

$$\langle e_j|v\rangle = \langle e_j|\sum v_i|e_i\rangle)$$

$$\sum_i v_i \langle e_j|e_i\rangle$$

$$\sum_i v_i \delta_{ij} = v_j$$

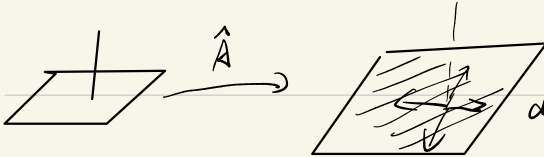
eigen values / eigenvectors



$$\hat{A}|v\rangle = \lambda|v\rangle$$

eigenspace of $\hat{A}$ w/ eigenvalue λ

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



degenerate.

$\lambda = 2$?

adjoint

Def An $n \times n$ matrix has adjoint $A^\dagger = (A^*)^T = (A^T)^*$

ex
$$\begin{pmatrix} 1 & 2i \\ 1+i & 3i \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 1-i \\ -2i & 3i \end{pmatrix}$$

Def the adjoint $\hat{A} : V \to V$ is an operator $\hat{A}^\dagger \; V \to V$

such that

$$(|u\rangle, \hat{A}|v\rangle) = (\hat{A}^\dagger |u\rangle, |v\rangle) \qquad (|u\rangle, \hat{A}|v\rangle) = (\hat{A}^\dagger |u\rangle, |v\rangle)$$

$$(\langle u|(A|v\rangle))$$

$$\hat{C}, \hat{B}, \quad (|u\rangle, \hat{A}|v\rangle) = (\hat{C}|u\rangle, |v\rangle) = (\hat{B}|u\rangle, |v\rangle)$$

$$((\hat{B} - \hat{C})|u\rangle, |v\rangle) = (\hat{B}|u\rangle, v\rangle) - (\hat{C}|u\rangle, |v\rangle) = 0$$

$(|u\rangle, |v\rangle) = 0$ if $|u\rangle$ or $|v\rangle = 0$

$(\hat{A}^\dagger)^\dagger = \hat{A}$

$(|u\rangle, \hat{A}|v\rangle) = (\hat{A}^\dagger|u\rangle, |v\rangle) = (|v\rangle, \hat{A}^\dagger|u\rangle)^* = ((\hat{A}^\dagger)^\dagger|v\rangle, |u\rangle)^* = (|u\rangle, (\hat{A}^\dagger)^\dagger|v\rangle) \Rightarrow (\hat{A}^\dagger)^\dagger = \hat{A}$

$(|u\rangle, \hat{A}|v\rangle) = (\hat{A}^\dagger|u\rangle, |v\rangle) = (|v\rangle, \hat{A}^\dagger|u\rangle)^* = ((\hat{A}^\dagger)^\dagger|v\rangle, |u\rangle)^* = (|u\rangle, (\hat{A}^\dagger)^\dagger|v\rangle)$

$\hat{A} = (\hat{A}^\dagger)^\dagger$

10.18

$\hat{X}|x\rangle = x|x\rangle$

$|\vec{r}\rangle = |x, y, z\rangle$

$\hat{x}, \hat{y}, \hat{z}$

$\hat{x}|\vec{r}\rangle = x|\vec{r}\rangle$

$\hat{y}|\vec{r}\rangle = y|\vec{r}\rangle$

$\hat{z} \ldots$

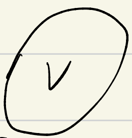
$\langle x|\psi\rangle^2$

$1 = \int_{-\infty}^{+\infty} dx\, |\psi(x)|^2$

$1 = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \quad |\psi(x,y,z)|^2$

$\downarrow$

$\langle x,y,z|\psi\rangle^2$



$\text{Prob}(V) = \underset{V}{\iiint} d^3\vec{r}\, |\psi(x,y,z)|^2$

$\hat{P}_x, \hat{P}_y, \hat{P}_z$

$\downarrow \quad \downarrow \quad \downarrow$ position representation

$-i\hbar\frac{\partial}{\partial x} \quad -i\hbar\frac{\partial}{\partial y} \quad -i\hbar\frac{\partial}{\partial z}$

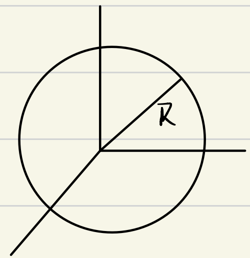
Let $\psi(x,y,z)$ What is

prob of $\quad x \in [x_0, x_0+dx]$

$y \in [y_0, y_0+dy]$

$z \ldots$

$\int_{-\infty}^{+\infty}\left[|\psi(x_0,y_0,z)|^2 dx\, dy\right]dz$



$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} |\psi(x,y,z)|^2\, dr\, d\theta\, d\phi \quad r^2\sin\theta$

$\hat{X}\psi(x_0,y_0,z_0) = x_0\psi(x_0,y_0,z_0)$

$= \langle x_0,y_0,z_0|\hat{X}|\psi\rangle = x_0\langle x_0,y_0,z_0|\psi\rangle = x_0\psi(x_0,y_0,z_0)$

$$([\hat{x},\hat{y}]\psi)(x_0,y_0,z_0)$$

$$=[(\hat{x}\hat{y}-\hat{y}\hat{x})\psi](\vec{r_0})$$

$$=\hat{x}_0\hat{y}_0\psi(\vec{r_0})-\hat{y}_0\hat{x}_0\psi(\vec{r_0})=0$$

$$[\hat{x},\hat{y}]=0$$

$$[\hat{x},\hat{z}]=0$$

$$[\hat{z},\hat{y}]=0$$

$$[\hat{p}_x,\hat{p}_y]\psi(x,y,z)$$

$$=[-i\hbar\frac{\partial}{\partial x},-i\hbar\frac{\partial}{\partial y}]\psi(x,y,z)=0$$

because $\frac{\partial^2}{\partial x\partial y}=\frac{\partial^2}{\partial y\partial x}$

$$|p_x,p_y,p_z\rangle=|\vec{p}\rangle$$

$$[\hat{x},\hat{p}_y]\psi(x,y,z)$$

$$=\hat{x}(-i\hbar\frac{\partial}{\partial y}\psi)+i\hbar\frac{\partial}{\partial y}(x\psi)=0$$

$$\psi(x,y,z)=Ne^{-\frac{x^2+y^2+z^2}{2a}}$$

(a) Normalize

$$\langle\psi|\psi\rangle=N^2\iiint dx\,dy\,dz\;e^{\frac{-x^2}{a}}e^{-\frac{y^2}{a}}e^{-\frac{z^2}{a}}$$

$$=N^2\cdot(a\pi)^{\frac{3}{2}}=1$$

$$\Rightarrow N=(\pi a)^{-\frac{3}{4}}$$

$$\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\int_0^{\infty}dr\;r^2\sin\theta\,e^{-\frac{r^2}{a}}$$

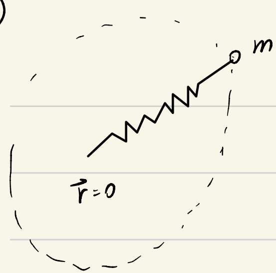
$$=4\pi\int_0^{\infty}r^2e^{-\frac{r^2}{a}}dr=4\pi\cdot\frac{1}{4}\sqrt{\pi}\,a^{\frac{3}{2}}=(\pi a)^{\frac{3}{2}}$$

(b) $\langle r\rangle=\langle\sqrt{x^2+y^2+z^2}\rangle$

$$=N^2\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\int_0^{\infty}dr\;r^3\sin\theta\,e^{-\frac{r^2}{a}}$$

$$=4\pi\frac{a^2}{2}(\pi a)^{-\frac{3}{2}}=2\sqrt{\frac{a}{\pi}}$$

⑤



$\vec{F} = -k\vec{r}$

$m\frac{d^2\vec{r}}{dt^2}$

$E = \frac{1}{2}m\vec{v}^2 + \frac{1}{2}k\vec{x}^2$

$\hat{H} = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{P_z^2}{2m} + \frac{1}{2}m\omega^2(x^2+y^2+z^2)$

$\hat{H}\psi = E\psi$

$\to -\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) + \frac{1}{2}m\omega^2(x^2+y^2+z^2)]\psi(x,y,z) = E\psi(x,y,z)$

$\psi(x,y,z) = X(x)\,Y(y)\,Z(z)$

$\frac{-\hbar^2}{2m}[X''YZ + XY''Z + XYZ''] + \frac{1}{2}m\omega^2(x^2+y^2+z^2)XYZ = EXYZ$

$\Rightarrow \frac{-\hbar^2}{2m}(\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}) + \frac{1}{2}m\omega^2 \underset{(x^2+y^2+z^2)}{\wedge} = E$

$\underbrace{(-\frac{\hbar^2}{2m}\frac{X''}{X} + \frac{1}{2}m\omega^2x^2)}_{f} + \underbrace{(\quad)}_{g} + \underbrace{(\quad)}_{h} = \underbrace{E}_{const}$

f, g, h are constants

$E_x = -\frac{\hbar^2}{2m}\frac{X''}{X} + \frac{1}{2}m\omega^2x^2$

$\Rightarrow E_x X = -\frac{\hbar^2}{2m}X'' + \frac{1}{2}m\omega^2x^2X$

$\Rightarrow X_n(x) \qquad E_{x,n} = \hbar\omega(n_x + \frac{1}{2})$

$n = 0, 1, 2 \cdots$

11.1

Warm up

$E_1 = -13.6\,eV = \frac{1}{2}\mu c^2 \alpha^2$

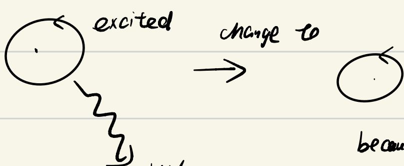
$\frac{1}{\mu} = \frac{1}{m_e} + \frac{1}{m_p}$

if 2 me $\Rightarrow \mu = \frac{1}{2} m_e \Rightarrow E_1 \Big|_{\text{two electron system}} = \frac{1}{2}\cdot\frac{1}{2} m_e c^2 \alpha^2$

$R_\infty = \frac{E}{hc} = \frac{13.6\,eV}{1240\,eV\cdot nm} = 0.0109\ nm^{-1}$

$\lambda = \frac{hc}{E} \longrightarrow$

$\lambda = \frac{h}{p}$



because $\left(\frac{1}{n^2} - \frac{1}{1^2}\right)$

Start

$|\Phi(0)\rangle = \frac{1}{\sqrt{2}}|1,0,0\rangle + \frac{1}{\sqrt{2}}|2,1,1\rangle$

$|n, l, m\rangle$

$|\Phi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\frac{E_1 t}{\hbar}} |1,0,0\rangle + \frac{1}{\sqrt{2}} e^{-i\frac{E_2 t}{\hbar}} |2,1,1\rangle$

E will not change with time $\langle \hat{H} \rangle$

① 50% E_1, 50% E_2 $\Rightarrow \langle H \rangle = \frac{E_1 + E_2}{2} = \frac{5}{8} E_1$

② $\langle \hat{H} \rangle = \langle \Psi | \hat{H} | \Psi \rangle$

$H|\Psi\rangle = \frac{1}{\sqrt{2}} \hat{H}|1,0,0\rangle + \frac{1}{\sqrt{2}} \hat{H}|2,1,1\rangle$

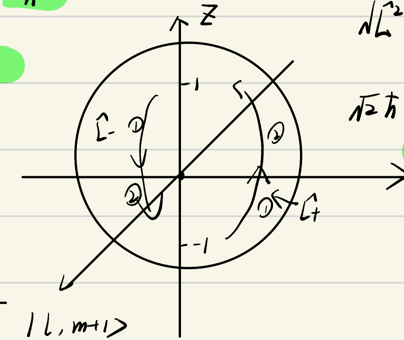
③ $\hat{H} \to -\frac{\hbar^2}{2m}\nabla^2 + V(x,y,z)$

$\langle L^2 \rangle$ $\quad \frac{1}{\sqrt{2}} e^{-i\frac{E_1 t}{\hbar}} |1,0,0\rangle + \frac{1}{\sqrt{2}} e^{-i\frac{E_2 t}{\hbar}} |2,1,1\rangle$

$\Rightarrow l(l+1)\hbar^2 \quad \frac{1}{2}\times 0 + \frac{1}{2}\times 2\hbar^2 = \hbar^2$

$\langle L_z \rangle \Rightarrow m\hbar \to \frac{\hbar}{2}$

$\langle L_x \rangle$



$\sqrt{L^2} = R$

$\sqrt{2}\hbar$

l=1 case

$L_+ = L_x + i L_y$

$L_- = L_x - i L_y$

$L_+|l,m\rangle = \hbar\sqrt{l(l+1) - m(m+1)}\,|l, m+1\rangle$

$L_-|l,m\rangle = \hbar\sqrt{l(l+1) - m(m-1)}\,|l, m-1\rangle$

$\Rightarrow L_x = \frac{1}{2}(L_+ + L_-)$

$\langle L_x \rangle = \langle \Psi | L_x | \Psi \rangle$

$L_+|\Psi\rangle = \cancel{\frac{1}{\sqrt{2}} L_+|1,0,0\rangle} + \cancel{\frac{1}{\sqrt{2}} L_+|2,1,1\rangle}$

$L_-|\Psi\rangle = \cancel{\frac{1}{\sqrt{2}} L_-|1,0,0\rangle} + \frac{1}{\sqrt{2}} L_-|2,1,1\rangle = \frac{1}{\sqrt{2}}\hbar\sqrt{2}|2,1,0\rangle$

$E_n \Rightarrow |n, l, m\rangle$ for each n, degenaracy

$l = 0, 1, \cdots, n-1 \rightarrow \sum_{l=0}^{n-1}(2l+1) = 2\sum_{l=0}^{n-1} l + n = 2\frac{(n-1)n}{2} + n$

for each l $= n^2$

$m = -l, \cdots, l$

approximate

$\Rightarrow l$ 分裂 s,p,d,f　　m ↑↓ 分裂

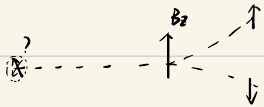
$\Rightarrow L_x|\Psi\rangle = \frac{1}{2}\hbar|2,1,0\rangle$

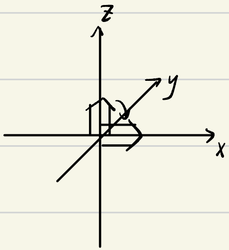
$\Rightarrow \langle L_x\rangle = \langle\Psi|L_x|\Psi\rangle = \langle\Psi|2,1,0\rangle = 0$

所有 $\langle n, l, m | n', l', m'\rangle = \delta_{nn'}\delta_{ll'}\delta_{mm'}$

11.8 Rotation Operators

①





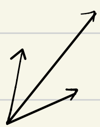
$\hat{R}(\frac{\pi}{2}\hat{y})|+z\rangle = |+x\rangle$

$\hat{R}(\frac{\pi}{2}\hat{y})|-z\rangle = |-x\rangle$

$\langle z+|\hat{R}^\dagger(\frac{\pi}{2}\hat{y}) = \langle +x|$

$\langle z-|\hat{R}^\dagger(\frac{\pi}{2}\hat{y}) = \langle -x|$

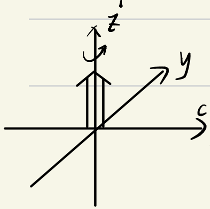
②



$R(\frac{\pi}{2}\hat{y})|\psi\rangle = R(\frac{\pi}{2}\hat{y})[c_+|+z\rangle + c_-|-z\rangle]$

$1 = \langle +x|+x\rangle = \langle +z|R^\dagger(\frac{\pi}{2}\hat{y})\, R(\frac{\pi}{2}\hat{y})|+z\rangle = \langle +z|+z\rangle$

⑤ $\hat{R}(\phi\hat{z})|+z\rangle$ = (phase) $|+z\rangle$

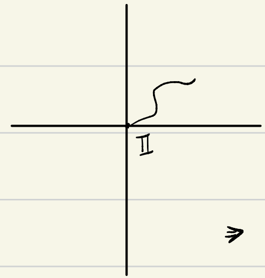


↳ $\hat{R}(\phi\hat{z})|+z\rangle = (\mathbb{1} + (\frac{-iJ_z\phi}{\hbar}) + \frac{1}{2!}(\frac{-iJ_z\phi}{\hbar})^2 + \cdots)|+z\rangle$

claim $J_z|+z\rangle = (\#)|+z\rangle$ Why this is true

$J_z|+z\rangle = c_+|+z\rangle + c_-|-z\rangle$

③ $R(d\phi\hat{z}) = \mathbb{1} - \frac{i}{\hbar}\hat{J}_z\, d\phi + O(d\phi^2)$



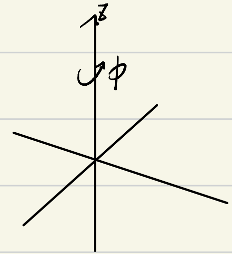
claim $\hat{J}_z$ is Hermitian

$R^\dagger(d\phi\hat{n}) = \mathbb{1} + \frac{i}{\hbar}\hat{J}_z^\dagger\, d\phi + O(d\phi^2)$

$R^\dagger R = \mathbb{1}$

$\Rightarrow (\mathbb{1} + \frac{i}{\hbar}\hat{J}_z^\dagger d\phi)(\mathbb{1} - \frac{i}{\hbar}J_z d\phi) = \mathbb{1} + \underbrace{\frac{i}{\hbar}(\hat{J}_z^\dagger - \hat{J}_z)\, d\phi}_{J_z = J_z^\dagger} + O(d\phi^2)$

④



$d\phi = \frac{\phi}{N}$

$\hat{R}(\phi\hat{z}) = \lim_{N\to\infty}[\mathbb{1} - \frac{i}{\hbar}J_z\frac{\phi}{N}]^N$

$\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$

$= \exp[-\frac{i}{\hbar}J_z\phi]$

⑥ $J_z|+z\rangle = \frac{\hbar}{2}|+z\rangle$

$$\hat{J}_z^2|+z\rangle = \left(\frac{\hbar}{2}\right)^2$$

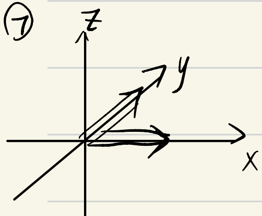
$$\hat{R}(z\phi)|+z\rangle = \left(\mathbb{I} + \left(-\frac{i J_z\phi}{\hbar}\right) + \frac{1}{2!}\left(\frac{-iJ_z\phi}{\hbar}\right)^2 + \cdots\right)|+z\rangle$$

$$\left(1 + \frac{-i\phi}{2} + \frac{1}{2!}\left(\frac{-i\phi}{2}\right)^2 + \cdots\right)|+z\rangle$$

$$= e^{-i\phi/2}|+z\rangle$$

$$J_z|-z\rangle = -\frac{\hbar}{2}|-z\rangle \Rightarrow R(\hat{z}\phi)|-z\rangle = e^{i\phi/2}|-z\rangle$$

⑦



$$|+x\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$$

$$|+y\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{i}{\sqrt{2}}|-z\rangle$$

$$\hat{R}\left(\frac{\pi}{2}\hat{z}\right)|+x\rangle = \frac{1}{\sqrt{2}}R\left(\frac{\pi}{2}\hat{z}\right)|+z\rangle + \frac{1}{\sqrt{2}}R\left(\frac{\pi}{2}\hat{z}\right)|-\hat{z}\rangle$$

$$= \frac{1}{\sqrt{2}}e^{-i\pi/4}|+z\rangle + \frac{1}{\sqrt{2}}e^{+i\pi/4}|-\hat{z}\rangle$$

$$= e^{-i\pi/4}\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{i}{\sqrt{2}}|-z\rangle\right)$$

$$= e^{-i\pi/4}|+y\rangle$$

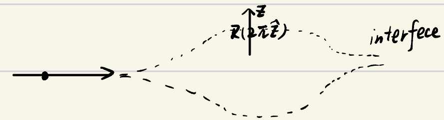
⑧ J_z is angular momentum operator.

$$R(2\pi\hat{z})|+z\rangle = e^{-i\pi}|+z\rangle = -|+z\rangle$$

$$R(2\pi\hat{z})|-z\rangle = e^{i\pi}|+z\rangle = -|-z\rangle$$

$$R(2\pi\hat{n})|\psi\rangle = -|\psi\rangle$$

⑨



⑩ Townsend: QM

$$J_z \xrightarrow{z} \begin{bmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{bmatrix}$$

$$|+z\rangle \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad R(\phi\hat{z}) \rightarrow \begin{bmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{bmatrix}$$

$$|-z\rangle \Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$R(\phi\hat{z})|+z\rangle = e^{-i\phi/2}|+z\rangle$$

$$R(\phi\hat{z})|-z\rangle = e^{i\phi/2}|-z\rangle$$

$$\exp(iJ_z\phi/\hbar) = \sum_{n=0}^{\infty}\left(\frac{i\phi}{\hbar}\frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^n\right)\frac{1}{n!}$$

$$= \sum_{n=0}^{\infty}\begin{bmatrix} (i\phi/2)^n\frac{1}{n!} & 0 \\ 0 & (-i\phi/2)^n/n! \end{bmatrix}$$

$$= \begin{bmatrix} e^{i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{bmatrix}$$

11.15

① A magnetic field $\vec{B}$

$B_0\hat{z} = \vec{B}$

$$H = -\gamma \vec{B}\cdot\vec{S} = -\gamma B_0 S_z$$

$$|\psi(0)\rangle = c_+|+z\rangle + c_-|-z\rangle$$

$$|\psi(t)\rangle = c_+ e^{-i\frac{E_+ t}{\hbar}}|+z\rangle + c_- e^{-i\frac{E_- t}{\hbar}}|-z\rangle$$

$$= \underbrace{\begin{bmatrix} e^{-i\frac{E_+ t}{\hbar}} & 0 \\ 0 & e^{-i\frac{E_- t}{\hbar}} \end{bmatrix}}_{U(t)} \begin{bmatrix} c_+ \\ c_- \end{bmatrix}$$

$$H|+z\rangle = -\gamma B_0 S_z|+z\rangle = \underbrace{-\frac{\gamma B_0 \hbar}{2}}_{E_+}|+z\rangle$$

$$H|-z\rangle = \underbrace{\frac{\gamma B_0 \hbar}{2}}_{E_-}|-z\rangle$$

$$U(t) = \begin{bmatrix} e^{\frac{i\gamma B_0 t}{2}} & 0 \\ 0 & e^{-\frac{i\gamma B_0 t}{2}} \end{bmatrix}$$

$$R(\phi\hat{z}) = \begin{bmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{bmatrix}$$

$$U(t) = R(-\gamma B_0 t\,\hat{z})$$

② S_x, S_y spin-$\frac{3}{2}$, in z basis

$$S_\pm = S_x \pm i S_y$$

$$S_\pm|s,m\rangle = \hbar\sqrt{s(s+1)-m(m\pm1)}\,|s,m\pm1\rangle$$

$$\begin{array}{cl} \begin{pmatrix} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \end{pmatrix} & \begin{array}{l} \leftarrow |\frac{3}{2},\frac{3}{2}\rangle \\ \leftarrow |\frac{3}{2},\frac{1}{2}\rangle \\ \leftarrow |\frac{3}{2},-\frac{1}{2}\rangle \\ \leftarrow |\frac{3}{2},-\frac{3}{2}\rangle \end{array} \end{array}$$

$$S_-|\tfrac{3}{2},\tfrac{3}{2}\rangle = \hbar\sqrt{\tfrac{3}{2}\cdot\tfrac{5}{2} - \tfrac{3}{2}(\tfrac{1}{2})}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle$$

$$= \sqrt{3}\hbar\,|\tfrac{3}{2},\tfrac{1}{2}\rangle$$

$$S_-|\tfrac{3}{2},\tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{15}{4} + \tfrac{1}{2}\cdot\tfrac{1}{2}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle$$

$$= 2\hbar\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle$$

$$S_-|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{15}{4} - \tfrac{1}{2}\times\tfrac{3}{2}}\,|\tfrac{3}{2},-\tfrac{3}{2}\rangle$$

$$= \sqrt{3}\hbar\,|\tfrac{3}{2},-\tfrac{3}{2}\rangle$$

$$S_-\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle = 0$$

$$S_- = \hbar\begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{3}\hbar \\ 2\hbar \\ \sqrt{3}\hbar \end{pmatrix} \Rightarrow S_+ = \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$S_+ = S_x + iS_y$$

$$S_- = S_x - iS_y$$

$$\Rightarrow S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

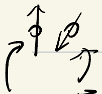
$$S_z \Rightarrow \begin{bmatrix} \frac{3\hbar}{2} & & & \\ & \frac{\hbar}{2} & & \\ & & -\frac{\hbar}{2} & \\ & & & -\frac{3\hbar}{2} \end{bmatrix}$$

$$S_x = \frac{1}{2}(S_+ + S_-)$$

$$S_y = \frac{i\hbar}{2}\begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$S_y = \frac{1}{2i}(S_+ - S_-)$$

2个 spin $\frac{1}{2}$



$\vec{S}_1$ $\quad$ $\vec{S}_2$

$[\vec{S}_1, \vec{S}_2] = 0$

$\mathcal{E}_1 = \text{span}\{|\uparrow\rangle_1, |\downarrow\rangle_1\}$

$\mathcal{E}_2 = \text{span}\{|\uparrow\rangle_2, |\downarrow\rangle_2\}$

$\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$

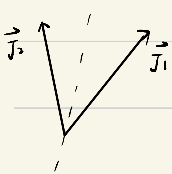
$= \text{span}\{|\uparrow\rangle_1 |\uparrow\rangle_2, |\uparrow\rangle_1 |\downarrow\rangle_2, |\downarrow\rangle_1 |\uparrow\rangle_2, |\downarrow\rangle_1 |\downarrow\rangle_2\}$

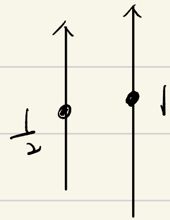
$S_1 = S_1 \otimes \mathbb{I}$

$S_2 = \mathbb{I} \otimes S_2$

③ $j_1 \quad j_2$

$j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2|$





$|\frac{1}{2}, \frac{1}{2}\rangle \quad |1, 1\rangle$

$|\frac{1}{2}, -\frac{1}{2}\rangle \quad |1, 0\rangle$

$|1, -1\rangle$

$\vec{S}_{tot} = \vec{S}_1 + \vec{S}_2$

$\vec{S}_{tot}^{\,2} \rightarrow s(s+1)\hbar^2$

$s = \frac{3}{2} \leftarrow$ 4 dims

or

$s = \frac{1}{2} \leftarrow$ 2 dims

Composite space

$|\frac{1}{2}, \frac{1}{2}\rangle \, |1, 1\rangle$

$|\frac{1}{2}, -\frac{1}{2}\rangle \, |1, 1\rangle$

$|\frac{1}{2}, \frac{1}{2}\rangle \, |1, 0\rangle$

$|\frac{1}{2}, -\frac{1}{2}\rangle \, |1, 0\rangle$

$|\frac{1}{2}, \frac{1}{2}\rangle \, |1, -1\rangle$

$|\frac{1}{2}, -\frac{1}{2}\rangle \, |1, -1\rangle$

6 dims

$S_\pm$, S^2, S_z

$|S, M\rangle$, $|S_1, m_1\rangle |S_2, m_2\rangle$

$|\frac{3}{2}, \frac{3}{2}\rangle$

$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$

$\Downarrow$

$\exists j$, $J^2 = j(j+1)\hbar^2$

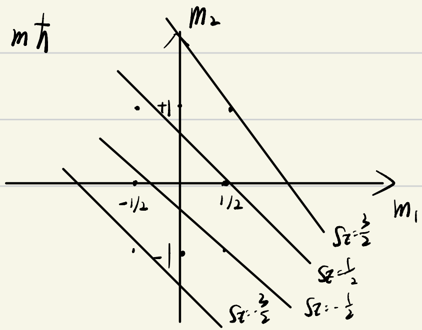
$J_z = m\hbar$

$\vec{S}_{tot} = \vec{S}_1 + \vec{S}_2$

$$
\begin{aligned}
[S_x, S_y] &= [S_{1x} + S_{2x}, S_{1y} + S_{2y}] \\
&= [S_{1x}, S_{1y}] + [S_{2x}, S_{2y}] \\
&= i\hbar S_{1z} + i\hbar S_{2z} \\
&= i\hbar S_z
\end{aligned}
$$

$S^2_{tot} = S(S+1)\hbar^2$

$S_z = m\hbar$

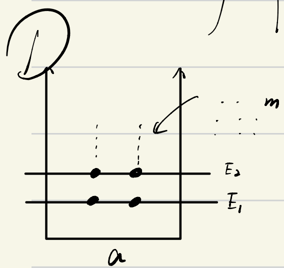


11.22

Identical particles:

20 identical particles in a box of width a

Non-interacting. $\begin{cases} \text{case 1} & \text{Bosons (no spin)} \\ \text{case 2} & \text{Fermions (spin } \frac{1}{2}) \end{cases}$



case 1   20 all in E_1 (ground state) $E_{tot} = 20\,E_1$

case 2 $\Rightarrow E_{tot} = 2E_1 + 2E_2 + \cdots + 2E_{10}$

$= 2E_1\,(1^2+2^2+3^2+\cdots+10^2) = 770\,E_1$ spin $\frac{1}{2}$

"No two Fermions can occupy the same state"

particle	spin
electron	$\frac{1}{2}$
proton	$\frac{1}{2}$
neutron	$\frac{1}{2}$
quarks	$\frac{1}{2}$
muons	$\frac{1}{2}$
neutrinos	$\frac{1}{2}$
Δ^+ baryon	$\frac{3}{2}$

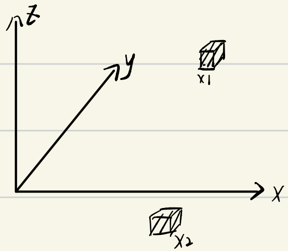
②

A two particle quantum system has a wave function

$\Psi(x_1, x_2, t)$ x_1: particle 1's position

x_2: - - - 2's - - -

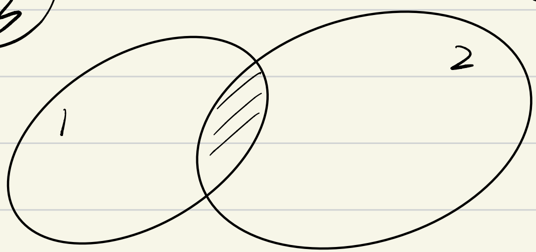
$d\vec{x}\,|\Psi(\vec{x}_1, \vec{x}_2)|^2$ 在 d^3x 中找到 particle 1 和 2 概率



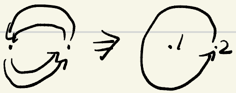
对于特殊的 ⇒ $\Psi(x_1, x_2) = \Psi(x_1)\Psi(x_2)$ "product wave function" independent

$\Psi(x_1, x_2) = e^{-(\vec{x}_1+\vec{x}_2)^2} + e^{-2(\vec{x}_1+\vec{x}_2)^2}$ ✗

③



$|\Psi(x_1, x_2)|^2 = |\Psi(x_2, x_1)|^2$ (identical)



$\Psi(x_1, x_2) = \underset{\pm 1}{e^{i\theta}}\,\Psi(x_2, x_1) = e^{2i\theta}\Psi(x_1, x_2)$

All composite wavefunctions are

symmetric under exchange of

bosons. $+1$

anti-symmetric under exchage of

fermions. -1

Su(2)

Fermions —— belt trick

4

$\Psi(x_1, x_2) = e^{-(x_1+x_2)^2} + e^{-2(x_1+x_2)^2}$

- If distinguishable ✓ 不能 $\psi(x_1)\psi(x_2)$
- If bosons ✓ swipe goes back
- If fermions ✗

If $\Psi(x_1, x_2) = e^{-x_1^2} \cdot e^{-2x_2^2}$

- If distinguishable ✓
- If bosons ✗ $\Rightarrow e^{-x_1^2-2x_2^2} + e^{-x_2^2-2x_1^2}$
- If fermions ✗ $\Rightarrow e^{-x_1^2-2x_2^2} - e^{-x_2^2-2x_1^2}$

(5)

Let $\psi_a(x)$ and $\psi_b(x)$ be orthonormal

$$\Psi(x_1, x_2) = \psi_a(x_1)\psi_b(x_2)$$

then $\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left(\psi_a(x_1)\psi_b(x_2) + \psi_a(x_2)\psi_b(x_1)\right)$

$$\int dx_2 \int dx_1 \, |\Psi(x_1, x_2)|^2 = \frac{1}{2} \int dx_1 \int dx_2 \, |\psi_a(x_1)|^2 |\psi_b(x_2)|^2 + \int dx_1 \int dx_2 \, \psi_a(x_1)\psi_a(x_2)\psi_b(x_1)\psi_b(x_2) = 1$$

(6) $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$

Distinguishable

min $\Psi(x_1, x_2) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x_1}{L}\right) \sin\left(\frac{\pi x_2}{L}\right) \sqrt{\frac{2}{L}}$ $\qquad 2E_1$

$\Psi(x_1, x_2) = \frac{2}{L} \sin\left(\frac{2\pi x_1}{L}\right) \sin\left(\frac{\pi x_2}{L}\right)$ $\qquad 5E_1$

$\frac{2}{L} \sin\left(\frac{\pi x_1}{L}\right) \sin\left(\frac{2\pi x_2}{L}\right)$ $\qquad 5E_1$

7 Bosons

$$\Psi(x_1, x_2) = \frac{2}{L}\sin\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L} \qquad 2E_1$$

$$\Downarrow$$

$$\Psi(x_1, x_2) = \frac{2}{L}\frac{1}{\sqrt{2}}\left(\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) + \sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\right) \qquad 5E_1$$

$$\sim \qquad 8E_1$$

8 Fermions

$\Psi(x_1, x_2) = 0$ 都最低造不出来，一出就对称 1,1 ✗

$$\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}\frac{2}{L}\left(\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) - \sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\right) \qquad 5E_1 \qquad 1,2$$

$$\frac{1}{\sqrt{2}}\frac{2}{L}\left(\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) - \sin\left(\frac{3\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\right) \qquad 10E_1 \qquad 1,3$$

$$13E_1 \qquad 2,3$$

11.29 $(\hat{S}_1+\hat{S}_2)^2 |s_1 m_1 ; s_2 m_2\rangle \Rightarrow \hat{S}^2 |(s_1 s_2) S M_S\rangle$

$S^2 |sm\rangle = \hbar^2 s(s+1) |sm\rangle \quad s = 0, 1, 2$

$s = |s_1 - s_2| \Rightarrow s_1 + s_2$