Space-Time Preconditioners

Consider Riesz bases for the spaces

$$\mathcal{X} := (L_2(0,T) \otimes V) \cap (H^1(0,T) \otimes V'),$$
$$\mathcal{Y} := (L_2(0,T) \otimes V),$$

where $V = H^1$.

The appropriate scaling of a basis function $bf(t,x) := \theta(t)\sigma(x)$ is then as follows:

- in $\mathcal{X}$:

$$\text{RightPreconditioner2D} = \frac{1}{\sqrt{\|\theta\|_{L_2}^2 \|\sigma\|_{H^1}^2 + \|\theta\|_{H^1}^2 \|\sigma\|_{H^{-1}}^2}}$$

- in $\mathcal{Y}$:

$$\text{LeftPreconditioner2D} = \frac{1}{\sqrt{\|\theta\|_{L_2}^2 \|\sigma\|_{H^1}^2}}$$

See also [Schwab, Stevenson, 2009].

The required norms can be computed in different ways. The different possibilites are presented for the space part $\sigma(x)$ of the basis function, but apply to the time part $\theta(t)$ as well.

- **$\mathbf{L_2}$-Norm**:

  (i) Normalization in L_2 **[N]**:

$$\|\sigma\|_{L_2}^2 = 1$$

(ii) Integral **[I]**:

$$\|\sigma\|_{L_2}^2 = \int_\Omega \sigma^2(x)dx$$

- **$\mathbf{H^1}$-Norm**:

(i) Scaling using normalization in L_2 **[Sc_N]**:

$$\|\sigma\|_{H^1}^2 = 2^{2j} \cdot \|\sigma\|_{L_2}^2 = 2^{2j} \cdot 1$$

(ii) Scaling using integral **[Sc_I]**:

$$\|\sigma\|_{H^1}^2 = 2^{2j} \cdot \|\sigma\|_{L_2}^2 = 2^{2j} \int_\Omega \sigma^2(x)dx$$

(iii) Integral **[I]**:

$$\|\sigma\|_{H^1}^2 = \int_\Omega \sigma^2(x)dx + \int_\Omega \sigma_x^2(x)dx$$

- **$\mathbf{H^{-1}}$-Norm**:

(i) Scaling using normalization in L_2 **[Sc_N]**:

$$\|\sigma\|_{H^{-1}}^2 = 2^{-2j} \cdot \|\sigma\|_{L_2}^2 = 2^{-2j} \cdot 1$$

(ii) Scaling using L_2-Integral **[Sc_L_2I]**:

$$\|\sigma\|_{H^{-1}}^2 = 2^{-2j} \cdot \|\sigma\|_{L_2}^2 = 2^{-2j} \int_\Omega \sigma^2(x)dx$$

(iii) Scaling using H^1-Integral **[Sc_H^1I]**:

$$\|\sigma\|_{H^{-1}}^2 = 2^{-2j} \cdot \|\sigma\|_{L_2}^2 = 2^{-4j} \left(\int_\Omega \sigma^2(x)dx + \int_\Omega \sigma_x^2(x)dx \right)$$

(iv) Duality Properties **[D]**:

$$\|\sigma\|_{H^{-1}}^2 \simeq \frac{1}{\|\sigma\|_{H^1}^2} = \frac{1}{\int_\Omega \sigma^2(x)dx + \int_\Omega \sigma_x^2(x)dx}$$

The following tables are overviews over the implemented combinations of norm calculations.

LeftNormPreconditioners	Time L_2	Space H^1
LeftNormPreconditioner2D	**I**	**I**

RightNormPreconditioners	Time		Space	
	L_2	H^1	H^1	H^{-1}
(not working)	**N**	**I**	**I**	**Sc_N**
RightNormPreconditioner2D	**N**	**I**	**I**	**Sc_**L_2**I**
RightNormPreconditioner2D_a	**I**	**I**	**I**	**Sc_**L_2**I**
RightNormPreconditioner2D_b	**I**	**I**	**I**	**D**
RightNormPreconditioner2D_c	**I**	**Sc_I**	**I**	**D**

References

[Schwab, Stevenson, 2009] C. Schwab and R. Stevenson. *Space-time adaptive wavelet methdos for parabolic evolution problems.* Math.Comp., 78 (267), pp. 1293-1318 (2009)