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Crises, noise, and tipping in the Hassell population model

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We consider a problem of the analysis of the noise-induced tipping in population systems. To study this phenomenon, we use Hassell-type system with Allee effect as a conceptual model. A mathematical investigation of the tipping is connected with the analysis of the crisis bifurcations, both boundary and interior. In the parametric study of the abrupt changes in dynamics related to the noise-induced extinction and transition from order to chaos, the stochastic sensitivity function technique and confidence domains are used. The effectiveness of the suggested approach to detect early warnings of critical stochastic transitions is demonstrated. *Published by AIP Publishing.*
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A problem of the prediction of the abrupt changes in system dynamics is topical for science and society. Currently, a huge number of examples demonstrating tipping points of various nature is discovered. So, the elaboration of the general constructive mathematical methods to predict the potential tipping points is a challenging problem of the modern nonlinear dynamics. In deterministic models, tipping points are related to the multistability and bifurcations. Inevitable random disturbances cause noise-induced transitions in multistable systems and shift bifurcation points. In population systems, such transitions can lead to ecological catastrophes connected with the extinction of some species or whole population systems. To recognize early warnings of such tipping points, we suggest a constructive semi-analytical approach based on the stochastic sensitivity function technique and confidence domains method. The effectiveness of this approach is demonstrated by the stochastically forced Hassell-type model with the embedded Allee effect. This simple conceptual discrete model exhibits a rich variety of regular and chaotic regimes, with border and interior crisis bifurcations. A study of this model allows us to clarify mechanisms of the noise-induced tipping in multistable nonlinear systems and discuss the mathematical approach to the analysis of such tipping.

I. INTRODUCTION

In the behavior of some systems, there are situations where inessential changes in one factor lead to a sharp, irreversible, often catastrophic, change in another factor. This kind of transformation is often observed in fairly narrow parametric zones, the position of which is associated with some special so-called tipping points. The passage of such points in itself does not seem extraordinary, but causes unexpected and unpredictable results. Such phenomena are observed in different fields of science and attract the interest of many researchers (see Ref. 1 and bibliography therein). Tipping points in climate dynamics were discovered and widely studied (see, e.g., Refs. 2–6) In ecology, the tipping point is a threshold beyond which an abrupt shift of the population dynamic regime occurs. Beyond the tipping points, ecosystems may change into qualitatively different states or even collapse.[7–9] In the analysis of complex systems, it is extremely important to find and investigate tipping points. Theoretically, in the framework of nonlinear dynamic models, tipping points are attributed to multistability[10–12] and various bifurcations, especially global and crisis ones.[13–17] In Ref. 18, a variety of tipping points were considered, bifurcation-type or noise-induced tipping, for example.

In this paper, we study some general variants of tipping points in population dynamic models. The Allee effect is a classic example of the tipping in population systems.[19–21] The Allee threshold is a critical population size below which the population goes into extinction. So, this Allee threshold plays a role of the tipping point in such systems. However, in the behavior of the population system, tipping points appear not only with the changes of initial states, but also under the variation of other biological parameters. At the same time, crucial changes of population dynamics can be associated with both deterministic bifurcations and stochastic effects.[22–24] Here, the analysis of ecological shifts caused by environmental noise is an important subject.[7,8]

The aim of this paper is to study how tipping points in population systems are related to Allee effect and crisis bifurcations in presence of the inevitable random environmental noise. We use Hassell-type discrete model with embedded Allee effect as a deterministic skeleton. The initial well-known discrete Hassell system[25] was used as a conceptual model for study of the complex phenomena in population dynamics.[26–28] In the Hassell-type bistable model with Allee effect, there exist a trivial stable equilibrium corresponding to the extinction regime and the nontrivial attractor corresponding to the persistence in the form of equilibrium, periodic, and chaotic regimes. So, this model is quite representative in the analysis of the various tipping points.

In Sec. II, persistence zones are described in the space of population size and system parameters. Borders of the persistence zone are defined by the Allee threshold (unstable equilibrium), and boundary crisis and saddle-node bifurcation points.

In Sec. III, we discuss how noise changes these tipping points. As a tipping point indicator, we use a mutual arrangement of the confidence domains and separatrices which detach the basin of attraction of the trivial equilibrium (extinction zone) and the basin of non-trivial attractor (persistence zone). These confidence domains are constructed on the base of the stochastic sensitivity function technique,[29,30] covering both regular and chaotic attractors. It is shown how increasing noise compresses the persistence zone and causes the transitions from order to chaos.

In the Appendix, a theoretical background of the stochastic sensitivity function technique is presented.

II. DETERMINISTIC MODEL

Consider Hassell-type population model[25] with Allee effect

$$x_{t+1} = \frac{\alpha x_t^2}{(\beta + x_t)^6}. \tag{1}$$

Here, x_t is a density of the population at the time t and positive parameters α and β define the intrinsic growth rate and carrying capacity of the environment. This system has the trivial equilibrium $\bar{x}_0 = 0$ which is stable for any parameters. Along with $\bar{x}_0 = 0$, the system (1) can possess two more equilibria $\tilde{x}, \bar{x} : \bar{x}_0 < \tilde{x} < \bar{x}$. The equilibrium $\tilde{x}$ is always unstable, and the stability of $\bar{x}$ depends on parameters. The case when $\bar{x}$ is stable is presented in Fig. 1(a), where a cobweb plot for system (1) with $\alpha = 1$, $\beta = 0.56$ is shown. Here, the points $\bar{x}_0$, $\tilde{x}$, and $\bar{x}$ are plotted by red filled circle, empty circle, and green filled circle, respectively.

The Allee effect is usually associated with the existence of some threshold population density. If the population size is less than this threshold, then the population is extinct, otherwise, the population exists in various regimes. In system (1), the unstable equilibrium $\tilde{x}$ plays a role of this Allee threshold. Indeed, if the initial value x_0 is less than $\tilde{x}$, then the solution x_t vanishes, and the population is extinct [see red iterations in Fig. 1(a)]. If $x_0 > \tilde{x}$, then the solution x_t tends to the stable equilibrium $\bar{x}$ (green iterations). If the equilibrium $\bar{x}$ is unstable, then for $x_0 > \tilde{x}$ the solution x_t tend to the non-equilibrium regime (cycle or chaotic attractor). Such chaotic attractor is shown in Fig. 1(b) for $\alpha = 1$, $\beta = 0.33$. So, in the system (1), the Allee threshold $\tilde{x}$ plays a role of the tipping point which separates two different dynamic regimes (extinction and persistence) under the variation of the initial value of the population size.

Consider now what are the tipping points under the variation of the system parameters. In this paper, we fix $\alpha = 1$ and study system dynamics under the variation of the parameter β only. In Fig. 2, the bifurcation diagram of the Hassell model (1) is plotted. Here, the red dashed line marks the tipping points connected with the unstable equilibrium $\tilde{x}$. As one can

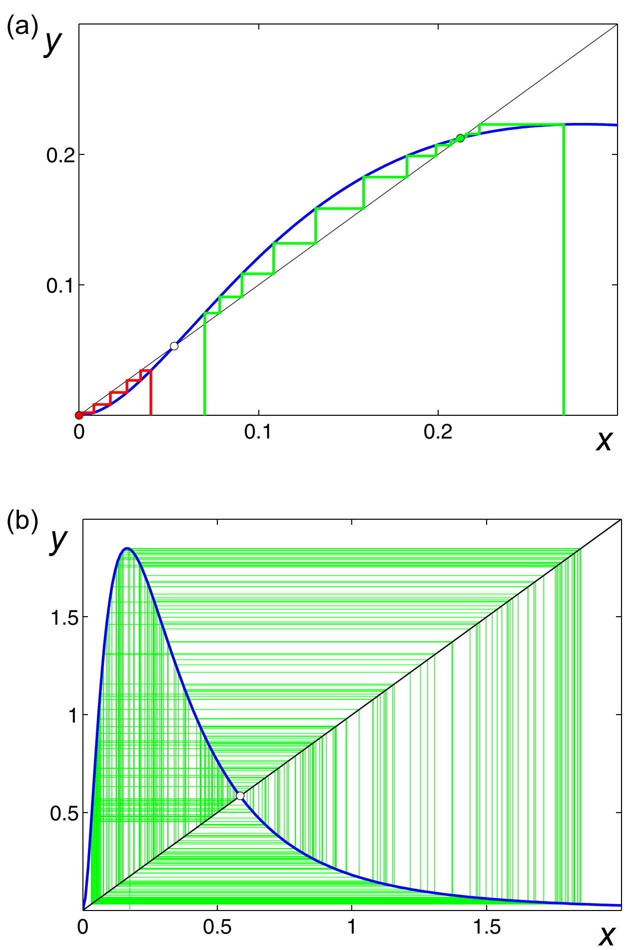

FIG. 1. Iterations of deterministic model with (a) $\beta = 0.56$ and (b) $\beta = 0.33$.

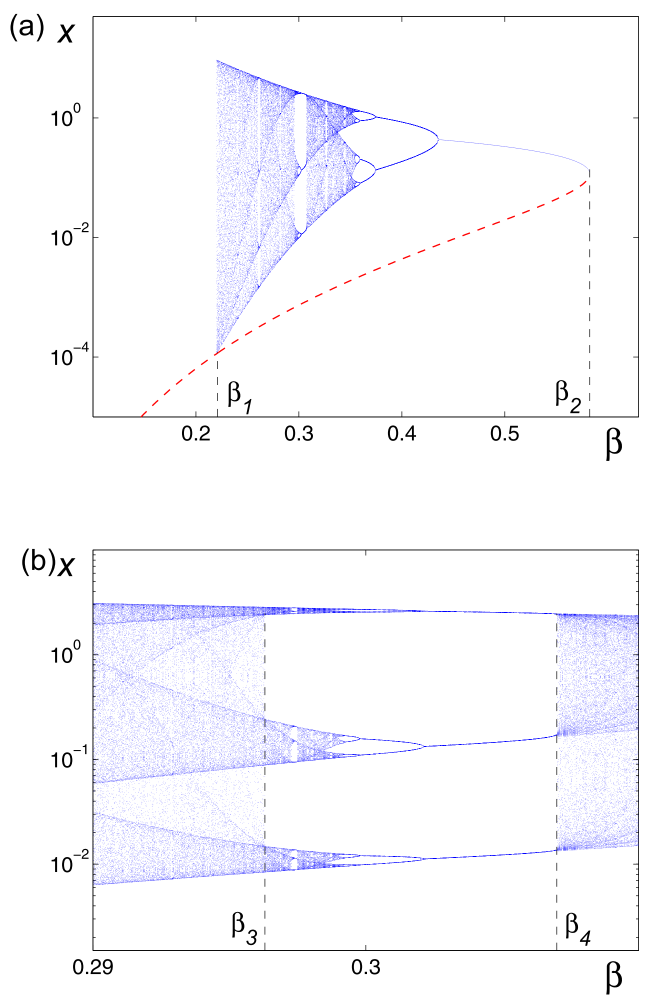

FIG. 2. Bifurcation diagram of Hassell model (1).

see, for system (1), the persistence β-zone is bounded by the interval $\beta_1 < \beta < \beta_2$. Here, $\beta_1 = 0.2202$, $\beta_2 = 0.5824$. For any $\beta < \beta_1$ or $\beta > \beta_2$, the population is extinct for all initial values x_0.

Mathematically, tipping points β_1 and β_2 are defined by corresponding bifurcations. The point β_1 is a boundary crisis bifurcation point:[15,31] when the parameter β passes through β_1 from the right to left, the chaotic attractor touches the unstable equilibrium $\tilde{x}$ and annihilates. The point β_2 is a saddle-node bifurcation point where the stable and unstable equilibria merge and disappear when the parameter passes β_2 from the left to right.

Inside the persistence zone, under the variation of β, another type of tipping points can be observed. Such points of the abrupt changes of dynamic regimes correspond to the interior crises. For example, such crises occur at $\beta_3 = 0.2963$, and $\beta_4 = 0.307$, near the β-zone corresponding to the window of three-periodicity. As the parameter β passes through β_4 from the left to right, a saddle-node bifurcation of the merging of stable and unstable 3-cycles occurs, and the chaotic attractor suddenly appears. As the parameter β passes through β_3 from the right to left, the small 3-piece chaotic attractor touches the unstable 3-cycle, and the large one-piece chaotic attractor appears.

Further, we consider how noise deforms the persistence zones and changes the tipping points.

III. NOISE-INDUCED TIPPING

Along with the system (1), consider the following stochastic system:

$$x_{t+1} = \frac{\alpha x_t^2}{(\beta + \varepsilon\xi_t + x_t)^6}, \tag{2}$$

where the parameter β of the carrying capacity is randomly forced. Here, ξ_t is a standard uncorrelated scalar random process with parameters $\mathrm{E}(\xi_t) = 0$, $\mathrm{E}(\xi_t^2) = 1$, and ε is a noise intensity. Random perturbations of the parameter β allow us to model an important factor of the stochastic environment that inevitably presents in any population system.

An influence of random disturbances on the deterministic attractors of system (1) in (β, x)-plane is shown in Fig. 3(a). Here, for three values of the noise intensity $\varepsilon = 0.01$ (red), $\varepsilon = 0.03$ (green), and $\varepsilon = 0.04$ (black), we plot 200 iterations of system (2) started from the deterministic attractors after the transient 10^4 steps. In the presence of weak noise, random states are located near the deterministic attractor. For increasing noise, the random trajectory can cross the Allee threshold (the unstable equilibrium $\tilde{x}$) and fall down to zero. This means that the population is extinct. As one can see, such noise-induced extinction is more probable near the borders β_1 and β_2 of the persistence zone. So, in the presence of noise, the persistence zone is shrinking. It should be noted that the left border corresponding to the crisis bifurcation is more sensitive to noise comparing with the right border.

This noise-induced compression of the persistence zone can be clearly seen in Fig. 3(b), where plots of mean values $m(\varepsilon)$ of the random states are shown by the same color. The

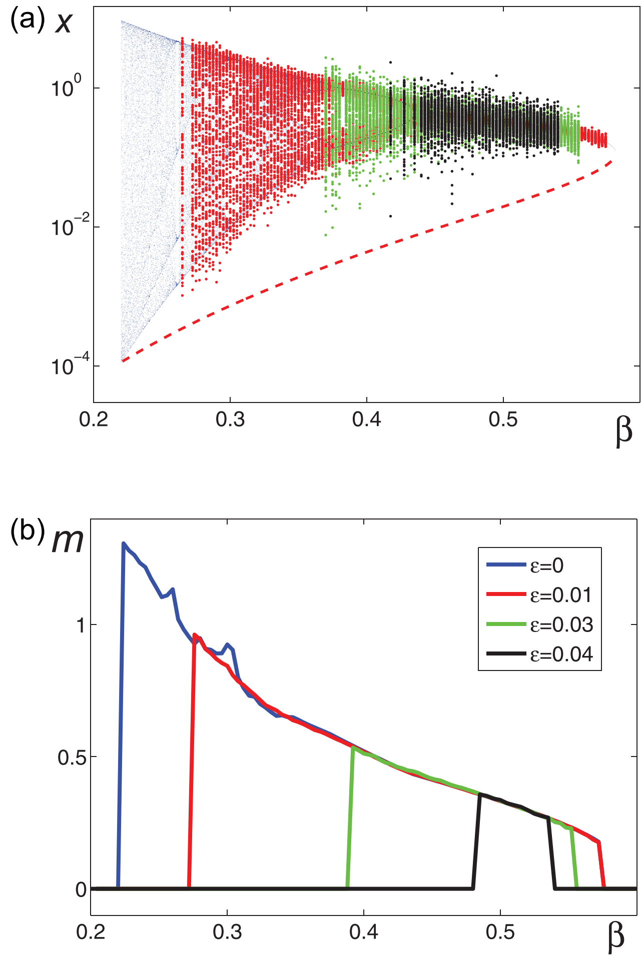

FIG. 3. Random states (a) and mean (b) of random states of the stochastic Hassel model for $\varepsilon = 0.01$ (red), $\varepsilon = 0.03$, (green), and $\varepsilon = 0.04$ (black).

details of the noise-induced extinction for different values of the parameter β under increasing ε are shown in Fig. 4.

As one can see, chaotic large-amplitude oscillation modes are the most susceptible to the noise, whereas the equilibrium regime near $\beta = 0.48$ is the most stable. In the parametric investigation of the noise-induced extinction, one has to take into account three factors, namely, the noise intensity, the stochastic sensitivity of attractors, and the distance between attractor and Allee threshold (the unstable equilibrium $\tilde{x}$ in Hassell model).

To study this phenomenon analytically, we will use the stochastic sensitivity function technique and the method of confidence intervals. The mathematical background of our technique is briefly presented in the Appendix. We will show how it works for the case $\beta = 0.48$ (stable equilibrium) and $\beta = 0.23$ (chaotic attractor).

First, consider $\beta = 0.48$. Here, the stable equilibrium $\bar{x} = 0.36561$, and the unstable equilibrium $\tilde{x} = 0.01465$. The stochastic sensitivity of $\bar{x}$ is defined [see (A4)] by the following formula:

$$M(\bar{x}) = \frac{s(\bar{x})}{1 - q(\bar{x})},$$

where

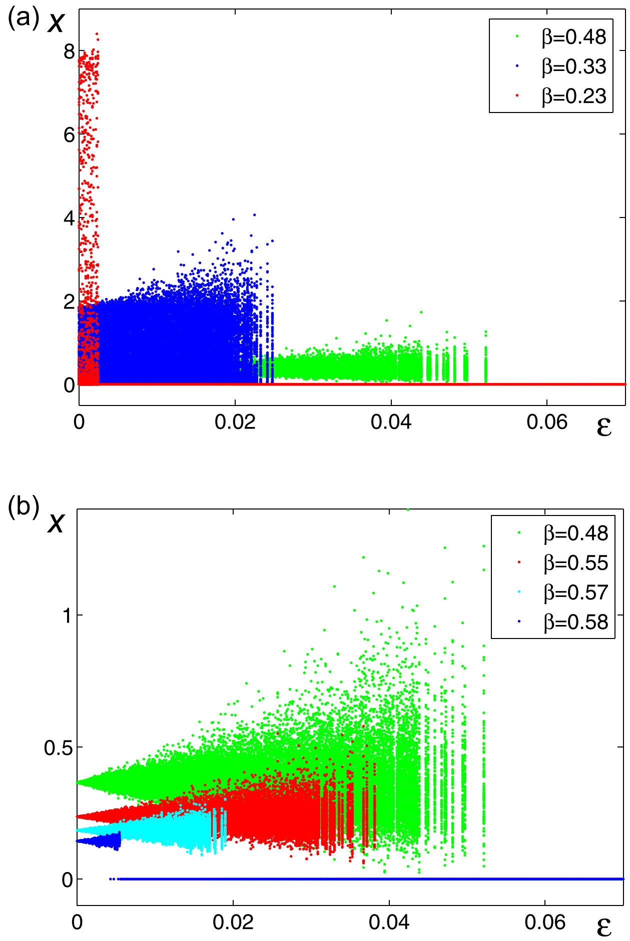

FIG. 4. Random states of stochastic Hassel model with $a = 1$ and noise in β.

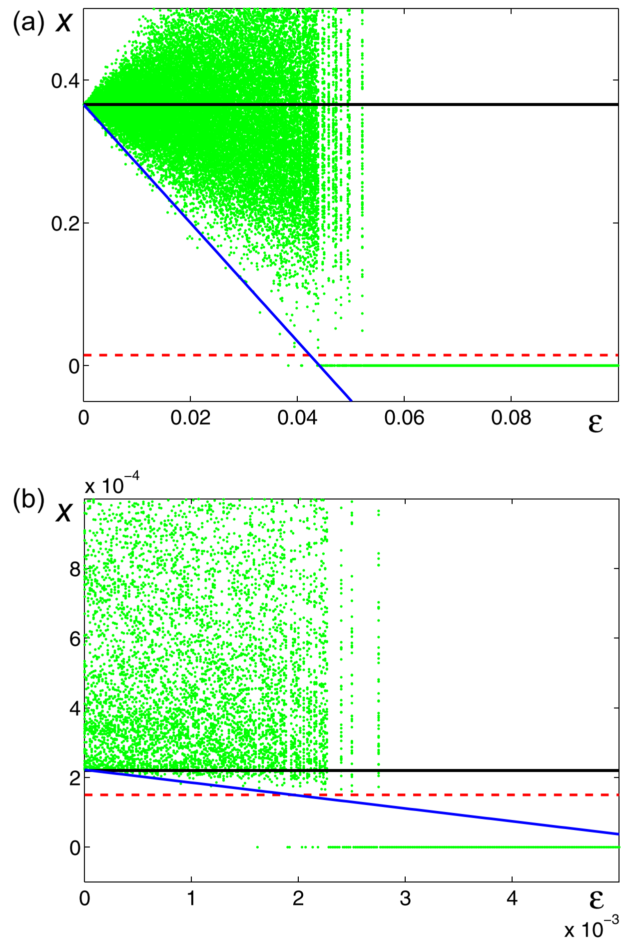

FIG. 5. Noise-induced extinction: (a) $\beta = 0.48$ and (b) $\beta = 0.23$.

$$s(x) = \left[\frac{\partial f}{\partial \eta}(x,0)\right]^2, \quad q(x) = \left[\frac{\partial f}{\partial x}(x,0)\right]^2,$$

$$f(x,\eta) = \frac{\alpha x^2}{(\beta+\eta+x)^6},$$

$$\frac{\partial f}{\partial x}(x,0) = \frac{2\alpha x(\beta-2x)}{(\beta+x)^7}, \quad \frac{\partial f}{\partial \eta}(x,0) = -\frac{6\alpha x^2}{(\beta+x)^7}.$$

Here, for the equilibrium $\bar{x}$ in system with $\beta = 0.48$, the stochastic sensitivity is $M(\bar{x}) = 10.402$. Using this value $M(\bar{x})$, one can find a confidence interval around the point $\bar{x}$. The lower border $\hat{x}(\varepsilon)$ of this interval is as follows:

$$\hat{x}(\varepsilon) = \bar{x} - \varepsilon\sqrt{2M(\bar{x})}\,\mathrm{erf}^{-1}(P), \quad \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt,$$

where P is a fiducial probability. An intersection of this border with $\tilde{x}$ marks the tipping point corresponding to the onset of noise-induced extinction. In Fig. 5(a), random states (green) of system (2) with $\beta = 0.48$ are plotted versus noise intensity ε. Here, the stable equilibrium $\bar{x}$ is plotted by black solid line, and the unstable equilibrium $\tilde{x}$ is plotted by red dashed line, and the lower border $\hat{x}(\varepsilon)$ of the confidence interval with $P = 0.99$ is shown by blue solid line. As one can see, the intersection point of $\hat{x}(\varepsilon)$ with $\tilde{x}$ well agree with the results of the direct numerical simulation and marks the tipping point of the sharp transition from the persistence to extinction.

Second, consider $\beta = 0.23$, where the persistence regime of the population dynamics has a form of large-amplitude chaotic oscillations. The borders a and b of the corresponding deterministic chaotic attractor $\mathcal{A}$ are $a = 2.222 \cdot 10^{-4}$, $b = 7.843$, and the unstable equilibrium is $\tilde{x} = 1.486 \cdot 10^{-4}$. The stochastic sensitivity of the lower border a is defined by the following formula [see (A8)]:

$$M(a) = q(b)s(c) + s(b), \quad c = \mathrm{argmax} f(x,0) = \frac{\beta}{2}.$$

For $\beta = 0.23$, we have $M(a) = 2.19 \cdot 10^{-4}$. Using the value $M(a)$, one can construct a lower border $\hat{x}(\varepsilon)$ of the confidence interval for random states which fall down from $\mathcal{A}$ through the point a,

$$\hat{x}(\varepsilon) = a - \varepsilon\sqrt{2M(a)}\,\mathrm{erf}^{-1}(P).$$

In Fig. 5(b), this border is plotted by blue solid line, the points a and $\tilde{x}$ are shown by black solid and red dashed lines, respectively. Random states of system (2) are plotted versus noise intensity ε by green points. Note that for this chaotic attractor, the noise intensity tipping point which marks an onset of the extinction is $\varepsilon \approx 2 \cdot 10^{-4}$. This is more than two order lesser than the critical noise intensity $\varepsilon \approx 4 \cdot 10^{-2}$ in the case $\beta = 0.48$ with the stable equilibrium.

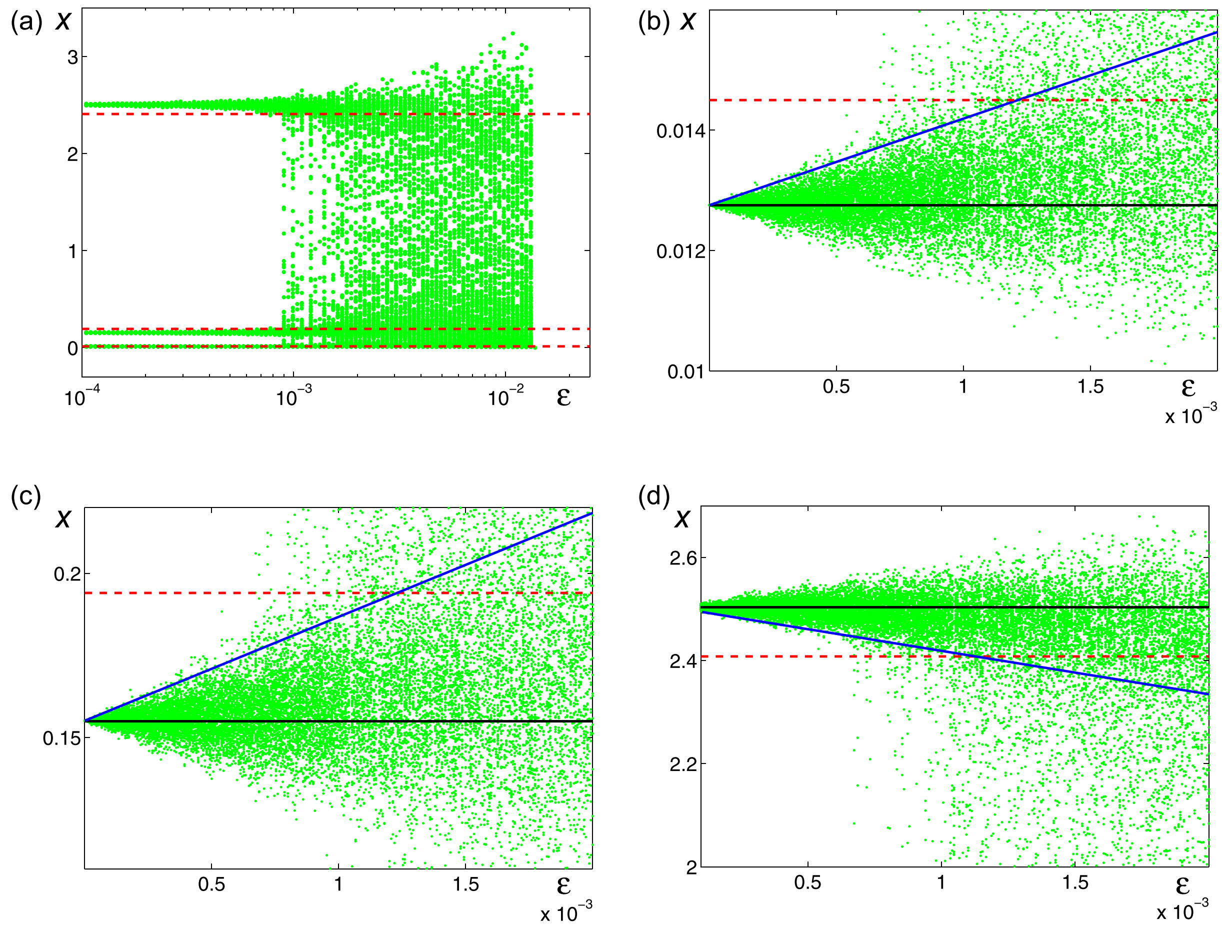

FIG. 6. Noise-induced interior crisis for $\beta = 0.306$. Unstable 3-cycle (dashed red), stable 3-cycle (black), random states (green), and borders of confidence intervals (blue).

So, the considered here tipping, connected with the noise-induced extinction, is studied mathematically by the noise-induced order crisis. Consider now another variant of the noise-induced tipping connected with the interior crisis. In the deterministic Hassell model (1), the interior crises are observed at the points β_3, β_4 (see Fig. 2, right panel).

Consider a response of the stochastic system (2) on the noise for $\beta = 0.306$ close to β_4 (see Fig. 6). For $\beta = 0.306$, the deterministic system (1) has the exponentially stable 3-cycle with states $\bar{x}_1 = 0.0128$, $\bar{x}_2 = 0.1551$, $\bar{x}_3 = 2.503$. Near this stable 3-cycle, the states $\tilde{x}_1 = 0.0145$, $\tilde{x}_2 = 0.194$, $\tilde{x}_3 = 2.408$ of the unstable 3-cycle are arranged. States of the stable cycle are shown in Fig. 6 by black solid lines, and states of the unstable cycle are plotted by red dashed lines. Here, random states of system (2) are shown by green.

For weak noise, random states are located near the stable 3-cycle. As noise intensity increases, the dispersion of random states increases too, the random solutions jump over the separatrix, and the onset of the stochastic mixing is observed. As one can see, the noise generates an appearance of the one-piece large attractor. Similar to the above considered cases of the equilibrium $\bar{x}$ and the chaotic attractor $\mathcal{A}$, we can find [see (A6)] the stochastic sensitivity of this 3-cycle, and construct confidence intervals around its states. The values of the stochastic sensitivity of the points $\bar{x}_1, \bar{x}_2, \bar{x}_3$ are as follows: $M_1 = 3.13 \cdot 10^{-1}$, $M_2 = 1.52 \cdot 10^2$, $M_3 = 1.074 \cdot 10^3$. Corresponding borders of confidence intervals are shown in Figs. 6(b)–6(d) by red dashed lines. As one can see, the intersections of these borders with states $\tilde{x}_{1,2,3}$ of the unstable 3-cycle well localize the tipping point corresponding to the onset of the noise-induced interior crisis.

Consider how noise causes the interior crisis near β_3. In Fig. 7, random states of system (2) with $\beta = 0.2965$ are plotted versus noise intensity ε. For $\varepsilon = 0$, the system possesses 3-piece chaotic attractor. Points of this deterministic attractor

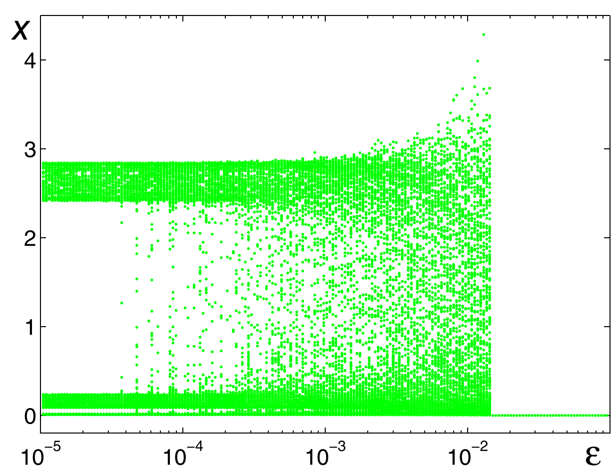

FIG. 7. Random states of system with $\beta = 0.2965$.

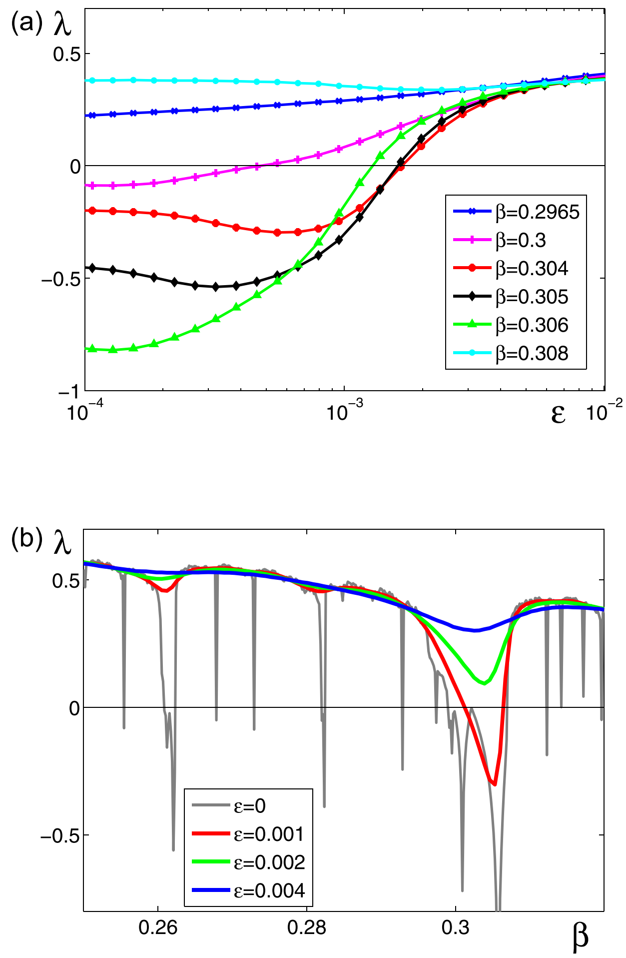

FIG. 8. Lyapunov exponents.

belong to three quite small separated intervals (see Fig. 2, right panel), but the growing noise, similar to the case of 3-cycle, causes the intermixing and generates large one-piece attractor.

It is worth noting that the transition from noisy 3-cycle to one-piece mixed attractor is accompanied by the transformation of regular dynamics to chaotic one. Such noise-induced chaos is illustrated in Fig. 8, where the Lyapunov exponent $\lambda(\varepsilon)$ is plotted for different values of the parameter from the considered β-zone. Here, a change of the sign of the Lyapunov exponent from minus to plus is a standard criterion of the transition from order to chaos. As one can see, for $\beta = 0.306$, this transition from order to chaos occurs near the tipping point $\varepsilon \approx 0.001$ that is close to the noise-induced interior bifurcation of the birth of the chaotic one-piece attractor. It also can be seen that under the noise, a thin fractal structure with the alternation of the regular and chaotic attractors of the deterministic system is smoothed out, and the system becomes chaotic in the whole β-zone.

IV. CONCLUSION

In this paper, a new approach to the study of the noise-induced tipping is suggested. To demonstrate the mathematical technique and constructive abilities of this approach, we used Hassell-type population model with embedded Allee effect. Due to Allee effect, this model is bistable. Despite of simplicity, this single species model is quite representative and exhibits various equilibrium, periodic, and chaotic regimes of the population dynamics. In the deterministic variant, tipping points are defined by the persistence zone borders: the Allee threshold, the crisis and saddle-node bifurcation points. We show how noise shifts these borders, and compresses the persistence zone. A parametric analysis of such noise-induced tipping points was carried out on the base of the theory of the stochastic sensitivity of attractors, and confidence domains method. It was shown how this technique can be used in the analysis of the noise-induced interior crisis that generates a transition from order to chaos. It is worth noting that main ideas and tools of our approach can be extended to the prediction of tipping points in more general dynamic models.

ACKNOWLEDGMENTS

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APPENDIX: STOCHASTIC SENSITIVITY

Consider a discrete-time one-dimensional nonlinear system

$$x_{t+1} = f(x_t, \eta_t), \quad \eta_t = \varepsilon\xi_t, \tag{A1}$$

where ξ_t is a scalar uncorrelated random process with parameters $\mathrm{E}\xi_t = 0$, $\mathrm{E}\xi_t^2 = 1$, and ε is a scalar parameter of the noise intensity.

Consider a solution $\bar{x}_t$ of the corresponding deterministic system (A1) with $\varepsilon = 0$ $(\bar{x}_{t+1} = f(\bar{x}_t, 0))$. Let x_t^ε be a solution of the stochastic system (A1): $x_t^0 = \bar{x}_t$. A sensitivity of the solution $\bar{x}_t$ to the random disturbances is defined by the variable

$$z_t = \left.\frac{\partial x_t^\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} = \lim_{\varepsilon\to 0}\frac{x_t^\varepsilon - \bar{x}_t}{\varepsilon}.$$

Dynamics of z_t is governed by the stochastic equation

$$z_{t+1} = \frac{\partial f}{\partial x}(\bar{x}_t, 0)\, z_t + \frac{\partial f}{\partial \eta}(\bar{x}_t, 0)\, \xi_t. \tag{A2}$$

Corresponding moments $M_t = \mathrm{E}z_t^2$ satisfy the following equation:

$$M_{t+1} = q(\bar{x}_t)M_t + s(\bar{x}_t), \quad q(x) = \left[\frac{\partial f}{\partial x}(x,0)\right]^2, \quad s(x) = \left[\frac{\partial f}{\partial \eta}(x,0)\right]^2. \tag{A3}$$

For the small noise intensity, it holds that $\mathrm{E}(x_t^\varepsilon - \bar{x}_t)^2 \approx \varepsilon^2 M_t$.

1. Stochastic sensitivity of stable equilibria

Let $\bar{x}$ be an exponentially stable equilibrium of the deterministic system (A1) with $\varepsilon = 0$. Due to stability of $\bar{x}$, it holds that $q(\bar{x}) < 1$, and system (A3) for $\bar{x}_t \equiv \bar{x}$ has a unique stable stationary solution $M_t \equiv M$, where

$$M = \frac{s(\bar{x})}{1 - q(\bar{x})}. \tag{A4}$$

The value M is called the stochastic sensitivity of the equilibrium $\bar{x}$.

2. Stochastic sensitivity of k-cycles

Let $\bar{x}_1, \ldots, \bar{x}_k$ be an exponentially stable k-cycle of the deterministic system (A1) with $\varepsilon = 0$. It holds that $\bar{x}_{t+1} = f(\bar{x}_t, 0)$ $(t = 1, 2, \ldots, k-1)$, $\bar{x}_1 = f(\bar{x}_k, 0)$. The necessary and sufficient condition of the exponential stability of this k-cycle is $|\frac{\partial f}{\partial x}(\bar{x}_1, 0) \cdot \ldots \cdot \frac{\partial f}{\partial x}(\bar{x}_k, 0)| < 1$. Due to cycle's stability, system (A3) has a unique stable k-periodic solution M_t satisfying the equation

$$M_{t+1} = q_t M_t + s_t, \quad q_t = q(\bar{x}_t), \quad s_t = s(\bar{x}_t). \tag{A5}$$

The set $\{M_1, \ldots, M_k\}$ define of the stochastic sensitivity of the cycle $\bar{x}_1, \ldots, \bar{x}_k$.[29] Here, the element M_1 is a solution of the equation

$$M_1 = q_1 \cdot \ldots \cdot q_k M_1 + r_{k+1}, \tag{A6}$$

where r_{k+1} is found by iterations $r_{t+1} = q_t r_t + s_t$, $t = 1, \ldots, k$, $r_1 = 0$. The rest elements $M_2, \ldots, M_k$ of the k-periodic solution M_t can be found from Eq. (A5) recurrently.

3. Stochastic sensitivity of chaotic attractors

Let $\mathcal{A}$ be a chaotic attractor of the deterministic system (A1) with $\varepsilon = 0$. We consider a case of the one-band chaotic attractor: $\mathcal{A} = (a, b)$, and the function $f(x, 0)$ has a single maximum at the point $c \in (a, b)$: $\max f(x, 0) = f(c, 0)$, $\frac{\partial f}{\partial x}(c, 0) = 0$. The borders a and b of the attractor $\mathcal{A}$ are connected with the point c: $b = f(c, 0)$, $a = f(b, 0) = f(f(c, 0), 0)$.

The stochastic sensitivity of the borders a and b of the chaotic attractor $\mathcal{A}$[30] is defined by the stochastic sensitivity of the corresponding points of the solution of the deterministic system (A1) with $\varepsilon = 0$, passing through these borders. Consider the solution $\bar{x}_t$ with the initial state $\bar{x}_1 = c$. The first iterations give us $\bar{x}_2 = f(\bar{x}_1) = b$, $\bar{x}_3 = f(\bar{x}_2) = a$. The stochastic sensitivity of the right border $x = b$ of the attractor $\mathcal{A}$ coincides with the stochastic sensitivity M_2 of the state $\bar{x}_2$ of the considered solution $\bar{x}_t$: $M(b) = M_2$. Similarly, for the stochastic sensitivity $M(a)$ of the left border, we have $M(a) = M_3$, where M_3 is the stochastic sensitivity of M_3. It follows from the general system (A3) that

$$M_2 = q_1 M_1 + s_1, \quad M_3 = q_2 M_2 + s_2, \tag{A7}$$

where

$$q_1 = \left[\frac{\partial f}{\partial x}(c, 0)\right]^2, \quad s_1 = s(c), \quad q_2 = \left[\frac{\partial f}{\partial x}(b, 0)\right]^2,$$
$$s_2 = s(b).$$

It follows from $\frac{\partial f}{\partial x}(c, 0) = 0$ that

$$M(a) = \left[\frac{\partial f}{\partial x}(f(c, 0), 0)\right]^2 s(c) + s(f(c, 0)), \quad M(b) = s(c). \tag{A8}$$

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