Answers to Chapter 4

§ **4.1**

1. $[T]_{D,B} = \begin{bmatrix} -1/3 & 1/3 \\ 0 & 1 \\ 2/3 & -2/3 \end{bmatrix}$; $[T]_{D,C} = \begin{bmatrix} 1/3 & 1/3 \\ 2 & 3 \\ -2/3 & -2/3 \end{bmatrix}$. **2(a)-(b)** $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. **(c)** $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

3. $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$. **4.** $\begin{bmatrix} -1 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. **5.** Let $[T]_{B,B} = [a_{ij}]$. Then $Tu_j = \sum_{k=1}^n a_{kj}u_k$. Since B is orthonormal, $\langle Tu_j, u_i \rangle = \sum_{k=1}^n a_{kj}\delta_{ki} = a_{ij}$.

§ **4.2**

1. The same equalities hold for linear transformations. **2(a)** The jth column of B is equal to Be_j. Use associativity of matrix product. **(b)** The (i,j)th entry of AB is equal to the ith row of A times the jth column of B.

3(a) If $A = \begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix}$ and $v = [b_1 \ \cdots \ b_n]^t$, then $Av = b_1A_1 + \cdots + b_nA_n$.

(b) If $B = \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix}$ and $u = [a_1 \ \cdots \ a_n]$, then $uB = a_1B_1 + \cdots + a_nB_n$.

4. Use induction on n with $A^0 = I$.

5. $0 = [a_{ij}]$ with each $a_{ij} = 0$, and $-[a_{ij}] = [-a_{ij}]$.

6. $[a_{ij}] = \sum_i \sum_j a_{ij}E_{ij}$. And this equals 0, when each a_{ij} is 0. **7.** Yes.

8. Bases for U: $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. V: $\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

(Throwing away lin. dep. vectors from basis for U union basis for V.)

$U+V$: $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. $U \cap V$: $\left\{ \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

$\left(\begin{bmatrix} a & -a \\ b & c \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\alpha & \gamma \end{bmatrix} \text{ leads to } \alpha = a,\ \beta = -a,\ b = -a,\ \gamma = c. \right)$

10. Basis: $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1+\sqrt{5} & 2 \\ -2 & 1-\sqrt{5} \end{bmatrix} \right\}$.

11. Easy to verify the subspace conditions.

12. Ae_i = the ith column of A and $R(A) = \operatorname{span}\{Ae_1, \ldots, Ae_n\}$.

14. Let $B = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}$. Then $Bw_1 = v_1, \ \ldots, \ Bw_n = v_n$. Since B is invertible, the conclusion follows.

15. $\operatorname{rank}(A) \le m < n$. So, $\operatorname{null}(A) \ge 1$. take a nonzero $(\alpha_1, \ldots, \alpha_n) \in N(A)$.

17(a) For all α. $\begin{bmatrix} 0 & 1 \\ 1 & -\alpha \end{bmatrix}$. **(b)** For no α. **(c)** For $\alpha \ne 0$. $\frac{1}{\alpha}\begin{bmatrix} 0 & \alpha \\ 1 & -1 \end{bmatrix}$.

(d) For any $\alpha \ne 1$. $\frac{1}{\alpha-1}\begin{bmatrix} \alpha & -1 \\ -1 & 1 \end{bmatrix}$. **18.** Yes. **19.** $\operatorname{rank}(T) = \operatorname{null}(T) = 2$.

20. $\{T_{ij} : 1 \le i \le m,\ 1 \le j \le n\}$, where $T_{ij}(v_j) = w_i$ and $T_{ij}(v_k) = 0$ for $k \ne j$.

21. Write $X \in \mathbb{F}^{n \times p}$ as $[X_1 \cdots X_p]$. $N(T) = \{[X_1 \cdots X_p] : Ax_1 = 0, \ldots, AX_p = 0\}$. So, $\operatorname{null}(T) = p \cdot \operatorname{null}(A)$. Then $\operatorname{rank}(T) = np - \operatorname{null}(T) = p \cdot \operatorname{rank}(A)$.

§ **4.3**

1(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. **(b)** $\begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 6 \end{bmatrix}$. **(c)** $\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$. **2.** No; $A^* = A \not\Rightarrow (iA)^* = (iA)$.

3. Yes. **4.** $A^* = A, B^* = B \Rightarrow (A + \alpha B)^* = A + \alpha B$ for real α; Basis: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

5(a) $\dim_{\mathbb{F}}(\mathbb{F}^{n\times n}) = n^2$; $\dim_{\mathbb{R}}(\mathbb{C}^{n\times n}) = 2n^2$. **(b)** In $\mathbb{F}^{n\times n}$, $\dim_{\mathbb{F}}$ is $(n^2+n)/2$. In $\mathbb{C}^{n\times n}$, $\dim_{\mathbb{R}}$ is n^2+n.
(c) In $\mathbb{F}^{n\times n}$, $\dim_{\mathbb{F}}$ is $(n^2-n)/2$. In $\mathbb{C}^{n\times n}$, $\dim_{\mathbb{R}}$ is n^2-n.
(d) In $\mathbb{C}^{n\times n}$, Basis over $\mathbb{R}$: the n matrices with a single 1 on the diagonal, the $n(n-1)/2$ matrices with a single pair of 1s at corresponding off-diagonal elements and the $n(n-1)/2$ matrices with a single pair of i and i at corresponding off-diagonal elements. Thus $\dim_{\mathbb{R}}$ is n^2. In $\mathbb{R}^{n\times n}$, $\dim_{\mathbb{R}}$ is $(n^2+n)/2$.
(e) In $\mathbb{F}^{n\times n}$, $\dim_{\mathbb{F}}$ is $(n^2+n)/2$. In $\mathbb{C}^{n\times n}$, $\dim_{\mathbb{R}}$ is n^2+n. **(f)** In $\mathbb{F}^{n\times n}$, $\dim_{\mathbb{F}}$ is n. In $\mathbb{C}^{n\times n}$, $\dim_{\mathbb{R}}$ is $2n$. **(g)** In $\mathbb{F}^{n\times n}$, $\dim_{\mathbb{F}}$ is 1. In $\mathbb{C}^{n\times n}$, $\dim_{\mathbb{R}}$ is 2.
6(a) $(A^t)^{-1} = (A^{-1})^t$. **(b)** $(A^*)^{-1} = (A^{-1})^*$. **(c)** Let $A^{-1} = [y_1 \cdots y_n]$. Now, $Ay_k = e_k$. A is lower triangular with nonzero entries on the diagonal. $A = [a_{ij}]$ and $y_k = (b_1, \ldots, b_n)^t$ implies $a_{11}b_1 = 0$, $a_{21}b_1 + a_{22}b_2 = 0$, ... Then $b_1 = 0$, $b_2 = 0$, $\ldots, b_{k-1} = 0$. So, A^{-1} is lower triangular. **(d)** Use A^t and (c).
7. Use $(AB)^* = B^*A^*$. **8(a)-(b)** Use $(AB)^* = B^*A^*$. **(c)** Use $(AB)^* = B^*A^*$ and $(B^*)^{-1} = (B^{-1})^*$.
9. $A^* + A = 0 = B^* + B \Rightarrow (A+\alpha B)^* + (A+\alpha B) = 0$ for real α. Basis: $\left\{\begin{bmatrix} i & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix} 0 & 0\\ 0 & i\end{bmatrix}, \begin{bmatrix} 0 & -1\\ 1 & 0\end{bmatrix}, \begin{bmatrix} 0 & i\\ i & 0\end{bmatrix}\right\}$.
10. $\begin{bmatrix} 1 & 1+i & 1\\ -1+i & 1 & 1\\ -1 & -1 & 1\end{bmatrix}$. **11.** $\begin{bmatrix} 1 & 1 & 0\\ 0 & 1 & 1\\ 1 & 0 & 1\end{bmatrix}$.

§ **4.4**
1. $tr(A+\alpha B) = \mathrm{tr}(A) + \alpha\mathrm{tr}(B)$; so V is a subspace of $\mathbb{F}^{n\times n}$.
2. $tr(A+\alpha B) = \mathrm{tr}(A) + \alpha\mathrm{tr}(B)$; with V as in Q.1, $\mathrm{null}(T) = \dim_{\mathbb{F}}(V) = n^2-1$.
3. $A = \begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}$. **4(a)** No; $\mathrm{tr}(-I+(-I)) < 0$. **(b)** Yes. **5.** $A = I = B$.
6. $\mathrm{tr}(AB) = \sum_{j=1}^n\sum_{k=1}^n a_{jk}b_{kj} = \sum_{k=1}^n\sum_{j=1}^n a_{kj}b_{jk} = \sum_{j=1}^n\sum_{k=1}^n b_{jk}a_{kj} = \mathrm{tr}(BA)$.
7. $\mathrm{tr}(AB-BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0 \neq \mathrm{tr}(I)$. **8.** Let $C = \begin{bmatrix} a & b\\ c & -a\end{bmatrix}$. If $a = 0$, take $A = \begin{bmatrix} 0 & -b\\ c & 0\end{bmatrix}$, $B = \begin{bmatrix} 1 & 0\\ 0 & 0\end{bmatrix}$.
If $a \neq 0$, take $A = \begin{bmatrix} 0 & a\\ 0 & c\end{bmatrix}$, $B = \begin{bmatrix} 0 & 0\\ 1 & b/a\end{bmatrix}$.
9. $(y^{\mathrm{t}}Ax)^{\mathrm{t}} = y^{\mathrm{t}}Ax$ implies $a_{12} = a_{21}$. Next, $x^{\mathrm{t}}Ax \geq 0$ gives a quadratic. Complete the square and argue.
10. $\mathrm{tr}(A) = \sum_i\sum_j |a_{ij}|^2$.
11. $A^*A = A^2 \Rightarrow AA^* = (A^*)^2$. Then $(A^*-A)^*(A^*-A) = AA^* - A^*A$. So, $\mathrm{tr}[(A^*-A)^*(A^*-A)] = 0$. By Q.10, $A^* - A = 0$. **13.** $\det(A)\times\det(B) = \det(AB) = \det(2C) = 2^4\det(C) = 16$. Since A has only integers entries, $\det(A)$ is an integer. Thus the pair $(\det(A), \det(B))$ can be $(\pm1, \mp16)$, $(\pm2, \mp8)$, $(\pm4, \mp4)$, or $(\pm8, \mp2)$, or $(\pm16, \mp1)$. Then $max(det(A) + \det(B))$ is $16+1 = 17$.
14. $AE_{ij} = E_{ij}A \Rightarrow a_{ij} = 0$ for $i \neq j$ and $a_{ii} = a_{jj}$. Then $A = a_{11}I$.
15(a)-(b) Multiply and see. **(c)** Then $A = a_{11}I$.
16(a) $\det(A)\det(B) = \det(AB) = \det(I) = 1 \Rightarrow \det(A) \neq 0$. A^{-1} exists. Now, $AB = I \Rightarrow B = A^{-1}$. Then $AB = I$. **(b)** $A = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$, $B = \begin{bmatrix} 1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}$.

§ **4.5**
1. $\begin{bmatrix} 1 & 1 & -1\\ -1 & 1 & 1\\ 1 & -1 & 1\end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}$. **(a)** $[I]_{N,O} = \frac{1}{2}\begin{bmatrix} 2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix}$. **(b)** $[T]_{N,O} = \frac{1}{2}\begin{bmatrix} 2 & 3 & 3\\ 2 & 1 & 3\\ 0 & 0 & 2\end{bmatrix}$.
(c) $\begin{bmatrix} 1\\ 2\\ 3\end{bmatrix}_O = \begin{bmatrix} 1\\ 0\\ 2\end{bmatrix}$, $\begin{bmatrix} 1\\ 2\\ 3\end{bmatrix}_N = \frac{1}{2}\begin{bmatrix} 4\\ 3\\ 5\end{bmatrix}$, $\left[T\begin{bmatrix} 1\\ 2\\ 3\end{bmatrix}\right]_N = \begin{bmatrix} 4\\ 4\\ 2\end{bmatrix}$.
2(a) $Q = \begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}$, $R = \begin{bmatrix} 1 & 0\\ -4 & -1\end{bmatrix}$. **(b)** $P = \begin{bmatrix} 1 & 1\\ -1 & -3\end{bmatrix}$. **(c)** $S = \begin{bmatrix} 1 & 0\\ -4 & -1\end{bmatrix}$.
(d) $PQP^{-1} = [I]_{N,O}[A]_{O,O}[I]N,O^{-1} = [A]_{N,O}[I]_{O,N} = [A]_{N,N} = S$.

3. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$; then $[v]_B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $A[v]_B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $[Av]_B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.
4. If $Tv_i = a_{i1}w_1 + \cdots a_{im}w_m$ for $1 \le i \le n$, then $T = \sum_{i=1}^n \sum_{j=1}^m a_{ij}T_{ij}$. Next, this equals 0 implies $Tv_i = 0$. As $\{w_j\}$ lin. ind., $a_{i1} = \cdots = a_{im} = 0$. Conclude that $\{T_{ij}\}$ is lin. ind.
5(a) T is one-one iff $\text{null}(T) = \{0\}$ iff $\text{null}([T]_{E,B}) = 0$ iff $\text{rank}([T]_{E,B}) = n$. **(b)** T is onto iff $[T]_{E,B}$ is onto iff $\text{rank}([T]_{E,B}) = m$.
6. Both $\mathcal{L}(V,W)$ and $\mathbb{F}^{m\times n}$ are vector spaces. Use Exercise 5 and show that $[\alpha T]_{E,B} = \alpha[T]_{E,B}$ and $[S+T]_{E,B} = [S]_{E,B} + [T]_{E,B}$.
7. Since the map $T \mapsto [T]_{E,B}$ is an isomorphism, it maps a basis onto a basis.
8(a) Write $C_j :=$ the jth column of $A = [\langle u_1, u_j\rangle, \ldots, \langle u_n, u_j\rangle]^t$. Suppose for scalars $b_1, \ldots, b_n$, $\sum_j b_j C_j = 0$. Its ith component gives $\sum_j b_j \langle u_i, u_j\rangle = 0$. That is, for each i, $\langle \sum_j b_j u_j, u_i\rangle = 0$. Since $\{u_i\}$ is a basis, for each $v \in V$, $\langle \sum_j b_j u_j, v\rangle = 0$. In particular, $\langle \sum_j b_j u_j, \sum_j b_j u_j\rangle = 0$. Or, $\sum_j b_j u_j = 0$. Due to lin ind. of $\{u_j\}$, each $b_j = 0$. So, the columns of A are lin. ind.
(b) Since $\{C_1, \ldots, C_n\}$ is a basis for $\mathbb{F}^{n\times 1}$, there exist unique scalars $b_1, \ldots, b_n$ such that $[\overline{\alpha}_1, \ldots, \overline{\alpha}_n]^t = b_1 C_1 + \cdots + b_n C_n$. Comparing the components, we have $\overline{\alpha}_i = \langle u_i, \sum_j b_j u_j\rangle$. So, $\alpha_i = \langle \sum_j b_j u_j, u_i\rangle$.
9. $[T]_{C,C} = [I]_{C,B}[T]_{B,B}[I]_{B,C}$. Thus we show that if $R = P^{-1}QP$, then $\text{tr}(R) = \text{tr}(P)$ for $n\times n$ matrices P, Q, R, with P invertible. For this, use $\text{tr}(M_1M_2) = \text{tr}(M_2M_1)$. Similarly, do for the determinant.
10. $x = \sum_i \langle x, u_i\rangle u_i$, $y = \sum_j \langle y, u_j\rangle u_j \Rightarrow \langle x, y\rangle = \sum_i \langle x, u_i\rangle \sum_j \langle u_j, y\rangle \langle u_i, u_j\rangle$. This proves the first part. Next, define $T : V \to \mathbb{F}^n$ by $T(u_k) = e_k$ for $k = 1, \ldots, n$. Since $x = \sum_k \langle x, u_k\rangle u_k$, $Tx = \sum_k \langle x, u_k\rangle e_k$. Using first part, $\|Tx\|^2 = \sum_k |\langle x, u_k\rangle|^2 = \|x\|^2$.

§ **4.6**
1. For $A = [a_{ij}]$, write $\overline{A} = [\overline{a}_{ij}]$. See that $\text{rank}(\overline{A} = \text{rank}(A)$. Then use $\text{rank}(B^t) = \text{rank}(B)$.
2. If $\text{rank}(A) = r = \text{rank}(B)$, then $A = Q^{-1}E_rP$ and $B = M^{-1}E_rS$. So, $B = M^{-1}QAP^{-1}S$.
3(a) $R(AB) = \{ABx : x \in \mathbb{F}^{k\times 1}\} \subseteq \{Ay : y \in \mathbb{F}^{n\times 1}\} = R(A)$. **(b)** From (a), $\text{rank}(AB) \le \text{rank}(A)$. Next, $\text{rank}((AB)^t) = \text{rank}(B^tA^t) \le \text{rank}(B^t) = \text{rank}(B)$.
4. Let $A = DE$ and $B = FG$ be the full rank factorizations of A and B. Now, $A + B = \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} E \\ G \end{bmatrix}$. By Exercise 3, $\text{rank}(A+B) \le \text{rank}\left(\begin{bmatrix} E \\ G \end{bmatrix}\right) \le \text{rank}(E) + \text{rank}(G) = \text{rank}(A) + \text{rank}(B)$.
5(a) Since $A = BC$, each column of A is a linear combination of columns of B. Since B has full rank,the columns of B are lin. ind. **(b)** Use (a) on $A^t = C^tB^t$.
6. The columns of A are unique linear combinations of columns of A. The coefficients in these linear combinations give the matrix C. Thus C is a unique matrix. **7.** Since D is invertible, $\text{rank}(BD) = \text{rank}(B)$.
8. From Exercise 5(a), columns of B_1 form a basis for $R(A)$. Also, the columns of B_2 form a basis for $R(A)$. The isomorphism that maps the columns of B_1 to columns of B_2 provides such a D. Then use Exercise 6.