Department of Mathematics, IIT Madras
Quiz-1 Solution MA2031 03/09/2018
Linear Algebra for Engineers
Time : 8:00-8:50, Marks: 20
Answer all questions.

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1. Let J be the set of all irrational numbers and let $V = J \cup \{0, 1, -1\}$.
   Is V a subspace of $\mathbb{R}$? Justify your answer. [2]
   Sol: $1 \pm \sqrt{2} \in V$ but their sum $2 \notin V$. So, V is not a subspace.
   Aliter: $1 \in V$ but $2 \cdot 1 = 2 \notin V$. Or, some such example. [0 or 2]

2. Let A and B be subsets of a vector space V. Prove or give a counter example for each of the following statements:

   (a) If $A \subseteq B$, then $\operatorname{span}(A) \subseteq \operatorname{span}(B)$. [2]
   Sol: Suppose $A \subseteq B$. If $A = \varnothing$, then $\operatorname{span}(A) = \{0\} \subseteq \operatorname{span}(B)$. If $A \neq \varnothing$, then any linear combination of vectors from A is a linear combination of vectors from B. So, $\operatorname{span}(A) \subseteq \operatorname{span}(B)$. [0 or 2]

   (b) $\operatorname{span}(A \cap B) \subseteq \operatorname{span}(A) \cap \operatorname{span}(B)$. [2]
   Sol: $A \cap B \subseteq A$. Using (a), $\operatorname{span}(A \cap B) \subseteq \operatorname{span}(A)$. Similarly, $\operatorname{span}(A \cap B) \subseteq \operatorname{span}(B)$. Then $\operatorname{span}(A \cap B) \subseteq \operatorname{span}(A) \cap \operatorname{span}(B)$. [0 or 2]

   (c) $\operatorname{span}(A) \cap \operatorname{span}(B) \subseteq \operatorname{span}(A \cap B)$. [2]
   Sol: Let $V = \mathbb{R}^2$. $A = \{(1,0),(0,1)\}$, $B = \{(1,1),(1,2)\}$. Then $\operatorname{span}(A \cap B) = \operatorname{span}(\varnothing) = \{0\}$. $\operatorname{span}(A) \cap \operatorname{span}(B) = \mathbb{R}^2 \not\subseteq \{0\}$. Or, some such example. [0 or 2]

3. In $\mathbb{R}_4[t]$, let $B = \{1 - 2t + t^3,\ 3t - t^2 - t^4,\ 1 + t^3 + t^4,\ 4t - 2t^2 - 3t^4\}$. Either show that B is linearly independent or express one of the elements of B as a linear combination of the others. [5]
   Suppose $a(1-2t+t^3)+b(3t-t^2-t^4)+c(1+t^3+t^4)+d(4t-2t^2-3t^4) = 0$. Then $a+c = 0$, $-2a + 3b + 4d = 0$, $b + 2d = 0$, $-b + c - 3d = 0$. [2]
   Solving, we obtain $c = d = -a,\ b = 2a$. [1]
   So, $4t - 2t^2 - 3t^4 = (1 - 2t + t^3) + 2(3t - t^2 - t^4) - (1 + t^3 + t^4)$. [2]

4. Consider $U = \operatorname{span}\{(1,2,3),\ (2,1,1)\}$ and $W = \operatorname{span}\{(1,0,1),\ (3,0,-1)\}$ as subspaces of $\mathbb{R}^3$. Construct a basis for $U + W$ and determine the dimension of $U \cap W$. [5]
   Sol: $U + W = \operatorname{span}\{(1,2,3),\ (2,1,1),\ (1,0,1),\ (3,0,-1)\}$. [1]
   $$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \\ 3 & 0 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \qquad [2]$$
   Thus a basis for $U + W$ is $\{(1,0,0),\ (0,1,0),\ (0,0,1)\}$. [1]
   Then $\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U + W) = 2 + 2 - 3 = 1$. [1]
   Aliter: $U + W = \operatorname{span}\{(1,2,3),\ (2,1,1),\ (1,0,1),\ (3,0,-1)\}$. [1]
   Taking the vectors as columns,
   $$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \qquad [2]$$
   Thus a basis for $U + W$ is $\{(1,2,3),\ (2,1,1),\ (1,0,1)\}$. [1]
   Then $\dim(C \cap W) = \dim(U) + \dim(W) - \dim(U + W) = 2 + 2 - 3 = 1$. [1]

5. In $\mathbb{R}^2$, define $\langle (a,b),(c,d) \rangle = ac - bd$. Determine whether $\langle\ ,\ \rangle$ is an inner product or not. [2]
   Sol: $\langle (1,1),(1,1) \rangle = 1 - 1 = 0$. It contradicts $\langle x, x \rangle = 0$ iff $x = 0$. Hence it is not an inner product. (Or, some such example.) [0 or 2]
