Department of Mathematics, IIT Madras
Quiz-1 Solution MA2031 03/09/2018
Linear Algebra for Engineers
Time : 8:00-8:50, Marks: 20
Answer all questions.

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1. Let J be the set of all irrational numbers and let $V = J \cup \{0, 1, -1\}$.
Is V a subspace of $\mathbb{R}$? Justify your answer. [2]
Sol: $1 \pm \sqrt{2} \in V$ but their sum $2 \notin V$. So, V is not a subspace.
Aliter: $1 \in V$ but $2 \cdot 1 = 2 \notin V$. Or, some such example.

2. Let A and B be subsets of a vector space V. Prove or give a counter example for each of the following statements:
   (a) If $A \subseteq B$, then $\text{span}\,(A) \subseteq \text{span}\,(B)$. [2]
   Sol: Suppose $A \subseteq B$. If $A = \varnothing$, then $\text{span}\,(A) = \{0\} \subseteq \text{span}\,(B)$. If $A \neq \varnothing$, then any linear combination of vectors from A is a linear combination of vectors from B. So, $\text{span}\,(A) \subseteq \text{span}\,(B)$.
   (b) $\text{span}\,(A \cap B) \subseteq \text{span}\,(A) \cap \text{span}\,(B)$. [2]
   Sol: $A \cap B \subseteq A$. Using (a), $\text{span}\,(A \cap B) \subseteq \text{span}\,(A)$. Similarly, $\text{span}\,(A \cap B) \subseteq \text{span}\,(B)$. Then $\text{span}\,(A \cap B) \subseteq \text{span}\,(A) \cap \text{span}\,(B)$.
   (c) $\text{span}\,(A) \cap \text{span}\,(B) \subseteq \text{span}\,(A \cap B)$. [2]
   Sol: Let $V = \mathbb{R}^2$. $A = \{(1, 0), (0, 1)\}$, $B = \{(1, 1), (1, 2)\}$. Then $\text{span}\,(A \cap B) = \text{span}\,(\varnothing) = \{0\}$. $\text{span}\,(A) \cap \text{span}\,(B) = \mathbb{R}^2 \not\subseteq \{0\}$. Or, some such example.

3. In $\mathbb{R}_4[t]$, let $B = \{1 - 2t + t^3,\ 3t - t^2 - t^4,\ 1 + t^3 + t^4,\ 4t - 2t^2 - 3t^4\}$. Either show that B is linearly independent or express one of the elements of B as a linear combination of the others. [5]
Suppose $a(1 - 2t + t^3) + b(3t - t^2 - t^4) + c(1 + t^3 + t^4) + d(4t - 2t^2 - 3t^4) = 0$. Then $a + c = 0$, $-2a + 3b + 4d = 0$, $b + 2d = 0$, $-b + c - 3d = 0$.
Solving, we obtain $c = d = -a,\ b = 2a$.
So, $4t - 2t^2 - 3t^4 = (1 - 2t + t^3) + 2(3t - t^2 - t^4) - (1 + t^3 + t^4)$.

4. Consider $U = \text{span}\,\{(1, 2, 3),\ (2, 1, 1)\}$ and $W = \text{span}\,\{(1, 0, 1),\ (3, 0, -1)\}$ as subspaces of $\mathbb{R}^3$. Construct a basis for $U + W$ and determine the dimension of $U \cap W$. [5]
Sol: $U + W = \text{span}\,\{(1, 2, 3),\ (2, 1, 1),\ (1, 0, 1),\ (3, 0, -1)\}$.
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \\ 3 & 0 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$
Thus a basis for $U + W$ is $\{(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)\}$.
Then $\dim\,(U \cap W) = \dim\,(U) + \dim\,(W) - \dim\,(U + W) = 2 + 2 - 3 = 1$.
Aliter: $U + W = \text{span}\,\{(1, 2, 3),\ (2, 1, 1),\ (1, 0, 1),\ (3, 0, -1)\}$.
Taking the vectors as columns,
$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$
Thus a basis for $U + W$ is $\{(1, 2, 3),\ (2, 1, 1),\ (1, 0, 1)\}$.
Then $\dim\,(C \cap W) = \dim\,(U) + \dim\,(W) - \dim\,(U + W) = 2 + 2 - 3 = 1$.

5. In $\mathbb{R}^2$, define $\langle (a, b), (c, d) \rangle = ac - bd$. Determine whether $\langle\ ,\ \rangle$ is an inner product or not. [2]
Sol: $\langle (1, 1), (1, 1) \rangle = 1 - 1 = 0$. It contradicts $\langle x, x \rangle = 0$ iff $x = 0$. Hence it is not an inner product. (Or, some such example.)
