Answers to Chapter 1

§ **1.2**

1(a) It is; $0 = (0,0)$, $-(a,0) = (-a,0)$. **(b)** It is. **(c)** It is not. **(d)** It is not; $1(1,1) + 2(1,1) \neq 3(1,1)$. **(e)** It is not; $1(1,1) \neq (1,1)$. **(f)** It is not; $1(1,1) + 1(1,1) \neq 2(1,1)$. **(g)** It is; $0 = 1$, $-x = 1/x$. **(h)** It is not; $\sqrt{2} \cdot \sqrt{2} \notin V$. **(i)** It is not; $(1+1) \odot 1 \neq 1 \odot 1 \oplus 1 \odot 1$. **(j)** It is; $0 = w$, $-x = w - x$. **(k)** It is; $0 = (1,0)$, $-(a,b) = (2-a,-b)$.
2. No. $t^5 + (-t^5)$ is not in the set. **3(a)** Add $-y$. **(b)** Follows from Theorem 1.2(3e). **(c)** Follows from (b) as there are infinite number of scalars.

§ **1.3**

1(a) It is not; $(0,0) \notin U$. **(b)** It is. **(c)** It is. **(d)** It is. **(e)** It is. **(f)** It is not; $(1/2)t \notin U$. **(g)** It is; taking zero function as even. **(h)** It is not; $f(t) = t^2$ does not have an additive inverse in U.
2. $\mathbb{R}^2 : \{(0,0\}$, $\mathbb{R}^2$, and all straight lines passing through the origin.
$\mathbb{R}^3 : \{(0,0,0)\}$, $\mathbb{R}^3$, all straight lines passing through the origin, and all planes passing through the origin.
3. Only for $b = 0$. **4.** Yes. **5.** Not closed under addition (scalar multiplication). **6.** Yes. **7.** Yes. **8(a,b,c)** Yes. **9.** It is a subspace of the space of all functions from S to $\mathbb{R}$.

§ **1.4**

1. No. $t^3 + 2t^2 + 2t$ is not in the span. **2.** $\{a_0 + a_1t + a_2t^2 + \cdots + a_nt^n : a_{2i-1} = 0\}$. **3.** $\mathbb{F}^3$. **4.** $u \in \operatorname{span}(S_2)$. So, $\operatorname{span}(S_2) \subseteq \operatorname{span}(S_1)$ implies $u \in \operatorname{span}(S_1)$. Next, $u \in \operatorname{span}(S_1) \Rightarrow S_2 \subseteq \operatorname{span}(S_1) \Rightarrow \operatorname{span}(S_2) \subseteq \operatorname{span}(S_1)$.
5. S is a subspace iff it equals the minimal subspace containing it.
6. $1 = u_1$, $t^i = u_i - \sum_{j=0}^{i-1} u_j$; and $\mathbb{F}_{n-1}[t] = \operatorname{span}\{1, t, \ldots, t^{n-1}\}$. Next, $\mathbb{F}[t] = \cup_{n=0}^{\infty} F_n[t]$.
7. The intersection is a (the) minimal subspace containing S.
8. Both $\varnothing$ and $\{0\}$ span $\{0\}$. If $V \neq \{0\}$, then both V and $V \setminus \{0\}$ span V.
9. No; any linear combination of $1, t, t^2, \ldots$ is a polynomial in t.
10. $\{f, g\}$, where $f(1) = 1, f(2) = 0$; $g(1) = 0, g(2) = 1$.
11(a) A linear combination of linear combinations is a linear combination.
(b) A linear combination of vectors from A is also a linear combination of vectors from B.
(c) $\operatorname{span}(A \cap B) \subseteq \operatorname{span}(A)$ and $\operatorname{span}(A \cap B) \subseteq \operatorname{span}(B)$. **(d)** False: $V = \mathbb{R}$, $A = \{1\}$, $B = \{2\}$. **(e)** False: $V = \mathbb{R}^2$, $A = \{(1,0),(0,1)\}$, $B = \{(1,0)\}$. **(f)** False: $V = \mathbb{R}^2$, $A = \{(1,0),(0,1)\}$, $B = \{(1,1)\}$.
12. $U = x$-axis, $V = y$-axis, $W =$ the line $y = x$.
13. If $x = u + w = u' + w'$ for $u, u' \in U$ and $w, w' \in W$, then $u - u' = w - w'$. So both $u - u'$, $w - w' \in U \cap W$.
14(a) $V \subseteq V + W$. So, $U \cap V \subseteq U \cap (V + W)$. Similarly, $U \cap W \subseteq U \cap (V + W)$.
(b) Take $X = \mathbb{R}^2$, $U = \operatorname{span}\{(1,1)\}$, $V = \operatorname{span}\{(1,0)\}$, $W = \operatorname{span}\{(0,1)\}$.
(c) $V \cap W \subseteq V$; so $U + (V \cap W) \subseteq U + V$. $V \cap W \subseteq W$; so $U + (V \cap W) \subseteq U + W$. Therefore, $U + (V \cap W) \subseteq (U + V) \cap (U + W)$. **(d)** Take U, V, W, X as in (b).
15. c_{00}, the set of all sequences each having finitely many nonzero terms.

§ **1.5**

1(a) Lin Ind. **(b)** $(7,8,9) = 2(4,5,6) - (1,2,3)$. **(c)** Lin. Ind. **(d)** 4th $= 7/11$ times 1st $+8/11$ times 2nd $+13/11$ times 3rd. **(e)** Lin. Ind. **(f)** Lin. Ind. **(g)** Lin. Ind. **(h)** $2 = 2\sin^2 t + 2\cos^2 t$. **(i)** Lin. Ind.
(j) Lin. Ind. **2.** Yes; No. **3.** Not necessarily; In $\mathbb{F}_2[t]$, $\{1, 2, t\}$. **4. 4.** $(1,0)$, $(0,1)$, $(1,1)$.
5. If $(a,b) = \alpha(c,d)$, then $ad - bc = 0$. If $(c,d) = \alpha(a,b)$, then $ad - bc = 0$. If $ad = bc$, then $(a,b) = (0,0)$ or $(c,d) = (0,0)$ or $(a,b) = \alpha(c,d)$ for some nonzero α.
6(a) If a linear combination of vectors from a subset is zero, then the same shows that a linear combination of vectors from the superset is also zero. **(b)** Follows from (a). **(c)** Follows from (a). **(d)** Follows from (c). **7(a)** $\{(1,0)\}$ is a lin. ind subset of the lin. dep. set $\{(1,0),(0,1)\}$. **(b)** Take the sets in (a). **(c)** $\{(1,0)\}$ and $\{(2,0)\}$ are each lin. ind. but their union is not. **(d)** $\{(1,0),(2,0)\}$ and $\{(1,0),(3,0)\}$ are each lin. dep. but their intersection is not. **8(a)** Not necessarily. $A = \{(1,0),(2,0)\}$, $B = \{(0,1)\}$.
(b) If $v \neq 0$ and $v = a_1u_1 + \cdots + a_nu_n = b_1v_1 + \cdots + b_mv_m$ for nonzero a_i, b_j and $u_i \in A$, $v_j \in B$, then $a_1u_1 + \cdots + a_nu_n - b_1v_1 - \cdots - b_mv_m = 0$ shows that $A \cup B$ is lin. dep. **9.** $\mathbb{R}^2$ is spanned by two vectors.

10. $\mathbb{F}_2[t]$ is spanned by three vectors.
11. Suppose $ae^t + bte^t + ct^3e^t = 0$. Evaluate it at $t = -1, 0, 1$. Solve for a, b, c.
12. Yes. Let $f(t) = \sum_{k=1}^{n} a_k \sin kt$. $f(t) = 0 \Rightarrow \int_{-\pi}^{\pi} \sin mt f(t) dt = 0$ for any m. Evaluate the integral and conclude that $a_m = 0$ for $1 \le m \le n$.
13. Otherwise, a higher degree polynomial is a linear combination of some lower degree polynomials. Differentiate the equation to get a contradiction.

§ **1.6**
1(a) Basis. **(b)** Basis. **(c)** Not a basis. **(d)** Basis. **2.** Yes. **3(a)** Yes. **(b)** No.
4. $\{(1,0,-1),(0,1,-1)\}$. **5.** $\{(1,0,0,0,1),(0,1,0,1,0),(0,0,1,0,1)\}$.
6. $\{t-2,\ t^2-2t-2\}$. **7.** $\{1+t^2,\ 1-t^2,\ t,\ t^3\}$. **8.** Yes; Yes.
9. $\{e_1,e_2,e_3\}$, $\{e_1+e_2, e_1+e_3, e_2+e_3\}$, $\{e_1+2e_2, e_2+2e_3, e_3+2e_1\}$.

§ **1.7**
1. $\{0\}$, $\mathbb{R}^3$, straight lines passing through the origin, and planes passing through the origin.
2(a) Basis: $\{(0,1,0,0,0),\ (1,0,1,0,0),\ (1,0,0,1,0),\ (0,0,0,0,1)\}$.
(b) Basis: $\{(1,0,0,0,-1),\ (0,1,1,1,0)\}$. **(c)** Basis: $\{(1,-1,0,2,1),\ (2,1,-2,0,0),\ (2,4,1,0,1)\}$
3. 3; It is $\operatorname{span}\{1+t^2, -1+t+t^2, t^3\}$.
4. $\dim(U\cap W) = \dim(U)$ and $U\cap W$ is a subspace of U implies $U\cap W = U$.
5. If $U\cap W = \{0\}$, then $\dim(U)+\dim(W) = \dim(U+W) \le 9$; a contradiction.
6. $\{(-3,0,1)\}$ is a basis for $U\cap W$; $\dim(U+W) = 3$.
7. $U+V = \{(a_i)\in\mathbb{R}^{50} : 12/i\}$, $\dim(U) = 34$, $\dim(V) = 38$, $\dim(U+V) = 46$, $\dim(U\cap V) = 26$.
8. No: $\dim(\operatorname{span}\{t,t^2,t^3,t^4,t^5\}) \le \dim(V) = 3$.
9. $\{f_1,\ldots,f_n\}$ is a basis where $f_i(i) = 1,\ f_i(j) = 0$ for $j \ne i$.
10. If S is a linearly dependent spanning set, systematically delete vectors to get a basis; contradicting $|S| = \dim(V)$. For Theorem 1.28, adjoin to a basis of U the vectors from a spanning set of V and delete all linearly dependent vectors from the new ones. **11.** $\mathbb{R}[t] \subseteq V$.

§ **1.8**
1. 3. **2.** Basis: $\{(1,-1,0,2,1),\ (0,3,-2,-4,-2),\ (0,0,5,4,2),\ (0,0,0,0,1)\}$.
3(a) Bases for $U : \{(1,2,3),(2,1,1)\}$; $W : \{(1,0,1),(3,0,-1)\}$;
$U+W : \{(1,2,3),\ (0,3,5),(0,2,2)\}$; $U\cap W : \{(-3,0,1)\}$.
(b) Bases for $U : \{(1,0,2,0),\ (1,0,3,0)\}$; $W : \{(1,0,0,0),\ (0,1,0,0),\ (0,0,1,1)\}$;
$U+W : \{(1,0,2,0),\ (0,1,0,0),\ (0,1,0,0),\ (0,0,1,1)\}$; $U\cap W : \{(1,0,0,0)\}$.
(c) Bases for $U : \{(1,0,0,2),\ (3,1,0,2),\ (7,0,5,2)\}$; $W : \{(1,0,3,2),\ (1,1,-1,-1)\}$;
$U+W : \{(1,0,0,2),\ (0,1,0,-4),\ (0,0,3,0),\ (0,0,0,1)\}$; $U\cap W : \{(1,-12,15,14)\}$.
4. dim of $U : 3$; $W : 3$; $U+W : 4$; $U\cap W : 2$. **5.** Basis: $\{1+t^2,\ t+2t^2,\ t^3\}$.
6(a) Each $v_i \in \operatorname{span}\{v_1, v_2-v_1, \ldots, v_n-v_1\}$. **(b)** Due to (a) and $\dim(V) = n$.