Answers to Chapter 2

§ **2.1**

2(a) $\langle(0,1),(0,1)\rangle = 0$. **(b)** As in (a). **(c)** $\langle(1,1),(1,1)\rangle = 0$. **(d)** $\langle 1,1\rangle = 0$.
(e) As in (d). **(f)** $\langle 1,t\rangle = 0$. **(g)** For $f(t) = 0$ in $[0,1/2]$ and $f(t) = t$ in $(1/2,1]$, $\langle f,f\rangle = 0$.
3. With $y = \sum_{i=1}^{n} \alpha_i x_i$, $\langle y,y\rangle = 0$. **4.** $\langle x,y\rangle = \alpha + i\beta \Rightarrow \mathrm{Re}\langle ix,y\rangle = -\beta$.
5. (a)-(c) easy. **(d)** $\|x+\alpha y\|^2 = \|x-\alpha y\|^2$ iff $\mathrm{Re}(\overline{\alpha}\langle x,y\rangle) = 0$. Take $\alpha = \overline{\langle x,y\rangle}$.
(e) $\|x+y\|^2 = (\|x\|+\|y\|)^2$ iff $\mathrm{Re}\langle x,y\rangle = \|x\|\,\|y\|$ (Using $\mathrm{Re}\langle x,y\rangle \le |\langle x,y\rangle| \le \|x\|\,\|y\|$) iff $|\langle x,y\rangle| = \|x\|\,\|y\|$ (Cauchy-Schwartz) iff one is a scalar multiple of the other.

§ **2.2**

1. $W = \mathrm{span}\{(3,-1,3,0),(0,-1,3,3)\}$. **2.** $\langle x,y\rangle = \langle y,x\rangle \Rightarrow \langle x+y,x-y\rangle = \|x\|^2 - \|y\|^2$. **5.** Yes.
6. If $B = \{v_1,\ldots,v_n\}$ is an orthonormal set and for each x, $\|x\|^2 = \sum_{j=1}^{n} |\langle x,v_j\rangle|^2$, then B is an orthonormal basis. For, let $y = \sum_{j=1}^{n}\langle x,v_j\rangle v_j$. Then $\|x\|^2 = \|y\|^2$.
8(a) $x \in V^\perp \Rightarrow \langle x,v\rangle = 0$ for all $v \in V$. In particular, $\langle x,x\rangle = 0$. For the second equality, $\langle v,0\rangle = 0$ for all $v \in V$. **(b)** If $x \in S$, then $\langle x,y\rangle = 0$ for all $y \in S^\perp$.
9(a) $W \subseteq V$, $W^\perp \subseteq V$; so $W + W^\perp \subseteq V$. Let $v \in V$. Let $\{v_1,\ldots,v_n\}$ be an orthonormal basis of V. Write $x = \sum_{j=1}^{n}\langle x,v_j\rangle v_j$; $y = v - x$. Then $\langle y,v_j\rangle = 0$. Hence $y \in W^\perp$. **(b)** Let $x \in W \cap W^\perp$. Then $\langle x,x\rangle = 0$. **(c)** $W \subseteq W^{\perp\perp}$. Let $x \in W^{\perp\perp}$. Using (a), $x = w + y$, for some $w \in W$ and $y \in W^\perp$. Then $\langle w,y\rangle = 0 \Rightarrow 0 = \langle x,y\rangle = \langle y,y\rangle$. Then $x = w \in W$.
10. Let $x(t) := \sum_{j=1}^{n} a_j \sin(jt) = 0$. Compute $\int_0^{2\pi} x(t)\sin(mt)\,dt$ for $m = 1,2,\ldots,n$.

§ **2.3**

1(a) $(1,2,0)$, $(6/5,-3/5,0)$, $(0,0,1)$. **(b)** $(1,1,1)$, $(2/3,-4/3,2/3)$, $(1,0,-1)$.
(c) $(0,1,1)$, $(0,1,-1)$, $(-1,0,0)$. **2(a)** $(1/\sqrt{14})(1,-2,3)$. **(b)** $(1/\sqrt{74})(7,-4,3)$. **(c)** $(1/2\sqrt{5})(0,2,4)$.
3(a) $\mathrm{span}\{(-1,1,0,1),\ (0,0,1,0)\}$. **(b)** $\mathrm{span}\{(-6,1,5,2),\ (0,1,-1,1)\}$. **(c)** $\mathrm{span}\{(1,0,0,0),\ (0,0,1,1)\}$.
4. $(1/2)(i,1-i,-1)$. **5(a)** $1,\ t-1/2,\ t^2-t+1/6$. **(b)** $1,\ t,\ t^2-1/3$. **(c)** $1,\ t+1/2,\ t^2-5t-11/6$.
6(a) $\{-(b+c+d)+2bt+3ct^2+4dt^3 : a,b,c,d \in \mathbb{R}\}$. **(b)** $1,\ t-1/2,\ t^2-t+1/6,\ t^3-3t^2/2+3t/5-1/20$.

§ **2.4**

1. $(5/3,4/3,1/3)$. **2.** $(-1/3,2/3,-1/3)$. **3.** v since $v \in U$.
4. $-19/20 - 3t/5 + 3t^2/2$. **5.** $e^t - 9(e-/e) + 3t/e - 15(e-13/e)t^2/8$.