Answers to Chapter 3

§ 3.1

1(a) $T(0,0) \neq (0,0)$. **(b)** $T(2,2) = (2,4); 2T(1,1) = (2,2)$. **(c)** $T(\pi/2,0) = (1,0); 2T(\pi/4,0) = (\sqrt{2},0)$.
(d) $T(-1,0) = (1,0); (-1)T(1,0) = (-1,0)$. **(e)** $T(0,0) \neq (0,0)$. **(f)** $T(0,2) = (0,4); 2T(0,1) = (2,2)$.
2. $T(x) = \alpha x$ for some α. **3.** $T(2,3) = (5,11)$. T is one-one.
4. $TS(x) = 0$ and $ST(x) = x(1) - x(0)$. Both are linear transformations.
5. No. If $T(a,b) = (1,1)$, then $T(-a,-b) = (-1,-1)$, which is not in the co-domain square. Notice that $(a,b) \neq (0,0)$ and $(-a,-b)$ lies in the domain square.
6. Fix a basis $\{v_1, v_2\}$ for V. If $v = av_1 + bv_2$, define $T(v) = (a,b)$.

§ 3.2

1(a) No T since $T(2,-1) \neq 2T(1,1) - 3T(0,1)$. **(b)** $T(a,b) = (2a-b, a-b, 2a)$.
(c) No T as $T(-2,0,-6) \neq -2T(1,0,3)$. **(d)** $T(a,b,c) = (c, (b+c-a)/2)$.
(e) $T(a+bt+ct^2+dt^3) = b+c$ and many more. **(f)** This T itself.
(g) No T since $T(1+t) \neq T(1) + T(t)$. **(h)** This T is linear.
2. No. Let $T(1,1) = (a,b)$, $T(1,-1) = (c,d)$. Now, $-1 \leq a,c \leq 1$ and $0 \leq b,d \leq 2$. Then $T(-1,-1) = (-a,-b)$, $T(-1,1) = (-c,-d)$. Here, $0 \leq -b,-d \leq 2$ also. (Image points are inside the co-domain square.) It forces $b = d = 0$. So, $T(1,1) = (a,0)$, $T(1,-1) = (c,0)$. This gives $T(\alpha,\beta) = ((\alpha+\beta)a/2 + (\alpha-\beta)b/2, 0)$ for all $\alpha, \beta \in \mathbb{R}$. Then T cannot take any point to $(1,2)$.
3. Expand $\|T(u+v)\|^2$ and use $\|Tx\|^2 = \|x\|^2$ for all $x \in V$.

§ 3.3

1(a) $\operatorname{rank}(T) = 2$, $\operatorname{null}(T) = 0$. **(b)** $\operatorname{rank}(T) = 2$, $\operatorname{null}(T) = 1$. **(c)** $\operatorname{rank}(T) = 2$, $\operatorname{null}(T) = 0$.
(d) $\operatorname{rank}(T) = 2$, $\operatorname{null}(T) = 0$. **(e)** $\operatorname{rank}(T) = 2$, $\operatorname{null}(T) = 0$. **(f)** $\operatorname{rank}(T) = 3$, $\operatorname{null}(T) = 0$.
2. Set of all constant functions.
3(a) $N(T) = \{(a,a,-a) : a \in \mathbb{R}\}$, $R(T) = \{(a, a+b, b) : a,b \in \mathbb{R}\}$. **(b)** $S = \{(1,0,1) + x : x \in N(T)\}$.
4(a) $\operatorname{rank}(T) \leq \dim(V)$ and $R(T) \subseteq W$. **(b)** T is onto implies $\operatorname{rank}(T) = \dim(W)$. **(c)** T is one-one implies $\operatorname{rank}(T) = \dim(V)$. **(d)** Follows from (c). **(e)** Follows from (b).
5(a) $T(a,b) = (a-b, a-b)$. **(b)** $S(a,b) = (a,b)$, $T(a,b) = (b,a)$.
6. Let $\{u_1, \ldots, u_k\}$ be a basis for U. extend it to a basis $\{u_1, \ldots, u_k, v_1, \ldots, v_m\}$ for V.
(a) $T(u_i) = u_i$, $T(v_j) = 0$. **(b)** $T(u_i) = 0$, $T(v_j) = v_j$.
7. $R(T) = \operatorname{span}\{Tv_1, \ldots, Tv_n\}$. **(a)** T is one-one iff $R(T) = \dim(V)$ iff $\{Tv_1, \ldots, Tv_n\}$ is a basis of $R(T)$. Similarly, (b)-(c) follow. **8.** f is a linear transformation. f is one-one iff $N(f) = \{0\}$ iff $\alpha_1 v_1 + \cdots + \alpha_n v - n = 0$ implies $\alpha_1 = \cdots = \alpha_n = 0$.
9. $T(I-T) = (I-T)T$. Let $y \in R(T)$. Then for some x, $y = Tx$. So, $(I-T)Tx = (I-T)y = 0$. That is, $y \in N(I-T)$. Similarly other implications are proved.
10. $R(TS) \subseteq T(R(S))$. So, $\operatorname{rank}(TS) \leq \dim(R(S)) = \operatorname{rank}(S)$. And, $R(TS) \subseteq T(R(S)) \subseteq T(V) = R(T)$. So, $\operatorname{rank}(TS) \leq \operatorname{rank}(T)$.

§ 3.4

1. $T(v_i) = e_i$. $N(T) = \{0\}$ and $R(T) = \mathbb{F}^n$.
2. $T(a_0, a_1, \ldots, a_n) = (a_1 + a_2 t + \cdots + a_n t^{n-1} + a_0 t^n)$.
3. $R(T) = \operatorname{span}\{Tv_1, \ldots, Tv_n\}$. See Exercise 7 of § 3.3.
4. $\operatorname{null}(T) = \dim(V) - 1$ iff $\operatorname{rank}(T) = 1$ iff T is onto. **5.** Use Theorem 3.17.
6. (a) $\langle Tx, Tx \rangle = (Tx)^*(Tx) \geq 0$. (b) $\langle Tx, Tx \rangle = (Tx)^*(Tx) = 0$ iff $Tx = 0$ iff $x = 0$. Similarly, other conditions can be verified.
7(a) $ST = I_V$ implies T is one-one; then $\operatorname{null}(T) = 0$. $\dim(V) = \dim(W)$ implies $\operatorname{rank}(T) = \dim(V) = \dim(W)$. So, T is onto. Thus T is an isomorphism and S is its inverse.
(b) Define $T : \mathbb{R}^2 \to \mathbb{R}^3$, $S : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(a,b) = (a,b,0)$ and $S(a,b,c) = (a,b)$. Then $ST(a,b) = S(a,b,0) = (a,b)$. But $TS(a,b,c) = T(a,b) = (a,b,0)$.
8. $V = \mathbb{R}^\infty$; $T(a_1, a_2, a_3, \ldots) = (a_1, a_1, a_2, a_2, a_3, a_3, \ldots)$; $S(a_1, a_2, a_3, \ldots) = (a_1, a_3, a_5, \ldots)$.

§ **3.5**

1. $T^*\alpha - \alpha u$. **2.** $T^*(a_1,\ldots,a_n) = (a_2,\ldots,a_n,0)$.

3. $ST = TS = I \Rightarrow T^*S^* = S^*T^* = I$. So, T^* is invertible and its inverse is $S^* = (T^{-1})^*$.

4(a) (i) $U \subseteq V$, $U^\perp \subseteq V$; so $U + U^\perp \subseteq V$. (ii) Let $v \in V$. Let $\{v_1,\ldots,v_n\}$ be an orthonormal basis of V. Write $x = \sum_{j=1}^n \langle x, v_j\rangle v_j$; $y = v - x$. Then $\langle y, v_j\rangle = 0$. Hence $y \in U^\perp$. (iii) Let $x \in U \cap U^\perp$. Then $\langle x, x\rangle = 0$.
(b) $U \subseteq U^{\perp\perp}$. Let $x \in U^{\perp\perp}$. Using (a), $x = w + y$, for some $w \in U$ and $y \in U^\perp$. Then $\langle w, y\rangle = 0 \Rightarrow 0 = \langle x, y\rangle = \langle y, y\rangle$. Then $x = w \in U$.

5(a) $x \in N(T) \Rightarrow \langle Tx, y\rangle = 0 \Rightarrow \langle x, T^*y\rangle = 0\ \forall y \in W$. So, $x \in R(T^*)^\perp$. Also, $x \in R(T^*)^\perp \Rightarrow \langle x, T^*y\rangle = 0\ \forall y \in W \Rightarrow \langle Tx, y\rangle = 0\ \forall y \in W$. In particular, $\langle Tx, Tx,\rangle = 0$. So, $Tx = 0 \Rightarrow x \in N(T)$. Others are proved using T^* instead of T and Exercise 4.

6(a) $T : \mathbb{F}^2 \to \mathbb{F}^2$, $T(a,b) = (2a-3b, 3a-2b)$. $T^*(a,b) = (2a+3b, -3a+2b)$.
(b) The T in (a). **(c)** $T : \mathbb{F}^2 \to \mathbb{F}^2$, $T(a,b) = (-b,a)$, $T^*(a,b) = (b,-a)$. **(d)** $T = 2I$.

7. If $TT^* = T^*T = I$, then $\langle Tx, Tx\rangle = \langle x, x\rangle$. For the converse, use polarization identity in Exercise 6 of § 2.1 so that $\|Tx\| = \|x\|$ implies $\langle Tx, Ty\rangle = \langle x, y\rangle$. It gives $T^*T = TT^* = I$. **8.** $(1/3, 1/3, 1/3)$.

9. Let $\{v_1,\ldots,v_n\}$ be an orthonormal basis of V. Take $y = \sum_{i=1}^n \overline{f(v_i)}\, v_i$.

10. $\text{null}(T) = \text{null}(T^*T) = \text{null}(I) = 0$. Thus T is invertible. Then $T^*T = I \Rightarrow TT^*T = T \Rightarrow TT^* = I$.