Department of Mathematics, IIT Madras
Quiz-2 Solution MA2031 15/10/2018
Linear Algebra for Engineers
Time : 8:00-8:50, Marks: 20
Each question carries 4 marks. Answer all questions.

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1. Consider $\mathbb{R}^4$ as an inner product space with the inner product given by $\langle (a_1, a_2, a_3, a_4),\ (b_1, b_2, b_3, b_4)\rangle = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$.
   Let $W = \{x \in \mathbb{R}^4 : x \perp (1, -1, 0, 1),\ x \perp (1, -2, 1, 0)\}$. It is known that W is a subspace of $\mathbb{R}^4$. Find a basis for W.

   Sol: Let $(a, b, c, d) \in W$. Then $a-b+d = 0,\ a-2b+c = 0$. Solving, we get $d = -a+b,\ c = -a+2b$. Thus $W = \{(a, b, -a + 2b, -a + b) : a, b \in \mathbb{R}\} = \{a(1, 0, -1, -1) + b(0, 1, 2, 1) : a, b \in \mathbb{R}\} = \operatorname{span}\{(1, 0, -1, -1),\ (0, 1, 2, 1)\}$. Since $\{(1, 0, -1, -1),\ (0, 1, 2, 1)\}$ is lin. ind., it is a basis of W.

2. Consider the inner product space $\mathbb{R}_3[t]$ with the inner product as $\langle a_1 + a_2t + a_3t^2 + a_4t^3, b_1 + b_2t + b_3t^2 + b_4t^3\rangle = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$.
   Let $U = \operatorname{span}\{1 + t + t^2, t + t^2 + t^3\}$. Determine the best approximation of t^2 from U.

   Sol: Suppose $v = a(1 + t + t^2) + b(t + t^2 + t^3)$ is the best approximation. Then $t^2 - a(1+t+t^2) - b(t+t^2+t^3) \perp 1+t+t^2$ and $t^2 - a(1+t+t^2) - b(t+t^2t^3) \perp t+t^2+t^3$. That is, $3a + 2b = 1,\ 2a + 3b = 1$. Solving, we get $a = b = \frac{1}{5}$. Thus $v = \frac{1}{5}(1 + 2t + 2t^2 + t^3)$ is the required best approximation.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that satisfies $T(1, 1, 0) = (1, -2, 1),\ T(1, 0, 1) = (-1, 2, 1),\ T(0, 1, 1) = (2, 1, -1)$.
   For any $(a, b, c) \in \mathbb{R}^3$, determine $T(a, b, c)$.

   Sol: Suppose $(a, b, c) = \alpha(1, 1, 0) + \beta(1, 0, 1) + \gamma(0, 1, 1)$. Then $\alpha+\beta = a,\ \alpha+\gamma = b,\ \beta+\gamma = c$. Solving, we get $\alpha = \frac{1}{2}(a+b-c),\ \beta = \frac{1}{2}(a-b+c),\ \gamma = \frac{1}{2}(-a+b+c)$.
   $(a, b, c) = \frac{1}{2}(a + b - c)(1, 1, 0) + \frac{1}{2}(a - b + c)(1, 0, 1) + \frac{1}{2}(-a + b + c)(0, 1, 1)$.
   Therefore, $T(a, b, c) = \frac{1}{2}(a + b - c)(1, -2, 1) + \frac{1}{2}(a - b + c)(-1, 2, 1) + +\frac{1}{2}(-a + b + c)(2, 1, -1)$ $= (2b - a,\ \frac{1}{2}(-a - 3b + 5c),\ \frac{1}{2}(3a - b - c))$.

4. Define the linear transformation $T : \mathbb{R}_2[t] \to \mathbb{R}^4$ by $T(a + bt + ct^2) = (a + b,\ b + c,\ c + a,\ a + b + c)$. Determine bases for $R(T)$ and $N(T)$.

   Sol: $R(T) = \{(a + b,\ b + c,\ c + a,\ a + b + c) : a, b, c \in \mathbb{R}\}$
   $= \{a(1, 0, 1, 1) + b(1, 1, 0, 1) + c(0, 1, 1, 1) : a, b, c \in \mathbb{R}\}$
   $= \operatorname{span}\{(1, 0, 1, 1),\ (1, 1, 0, 1),\ (0, 1, 1, 1)\}$.
   The vectors $(1, 0, 1, 1),\ (1, 1, 0, 1)$ and $(0, 1, 1, 1)$ are lin. ind.
   So, a basis for $R(T)$ is $\{(1, 0, 1, 1),\ (1, 1, 0, 1),\ (0, 1, 1, 1)\}$.
   Let $a + bt + ct^2 \in N(T)$. Then $T(a + bt + ct^2) = (a + b,\ b + c,\ c + a,\ a + b + c) = (0, 0, 0, 0)$.
   Then $a + b = 0,\ b + c = 0,\ c + a = 0,\ a + b + c = 0$. Solving, we get $a = b = c = 0$. So, $N(T) = \{0\}$.
   Hence the basis of $N(T)$ is $\varnothing$.
   Aliter for basis of $N(T)$: One may use Rank-nullity theorem to get $\varnothing$ as a basis of $N(T)$.

5. Let $T : V \to W$ be a linear transformation. Let $\{v_1, \ldots, v_n\}$ be a basis of V. Prove that T is one-one if and only if the vectors $T(v_1), \ldots, T(v_n)$ are linearly independent.

   Sol: Suppose T is one-one. For liner independence of $Tv_1, \ldots, Tv_n$, suppose $c_1Tv_1 + \cdots c_nTv_n = 0$. Then $T(c_1v_1 + \cdots + c_nv_n) = 0$. As T is one-one, $c_1v_1 + \cdots + c_nv_n = 0$. Since $\{v_1, \ldots, v_n\}$ is a basis of V, $c_1 = \cdots = c_n = 0$. Hence the vectors $Tv_1, \ldots, Tv_n$ are lin. ind. Conversely, suppose T is not one-one. Then $N(T)$ contains a nonzero vector. Let $x \neq 0$ and $x \in N(T)$. There exist unique scalars a_i not all zero such that $x = a_1v_1 + \cdots + a_nv_n$. Then $Tx = 0$ implies $a_1Tv_1 + \cdots + a_nTv_n = 0$. Hence the vectors $Tv_1, \ldots, Tv_n$ are lin. dep.
   Aliter: Notice that $\dim(V) = n$ and $(R(T) = \operatorname{span}\{Tv_1, \ldots, Tv_n\})$. Thus, T is one-one iff $\operatorname{null}(T) = 0$ iff $\operatorname{rank}(T) = n$ iff $\dim\operatorname{span}\{Tv_1, \ldots, Tv_n\} = n$ iff $Tv_1, \ldots, Tv_n$ are lin. ind.
