Answers to Chapter 6

§ 6.1

1. Let λ be a diagonal entry of A. In $A-\lambda I$, the corresponding diagonal entry is 0. Look at the first 0 entry on the diagonal of $A-\lambda I$. (first from $(1,1)$th entry through (n,n)th.) If this occurs on the kth column, then the kth column is a linear combination of earlier columns. Thus $\operatorname{rank}(A-\lambda I) < n$. And, $\operatorname{null}(A-\lambda I) \geq 1$. Thus $Av = \lambda v$ for some $v \neq 0$.

2. For rows summing to α: $A[1 \ \cdots \ 1]^t = \alpha[1 \ \cdots \ 1]^t$. For columns summing to α, use $\operatorname{null}(A-\alpha I) = \operatorname{null}(A^t - \alpha I)$. **3(a)** $Tv = \lambda v \Rightarrow T^k v = \lambda^k v$. **(b)** $Tv = \lambda v \Rightarrow (T+\alpha I)v = (\lambda+\alpha)v$. **(c)** Follows from (a)-(b). Next, if $\mathbb{F} = \mathbb{C}$, then yes.

4. Let $Tv = \lambda v$. Then $T^{-1}Tv = v = (1/\lambda)\lambda v = (1/\lambda)Tv$, and $Tv \neq 0$ for $v \neq 0$.

5. Yes; $T = I$. **6.** Yes; $T = \lambda I$. **7.** x may not be a common eigenvector.

8. $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

§ 6.2

1(a) $(t-1)(t-2)$, $[1 \ -1]^t$, $[-2 \ 1]^t$. **(b)** $(t-i)(t+i)$, $[1 \ -2-i]^t$, $[1 \ i-2]^t$.
(c) $(t+2)(t-3)(t-5)$, $[5 \ 2 \ 0]^t$, $[0 \ 1 \ 0]^t$, $[3 \ -3 \ 7]^t$.
(d) $(t+1)(t-2)(t-3)$, $[1 \ -1 \ 1]^t$, $[1 \ 2 \ 4]^t$, $[1 \ 3 \ 9]^t$.

2. Eigenvalue is 0; eigenvector is 1. **3.** If $a = 0$, then A has lin. dep. columns. If $a \neq 0$, then A has lin. ind. columns. $\chi_A(t) = t^3 - ct^2 - bt - a$. Use Cayley-Hamilton theorem and verify that $A(1/a)(A^2 - cA - bI) = I$.

4. Let $\chi_A(t) = a_0 + a_1 t + \cdots + a_{n-1}t^{n-1} + t^n$. Then A is invertible iff $a_0 \neq 0$. And, $A = -(a_0)^{-1}(a_1 I + a_2 A + \cdots + a_{n-1}A^{n-2} + A^{n-1})$.

5. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has no real eigenvalue.

6. $x \perp y$. Now, $z := x \times y$ is orthogonal to both x and y. Then $\{x, y, z\}$ is an orthogonal basis of $\mathbb{R}^{3\times 1}$. Let $Ax = \alpha x$ and $Ay = \beta y$. Let $Az = ax + by + cz$. Then $\langle Az - cz, x\rangle = \langle Az, x\rangle - \langle cz, x\rangle = \langle z, Ax\rangle + 0 = \langle z, ax\rangle = 0$. Similarly, $\langle Az - cz, y\rangle = 0$. So, $\langle Az - cz, Az - cz\rangle = \langle Az - cz, ax + by\rangle = 0$. That is, $Az - cz = 0$.

§ 6.3

1. $\{(1/\sqrt{2}, -1/\sqrt{2})^t, \ (1/\sqrt{2}, 1/\sqrt{2})^t\}$; $[T] = \begin{bmatrix} -2 & 9 \\ 0 & 3 \end{bmatrix}$.

2(a) $P = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$; $U = \begin{bmatrix} 3 & -14 \\ 0 & 1 \end{bmatrix}$. **(b)** $P = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$;
$U = \begin{bmatrix} 2+i & -1+2i \\ 0 & 2-i \end{bmatrix}$. **(c)** $P = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & -2 & 2 \end{bmatrix}$; $U = \begin{bmatrix} 9 & 0 & 9 \\ 0 & 9 & 9 \\ 0 & 0 & 9 \end{bmatrix}$.

3. Notice that with $D = \operatorname{diag}(9,9,9)$, $(U-D)^2 = 0$. Using binomial theorem to compute $U^{50} = (D + (U - D))^{50}$ gives $U^{50} = D^{50} + 50D^{49}(U-D) = \begin{bmatrix} 9^{50} & 0 & 50\cdot 9^{50} \\ 0 & 9^{50} & 50\cdot 9^{50} \\ 0 & 0 & 9^{50} \end{bmatrix}$. Then $A^{50} = P^t U P = 9^{49}\begin{bmatrix} 209 & 400 & 400 \\ -50 & -91 & -100 \\ -50 & -100 & -91 \end{bmatrix}$.

4. Let $A = P^*UP$, where U is upper triangular and P is unitary. Now, $PA^*AP^* = PP^*U^*PP^*UPP = U^*U$. Then $\operatorname{tr}(A^*A) = \operatorname{tr}(PA^*AP^*) = \operatorname{tr}(U^*U) = \sum_i |u_{ii}|^2 + \sum_{i>j}|u_{ij}|^2 \geq \sum_i |u_{ii}|^2 = \sum_i |\lambda_i|^2$.

5. $|\det(A)| \leq (\Pi|\lambda_i|^2)^{1/2} \leq (\frac{1}{n}\sum|\lambda_i|^2)^{n/2} \leq (\frac{1}{n}\operatorname{tr}(A^*A))^{n/2} \leq (c^2 n)^{n/2}$.

6. $A - \lambda I = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$ where $B \in \mathbb{C}^{m\times m}$, $C \in \mathbb{C}^{m\times(n-m)}$, $0 \in \mathbb{C}^{(n-m)\times m}$ and $D \in \mathbb{C}^{(n-m)\times(n-m)}$. Further, B and D are upper triangular, all diagonal entries of B are 0, and each diagonal entry of D is nonzero. Due to these particular forms of B and D, we see that $B^m = 0$ and D^m is upper triangular with nonzero diagonal entries. Taking successive powers of $A - \lambda I$ we see that

$$(A-\lambda I)^m = \begin{bmatrix} B^m & \tilde{C} \\ 0 & D^m \end{bmatrix} = \begin{bmatrix} 0 & \tilde{C} \\ 0 & D^m \end{bmatrix}.$$

Here, $\tilde{C} \in \mathbb{C}^{m\times(n-m)}$ is some matrix. Clearly, $\text{rank}((A-\lambda I)^m)) = n-m$. Due to Rank-nullity theorem, $\text{null}((A-\lambda I)^m) = m$.

7. There exists an invertible $P \in \mathbb{C}^{n\times n}$ such that $B = P^{-1}AP$. Then $P^{-1}(A-\lambda I)P = B-\lambda I$. If $P^{-1}(A-\lambda I)^k P = (B-\lambda I)^k$, then $P^{-1}(A-\lambda I)^{k+1}P = P^{-1}(A-\lambda I)^k PP^{-1}(A-\lambda I)P = (B-\lambda I)^k(B-\lambda I) = (B-\lambda I)^{k+1}$. By induction we have $(A-\lambda I)^m$ is similar to $(B-\lambda I)^m$. So, $\text{null}((A-\lambda I)^m) = \text{null}((B-\lambda I)^m) = m$ by Exercise 6.