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Representation of linear transformations by Matrices

eg: $T : P_2 \to P_3$

$P_2 - B = \{1, 1+x, x^2\}$

$P_3 - B' = \{1, x, x^2, x^3\}$

Suppose,

$$T(\alpha_0 + \alpha_1 x + \alpha_2 x^2) = \alpha_2 x^3 + \alpha_1 x^2 + \alpha_0$$

$T(1) = 1 = 1(1) + 0(x) + 0(x^2) + 0(x^3)$

$T(1+x) = x^2+1 = 1(1) + 0(x) + 1(x^2) + 0(x^3)$

$T(x^2) = x^3 = 0(1) + 0(x) + 0(x^2) + 1(x^3)$

$\cancel{[T]}_{B,B}$ $[T]_{B,B'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{3\times 4}$

$F(T) = [T]_{B,B'}$

→ F is well defined function

, F is linear

, F is one-one

, F is onto

$F : \cancel{\,} L(V, W) \to R^{m\times n}$

→ Let $T : V \to V$

fix a basis B, $[T]_B$ ...

eg. 1) $T(x_1, x_2) = (x_1, 0)$

find $[T]_B$, where B is the standard basis.

Sol: $B = \{(1,0), (0,1)\}$

$T(1,0) = (1,0) = 1(1,0) + 0(0,1)$

$T(0,1) = (0,0) = 0(1,0) + 0(0,1)$

$\therefore [T]_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$

2) $D : P_3 \to P_3$, D is differentiation

$B = \{1, x, x^2, x^3\}$, find $[D]_B$

Sol:

$D(1) = 0 = 0(1) + 0(x) + 0(x^2) + 0(x^3)$

$D(x) = 1 = 1(1) + 0(x) + 0(x^2) + 0(x^3)$

$D(x^2) = 2x = 0(1) + 2(x) + 0(x^2) + 0(x^3)$

$D(x^3) = 3x^2 = 0(1) + 0(x) + 3(x^2) + 0(x^3)$

$[D]_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}_{4\times 4}$

Theorem:

$T : V \to V$

Let $\text{Dim}(V) = n$

fix a basis B, then

$[Tv]_B = [T]_B' [v]_B$

Pf:

let, $B = \{v_1, v_2, \dots, v_n\}$

fix a vector $v \in V$

let, $v = \alpha_1 v_1 + \dots + \alpha_n v_n$

$[v]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$

$Tv_1 = \omega_1$

$= a_{11} v_1 + \dots + a_{n1} v_n$

$$Tv_2 = a_{21}v_1 + \cdots + a_{2n}v_n$$

$$\vdots$$

$$Tv_n = a_{n1}v_1 + \cdots + a_{nn}v_n$$

$$\therefore [T]_B = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$Tv = \alpha_1 Tv_1 + \alpha_2 Tv_2 + \cdots + \alpha_n Tv_n$$

$$[Tv]_B = \begin{bmatrix} \alpha_1 a_{11} + \alpha_2 a_{21} + \cdots + \alpha_n a_{n1} \\ \alpha_1 a_{12} + \alpha_2 a_{22} + \cdots + \alpha_n a_{n2} \\ \vdots \\ \alpha_1 a_{1n} + \alpha_2 a_{2n} + \cdots + \alpha_n a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}_{n\times n} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}_{n\times 1}$$

$$[Tv]_B = [T]_B' [v]_B$$

→ Let, $T: V \to V$ be invertible. fix B, then $[T]_B$ is non-singular.

Proof: let $[T]_B$ is singular, then
Suppose, $[T]_B' \bar{x} = 0$ $(0 \neq x \in \mathbb{R}^n)$

$$B = \{v_1, v_2, \ldots, v_n\}$$

Consider, $v = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$

Where, $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$[Tv]_B = [T]_B' [v]_B$$

$$= [T]_B' \bar{x} = 0$$

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$\Rightarrow Tv = 0 \Rightarrow v = 0$ [$\because T$ is invertible]

But, if $v = 0$, $\tilde{\alpha} = 0$

$\hookrightarrow$ Contradiction

$\therefore [T]_B$ is non-singular

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$Dim(V) = n$

$T: V \to V$ linear

fix $B = \{v_1, \dots, v_n\}$

$[T]_B$; T is invertible $\Leftrightarrow$ is non-singular

$\rightsquigarrow$ Assume $[T]_B$ is non-singular

To prove: T is invertible

Pf: Let $Tv = 0$

$$v = \sum_{i=1}^{n} \alpha_i v_i$$

$$[v]_B = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$0 = [Tv]_B = [T]_B' [v]_B$$

non-singular

So, $[v]_B = 0 \Rightarrow v = 0$

$\to T^{-1}: V \to V$ is linear

Qn: $[T^{-1}]_B = ?$

$[T^{-1}]_B = [T_B]^{-1}$

$T(x_1, x_2) = (x_1, x_1 + x_2)$; T is 1-1

$B = \{(1,0), (1,1)\}$

$T(1,0) = (1,1)$; $[T]_B = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$

$T(1,1) = (1,2)$

$[T]_B^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

$T^{-1}(x_1, x_2) = (x_1, x_2 - x_1)$

$T^{-1}(1,0) = (1,-1) = 2(1,0) - 1(1,1)$

$T^{-1}(1,1) = (1,0) = 1(1,0) + 0(1,1)$

$[T^{-1}]_B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ → same

$T^{-1}v_1 = b_{11}v_1 + b_{12}v_2 + \cdots + b_{1n}v_n$

$T^{-1}v_2 = b_{21}v_1 + b_{22}v_2 + \cdots + b_{2n}v_n$

$\vdots$

$T^{-1}v_n = b_{n1}v_1 + b_{n2}v_2 + \cdots + b_{nn}v_n$

$$[T^{-1}]_B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

Prove $[T^{-1}]_B = [T]_B^{-1}$

Proof: $T^{-1}(Tv_1) = v_1 = T^{-1}(a_{11}v_1 + \cdots + a_{1n}v_n)$

$v_1 = a_{11}(T^{-1}v_1) + a_{12}(T^{-1}v_2) + \cdots + a_{1n}(T^{-1}v_n)$

$v_1 = a_{11}(b_{11}v_1 + b_{12}v_2 + \cdots + b_{1n}v_n) + \cdots + a_{1n}(b_{n1}v_1 + \cdots + b_{nn}v_n)$

$= (a_{11}b_{11} + \cdots + a_{1n}b_{n1})v_1 + \cdots + (\ldots)v_n$

$= 1 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n = v_1$

unique linear combinations

(or) $[T(T^{-1}v)]_B = [T_B]^T [T^{-1}v]_B$

$[v]_B = [T]_B^T [T^{-1}]_B^T [v]_B$ (linear)

$\tilde{x} = PQ\tilde{x}$

$$PQ = I \Rightarrow [T]_B [T^{-1}]_B = I$$

$$[T]_B = [T]_B^{-1}$$

→ $S, L : V \to V$ be two linear transformation

$$I = [T \cdot T^{-1}]_B = [T]_B [T^{-1}]_B$$

$$[S \circ L]_B = \cancel{[S]_B} [L]_B [S]_B$$

<u>Theorem:</u> Let $S, L : V \to V$ be linear, fix a basis B. then $[S \circ L]_B = [L]_B [S]_B$

<u>Proof:</u>

$$[S(L(v))] = [S \circ L(v)]_B$$

$$= [S(w)]_B$$

$$= [S]_B^T [w]_B$$

$$= [S]_B^T [L]_B^T [v]_B$$

$$[A]^T [v]_B = [Av]_B$$

$$[A]^T = [S]_B^T [L]_B^T$$

$$[S \circ L]_B = [A] = [L]_B [S]_B$$

→ $\text{Dim}(V) = n$

$$T : V \to V$$

fix B, B', $[T]_B ; [T]_{B'}$

there exists Q, an invertible $n \times n$ matrix such that, $[T]_B = Q [T]_{B'} Q^{-1}$

($[T]_B, [T]_{B'}$ are similar

$\text{Det}[T]_B = \text{Det}[T]_{B'}$)

Thm: $\text{Dim}(V) = n$

$B', B \rightarrow$ Basis

$T: V \to V$ is linear

$[T]_B = P[T]_{B'} P^{-1}$ for some P.

Pf:

Given, $B = \{v_1, \dots, v_n\}$

$B' = \{w_1, \dots, w_n\}$

To prove: $\exists$ a non-singular $n \times n$ matrix P, s.t.

$$[T]_B = P[T]_{B'} P^{-1}$$

Proof:

$$Tv_1 = a_{11} v_1 + \dots + a_{1n} v_n$$

$$Tv_2 = a_{21} v_1 + \dots + a_{2n} v_n$$

$$\vdots$$

$$Tv_n = a_{n1} v_1 + \dots + a_{nn} v_n$$

$$[T]_B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}_{n \times n} = A$$

Ily, let $[T]_{B'} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & & & \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix}_{n \times n} = L$

We know that,

If B, B' be bases, then $\exists$ an $n \times n$ matrix P, such that, $P[v]_B = [v]_{B'}$

$$\therefore [Tv]_B = [T]_B^T [v]_B = A^T [v]_B$$

$$= A^T Q [v]_{B'}$$

[Q is $n \times n$ matrix and non-singular]

Ily, $[Tv]_{B'} = L^T [v]_{B'}$

$$[Tv]_B = Q[Tv]_{B'} = QL^T [v]_{B'}$$

$\therefore A^T Q [V]_{B'} = Q L^T [V]_{B'}$

$\Rightarrow A^T Q = Q L^T$

$\Rightarrow A^T = Q L^T Q^{-1}$

$\Rightarrow A = Q^T L (Q^{-1})^T \Longrightarrow$ Hence proved

eg: 1) $T(x_1, x_2) = (x_1, 0)$

$B = \{(1,0), (0,1)\}$

$B' = \{(1,1), (2,1)\}$

S.T. $[T]_B$ and $[T]_{B'}$ are similar.

Sol:- $[T]_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$[T]_{B'} = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$

$[T]_B$, $[T]_{B'}$ are similar

$\rightarrow \exists Q$, s.t. $[T]_B = Q[T]_{B'} Q^{-1}$

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} Q = Q \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$

let, $Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$

$\Rightarrow a = -a - 2b \Rightarrow 2b = -2a \Rightarrow a = -b$

$b = a + 2b \Rightarrow a + b = 0$

$-c - 2d = 0 \Rightarrow -c = 2d$

$Q = \begin{bmatrix} a & -a \\ c & -c/2 \end{bmatrix}$; Take $a = 1$, $c = -2$

$\therefore Q = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$

$\therefore [T]_B$ and $[T]_{B'}$ are similar

2) let $\text{Dim}(V) = 2$

$T: V \to V$ is linear

$[T]_B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

P.T. $T^2 - (a+d)T + (ad-bc)I = 0$

Sol

$T^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$

$(a+b)T = \begin{bmatrix} a^2+ab & ab+b^2 \\ ac+bc & ad+bd \end{bmatrix}$

$\therefore T^2 - (a+b)T = \begin{bmatrix} bc-ab & bd-b^2 \\ bc-cd & \end{bmatrix}$ X

To prove: $[L]_B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $L = T^2 - (a+b)T + (ad-bc)I$

$\therefore [T^2 - (a+d)T + (ad-bc)I]_B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow [T^2]_B - (a+d)[T]_B + (ad-bc)[I]_B$

$= [T]_B^2 - (a+d)[T]_B + (ad-bc)I$

$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} - \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore T^2 - (a+d)T + (ad-bc)I = 0$

7. ↳ Given, A and B are similar
prove that $Tr(A) = Tr(B)$
↓
Trace

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Linear functionals

V is a vector space.

Any $T: V \to \mathbb{R}$ which is linear is called linear functional.

eg: 1. $T(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n$

2. $T(x) = 0, \ \forall x \in V$

3. $V = \mathbb{R}^{n \times n}$ and $f(A) = Tr(A)$

$g(A) = \sum_{ij} a_{ij}$

4. $V = C[a, b]$

$= \{f : [a,b] \to \mathbb{R} : f \text{ is continuous}\}$

$Tf = \int_a^b f(t)\, dt$

$L(V, \mathbb{R}) = V'$ space of all linear functionals defined on V.

let, $Dim(V) = n$,

then $Dim(L(V,\mathbb{R})) = Dim(V) = n$

$L(V, \mathbb{R}) =$ Dual-space of V.

Dual basis:

fix a basis for V. i.e $\{v_1, v_2, \dots, v_n\}$

$f'(v_1) = 1$

$f'(v_i) = 0 \quad$ if $i > 1$ $\quad$ [where $f'(x)$ is a $f': V \to \mathbb{R}$

i.e $f'(x) = f'(\sum_{i=1}^{n} \alpha_i v_i)$

$= \alpha_1 f'(v_1)$

$= \alpha_1$

Ily, let $f^2(v_1) = 0$

$f^2(v_2) = 1$

$f^2(v_3) = 0$

$\vdots$

$f^2(v_n) = 0$

Now there are n linear functionals

$f^1, f^2, \ldots, f^n$

Theorem: $\{f^1, f^2, \ldots, f^n\}$ is an ordered basis for $L(V, \mathbb{R})$

Proof: Suppose, $y_1 f^1 + y_2 f^2 + \cdots + y_n f^n = 0$

let, $y_1 f^1 + y_2 f^2 + \cdots + y_n f^n = T$

$T(v_1) = y_1 = 0$ [$\because$ T is a zero ~~transformation~~ linear functional]

$\vdots$

$T(v_n) = y_n = 0$

$\therefore \{f^1, f^2, \ldots, f^n\}$ is an ordered basis

Also called Dual basis.

→ let $g \in L(V, \mathbb{R})$

1. then, $g = c_1 f^1 + c_2 f^2 + \cdots + c_n f^n$

where, $c_1 = g(v_1)$

$c_2 = g(v_2)$

$\vdots$

$c_n = g(v_n)$.

2. $x \in V \Rightarrow$ ~~$x = c_1' f^1 + c_2' f^2 + c_3' f^3 + \ldots$~~

$\Rightarrow x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

where, $\alpha_1 = f^1(x)$

$\alpha_2 = f^2(x)$ and so on

IR]

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→ let, $V = P_2$

Consider, t_1, t_2, t_3 $(t_1 \neq t_2 \neq t_3)$

s.t. (Such that) $L_1(P) = P(t_1)$

$L_2(P) = P(t_2)$

$L_3(P) = P(t_3)$

$\Rightarrow L_1(a_2t^2 + a_3t + a_4) = a_2t_1^2 + a_3t_1 + a_4$

$L_1(P) = P(t_1)$

$L_1(Q) = Q(t_1)$

$\therefore L_1(P+Q) = (P+Q)(t_1) = P(t_1) + Q(t_1)$

$= L_1(P) + L_1(Q)$

$L_1(\alpha P) = \alpha P(t_1)$

$= \alpha L_1(P)$

$\therefore L_1, L_2, L_3$ are linear functionals

→ prove that $\{L_1, L_2, L_3\}$ is a Dual basis for $L(P_2, \mathbb{R})$

__Proof:__ Suppoce $\alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3 = 0$

i.e $\alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3$ is a zero linear functional

Basis for $P_2 = \{1, t, t^2\}$

$\therefore \alpha_1 L_1(1) + \alpha_2 L_2(1) + \alpha_3 L_3(1) = 0$

$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = 0$

$\alpha_1 L_1(t) + \alpha_2 L_2(t) + \alpha_3 L_3(t) = 0$

$\Rightarrow \alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_3 = 0$

$\alpha_1 L_1(t^2) + \alpha_2 L_2(t^2) + \alpha_3 L_3(t^2) = 0$

$\Rightarrow \alpha_1 t_1^2 + \alpha_2 t_2^2 + \alpha_3 t_3^2 = 0$

$$\therefore \quad \begin{array}{c} A \\ \begin{pmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \end{pmatrix} \end{array} \begin{array}{c} \alpha \\ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \end{array} = \begin{array}{c} 0 \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{array}$$

If $t_1 \neq t_2 \neq t_3$, $\text{Det}(A) \neq 0$

So, A is non-singular

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\therefore \{L_1, L_2, L_3\}$ is a Dual basis for $L(P_2, R)$

Qn: find a basis B for P_2 such that B^* is a dual basis corresponding to B.

Sol: Suppose $\{v_1, v_2, v_3\} = B$

Consider,

$L_1(v_1) = 1$;	$L_2(v_1) = 0$;	$L_3(v_1) = 0$
$L_1(v_2) = 0$	$L_2(v_2) = 1$	$L_3(v_2) = 0$
$L_1(v_3) = 0$	$L_2(v_3) = 0$	$L_3(v_3) = 1$
$\Downarrow$	$\Downarrow$	$\Downarrow$
$v_1(t_1) = 1$	$v_1(t_2) = 0$	$v_1(t_3) = 0$
$v_2(t_1) = 0$	$v_2(t_2) = 1$	$v_2(t_3) = 0$
$v_3(t_1) = 0$	$v_3(t_2) = 0$	$v_3(t_3) = 1$

$$\Rightarrow v_1 = \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)}$$

$$v_2 = \frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)}$$

$$v_3 = \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)}$$

$\rightsquigarrow$ If $\{v_1, v_2, v_3\}$ is a basis in P_2 and $\{L_1, L_2, L_3\}$ is a dual basis for $L(P_2, R)$.

then for any $p \in P_2$; $P = p(t_1)v_1 + p(t_2)v_2 + p(t_3)v_3$

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Inner product

$\mathbb{R}^2$ or $\mathbb{R}^3$

dot product of $\begin{pmatrix} 1+i \\ 1 \end{pmatrix} \cdot \begin{pmatrix} i \\ -1 \end{pmatrix} = (1+i)(-i) + 1(-1)$

Conjugate v.s

V is real or complex

Innerproduct in real VS: $f : V \times V \to \mathbb{R}$

f is called a innerproduct if

(1) $f(u,u) \geq 0$; $\forall u \in V$

(2) $f(u,u) = 0$; iff $u = 0$

(3) $f(u, \alpha v) = f(\alpha u, v) = \alpha f(u,v) = \alpha f(v,u)$
$(\forall \alpha \in \mathbb{R}$; $u, v \in V)$

(4) $f(u, v+w) = f(u,v) + f(u,w)$
$\forall u, v, w \in V$

eg: $\mathbb{R}^n$

(let $x, y \in \mathbb{R}^n$. Define $f(x,y) = x_1y_1 + x_2y_2 + \cdots + x_ny_n$
'f' is standard inner product.)

→1) $V = \mathbb{R}^n$

$f(x,y) = 2x_1y_1 + 3x_2y_2 + x_3y_3 + \cdots + x_ny_n$

Sol: i) $f(u,u) \geq 0$

ii) $f(u,u) = 0 \Rightarrow 2u_1^2 + 3u_2^2 + \cdots + u_n^2 = 0$
$\Rightarrow u = 0$

iii) $f(x, \alpha y) = 2x_1(\alpha y_1) + 3x_2(\alpha y_2) + \cdots + x_n(\alpha y_n)$
$= \alpha [f(x,y)] = f(\alpha x, y)$

2) $V = \mathbb{R}^{n \times n}$

$f(A,B) = \text{Trace}(AB^T)$

Prove that it is innerproduct

Let $A \in \mathbb{R}^{n\times m}$

$f(A,A) = \text{Trace}(AA^T) \geq 0$

If $f(A,A)=0 \Rightarrow A=0$

||ly for (iii), (iv)

3) $V = P_n$ (Polynomial)

let $x, y \in P_n$

$x(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$

$y(t) = b_n t^n + b_{n-1} t^{n-1} + \cdots + b_1 t + b_0$

$f(x,y) = a_n b_n + a_{n-1} b_{n-1} + \cdots + a_0 b_0$

Verify if it is inner product

4) $f(x,y) = \int_0^1 x(t)\, y(t)\, dt$

Verify that f is a inner product

Pf: i) $f(x,x) = \int_0^1 x^2(t)\, dt \geq 0$

ii) $f(x,x) = 0 = \int_0^1 x^2(t)\, dt \Rightarrow x(t) = 0$

iii) $f(x, y+z) = \int_0^1 x(t)\, [y(t) + z(t)]\, dt$

$= \int_0^1 x(t)\, y(t)\, dt + \int_0^1 x(t)\, z(t)\, dt$

$= f(x,y) + f(x,z)$

→ Complex inner product:

let V be a complex V.S

A function $f: V \times V \to \mathbb{C}$ is a inner product if

1) $f(u,u) \geq 0$; $\forall\, u \in V$

2) $f(u,u) = 0$ iff $u = 0$

3) $f(u,v) = \overline{f(v,u)}$; $u, v \in V$

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4) $f(\alpha u, v) = \alpha f(u,v)$; $\forall \alpha \in \mathbb{C}$

$$f(x, \alpha y) = \overline{f(\alpha y, x)} = \overline{\alpha f(y,x)}$$
$$= \bar{\alpha}\, \overline{f(y,x)}$$
$$= \bar{\alpha} f(x,y)$$

Engi

Pd50

5) $f(u+v, w) = f(u,w) + f(v,w)$ $\forall u, v, w \in V$

eg: $\mathbb{C}^n$

1) Define: $f(x,y) = x_1\bar{y_1} + x_2\bar{y_2} + \cdots + x_n\bar{y_n}$

Pf: 1) $f(x,x) \geq 0$ $(\because x \cdot \bar{x} = |x|^2)$

2) $f(x,x) = 0 \Rightarrow x = 0$

3) $f(x,y) = \overline{f(y,x)}$

4) $f(\alpha x, y) = \alpha f(x,y)$

5) $f(x, y+w) = f(x,y) + f(x,w)$

fd

2) $V = \mathbb{C}^{n\times n}$; $B^* = \overline{[B^T]}$

$f(A,B) = Tr(AB^*)$

3) $V = P_n(\mathbb{C})$; $x = a_0 + a_1 t + \cdots + a_n t^n$

$y = b_0 + b_1 t + \cdots + b_n t^n$

$f(x,y) = \sum_{i=1}^{n} a_i \bar{b_i}$

↳ Norm (length) : let ! $(V, \langle\cdot,\cdot\rangle)$ be an inner product Space (I.P.S). both real or complex.

let $x \in V$; Norm of $x = \|x\|$

$= \sqrt{\langle x, x\rangle}$

↳ $\|\alpha x\| = |\alpha| \|x\|$

↳ $\|x+y\| \leq \|x\| + \|y\|$

$\|x+y\| \leq \|x\| + \|y\|$; $(x, y \in V)$

$$\|x+y\|^2 = \langle x+y, x+y \rangle$$
$$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$$
$$= \|x\|^2 + \|y\|^2 + (\langle x,y\rangle + \langle y,x\rangle)$$
$$= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(\langle x,y\rangle)$$

→ Cauchy-schwarz theorem:

$$|\langle x,y\rangle| \leq \|x\|\,\|y\|$$ (If $x=0$ or $y=0$, then inequality is true. Now $x\neq 0; y\neq 0$)

Pf: let $t \in C$ (or R)

$$\langle x-ty, x-ty\rangle = \langle x,x\rangle + t^2\langle y,y\rangle - \bar{t}\langle x,y\rangle - t\langle y,x\rangle$$
$$= \|x\|^2 + |t|^2\|y\|^2 - \bar{t}(\langle x,y\rangle + t\langle y,x\rangle)$$

$$\|x\|^2 + t^2\|y\|^2 - \bar{t}(\langle x,y\rangle) + (t\langle y,x\rangle) \geq 0 \; ; \forall t$$ ($\because$ Norm is always non-negative)

$$\Rightarrow \|x\|^2 + t^2\|y\|^2 \geq 2t\,(\mathrm{Re}(\langle x,y\rangle))$$

$$t = \frac{\langle x,y\rangle}{\|y\|^2}$$

$$\Rightarrow \|x\|^2 + \frac{\langle x,y\rangle^2}{\|y\|^2} \geq \frac{\langle x,y\rangle}{\|y\|^2}(\langle x,y\rangle + \langle y,x\rangle)$$

$$\Rightarrow \|x\|^2\|y\|^2 \geq (\langle x,y\rangle\langle y,x\rangle)$$

$$\Rightarrow \|x\|^2\|y\|^2 \geq \langle x,y\rangle\overline{\langle x,y\rangle}$$

$$\Rightarrow |\langle x,y\rangle|^2 \leq \|x\|^2\|y\|^2$$

$$\Rightarrow |\langle x,y\rangle| \leq \|x\|\,\|y\|$$

→ Cauchy-Schwarz theorem:

$$|\langle x,y\rangle| \le \|x\|\,\|y\|$$

If $x=0$ or $y=0$, then inequality is true,
Now $x\neq 0$ and $y\neq 0$)

Pf: $t\in \mathbb{C}$ (or $\mathbb{R}$)

$$\langle x-ty, x-ty\rangle = \langle x,x\rangle + |t|^2\langle y,y\rangle - \bar{t}\langle x,y\rangle - t\langle y,x\rangle$$

$$= \|x\|^2 + |t|^2\|y\|^2 - \bar{t}\langle x,y\rangle - t\langle y,x\rangle \ge 0;\ \forall t$$

[∵ norm is non-negative]

$$t = \frac{\langle x,y\rangle}{\|y\|^2}$$

$$\Rightarrow \|x\|^2 + \frac{|\langle x,y\rangle|^2}{\|y\|^2} \ge \frac{|\langle x,y\rangle|^2}{\|y\|^2} + \frac{|\langle x,y\rangle|^2}{\|y\|^2}$$

$$\Rightarrow \|x\|^2\|y\|^2 \ge |\langle x,y\rangle|^2$$

$$\Rightarrow |\langle x,y\rangle| \le \|x\|\,\|y\|$$

Hence proved

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→ Equality holds iff x and y are L.D

[∵ $\langle x-ty, x-ty\rangle = 0$
$\Rightarrow x = ty$]

eg: 1. Consider, $V = \{f:\mathbb{R}\to\mathbb{R} : f \text{ is continuous}\}$

$$\therefore |\langle f,g\rangle| = \left|\int_0^1 f(t)\cdot g(t)\,dt\right| \le \left(\int_0^1 f^2(t)\,dt\right)^{1/2}\left(\int_0^1 g^2(t)\,dt\right)^{1/2}$$

2. $\mathbb{R}^n$; $x = (x_1, x_2, \dots x_n)$
$y = (y_1, y_2, \dots y_n)$

$$\langle x,y\rangle = \sum_{i=1}^{n} x_i y_i$$

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$$\left|\left(\sum x_i y_i\right)\right| \le \sqrt{\sum x_i^2}\sqrt{\sum y_i^2}$$

Def^n: **Orthogonal Vectors:**

Let $x, y \in V$. If $\langle x, y\rangle = 0$, then x and y are orthogonal vectors (o.g).

→ x is o.g to itself $\Leftrightarrow x = 0$

Thm: Let $\{v_1, v_2, \dots, v_m\}$ be o.g. Let $v_i \neq 0$ for any i, then $v_1, v_2, \dots v_m$ are L.I.

Pf: ($\{v_1, v_2, \dots, v_m\}$ is o.g $\Rightarrow$ ~~Any~~ Any two vectors are o.g to each other)

Suppose, $c_1 v_1 + c_2 v_2 + \dots + c_m v_m = 0$

$\Rightarrow \langle c_1 v_1 + c_2 v_2 + \dots + c_m v_m, v_1\rangle = 0$

$\Rightarrow \langle c_1 v_1, v_1\rangle = 0$

$\Rightarrow c_1 \langle v_1, v_1\rangle = 0$

$\because \langle v_1, v_1\rangle \neq 0 \Rightarrow c_1 = 0$

lly, $c_2 = c_3 = \dots = c_m = 0$

$\therefore v_1, v_2, \dots, v_m$ are L.I.

→ Span$\{v_1, v_2, \dots, v_m\}$ be o.g; and $\|v_i\| = 1$

let $x \in$ Span$\{v_1, \dots, v_m\}$

$x = k_1 v_1 + k_2 v_2 + \dots + k_m v_m$

$= \langle x, v_1\rangle v_1 + \langle x, v_2\rangle v_2 + \dots + \langle x, v_m\rangle v_m$

→ Orthonormal basis for V :

Let Dim(V) = n, then

$\{v_1, \dots, v_n\}$ is called an o.n. basis for V, if $\{v_1, \dots, v_n\}$ is o.g set and if $\|v_i\| = 1$ $\forall i$.

eg: 1. O.n basis for R^2 is

$$\left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\}$$

in general $\{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}$

2. $R^3 \to \{(\cos\theta, \sin\theta, 0), (-\sin\theta, \cos\theta, 0), (0,0,1)\}$

→ O.n. basis for span$\{x, y\}$; x, y are L.I

$$\left\{ x, \frac{1}{\|u\|}\left(y - \frac{\langle y, x\rangle}{\langle x, x\rangle} x\right) \right\}$$

$$u = y - \frac{\langle y, x\rangle}{\langle x, x\rangle} x$$

→ Consider span$\{x, y, z\}$, x, y, z are L.I

let $\left\{x, \frac{u}{\|u\|}, \frac{v}{\|v\|}\right\}$ is o.n basis

$$\therefore u = y - \frac{\langle y, x\rangle}{\langle x, x\rangle} x$$

$$v = z - \frac{\langle z, x\rangle}{\langle x, x\rangle} x - \frac{\langle z, u\rangle}{\langle u, u\rangle} u$$

$$= z - \frac{\langle z, x\rangle}{\|x\|^2} x - \frac{\langle z, u\rangle}{\|u\|^2} u$$

for span of $\{x, y, z, w\}$

let o.n basis $= \{x, u, v, m\}$

$$u = y - \frac{\langle y, x\rangle}{\|x\|^2} x$$

$$v = z - \frac{\langle z, x\rangle}{\|x\|^2} x - \frac{\langle z, u\rangle}{\|u\|^2} u$$

$$m = w - \frac{\langle w, x\rangle}{\|x\|^2} x - \frac{\langle w, u\rangle u}{\|u\|^2} - \frac{\langle w, v\rangle}{\|v\|^2} v$$

(Gram Schmidt process)

eg: Consider, span$\{(3,0,4), (-1,0,7), (2,9,11)\}$

o.n basis is $\left\{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|}\right\}$

$$v_1 = (3,0,4)$$

$$v_2 = (-1,0,7) - \frac{\langle (3,0,4), (-1,0,7)\rangle}{\|v_1\|^2} v_1$$

$$= (-1,0,7) - \frac{25}{25}(3,0,4) = (-4,0,3)$$

$$v_3 = (2,9,11) - \left(\frac{(2,9,11)\cdot(3,0,4)}{(25)}\right)(3,0,4) - \frac{(2,9,11)\cdot(-4,0,3)}{25}(-4,0,3)$$

$$= (2,9,11) - (6,0,8) - (-4,0,3)$$

$$= (0,9,0)$$

$\therefore$ o.n basis $= \left\{\frac{1}{5}(3,0,4); \frac{1}{5}(-4,0,3); \frac{1}{9}(0,9,0)\right\}$

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→ let $w \in V$, if $\{v_1, v_2, \dots, v_n\}$ is an orthogonal set and if $\|v_i\| = 1$, then

$$u = w - \langle w, v_1\rangle v_1 - \langle w, v_2\rangle v_2 - \dots \langle w_n, v_n\rangle v_n$$

is orthogonal to v_i for each i.

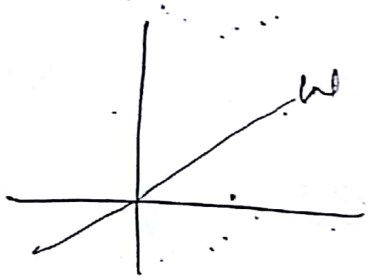
Pf: To P.T $u \perp v_i$, $\forall i$

$$\langle u, v_i\rangle = \langle w, v_i\rangle - \langle\langle w, v_i\rangle v_i, v_i\rangle$$
$$= \langle w, v_i\rangle - \langle w, v_i\rangle \|v_i\|^2$$
$$= 0$$

→ let $W \subseteq V$ be a subspace (or any non-empty set)

$$W^{\perp} = \{x \in V : \langle x, u\rangle = 0 \ \forall u \in W\}$$

$W^{\perp} \equiv$ Orthogonal complement of W.



$W = \text{Span}\{(1,1,0)\}$

$W^{\perp} = \text{Span}\{(0,0,1), (1,-1,0)\}$

→ $W^{\perp}$ is always subspace

Pf: let $x, y \in W^{\perp}$

$\Rightarrow \langle x, u\rangle = 0$, $\forall u \in W$

and $\langle y, u\rangle = 0$, $\forall u \in W$

$\Rightarrow \langle x+y, u\rangle = 0$; $u \in W$

(subspace)

Scalar multiplication:

let $x \in W^{\perp}$ and $\alpha \in \mathbb{C}$ (or $\mathbb{R}$)

$$\langle \alpha x, u\rangle = \alpha\langle x, u\rangle = 0$$

$W \cap W^{\perp} = (0)$

let. $x \in W \cap W^{\perp}$

$\langle x, x \rangle = 0$

only possible way $x = 0$

$\rightarrow$ $W = \text{Span}\{(1,0)\}$

$W^{\perp} = \text{Span}\{(0,1)\}$

$(x, y) = x(1,0) + y.(0,1)$

Theorem: let $\dim(V) = n$

Suppose W is a subspace of V, then

$v \in V \Rightarrow v = w + u$, where $w \in W$
$u \in W^{\perp}$

Further if. $v = x + y$, where $x \in W$ and
$y \in W^{\perp}$, then $x = w$, $y = u$

Note: Any Vector space has finite dim of orthonormal basis

Pf: let $\{w_1, w_2, \ldots, w_n\}$ be an orthonormal basis for W.

let $v \in V$, $P = v - u$

Put, $P = v - \langle v, w_1 \rangle w_1 - \langle v, w_2 \rangle w_2 - \cdots$
$\langle v, w_m \rangle v_m$

P is orthogonal to $w_1, w_2, \ldots, w_m$

$\Rightarrow P \in W^{\perp}$.

Suppose, $v = x + y$, $x \in W$, $y \in W^{\perp}$
$v = w + u$, $w \in W$, $u \in W^{\perp}$

$\Rightarrow 0 = (x - w) + (y - u)$

$\langle x - w, y - u \rangle = 0$

a, b orthogonal $\Rightarrow$ $\|a-b\|^2 = \|a+b\|^2 = \|a\|^2 + \|b\|^2$

$$0 = \|(x-w)+(y-u)\|^2$$
$$= \|x-w\|^2 + \|y-u\|^2$$
$$\Rightarrow x = w ;\ y = u$$

$\hookrightarrow$ Assume V is a real vector space

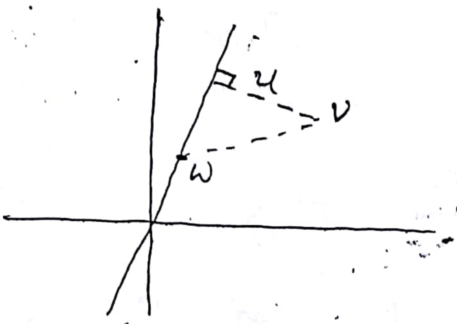
let W be a subspace

let $v \in V$

Best approximation of v in W is $u \in W$ iff

$\|v-u\| \leq \|x-u\|$. for any $x \in W$

$\rightarrow$ $\|v-u\|$ is min



5

Suppose u is the best approximation of v is W

then $\langle v-u, y\rangle = 0$, $\forall y \in W$

Pf: let $y \in W$

$$\langle v-u, v-u\rangle = \|v-u\|^2$$
$$\leq \|v-w\|^2 \text{ for any } w \in W$$

$$\|v-u\|^2 \leq \|v\|^2 - 2\langle v, w\rangle + \|w\|^2 \text{, for any } w.$$

$$w = u+y \in W$$

$$\|v-w\|^2 = \langle v-(u+y), v-(u+y)\rangle$$
$$= \langle (v-u)-y ; (v-u)-y\rangle$$
$$= \langle v-u, v-u\rangle + (-2\langle v-u, y\rangle + \|y\|^2)$$

$\therefore$ u is the best approximation

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$W \cap W^{\perp} = (0)$

let $x \in W \cap W^{\perp}$

$\langle x, x \rangle = 0$

only possible way $x = 0$

$\rightarrow$ $W = \text{Span}\{(1,0)\}$

$W^{\perp} = \text{Span}\{(0,1)\}$

$(x, y) = x(1,0) + y.(0,1)$

Theorem: let $\dim(V) = n$

Suppose W is a subspace of V, then

$v \in V \Rightarrow v = w + u$, where $w \in W$, $u \in W^{\perp}$

Further if $v = x + y$, where $x \in W$ and $y \in W^{\perp}$, then $x = w$, $y = u$

Note: Any Vector space has finite dim of ortho~~gonal~~ normal basis

Pf: let $\{w_1, w_2, \ldots, w_n\}$ be an orthonormal basis for W.

let $v \in V$, $P = v \perp u$

Put, $P = v - \langle v, w_1 \rangle w_1 - \langle v, w_2 \rangle w_2 - \cdots \langle v, w_m \rangle v_m$

P is orthogonal to $w_1, w_2, \ldots, w_m$

$\Rightarrow P \in W^{\perp}$

Suppose, $v = x + y$ | $x \in W$, $y \in W^{\perp}$

$v = w + u$, $w \in W$, $u \in W^{\perp}$

$\Rightarrow 0 = (x - w) + (y - u)$

$\langle x - w, y - u \rangle = 0$

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a, b orthogonal $\Rightarrow$ $\|a-b\|^2 = \|a+b\|^2 = \|a\|^2 + \|b\|^2$

$$0 = \|(x-w)+(y-u)\|^2$$
$$= \|x-w\|^2 + \|y-u\|^2$$
$$\Rightarrow x = w ; y = u$$

$\hookrightarrow$ Assume V is a real vector space

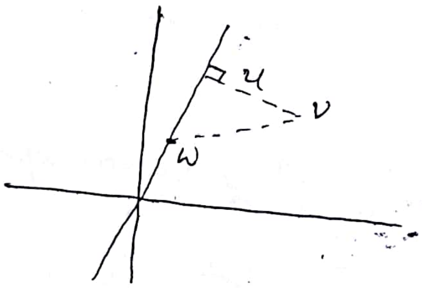
let W be a subspace.

let $v \in V$

Best approximation of v in W is $u \in W$ iff

$\|v-u\| \leq \|x-u\|$. for any $x \in W$

$\rightarrow$ $\|v-u\|$ is min



5

Suppose u is the best approximation of v is W

then $\langle v-u, y \rangle = 0$, $\forall y \in W$

Pf: let $y \in W$,

$$\langle v-u, v-u \rangle = \|v-u\|^2$$
$$\leq \|v-w\|^2 \text{ for any } w \in W$$

$\|v-u\|^2 \leq \|v\|^2 - 2\langle v, w \rangle + \|w\|^2$, for any w.

$w = u + y \in W$.

$$\|v-w\|^2 = \langle v-(u+y), v-(u+y) \rangle$$
$$= \langle (v-u)-y ; (v-u)-y \rangle$$
$$= \langle v-u, v-u \rangle + (2\langle v-u, y \rangle + \|y\|^2)$$

$\therefore u$ is the best approximation

$W = \text{Span}\{(3,12,-1)\}$
find the best approximation of $(-10,2,8)$ in W.

Sol: O.N. basis of $W = \left(\frac{3}{\sqrt{154}}, \frac{12}{\sqrt{154}}, \frac{-1}{\sqrt{154}}\right)$.

$\therefore u = \langle v, e_1\rangle e_1$

$= \left\langle (-10,2,8), \left(\frac{3}{\sqrt{154}}, \frac{12}{\sqrt{154}}, \frac{-1}{\sqrt{154}}\right)\right\rangle \left(\frac{3}{\sqrt{154}}, \frac{12}{\sqrt{154}}, \frac{-1}{\sqrt{154}}\right)$

$= \cancel{\frac{-14}{\sqrt{154}}}\ \frac{-14}{154}(3,12,-1)$

Theorem: $(W^\perp)^\perp = W$

Proof: i) let $x \in W$
let $v \in W^\perp$
$\Rightarrow \langle x, v\rangle = 0 \Rightarrow x \in (W^\perp)^\perp$
$\Rightarrow W \subseteq (W^\perp)^\perp$

ii) let $y \in (W^\perp)^\perp$
$y = u + v$, where $u \in W$, $v \in W^\perp$
$\Rightarrow y - u = v \in W^\perp$
$\Rightarrow (y-u) \in W^\perp$ — ①

Also, $\langle y, x\rangle - \langle u, x\rangle = 0$; $x \in W^\perp$
$\Rightarrow \langle y-u, x\rangle = 0$
$\Rightarrow y - u \in (W^\perp)^\perp$ — ②
$\Rightarrow$ from ① and ②
$y - u = 0 \Rightarrow y = u \in W$

$\therefore (W^{\perp})^{\perp} \subseteq W$

$\therefore$ from i) and ii) $\quad W = (W^{\perp})^{\perp}$

$\hookrightarrow \quad P_W : V \to V$

s.t. $P_W(v) = u \quad ; P_W$ is linear

(u is the best approximation v in W)

$$P_W(x) = \langle x, e_1 \rangle e_1 + \cdots + \langle x, e_m \rangle e_m$$

$$P_W(y) = \langle y, e_1 \rangle e_1 + \cdots + \langle y, e_m \rangle e_m$$

$$P_W(x+y) = \langle x+y, e_1 \rangle e_1 + \cdots + \langle x+y, e_m \rangle e_m$$

$$= P_W(x) + P_W(y)$$

$\text{||}^{ly} \quad P_W(\alpha x) = \alpha P_W(x)$

$$\text{Null}(P_W) = W^{\perp}$$

$$\text{Range}(P_W) = W$$

6/10/17

$T : V \to \mathbb{R}$

linear functionals

$$T(x_1, x_2, \ldots, x_n) = (a_1, a_2, \ldots, a_n) \cdot (x_1, x_2, \ldots x_n)$$

$$T(x) = \underline{a} \cdot \underline{x} = \langle a, x \rangle$$

$$T(x_1, x_2, x_3) = 2x_1 = (2, 0, 0) \cdot (x_1, x_2, x_3)$$

Theorem: Let $T : V \to \mathbb{R}$ be a linear functional (V - finite dimensional), then there exists a unique vector $a \in V$, s.t.

$$T(v) = \langle a, v \rangle \quad ; \quad (v \in V)$$

(Riesz Representation)

$W = \text{Span}\{(3, 12, -1)\}$

find the best approximation of $(-10, 2, 8)$ in W.

Sol :

O.N. basis of $W = \left(\frac{3}{\sqrt{154}}, \frac{12}{\sqrt{154}}, \frac{-1}{\sqrt{154}}\right)$.

$$\therefore u = \langle v, e_1 \rangle e_1$$

$$= \left\langle (-10, 2, 8), \left(\frac{3}{\sqrt{154}}, \frac{12}{\sqrt{154}}, \frac{-1}{\sqrt{154}}\right)\right\rangle \left(\frac{3}{\sqrt{154}}, \frac{12}{\sqrt{154}}, \frac{-1}{\sqrt{154}}\right)$$

$$= \cancel{\frac{-14}{\sqrt{154}}} \; \frac{-14}{154}(3, 12, -1)$$

Theorem: $(W^{\perp})^{\perp} = W$

Proof : i) let $x \in W$

let $v \in W^{\perp}$

$\Rightarrow \langle x, v \rangle = 0 \Rightarrow x \in (W^{\perp})^{\perp}$

$\Rightarrow \cancel{(W^{\perp})^{\perp}} \; W \subseteq (W^{\perp})^{\perp}$

ii) let $y \in (W^{\perp})^{\perp}$

$y = u + v$, where $u \in W$, $v \in W^{\perp}$

$\Rightarrow y - u = v \in W^{\perp}$

$\Rightarrow (y - u) \in W^{\perp}$ — ①

Also, $\langle y, x \rangle - \langle u, x \rangle = 0$; $x \in W^{\perp}$

$\Rightarrow \langle y - u, x \rangle = 0$

$\Rightarrow y - u \in (W^{\perp})^{\perp}$ — ②

$\Rightarrow$ from ① and ②

$y - u = 0 \Rightarrow y = u \in W$

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$\therefore (W^{\perp})^{\perp} \subseteq W$

$\therefore$ from i) and ii) $W = (W^{\perp})^{\perp}$

$\hookrightarrow$ $P_W : V \to V$

s.t. $P_W(v) = u$; P_W is linear

(u is the best approximation v in W)

$P_W(x) = \langle x, e_1\rangle e_1 + \cdots + \langle x, e_m\rangle e_m$

$P_W(y) = \langle y, e_1\rangle e_1 + \cdots + \langle y, e_m\rangle e_m$

$P_W(x+y) = \langle x+y, e_1\rangle e_1 + \cdots + \langle x+y, e_m\rangle e_m$

$= P_W(x) + P_W(y)$

ly $P_W(\alpha x) = \alpha P_W(x)$

$\text{Null}(P_W) = W^{\perp}$

$\text{Range}(P_W) = W$

6/10/17

$T : V \to \mathbb{R}$

linear functionals

$T(x_1, x_2, \ldots, x_n) = (a_1, a_2, \ldots, a_n) \cdot (x_1, x_2, \ldots x_n)$

$T(x) = \underline{a} \cdot \underline{x} = \langle a, x\rangle$

$T(x_1, x_2, x_3) = 2x_1 = (2, 0, 0) \cdot (x_1, x_2, x_3)$

Theorem: Let $T : V \to \mathbb{R}$ be a linear functional (V - finite dimensional), then there exists a unique vector $a \in V$, s.t.

$T(v) = \langle a, v\rangle$; $(v \in V)$

(Riesz Representation)

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To prove: $T^*(\alpha v_1 + v_2) = \alpha c_1 + c_2$; $\forall \alpha$

$\langle Tx, \alpha v_1 \rangle = \langle x, \alpha c_1 \rangle$; $\forall x$

$\langle Tx, v_2 \rangle = \langle x, c_2 \rangle$; $\forall x$

$\Rightarrow \langle Tx, \alpha v_1 + v_2 \rangle = \langle x, \alpha c_1 + c_2 \rangle$; $\forall x$

$T^*(\alpha v_1 + v_2) = \alpha c_1 + c_2$

So T^* is linear

eg: T is 1-1 ; Prove that T^* is 1-1

Pf: T is 1-1 (Nullity $= 0$)

let $T^* v = 0$ for some v

$\langle Tx, v \rangle = \langle x, T^* v \rangle$

$= \langle x, 0 \rangle$

$= 0$ ~~for~~ $\forall x$

$\Rightarrow \langle Tx, v \rangle = 0$; $\forall x$

$\exists p$ s.t $Tp = v$

$\langle v, v \rangle = 0 \Rightarrow v = 0$

$\hookrightarrow$ $V \rightarrow$ let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for V.

$T: V \rightarrow V$ be linear.

$$[T] = \begin{bmatrix} \langle Te_1, e_1 \rangle & \cdots & \langle Te_1, e_n \rangle \\ \vdots & & \\ \langle Te_n, e_1 \rangle & \cdots & \langle Te_n, e_n \rangle \end{bmatrix}$$

$Te_1 = \alpha_1 e_1 + \cdots + \alpha_n e_n$

$= \langle Te_1, e_1 \rangle e_1 + \cdots + \langle Te_n, e_n \rangle e_n$

$T: V \rightarrow V$ linear ; T^* be adjoint of T.

$$[T^*] = \begin{bmatrix} \langle T^* e_1, e_1 \rangle & \cdots & \langle T^* e_1, e_n \rangle \\ \vdots & & \\ \langle T^* e_n, e_1 \rangle & \cdots & \langle T^* e_n, e_n \rangle \end{bmatrix}$$

$[T^*]$ is the transpose ~~matrix~~ conjugate of $[T]$

$\hookrightarrow$ let $T: V \to V$ be linear

then $(T^*)^* = T$

Pf:

$$\langle Tx, v\rangle = \langle x, T^* v\rangle$$

$$\langle T^* v, x\rangle = \langle v, (T^*)^* x\rangle$$

$$\Rightarrow \langle x, T^* v\rangle = \langle (T^*)^* x, v\rangle$$

$$\langle Tx, v\rangle = \langle (T^*)^* x, v\rangle$$

$\hookrightarrow$ $T: V \to V$

$S: V \to V$

$(T+S)^* = T^* + S^*$

let $v \in V$

for any $x \in V$

$$\langle (T+S)x, v\rangle = \langle x, (T+S)^* v\rangle$$

$$\langle Tx + Sx, v\rangle = \langle Tx, v\rangle + \langle Sx, v\rangle$$

$$= \langle x, T^* v\rangle + \langle x, S^* v\rangle$$

$$= \langle x, (T^* + S^*) v\rangle$$

$$(S+T)^* = S^* + T^*$$

$\rightarrow$ $(\alpha T)^* = \bar{\alpha} T^*$

Pf: fix $v \in V$

then for all $x \in V$,

$$\langle (\alpha T)x, v\rangle = \langle x, (\alpha T)^* v\rangle \quad \text{— (1)}$$

$$\langle \alpha Tx, v\rangle = \alpha \langle Tx, v\rangle$$

$$= \alpha \langle x, T^* v\rangle \quad \text{— (2)}$$

$$\alpha \langle x, T^* v\rangle = \langle x, \bar{\alpha} T^* v\rangle$$

So $\langle x, (\alpha T)^* v\rangle = \langle x, \bar{\alpha} T^* v\rangle$

To prove: $T^*(\alpha v_1 + v_2) = \alpha c_1 + c_2$; $\forall \alpha$

$\langle Tx, \alpha v_1 \rangle = \langle x, \alpha c_1 \rangle$; $\forall x$

$\langle Tx, v_2 \rangle = \langle x, c_2 \rangle$; $\forall x$

$\Rightarrow \langle Tx, \alpha v_1 + v_2 \rangle = \langle x, \alpha c_1 + c_2 \rangle$; $\forall x$

$T^*(\alpha v_1 + v_2) = \alpha c_1 + c_2$

So T^* is linear

Q: T is 1-1 ; Prove that T^* is 1-1

Pf: T is 1-1 (Nullity = 0)

let $T^* v = 0$ for some v

$\langle Tx, v \rangle = \langle x, T^* v \rangle$

$= \langle x, 0 \rangle$

$= 0$ ~~for~~ $\forall x$

$\Rightarrow \langle Tx, v \rangle = 0$; $\forall x$

$\exists p$ s.t $Tp = v$

$\langle v, v \rangle = 0 \Rightarrow v = 0$

$\rightarrow$ $V \rightarrow$ let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for V.

$T: V \to V$ be linear.

$$[T] = \begin{bmatrix} \langle Te_1, e_1 \rangle & \cdots & \langle Te_1, e_n \rangle \\ \vdots & & \\ \langle Te_n, e_1 \rangle & \cdots & \langle Te_n, e_n \rangle \end{bmatrix}$$

$Te_1 = \alpha_1 e_1 + \cdots + \alpha_n e_n$

$= \langle Te_1, e_1 \rangle e_1 + \cdots + \langle Te_n, e_n \rangle e_n$

$T: V \to V$ linear ; T^* be adjoint of T.

$$[T^*] = \begin{bmatrix} \langle T^* e_1, e_1 \rangle & \cdots & \langle T^* e_1, e_n \rangle \\ \vdots & & \\ \langle T^* e_n, e_1 \rangle & \cdots & \langle T^* e_n, e_n \rangle \end{bmatrix}$$

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$[T^*]$ is the transpose matrix conjugate of $[T]$

→ let $T: V \to V$ be linear
then $(T^*)^* = T$

Pf:
$$\langle Tx, v\rangle = \langle x, T^* v\rangle$$
$$\langle T^* v, x\rangle = \langle v, (T^*)^* x\rangle$$
$$\Rightarrow \langle x, T^* v\rangle = \langle (T^*)^* x, v\rangle$$
$$\langle Tx, v\rangle = \langle (T^*)^* x, v\rangle$$

→ $T: V \to V$
$S: V \to V$
$(T+S)^* = T^* + S^*$

let for any $v \in V$, $x \in V$
$$\langle (T+S)x, v\rangle = \langle x, (T+S)^* v\rangle$$
$$\langle Tx + Sx, v\rangle = \langle Tx, v\rangle + \langle Sx, v\rangle$$
$$= \langle x, T^* v\rangle + \langle x, S^* v\rangle$$
$$= \langle x, (T^* + S^*) v\rangle$$
$$(S+T)^* = S^* + T^*$$

→ $(\alpha T)^* = \bar{\alpha} T^*$

Pf: fix $v \in V$
then for all $x \in V$,
$$\langle (\alpha T)x, v\rangle = \langle x, (\alpha T)^* v\rangle \quad \text{— } 1$$
$$\langle \alpha Tx, v\rangle = \alpha \langle Tx, v\rangle$$
$$= \alpha \langle x, T^* v\rangle \quad \text{— } 2$$
$$\alpha \langle x, T^* v\rangle = \langle x, \bar{\alpha} T^* v\rangle$$
So $\langle x, (\alpha T)^* v\rangle = \langle x, \bar{\alpha} T^* v\rangle$