Assignment 3

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CME 241

Problem 1

Bellman Equations for the Deterministic Policy $\pi_D : \mathcal{S} \rightarrow \mathcal{A}$:

$$\begin{aligned}
V^{\pi_D}(s) &= Q^{\pi_D}(s, \pi_D(s)), \ \ \forall s \in \mathcal{S} \\
Q^{\pi_D}(s,a) &= R(s,a) + \gamma \sum_{s' \in N} P(s,a,s') V^{\pi_D}(s'), \ \ \forall s \in \mathcal{S}, \ a \in \mathcal{A} \\
V^{\pi_D}(s) &= R(s, \pi_D(s)) + \gamma \sum_{s' \in N} P(s, \pi_D(s), s') V^{\pi_D}(s'), \ \ \forall s \in \mathcal{S} \\
Q^{\pi_D}(s,a) &= R(s,a) + \gamma \sum_{s' \in N} P(s,a,s') Q^{\pi_D}(s', \pi_D(s')), \ \ \forall s \in \mathcal{S}, \ a \in \mathcal{A}
\end{aligned}$$

Problem 2

We observe that the transition probabilities and the reward function are the same $\forall s \in \mathcal{S}$ and $a \in [0,1]$. The same holds true for the reward function. This means that the dynamics of the given infinite-state MDP are the same as the finite-state MDP with two states, let's say $s_1,\ s_2$ that has transition probabilities:

$$\mathbb{P}[s_2|s_1,a] = a, \mathbb{P}[s_1|s_1,a] = 1-a, \mathbb{P}[s_1|s_2,a] = a, \mathbb{P}[s_2|s_2,a] = 1-a,\ a \in [0,1]$$

and reward function:

$$R_T(s_1,a,s_2) = 1-a,\ R_T(s_1,a,s_1) = 1+a,\ R_T(s_2,a,s_1) = 1-a,\ R_T(s_2,a,s_2) = 1+a$$

Hence, in our initial MDP the Optimal Value Function V*(s) is the same $\forall s \in \mathcal{S}$ and it satisfies the Bellman Optimality Equations:

$$V^*(s) = \max_a \{R(s,a) + \gamma \sum_{s' \in S} P(s,a,s')V^*(s')\}$$

where

$$\begin{aligned} R(s,a) &= \mathbb{E}[R_{t+1}|S_t = s, A_t = a] \\ &= \sum_{s' \in S} P(s,a,s')R_T(s,a,s') \\ &= P(s,a,s+1)R_T(s,a,s+1) + P(s,a,s)R_T(s,a,s) \\ &= a(1-a) + (1-a)(1+a) \\ &= (1-a)(2a+1) \\ &= -2a^2 + a + 1 \end{aligned}$$

Therefore:

$$\begin{aligned} V^*(s) &= \max_{a \in [0,1]} \{-2a^2 + a + 1 + \gamma(aV^*(s) + (1-a)V^*(s)\} \\ \implies V^*(s) &= \max_{a \in [0,1]} \{-2a^2 + a + 1 + \gamma V^*(s)\} \\ \implies (1-\gamma)V^*(s) &= \max_{a \in [0,1]} \{-2a^2 + a + 1\} \\ \xRightarrow{\gamma=0.5} 0.5V^*(s) &= \max_{a \in [0,1]} \{-2a^2 + a + 1\} \\ \implies 0.5V^*(s) &= 9/8 \\ \implies V^*(s) &= 9/4 = 2.25, \quad \forall s \in \mathcal{S} \end{aligned}$$

The above maximization is achieved for $a = 1/4$, which means that the optimal policy $\forall s \in \mathcal{S}$ is $\pi^*(s) = 1/4$.

Problem 3

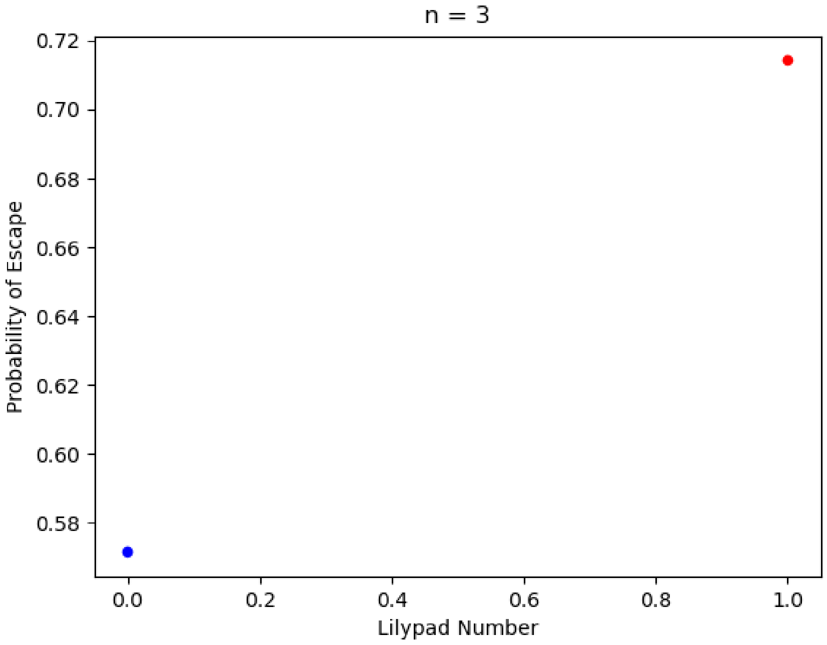
State Space: $\mathcal{S} = \{i | 0 \leq i \leq n\}$, $\mathcal{T} = \{0, n\}$, where i corresponds to the i-th lilypad.
Action Space: $\mathcal{A} = \{A, B\}$, where A and B are the sounds of the frog.
Transition Function for $1 \leq i \leq n-1$:

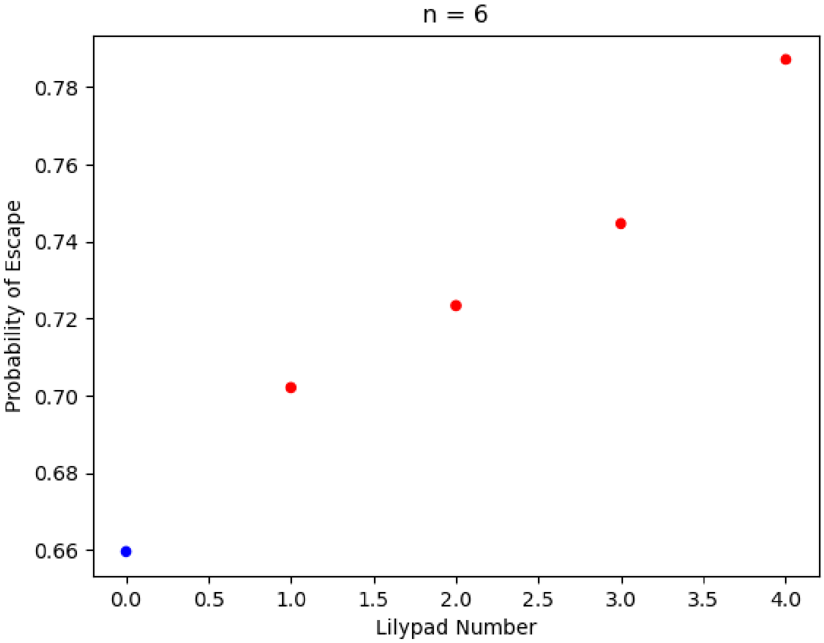
$$\begin{aligned} P[i'|i,A] &= \begin{cases} \frac{i}{n}, & i' = i-1 \\ \frac{n-i}{n}, & i' = i+1 \\ 0, & o.w. \end{cases} \\ P[i'|i,B] &= \begin{cases} \frac{1}{n}, & i' = 0, \ldots, i-1, i+1, \ldots, n \\ 0, & i' = i \end{cases} \end{aligned}$$

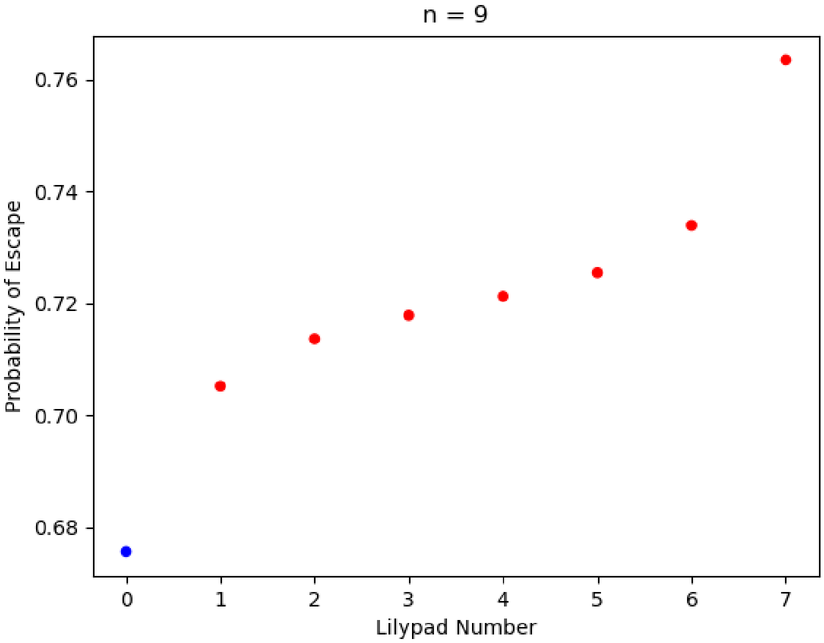
Since the goal is that the frog escapes the pond, i.e. reaches lilypad n then the Reward Function for $1 \leq i \leq n-1$ and $a \in \{A, B\}$ is:

$$R(i, a, i') = \begin{cases} 1, & i' = n \\ 0, & o.w. \end{cases}$$

See RL-book/_lkourti/Assign3/prob3.3.py for the implementation. All possible deterministic policies are compared to find the Optimal Policy and the corresponding Optimal Value Function. Additionally, the Optimal Escape Probability is derived from the Value Function and it's plotted in the following figures. What we observe is that for all values of n **Croak B** (**blue dots**) is the optimal action only for the first lilypad, for all other lilypads **Croak A** (**red dots**) is the optimal action.




n = 6
Probability of Escape
0.78
0.76
0.74
0.72
0.70
0.68
0.66
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
Lilypad Number


n = 9
Probability of Escape
0.76
0.74
0.72
0.70
0.68
0
1
2
3
4
5
6
7
Lilypad Number

Problem 4

Our goal is to to minimize the infinite-horizon Expected Discounted-Sum of Costs when $\gamma = 0$ (myopic case).

$$\begin{aligned}
V^*(s) &= \max_{\pi} V^{\pi}(s), \quad \forall s \in S \\
&= \max_{a} \{\mathbb{E}_{s' \sim N(s,\sigma^2)}[-e^{as'}|s]\} \\
&= \max_{a} \{-\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\sigma)^2}{2\sigma^2}} e^{ax}\, dx\} \\
&= \max_{a} \{-\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2-2xs+s^2-2\sigma^2 ax}{2\sigma^2}}\, dx\} \\
&= \max_{a} \{-\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2-2x(s+\sigma^2 a)+(s+\sigma^2 a)^2}{2\sigma^2}+as+\frac{\sigma^2 a^2}{2}}\, dx\} \\
&= \max_{a} \{-e^{as+\frac{\sigma^2 a^2}{2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-(s+\sigma^2 a))^2}{2\sigma^2}}\, dx\} \\
&= \max_{a} \{-e^{as+\frac{\sigma^2 a^2}{2}} \mathbb{E}_{x \sim N(s+\sigma^2 a,\sigma^2)}[1]\} \\
&= \max_{a} \{-e^{as+\frac{\sigma^2 a^2}{2}}\}
\end{aligned}$$

The optimal policy satisfies $a^* = \arg\max_a \{-e^{as+\frac{\sigma^2 a^2}{2}}\}$. Hence, we solve for:

$$\begin{aligned}
\frac{\partial\{-e^{as+\frac{\sigma^2 a^2}{2}}\}}{\partial a} = 0 \quad &\Longrightarrow \quad s + a^*\sigma^2 = 0 \\
&\Longrightarrow \quad a^* = -\frac{s}{\sigma^2}
\end{aligned}$$

which achieves the maximum, since the function is concave (2nd derivative is negative). Therefore, the Optimal Action (Deterministic Policy) is:

$$\pi^*(s) = -\frac{s}{\sigma^2}, \quad \forall s \in S$$

and by substituting a^*, we get that the Optimal Value Function is:

$$V^*(s) = -e^{-\frac{s^2}{\sigma^2}+\frac{s^2}{2\sigma^2}} = -e^{-\frac{s^2}{2\sigma^2}}, \quad \forall s \in S$$

which results in an Optimal Cost equal to $e^{-\frac{s^2}{2\sigma^2}}, \quad \forall s \in S$.