Assignment 6

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① $U(x) = x - \frac{\alpha x^2}{2}$, $x \sim N(\mu, \sigma^2)$

$$\left.\begin{array}{l} E[x] = \mu \\ E[x^2] - (E[x])^2 = \sigma^2 \end{array}\right\} \Rightarrow E[x^2] = \mu^2 + \sigma^2$$

- $E[U(x)] = E\left[x - \frac{\alpha x^2}{2}\right] = E[x] - \frac{\alpha}{2} E[x^2] \Rightarrow E[U(x)] = \mu - \frac{\alpha}{2}(\mu^2 + \sigma^2)$

- In class we showed:

$E[U(x)] = U(x_{CE}) \Rightarrow U'(\bar{x})(x_{CE} - E[x]) \approx \frac{1}{2} U''(\bar{x}) \sigma_x^2$

$$\Rightarrow U'(\bar{x}) x_{CE} \approx U'(\bar{x}) E[x] + \frac{1}{2} U''(\bar{x}) \sigma_x^2$$

$$\Rightarrow x_{CE} \approx E[x] + \frac{1}{2} \frac{U''(\bar{x})}{U'(\bar{x})} \sigma_x^2$$

$$\Rightarrow x_{CE} \approx \mu + \frac{\sigma^2}{2} \frac{-\alpha}{1 - \alpha\mu} \Rightarrow x_{CE} \approx \mu - \frac{\alpha\sigma^2}{2(1-\alpha\mu)}$$

since $U'(x) = 1 - \alpha x$, $U''(x) = -\alpha$

- $\pi_A = \bar{x} - x_{CE} = \mu - \mu + \frac{\alpha\sigma^2}{2(1-\alpha\mu)} \Rightarrow \pi_A = \frac{\alpha\sigma^2}{2(1-\alpha\mu)}$

Portfolio:

Wealth after one year is given by: (z in millions)

$W \sim N\left((1+\mu)z + (1+r)(1-z), z^2\sigma^2\right)$ which is $W \sim N\left(z(\mu - r) + 1 + r, z^2\sigma^2\right)$

$U(W) = W - \frac{\alpha W^2}{2}$

The goal is to maximize $E[U(W)]$

$$E[U(W)] = z(\mu - r) + 1 + r - \frac{\alpha}{2}\left[(z(\mu - r) + 1 + r)^2 + \sigma^2 z^2\right]$$

$$(E[U(W)])' = \mu - r - \alpha(\mu - r)[z(\mu - r) + 1 + r] - \alpha\sigma^2 z$$

$$= -\alpha z[(\mu - r)^2 + \sigma^2] + (\mu - r)[1 - \alpha(1 + r)]$$

So $(E[U(W)])' = 0 \Rightarrow z = \frac{(\mu - r)[1 - \alpha(1+r)]}{\alpha[(\mu - r)^2 + \sigma^2]}$

which maximizes $E[U(W)]$ since it is a concave function

Since z is in millions and the available capital is 1 million we require $z \in [0,1]$

- $z > 0 \Rightarrow \frac{(\mu-r)[1-\alpha(1+r)]}{\alpha[(\mu-r)^2+\sigma^2]} > 0 \Rightarrow \begin{cases} \alpha > 0 \\ \alpha > \frac{1}{1+r} > 0 \end{cases}$

- $z < 1 \Rightarrow \frac{(\mu-r)[1-\alpha(1+r)]}{\alpha[(\mu-r)^2+\sigma^2]} < 1$

$$\overset{\alpha>0}{\Rightarrow} (\mu-r) - \alpha(\mu-r)(1+r) < \alpha[(\mu-r)^2+\sigma^2]$$

$$\Rightarrow \alpha > \frac{\mu-r}{(\mu-r)^2+\sigma^2+(\mu-r)(1+r)}$$

where we assumed $\mu-r>0$.

We plot z as a function of $\alpha \in \left[\frac{1}{1+r}, \frac{\mu-r}{(\mu-r)^2+\sigma^2+(\mu-r)(1+r)}\right]$

and we observe that the larger the α, the larger our risk aversion.

(2) We want to maximize the expected utility of Wealth W.
As calculated in the book:

$$\log W \sim N\left(r + \pi(\mu - r) - \frac{\pi^2\sigma^2}{2}, \pi^2\sigma^2\right)$$

In our case $U(x) = \log(x) \Rightarrow U(W) = \log W$

$$E[U(W)] = E[\log W] = r + \pi(\mu - r) - \frac{\pi^2\sigma^2}{2}$$

which is a concave function of π.

$$(E[U(W)])' = 0 \Rightarrow \left[r + \pi^*(\mu - r) - \frac{\pi^{*2}\sigma^2}{2}\right]' = 0$$

$$\Rightarrow (\mu - r) - \frac{2\pi^*\sigma^2}{2} = 0 \Rightarrow \pi^* = \frac{\mu - r}{\sigma^2}$$

So we found the optimal allocation in the risky asset.

③ > The two returns after a single bet of fW_0:

- $(1+\alpha)fW_0$ w/ probability p
- $(1-\beta)fW_0$ w/ probability $(1-p)$

> Outcomes for W:

- $(W_0 - fW_0) + (1+\alpha)fW_0 = W_0(1-f) + fW_0 + \alpha f W_0$
  $= (1+\alpha f)W_0$
- $(1-\beta f)W_0$

> Outcomes for $U(W) = \log(W)$:

- $\log((1+\alpha f)W_0)$
- $\log((1-\beta f)W_0)$

> $E[\log(W)] = p\log[(1+\alpha f)W_0] + (1-p)\log[(1-\beta f)W_0]$

> $E'[\log(W)] = \dfrac{p\alpha}{1+\alpha f} - \dfrac{(1-p)\beta}{1-\beta f} = \dfrac{p\alpha}{1+\alpha f} + \dfrac{(1-p)\beta}{\beta f - 1}$

> $E''[\log(W)] = -\dfrac{p\alpha^2}{(1+\alpha f)^2} - \dfrac{(1-p)\beta^2}{(\beta f - 1)^2} < 0$

So maximization of $\log(W)$ is achieved for:

$E'[\log(W)] = 0 \Rightarrow \dfrac{p\alpha}{1+\alpha f^*} + \dfrac{(1-p)\beta}{\beta f^* - 1} = 0 \Rightarrow (\beta f^* - 1)p\alpha + (1+\alpha f^*)(1-p)\beta = 0$

$\Rightarrow \alpha\beta p f^* - \alpha p + \alpha\beta(1-p)f^* + \beta(1-p) = 0 \Rightarrow \alpha\beta f^* = \alpha p - \beta(1-p)$

$\Rightarrow f^* = \dfrac{\alpha p - \beta(1-p)}{\alpha\beta} \Rightarrow f^* = \dfrac{p}{\beta} - \dfrac{1-p}{\alpha}$

From this formula of f^* we make the following reasonable observations:

- As p (prob. of revenue) increases, the fraction of W_0 bet increases.
- As $(1-p)$ (prob. of loss) increases, the fraction of W_0 bet decreases.
- As β (amount of loss) increases, the fraction of W_0 bet decreases.
- As α (amount of revenue) increases, the fraction of W_0 bet increases.