Problem 2 - Assignment 9

LPT model for $t = 0, \dots, T-1$

$$P_{t+1} = P_t e^{z_t}$$

$$X_{t+1} = \rho X_t + \eta_t$$

$$Q_t = P_t(1 - \beta N_t - \theta X_t)$$

Let R_t be the remaining stocks at time t
and N_t be the number of shares sold at time t,
such that $R_t = R_{t-1} - N_{t-1}$, $R_0 = N$, $R_T = 0$
where N is the total number of shares to be sold.

The goal is to maximize : $E\left[\sum_{t=0}^{T-1} N_t Q_t\right]$
The Optimal Value Function satisfies the Bellman Equation :

$$V_t^*(P_t, R_t, X_t) = \max_{N_t}\left[E\left[N_t Q_t + V_{t+1}^*(P_{t+1}, R_{t+1}, X_{t+1})\right]\right]$$

Starting from $t = T-1$ and then recursively moving backwards, we obtain :

- $V_{T-1}^*(P_{T-1}, R_{T-1}, X_{T-1}) = \max_{N_{T-1}}\left[E[N_{T-1}Q_{T-1}]\right] = E[R_{T-1}Q_{T-1}] = R_{T-1}P_{T-1}(1 - \beta R_{T-1} - \theta X_{T-1})$

- $V_{T-2}^*(P_{T-2}, R_{T-2}, X_{T-2}) = \max_{N_{T-2}}\left[E\left[N_{T-2}Q_{T-2} + V_{T-1}^*(P_{T-1}, R_{T-1}, X_{T-1})\right]\right]$

$$= \max_{N_{T-2}}\left[E\left[N_{T-2}P_{T-2}(1 - \beta N_{T-2} - \theta X_{T-2}) + P_{T-1}(R_{T-2} - N_{T-2})(1 - \theta X_{T-1} - \beta(R_{T-2} - N_{T-2}))\right]\right]$$

$$= \max_{N_{T-2}}\left[N_{T-2}P_{T-2}(1 - \beta N_{T-2} - \theta X_{T-2}) + P_{T-2}E[e^{z_{T-2}}](R_{T-2} - N_{T-2})(1 - \theta\rho X_{T-2} - \beta(R_{T-2} - N_{T-2}))\right]$$

Let $q = E[e^{z_{T-2}}] = e^{\mu_z + \frac{\sigma_z^2}{2}}$, then :

$$V_{T-2}^* = \max_{N_{T-2}}\left[N_{T-2}P_{T-2}(1 - \beta N_{T-2} - \theta X_{T-2}) + qP_{T-2}(R_{T-2} - N_{T-2})(1 - \theta\rho X_{T-2} - \beta(R_{T-2} - N_{T-2}))\right]$$

The above is a function of N_{T-2} and it's concave(*), so by taking its derivative and setting it to 0, we can find N_{T-2}^* :

(*) The coefficient of N_{T-2}^2 is negative : $-\beta(1+q)P_{T-2} < 0$

$$N^*_{T-2} = C^{(1)}_{T-2} + C^{(2)}_{T-2} R_{T-2} + C^{(3)}_{T-2} X_{T-2}$$

where

$$C^{(1)}_{T-2} = \frac{q-1}{-2B(q+1)}, \quad C^{(2)}_{T-2} = \frac{q}{q+1}, \quad C^{(3)}_{T-2} = \frac{\theta(q\rho-1)}{2B(q+1)}$$

Substituting N^*_{T-2} back in V^*_{T-2}, we obtain the following:

$$V^*_{T-2}(P_{T-2}, R_{T-2}, X_{T-2}) = q P_{T-2} \Big[C^{(4)}_{T-2} + C^{(5)}_{T-2} R_{T-2} + C^{(6)}_{T-2} X_{T-2} + C^{(7)}_{T-2} R^2_{T-2} + C^{(8)}_{T-2} X^2_{T-2} + C^{(9)}_{T-2} R_{T-2} X_{T-2} \Big]$$

where

$$C^{(4)}_{T-2} = C^{(1)}_{T-2}(1 - B C^{(1)}_{T-2}) - q C^{(1)}_{T-2}(1 + B C^{(1)}_{T-2})$$

$$C^{(5)}_{T-2} = 2 C^{(2)}_{T-2}$$

$$C^{(6)}_{T-2} = (1-q) C^{(3)}_{T-2}$$

$$C^{(7)}_{T-2} = -B C^{(2)}_{T-2}$$

$$C^{(8)}_{T-2} = C^{(3)}_{T-2}(-B C^{(3)}_{T-2} - \theta) - q C^{(3)}_{T-2}(-\theta\rho + B C^{(3)}_{T-2})$$

$$C^{(9)}_{T-2} = -\theta(1+\rho) C^{(2)}_{T-2}$$

Continuing backwards in time in this manner gives:

$$V^*_{T-3}(P_{T-3}, R_{T-3}, X_{T-3}) = \max_{N_{T-3}} \Big[E[N_{T-3} Q_{T-3} + V^*_{T-2}(P_{T-2}, R_{T-2}, X_{T-2})] \Big]$$

$$\overset{\max}{=} E\Big[N_{T-3} P_{T-3}(1 - B N_{T-3} - \theta N_{T-3}) + q P_{T-2}\big(C^{(4)}_{T-2} + C^{(5)}_{T-2} R_{T-2} + C^{(6)}_{T-2} X_{T-2} + C^{(7)}_{T-2} R^2_{T-2} + C^{(8)}_{T-2} X^2_{T-2} + C^{(9)}_{T-2} X_{T-2} R_{T-2}\big)\Big]$$

$$= \max_{N_{T-3}} \Big[N_{T-3} P_{T-3}(1 - B N_{T-3} - \theta N_{T-3}) + q^2 P_{T-3}\big(C^{(4)}_{T-2} + C^{(5)}_{T-2}(R_{T-3} - N_{T-3}) + C^{(6)}_{T-2} \rho X_{T-3} + C^{(7)}_{T-2}(R_{T-3} - N_{T-3})^2 + C^{(8)}_{T-2} \sigma^2_\eta X^2_{T-3} + C^{(9)}_{T-2} \rho X_{T-3}(R_{T-3} - N_{T-3})\big)\big) \Big]$$

Taking its derivative and setting it to 0, yields the following after many messy calculations:

$$N^*_{T-3} = C^{(1)}_{T-3} + C^{(2)}_{T-3} R_{T-3} + C^{(3)}_{T-3} X_{T-3}$$

where

$$C^{(1)}_{T-3} = \frac{q C^{(5)}_{T-2} - 1}{2(-B + q C^{(7)}_{T-2})}$$

$$C^{(2)}_{T-3} = \frac{q C^{(7)}_{T-2}}{-B + q C^{(7)}_{T-2}}$$

$$C^{(3)}_{T-3} = \frac{q\rho C^{(9)}_{T-2} + \theta}{2(-B + q C^{(7)}_{T-2})}$$

Substituting N^*_{T-3} back in V^*_{T-3}, we obtain the following:

$$V^*_{T-3}(P_{T-3}, R_{T-3}, X_{T-3}) = qP_{T-3}\Big[C^{(4)}_{T-3} + C^{(5)}_{T-3}R_{T-3} + C^{(6)}_{T-3}X_{T-3} + C^{(7)}_{T-3}R^2_{T-3} + C^{(8)}_{T-3}X^2_{T-3} + C^{(9)}_{T-3}R_{T-3}X_{T-3}\Big]$$

where

$$C^{(4)}_{T-3} = C^{(1)}_{T-3}(1 - bC^{(1)}_{T-3}) + q(C^{(4)}_{T-2} + \sigma^2_\eta C^{(8)}_{T-2}) - qC^{(1)}_{T-3}(C^{(5)}_{T-2} - C^{(1)}_{T-3}C^{(7)}_{T-2})$$

$$C^{(5)}_{T-3} = C^{(2)}_{T-3} + q(1 - C^{(2)}_{T-3})C^{(5)}_{T-2}$$

$$C^{(6)}_{T-3} = q\rho C^{(6)}_{T-2} - C^{(3)}_{T-3}(qC^{(5)}_{T-2} - 1)$$

$$C^{(7)}_{T-3} = -bC^{(2)}_{T-3}$$

$$C^{(8)}_{T-3} = C^{(3)}_{T-3}(-bC^{(3)}_{T-3} - \theta) + q\rho^2 C^{(8)}_{T-2} - qC^{(3)}_{T-3}(\rho C^{(9)}_{T-2} - C^{(3)}_{T-3}C^{(7)}_{T-2})$$

$$C^{(9)}_{T-3} = -\theta C^{(2)}_{T-3} + q\rho C^{(9)}_{T-2}(1 - C^{(2)}_{T-3})$$

Continuing in this fashion, we obtain the following

- $N^*_t = C^{(1)}_t + C^{(2)}_t R_t + C^{(3)}_t X_t$

  where

  $$C^{(1)}_t = \frac{qC^{(5)}_{t+1} - 1}{2(-b + qC^{(7)}_{t+1})}, \quad C^{(2)}_t = \frac{qC^{(7)}_{t+1}}{-b + qC^{(7)}_{t+1}}, \quad C^{(3)}_t = \frac{q\rho C^{(9)}_{t+1} + \theta}{2(-b + qC^{(7)}_{t+1})}$$

- $V^*_t(P_t, R_t, X_t) = qP_t(C^{(4)}_t + C^{(5)}_t R_t + C^{(6)}_t X_t + C^{(7)}_t R^2_t + C^{(8)}_t X^2_t + C^{(9)}_t R_t X_t)$

  where

  $$C^{(4)}_t = C^{(1)}_t(1 - bC^{(1)}_t) + q(C^{(4)}_{t+1} + \sigma^2_\eta C^{(8)}_{t+1}) - qC^{(1)}_t(C^{(5)}_{t+1} - C^{(1)}_t C^{(7)}_{t+1})$$

  $$C^{(5)}_t = C^{(2)}_t + q(1 - C^{(2)}_t)C^{(5)}_{t+1}$$

  $$C^{(6)}_t = q\rho C^{(6)}_{t+1} - C^{(3)}_t(qC^{(5)}_{t+1} - 1)$$

  $$C^{(7)}_t = -bC^{(2)}_t$$

  $$C^{(8)}_t = C^{(3)}_t(-bC^{(3)}_t - \theta) + q\rho^2 C^{(8)}_{t+1} - qC^{(3)}_t(\rho C^{(9)}_{t+1} - C^{(3)}_t C^{(7)}_{t+1})$$

  $$C^{(9)}_t = -\theta C^{(2)}_t + q\rho C^{(9)}_{t+1}(1 - C^{(2)}_t)$$

where $q = e^{\mu_z + \frac{\sigma_z^2}{2}}$