Assignment 7

CME 241

Problem 1

The only difference compared to the modeling done in class is that the utility function is : $U(x) = \log(x)$ for the consumption

- Riskless Asset : $dR_t = rR_t dt$
- Risky Asset : $dS_t = \mu S_t dt + \sigma S_t dz_t$, $\mu > r > 0$, $\sigma > 0$
- Wealth at time t : $W_t \geq 0$
- Fraction of W_t allocated to the risky asset : $\pi(t, W_t) = \pi_t$
- ——— // ——— the riskless asset : $1 - \pi(t, W_t) = 1 - \pi_t$
- Wealth consumption per unit of time : $c(t, W_t) = c_t \geq 0$
- Process for wealth W_t :

$$dW_t = [(\pi_t(\mu - r) + r)W_t - c_t]dt + \pi_t \sigma W_t dz_t$$

The goal is to determine (π_t, c_t) $\forall t \in [0, t)$ to maximize :

$$E\left[\int_t^T e^{-\rho(s-t)} \log(c_s) ds + e^{-\rho(T-t)} \varepsilon^\gamma \log(W_T) \mid W_t\right]$$, where ε^γ is the beques with $0 < \varepsilon \ll 1$, $\gamma = 1$

$\rho \geq 0$ is the utility discount rate.

MDP Formulation :

> State at time t : (t, W_t)

> Action at time t : (π_t, c_t)

> Reward per unit of time at time t : $U(c_t) = \log(c_t)$

> Return at time t : $\int_t^T e^{-\rho(T-t)} \log(c_s) ds$

> Policy $(t, W_t) \to (\pi_t, c_t)$ that maximizes the Expected Return

So the Optimal Value Function $V^*(t, W_t)$ is :

$$V^*(t, W_t) = \max_{\pi, c} E_t\left[\int_t^T e^{-\rho(T-t)} \log(c_s) ds + e^{-\rho(T-t)} \varepsilon \log(W_T)\right]$$

which satisfies the following recursive formulation for $0 \leq t < t_1 < T$

$$V^*(t, W_t) = \max_{\pi, c} E_t\left[\int_t^{t_1} e^{-\rho(T-t)} \log(c_s) ds + e^{-\rho(t_1-t)} V^*(t_1, W_{t_1})\right]$$

$$\Rightarrow e^{-\rho t} V^*(t, W_t) = \max_{\pi, c} E_t\left[\int_t^{t_1} e^{-\rho s} \log(c_s) ds + e^{-\rho t_1} V^*(t_1, W_{t_1})\right]$$

In stochastic differential form:

$$\max_{\pi_t, c_t} E_t\left[d\left(e^{-\rho t} V^*(t, W_t)\right) + e^{-\rho t} \log(c_t)\, dt\right] = 0$$

$$\Rightarrow \max_{\pi_t, c_t} E_t\left[dV^*(t, W_t) + \log(c_t)\, dt\right] = \rho V^*(t, W_t)\, dt$$

Applying Itô's lemma and using the facts $E[dz]=0$ and $E[dz^2] = dt$ we obtain the HJB equation in PDE form:

$$\max_{\pi_t, c_t}\left[\frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial W_t}\left((\pi_t(\mu - r) + r)W_t - c_t\right) + \frac{\partial^2 V^*}{\partial W_t^2} \frac{\pi_t^2 \sigma^2 W_t^2}{2} + \log(c_t)\right] = \rho V^*(t, W_t)$$

Let us write the above equation more succinctly as:

$$\max_{\pi_t, c_t} \Phi(t, W_t; \pi_t, c_t) = \rho V^*(t, W_t) \qquad (1)$$

To find optimal π_t^*, c_t^* by taking partial derivatives of $\Phi(t, W_t; \pi_t, c_t)$ wrt π_t and c_t and equate them to 0 (1st order conditions for Φ)

- $\frac{\partial \Phi}{\partial \pi_t} = 0 \Rightarrow (\mu - r)\frac{\partial V^*}{\partial W_t} + \frac{\partial^2 V^*}{\partial W_t^2} \pi_t^* \sigma^2 W_t = 0$

$$\Rightarrow \pi_t^* = \frac{-\frac{\partial V^*}{\partial W_t}(\mu - r)}{\frac{\partial^2 V^*}{\partial W_t^2} \sigma^2 W_t} \qquad (2)$$

- $\frac{\partial \Phi}{\partial c_t} = 0 \Rightarrow -\frac{\partial V^*}{\partial W_t} + \frac{1}{c_t^*} = 0 \Rightarrow c_t^* = \frac{1}{\frac{\partial V^*}{\partial W_t}} \qquad (3)$

$$\text{So } (1) \overset{(2)}{\underset{(3)}{\Longrightarrow}} \frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial W_t}(\mu - r) W_t \pi_t^* + \frac{\partial V^*}{\partial W_t} r W_t - \frac{\partial V^*}{\partial W_t} c_t^* + \frac{\partial^2 V^*}{\partial W_t^2} \frac{\sigma^2 W_t^2}{2} \pi_t^{*2} + \log(c_t^*) = \rho V^*(t, W_t)$$

$$\Rightarrow \frac{\partial V^*}{\partial t} - \frac{(\mu - r)^2}{2\sigma^2} \frac{\left(\frac{\partial V^*}{\partial W_t}\right)^2}{\frac{\partial^2 V^*}{\partial W_t^2}} + \frac{\partial V^*}{\partial W_t} r W_t - 1 - \log\left(\frac{\partial V^*}{\partial W_t}\right) = \rho V^*(t, W_t) \qquad (4)$$

The boundary condition is $V^*(T, W_T) = \varepsilon \log(W_T)$

We surmise with a guess solution $V^*(t,W_t) = f(t) + \frac{1}{\rho}\log(W_t)$

Then $\cdot\ \frac{\partial V^*}{\partial t} = f'(t)$

$\cdot\ \frac{\partial V^*}{\partial W_t} = \frac{1}{\rho W_t}$

$\cdot\ \frac{\partial^2 V^*}{\partial W_t^2} = -\frac{1}{\rho W_t^2}$

Hence by subsituting the guess solution in the PDE we get

$$f'(t) - \frac{(\mu-r)^2}{2\sigma^2}\frac{\frac{1}{\rho^2 W_t^2}}{-\frac{1}{\rho W_t^2}} + \frac{1}{\rho W_t} r W_t - 1 - \log\left(\frac{1}{\rho W_t}\right) = \rho f(t) + \log(W_t)$$

$$\Rightarrow f'(t) + \frac{(\mu-r)^2}{2\rho\sigma^2} + \frac{r}{\rho} - 1 + \log(\rho) + \cancel{\log(W_t)} = \rho f(t) + \cancel{\log(W_t)}$$

$$\Rightarrow f'(t) = \rho f(t) - \left[\frac{(\mu-r)^2}{2\rho\sigma^2} + \frac{r}{\rho} + \log(\rho) - 1\right]$$

Let $\boxed{v = \frac{(\mu-r)^2}{2\rho\sigma^2} + \frac{r}{\rho} + \log(\rho) - 1}$

Then we get the following PDE

$$f'(t) = \rho f(t) - v \Rightarrow \frac{df}{dt} = \rho f - v \Rightarrow \frac{df}{\rho f - v} = dt \Rightarrow \int \frac{df}{\rho f - v} = \int dt + C_1$$

$$\overset{\rho \neq 0}{\Rightarrow} \frac{1}{\rho}\log(\rho f - v) = t + C_1 \Rightarrow \log(\rho f - v) = \rho(t + C_1)$$

$$\Rightarrow \rho f - v = e^{\rho(t+C_1)} \Rightarrow f = \frac{v + e^{\rho(t+C_1)}}{\rho}$$

C_1 is found from the boundary condition:

$V^*(T,W_T) = \varepsilon\log(W_T)$

$\Rightarrow f(T) + \frac{1}{\rho}\log(W_T) = \varepsilon\log(W_T)$

$\Rightarrow f(T) = \log(W_T^{\varepsilon}) - \log(W_T^{1/\rho})$

$\Rightarrow f(T) = \log(W_T^{\varepsilon - 1/\rho})$

$\Rightarrow \frac{v + e^{\rho(T+C_1)}}{\rho} = \log(W_T^{\varepsilon - 1/\rho}) \Rightarrow e^{\rho C_1} = \frac{\rho\log(W_T^{\varepsilon - 1/\rho}) - v}{e^{\rho T}}$

$$\Rightarrow \boxed{C_1 = \frac{1}{\rho}\log\left[\frac{\rho\log(W_T^{\varepsilon - 1/\rho}) - v}{e^{\rho T}}\right]}$$

Hence the control problem is solved:

- $V^*(t, W_t) = \dfrac{v + e^{\rho(t+C_1)}}{\rho} + \dfrac{1}{\rho}\log(W_t)$

  where
  - $v = \dfrac{(\mu - r)^2}{2\rho\sigma^2} + \dfrac{r}{\rho} + \log(\rho) - 1$
  - $C_1 = \dfrac{1}{\rho}\log\left[\dfrac{\rho\log\left(W_T^{\varepsilon^{-1}/\rho}\right) - v}{\varepsilon^{\rho T}}\right]$

- $\pi^*(t, W_t) = \dfrac{-\frac{1}{\rho W_t}\cdot(\mu - r)}{-\frac{1}{\rho W_t^2}\sigma^2 W_t} \Rightarrow \pi^*(t, W_t) = \dfrac{\mu - r}{\sigma^2}$

- $c^*(t, W_t) = \dfrac{1}{\frac{1}{\rho W_t}} \Rightarrow c^*(t, W_t) = \rho W_t$