Problem 1 - Assignment 10

- $dS_t = \sigma dz_t \Rightarrow S_T \sim N(S_t, \sigma(T-t))$, where we let $\mu_S = S_t$, $\sigma_S = \sigma(T-t)$

$$V(t,S_t,W,I) = E\left[-e^{-\gamma(W+IS_T)} \mid (t,S_t)\right]$$
$$= -e^{-\gamma W} E\left[e^{-\gamma I S_T} \mid (t,S_t)\right]$$
$$= -e^{-\gamma W} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\gamma I s} e^{-\frac{(s-\mu_S)^2}{2\sigma_S^2}} ds$$
$$= -e^{-\gamma W} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{2\gamma I \sigma_S^2 s + (s-\mu_S)^2}{2\sigma_S^2}\right) ds$$

The enumerator of the exponential can be written as follows:

$$2\gamma I\sigma_S^2 s + (s-\mu_S)^2 = 2\gamma I \sigma_S^2 s + s^2 - 2\mu_S s + \mu_S^2$$
$$= s^2 - 2(\mu_S - \gamma I \sigma_S^2)s + \mu_S^2$$
$$= s^2 - 2(\mu_S - \gamma I\sigma_S^2)s + (\mu_S - \gamma I \sigma_S^2)^2 + \mu_S^2 - (\mu_S - \gamma I \sigma_S^2)^2$$
$$= [s - (\mu_S - \gamma I \sigma_S^2)]^2 + \mu_S^2 - \mu_S^2 + 2\mu_S \gamma I \sigma_S^2 - \gamma^2 I^2 \sigma_S^4$$
$$= [s - (\mu_S - \gamma I\sigma_S^2)]^2 + 2\mu_S\gamma I \sigma_S^2 - \gamma^2 I^2 \sigma_S^4$$

So, $$V(t,S_t,W,I) = -e^{-\gamma W} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{[s-(\mu_S - \gamma I \sigma_S^2)]^2}{2\sigma_S^2}\right) ds \cdot \exp\left(\frac{-2\mu_S\gamma I\sigma_S^2 + \gamma^2 I^2 \sigma_S^4}{2\sigma^2}\right)$$
$$= -e^{-\gamma W} \cdot 1 \cdot e^{-\mu_S \gamma I} \cdot e^{\gamma^2 I^2 \sigma_S^2/2}$$

which is $\boxed{V(t,S_t,W,I) = -e^{-\gamma W} \cdot e^{-S_t\gamma I} \cdot e^{\frac{\gamma^2 I^2 \sigma^2 (T-t)}{2}}}$

- For ease of exposition we denote: $Q^{(b)}(t,S_t,I) = Q^{(b)}$, $Q^{(a)}(t,S_t,I) = Q^{(a)}$

> $V(t,S_t,W-Q^{(b)}, I+1) = -\exp(-\gamma(W-Q^{(b)}))\exp(S_t\gamma(I+1))\exp\left(\frac{\gamma^2(I+1)^2\sigma^2(T-t)}{2}\right)$

> $V(t,S_t,W-Q^{(b)}, I+1) = -\exp(-\gamma W)\exp(-\gamma S_t I)\exp\left(\frac{\gamma^2 I^2\sigma^2(T-t)}{2}\right)$

Hence,

$$-\exp(-\gamma W)\exp(-\gamma S_t I)\exp\left(I^2 \frac{\gamma^2\sigma^2(T-t)}{2}\right) = -\exp(-\gamma(W-Q^{(b)}))\exp(-\gamma S_t(I+1))\exp\left((I+1)^2\frac{\gamma^2\sigma^2(T-t)}{2}\right)$$
$$\Rightarrow \exp\left(I^2\frac{\gamma^2\sigma^2(T-t)}{2}\right) = \exp(\gamma Q^{(b)})\exp(-\gamma S_t)\exp\left((I+1)^2\frac{\gamma^2\sigma^2(T-t)}{2}\right)$$
$$\Rightarrow I^2\frac{\gamma^2\sigma^2(T-t)}{2} = \gamma Q^{(b)} - \gamma S_t + (I+1)^2\frac{\gamma^2\sigma^2(T-t)}{2}$$
$$\Rightarrow \gamma Q^{(b)} = (I^2 - (I+1)^2)\frac{\gamma^2\sigma^2(T-t)}{2} + \gamma S_t$$
$$\Rightarrow Q^{(b)} = S_t + (-2I-1)\frac{\gamma\sigma^2(T-t)}{2} \Rightarrow \boxed{Q^{(b)}(t,S_t,I) = S_t + (-2I-1)\frac{\gamma\sigma^2(T-t)}{2}}$$

Similarly,

$$V(t, S_t, W; I) = V(t, S_t, W + Q^{(a)}, I-1)$$

$$\Rightarrow -\exp(-\gamma W)\exp(-\gamma S_t I)\exp\left(I^2 \frac{\gamma^2\sigma^2(T-t)}{2}\right) = -\exp(-\gamma(W+Q^{(a)}))\exp(-\gamma S_t(I-1))\exp\left((I-1)^2\frac{\gamma^2\sigma^2(T-t)}{2}\right)$$

$$\Rightarrow \exp\left(I^2 \frac{\gamma^2\sigma^2(T-t)}{2}\right) = \exp(-\gamma Q^{(a)})\exp(\gamma S_t)\exp\left((I-1)^2\frac{\gamma^2\sigma^2(T-t)}{2}\right)$$

$$\Rightarrow I^2 \frac{\gamma^2\sigma^2(T-t)}{2} = -\gamma Q^{(a)} + \gamma S_t + (I-1)^2\frac{\gamma^2\sigma^2(T-t)}{2}$$

$$\Rightarrow \gamma Q^{(a)} = \left((I-1)^2 - I^2\right)\frac{\gamma^2\sigma^2(T-t)}{2} + \gamma S_t$$

$$\Rightarrow Q^{(a)} = (-2I+1)\frac{\gamma\sigma^2(T-t)}{2} + S_t \Rightarrow \boxed{Q^{(a)}(t, S_t, I) = S_t + (-2I+1)\frac{\gamma\sigma^2(T-t)}{2}}$$