Assignment 8

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CME 241

Problem 1

To form the required mathematical modeling, we will follow closely the series of events that happen during one day (time=t):

- Customers make deposits d_t.
- Bank liquidates its investments from the previous day, and has available cash a_t.
- The bank borrows amount $y_t > 0$ or pays off some of the liabilities $y_t < 0$. This increases, or respectively decreases, next day's liabilities l_{t+1} by $y_t(1+R)$.
- Current (w_t) and pending (u_{t-1}) withdrawal requests are satisfied to the best possible extent, and unfulfilled withdrawal requests are logged as u_t.
- The bank buys q_t shares of stock at price s_t.
- If the available cash c_t satisfies $c_t < C$, the bank pays $K \cdot cot(\frac{\pi c_t}{2C})$ to the regulator.

The *State* is observed everyday after the liquidation of the investment assets and it contains:

- Time $t \in [1, \dots, T]$
- The cash a_t available to the bank, after selling the assets bought the previous day
- The bank's liabilities l_t for loans taken up to the previous day
- The amount u_t of unfulfilled withdrawal requests
- The current stock price s_t for the investment asset under consideration

Each day $t = 0, \dots, T-1$, the bank decides two things: (a) the quantity of stock q_t to invest in, and (b) the amount y_t to borrow or pay off. Hence, the *Action Space* is the pair (q_t, y_t) for $t = 0, \dots, T-1$, assuming also $q_T = 0, y_T = 0$

Since the goal is to maximize the Expected Utility U of assets less liabilities at the end of a T-day horizon, the *Reward* is 0 for $t = 1, \dots, T-1$ and on the last day $U(a_T - l_T)$.

Recursively, we get the following *State Transitions*:

$$a_{t+1} = max\left(a_t + y_t - q_t s_t - K \cdot cot\left(\frac{\pi \cdot min(a_t + y_t - q_t s_t,\ C)}{2C}\right) + d_{t+1} - w_{t+1} - u_t,\ 0\right) + q_t s_{t+1}$$

$$l_{t+1} = (l_t + y_t)(1+R)$$

$$u_{t+1} = max\left(w_{t+1} + u_t - \left(a_t + y_t - q_t s_t - K \cdot cot\left(\frac{\pi \cdot min(a_t + y_t - q_t s_t,\ C)}{2C}\right) + d_{t+1}\right), 0\right)$$

s_{t+1} known from the stochastic process that the price follows

We also note that the distributions for d_t and w_t are assumed to be known. Additionally, the following constraints are imposed:

- If the bank borrows money, then $y_t > 0$. If the bank pays off money, then $y_t < 0$, but since we cannot pay back more than what is owed, we require $y_t \geq l_t$.
- The bank must borrow sufficient amount of money to ensure they can pay the regulator. If $c_t < C$, then it is required that $c_t \geq K \cdot cot(\frac{\pi c_t}{2C})$. This defines a lower limit for c_t, let's call that c_{lo}. Hence, it is required that $a_t + y_t \geq c_{lo}$
- The bank can make investments only if they have already ensured they possess the amount required by the regulator, i.e. $a_t + y_t - q_t s_t \geq c_{lo}$

Having modeled the problem as an MDP, it is clear that we are working under the continuous-state, continuous-action, finite-horizon setting. Thus, we can use Approximate Value Iteration and sample both the state space and the action space. The optimal solution can be obtained with a simple backward induction from final time step back to time step 0.

Problem 2

The goal is to identify the optimal supply S that minimizes your Expected Cost $c(S)$, given by the following:

$$\begin{aligned} c(S) &= p\mathbb{E}[(x-S)^+] + h\mathbb{E}[(S-x)^+] \\ &= \mathbb{E}[h(S-x)^+ + p(x-S)^+] \end{aligned}$$

Let $g(y) = hy^+ + py^-$, then $c(S) = \mathbb{E}[g(S-x)]$, where g is convex and also convexity is preserved by linear transformation. So, by the expectation operator we get that c(s) is convex, too.

By Jensen's inequality, $\mathbb{E}[g(S-x)] \geq g(S-\bar{x})$, where $\bar{x} = \int_{-\infty}^{\infty} uf(u)du$ and the minimum of $g(S-\bar{x})$ is achieved at $\bar{x}$.

If the distribution of x is continuous, we can find an optimal solution by taking the derivative of $c(S)$ and setting it to zero. Since we can interchange the derivative and the expectation operators, it follows that:

$$c'(S) = h\mathbb{E}[\delta(S-x)] - p\mathbb{E}[\delta(x-S)]$$

where $\delta(u) = \begin{cases} 1, & u \geq 0 \\ 0, & \text{otherwise} \end{cases}$ Consequently,

$$\begin{aligned} c'(S) &= h\mathbb{P}[S-x \geq 0] - p\mathbb{P}[x-S \geq 0] \\ &= h\mathbb{P}[x \leq S] - p\mathbb{P}[x \geq S] \\ &= h\mathbb{P}[x \leq S] - p(1-\mathbb{P}[x \leq S]) \\ &= (h+p)\mathbb{P}[x \leq S] - p \end{aligned}$$

Setting the derivative to zero reveals that:

$$c'(S) = 0 \implies \mathbb{P}[x \leq S] = \frac{p}{h+p}$$

If cdf F is continuous then $S^* = inf\{S \geq 0 : \mathbb{P}[x \leq S] \geq \frac{p}{h+p}\}$. And if F is strictly increasing then F has an inverse and there is a unique optimal solution given by:

$$S^* = F^{-1}\left(\frac{p}{h+p}\right)$$

This problem can be expressed in terms of a call/put options portfolio problem. If we wanted to sell a call and a put, then the formulation of the above problem is equivalent to finding the optimal strike price K to minimize the payoff for a call with payoff $\mathbb{E}[(x-K)^+]$ and a put with payoff $\mathbb{E}[(K-x)^+]$, where p, h denote the slopes of the two payoffs, respectively.