Convex optimization exercises pt. 0 - math

Fact. *Let M be a real symmetric matrix. Then eigenvalues of M are all real. Moreover, there exists an orthonormal basis consisting of eigenvectors of M.*

Exercise 1. *Show that a real symmetric matrix M can be decomposed as $M = A^T DA$, where A is orthogonal and D diagonal.*

Exercise 2. *Show that for a real symmetric matrix M the following are equivalent:*

- *all eigenvalues of M are non-negative;*
- *$x^T Mx \geq 0$ for all $x \in \mathbb{R}^n$;*
- *$M = B^T B$ for some matrix B.*

Either of the above conditions defines a *positive semi-definite (PSD)* matrix. We denote the fact that A is PSD by $A \succeq 0$. The set of all $n \times n$ positive semi-definite matrices is denoted by S^n_+. Matrices with strictly positive eigenvalues are called *positive definite* and denoted S^n_{++}. We will often use notation $A \succeq B$ to mean $A - B \succeq 0$

Exercise 3. *Show that $M \succeq \lambda I$ iff all eigenvalues of M are $\geq \lambda$.*

A matrix is *diagonally dominant* if, for each row, the absolute value of the diagonal entry is greater or equal to sum of absolute values of all the other entries.

Exercise 4. *Show that a symmetric diagonally dominant matrix with non-negative diagonal entries is PSD.*

For any real function f and $M = A^T DA$ (with A orthogonal and D diagonal, $D = \text{diag}(\lambda_1, \dots, \lambda_n)$) we can define $f(M) = A^T f(D)A$, where $f(D) = \text{diag}(f(\lambda_1), \dots, f(\lambda_n))$, whenever $f(\lambda_i)$ is well-defined for all eigenvalues λ_i of M.

Exercise 5. *Verify that the definitions of M^2 and M^{-1} (if it exists) coincide with the standard ones. Verify that $M^\alpha M^\beta = M^{\alpha+\beta}$ for any real α, β (if they exist).*

Exercise 6. *Show that if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, then $f(M) \preceq g(M)$ (if both exist).*

Exercise 7. *For a positive-definite M, verify that $\{x : (x-z)^T M^{-1}(x-z) \leq 1\}$ describes an ellipsoid. What is its center? What are the semi-axes and their lengths?*

Exercise 8. *Consider an ellipsoid $x^T M^{-1}x \leq 1$. Represent the left side of this inequality as $||y||$ for some y. Use this to derive another representation of the same ellipsoid.*

Exercise 9. *Compute the gradient and Hessian of the function $f(x) = x^T Ax + b^T x + c$.*