Exercises: duality

1. **Mohar-Poljak Theorem for MaxCut .** Recall from the lecture that in the MaxCut problem, we are given a graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}$ and the goal is to find a partition $V = A \uplus B$ which maximizes $w(A, B) = \sum_{a\in A, b\in B} w(ab)$.

   This is equivalent to the following (non-convex) quadratic program:

   $$\begin{array}{lll} \text{maximize} & \frac{1}{2}\sum_{ij\in E} w(ij)(1 - x_i x_j) & \\ \text{subject to} & x_i^2 = 1 & \forall\, i \in V \end{array}$$

   Let $n = |V|$. The *Laplacian* of (G, w) is the $n \times n$ matrix defined as follows:

   $$L_{ij} = \begin{cases} \sum_{ik\in E} w(ik) & i = j \\ -w(ij) & i \neq j,\ ij \in E \\ 0 & \text{otherwise.} \end{cases}$$

   Hence, our quadratic program can be rewritten as:

   $$\begin{array}{lll} \text{maximize} & \frac{1}{4}x^T L x & \\ \text{subject to} & x_i^2 = 1 & \forall\, i \in V \end{array}$$

   Using Lagrange duality, show that the weight of maximum cut can be upper-bounded by

   $$\frac{\lambda_{\max}(L)}{4} n,$$

   where $\lambda_{\max}(L)$ is the largest eigenvalue of L.

2. **The sum of the largest elements of a vector.** For a given integer $r = 1, \ldots, n$ define $f : \mathbb{R}^n \to \mathbb{R}$ as

   $$f(x) = \sum_{i=1}^{r} x_{[i]},$$

   where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ are the components of x.

   a) Prove that the constraint $f(x) \leq \alpha$ is convex.
   Hint: Showing that it is equivalent to $\binom{n}{r}$ linear inequalities.

   b) Note that given $x \in \mathbb{R}^n$, the value of $f(x)$ is the optimum value of the following LP:

   $$\begin{array}{ll} \text{minimize} & x^T y \\ \text{subject to} & 0 \leq y \leq 1 \\ & \sum_{i=1}^{n} y_i = r. \end{array} \tag{1}$$

   Derive the dual of the LP above and use it to emulate the constraint $f(x) \leq \alpha$ using only $O(n)$ linear inequalities.

   c) To see an application of f, extend the convex program Markovitz portfolio optimization from the lecture:

   $$\begin{array}{ll} \text{minimize} & x^T Q x \\ \text{subject to} & \sum_i x_i = B \\ & \mu^T x \geq M \\ & x \geq 0 \end{array}$$

   by a constraint saying that no more than 80% of the total budget can be invested in any 10% of the assets.

3. **Separating hyperplane between two polyhedra.** Formulate the following problem as an LP: find a separating hyperplane that strictly separates two polyhedra $\mathcal{P}_1 = \{x \mid Ax \leq b\}$ and $\mathcal{P}_2 = \{x \mid Cx \leq d\}$, i.e., find a hyperplane $a^T x = \gamma$ such that $a^T x \geq \gamma$ for $x \in \mathcal{P}_1$ and $a^T x \leq \gamma$ for $x \in \mathcal{P}_2$.

   Hint: use LP duality to find γ and a such that

   $$\inf_{x\in\mathcal{P}_1} a^T x \geq \gamma \geq \sup_{x\in\mathcal{P}_2} a^T x.$$