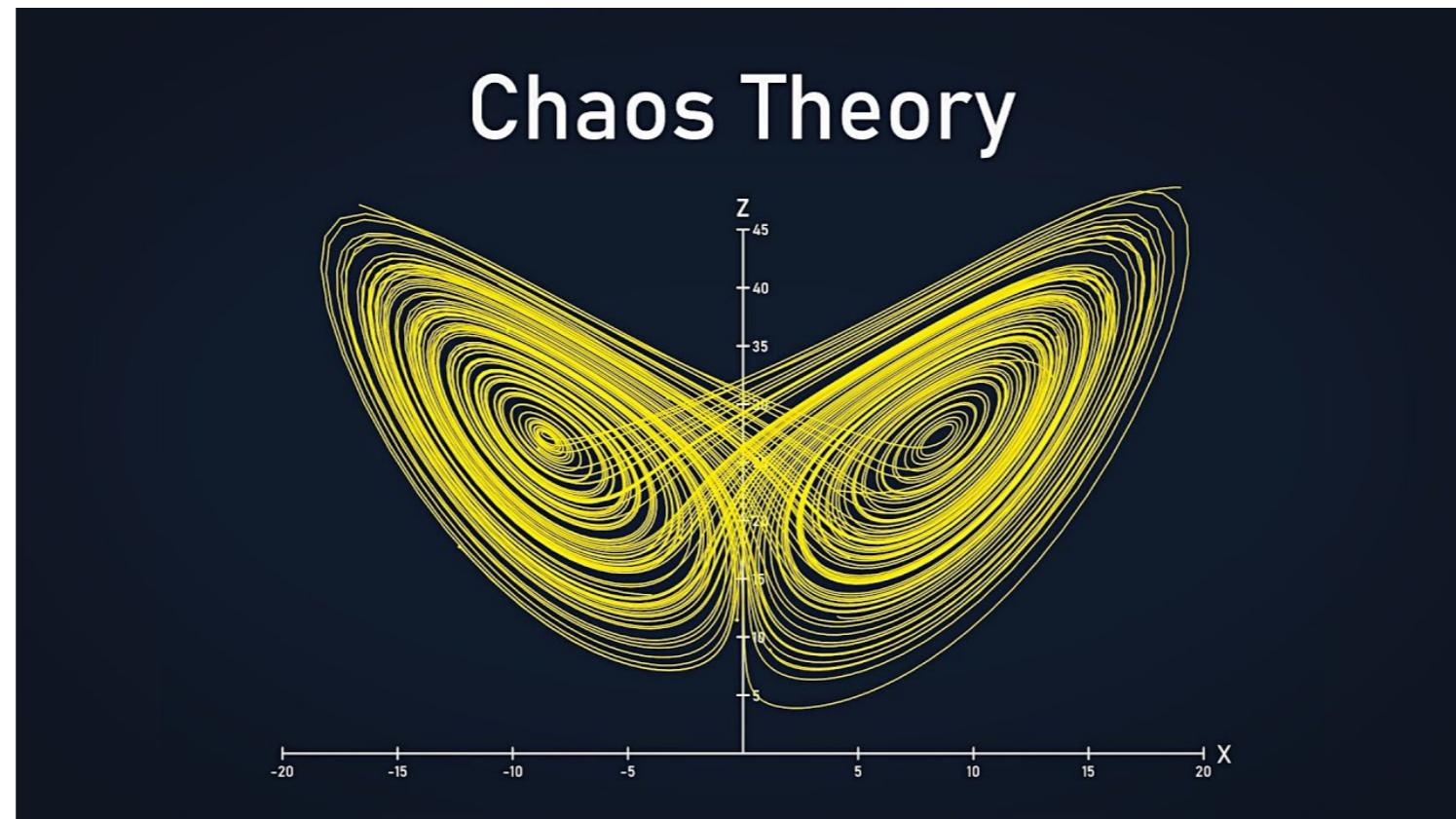


Chaos

Lecture 5



PHYS 246 class 4

Fall 2025

J Noronha-Hostler

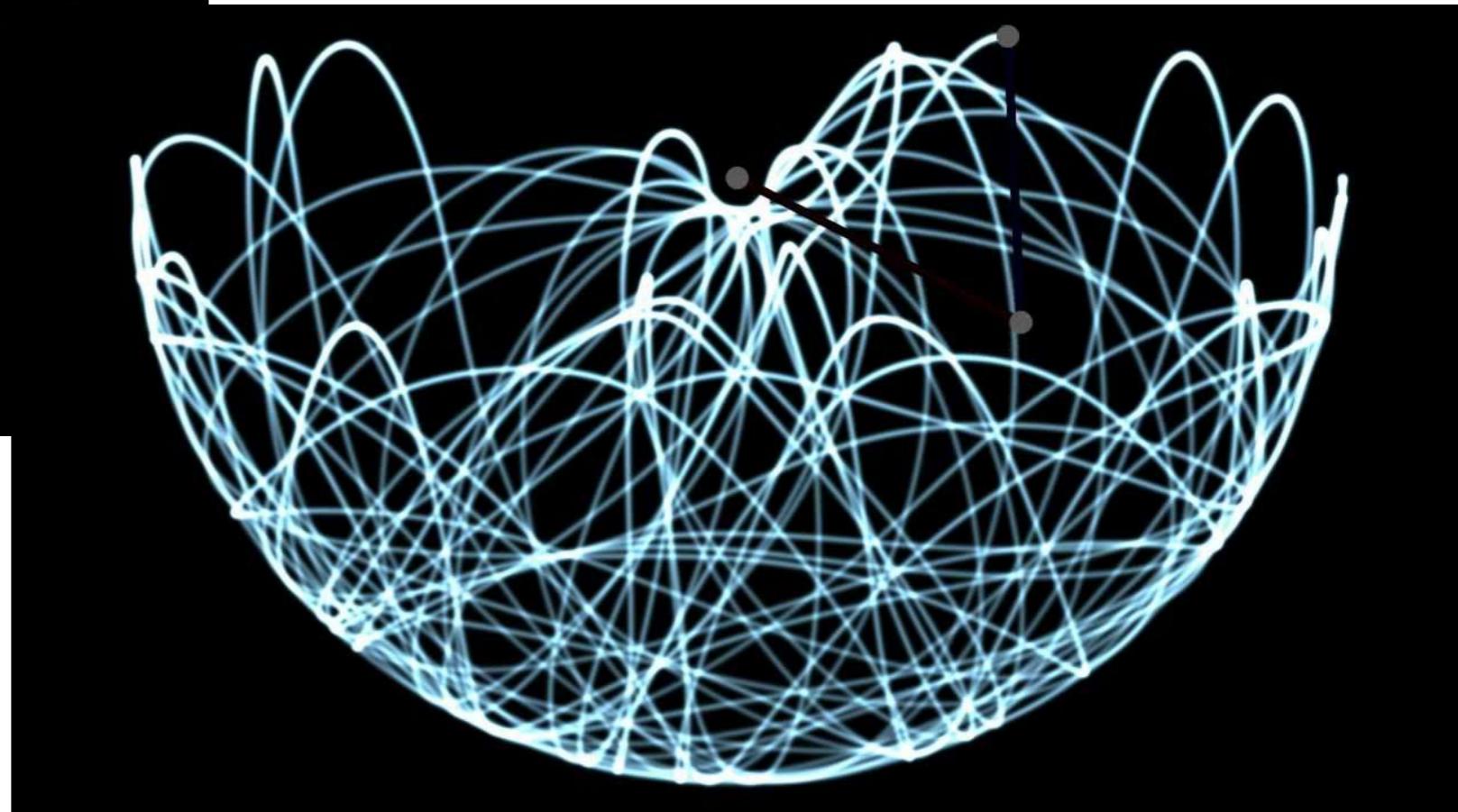
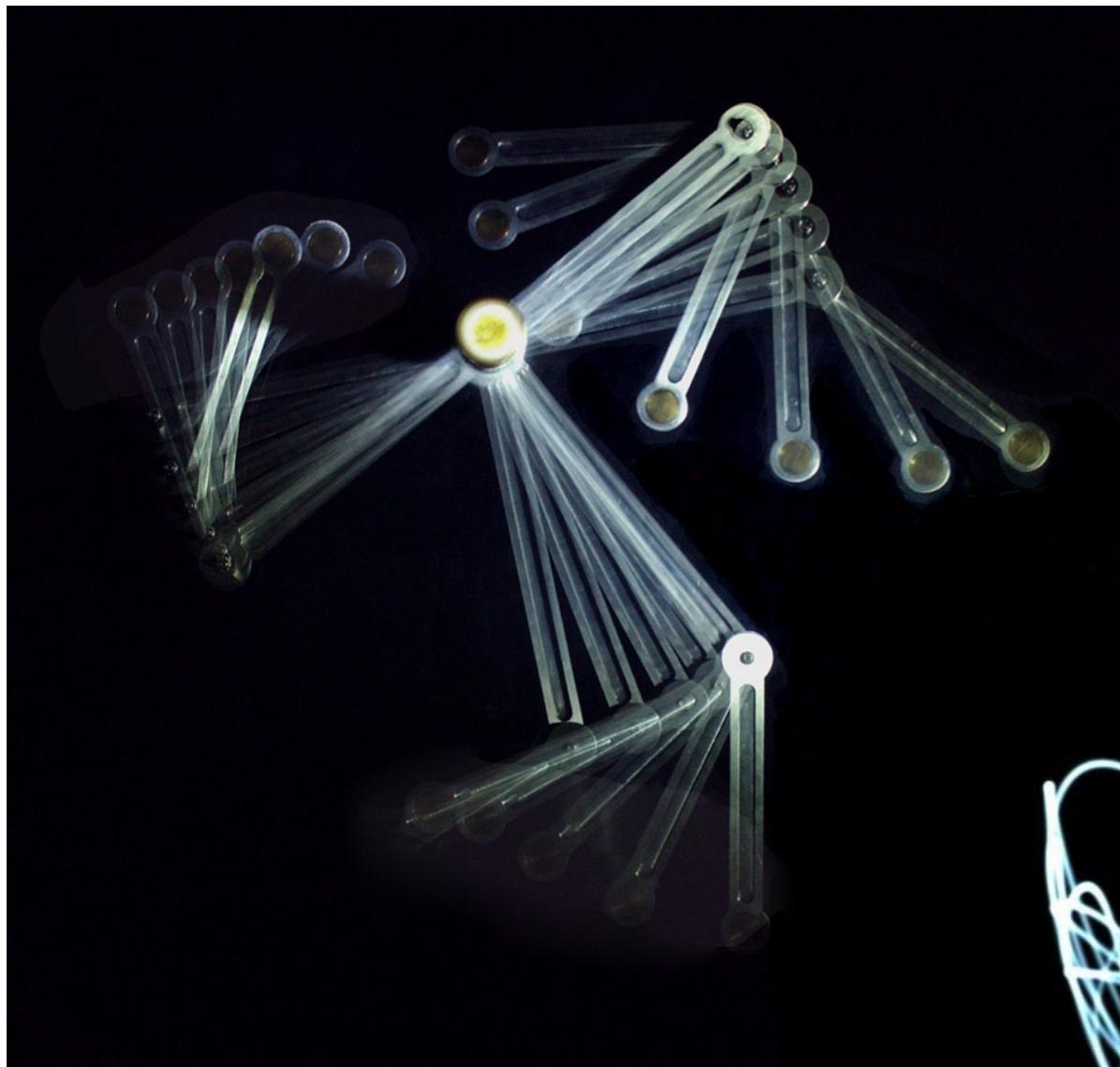
<https://jnoronhahostler.github.io/IntroductionToComputationalPhysics/intro.html>

Announcements/notes

- There are still some issues with the PDFs, starting for tonight's assignment (assigned last week), you WILL lose points (10% off)!
- For Animate functions in Chaos Assignment, it is expecting a 2D array of positions that needs to be passed as an argument as a list [positions]. If your positions array is 1D (can check using `positions.shape`), you can add a second axis using `positions[:,np.newaxis]` to get a (N,1) array.

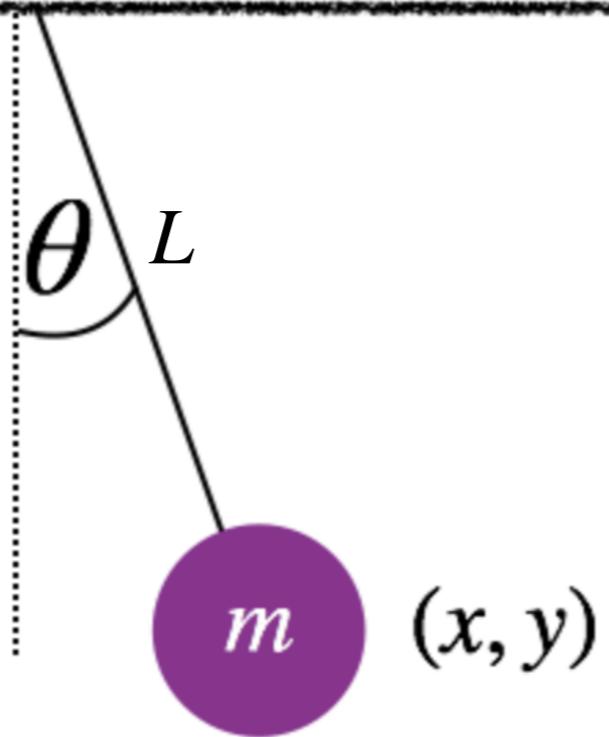
Chaos and pendulums

Starting from one pendulum and diving into chaos!



Pendulum

Going from $(x, y) \rightarrow \theta$



Bottom of arc

$$\theta = 0$$

Angular acceleration

Angular velocity

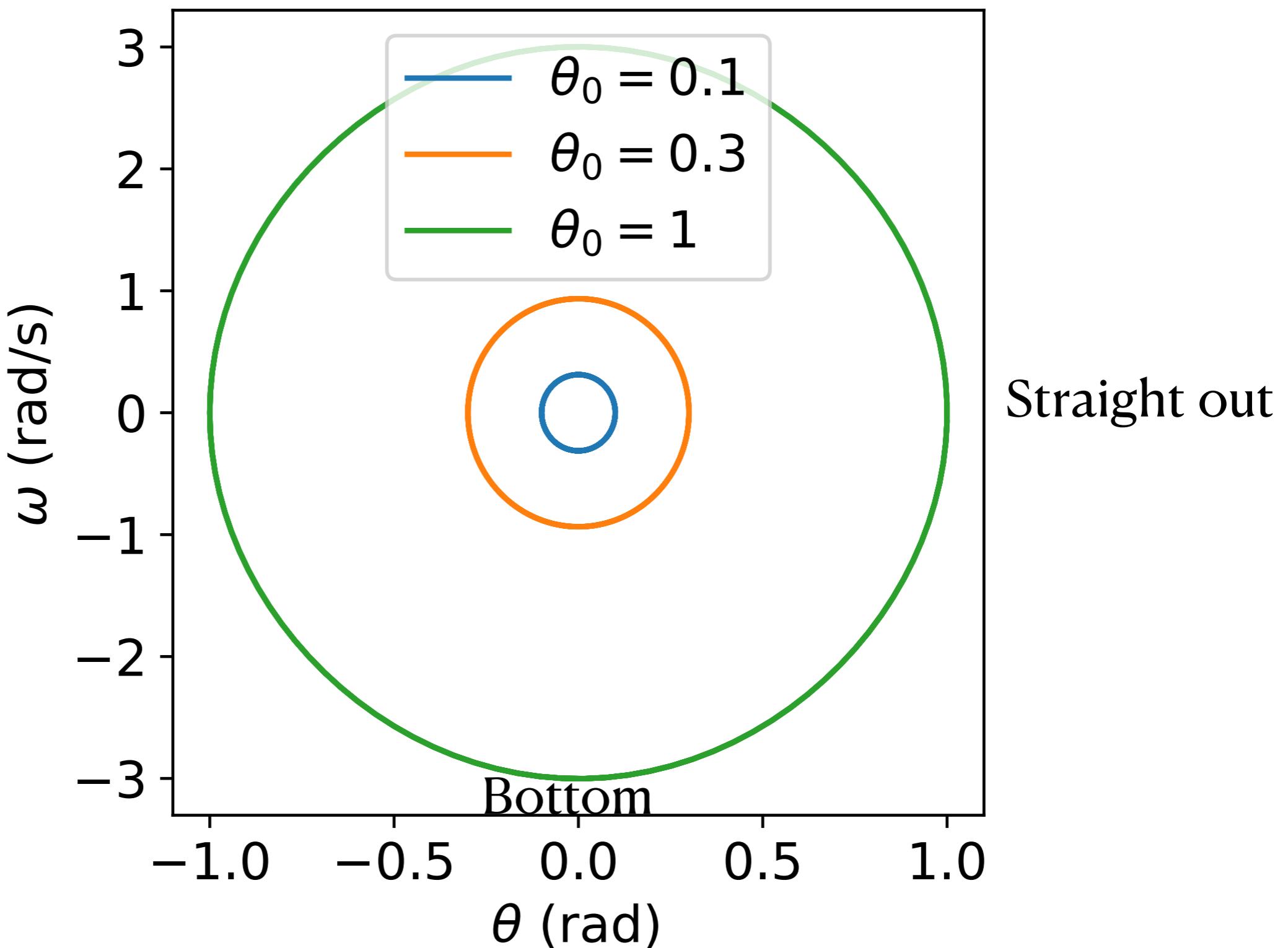
Angular displacement

$$\frac{d\omega}{dt} = \alpha(\theta) = -\frac{g \sin(\theta)}{L}$$

$$\omega(t + \Delta t) \sim \omega(t) + \frac{d\omega}{dt} \Delta t$$

$$\theta(t + \Delta t) \sim \theta(t) + \omega \Delta t$$

Angular velocity vs displacement



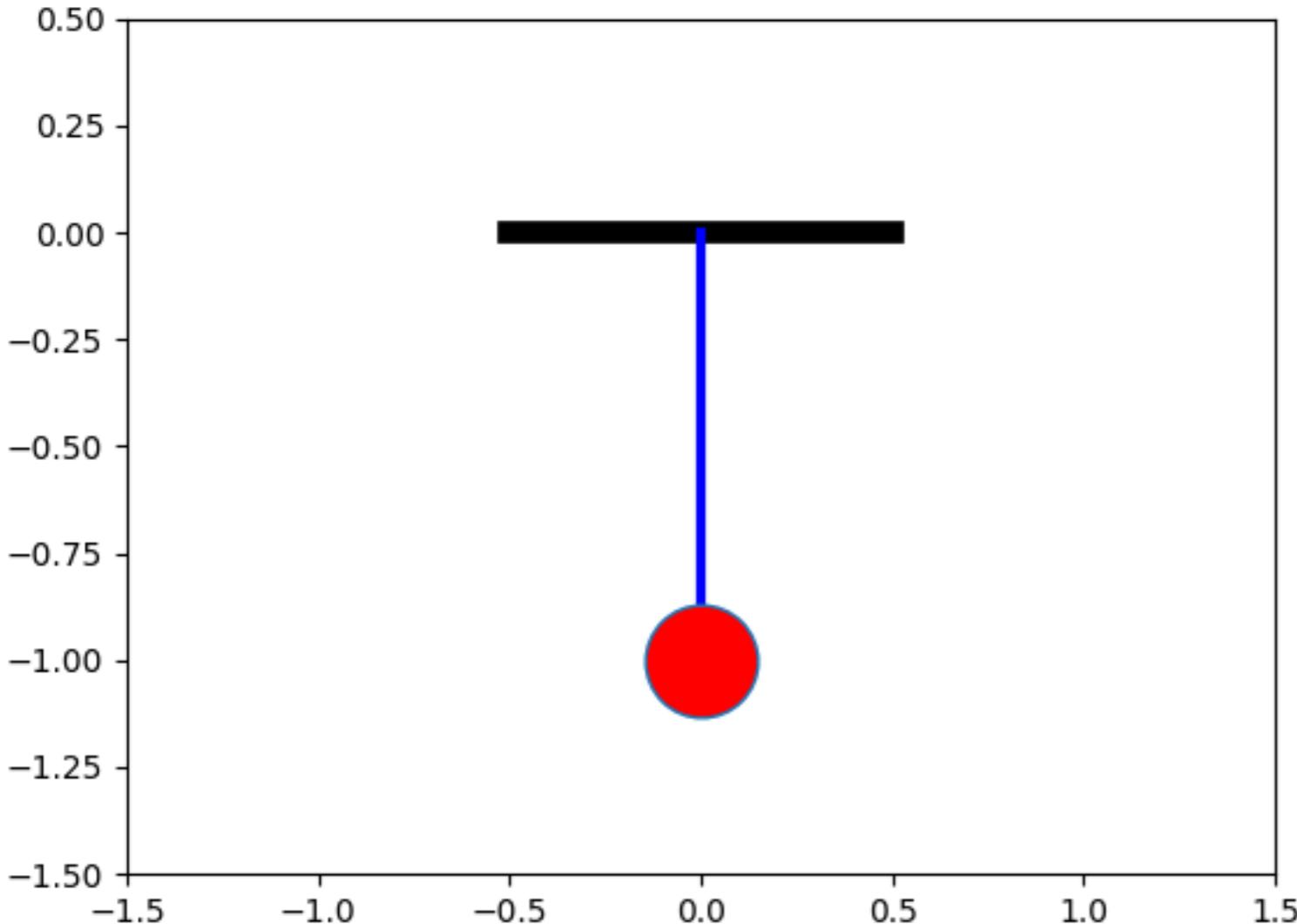
Small Angle approximation

When does it break down?

- Small angle approximation: $\theta \sim \sin\theta$
 - Sanity check: $\theta = 0, \quad 0 = \sin 0$
 - Can compare:
 - $\alpha(\theta) = -\frac{g \sin(\theta)}{L}$ vs $\alpha(\theta) = -\frac{g\theta}{L}$
 - When does this break down?

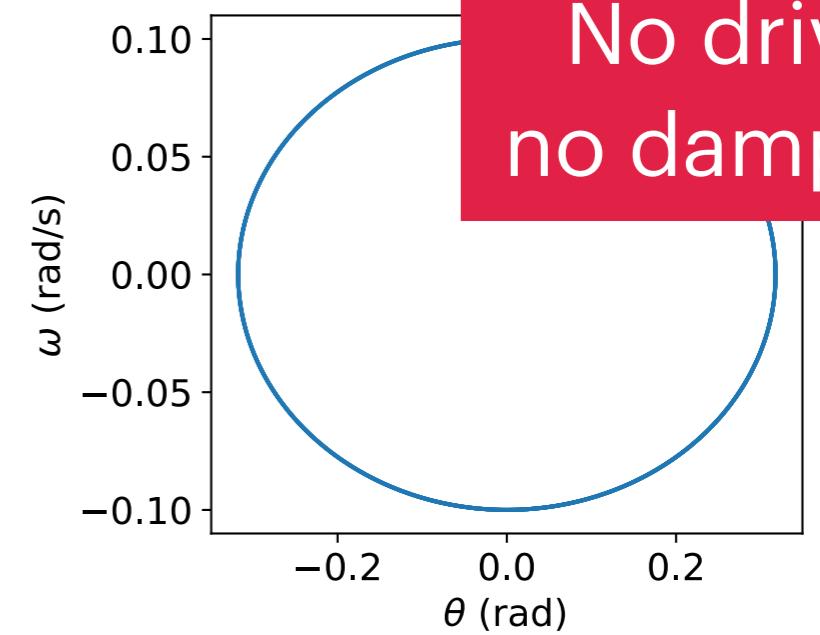
Driving Pendulum

Overdamped

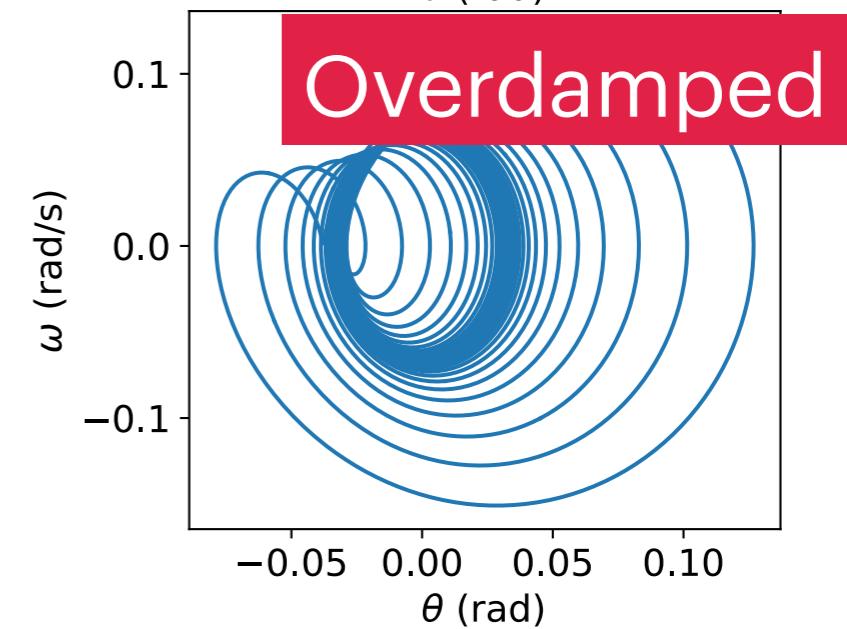


Underdamped (goes above start position, then slows)

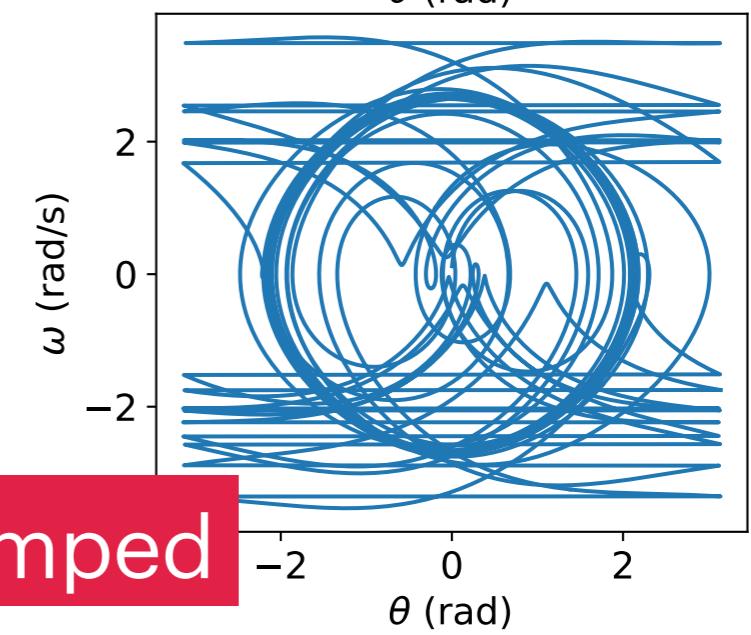
No drive,
no damping



Overdamped

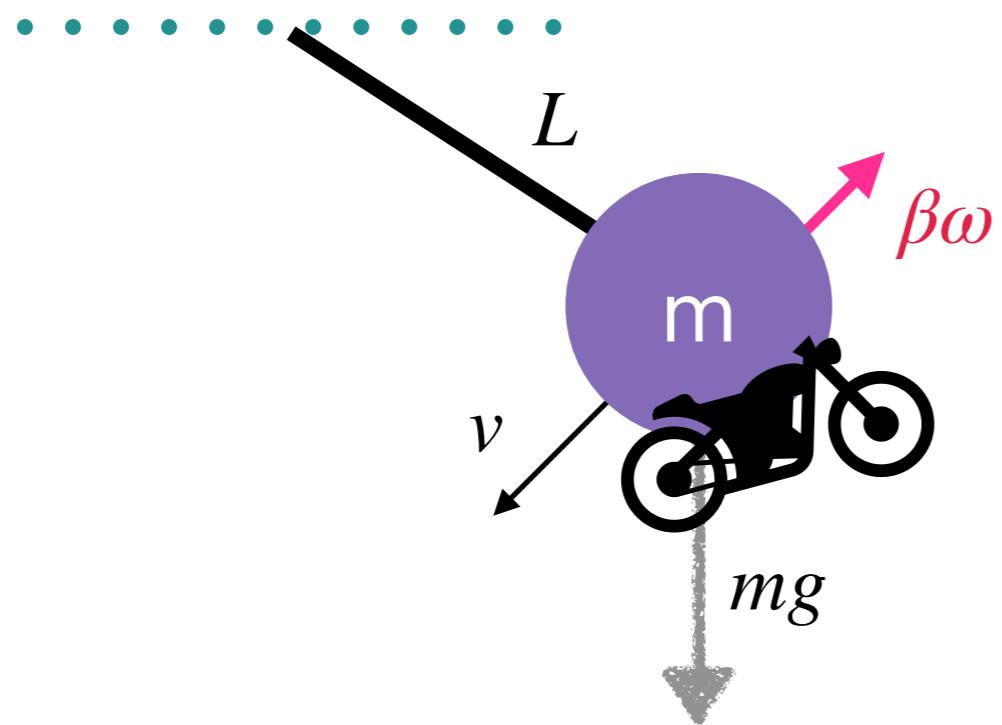


Underdamped



Damped vs driven a pendulum

Attach ball of mass m to a rod (not string!) of length L



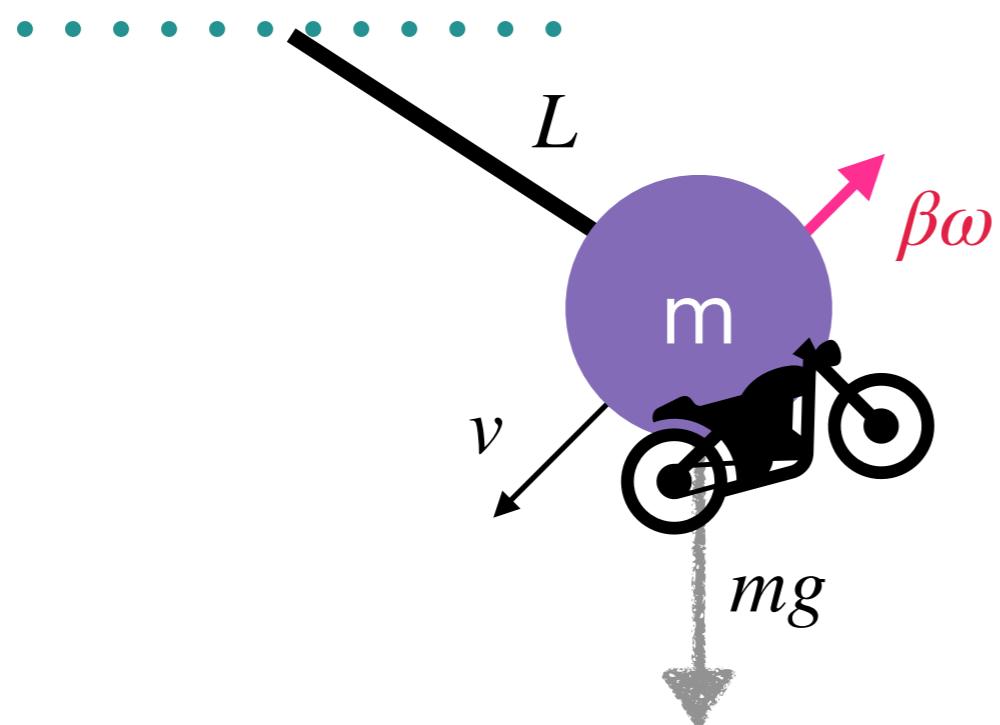
There's an angular velocity-dependent drag force $\beta\omega$ from air resistance

Pretend some engine is attached that provides an additional force (torque) $\tau = \gamma \sin \Omega_{ext} t$

$$\tau = I\alpha = mL^2 \frac{d^2\theta}{dt^2}$$

$$\frac{d\omega}{dt} = -A\omega - B \sin \theta + C \sin \Omega_{ext} t$$

Damped vs driven a pendulum



$$\tau = I\alpha = mL^2 \frac{d^2\theta}{dt^2}$$
$$= \underbrace{-mgL \sin \theta}_{F_g} + \underbrace{-\beta \frac{d\theta}{dt}}_{F_{drag}} + \underbrace{\gamma \sin \Omega_{ext} t}_{F_{ext}}$$

After some algebra...

$$\frac{d\omega}{dt} = -A\omega - B \sin \theta + C \sin \Omega_{ext} t$$

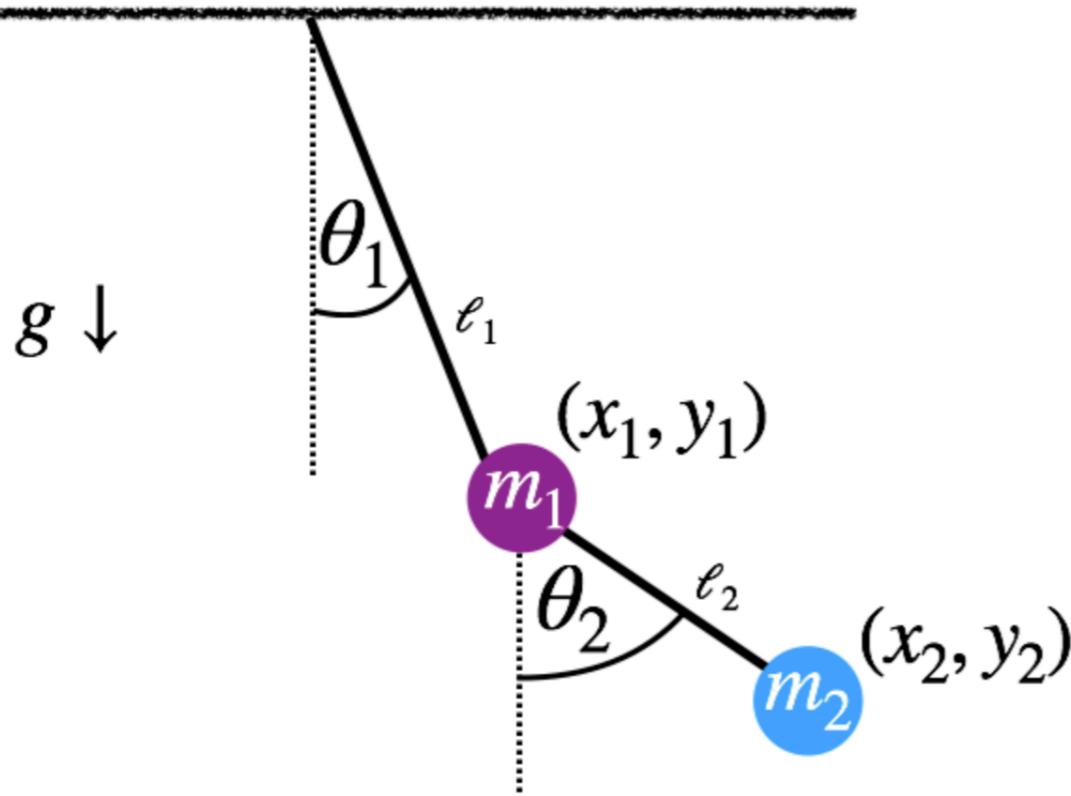
$$\omega = \frac{d\theta}{dt}$$

Double Pendulum

Yikes!

$$\alpha_1(\theta_1, \theta_2) := \frac{l_2}{l_1} \left(\frac{m_2}{m_1 + m_2} \right) \cos(\theta_1 - \theta_2)$$

$$\alpha_2(\theta_1, \theta_2) := \frac{l_1}{l_2} \cos(\theta_1 - \theta_2)$$



$$f_1(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) := -\frac{l_2}{l_1} \left(\frac{m_2}{m_1 + m_2} \right) \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - \frac{g}{l_1} \sin \theta_1$$

$$f_2(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) := \frac{l_1}{l_2} \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - \frac{g}{l_2} \sin \theta_2$$

$$g_1 := \frac{f_1 - \alpha_1 f_2}{1 - \alpha_1 \alpha_2}$$

$$g_2 := \frac{-\alpha_2 f_1 + f_2}{1 - \alpha_1 \alpha_2}$$

$$\frac{d}{dt} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ g_1(\theta_1, \theta_2, \omega_1, \omega_2) \\ g_2(\theta_1, \theta_2, \omega_1, \omega_2) \end{pmatrix}$$

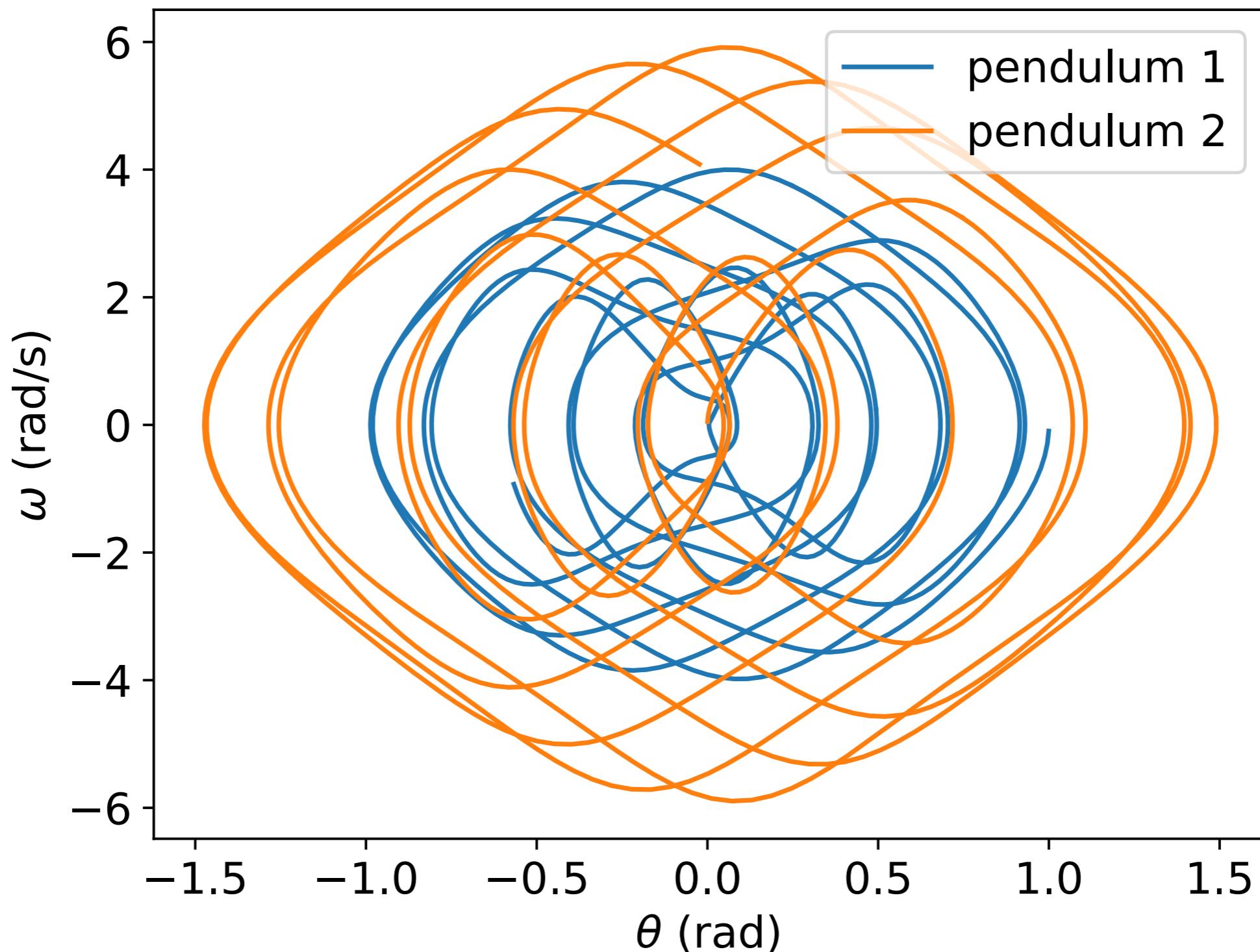
A bit difficult to work out forces!

Easier to use Euler-Lagrange method (PHYS 325) (principle of least action)

<https://dassencio.org/33>

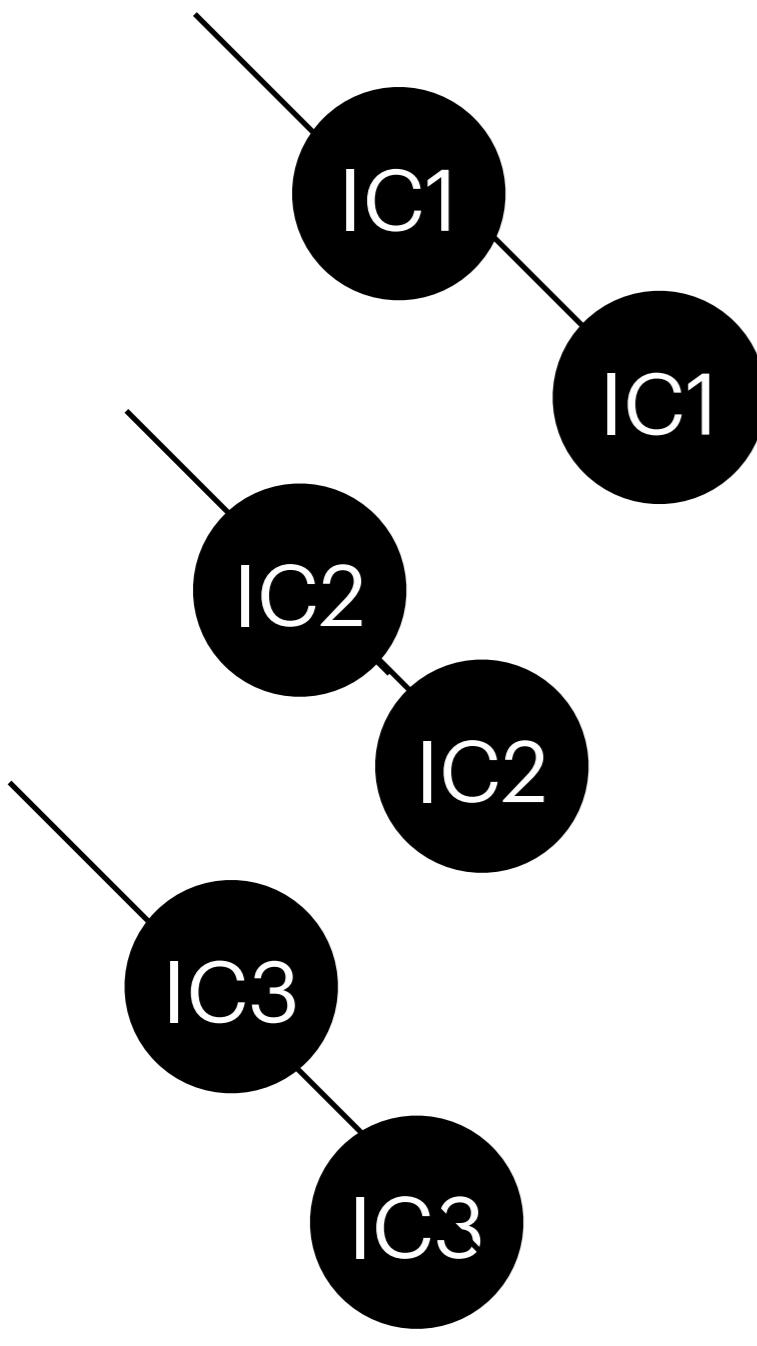
Double pendulum

Phase space

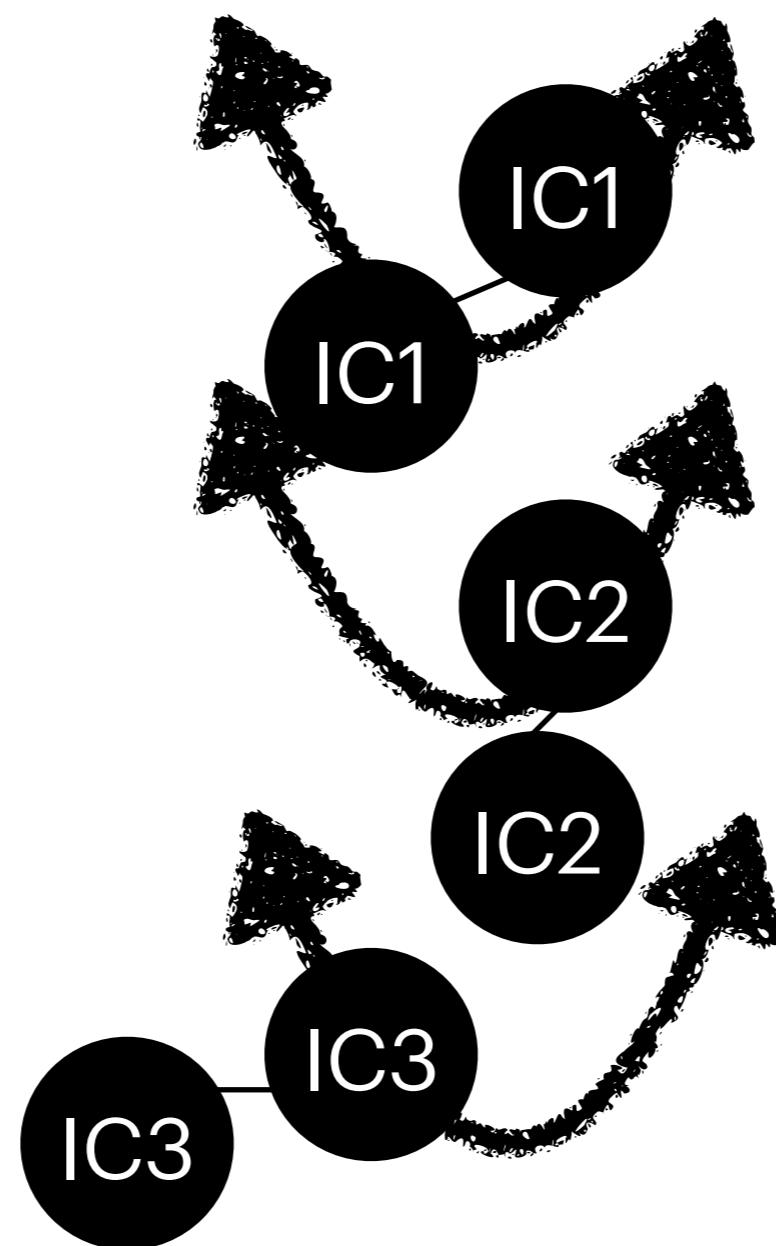


Chaos

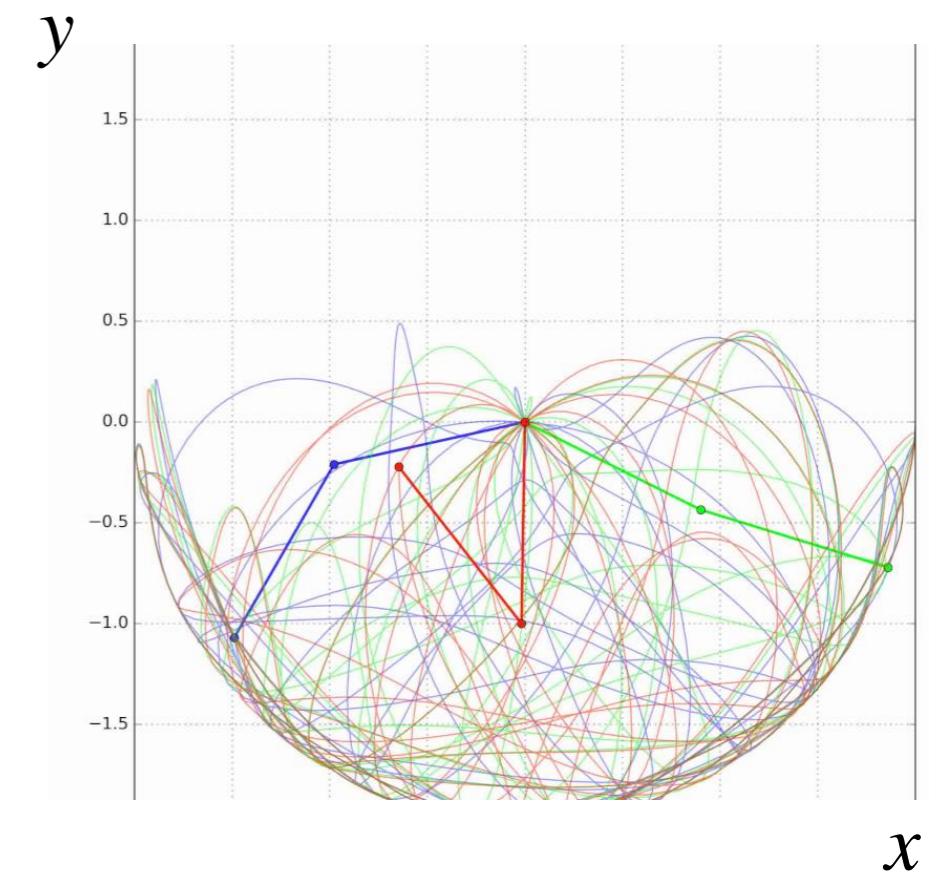
Initial conditions matter (a lot)!



t_0



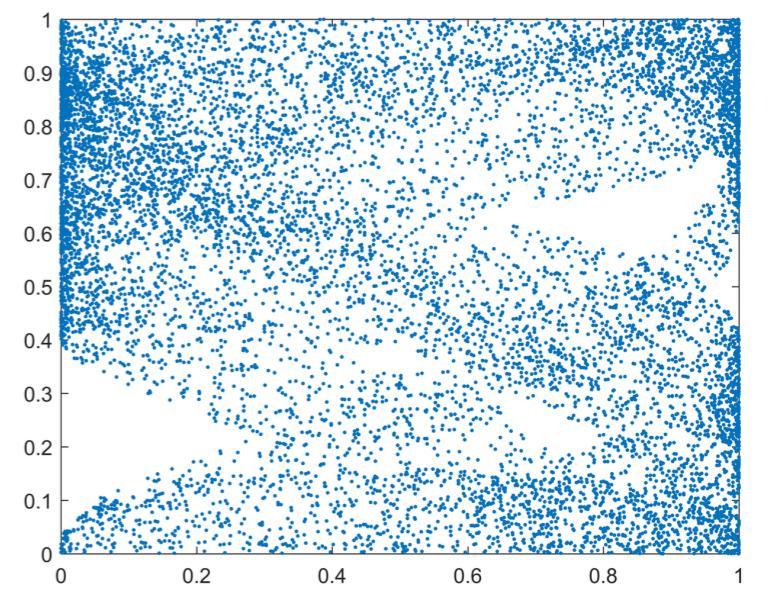
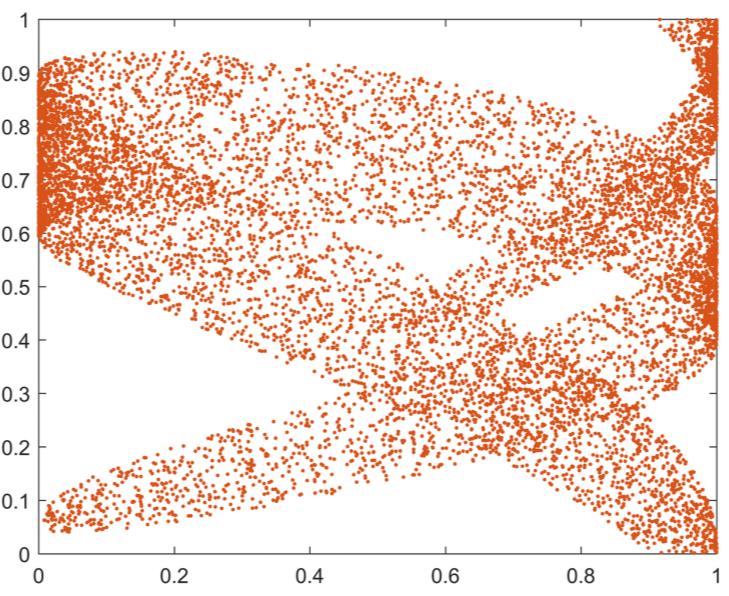
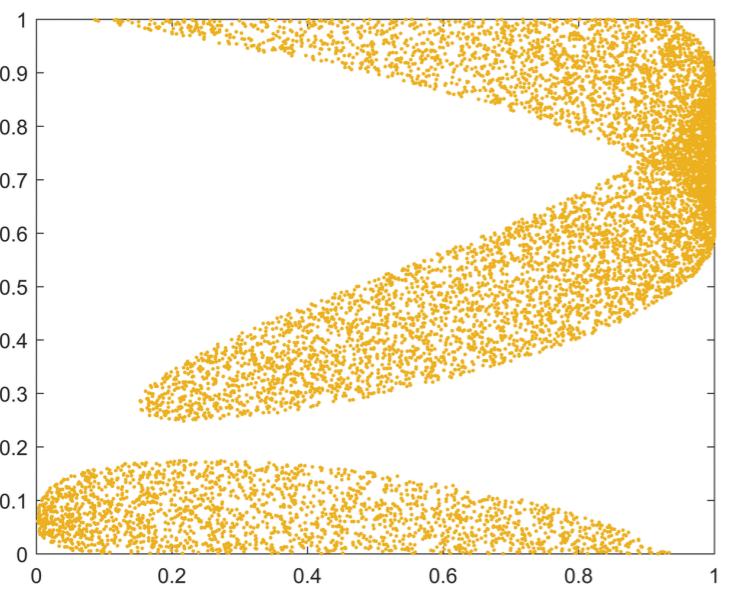
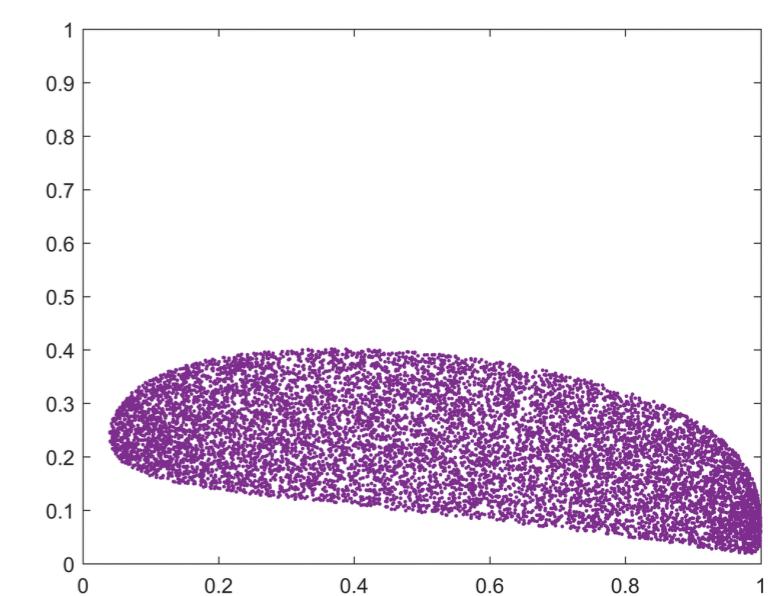
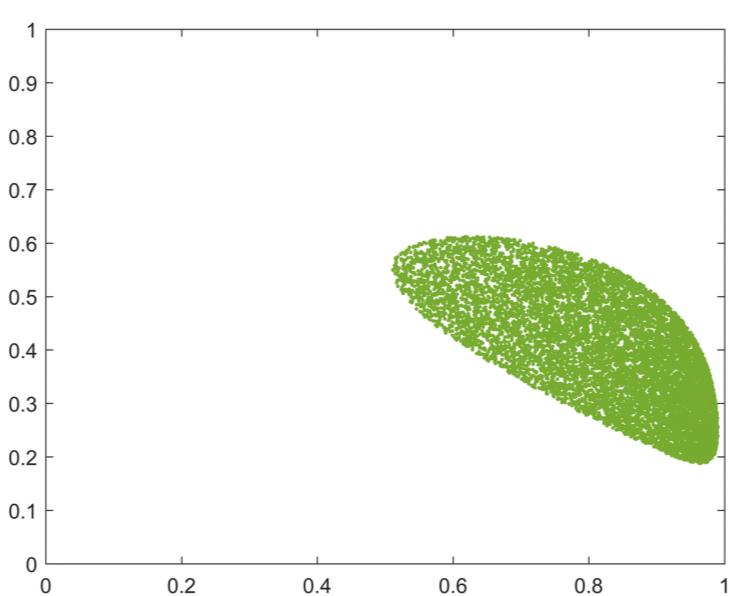
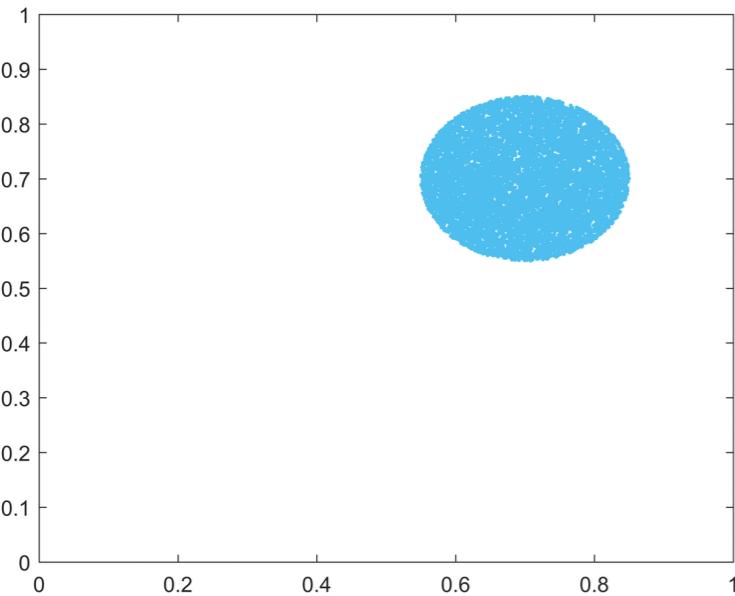
$t_0 = 10$



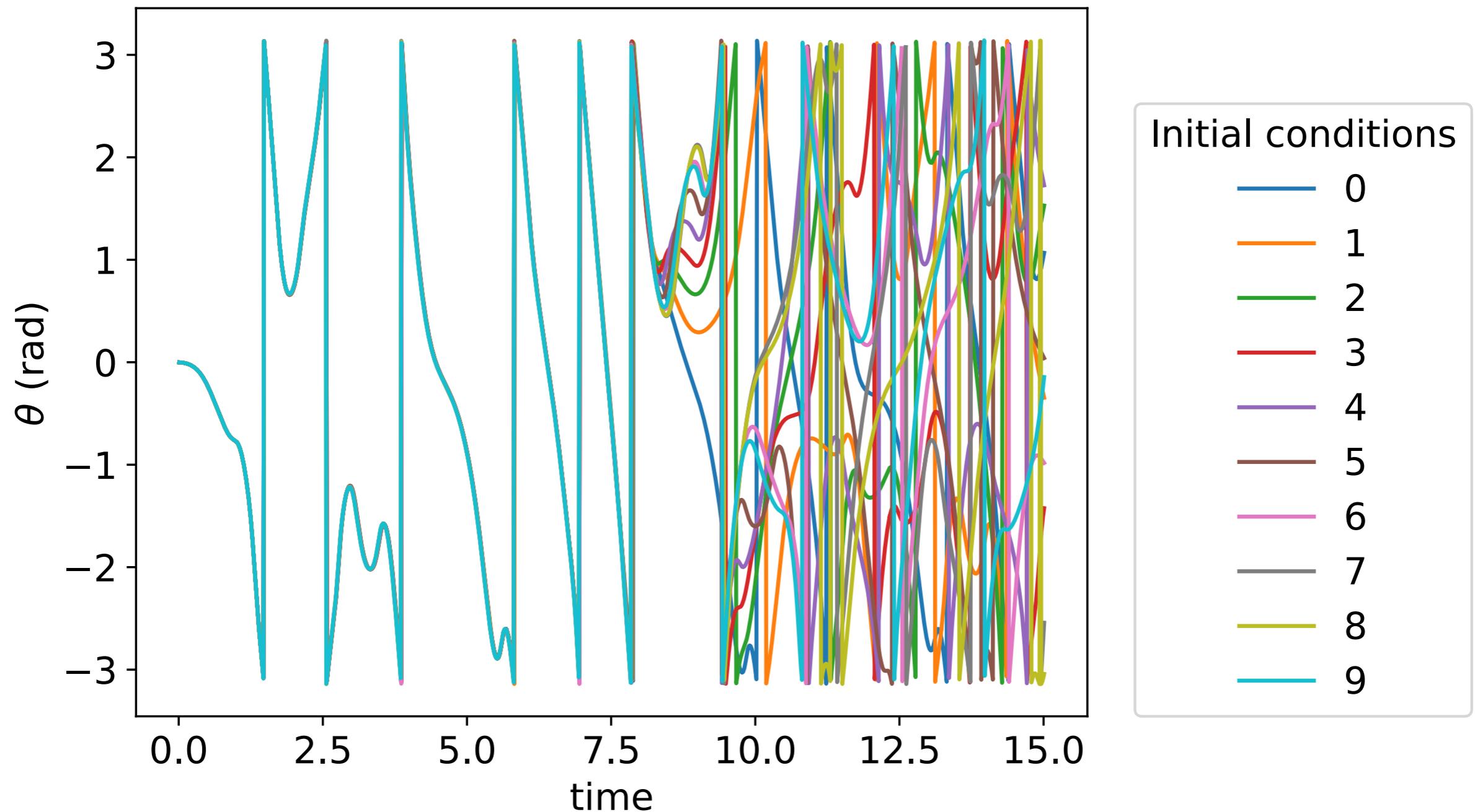
$t_0 = 100$

Chaos - general concept

By N3kromancer732 - Own work, CC BY-SA 4.0, <https://commons.wikimedia.org/w/index.php?curid=50379199>



Initial conditions over time



Checks for your code

Undamped, undriven pendulum:

- Small angles should match analytic result
- Energy should be conserved (phase space should be an ellipse)

Driven pendulum

Large driving force should make it unstable.

Small driving force plus damping --> should go to (0,0)

Double pendulum

Small initial angles/velocities, small angle approximation should still hold

Python suggestions

animate is meant to visualize multiple runs

```
def animateMe_single_pendulum(positions, params):
    """
    positions [run, time, theta]
    params {'l1': length of the pendulum}
    """
    ...
```

Problems with global variables

```
a=5
[7] ✓ 0.0s
```

```
More... def f(x):
                    return x * a
print(f(2))

[6] ✓ 0.0s
```

... 14

Magically wrap the positions to (-pi,pi)

```
... pos -= pos - 2*np.pi*np.ceil((pos-np.pi)/(2*np.pi))
```

```
a=7
[5] ✓ 0.0s
```

```
print(f(2))

[8] ✓ 0.0s
```

... 10