Chaos

https://lkwagner.github.io/IntroductionToComputationalPhysics/intro.html

ANNOUNCEMENTS/NOTES

- 'Exoplanets' was due today.
- Make sure you get the ipynb file that's available at class time! OK to look earlier but we put in bugfixes (and sometimes questions) up to the last minute. We won't change anything after class starts on the Thursday before it's due.

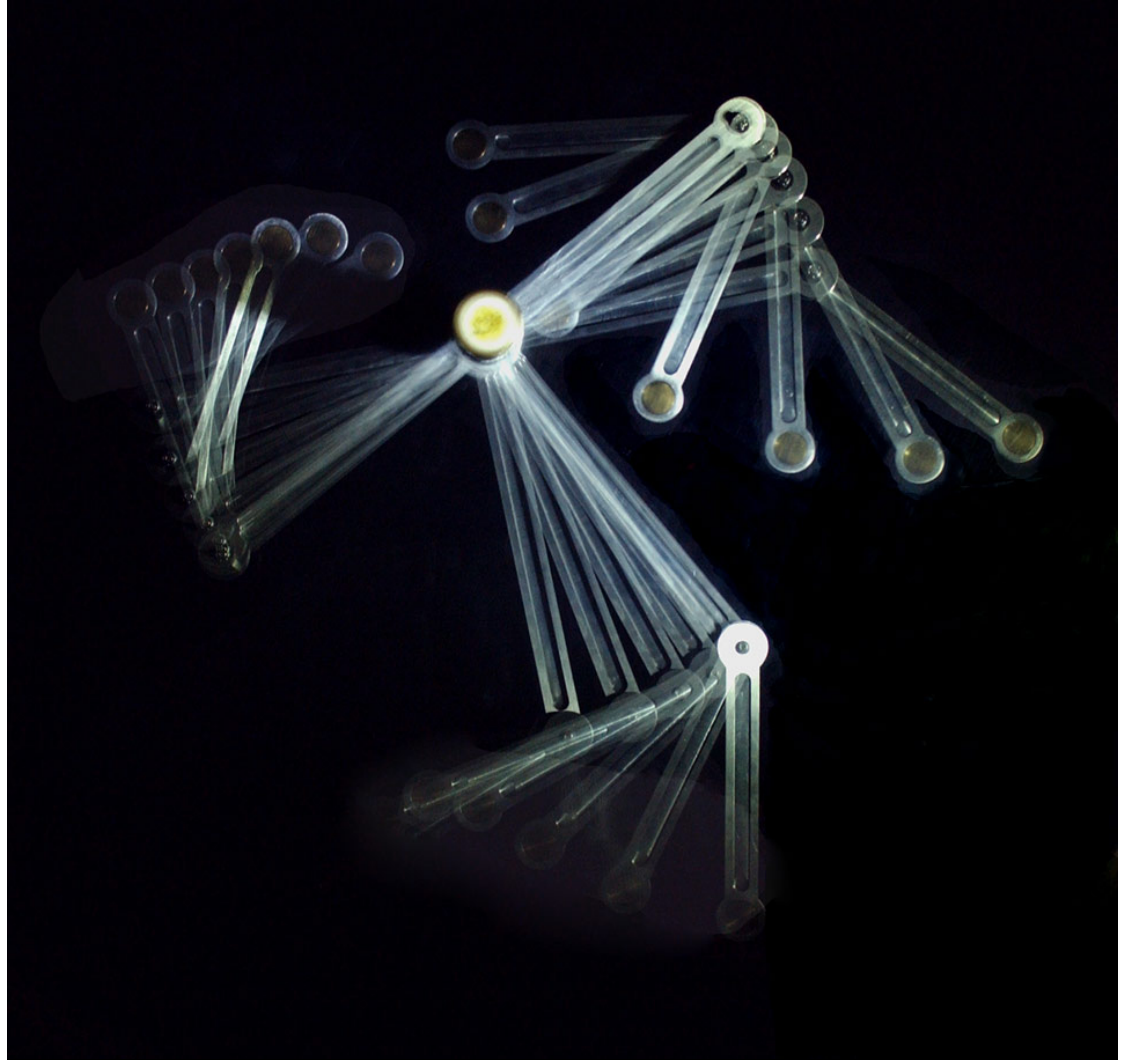
Communication (I promise this will help you...):

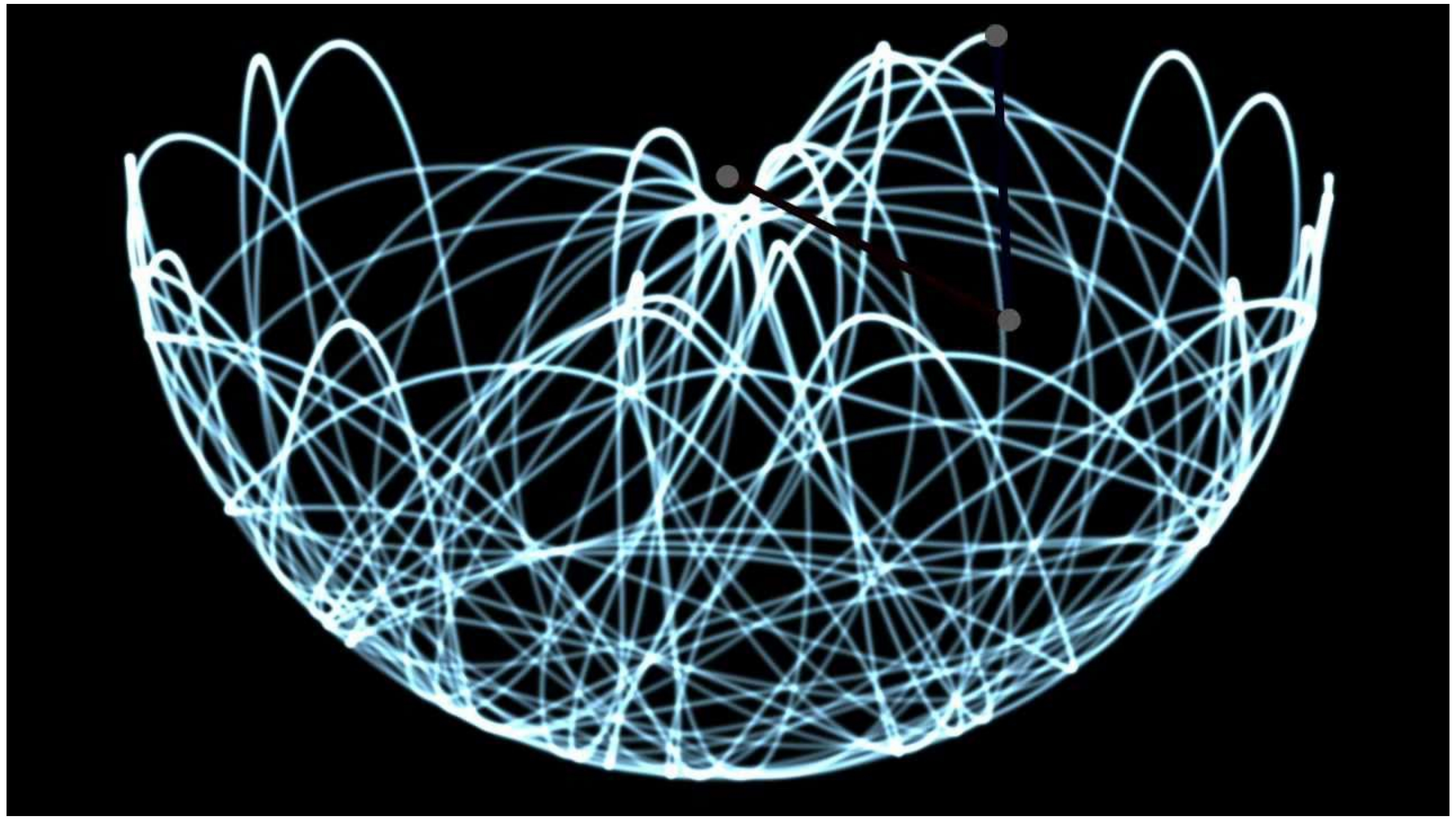
- Labels and units are required for graphs (graphs are not complete unless they have this!)
- When outputting numbers, also indicate what they are and the units.
- Docstrings for all functions.
- Document units as well.

SUMMARY

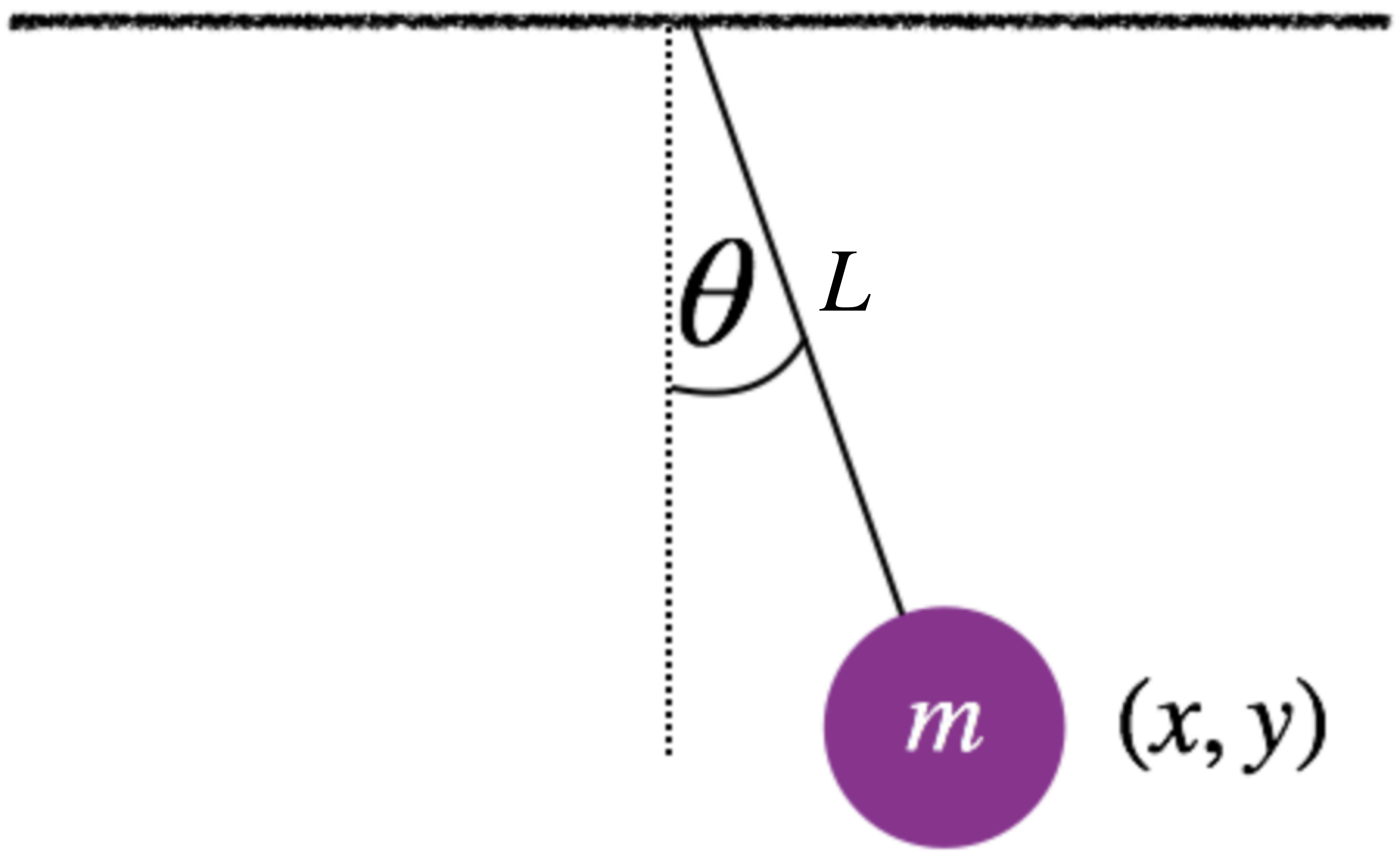
Going to consider 3 situations:

- Just a pendulum
- A driven pendulum with damping
- A double pendulum





PENDULUM



Bottom of arc

Angular acceleration

$$\frac{d\omega}{dt} = \alpha(\theta, \omega) = -\frac{g\sin(\theta)}{L}$$

We will do the midpoint method:

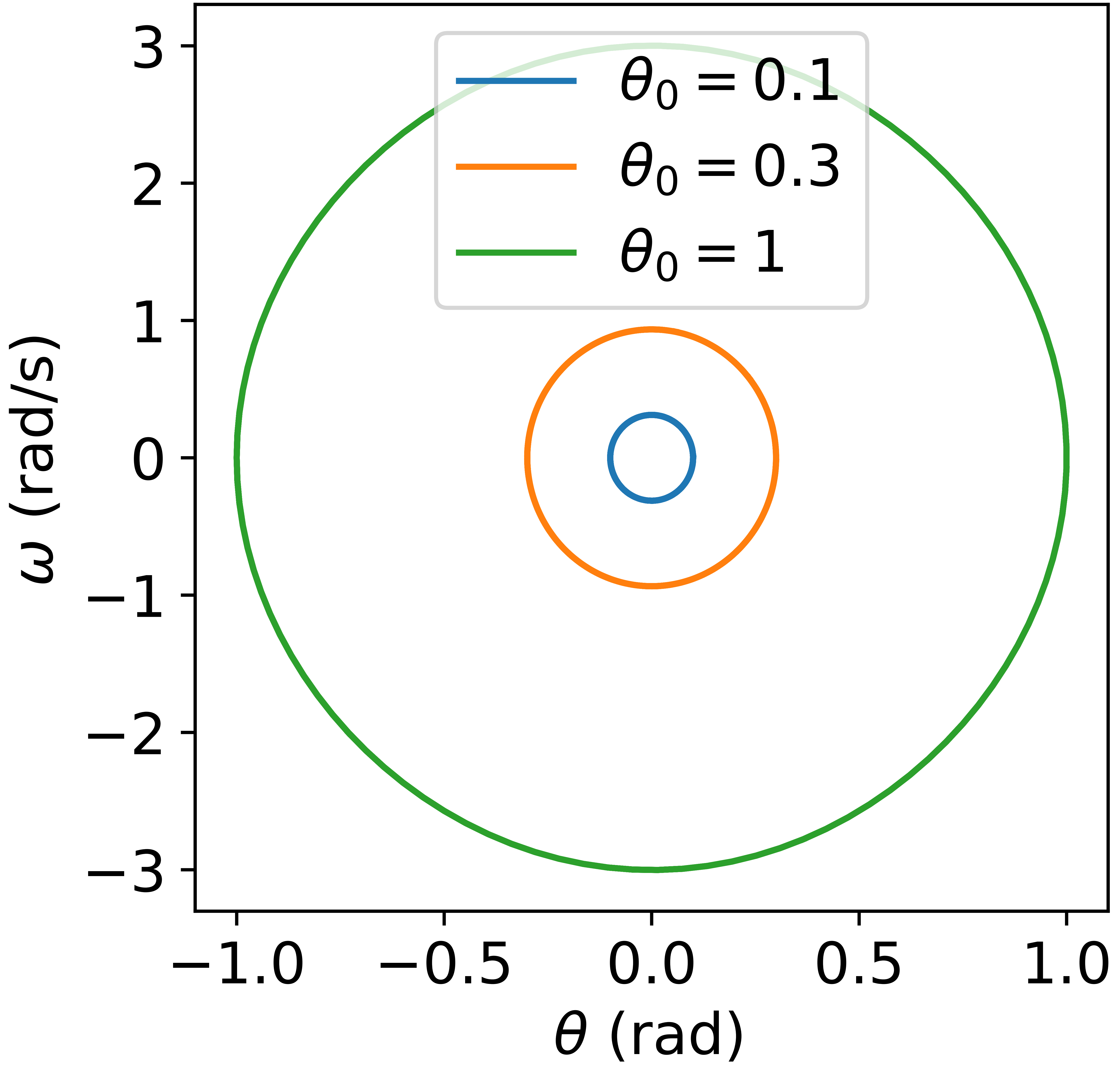
Angular velocity

$$\omega(t + \Delta t) \sim \omega(t) + \frac{d\omega}{dt}\Delta t$$

Angular displacement

$$\theta(t + \Delta t) \sim \theta(t) + \omega\Delta t$$

Pendulum: phase space



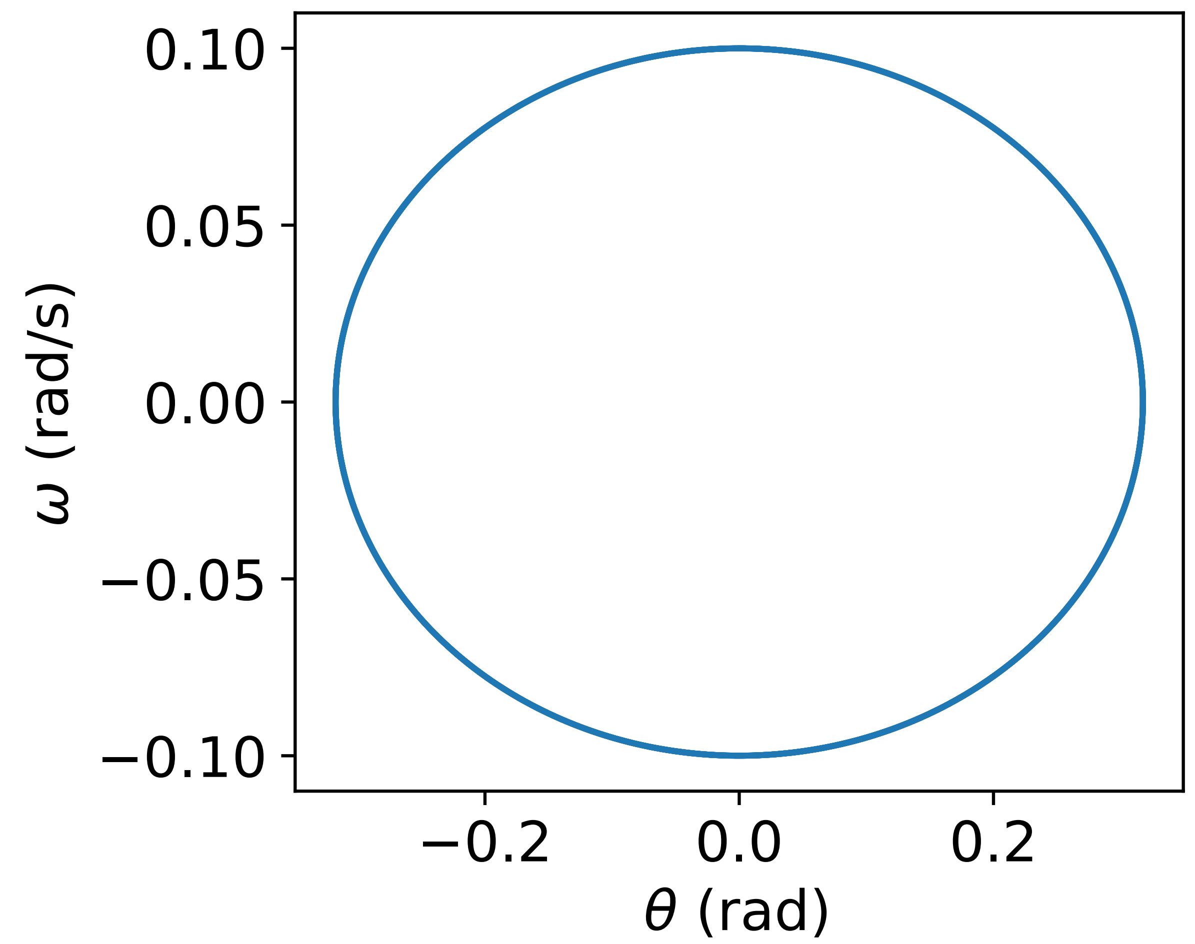
Small Angle approximation

- Small angle approximation: $\theta \sim sin\theta$
  - Sanity check: $\theta = 0, \quad 0 = \sin 0$
- Can compare:
  - $$\alpha(\theta) = -\frac{g\sin(\theta)}{L} \qquad \text{vs} \qquad \alpha(\theta) = -\frac{g\theta}{L}$$
- When does this break down?

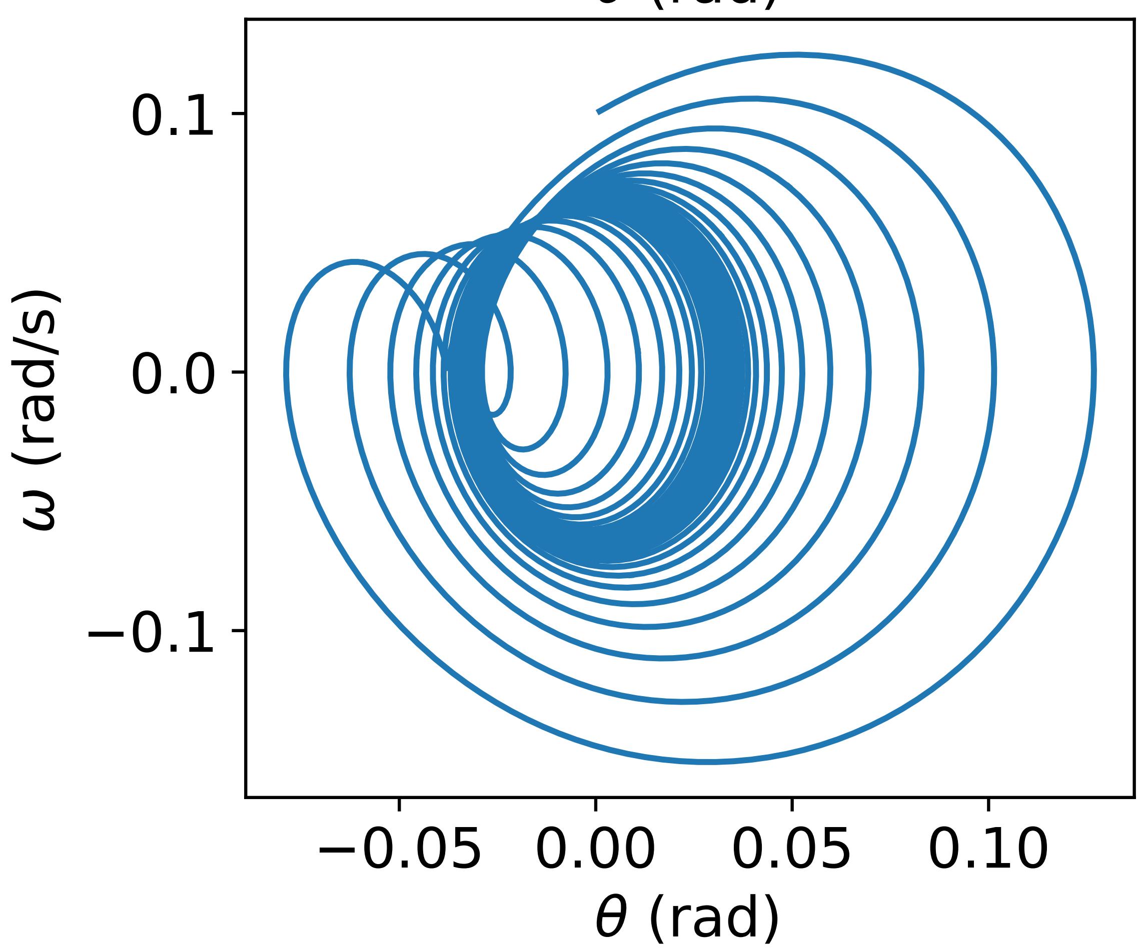
DRIVEN PENDULUM

$$\frac{d\omega}{dt} = -A\omega - B\sin\theta + C\sin\Omega_{ext}t$$

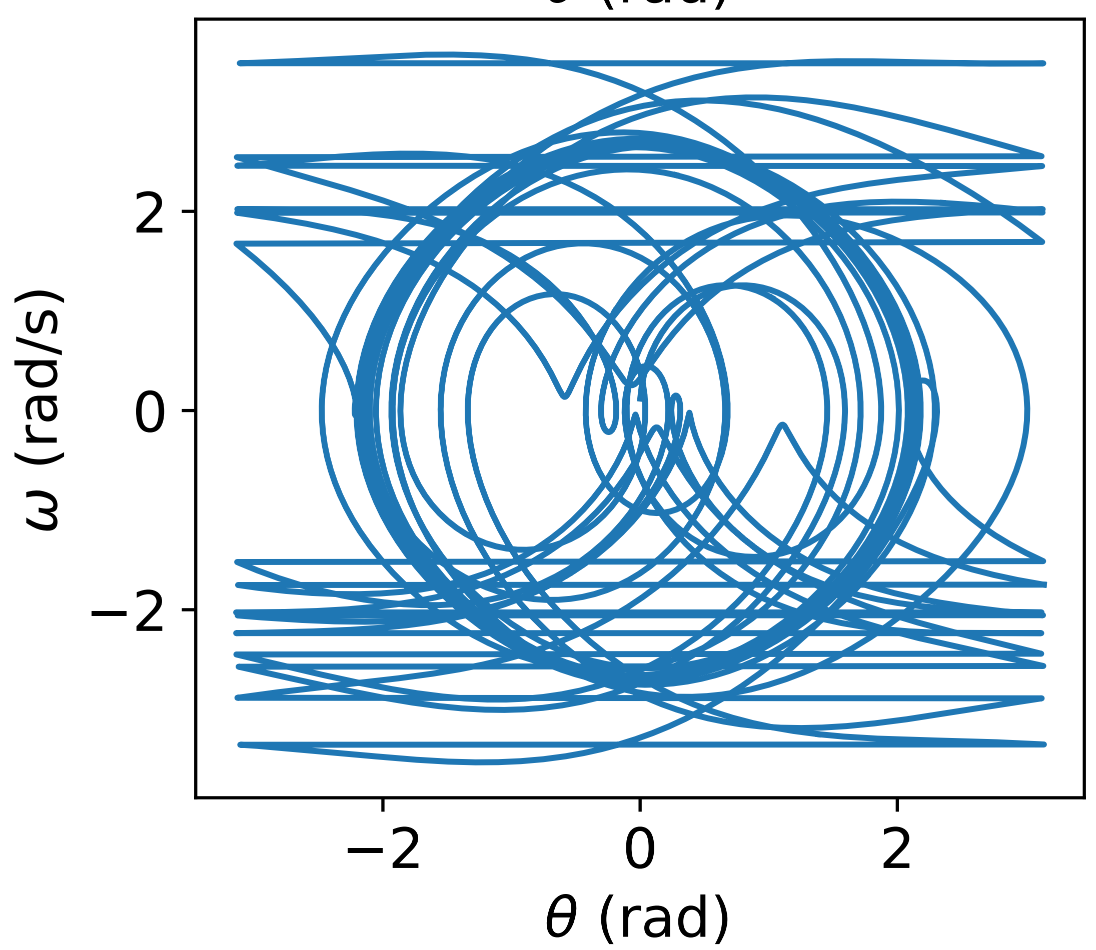
No drive,
no damping



Overdamped

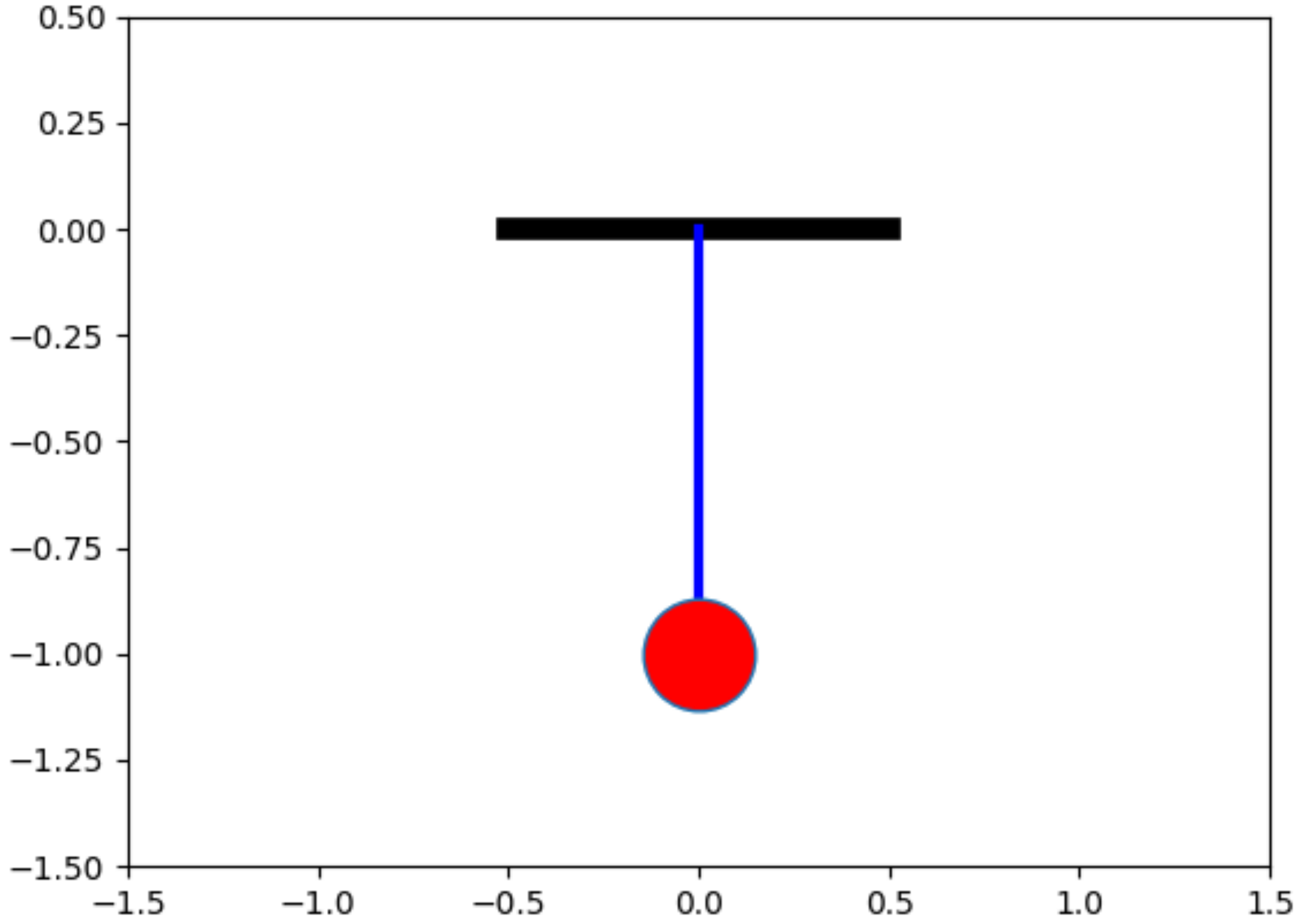


Underdamped

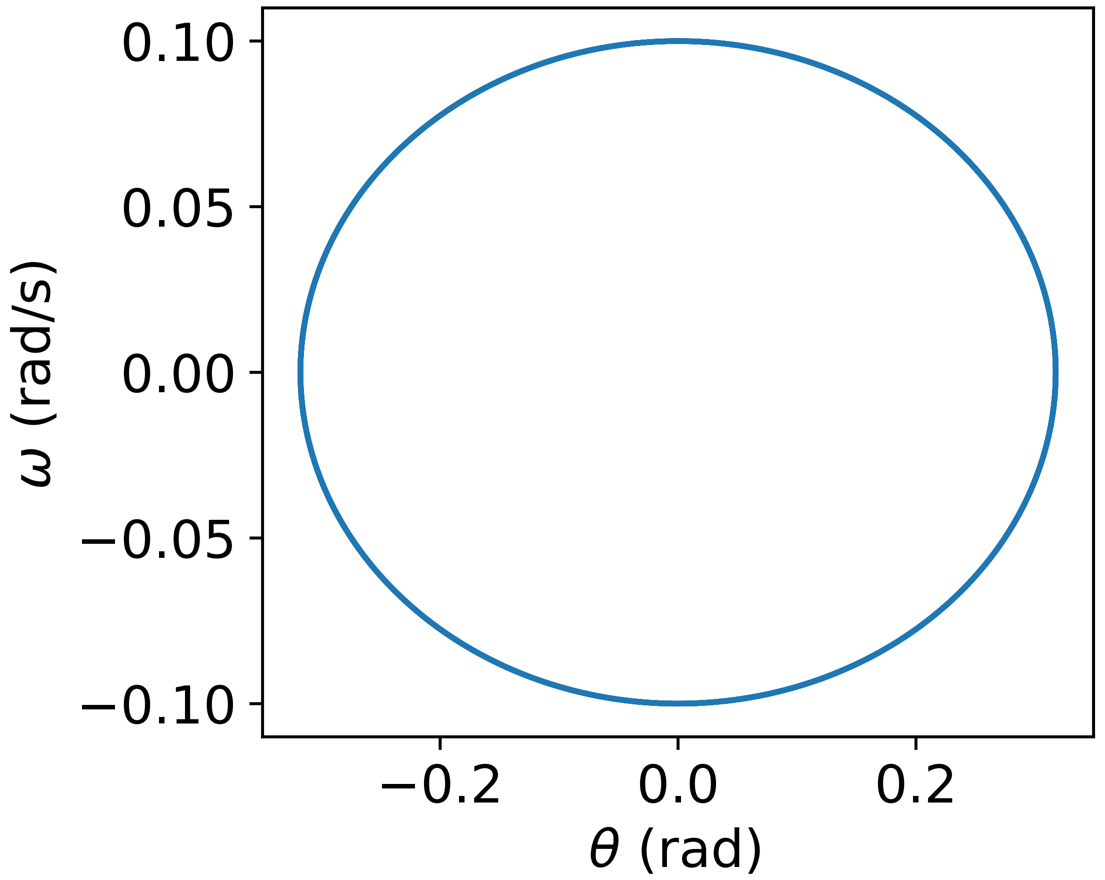


DRIVING PENDULUM

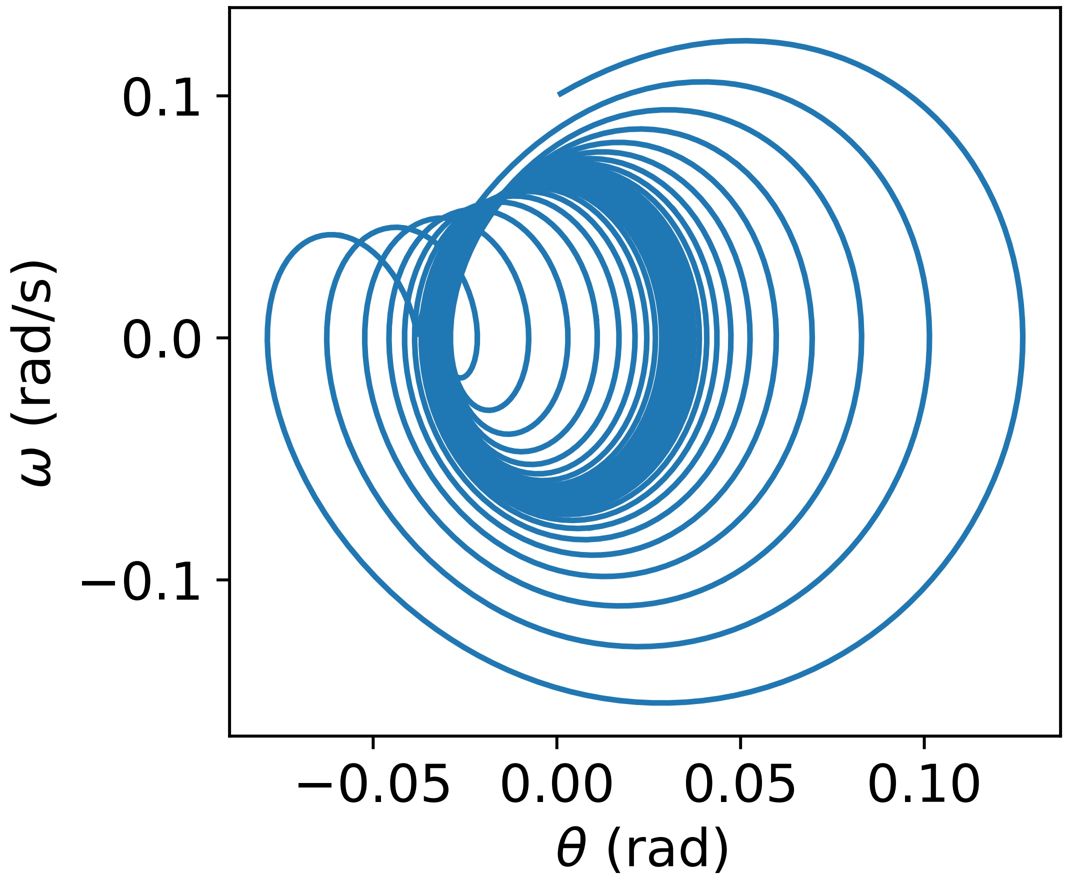
Overdamped



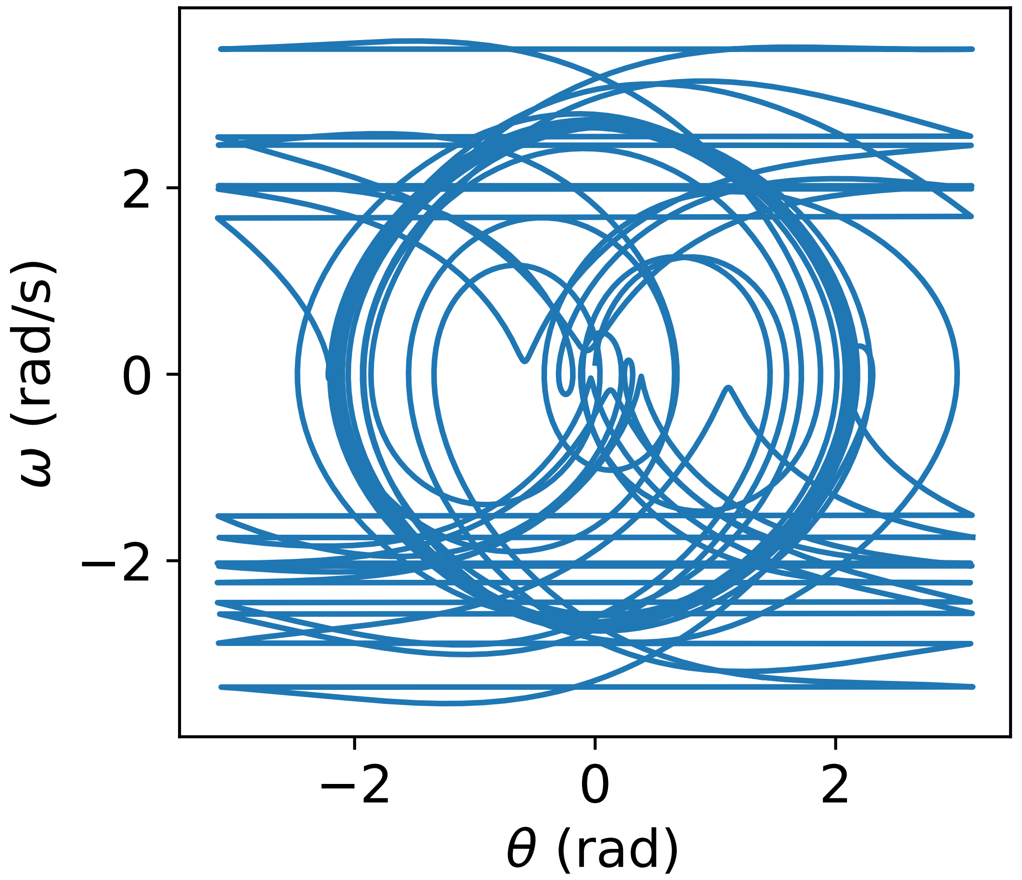
Underdamped (goes above start position, then slows)



No drive, no damping

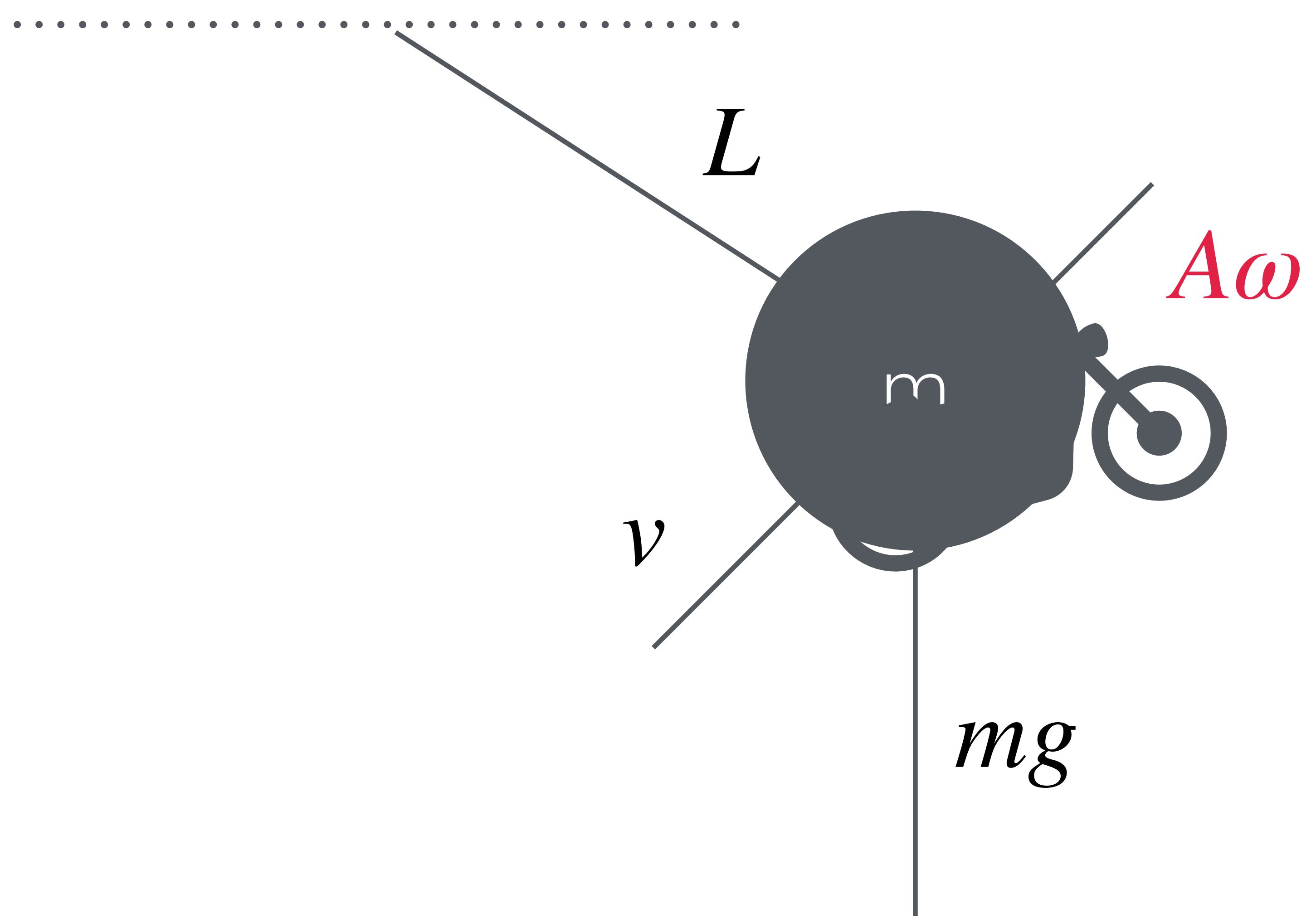


Overdamped



Underdamped

DAMPED VS DRIVEN A PENDULUM



Attach ball of mass m to a rod (not string!) of length L

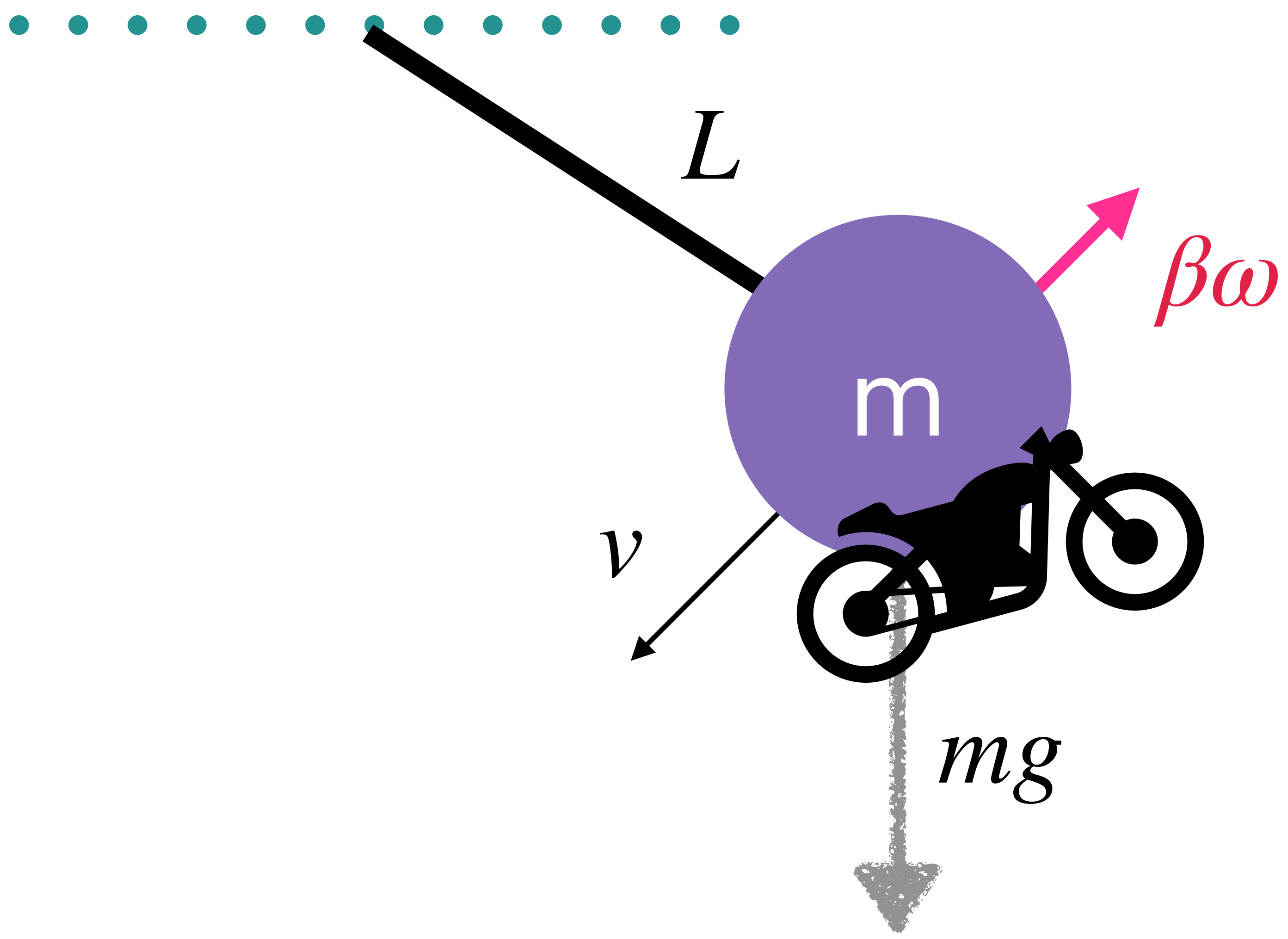
There's an angular velocity-dependent drag force $A\omega$ from air resistance

Pretend some engine is attached that provides an additional force (torque) $\tau = \gamma \sin \Omega_{ext} t$

$$\tau = I\alpha = mL^2 \frac{d^2\theta}{dt^2}$$

$$\frac{d\omega}{dt} = -A\omega - B\sin\theta + C\sin\Omega_{ext}t$$

DAMPED VS DRIVEN A PENDULUM



$$\tau = I\alpha = mL^2\frac{d^2\theta}{dt^2}$$

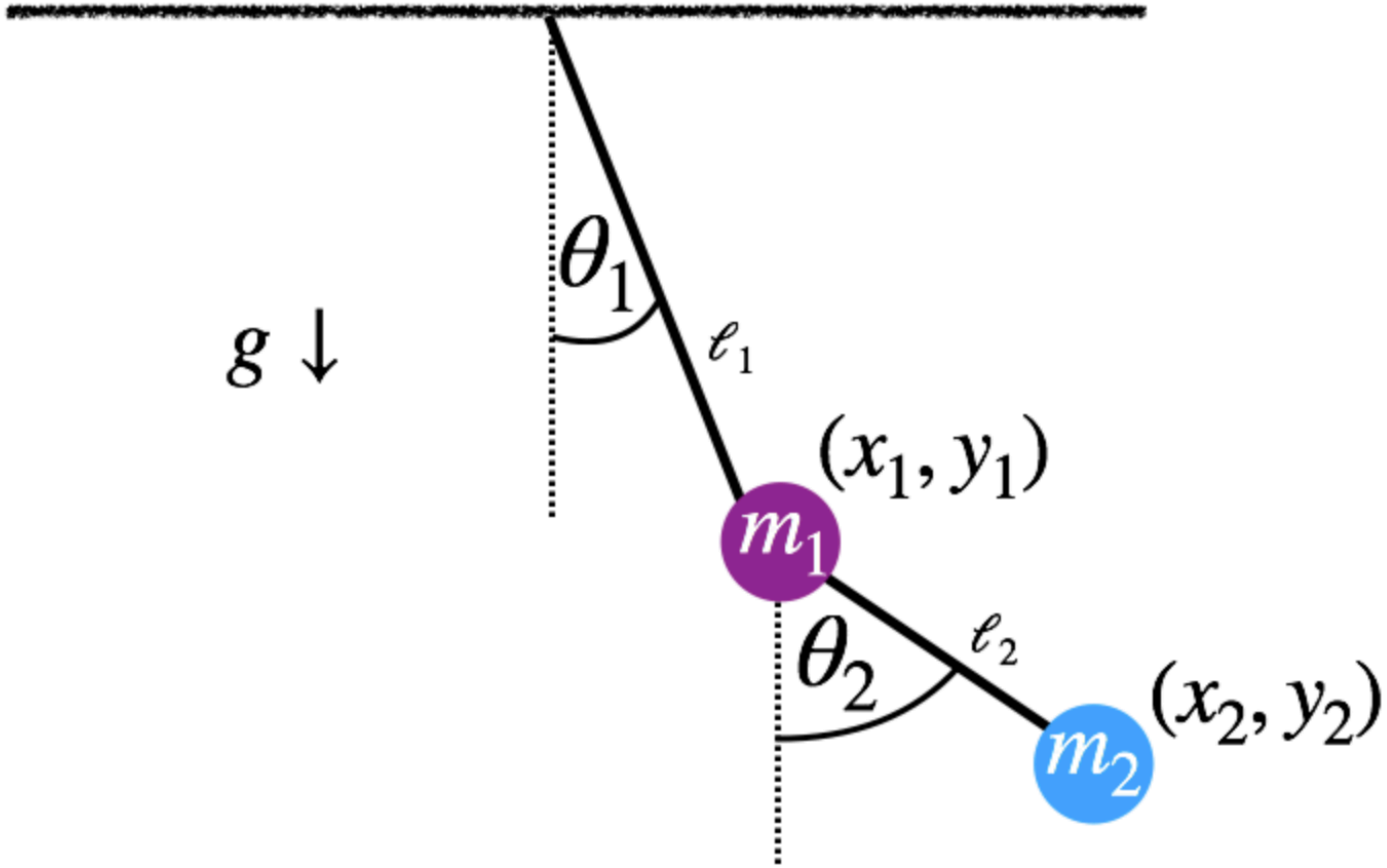
$$= \underbrace{-mgL\sin\theta}_{F_g} + \underbrace{-\beta\frac{d\theta}{dt}}_{F_{drag}} + \underbrace{\gamma\sin\Omega_{ext}t}_{F_{ext}}$$

After some algebra...

$$\frac{d\omega}{dt} = -A\omega - B\sin\theta + C\sin\Omega_{ext}t \qquad \omega = \frac{d\theta}{dt}$$

Double Pendulum

Double pendulum



A bit difficult to work out forces!

Easier to use Euler-Lagrange method (PHYS 325) (principle of least action)

$$\alpha_1(\theta_1, \theta_2) := \frac{l_2}{l_1}\left(\frac{m_2}{m_1 + m_2}\right)\cos(\theta_1 - \theta_2)$$

$$\alpha_2(\theta_1, \theta_2) := \frac{l_1}{l_2}\cos(\theta_1 - \theta_2)$$

$$f_1(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) := -\frac{l_2}{l_1}\left(\frac{m_2}{m_1 + m_2}\right)\dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - \frac{g}{l_1}\sin\theta_1$$

$$f_2(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) := \frac{l_1}{l_2}\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - \frac{g}{l_2}\sin\theta_2$$

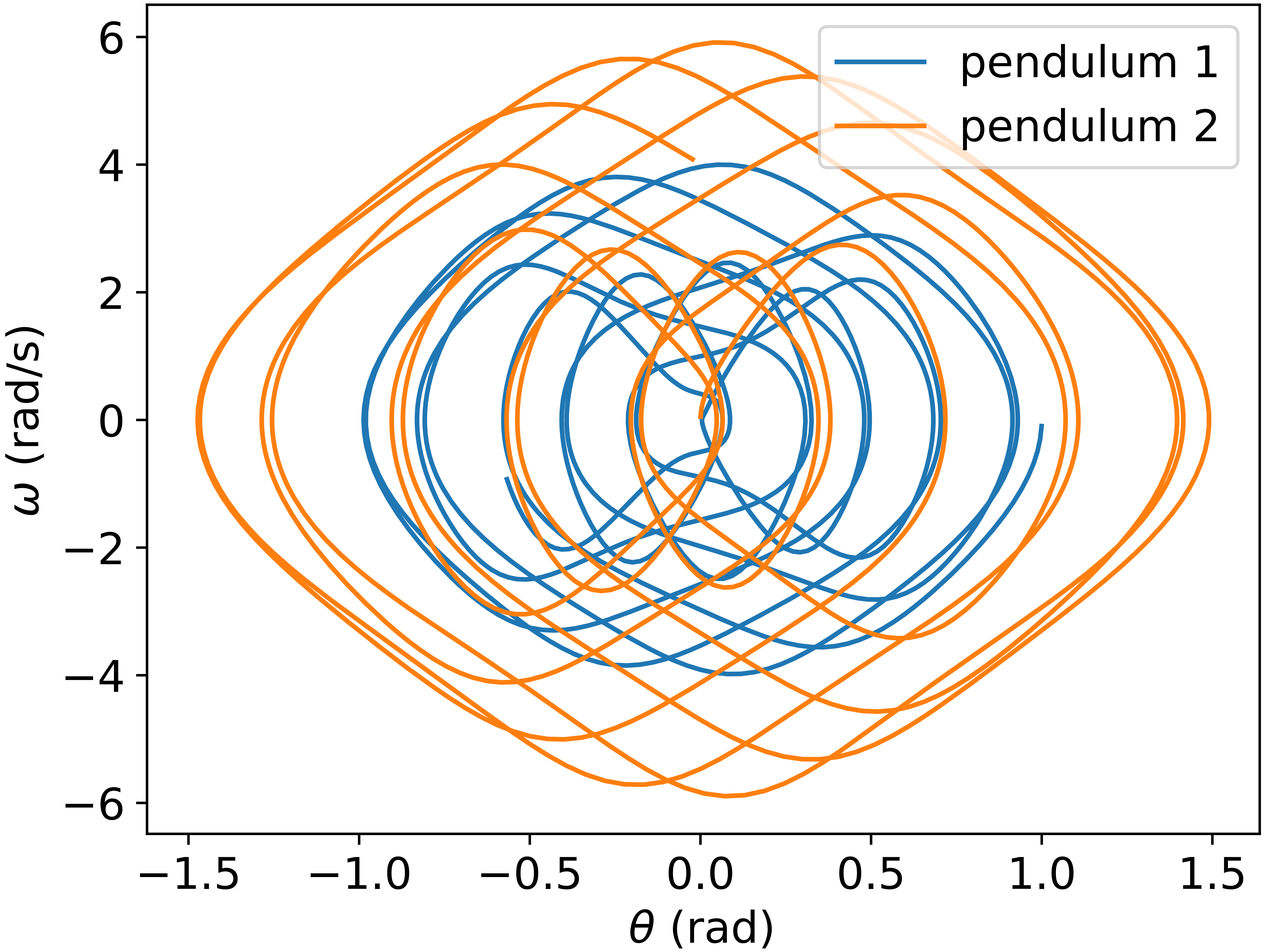
$$g_1 := \frac{f_1 - \alpha_1 f_2}{1 - \alpha_1 \alpha_2}$$

$$g_2 := \frac{-\alpha_2 f_1 + f_2}{1 - \alpha_1 \alpha_2}$$

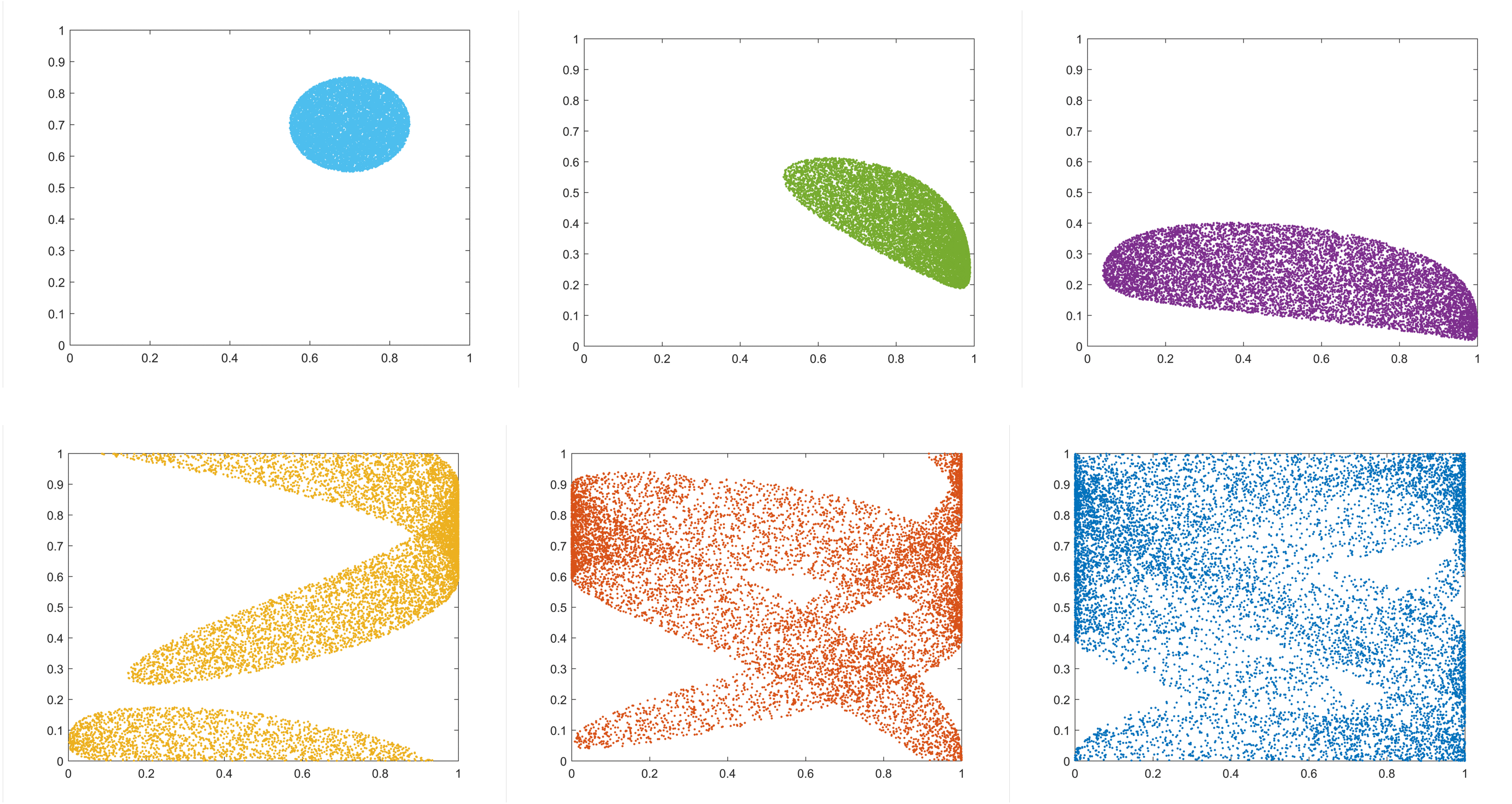
$$\frac{d}{dt}\begin{pmatrix}\theta_1\\ \theta_2\\ \omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix}\omega_1\\ \omega_2\\ g_1(\theta_1, \theta_2, \omega_1, \omega_2)\\ g_2(\theta_1, \theta_2, \omega_1, \omega_2)\end{pmatrix}$$

https://dassencio.org/33

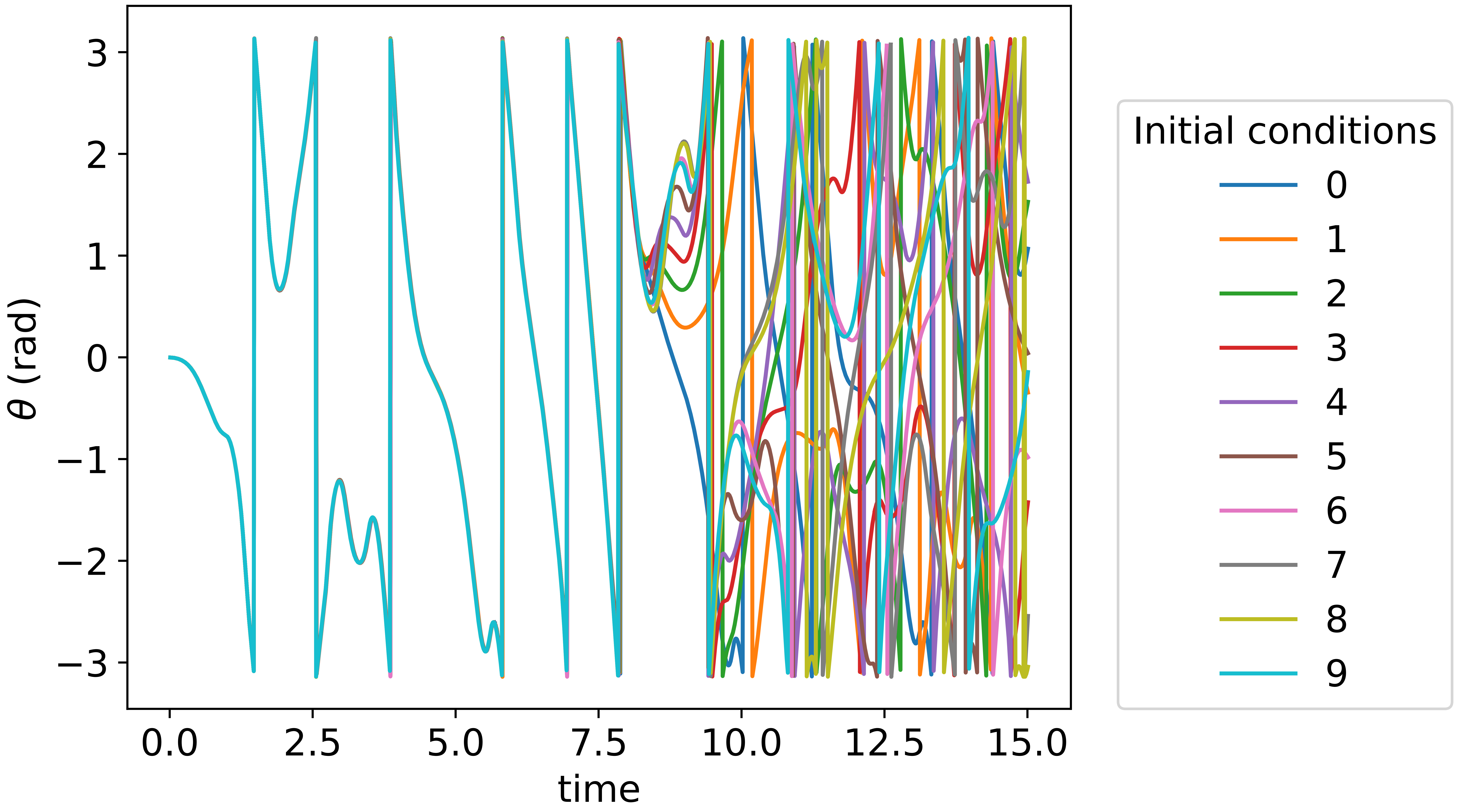
Double pendulum phase space



CHAOS -- GENERAL CONCEPT



Chaos



Initial conditions
0
1
2
3
4
5
6
7
8
9
θ (rad)
time
3
2
1
0
−1
−2
−3
0.0
2.5
5.0
7.5
10.0
12.5
15.0

Checks for your solutions

Undamped, undriven pendulum:

- Small angles should match analytic result
- Energy should be conserved (phase space should be an ellipse)

Driven pendulum

Large driving force should make it unstable.

Small driving force plus damping --> should go to (0,0)

Double pendulum

Small initial angles/velocities should

PYTHON STUFF

Why I don't recommend global variables

```
a=5
[7] ✓ 0.0s
```

```
def f(x):
    return x * a

print(f(2))
[6] ✓ 0.0s
... 14
```

```
a=7
[5] ✓ 0.0s
```

```
print(f(2))
[8] ✓ 0.0s
... 10
```

animate is meant to visualize multiple runs

```
def animateMe_single_pendulum(positions, params):
    """
    positions [run, time, theta]
    params {'l1': length of the pendulum}
    """
```

Magically wrap the positions to (-pi,pi)

```
    pos = pos - 2*np.pi*np.ceil((pos-np.pi)/(2*np.pi))
```