Labs 5–6: Mechanical Resonance

January 19, 2025

1 Goals

- Investigate the transient behaviour of a mechanical oscillator with damping.
- Investigate resonance properties of a mechanical oscillator with damping.
- Collect and analyze data suitable for inclusion in a formal report.

1.1 References

- Sections on damped and forced harmonic motion in your first-year or PHYS 255 textbook. See also the Purcell chapter on Canvas.
- Eddy Currents and Magnetic Damping, LibreText College Physics 1e (OpenStax), Chapter 23.7.

1.2 Introduction

In this experiment, you will investigate the mechanical resonance of a mass attached to a hacksaw blade. We attach a tiny accelerometer to the end of the blade to measure the motion. To add damping, we have included a small but powerful magnet at the end of the blade. Placing a good conductor (bar of copper) over it leads to eddy damping.

First, you will study the behavior of the oscillator when it has damping and is also forced by a periodic external force. Then you will observe the damped simple harmonic oscillations of this system and compare to simple Newtonian theory.

The structure of this lab is similar to that of the AC circuits investigated previously. In those labs, we constructed electrical circuits where we gave an input (a sine wave) and measured the output (another sine wave). Here, we have a mechanical system where we give an input (either a sine wave or an initial condition), and we measure a response (acceleration of the hacksaw blade). Both experiments supply an input and measure an output. Indeed, the LabVIEW programs we developed for the AC circuit will be used here, too, with small modifications.

The similarities are even deeper when we notice that the equations of motion describing a damped, driven harmonic oscillator are the same as those used to describe an LCR circuit. Thus, what we have learned about LCR circuits can be applied to mechanical resonance, if we make appropriate substitutions: $m \to L$, $k \to \frac{1}{C}$, $2\gamma \to R$, etc.

When you look carefully at the behaviour of both of these systems—the series LCR circuit and the mechanical resonator—they only approximately follow the simple model predictions. Since their underlying physics is rather different, it should not be too surprising that the ways that they deviate from "ideal" behaviour are very different. It is a bit like Tolstoy's famous dictum: "Happy families are all alike; every unhappy family is unhappy in its own way."[1] That is, ideal second-order systems are all alike; every physical realization is different in its own way. The charm and the challenge of this experiment is both to understand the underlying second-order system *and* appreciate its non-ideal features.

2 Prelab questions

In general, you should review the equations and solution for a forced, damped oscillator. Think about the implications of measuring the acceleration of the motion rather than the amplitude. Make sure you understand the notion of resonance, the difference between the natural and forcing frequencies, and the difference between the transient and steady-state solutions of an inhomogeneous differential equation. Then do the following:

Prelab for Week 1. The forced harmonic oscillator obeys

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \left(\frac{F}{m}\right) \mathrm{e}^{i\omega t}\,, \tag{1}$$

where $F(t)$ is applied force, and where we implicitly take the real part of the equation for a real displacement. Here, the driving frequency ω is set by the experimentalist. Substituting a trial solution of the form $x(t) = x_0(\omega)\,\mathrm{e}^{i\omega t}$ gives a steady-state solution of the form

$$x_0(\omega) = \frac{F_0(\omega)/m}{\omega_0^2 - \omega^2 + 2i\gamma\omega}\,, \tag{2}$$

with $F_0(\omega)$ the (Fourier) amplitude of $F(t)$ at angular frequency ω.

Since $a(t) = \ddot{x}(t)$, the acceleration is $a(t) = a_0\,\mathrm{e}^{i\omega t}$, with $a_0 = (i\omega)^2 x_0 = -\omega^2 x_0$. Note that when we write the magnitude of the acceleration $|a_0(\omega)|$, it is defined to be a positive number. The minus sign is then associated with a $+\pi$ phase shift ($+\pi$ because each time derivative brings a factor i, which is a $\pi/2$ lead).

The magnitude of the acceleration is then given by

$$|a_0(\omega)| = \frac{\omega^2(F_0/m)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}\,. \tag{3}$$

[1]Opening line to the novel *Anna Karenina*, 1878, Leo Tolstoy.

Since the applied force $F(t)$ might have varying amplitudes F_0 at different frequencies ω, it makes more sense experimentally to focus on the ratio of acceleration to force at ω,

$$\frac{|a_0(\omega)|}{|F_0(\omega)/m|} = \frac{\omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}\,. \tag{4}$$

Question 1. (i) Plot the magnitude response of the acceleration as a function of frequency, for an undamped-oscillator frequency $\omega_0/\,2\pi = 12$ Hz and a Q factor of 10. (As discussed below, $Q \equiv \omega_0/\,2\gamma$.) For the amplitude, produce a log-log plot, as well. (ii) Show that the ratio of the resonance peak height to the high-frequency asymptote is $\approx Q$. This gives another easy way to estimate Q, in addition to comparing the ratio of peak height to width discussed in the analysis. (iii) Compare the curves to the $V_\mathrm{R}/V_\mathrm{in}$ curves you measured last week. Explain why they are different.

Question 2. Following the analysis from the AC labs, find the phase shift between the acceleration and force, $\tan\phi = \mathrm{Im}\,a/\,\mathrm{Re}\,a$, and plot ϕ as a function of frequency. Compare to the phase curves you measured last week. Explain why they are different. Hint: Remember to use the `np.atan2` function!

Prelab for Week 2. Consider a damped oscillator described by the equation of motion

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0\,. \tag{5}$$

Question 1. (i) If the initial velocity of the oscillator is zero and if the motion is underdamped ($\gamma < \omega_0$), show that the displacement $x(t)$ follows a decaying oscillation,

$$x(t) = x_0\ \mathrm{e}^{-\gamma t}\ \cos\omega_1 t\,, \tag{6}$$

where x_0 is the initial position and $\omega_1 = \sqrt{\omega_0^2 - \gamma^2}$. Note that $\omega_1 = \omega_0$ when $\gamma = 0$. (ii) For small damping ($\gamma \ll \omega_0$), show that the oscillation amplitude decays by a factor $1/\mathrm{e}$ after about Q/π oscillation periods. Thus, yet another way to estimate Q is to count the number of periods until the amplitude is ≈ 0.63 times its original value (and then multiply by π). This works better for larger Q values.

In this experiment, we do not measure the position of the hacksaw blade tip $x(t)$ but rather its acceleration $a(t) = \ddot{x}(t)$.

Question 2. Show that the acceleration $\ddot{x}(t) = a(t)$ obeys an equation of the form

$$a(t) = a_0 \, \mathrm{e}^{-\gamma t} \cos(\omega_1 t + \varphi) \, , \tag{7}$$

where a_0 is the initial acceleration and φ is a phase shift. Hint: Differentiate Eq. 6 twice and use a trig identity.

The full solution to the driven oscillator has *two* terms, a decaying oscillation at ω_1 (Eq. 6) plus a steady-state oscillation at the driving angular frequency ω (Eq. 2). When analyzing forced oscillations, you should first let the transient part of the solution decay. This can take tens of seconds for low damping, less at higher damping.

Question 3. Plot the amplitude and phase data for the maximal-damping response that you measured in Week 1. Include with a careful curve-fit (plotted with enough points, say 1000, to make the fit curves look smooth). Extend the frequency range of the plots so that the phase curves get close to the expected asymptotic values of π and 0. You should compute the χ^2, using appropriate errors to weight the sum, and also plot residuals for each fit. Are the fits are good, in the statistical sense? Since the two fits are based on the same data, one might hope that the parameters agree with each other, to within statistical uncertainties. Do they? Discuss the result.

3 Experiments

The experiments described below are to be done over two weeks. You will turn in your lab book when both experiments are done (i.e., not after Week 1). Note that Prelab Question 2 in Week 2 is based on your results from Week 1. The data from these two labs will also form the basis for your technical report.

3.1 Some preliminaries

Before starting, we discuss a few aspects of the experimental apparatus itself. A hacksaw blade is clamped to a heavy block, as shown in Fig. 1. Part of the blade extends beyond the block and is free to oscillate, as indicated by the back-and-forth arrows. At the end of the hacksaw blade is a "T bar" with a flat surface for mounting an accelerometer and a magnet.

3.1.1 Components

We introduce some of the components of the apparatus.

1. *Accelerometer*. Look closely at the little circuit board screwed onto the metal "T": the small, black, rectangular object is the accelerometer chip (model ADXL335 from Analog Devices). Such chips are found in smart phones and in video-game controllers

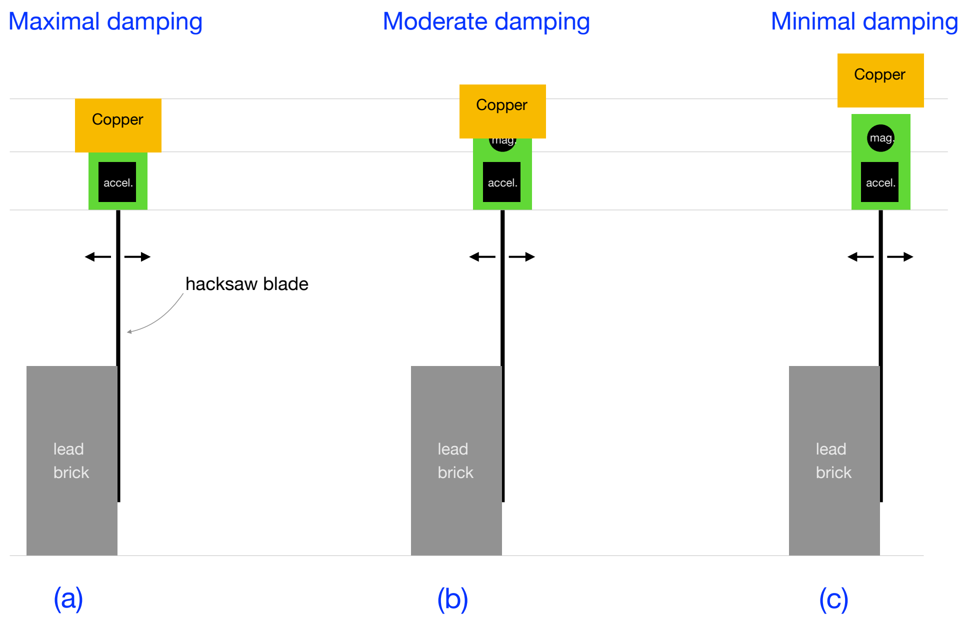

Figure 1: Top view of the mechanical-resonance setup The oscillating mass consists of an accelerometer and magnet attached to the end of a hacksaw blade that is anchored at one end. The blade is free to oscillate, as shown by the arrows. A block of copper placed over the magnet provides eddy-current damping. (a), (b), (c) show three positions for the copper block, corresponding to three different damping levels. The mechanical driver is not shown.

with "motion sensing" capabilities. How it works is illustrated in Fig. 2. One nice feature is that the signal processing is built in. Inside the chip is an amplifier to boost the signal and a low-pass filter with a cutoff frequency ≈ 50 Hz, to reduce noise and also the sampling rate needed to avoid aliasing. The output is then easy to measure with the DAQ. The accelerometer actually has three separate axes (aligned along the axes of the chip body), although we use only one for our experiments.

There are a few technical things to note about the accelerometer. (See the data sheet for more.) First, the supply voltage is around 3 V. We use a pair of 1.5 V batteries in series. Please use the on/off switch on the battery case to disconnect the batteries at the end of each lab session, to save battery power. But it is okay to leave them on for the whole afternoon. The output voltage of the accelerometer is scaled to the supply voltage, with zero acceleration being near (but not exactly at) the middle of the range. Thus, when the blade is stationary, you should see a voltage output of ≈ 1.5 V.

2. *Eddy damping.* A copper block placed close to – but not touching! – the little magnet will damp the hacksaw blade motion. As the magnet moves, it induces currents in the copper block whose *back action* opposes motion, leading to damping. (Because there is no mechanical contact, the damping is much more "ideal" than damping due

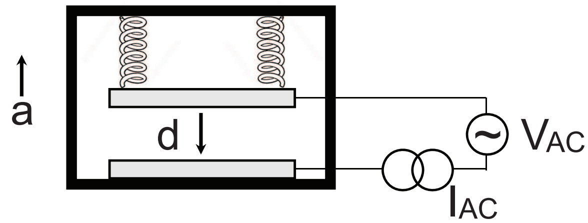

Figure 2: Illustration of accelerometer principle. A capacitor has a fixed plate and a movable plate attached to springs. An acceleration a causes a displacement in the movable plate, which changes the capacitance – measured by imposing an AC current I_{AC} and measuring the resultant AC voltage V_{AC}. Recall that for a parallel plate capacitor, capacitance $C = \varepsilon A/d$, where A is the plate area, d the plate spacing, and ε the dielectric coefficient of the medium between the plates. V_{AC} is thus *linear* in d. The actual capacitor geometry and circuits are more complicated than what is shown here.

to mechanical friction.[2]) Qualitatively, eddy damping depends on the number and density of field lines that penetrate the conductor. Thus, you can reduce the damping either by lifting the conductor up away from the magnet or, more conveniently, by simply sliding the ring stand holder horizontally, uncovering part of the magnet.

3. *Vibration driver.* You will use a (Pasco Mechanical Wave Driver, Model SF-9324) to excite forced oscillations at various frequencies in order to observe the resonance phenomenon.

3.2 Week 1: Forced oscillations with magnetic damping

1. **Accelerometer calibration**. Before you start your oscillator measurements, you will need to calibrate the accelerometer signal. From the data sheet, the output signal is expected to be linear in the acceleration (for each axis, although we just use one) and proportional to the supply voltage V_{s} given by the batteries, which may slowly change over time. The accelerometer signal $V_{\mathrm{acc}}(t)$ is then 0 to V_{s}, with zero acceleration approximately $\frac{1}{2}V_{\mathrm{s}}$, positive acceleration $V > \frac{1}{2}V_{\mathrm{s}}$ and negative acceleration $V < \frac{1}{2}V_{\mathrm{s}}$. To get a "two-point calibration" (assuming the response is linear), measure 0 acceleration by measuring V_{acc} when the accelerometer and circuit board are flat. Then measure the V_{acc} corresponding to g by tilting it vertically. The two numbers are enough to convert the output signal into an absolute acceleration. Are you measuring $+g$ or $-g$? Derive a formula $a(V)$ that outputs the acceleration (in units of g or m/s^2) given an input voltage. To test your calibration (and understanding), predict the voltage for

[2]Electric cars also use magnetic damping for braking. However, in cars, the induced currents are used to recharge the battery ("regenerative braking"). In this experiment, by contrast, the induced currents just heat up the copper block (get "dissipated" via the electrical resistance of the copper).

when you reverse the tilt. The disagreement between your prediction and the measured voltage gives an estimate of the calibration error in your formula.

2. **Free oscillation frequency**. Fix the hacksaw blade so that its free oscillation frequency ≈ 12 Hz.

   (a) Mount the hacksaw blade so that the blade extends beyond the brick. With the copper damping block far from the magnet, test the natural oscillation frequency of the blade/sensor combination by pulling the oscillator mass to one side and releasing it. Is there more than one oscillation frequency present? Focus on oscillations perpendicular to the flat part of the blade.

   (b) Observe the accelerometer signal on an oscilloscope. Check the oscillation frequency, and adjust the blade length extending beyond the brick until the oscillation frequency is about 12 Hz. You should be able to get a signal that looks like a clean damped sine wave with slowly decreasing amplitude. If the signal looks more complicated, trying waiting to see if a transient dies away, leaving a "clean" sine wave. Or try different ways of releasing the blade so that you do not impose unwanted initial conditions. Note that because there is damping even without the magnet, the free-oscillation frequency is close to but slightly less than ω_0.

3. **Mechanical driver**. In order to explore driven motion, we need to attach the mechanical driver. The driver should be positioned so that its movable shaft is perpendicular to the blade. The blade should be pushed enough against the blade that it always remains in contact. Test this by attaching cables from a function generator to the driver and send in a signal to test (near the natural frequency). You should explore where to position the oscillating driver shaft and how hard it should press against the blade. Be sure that the accelerometer signal is a clean sine wave of reasonably large amplitude. If the blade goes out of contact, you might hear a clicking sound. (Drive at or near the resonant frequency to best test for this.) A driving amplitude ≈ 1 V is a good start, but adjust as needed.

   Show your setup to one of the instructors before moving on to the next step.

4. **Maximal damping**. Start the experiments with the highest ("maximal") level of damping. To do so, bring the copper damping plate into position, covering the magnet as shown in Fig. 1(a). As you do this, position the block as close as possible above the magnet but <u>not touching</u>. Try to make the two surfaces as parallel as possible. The alignment is challenging, as the ring stand that holds the copper block does not allow for adjustment of all orientations of the block. Do the best you can. As a test, pull away the mechanical driver to look at free oscillations. (But remember where it was!) Release the hacksaw blade by hand. It should come to rest quickly, after just a few oscillation cycles, without any scraping. Then put the driver back into its original position. The accelerometer motion should again be a nice sinusoidal signal.

5. **Forced motion, with maximal damping**. Now that you have adjusted the copper block to provide maximal damping, added the driver, and verified that you have a clean sine wave of reasonable amplitude (when near resonance), you are ready to investigate the acceleration response as a function of frequency. The procedure is much the same as for the LCR circuit. One difference is that the initial transients can last long enough that they complicate the sinusoidal curve. You will need to learn how long to wait (or how much data to eliminate) before the response is sinusoidal. Use fits to be sure that you are waiting long enough. (Hint: The lower the damping, the longer transients will take to decay.)

   Measure $V_{\mathrm{acc}}(t)$ and $V_{\mathrm{in}}(t)$ for a variety of frequencies, recording the sinusoidal signals to disk for each frequency. As you measure the points, use cursors to quickly get the amplitudes and phase differences. Using Python, plot the amplitudes of $V_{\mathrm{acc}}(t)$ and $V_{\mathrm{in}}(t)$. We might expect the amplitude V_{in} to be independent of frequency if the function generator amplitude is constant. Is it? Why not?[3] To deal with a varying amplitude, we will focus on their ratio[4] (see Eq. 4). Also, plot the phase difference as a function of frequency. Make sure you get enough points to nicely trace out the response curves. If there are big gaps, take more data. Beware of spurious jumps in the phase.

   Once you have your data, perform curve fits to the amplitude-ratio response to extract ω_0, γ, and (as a result), Q. Do the same for the phase response. If there are big differences, look for issues with signs, etc. (and discuss with your instructor). This work will form part of Prelab Question 2 for Week 2.

[3] The physics of the mechanical driver is not our focus for this lab, but you can start to understand its behaviour by looking at Sec. 2.4.2 of the Reference Manual.

[4] It is best to look at the ratio of voltage amplitudes, $V_{\mathrm{acc}}/V_{\mathrm{in}}$. Even though we have calibrated the accelerometer, we have not calibrated the force. So your formula will still have an arbitrary calibration constant. Remember that you want sine-wave amplitudes (separated from any offset).

3.3 Week 2: Forced and free oscillations

This week, we will explore transients and lower damping. The lower the damping, the trickier the measurements, as we will see.

1. *Maximal damping*: If you did not finish measuring the driven response from last week, please take more measurements. Make sure that any measurements you did last week have not changed. If you start to find different values, just take a new set. Once you have completed your set of forced oscillations for maximum damping, measure the transient by pulling back the driver and moving the oscillator mass away from equilibrium with your finger. Collect data for a suitable amount of time to see several decay times and save it. Either by looking directly at the transient data or by fitting the decaying acceleration and looking at the residuals, test whether there are signs of higher modes. If so, estimate the frequency from the residuals. But also try to find ways of releasing the hacksaw blade that minimize or even eliminate such motion.

2. Repeat for intermediate and minimal damping. See Fig. 1(b,c). When you change the damping, be sure to move only the copper plate, while keeping the hacksaw-blade apparatus and driver fixed. Note that the transient times for minimal damping can be very long. You will likely need to modify your LabVIEW program because the DAQ Assistant has a default timeout of 10 s, meaning that it will give an error if you ask it to collect data for a longer period. You can add a control (or constant) to specify the timeout in such cases. Make it a bit longer than the longest requested data acquisition. (Use the `Context help` feature in LabVIEW to figure out where to set the timeout.) **Useful tips:** Consider starting the oscillator moving just before you start the data acquisition. This gives cleaner data since it allows some of the non-ideal transients (torsional or higher-order modes etc.) to decay away. Listen for any "clicking sounds" at large response amplitudes (near resonance), indicating that the blade is detaching from the driver piston. Adjust to eliminate the clicking.

3. **Bonus issues:** the "spring force" in this lab is not an ideal Hooke's Law spring. The restoring force should become nonlinear at high amplitudes. Can you think of a way to investigate nonlinear effects for the forced oscillations at high amplitude? Nonlinear effects generally induced higher-order harmonics. Can you test for the presence of these? Also, nonlinear oscillators have an amplitude-dependent frequency. Do you see evidence of this in the transient oscillations? You can ask your instructor for hints....

3.4 Analysis

As much as possible, do the analysis below at the appropriate time *while* you are taking data. At a minimum, do enough to see that things are working as expected. Experiments should always be a repeated cycle of data taking and analysis. The analysis is based on the data, but it also tells you what new data need to be taken.

1. Perform curve fits to the sinusoidal voltage measurements in order to extract accurate measurements of the amplitudes and phase shifts at the different frequencies for the different damping values. The fit parameters should be consistent with any cursor measurements you make.

2. Plot the acceleration amplitude and phase shift of the system as a function of frequency for the different damping values. Fit the resonance curve and the phase curve to determine the natural resonance frequency and the damping constant. (For the phase curve, use the arctan function, `numpy.arctan2` – note the previous comment about `numpy.arctan2` having two arguments). Estimate the uncertainties in the parameters you have measured. For evaluating the fits, you should probably make three separate linear graphs with accompanying normalized residual plots. For summarizing the amplitude response, you will want to look at both linear and log-log plots, with all three data sets plotted on one graph. (No need to have residuals for this plot, as the emphasis is on comparing the results of the three datasets.)

3. Perform curve fits to your transient decay data sets for the three different values of damping. You will need to restrict the range of the fit using Python's slicing capability. Start with a small range of only a few periods. Compare the parameters taken at large amplitudes with those from a fit where the amplitude is small. Which parameters depend on the range of the local amplitude and why? Also, look carefully at the residuals. Are there traces of higher frequencies? If so, what is the frequency you see (as a ratio with respect to $f_0 = \omega_0/2\pi$)? If you have time, can you take more data to eliminate the higher modes?

4. Resonance is often characterized by a *quality factor* Q, which, for a mechanical oscillator, we define as $Q \equiv \omega_0/2\gamma$. Alternatively,

$$Q \approx \frac{\text{frequency where amplitude is maximum}}{\text{full-width at half-maximum power}}\,. \tag{8}$$

   Here, "full-width at half-maximum power" (FWHM) refers to the frequencies on either side of the peak where the amplitude is $\frac{1}{\sqrt{2}} \approx 0.7\times$ the maximum. The approximation becomes better for large Q. So use FWHM to estimate Q when you only need a rough estimate of Q. (As our approximation of $\frac{1}{\sqrt{2}}$ suggests, you can think of it as a "±10%" estimate.) You can easily estimate the FWHM from a graph or with cursors on an oscilloscope, using either Eq. (8) or the ratio of peak to asymptotic amplitude (Prelab W1, Q1). The expression $Q = \omega_0/2\gamma$ should be used for a more precise determination.

Determine the quality factor Q from the resonance curves. Compare to Q calculated from the decay constant and natural oscillation frequency. (You have two sets of parameters you can use – one from the forced oscillations and one from the free oscillations.)

5. There are subtleties with the phase delay. As we have discussed, the two channels you measure, $V_{\mathrm{in}}(t)$ and $V_{\mathrm{acc}}(t)$, are not measured at the same time. Also, there may be a phase shift between the voltage applied to the mechanical driver and the current it outputs to move the piston. Think about the consequences of these effects for your phase measurement. Can you think of ways to test for their effects that do not involve the hacksaw blade and its response?

6. Larger issues to think about in this experiment: What happens as you increase the driving amplitude at a fixed frequency? What amplitude should you use? Would it be better to try different amplitudes at different frequencies? Do you see any signs of other resonance frequencies? If so, why might they occur? Can you get rid of them?

7. You have obtained fit parameters by different methods: forced oscillations, amplitude and phase fits, and free oscillations. Are the values consistent with each other?

8. More generally, is the model of a simple harmonic oscillator a good one for your system? What does it successfully describe about the behavior that you see? What does it not describe well? Can you understand physically what is going on when the model fails? Thinking about issues like these, and their implications for your analysis, are part of what makes up a good report.