Lab 5 - Mechanical Resonance

02/04/2025

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Equipment

- Function Generator
- Oscilloscope
- Breadboard
- Wires
- Damper
- Mechanical Driver
- Mechanical Oscillator Apparatus
  - Hacksaw blade connected to an accelerometer.
- 3V Battery
- DAQ

Goal: Investigate transient behaviour of mechanical oscillator with damping. Investigate resonance properties of a mechanical oscillator with damping.

Prelab:

Forced Harmonic Oscillator Equation:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \left(\frac{F}{m}\right)e^{i\omega t}$$

Steady State Solution

$$x_0(\omega) = \frac{F_0(\omega)/m}{\omega_0^2 - \omega^2 + 2i\gamma\omega}$$

$F_0(\omega)$: Fourier amplitude of $F(t)$ at ω

Acceleration

$a(t) = a_0 \exp(i\omega t)$ $a_0 = (i\omega)^2 x_0 = -\omega^2 x_0$

$$|a_0(\omega)| = \frac{\omega^2 (F_0/m)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}$$

$$\frac{|a_0(\omega)|}{|F_0(\omega)/m|} = \frac{\omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}$$

1 (i) Plot of $|a_0(\omega)|$ for undamped-oscillator freq.

$f = \frac{\omega_0}{2\pi} = 12$ Hz $Q = \frac{\omega_0}{2\gamma} = 10$

Log/log plot for amplitude aswell.

Plots on crowdmark

(ii) Show $\frac{\text{Peak height}}{\text{high freq asymptote}} \approx Q$

At Resonance $\omega = \omega_0$ So

Peak height $= |a_0(\omega)| = \frac{\omega^2 (F_0/m)}{\sqrt{\cancel{(\omega^2 - \omega_0^2)^2} + 4\gamma^2\omega^2}} = \frac{\omega^{\cancel{2}}}{2\gamma \cancel{\omega}} \frac{F_0}{m} = \frac{\omega}{2\gamma} \frac{F_0}{m} = Q \frac{F_0}{m}$

In Code $\frac{F_0}{m} = 1$ So Peak height $= Q(1) = Q$

High freq Asymptote: $|a_0(\omega >> \omega_0)| \longrightarrow \frac{F_0}{m}$ $\therefore$ $|a_0(\omega >> \omega_0)| = 1$

$\therefore$ Ratio $\boxed{\frac{\text{Peak Height}}{\text{High freq Asym.}} = \frac{Q}{1} = Q}$

(iii) The acceleration plot has a linear increase at low frequencies and expriencies exponential growth to peak near resonance, after resonance it decays to a constant value at high frequencies. (flat Aysmptote at high f)

The $\frac{V_R}{V_{in}}$ plot shows a peak at resonance and decay towards both sides of the peak. (No flat Aysmptote at high f).

The difference is due to the extra ω^2 factor in $|a_0(\omega)|$ numerator causing a flat asymptote at high frequencies which is unseen in $\frac{V_R}{V_{in}}$ plot.

2 Find phase shift between acceleration and force.

$\tan\phi = \text{Im } a / \text{Re } a$

Plot ϕ as a function of ~~frequencies~~ frequency. Compare to phase curves from last week and explain differencies.

$a_0 = -\omega^2 x_0$ $\qquad$ $x_0 = \frac{F_0(\omega)/m}{\omega_0^2 - \omega^2 + 2i\gamma\omega}$

$a_0 = \frac{-\omega^2 \left(F_0(\omega)/m\right)}{\omega_0^2 - \omega^2 + 2i\gamma\omega}$ $\qquad$ $\phi = \tan^{-1}\left(\frac{\text{Im } a_0}{\text{Re } a_0}\right)$

Phase plot on crowdmark.

At low freq $\phi \simeq 180°$

At Resonance $\phi \approx 90°$ $\quad$ Phase flips

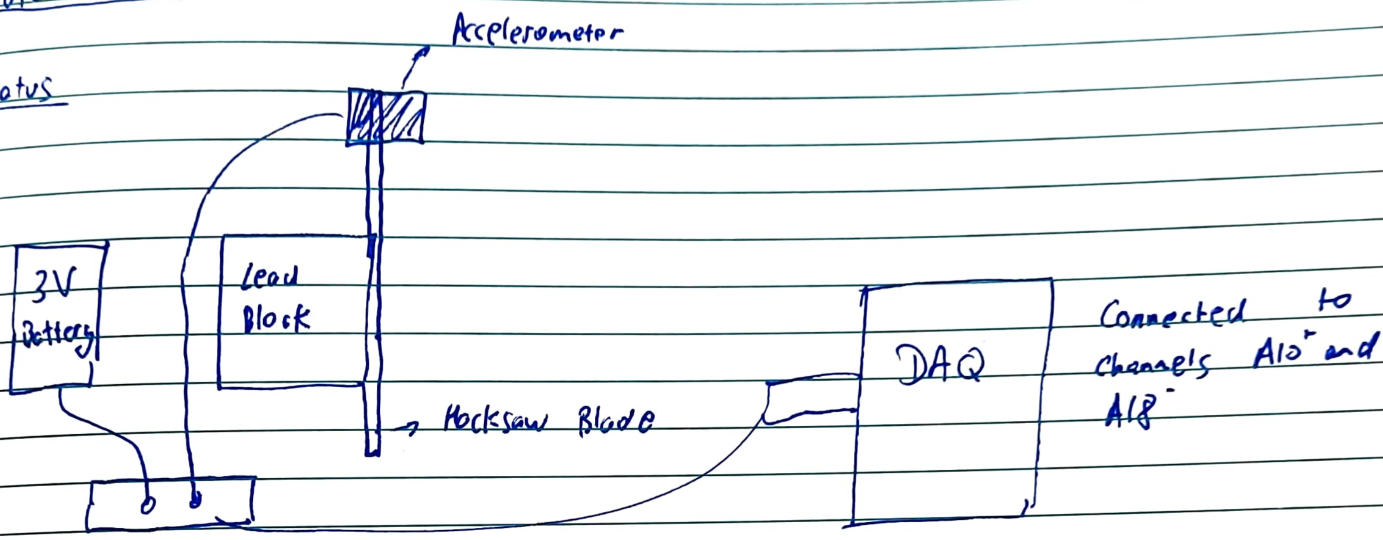
At high freq $\phi \simeq 0°$

Phase plot from last week are different due to ω^2 factor as they were $\phi \simeq 0°$ at low, $-90°$ at resonance and $-180°$ at hight f.

Experiments

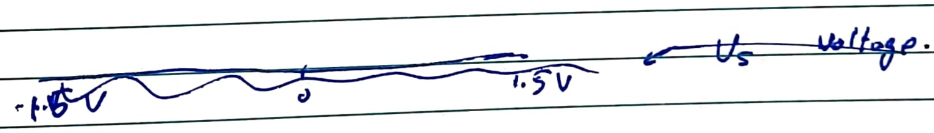
① Calibrate Accelerometer

• Apparatus



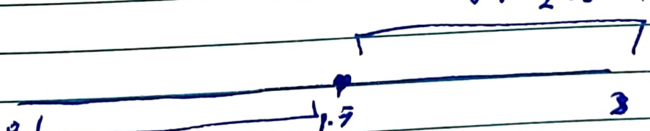
Goal: ~~Measure~~ Calibrate Accelerometer by making measurements of V_{acc} at "0" V.

$V_{acc} \propto V_s$ where $V_s \sim 3V$ with V_{acc} zeroed at $\frac{1}{2}V_s$



V_{acc} zero : $\frac{1}{2}V_s = \frac{1}{2}(3) = \frac{3}{2}V = 1.5\ V$

$V > \frac{1}{2}V_s$ (Positive acceleration)



$V < \frac{1}{2}V_s$ V_{acc} zero

(Negative Acceleration)

Battery Output = ~2.98 V

(Voltage Measurement at "0" Volts at V_s)

After connecting the system as depicted, Labview ran and the subsequent measurement of the wave recorded.

LVM data extracted to python with and the mean amplitude and Voltage Mean ↑ calculated from the data points. ~~Voltage mean found~~

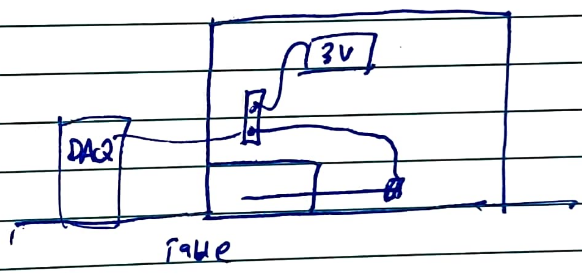
- Voltage mean : mean(Votage data)
- Amplitude (v) : | Voltage data - Voltage mean |
- Mean Amplitude : Mean (amplitude)

Results

Mean Votage = 1.502134 V with amplitude : 0.003807 V
↳ Noise

Voltage Measurement at (+g)

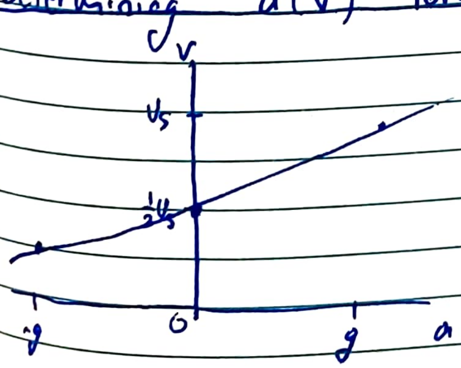
The Apparatus tilted 90° and in a similar method to before it's mean Voltage and amplitude calculated and plotted.



$V_{mean} = 1.799403\,V$ Amplitude (Noise) : 0.001537 V

The blade towards the earth taken as positive so this orientation taken as positive, hence +g.

Determining a(v) formula



$$\Delta V = m \Delta a$$

$$V_g - V_0 = m(a_g - a_0)$$

$$m = \frac{V_g - V_0}{a_g - a_0} = \frac{1.799403 - 1.502134}{9.81 - 0}$$

$$= 0.0303026 5 \approx 0.03$$

$$\boxed{\Delta a(v) = \frac{\Delta V}{0.03}}$$

V_g Prediction

$\Delta a(v) = \frac{\Delta V}{0.03}$ $\quad a(V_0) - a(V_g) = \frac{V_0 - V_g}{0.03}$

$0.03(0 - (-9.81)) = 1.502134V - V_g$

$0.03(9.81) - 1.502134 = -V_g$

$V_g = 1.502134 - 0.03(9.81) = 1.207834 \approx \boxed{1.21V}$

V_g Measurement

In a similar process to before the V_g data is collected in Labview, from the LVM file, the mean voltage along the amplitude is computed in Python.

$V_{g_{mean}} = 1.198823 \ V \approx \boxed{1.20V}$

Amplitude (Noise): 0.001242 V

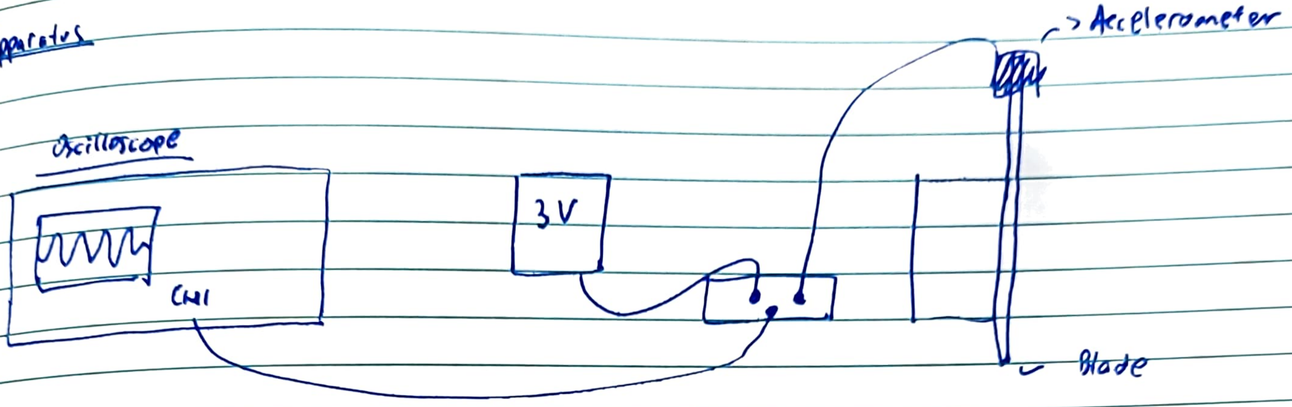
The predicted and the measured voltage are approximately identical with the minor disagreement being around 0.01V.

So $\boxed{\text{Calibration error} \approx 0.01V}$

* Board also clamped to table now to decrease any noise frequency.

Free Oscillation Frequency

· Apparatus



- The accelerometer connected straight to the oscilloscope to measure its frequency. The blade length adjusted repeatedly along the lead block ~~to~~ until the measured frequency ~ 12 Hz.

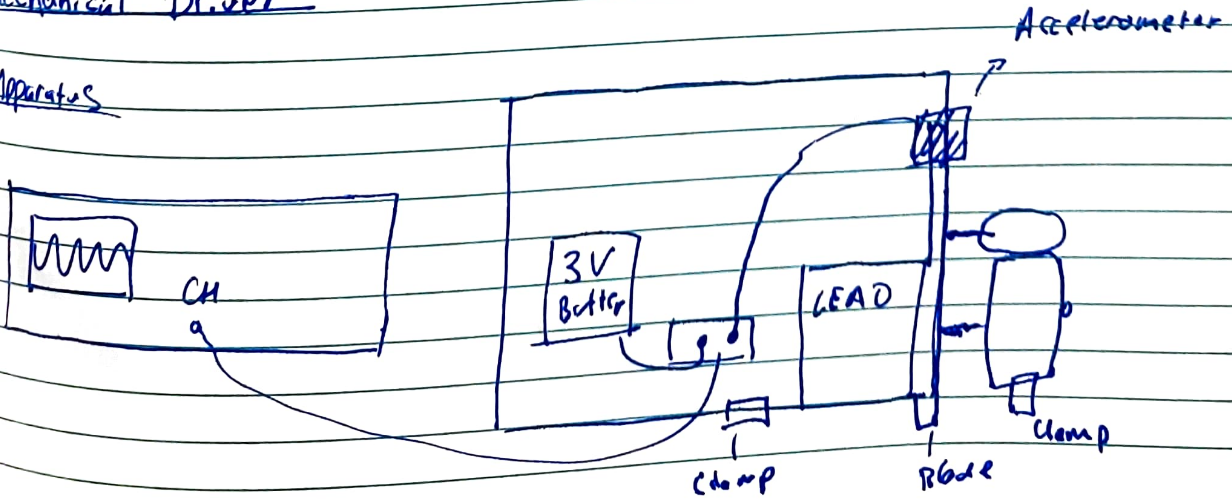
More than 1 oscillation present?
- NO, beat phenomena not observed and the observed frequency is a single sinusodal wave with decreasing amplitude as time goes on.

Blade Adjusted until measured frequency ≈ $\boxed{12.16\text{ Hz}}$ ± 1 Hz

$\boxed{\text{Free Oscillation frequency} \simeq 12.16\text{ Hz}}$

Mechanical Driver

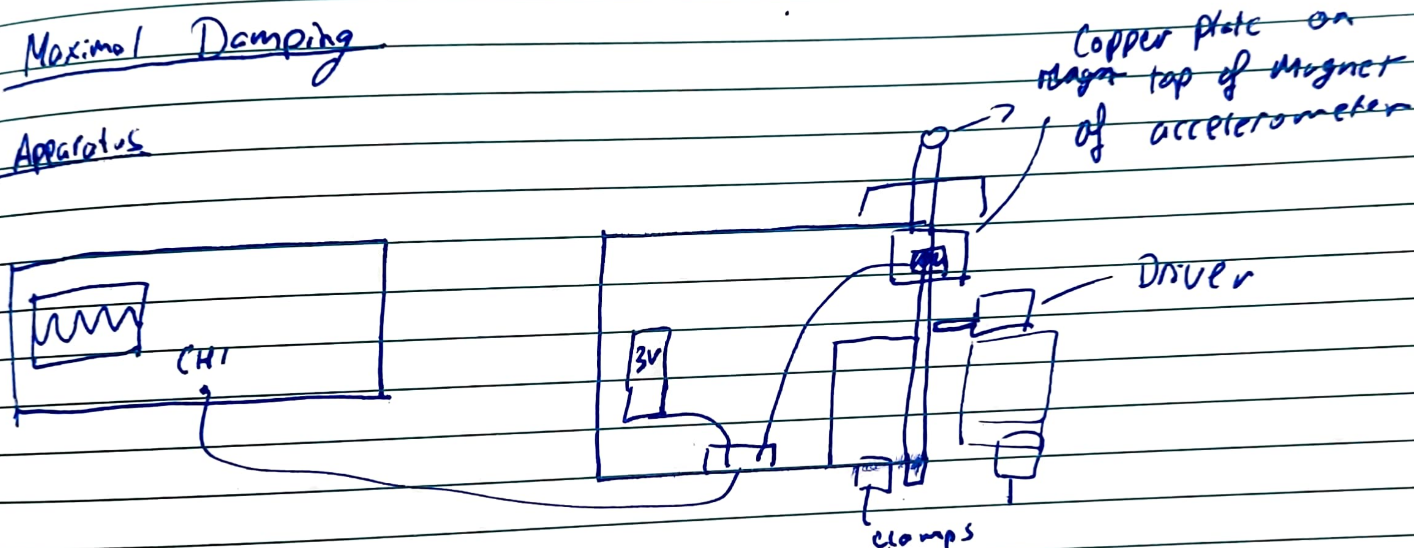
Apparatus



The position of the driver ensures a nice sine wave at 12 Hz, 20 Vpp but the quality of wave drops around higher frequencies of 20 - 30 Hz. The quality did improve through adjusting the output amplitude and the resolution of oscilloscope so apparatus left as is. Also seems to generate good waves below resonance.

Maximal Damping

- Apparatus



Copper plate moved straight above the magnet of the accelerometer and induced maximal damping. The blade displaced from equilibrium by hand, stopped after couple oscillations.

Plate removed and oscillations continued as usual.

Forced motion with maximal damping

Apparatus

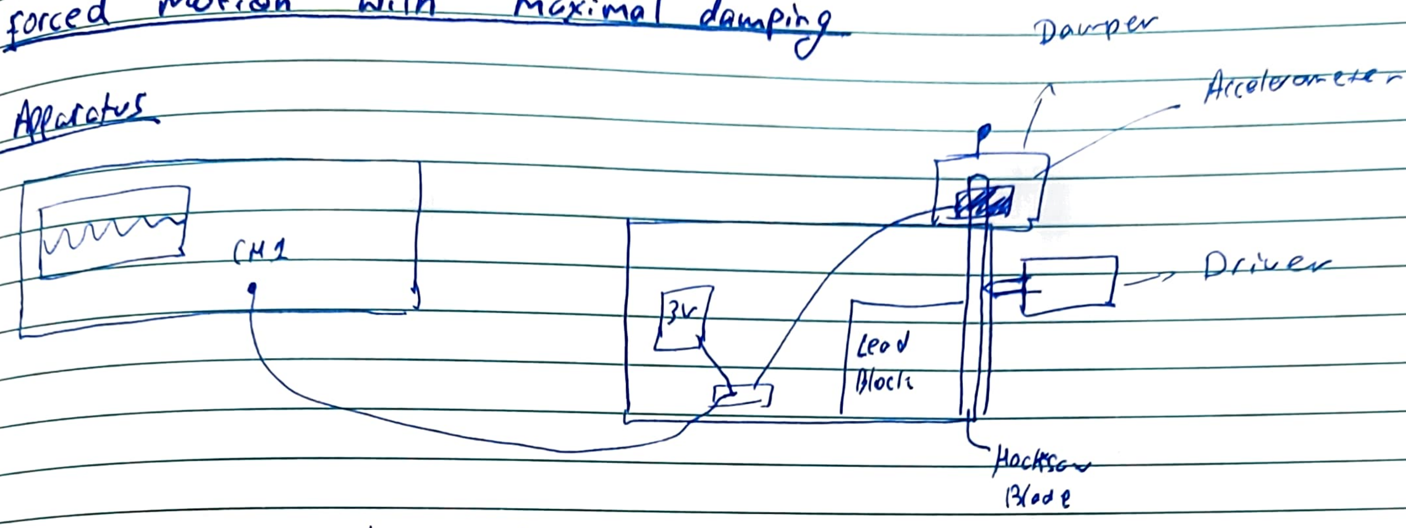


Plate set in max damping position and mechanical driver activated aswell allowing it to continuously run the system.

- Measure $V_{acc}(t)$ and $V_{in}(t)$ for variety of frequencies:

V_{acc} channel AI0 and AI8
V_{in} " AI1 and AI9

Frequency	Amplitudes V_{acc}	V_{in}	Phase Difference
19	+0.0445	0.7221	1.65
19.1	0.051	0.725	1.155
19.2			
19.3			
19.4			
19.5			
19.6			
19.7			
19.8			
19.9			

Amplitudes

1k.

Frequency (Hz)	V_{acc}	V_{in}	Phase Difference (V_{in})
1.0	0.002499	0.7185	-0.8141
2.0	0.01485	0.7223	-0.7111
3.0	0.01587	0.7223	-0.0606
4.0	0.01798	0.72266	-0.0712
5.0	0.020206	0.7227	-0.2866
~~6~~ 7.0	0.02255	0.72279	-0.2727
9.0	0.02535	0.7227	-0.4735
9.5	0.02716	0.72286	-0.5139
10.0	0.02879	0.72299	-0.5778
10.2	0.03056	0.7230	-0.6010
10.4	0.03267	0.7229	-0.6574
10.6	0.03446	0.7230	-0.7062
10.8	0.03715	0.7230	-0.7743
11.0	0.03874	0.7237	-0.8528
11.1	0.0441	0.7227	-0.8951
11.2	0.04309	0.7228	-0.9367
11.3	0.0445	0.7221	-0.9787
11.4	0.04637	0.72246	-1.0338
11.5	0.04801	0.72260	-1.1104
11.6	0.04869	0.7229	-1.1624
11.7	0.04890	0.7231	-1.2313
11.8	0.04879	0.7229	-1.3192
11.9	0.04820	0.7238	-1.4061
12.0 Resonance	0.04717	0.72379	-1.4926
12.1	0.04585	0.7237	-1.5921
12.2	0.04460	0.72365	-1.6877
12.3	0.0432	0.7236	-1.7878
12.4	0.041799	0.72372	-1.8891
12.5	0.0399	0.7235	-1.9832
12.6	0.03845	0.7234	-2.0870
12.7	0.03678	0.7233	-2.1732
12.8	0.03555	0.7237	-2.2564
12.9	0.03412	0.7232	-2.3398
13.0	0.0327	0.7234	-2.4101
13.1	0.03190	0.7233	-2.4788
13.2	0.0360	0.7234	-2.5413
13.3	0.02971	0.72291	-2.5866

Frequency (Hz)	Vocc	Vin	Phase Difference
13.4	0.02745	0.7231	-2.6413
13.5	0.02447	0.7234	-2.6851
13.6	0.0232	0.7234	-2.7981
13.7	0.02210	0.7234	-2.7694
13.8	0.0182	0.7234	-2.7936
13.9	0.0157	0.7237	-2.8315
14.0	0.01387	0.7237	-2.8573
14.2	0.01277	0.7240	-2.9143
14.6	0.00071	0.7208	-2.9989
14.8	0.01154	0.7246	-3.0264
15.0	0.0108	0.7245	-3.0522
16.0	0.0104	0.7253	-3.1771
17.0	0.00072	0.7214	-3.2470
18.0	0.00114	0.7211	-3.3157
19.0	0.0017	0.7216	-3.3363
20.0	0.0039	0.7219	-3.3733
21.0	0.0092	0.7222	-3.4163
~~22.0~~ 22.0	0.0115	0.7223	-3.4483

The phase plot runs from 0 to $-\pi$ instead of standard behaviour of π to 0.

Possible reason: Input and ground wires were backwards on DAQ which causes a π difference on phase.

Curve fit Parameters $\left(\frac{V_{occ}}{V_{in}} \text{ vs } f\right)$

Scaling factor A: 1.629

Natural frequency: $\omega_0 = 12.68$ Hz

Damping Coefficient: $\gamma = 1.95$ Hz

$\frac{\omega_0}{\gamma} = Q = 6.49$

Vin amplitude changes but is roughly the same for all frequencies.

Lab 6 - Continued

Prelab

Damped Oscillator

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

① If $v_i = 0$ and if motion is underdamped ($\gamma < \omega_0$), show $x(t)$ follows

$$x(t) = x_0 e^{-\gamma t} \cos \omega_1 t.$$

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

Characteristic Equation

$$r^2 + 2\gamma r + \omega_0^2 = 0$$

$$r = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

Underdamped so $\gamma < \omega_0$,

$$\gamma^2 - \omega_0^2 < 0 \rightarrow \sqrt{\gamma^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \gamma^2} = i\omega_1$$

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2}$$

General Solution

$$x(t) = e^{-\gamma t}\left[A\cos(\omega_1 t) + B\sin(\omega_1 t)\right]$$

Initial Conditions

$x(0) = x_0$

$\dot{x}(0) = 0$

$$x(0) = e^{0}\left[A\cos(0) + B\sin(0)\right] = x_0$$

$$A = x_0$$

$$\dot{x}(t) = e^{-\gamma t}\left[-\gamma\left(A\cos\omega_1 t + B\sin\omega_1 t\right)\right] + e^{-\gamma t}\left[-A\omega_1\sin(\omega_1 t) + B\omega_1\cos(\omega_1 t)\right]$$

$$\dot{x}(0) = e^{0}\left(-\gamma A\cos(0) - \gamma B\sin(0)\right) + e^{0}\left[-A\omega_1\sin(0) + B\omega_1\cos(0)\right] = 0$$

$$\dot{x}(0) = -\gamma A + B\omega_1 = 0$$

$$-\gamma x_0 + B\omega_1 = 0$$

$$B = \frac{\gamma}{\omega_1} x_0$$

$$\boxed{x(t) = x_0 e^{-\gamma t}\left[\cos(\omega_1 t) + \frac{\gamma}{\omega_1}\sin(\omega_1 t)\right]}$$

Can use $A\cos(\omega_1 t) + B\sin(\omega_1 t) = R\cos(\omega_1 t - \phi)$ to write equation similar to one asked but x_0 will just incorporate the shifted phase

(ii) For small damping $\gamma << \omega_0$

$$Q = \frac{\omega_0}{2\gamma} \qquad A(t) = x_0 e^{-\gamma t}$$

# of periods to where Amplitude decays by $\frac{1}{e}$

$$T \approx \frac{2\pi}{\omega_1} \approx \frac{2\pi}{\omega_0}$$

After n oscillations

$$A(nT) = x_0 \exp(-\gamma n T)$$

$$\frac{A(nT)}{x_0} = e^{-\gamma n T} = \frac{1}{e} \rightarrow \gamma n T = 1$$

$$\ln(e^{-\gamma n T}) = \ln\left(\frac{1}{e}\right)$$

$$-\gamma n T = \ln(1) - \ln(e)$$

$$-\gamma n T = 0 - 1$$

$$-\gamma n T = -1$$

$$\gamma n T = 1$$

$$n = \frac{1}{\gamma T}$$

$$\left(T \approx \frac{2\pi}{\omega_0}\right)$$

$$n \approx \frac{1}{\gamma}\frac{\omega_0}{2\pi} = \frac{\omega_0}{2\gamma}\frac{1}{\pi}$$

$$n = \frac{\omega_0}{2\gamma}\frac{1}{\pi} = Q\frac{1}{\pi} = \boxed{\frac{Q}{\pi}}$$

$$\boxed{Q = n\pi}$$

So after $n = \frac{Q}{\pi}$ periods oscillation decays by a factor of $\frac{1}{e}$

(2) $x(t) = x_0 e^{-\gamma t} \cos(\omega_1 t)$

$$\dot{x}(t) = x_0 e^{-\gamma t}\left[-\gamma \cos(\omega_1 t) - \omega_1 \sin(\omega_1 t)\right]$$

$$\ddot{x}(t) = x_0\left[(-\gamma e^{-\gamma t})(-\gamma\cos(\omega_1 t) - \omega_1 \sin(\omega_1 t)) + e^{-\gamma t}(\gamma\omega_1 \sin(\omega_1 t) - \omega_1^2 \cos(\omega_1 t))\right]$$

$$= x_0 e^{-\gamma t}\left[-\gamma(-\gamma\cos(\omega_1 t) - \omega_1 \sin(\omega_1 t)) + \gamma\omega_1 \sin(\omega_1 t) - \omega_1^2\cos(\omega_1 t)\right]$$

$$= x_0 e^{-\gamma t}\left[\gamma^2\cos(\omega_1 t) + \gamma\omega_1 \sin(\omega_1 t) + \gamma\omega_1 \sin(\omega_1 t) - \omega_1^2 \cos(\omega_1 t)\right]$$

$$= x_0 e^{-\gamma t}\left[(\gamma^2 - \omega_1^2)\cos(\omega_1 t) + 2\gamma\omega_1 \sin(\omega_1 t)\right]$$

$$\ddot{x}(t) = x_0 e^{-\gamma t}\left[(\gamma^2-\omega_1^2)\cos(\omega_1 t) + 2\gamma\omega_1 \sin(\omega_1 t)\right]$$

$A = \gamma^2 - \omega_1^2 \qquad B = 2\gamma\omega_1$

$|R| = \sqrt{(\gamma^2-\omega_1^2)^2 + (2\gamma\omega_1)^2}$

~~ph~~ Phase Shift : ϕ

$A\cos(\omega_1 t) + B\sin(\omega_1 t) = R\cos(\omega_1 t - \delta)$

$R\cos(\omega_1 t + \phi)$

$R = \sqrt{A^2+B^2} \qquad \delta = \tan^{-1}\left(\frac{B}{A}\right)$

$A\cos\theta + B\sin\theta = \sqrt{A^2+B^2}\cos(\theta - \delta)$

$\therefore \ddot{x}(t) = x_0 e^{-\gamma t} R\cos(\omega_1 t + \phi)$

$a_0 = x_0 R$

$\rightarrow \ddot{x}(t) = \boxed{a(t) = a_0 e^{-\gamma t}\cos(\omega_1 t + \phi)}$

(3) Due to time constraints phase data was accidentally not recorded while varying frequency around the resonance. When phase was extracted from each frequency file, the plots were very far from ideal behaviour suggesting analysis error. And due to not being able to take data over the break, dummy data was generated for the purpose of answering this prelab question.

Plots and code on Crowdmark.

χ^2_{phase} : 16.32 Reduced χ^2_{phase} : 0.71

Reduced $\chi^2 < 1$ so suggests good fit.
The residuals are also appear random indicating good fit.

$\chi^2_{amplitude}$: 20.71 Reduced $\chi^2_{amplitude}$: 0.88

Reduced $\chi^2 < 1$ and residuals random on plot indicating good fit.

The parameters also agree and are within the uncertainity range.

- Redid the amplitude measurements to check if the apparatus is consistent with measurements from last lab.

Frequency (Hz)	Vocc (v)	Vin (v)	Phase diff (rad)
11.9 Hz	0.0475	0.722	1.13133
12.1	0.051	0.725	1.6557
12.2	0.0570	0.7250	1.7100
12.3	0.060	0.7400	1.6202
12.4	0.0612	0.7452	1.8096
12.5	0.0570	0.7180	1.9788
11.4	0.0315	0.7211	1.0663
11.5	0.0380	0.7215	1.0659
11.6	0.0415	0.7218	1.1313
11.7	0.0380	0.7187	1.1286
11.8	0.0445	0.4457	1.3212

- Measurements are consistents ✓

Measure transients at max damping.

- Using the max damping apparatus from last week, decoupled the driver and pulled the blade to one side and let it go. The resulting motion is erratic for short term before becoming damped and stopped. Ran this for a few times and extracted the DAQ data to python. The residual plot is not random instead showing oscillatory pattern, this indicates there is a higher order frequency affecting the system.

Fourier transform used on the residuals to estimate the higher order frequency.
Output & Code on crowdmark.

This process was repeated for min and moderate damping, where the copper plate position was adjusted for the required situation. The residuals for these were also not random and the higher order frequency est. using FFT.

Analysis

- For V_{acc}, V_{in} measurement at diff frequencies, all data extracted to python and curve fitted, the fit revealed correct amplitude values and phases.

- Phase difference and Amplitude ratio plotted vs frequency and fitted using curve fit.

Curve fit Parameters

$\omega_0 = 77.948 \pm 0.276$ rad/s

$\gamma = 5.911 \pm 0.253$ rad/s

$\phi_{offset} = -3.446 \pm 0.024$ rad

$Q = 6.593$

Phase

~~Amplitude~~

$A_{fit} = 0.0109 \pm 0.0001$

$f_0 = 12.321 \pm 0.006$ Hz

$\gamma = 0.990 \pm 0.008$ Hz

$Q = 6.222$

Amplitude

These fit values and uncertainties were calculated at max damping. Moderate or min damping values were not recorded for steady state measurements of Vin and Vacc due to ambigious instructions. Due to instructions not explicitely stating to remeasure the Lab 5 values it was presumed that remeasure for min and moderate were for transient measurements in part (a) for Lab 6.

- γ appears to depend on the amplitude, it's larger at high amplitudes possibly due to friction or velocity dependent energy losses being greater. Residuals are also not random and show oscillatory patterns indicating traces of higher frequencies, these were calculated using fft and shown on crowdmark.

Q-Values

Resonance Curves ($\frac{V_{acc}}{V_{in}}$ vs f)

$Q = 6.222$

$Q_{phase} = 6.59$

$\boxed{\text{Free decay}}$ (Max Damping)

Instead of slicing the time interval, the whole curve fitted at once to compute Q.

$A = -0.8267 \pm 0.0037$ $\quad \phi = -0.882 \pm 0.0041$
$\gamma = 4.151 \pm 0.026$ $\quad$ offset $: 1.46 \pm 0.00048$
$\omega = 75.56 \pm 0.025$ rad/s $\quad f_0 = 12.03\,Hz$

$\boxed{Q_{free} = 9.102}$

$Q_{free} \neq Q_{resonance} / Q_{phase}$ indicating damping is amplitude dependent. At larger amplitudes system loses less energy (higher Q) so transient has higher Q than driven, damped system which to keep consistent amplitude.

- Time offset in DAQ's data collection could impart a permanent ϕ to all data collected causing entire phase v freq graph to shift.
  - To test this you could connect the same source to both channels so they should be in perfect phase and note any measured offset.

- When response is linear, driving amplitude scales output proportionally, if non linear can cause a shift in resonance / excite higher modes (non linear)
  - Better to use small driving amplitudes to keep response linear.

Testing differing amplitudes at differing frequencies, allow you to test whether system linear or not. Other resonance frequencies show up at small peaks in amp ratio, phase shifts, ordered residuals etc. You can try to remove them by damping or physically constraining the system.

- The fit parameters are consistent any disagreements are explained by nature of the system.

- SHO model captures the general shape of resonance peak and the π shift around resonance and the exponential decay quite well.
  - For moderate amplitudes, predicts behaviour well.

- Misses amplitude - dependent damping as real friction more complicated than drag term (γ). The single ω_0 of model can't account for other resonances.

The model fails at large amplitudes as it has multiple d.o.f.

Summary

Hacksaw blade driven at various f and made to observe forced oscillations. Amplitude ratio $\frac{V_{osc}}{V_{in}}$ and phase difference measured. From them data

fitted and identified resonance peaks, ω_0, γ and Q. Then driver disconnected and transient measured and its ω_0, γ and Q extracted.

Results conclude both methods yield similar values but not identical (Q) suggesting amplitude dependent damping or higher order modes. This showcases SHO (Simple harmonic Oscillator) works well at moderate amplitudes, capturing main resonance and phase shift but fails to describe high-amplitude or multi-mode behaviour.