Fri. Feb 10 2023

1 Vector Spaces

Suppose that V is a finite dimensional vector space over F, with $\dim(V) = n$.

V may have *many different* bases, we know that they all have the same size n.

Say $\mathcal{B} = \{\alpha_1, ..., \alpha_n\}$ is a basis fix the ordering of $\mathcal{B}$.

Fix the ordering of $\mathcal{B}$.

Theorem

For any $\alpha \in V$, there is a unique n tuple $(x_1, ..., x_n) \in F^n$ such that

$$\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n$$

Proof

Existence is immediate, since $\mathcal{B}$ is a basis, thus $\mathcal{B}$ spans V.

Uniqueness

Say $\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n$ and $\alpha = y_1\alpha_1 + \cdots + y_n\alpha_n$.

Then we have that

$x_1\alpha_1 + \cdots + x_n\alpha_n - y_1\alpha_1 + \cdots + y_n\alpha_n = 0$, so $(x_1 - y_1)\alpha_1 + \cdots + (x_n - y_n)\alpha_n = 0$

But since $\{\alpha_1, ..., \alpha_n\}$ is linearly independent, all coefficients must be 0.

∎

What this means is that, for a vector space V, there is an associated mapping in F^n. Notice that we know nothing about the vectors α_i.

We define $[\alpha]_{\mathcal{B}}$ to be the *coordinates* of α with respect to $\mathcal{B}$.

Check: The mapping $\alpha \mapsto [\alpha]_{\mathcal{B}} \in F^n$ satisfies

1. One to one-ness
2. Onto-ness
3. "Additive", for any $\alpha, \beta \in V$, if $\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n$ and $\beta = y_1\alpha_1 + \cdots + y_n\alpha_n$. Then

$$[\alpha + \beta]_{\mathcal{B}} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = [\alpha]_{\mathcal{B}} + [\beta]_{\mathcal{B}}$$

4. $[c\alpha]_{\mathcal{B}} + c[\alpha]_{\mathcal{B}}$

There exists an *isomorphism* between V and F^n.

Example

Let $\mathcal{P}$ be the space of al polynomials. Let $f(x) = x^3$, and $g(x) = x^5$. Then, let

$$V = \text{Span}\{f, g\} = \{\text{all } ax^3 + bx^5 : a, b \in F\}$$

then, $\dim(V) = 2$, since f and g are linearly independent.

Typical $h(x) \in V$, say $h(x) = 10x^3 - 2x^5$.

$$[h]_{\mathcal{B}} = \begin{bmatrix} 10 \\ -2 \end{bmatrix}$$

$\langle$ $[h]_{\mathcal{B}}$ is the mapping of h to F^n. **TODO** is this right? $\rangle$

Now let $k(x) = 2x^3 + 4x^5$ and $l(x) = x^3 + 3x^5$. Since k, l are linearly independent, they form another basis of V.

$$\mathcal{B}' = \{k(x), l(x)\}$$

1.1 Change of Basis

Given $\mathcal{B} = \{\alpha_1, ..., \alpha_n\}$, and $\mathcal{B}' = \{\alpha'_1, ..., \alpha'_n\}$ bases for V.

We want to describe the map going from $[\alpha]_{\mathcal{B}} \mapsto [\alpha]_{\mathcal{B}'}$.

$\langle$ We want to find The $\mathcal{B}$ coordinate of $\alpha \mapsto$ the $\mathcal{B}'$ coordinate of α $\rangle$

Step 1.

Compute the $\mathcal{B}$ coordinate of $\alpha'_1, ..., \alpha'_n$, *old* coordinates of the *new* basis elements.

Step 2.

For an $n \times m$ matrix

$$P = \Big[[\alpha'_1]_{\mathcal{B}}, ..., [\alpha'_n]_{\mathcal{B}}\Big]$$

Check: for any $\alpha \in V$

$$[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$$

Ans: This is what we actually want

$$[\alpha]_{\mathcal{B}'} = P^{-1}[\alpha]_{\mathcal{B}}$$

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TODO Missing *some* info

Want: Describe the mapping $T : F^n \to F^n$

$$T([\alpha]_{\mathcal{B}_{\text{old}}}) = [\alpha]_{\mathcal{B}'_{\text{new}}}$$

$\langle$ If we switch the basis for some reason, we want to see what the new coordinates are. $\rangle$

To do this: For each α'_j, compute $[\alpha'_j]_{\mathcal{B}_{\text{old}}}$. Let

$$P = \left[[\alpha'_1]_{\mathcal{B}_{\text{old}}} \cdots [\alpha'_n]_{\mathcal{B}_{\text{old}}} \right]$$

be an $n \times n$ matrix.

Claim: For any $\alpha \in V$

$$P \cdot [\alpha]_{\mathcal{B}'_{\text{new}}} = [\alpha]_{\mathcal{B}_{\text{old}}}$$

How?

$$P \cdot [\alpha'_1]_{\mathcal{B}'_{\text{new}}} = P \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [\alpha'_1]_{\mathcal{B}_{\text{old}}}$$

This is the 1^{st} column of P, and similarly for all columns.

Thus: For any $\alpha \in V$,

$$[\alpha]_{\text{new}} = P^{-1} \cdot [\alpha]_{\text{old}}$$

Example

In practice, we have the following.

$V = \text{Span}(\{x^3, x^5\})$ subspace of $\mathcal{P}$ = all polynomials. Let $f(x) = x^3, g(x) = x^5, \mathcal{B} = [x^3, x^5]$. Let $h(x) = 10x^3 - 2x^5 \in V$.

Question: What are the coordinates of h with respect to $\mathcal{B}$?

Answer:

$$[h]_{\mathcal{B}} = \begin{bmatrix} 10 \\ -2 \end{bmatrix}$$

Example

Let $k(x) = 2x^3 + 5x^5$, $l(x) = x^3 + 3x^5$.

Let $\mathcal{B}' = \{k(x), l(x)\}$ be another basis of V.

Question: What are the coordinates of h with respect to $\mathcal{B}'$?

Answer:

$[k(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $[l(x)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

So

$$P = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

Check:

$$P^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

Then

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ -2 \end{bmatrix} = \begin{bmatrix} 32 \\ -54 \end{bmatrix}$$

This means:

$$h(x) = 32k(x) - 54l(x) = 10x^3 - 2x^5$$

Which is what we expect.

Example

Let $V = \mathbb{R}^2$. Standard basis $\mathcal{B} = \{\varepsilon_1, \varepsilon_2\} = \{(1,0),(0,1)\}$

$$[(5,4)]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Fix angle θ, Let

$$\mathcal{B}' = \{(\cos(\theta), \sin(\theta)), (-\sin(\theta), \cos(\theta))\}$$

Question: What is $\begin{bmatrix} 5 \\ 4 \end{bmatrix}_{\mathcal{B}'_{\text{new}}}$?

Answer:

1. Form P

$$[(\cos(\theta), \sin(\theta))]_{\mathcal{B}} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

$$[(-\sin(\theta), \cos(\theta))]_{\mathcal{B}} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

Then

$$P = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Fact:

$$P^{-1} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

so we have

$$\begin{aligned} [(5,4)]_{\mathcal{B}'_{\text{new}}} &= P^{-1} \begin{bmatrix} 5 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 5\cos(\theta) & 4\sin(\theta) \\ -5\sin(\theta) & 4\cos(\theta) \end{bmatrix} \end{aligned}$$

2 Chapter 3

Say V, W are both vector spaces over the same field F.

Definition

A **Linear Transformation** $T : V \to W$ is a function satisifying two rules

1. For all $\alpha, \beta \in V$,

$$T(\alpha + \beta) = T(\alpha) + T(\beta)$$

   Note that the first $+$ is addition in V, but the second is addition in W.

2. For all $\alpha \in V$ and $c \in F$,

$$T(c\alpha) = cT(\alpha)$$

$\langle$ The book combines these two into one. $\rangle$

Lots of examples to come

Two basic facts:

Suppose that $T : V \to W$ is a linear transformation

1. $T(0) = 0$
   Proof:
   $T(0 + 0) = T(0) + T(0)$ thus $T(0) = 0$.
   $\langle$ Always be aware of where the 0 lives $\rangle$
   TODO Not super clear

2. For all $\{\alpha_1, ..., \alpha_n\} \subseteq V$, all $\{c_1, ..., c_n\} \in F$,

$$c_1T(\alpha_1) + \cdots + c_nT(\alpha_n)$$

   Proof Easy induction on n.

Example

Take $A \in F^{m\times n}$ an $m \times n$ matrix with entries in F.

Then $T_A : F^n \to F^m$ given by $T_A(x) = A \cdot X$ is a linear transformation.

Check

Chose any $X, Y \in F^n$, then

$$T_A(X + Y) = A \cdot (X + Y) = A \cdot X + A \cdot Y = T_A(X) + T_A(Y)$$

For $c \in F$, have

$$T_A(cX) = A \cdot (cX) = cAX = cT_A(X)$$

which is what we expect.

Example

Consider $\mathcal{P}$ the set of all polynomials $a_0 + a_1 x + \cdots + a_n x^n$.

Differentiation

$$D : \mathcal{P} \to \mathcal{P}$$

$F(f) = f'$, the **derivative**

Claim: $D : \mathcal{P} \to \mathcal{P}$ is a linear transformation.

Check:

$$D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$$

and for $c \in F$,

$$D(cf) = (cf)' = c \cdot f' = c \cdot D(f)$$

which is what we expect.

Example

Let $C(\mathbb{R})$ be all combinations of all functions $f : \mathbb{R} \to \mathbb{R}$.

Define $I : C(\mathbb{R}) \to C(\mathbb{R})$ the **integral**

$$I(f) = \int_0^x f(t)dt$$

$\langle$ Note that the integral exists because you can always integrate a contiunous function. $\rangle$

The result is also continuous and differentiable by the Fundamental Theorem of Calculus.

$$D(I(f)) = f$$

Is the **Fundamental Theorem of Calculus**.

Therefore $I(f)$ really *is* continuous, $I(f) \in C(\mathbb{R})$.

Question: Is it really linear?

Check:

$$\begin{aligned} I(f+g) = &\int_0^x (f(t) + g(t))dt \\ = &\int_0^x f(t)dt + \int_0^x g(t)dt \\ = &I(f) + I(g) \end{aligned}$$

and

$$I(cf) = \int_0^x cf(t)dt = c\int_0^x f(t)dt = cI(f)$$

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Recall: A linear transformation $T : V \to W$ is a function between two vector spaces over the same field F, satisifying

1. For all $\alpha, \beta \in V$,

$$T(\alpha + \beta) = T(\alpha) + T(\beta)$$

   Note that the first + is addition in V, but the second is addition in W.

2. For all $\alpha \in V$ and $c \in F$,

$$T(c\alpha) = cT(\alpha)$$

For all $\alpha_1, ..., \alpha_k \in V$, and $c_1, ..., c_k \in F$, it breaks nicely into

$$T(c_1\alpha_1 + \cdots + c_k\alpha_k) = c_1T(\alpha_1) + \cdots + c_kT(\alpha_k)$$

Example

$I^* : C(\mathbb{R}) \to \mathbb{R}$ (all continuous functions from $\mathbb{R}$ to $\mathbb{R}$)

$$I^*(f) = \int_0^1 f(x)dx$$

$$I^*(x^2) = \int_0^1 x^2 dx = \frac{x^3}{x}\Big|_0^1 = \frac{1}{3}$$

Note that the output of $I*$ is just a number here. Additionally, I^* is linear: you can split integrals up for polynomials, and you can take constants outside.

For any V, W, we also have

$$X : V \to W$$

Is the zero transformation. It takes any $\alpha \in V$ to the 0 of W. We'll use this to prove theorems about linear transformations later.

Theorem

1. Linear Transformations $T : V \to W$ are **determined** by their behavior on a basis $\mathcal{B}$ of V. More precisely,

   Suppose that $\mathcal{B} = \{\alpha_1, ..., \alpha_n\}$ is a basis for V and suppose that $T, U : V \to W$ are both linear tranformations (and they agree on a basis), such that

$$T(\alpha_1) = U(\alpha_1), T(\alpha_2) = U(\alpha_2), ..., T(\alpha_n) = U(\alpha_n)$$

   Then $T = U$

2. For **any map** $T_0 : \mathcal{B} \to W$, there s a unique linear transformation $T : V \to W$ with $T \supseteq T_0$. In other words,

   Let $\mathcal{B} = \{\alpha_1, ..., \alpha_n\}$ be **any basis** for V and let $\beta_1, ..., \beta_n$ be **any vectors** in W.

   Then, there is a **unique** linear transformation $T : V \to W$ such that

$$T(\alpha_1) = \beta_1, T(\alpha_2) = \beta_2, ..., T(\alpha_n) = \beta_n$$

PROOF

1. **Uniqueness**: Chose any $\alpha \in V$, since $\mathcal{B}$ is a basis,

   $\langle$ Will show that $T = U \Leftrightarrow$ For any $\alpha \in V$, $T(\alpha) = U(\alpha)$ $\rangle$

$$\alpha = c_1\alpha_1 + \cdots + c_n\alpha_n$$

   for some **unique** $c_1, ..., c_n \in F$.

   Since T is a linear transformation,

$$T(\alpha) = c_1T(\alpha_1) + \cdots + c_nT(\alpha_n)$$

   Likewise with U,

$$U(\alpha) = c_1U(\alpha_1) + \cdots + c_nU(\alpha_n)$$

   But, since $T(\alpha_1) = U(\alpha_1), ..., T(\alpha_n) = U(\alpha_n)$, $T(\alpha) = U(\alpha)$.

   $\langle$ Essentially, if T, U work the same for all α_i, then their sum will obviously be the same, and so they'll give the same result for the same α. $\rangle$

   Note that this theorem *still* works for infinite dimensional vector spaces.

2. **Existence**: Chose any $\alpha \in V$. $\langle$ We must define $T(\alpha)$ $\rangle$

   Since $\mathcal{B}$ is a basis, we can write

$$\alpha = c_1\alpha_1 + \cdots + c_n\alpha_n$$

   which is unique.

   Define

$$T(\alpha) := c_1\beta_1 + \cdots + c_n\beta_n \in W$$

   Check: Is T linear?

   Say $\gamma = d_1\alpha_1 + \cdots + d_n\alpha_n$, $\delta = e_1\alpha_1 + \cdots + e_n\alpha_n$.

   In V, we have that $\gamma + \delta = (d_1 + e_1)\alpha_1 + \cdots + (d_n + e_n)\alpha_n$.

   By our definition of T, we have

$$\begin{aligned} T(\gamma + \delta) =& (d_1 + e_1)\beta_1 + \cdots + (d_n + e_n)\beta_n \\ =& (d_1\beta_1 + \cdots + d_n\beta_n) + (e_1\beta_1 + \cdots + e_n\beta_n) \\ =& T(\gamma) + T(\delta) \end{aligned}$$

   Check: $T(c\gamma) = cT(\delta)$

   So such a tranformation T exists. Additionally by part (1), it is unique.

■

Let $T : V \to W$ be a linear transformation.

Definition

$\text{Range}(T) = \{T(\alpha) : \alpha \in V\} \subseteq W$ is the set of all vectors in W hit by T.

Fact: $\text{Range}(T)$ is a **subspace** of W.

1. 0 is in it. This is because $T(0) = 0$, obviously.
2. **Combinations of α_i are in it**
   Say that $\beta_1, \beta_2 \in \text{Range}(T)$. $\langle$ must show that $\beta_1 + \beta_2 \in \text{Range}(T)$ $\rangle$
   Since $\beta_1 \in \text{Range}(T)$, there is some $\alpha_1 \in V$ such that
   $$T(\alpha_1) = \beta_1$$
   similarly for β_2. Now $T(\alpha_1 + \alpha_2) = T(\alpha_1) + T(\alpha_2) = \beta_1 + \beta_2$, since T is linear. So $T(\alpha_1 + \alpha_2) = \beta_1 + \beta_2$ so $\beta_1 + \beta_2 \in \text{Range}(T)$ $\langle$ since $\alpha_1, \alpha_2 \in V$ means that $\alpha_1 + \alpha_2 \in V$, because it's a vector space! $\rangle$
3. **Scaling Works:** Say $\beta \in \text{Range}(T)$, and $c \in F$. Chose $\alpha \in V$ such that $T(\alpha) = \beta$. Then $T(c\alpha) = cT(\beta)c\beta$, therefore $c\alpha \in \text{Range}(T)$.