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# 1 Vector Spaces

Suppose that  $V$  is a finite dimensional vector space over  $F$ , with  $\dim(V) = n$ .

$V$  may have *many different* bases, we know that they all have the same size  $n$ .

Say  $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$  is a basis fix the ordering of  $\mathcal{B}$ .

Fix the ordering of  $\mathcal{B}$ .

## THEOREM

For any  $\alpha \in V$ , there is a unique  $n$  tuple  $(x_1, \dots, x_n) \in F^n$  such that

$$\alpha = x_1\alpha_1 + \dots + x_n\alpha_n$$

### PROOF

Existence is immediate, since  $\mathcal{B}$  is a basis, thus  $\mathcal{B}$  spans  $V$ .

### Uniqueness

Say  $\alpha = x_1\alpha_1 + \dots + x_n\alpha_n$  and  $\alpha = y_1\alpha_1 + \dots + y_n\alpha_n$ .

Then we have that

$$x_1\alpha_1 + \dots + x_n\alpha_n - y_1\alpha_1 - \dots - y_n\alpha_n = 0, \text{ so } (x_1 - y_1)\alpha_1 + \dots + (x_n - y_n)\alpha_n = 0$$

But since  $\{\alpha_1, \dots, \alpha_n\}$  is linearly independent, all coefficients must be 0.



What this means is that, for a vector space  $V$ , there is an associated mapping in  $F^n$ . Notice that we know nothing about the vectors  $\alpha_i$ .

We define  $[\alpha]_{\mathcal{B}}$  to be the *coordinates* of  $\alpha$  with respect to  $\mathcal{B}$ .

**Check:** The mapping  $\alpha \mapsto [\alpha]_{\mathcal{B}} \in F^n$  satisfies

1. One to one-ness
2. Onto-ness
3. "Additive", for any  $\alpha, \beta \in V$ , if  $\alpha = x_1\alpha_1 + \dots + x_n\alpha_n$  and  $\beta = y_1\alpha_1 + \dots + y_n\alpha_n$ . Then

$$[\alpha + \beta]_{\mathcal{B}} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = [\alpha]_{\mathcal{B}} + [\beta]_{\mathcal{B}}$$

4.  $[c\alpha]_{\mathcal{B}} = c[\alpha]_{\mathcal{B}}$

There exists an *isomorphism* between  $V$  and  $F^n$ .

## EXAMPLE

Let  $\mathcal{P}$  be the space of all polynomials. Let  $f(x) = x^3$ , and  $g(x) = x^5$ . Then, let

$$V = \text{Span}\{f, g\} = \{\text{all } ax^3 + bx^5 : a, b \in F\}$$

then,  $\dim(V) = 2$ , since  $f$  and  $g$  are linearly independent.

Typical  $h(x) \in V$ , say  $h(x) = 10x^3 - 2x^5$ .

$$[h]_{\mathcal{B}} = \begin{bmatrix} 10 \\ -2 \end{bmatrix}$$

$\langle [h]_{\mathcal{B}}$  is the mapping of  $h$  to  $F^n$ . **TODO** is this right?  $\rangle$

Now let  $k(x) = 2x^3 + 4x^5$  and  $l(x) = x^3 + 3x^5$ . Since  $k, l$  are linearly independent, they form another basis of  $V$ .

$$\mathcal{B}' = \{k(x), l(x)\}$$

## 1.1 Change of Basis

Given  $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ , and  $\mathcal{B}' = \{\alpha'_1, \dots, \alpha'_n\}$  bases for  $V$ .

We want to describe the map going from  $[\alpha]_{\mathcal{B}} \mapsto [\alpha]_{\mathcal{B}'}$ .

$\langle$  We want to find The  $\mathcal{B}$  coordinate of  $\alpha \mapsto$  the  $\mathcal{B}'$  coordinate of  $\alpha$   $\rangle$

**Step 1.**

Compute the  $\mathcal{B}$  coordinate of  $\alpha'_1, \dots, \alpha'_n$ , *old* coordinates of the *new* basis elements.

**Step 2.**

For an  $n \times m$  matrix

$$P = \begin{bmatrix} [\alpha'_1]_{\mathcal{B}}, \dots, [\alpha'_n]_{\mathcal{B}} \end{bmatrix}$$

**Check:** for any  $\alpha \in V$

$$[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$$

**Ans:** This is what we actually want

$$[\alpha]_{\mathcal{B}'} = P^{-1}[\alpha]_{\mathcal{B}}$$

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**TODO** Missing *some* info

**Want:** Describe the mapping  $T : F^n \rightarrow F^n$

$$T([\alpha]_{\mathcal{B}_{\text{old}}}) = [\alpha]_{\mathcal{B}'_{\text{new}}}$$

$\langle$  If we switch the basis for some reason, we want to see what the new coordinates are.  $\rangle$

**To do this:** For each  $\alpha'_j$ , compute  $[\alpha'_j]_{\mathcal{B}_{\text{old}}}$ . Let

$$P = \begin{bmatrix} [\alpha'_1]_{\mathcal{B}_{\text{old}}} & \cdots & [\alpha'_n]_{\mathcal{B}_{\text{old}}} \end{bmatrix}$$

be an  $n \times n$  matrix.

**Claim:** For any  $\alpha \in V$

$$P \cdot [\alpha]_{\mathcal{B}'_{\text{new}}} = [\alpha]_{\mathcal{B}_{\text{old}}}$$

**How?**

$$P \cdot [\alpha'_1]_{\mathcal{B}'_{\text{new}}} = P \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [\alpha'_1]_{\mathcal{B}_{\text{old}}}$$

This is the  $1^{\text{st}}$  column of  $P$ , and similarly for all columns.

**Thus:** For any  $\alpha \in V$ ,

$$[\alpha]_{\text{new}} = P^{-1} \cdot [\alpha]_{\text{old}}$$

### EXAMPLE

In practice, we have the following.

$V = \text{Span}(\{x^3, x^5\})$  subspace of  $\mathcal{P}$ , the set of all polynomials. Let  $f(x) = x^3, g(x) = x^5, \mathcal{B} = \{x^3, x^5\}$ . Let  $h(x) = 10x^3 - 2x^5 \in V$ .

**Question:** What are the coordinates of  $h$  with respect to  $\mathcal{B}$ ?

**Answer:**

$$[h]_{\mathcal{B}} = \begin{bmatrix} 10 \\ -2 \end{bmatrix}$$

Let's now see what happens when we create a new basis  $\mathcal{B}'$ .

### EXAMPLE

Let  $k(x) = 2x^3 + 5x^5, l(x) = x^3 + 3x^5$ .

Let  $\mathcal{B}' = \{k(x), l(x)\} = \{2x^3 + 5x^5, x^3 + 3x^5\}$  be another basis of  $V$ , still with  $\mathcal{B} = \{f(x), g(x)\} = \{x^3, x^5\}$ .

**Question:** What are the coordinates of  $h(x) = 10x^3 - 2x^5$  with respect to  $\mathcal{B}'$  now?

**Answer:**

Well we know that  $[k(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$  and  $[l(x)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ , these are just the coordinates of  $k$ , and  $l$  with respect to  $\mathcal{B}$ .

So now we can construct our  $P$  matrix

$$P = \begin{bmatrix} [k(x)]_{\mathcal{B}} & [l(x)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

notice that  $P$ 's columns are constructed from  $k(x)$  and  $l(x)$ , expressed in terms of our standard basis  $\mathcal{B}$ .

**Check:**

$$P^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

Then

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ -2 \end{bmatrix} = \begin{bmatrix} 32 \\ -54 \end{bmatrix}$$

**This means:**

$$h(x) = 32k(x) - 54l(x) = 10x^3 - 2x^5$$

Which is what we expect.

## EXAMPLE

Let  $V = \mathbb{R}^2$ . Standard basis  $\mathcal{B} = \{\varepsilon_1, \varepsilon_2\} = \{(1, 0), (0, 1)\}$

$$[(5, 4)]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Fix angle  $\theta$ , Let

$$\mathcal{B}' = \{(\cos(\theta), \sin(\theta)), (-\sin(\theta), \cos(\theta))\}$$

**Question:** What is  $\begin{bmatrix} 5 \\ 4 \end{bmatrix}_{\mathcal{B}'_{\text{new}}}$  ?

**Answer:**

1. Form  $P$

$$[(\cos(\theta), \sin(\theta))]_{\mathcal{B}} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

$$[(-\sin(\theta), \cos(\theta))]_{\mathcal{B}} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

Then

$$P = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

**Fact:**

$$P^{-1} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

so we have

$$\begin{aligned}
[(5, 4)]_{\mathcal{B}'_{\text{new}}} &= P^{-1} \begin{bmatrix} 5 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 5 \cos(\theta) & 4 \sin(\theta) \\ -5 \sin(\theta) & 4 \cos(\theta) \end{bmatrix}
\end{aligned}$$

## 2 Linear Transformations

Say  $V, W$  are both vector spaces over the same field  $F$ .

### DEFINITION

A **Linear Transformation**  $T : V \rightarrow W$  is a function satisfying two rules

1. For all  $\alpha, \beta \in V$ ,

$$T(\alpha + \beta) = T(\alpha) + T(\beta)$$

Note that the first  $+$  is addition in  $V$ , but the second is addition in  $W$ .

2. For all  $\alpha \in V$  and  $c \in F$ ,

$$T(c\alpha) = cT(\alpha)$$

The book combines the two definitions above into one, like this,

$$T(c\alpha + \beta) = cT(\alpha) + T(\beta)$$

Let's quickly take some time to understand what  $V$  and  $W$  are here. Suppose we have a transformation  $T : V \rightarrow W$ , then  $V$  is the **domain** and  $W$  is the **codomain**.

Here,  $T$  is just a function, which means that it *must* use all of  $V$ , but it *does not* have to use all of  $W$ . For example, the following is a perfectly valid transformation.

### EXAMPLE

Let  $T : \mathcal{P}^3 \rightarrow \mathcal{P}^2$  be the transformation that takes all degree 3 polynomials to the space of degree 2 polynomials, with

$$T(f) = \mathbf{0}$$

for all  $f \in \mathcal{P}^3$ .

It's obvious that there are more degree 2 polynomials in the world than just the  $\mathbf{0}$  polynomial. So here, we say that the  $\text{Range}(T) = \{\mathbf{0}\}$ , and that

$$\text{Range}(T) \subseteq W$$

but maybe we are getting ahead of ourselves.

## 2.1 Basic Facts

Suppose that  $T : V \rightarrow W$  is a linear transformation

1.  $T(0) = 0$

**Proof:**

$$T(0 + 0) = T(0) + T(0) = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

**Note:**  $0$  lives in the field, and  $\mathbf{0}$  lives in  $W$ , the **codomain** of the transformation  $T$ .

⟨ Always be aware of where the  $0$  lives ⟩

2. For all  $\{\alpha_1, \dots, \alpha_n\} \subseteq V$ , all  $\{c_1, \dots, c_n\} \in F$ ,

$$c_1 T(\alpha_1) + \dots + c_n T(\alpha_n)$$

**Proof** Easy induction on  $n$ , just follows from part (2) of the definition.

## 2.2 Examples

Let's look at multiple examples of linear transformations to get an idea of how they behave.

### EXAMPLE

We already know that each matrix  $A$  has an associated linear transformation  $T_A$ . Let's look at this in more detail now.

Let  $A \in F^{m \times n}$  be an  $m \times n$  matrix with entries from a field  $F$ .

Then, let  $T_A : F^n \rightarrow F^m$  be defined by

$$T_A(\mathbf{x}) = A\mathbf{x}$$

where  $\mathbf{x}$  is a vector in  $F^n$ .

Let's check that this is indeed a linear transformation.

Chose any  $\mathbf{x}, \mathbf{y} \in F^n$ , then

1. ⟨ Check that  $T_A(\mathbf{x} + \mathbf{y}) = T_A(\mathbf{x}) + T_A(\mathbf{y})$  ⟩

Let  $\mathbf{x}, \mathbf{y} \in V$ , then

$$T_A(\mathbf{x} + \mathbf{y}) = A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T_A(\mathbf{x}) + T_A(\mathbf{y})$$

so this works as we expect.

2. ⟨ Check that  $T_A(c\mathbf{x}) = cT_A(\mathbf{x})$  for  $c \in F$ . ⟩

let  $c \in F$ , then we have

$$T_A(cX) = A \cdot (cX) = cAX = cT_A(X)$$

which is also what we expect.

so we have proved that  $T_A$  is a linear transformation!

### EXAMPLE

Consider  $\mathcal{P}$  the set of all polynomials  $a_0 + a_1x + \dots + a_nx^n$ .

Let's define  $D : \mathcal{P} \rightarrow \mathcal{P}$  which takes a function  $f \in \mathcal{P}$  to  $f' \in \mathcal{P}$ , where  $f'$  is the *derivative* of  $f$ .

$$D(f) = f'$$

**Claim:**

$D$  is a linear transformation.

**Proof:**

Take two functions  $f, g \in \mathcal{P}$ , then by definition of  $D$ , we have

$$D(f + g) = (f + g)' = f' + g' = D(f) + D(g)$$

and for  $c \in F$ ,

$$D(cf) = (cf)' = c \cdot f' = cD(f)$$

so the derivative is a linear transformation!

## EXAMPLE

Let  $C(\mathbb{R})$  be the set of all continuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

Let's define  $I : C(\mathbb{R}) \rightarrow C(\mathbb{R})$  which takes a function  $f \in C(\mathbb{R})$  to  $F \in C(\mathbb{R})$ , where  $F$  is the *antiderivative* of  $f$ .

$$I(f) = \int_0^x f(t)dt$$

⟨ Note that the integral exists because you can always integrate a continuous function. ⟩

The result is also continuous and differentiable by the Fundamental Theorem of Calculus.

$$D(I(f)) = f$$

Is the **Fundamental Theorem of Calculus**.

Therefore  $I(f)$  really *is* continuous,  $I(f) \in C(\mathbb{R})$ .

**Claim:**

$I$  is a linear transformation.

**Proof:**

Take two functions  $f, g \in \mathcal{P}$ , then by definition of  $I$ , we have

$$\begin{aligned} I(f + g) &= \int_0^x (f(t) + g(t))dt \\ &= \int_0^x f(t)dt + \int_0^x g(t)dt \\ &= I(f) + I(g) \end{aligned}$$

and



$$I(cf) = \int_0^x cf(t)dt = c \int_0^x f(t)dt = cI(f)$$

so the integral is a linear transformation!

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Recall: A linear transformation  $T : V \rightarrow W$  is a function between two vector spaces over the same field  $F$ , satisfying

1. For all  $\alpha, \beta \in V$ ,

$$T(\alpha + \beta) = T(\alpha) + T(\beta)$$

Note that the first  $+$  is addition in  $V$ , but the second is addition in  $W$ .

2. For all  $\alpha \in V$  and  $c \in F$ ,

$$T(c\alpha) = cT(\alpha)$$

For all  $\alpha_1, \dots, \alpha_k \in V$ , and  $c_1, \dots, c_k \in F$ , it breaks nicely into

$$T(c_1\alpha_1 + \dots + c_k\alpha_k) = c_1T(\alpha_1) + \dots + c_kT(\alpha_k)$$

### EXAMPLE

$I^* : C(\mathbb{R}) \rightarrow \mathbb{R}$  (all continuous functions from  $\mathbb{R}$  to  $\mathbb{R}$ )

$$I^*(f) = \int_0^1 f(x)dx$$

$$I^*(x^2) = \int_0^1 x^2 dx = \left. \frac{x^3}{3} \right|_0^1 = \frac{1}{3}$$

Note that the output of  $I^*$  is just a number here. Additionally,  $I^*$  is linear: you can split integrals up for polynomials, and you can take constants outside.

For any  $V, W$ , we also have

$$X : V \rightarrow W$$

Is the zero transformation. It takes any  $\alpha \in V$  to the 0 of  $W$ . We'll use this to prove theorems about linear transformations later.

### THEOREM

Let's prove existence and uniqueness of linear transformations.

1. Linear Transformations  $T : V \rightarrow W$  are **determined** by their behavior on a basis  $\mathcal{B}$  of  $V$ . More precisely,

Suppose that  $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$  is a basis for  $V$  and suppose that  $T, U : V \rightarrow W$  are both linear transformations (and they agree on a basis), such that

$$T(\alpha_1) = U(\alpha_1), T(\alpha_2) = U(\alpha_2), \dots, T(\alpha_n) = U(\alpha_n)$$

Then  $T = U$

2. For **any map**  $T_0 : \mathcal{B} \rightarrow W$ , there is a unique linear transformation  $T : V \rightarrow W$  with  $T \supseteq T_0$ . In other words,

Let  $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$  be **any basis** for  $V$  and let  $\beta_1, \dots, \beta_n$  be **any vectors** in  $W$ .

Then, there is a **unique** linear transformation  $T : V \rightarrow W$  such that

$$T(\alpha_1) = \beta_1, T(\alpha_2) = \beta_2, \dots, T(\alpha_n) = \beta_n$$

## PROOF

1. **Uniqueness:** Chose any  $\alpha \in V$ , since  $\mathcal{B}$  is a basis,

⟨ Will show that  $T = U \Leftrightarrow$  For any  $\alpha \in V$ ,  $T(\alpha) = U(\alpha)$  ⟩

$$\alpha = c_1\alpha_1 + \dots + c_n\alpha_n$$

for some **unique**  $c_1, \dots, c_n \in F$ .

Since  $T$  is a linear transformation,

$$T(\alpha) = c_1T(\alpha_1) + \dots + c_nT(\alpha_n)$$

Likewise with  $U$ ,

$$U(\alpha) = c_1U(\alpha_1) + \dots + c_nU(\alpha_n)$$

But, since  $T(\alpha_1) = U(\alpha_1), \dots, T(\alpha_n) = U(\alpha_n)$ ,  $T(\alpha) = U(\alpha)$ .

⟨ Essentially, if  $T, U$  work the same for all  $\alpha_i$ , then their sum will obviously be the same, and so they'll give the same result for the same  $\alpha$ . ⟩

Note that this theorem *still* works for infinite dimensional vector spaces.

2. **Existence:** Chose any  $\alpha \in V$ . ⟨ We must define  $T(\alpha)$  ⟩

Since  $\mathcal{B}$  is a basis, we can write

$$\alpha = c_1\alpha_1 + \dots + c_n\alpha_n$$

which is unique.

Define

$$T(\alpha) := c_1\beta_1 + \dots + c_n\beta_n \in W$$

**Check:** Is  $T$  linear?

Say  $\gamma = d_1\alpha_1 + \dots + d_n\alpha_n$ ,  $\delta = e_1\alpha_1 + \dots + e_n\alpha_n$ .

In  $V$ , we have that  $\gamma + \delta = (d_1 + e_1)\alpha_1 + \dots + (d_n + e_n)\alpha_n$ .

By our definition of  $T$ , we have

$$\begin{aligned} T(\gamma + \delta) &= (d_1 + e_1)\beta_1 + \dots + (d_n + e_n)\beta_n \\ &= (d_1\beta_1 + \dots + d_n\beta_n) + (e_1\beta_1 + \dots + e_n\beta_n) \\ &= T(\gamma) + T(\delta) \end{aligned}$$

**Check:**  $T(c\gamma) = cT(\gamma)$

So such a transformation  $T$  exists. Additionally by part (1), it is unique.



Let  $T : V \rightarrow W$  be a linear transformation.

## DEFINITION

$\text{Range}(T) = \{T(\alpha) : \alpha \in V\} \subseteq W$  is the set of all vectors in  $W$  hit by  $T$ .

**Fact:**  $\text{Range}(T)$  is a **subspace** of  $W$ .

1. 0 is in it. This is because  $T(0) = 0$ , obviously.

2. **Combinations of  $\alpha_i$  are in it**

Say that  $\beta_1, \beta_2 \in \text{Range}(T)$ .  $\langle$  must show that  $\beta_1 + \beta_2 \in \text{Range}(T)$   $\rangle$

Since  $\beta_1 \in \text{Range}(T)$ , there is some  $\alpha_1 \in V$  such that

$$T(\alpha_1) = \beta_1$$

similarly for  $\beta_2$ . Now  $T(\alpha_1 + \alpha_2) = T(\alpha_1) + T(\alpha_2) = \beta_1 + \beta_2$ , since  $T$  is linear. So  $T(\alpha_1 + \alpha_2) = \beta_1 + \beta_2$  so  $\beta_1 + \beta_2 \in \text{Range}(T)$   $\langle$  since  $\alpha_1, \alpha_2 \in V$  means that  $\alpha_1 + \alpha_2 \in V$ , because it's a vector space!  $\rangle$

3. **Scaling Works:** Say  $\beta \in \text{Range}(T)$ , and  $c \in F$ . Chose  $\alpha \in V$  such that  $T(\alpha) = \beta$ . Then  $T(c\alpha) = cT(\alpha) = c\beta$ , therefore  $c\beta \in \text{Range}(T)$ .

In other books this space is also called the **image** of  $T$ .

## DEFINITION

The **Null Space** of  $T : V \rightarrow W$  is the set

$$\text{Null}(T) = \{\alpha \in V \mid T(\alpha) = \mathbf{0}\}$$

$\langle$  In other words, this is the set of all vectors  $\alpha$  in  $V$  that, after a transformation  $T$  is applied, go to  $\mathbf{0}$ . Note that  $\mathbf{0}$  here is the zero of the vector space  $W \subseteq V$ .  $\rangle$

This is also sometimes called the **Kernel** of  $T$ .

## THEOREM

Let  $T : V \rightarrow W$  be a linear transformation.  $\text{Null}(T)$  is a subspace of  $V$ .

### PROOF

Let  $\alpha, \beta \in \text{Null}(T)$  and  $c \in F$ . Then,

$$T(c\alpha + \beta) = cT(\alpha) + T(\beta) = c\mathbf{0} + \mathbf{0} = \mathbf{0} \Rightarrow c\alpha + \beta \in \text{Null}(T)$$



It's pretty easy to see from this (and it should make sense) that the Null Space for a transformation  $T$  is itself a vector space.

## DEFINITION

The **Nullity** of  $T$  is the dimension of the Null space of  $T$ .

## DEFINITION

The **Rank** of  $T$  is the dimension of  $\text{Range}(T)$ . If this is equal to the dimension of  $W$ ,  $T$  is said to have **full rank**.

Note again that this comes back to our definition of  $W$  for our transformation  $T$ . Earlier, we saw that  $W$  was the *codomain* of  $T$ . If you think about how functions behave, this is like having a *surjective* function.

## EXAMPLE

Let  $\mathcal{P}_2$  be the set of all polynomials of degree 2 or less over a field  $F$ . Then, we have  $\dim(\mathcal{P}_2) = 3$ . Consider the linear transformation  $D : \mathcal{P}_2 \rightarrow \mathcal{P}_2$ , the differentiation operator. Then

$$\text{Range}(D) = \text{Span}(\{D(1), D(x), D(x^2)\}) = \text{Span}(\{1, 2x\}) \Rightarrow \text{Rank}(D) = 2$$

In other words, the Range of  $D$  is the Span of a basis of  $\mathcal{P}_2$  (in this case  $\{1, x, x^2\}$ ) after being evaluated through  $D$ , so  $\{1, 2x\}$ . So the rank of  $D$  here is 2.

For the Null Space of  $D$ , we have that

$$\text{Null}(D) = \{c \in F\} \Rightarrow \text{Nullity}(D) = 1$$

The Null Space is the set of all constant functions since those are the function that, on  $D$ , go to  $\mathbf{0}$ .

## 2.3 The Rank-Nullity Theorem

### RANK-NULLITY THEOREM

Let  $V$  be a vector space with  $\dim V = n$ . Let  $T : V \rightarrow W$ .

$$\text{Rank}(T) + \text{Nullity}(T) = \dim V = n$$

### PROOF

First, choose  $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$  to be a basis for  $\text{Null}(T)$ . This set is necessarily linearly independent in  $V$ . So, we can choose an additional  $\{\alpha_{k+1}, \dots, \alpha_n\}$  so that  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  is a basis of  $V$ .

Certainly,  $k \leq n$ , since  $\text{Null}(T)$  is a subspace of  $V$ .

We claim  $A = \{T(\alpha_{k+1}), \dots, T(\alpha_n)\}$  is a basis for  $\text{Range}(T)$ . From this we have our theorem.

Clearly,  $A \subseteq \text{Range}(T)$ . We also have, that since  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  is a basis of  $V$ ,  $\{T(\alpha_i)\}$  spans  $\text{Range}(T)$ .

However,  $T(\alpha_1) = T(\alpha_2) = \dots = T(\alpha_k) = \mathbf{0}$ , since they are in the null space, and hence do not contribute to the span. Thus,  $A$  spans  $\text{Range}(V)$ . Now we need only show  $A$  is linearly independent. We choose constants such that

$$c_{k+1}T(\alpha_{k+1}) + \dots + c_nT(\alpha_n) = \mathbf{0}$$

Let

$$\alpha^* = c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n \in V$$

We then have

$$T(\alpha^*) = c_{k+1}T(\alpha_{k+1}) + \cdots + c_nT(\alpha_n) = \mathbf{0} \Rightarrow \alpha^* \in \text{Null}(T)$$

So, we then have that, since  $\alpha^*$  is in the null space,

$$\alpha^* = d_1\alpha_1 + d_2\alpha_2 + \cdots + d_k\alpha_k = c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n$$

$$d_1\alpha_1 + d_2\alpha_2 + \cdots + d_k\alpha_k - c_{k+1}\alpha_{k+1} - \cdots - c_n\alpha_n = \mathbf{0} \in V$$

But since  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  is a basis of  $V$ , all the constants are zero, and in particular all of the  $c_i$  are zero. So,  $\{T(\alpha_{k+1}), \dots, T(\alpha_n)\}$  is linearly independent and is thus a basis of  $\text{Range}(T)$ .

■

Now that we have the rank-nullity theorem, we can analyze transformations and their matrices.

## DEFINITION

Let  $A$  be a matrix in  $F^{m \times n}$ .

The **Column Space** is the vector space spanned by the  $n$  columns of  $A$ . This is precisely  $\text{Range}(T_A)$ .

The **Row Space** is the vector space spanned by the  $m$  rows of  $A$ .

## THEOREM

Let  $A$  be a matrix, that when row-reduced has  $n$  unknowns and  $r$  non-zero rows.  $\text{Nullity}(T_A) = n - r$

## PROOF

This follows from the fact that elementary row operations preserve the row space, and that solving a linear system in  $r$  equations with  $n$  unknowns will have  $n - r$  degrees of freedom.

**TODO** I guess I can believe this but some more info would be nice.

■

## NOTE

Let  $A$  be a matrix. Then the following are equal

- The dimension of the row space of  $A$
- The dimension of the column space of  $A$
- The number of nonzero rows in the row-reduced form of  $A$
- $\text{Rank}(T_A)$

This follows immediately from the above and the Rank-Nullity Theorem.

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Suppose that  $A$  is an  $m \times n$  matrix. Now suppose that we row reduce  $A$ , let's call this matrix  $A^{rr}$ . Then we have that

$$\text{RowSpace}(A) = \text{RowSpace}(A^{rr})$$

And we know that  $\text{Rank}(A)$  is the number of non-zero rows of  $A^{rr}$  which we call  $r$ .

Moreover, the solution set of the homogeneous system  $A\mathbf{x} = \mathbf{0}$  has dimension  $n - r$ , where  $n$  is the number of columns subtract the number of redundant equations.

Now, we know that for a matrix  $A$ , there is an associated linear transformation  $T_A : F^n \rightarrow F^m$ .

Last time, we also saw that

1.  $\text{Range}(T_A) = \text{ColSpace}(A)$ ,
2.  $\text{Null}(T_A)$  is the solution set of  $A\mathbf{x} = \mathbf{0}$ .

Now we can put everything together. Recall the Rank-Nullity theorem, then we have that, for any linear transformation  $T_A$ ,

1.  $\text{Rank}(T_A) + \text{Nullity}(T_A) = \dim(F^n) = n$
2.  $\text{Rank}(T_A) := \dim(\text{Range}(T_A))$
3.  $\text{Nullity}(T_A) = \dim(\text{Null}(A)) = n - r$ , which is exactly the dimension of the set of all solutions to the homogeneous.
4. Finally we have that

$$\begin{aligned}\text{Rank}(A) &= \dim(\text{RowSpace}(A)) = \dim(\text{ColSpace}(A)) \\ &= \dim(\text{RowSpace}(A^{rr})) \\ &= \text{Rank}(T_A) \\ &= r\end{aligned}$$

Recall also that  $\text{Nullity}(T_A) = \dim(\text{Null}(T_A)) = n - r$ .

Consider a matrix  $A$  where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Then

$$A^{rr} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Is the row reduced matrix.

A basis for the row space of  $A$  is

$$\{(1, 0, 1, 1), (0, 1, 1, 1/3)\}$$

but another is

$$\{(1, 2, 3, 4), (1, 0, 1, 1)\}$$

We have  $T_A : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ , and  $\text{Rank}(T_A) = 2$ .

Basis for  $\text{Range}(T_A)$  equals the basis for  $\text{Col Space}(A)$

There are many more linear transformations than the ones given by a matrix, for instance the derivative or integrals.

Let  $T : V \rightarrow W$  be a linear transformation. From here there are two questions we can now ask.

1. Is  $T$  onto?

It is if and only if  $\text{Range}(T) = W$ . We saw this earlier. In terms of dimension, this means that  $\text{Rank}(T) = \dim(W)$ .

Note that here,  $V, W$  must be **finite dimensional**.

2. Is  $T$  one to one?

This requires some more work.

## THEOREM

$T : V \rightarrow W$  is one to one if and only if  $\text{Null}(T) = \{\mathbf{0}\}$ .

⟨ In other words, the Null space must only contain the zero vector. ⟩

### PROOF

Assume that  $T$  is one to one. We know that  $T(\mathbf{0}_V) = \mathbf{0}_W$ . Chose any  $\alpha \in \text{Null}(T)$ , then  $T(\alpha) = \mathbf{0}_W$ , by definition of being in the Null Space. Since  $T$  is one to one,  $\alpha$  must equal  $\mathbf{0}_V$ .

Now assume that  $\text{Null}(T)$  is just  $\mathbf{0}_W$ . To see that  $T$  is one to one, chose any  $\alpha, \alpha' \in V$ , with  $T(\alpha) = T(\alpha')$ . Then  $T(\alpha - \alpha') = T(\alpha) - T(\alpha')$  by linearity, but then since  $\alpha = \alpha'$ ,  $T(\alpha - \alpha') = \mathbf{0}$  so  $T(\alpha - \alpha')$  must be in the Null space of  $T$ , and since  $\text{Null}(T) = \{\mathbf{0}\}$ , and  $\alpha - \alpha' = \mathbf{0}$ , so  $\alpha = \alpha'$  and thus  $T$  is one to one.



### DEFINITION

$T$  is called **non-singular** if  $T$  is one to one.

This is just another term for something we already know.

## THEOREM

Now suppose that  $T : V \rightarrow W$  is a linear transformation with  $\dim(V) = \dim(W)$ . Then  $T$  is one to one if and only if  $T$  is onto.

### PROOF

By the Rank-Nullity theorem from last time, we have that

$$\text{Rank}(T) + \text{Nullity}(T) = \dim(V)$$

Now, assume that  $T$  is one to one, then  $\text{Nullity}(T) = 0$ , but then  $\text{Rank}(T) = \dim(V) = \dim(W)$ .

Now conversely, assume that  $T$  is onto. Then



$$\text{Rank}(T) = \dim(W) = \dim(V)$$

Therefore  $\text{Nullity}(T) = 0$ , and so  $T$  is one to one.



We are now starting to get a pretty good understanding of linear transformations, but suppose that we now want to combine them.

## 2.4 Combining Linear Transformations

Say  $T : V \rightarrow W$  and  $U : W \rightarrow Y$  are linear transformations over  $F$ .

⟨ then  $U \circ T : V \rightarrow Y$  is a function. ⟩

**Check the following:**

1.  $U \circ T$  is a linear transformation.  
 ⟨ You know how to do this, just check that they scale and add as we expect. ⟩
2. If both  $T$  and  $U$  are one to one, then the composition is also one to one.
3. If both  $T$  and  $U$  are onto, the composition is also onto.

### NOTE

$T \circ U$  would **not** be a linear transformation, assuming that  $Y$  and  $V$  are not the same vector space.

⟨ Linear transformations don't commute nicely like that. ⟩

Let's now look at  $T$  again.

### DEFINITION

A linear transformation  $T : V \rightarrow W$  is called **invertible** if there is a linear transformation  $U : W \rightarrow V$  such that

1.  $U \circ T : V \rightarrow V$  is the identity from  $V$ . In other words

$$U(T(\alpha)) = \alpha$$

For any  $\alpha \in V$ .

2.  $T \circ U : W \rightarrow W$

$$T(U(\alpha)) = \alpha$$

For any  $\alpha \in W$ .

### NOTE

It might be interesting for you to prove that, if one of the above applies, the other automatically applies as well.

If  $T$  is invertible, we call such a  $U$   $T^{-1}$ , the inverse transformation of  $T$ .

## NOTE

Inverse transformations are unique, if they exist.

⟨ We didn't talk about this in class but it *has* to be true. ⟩

**Proposition:** If  $T : V \rightarrow W$  is an *invertible* linear transformation if and only if  $T$  is both one to one and onto.

## NOTE

If  $T$  has an inverse, then it must be the case that  $\dim(V) = \dim(W)$ .

If this is surprising, just consider that this follows from the fact that  $T$  must be both one to one, and onto in order to have an inverse.

Today was exam review. As such, everything for today is written as examples.

### What to expect

1. Short answer **True / False**. Then write a sentence explaining your choice, doesn't need to be a proof.
2. Matrix stuff. Row Space, Col Space, Rank of Matrix, Solution set of homogeneous system, etc...
3. Linear Transformations. Polynomials,  $\{e^{ix}, e^{-ix}\}$  stuff, etc...
4. Full Proofs of statements. Rather straightforward, just involving linear independence, or spanning, or dimensions, etc...

The exam is 50 minutes, problems *will* be reasonable. At this point, we have definitely covered everything that will be on exam 1.

## 3 Example Problems

### EXAMPLE

#### Problem 3.1.4

Want: A linear transformation  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  such that  $T(1, -1, 1) = (1, 0)$ , and  $T(1, 1, 1) = (0, 1)$

Firstly, *is there one?*

The two vectors passed to  $T$  are linearly independent, so **they can be expanded to a basis**. Say for example,

$$\mathcal{B} = \{(1, -1, 1), (1, 1, 1), (0, 0, 1)\}$$

Where  $\mathcal{B}$  is a basis of  $\mathbb{R}^3$ . Now, there will be a *unique* linear transformation that will take it to any two points in  $\mathbb{R}^2$  (even if those two points are not linearly independent.)

#### Fundamental Fact:

If  $V, W$  are vector spaces, and  $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$  is a basis of  $V$ , then for any  $\beta_1, \dots, \beta_n \subseteq W$ , there is *exactly* one linear transformation  $T$  for which

$$T(\alpha_1) = \beta_1, \dots, T(\alpha_n) = \beta_n$$

NOTE:  $\beta_i$  do *not* have to be linearly independent! For instance consider the 0 transformation, then  $\beta_1 = \dots = \beta_n = \mathbf{0} \in W$ .

**Question:** How do we define such a  $T$ ?

Take any  $\alpha^* = c_1\alpha_1 + \dots + c_n\alpha_n$ . Then by linearity,

$$T(\alpha^*) = c_1\beta_1 + \dots + c_n\beta_n$$

NOTE: This shows that such a  $T$  *does* exist, but it's not *constructive*. We don't *actually* know what it is; we only know what properties it has, and that it exists.

NOTE: Before we expanded  $\{(1, -1, 1), (1, 1, 1)\}$  to a basis, there were an *infinite* family of linear transformations. This is because we got to choose the last vector of  $\mathcal{B}$ , in our case  $(0, 0, 1)$ . If we have chosen, say,  $(0, 1, 0)$  instead, then we would have gotten an entirely different transformation  $T$ .

## EXAMPLE

### Problem 2.4.6a

**Question:** How do we show that  $e^{ix}$  and  $e^{-ix}$  are linearly independent?

Let  $f_1(x) = 1, f_2(x) = e^{ix}, f_3(x) = e^{-ix}$ .

**Claim:**  $f_1, f_2, f_3$  are *linearly independent*.

Recall what this means. For *any*  $x$ ,

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = \mathbf{0}$$

Implies that  $c_1, c_2, c_3$  must all be 0. Where  $\mathbf{0}$  is the zero function:  $z(x) = 0$  for all  $x$ .

$$c_1 + c_2 e^{ix} + c_3 e^{-ix} = \mathbf{0}$$

Now let  $x = -100i$ , then  $e^{i(-100i)} = e^{100}$ , then  $e^{-i(-100i)} = e^{-100}$ , so we have

$$c_1 + c_2 e^{100} + c_3 e^{-100} = \mathbf{0}$$

Therefore,  $c_1, c_2, c_3$  *must* all be 0, since clearly all the functions are positive for  $x = -100i$ . This shows that the functions are linearly independent.

Since  $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = \mathbf{0}$  *has* to work for *any* value of  $x$ , if you just find *one* counterexample (like  $x = -100i$  above), that shows that they are linearly independent.

## EXAMPLE

**Question:** Suppose that the field  $F$  is  $\mathbb{C}$ . Is

$$W = \{g_1(x) = 1, g_2(x) = \sin x, g_3(x) = \cos x\}$$

Linearly independent?

**Answer:** Yes!

Suppose that for some  $x = \theta$  we have

$$\begin{aligned} c_1 g_1(x) + c_2 g_2(x) + c_3 g_3(x) &= \mathbf{0} \\ c_1 + c_2 \sin x + c_3 \cos x &= \mathbf{0} \end{aligned}$$

Then

$$c_2 \sin x + c_3 \cos x = -c_1$$

Then look at  $\theta + \pi$ ,

$$c_2 \sin(\theta + \pi) + c_3 \cos(\theta + \pi) = -c_1$$

So then we have the set of equations

$$c_2 \sin \theta + c_3 \cos \theta = c_2 \sin(\theta + \pi) + c_3 \cos(\theta + \pi)$$

But since  $\sin \theta = -\sin(\theta + \pi)$  and  $\cos \theta = -\cos(\theta + \pi)$ , we have

$$\begin{aligned} c_2 \sin \theta + c_3 \cos \theta &= -c_2 \sin \theta - c_3 \cos \theta \\ 2c_2 \sin \theta + 2c_3 \cos \theta &= 0 \end{aligned}$$

But in order for this to be true for all  $\theta$ ,  $c_1$  and  $c_2$  must both be 0.

NOTE: We know that  $\sin^2 x + \cos^2 x = 1$ , but we're not allowed to square  $g_2$  and  $g_3$  here. That's not a linear operation, so it's not relevant. However, it *is* true that  $\{1, \cos^2 x, \sin^2 x\}$  are linearly dependent.

NOTE:  $f_2 \in \text{Span}(\{g_1, g_2, g_3\})$ .

$$\begin{aligned} e^{-ix} &= \cos(-x) + i \sin(-x) \\ e^{-ix} &= \cos(x) - i \sin(x) \end{aligned}$$

So  $f_1 \in \text{Span}(\{1, \sin x, \cos x\})$ , but  $\dim(W) = 3$ , and we already know that  $f_1, f_2, f_3$  are linearly independent from the previous example, so we just found another basis for  $W$

$$\{f_1, f_2, f_3\} = \text{Span}(W)$$

## EXAMPLE

### Problem 2.4.4d

Let  $W = \text{Span}(\{(1, 0, i), (1 + i, 1, -1)\})$ .

**Question:** Is this set linearly independent?

Yes, neither is 0, or a multiple of the other.

**Question:** What is  $\dim(W)$ ?

2

So a basis could be

$$\mathcal{B} = \{(1, 0, i), (1 + i, 1, -1)\}$$

The vectors themselves.

**Question:** Let  $\beta_1 = (1, 1, 0)$ . Is  $\beta_1 \in W$ ?

Let's try to make it.

$$(1, 1, 0) = c_1(1, 0, i) + c_2(1 + i, 1, -1)$$

Need:  $1 = c_1 + c_2(i + i)$ , and  $1 = c_1(0) + c_2(1)$  so  $c_2 = 1$ , and finally,  $0 = c_1(i) + c_2(-1)$ .

So  $\beta_1 = (-i)(1, 0, -i) + 1(1 + i, 1, -1)$

## EXAMPLE

### 4d on Homework 3

If  $\dim(V) = \dim(W)$ , and  $\mathcal{B} = \{\alpha_1, \dots, \alpha_k\}$  is a basis for  $V$ , must

$$T(\alpha_1) + \dots + T(\alpha_k)$$

Be a basis for  $W$ ?

**Answer:** No!

Let  $T$  be the 0 transformation, then everything is mapped to 0.

NOTE: Here it's worth reiterating the difference between the *codomain* and the *range* of  $T$ .

Suppose that  $T : V \rightarrow W$  is a linear transformation that maps vectors from  $V$  to  $W$ . Then, we say that  $V$  is the *domain* of  $T$ , and that  $W$  is the *codomain* of  $T$ .

We know that a linear transformation is just a function, and one thing that we know about functions is that they *must* use their entire domain (so everything from  $V$  has to map *somewhere* in  $W$ ), but they *don't* have to use all of their codomain.

The nuance here is the word "*somewhere*". Suppose that we have the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  with  $f(x) = 5$ . Then the domain of  $f$  is  $\mathbb{R}$  and so is the codomain. Here, we can see that 5 is *somewhere* in the codomain of  $f$ , but it's certainly *not all* of it. Then, we say that the *range* of  $f$  is  $\{5\}$ . This is all just by definition.

Let's look at another example. Suppose that  $g : \mathbb{R} \rightarrow \mathbb{Z}$  with  $g(x) = \lfloor x \rfloor$  this time. In this case,  $g$  *does* use up all of its codomain, since every value in  $\mathbb{R}$  will be mapped to a value in  $\mathbb{Z}$ .

Linear transformation behave in exactly the same way. When we talk about  $\dim(W)$ , we're talking about the size of a basis of  $W$ , but  $T$  makes no promises about mapping elements from  $V$  to all of it.