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1 Vector Spaces

Suppose that V is a finite dimensional vector space over F , with $\dim(V) = n$.

V may have *many different* bases, we know that they all have the same size n .

Say $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ is a basis fix the ordering of \mathcal{B} .

Fix the ordering of \mathcal{B} .

THEOREM

For any $\alpha \in V$, there is a unique n tuple $(x_1, \dots, x_n) \in F^n$ such that

$$\alpha = x_1\alpha_1 + \dots + x_n\alpha_n$$

PROOF

Existence is immediate, since \mathcal{B} is a basis, thus \mathcal{B} spans V .

Uniqueness

Say $\alpha = x_1\alpha_1 + \dots + x_n\alpha_n$ and $\alpha = y_1\alpha_1 + \dots + y_n\alpha_n$.

Then we have that

$$x_1\alpha_1 + \dots + x_n\alpha_n - y_1\alpha_1 - \dots - y_n\alpha_n = 0, \text{ so } (x_1 - y_1)\alpha_1 + \dots + (x_n - y_n)\alpha_n = 0$$

But since $\{\alpha_1, \dots, \alpha_n\}$ is linearly independent, all coefficients must be 0.



What this means is that, for a vector space V , there is an associated mapping in F^n . Notice that we know nothing about the vectors α_i .

We define $[\alpha]_{\mathcal{B}}$ to be the *coordinates* of α with respect to \mathcal{B} .

Check: The mapping $\alpha \mapsto [\alpha]_{\mathcal{B}} \in F^n$ satisfies

1. One to one-ness
2. Onto-ness
3. "Additive", for any $\alpha, \beta \in V$, if $\alpha = x_1\alpha_1 + \dots + x_n\alpha_n$ and $\beta = y_1\alpha_1 + \dots + y_n\alpha_n$. Then

$$[\alpha + \beta]_{\mathcal{B}} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = [\alpha]_{\mathcal{B}} + [\beta]_{\mathcal{B}}$$

4. $[c\alpha]_{\mathcal{B}} = c[\alpha]_{\mathcal{B}}$

There exists an *isomorphism* between V and F^n .

EXAMPLE

Let \mathcal{P} be the space of all polynomials. Let $f(x) = x^3$, and $g(x) = x^5$. Then, let

$$V = \text{Span}\{f, g\} = \{\text{all } ax^3 + bx^5 : a, b \in F\}$$

then, $\dim(V) = 2$, since f and g are linearly independent.

Typical $h(x) \in V$, say $h(x) = 10x^3 - 2x^5$.

$$[h]_{\mathcal{B}} = \begin{bmatrix} 10 \\ -2 \end{bmatrix}$$

$\langle [h]_{\mathcal{B}}$ is the mapping of h to F^n . **TODO** is this right? \rangle

Now let $k(x) = 2x^3 + 4x^5$ and $l(x) = x^3 + 3x^5$. Since k, l are linearly independent, they form another basis of V .

$$\mathcal{B}' = \{k(x), l(x)\}$$

1.1 Change of Basis

Given $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$, and $\mathcal{B}' = \{\alpha'_1, \dots, \alpha'_n\}$ bases for V .

We want to describe the map going from $[\alpha]_{\mathcal{B}} \mapsto [\alpha]_{\mathcal{B}'}$.

\langle We want to find The \mathcal{B} coordinate of $\alpha \mapsto$ the \mathcal{B}' coordinate of α \rangle

Step 1.

Compute the \mathcal{B} coordinate of $\alpha'_1, \dots, \alpha'_n$, old coordinates of the new basis elements.

Step 2.

For an $n \times m$ matrix

$$P = \begin{bmatrix} [\alpha'_1]_{\mathcal{B}}, \dots, [\alpha'_n]_{\mathcal{B}} \end{bmatrix}$$

Check: for any $\alpha \in V$

$$[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$$

Ans: This is what we actually want

$$[\alpha]_{\mathcal{B}'} = P^{-1}[\alpha]_{\mathcal{B}}$$

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TODO Missing *some* info

Want: Describe the mapping $T : F^n \rightarrow F^n$

$$T([\alpha]_{\mathcal{B}_{\text{old}}}) = [\alpha]_{\mathcal{B}'_{\text{new}}}$$

\langle If we switch the basis for some reason, we want to see what the new coordinates are. \rangle

To do this: For each α'_j , compute $[\alpha'_j]_{\mathcal{B}_{\text{old}}}$. Let

$$P = \begin{bmatrix} [\alpha'_1]_{\mathcal{B}_{\text{old}}} & \cdots & [\alpha'_n]_{\mathcal{B}_{\text{old}}} \end{bmatrix}$$

be an $n \times n$ matrix.

Claim: For any $\alpha \in V$

$$P \cdot [\alpha]_{\mathcal{B}'_{\text{new}}} = [\alpha]_{\mathcal{B}_{\text{old}}}$$

How?

$$P \cdot [\alpha'_1]_{\mathcal{B}'_{\text{new}}} = P \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [\alpha'_1]_{\mathcal{B}_{\text{old}}}$$

This is the 1^{st} column of P , and similarly for all columns.

Thus: For any $\alpha \in V$,

$$[\alpha]_{\text{new}} = P^{-1} \cdot [\alpha]_{\text{old}}$$

EXAMPLE

In practice, we have the following.

$V = \text{Span}(\{x^3, x^5\})$ subspace of \mathcal{P} , the set of all polynomials. Let $f(x) = x^3, g(x) = x^5, \mathcal{B} = \{x^3, x^5\}$. Let $h(x) = 10x^3 - 2x^5 \in V$.

Question: What are the coordinates of h with respect to \mathcal{B} ?

Answer:

$$[h]_{\mathcal{B}} = \begin{bmatrix} 10 \\ -2 \end{bmatrix}$$

Let's now see what happens when we create a new basis \mathcal{B}' .

EXAMPLE

Let $k(x) = 2x^3 + 5x^5, l(x) = x^3 + 3x^5$.

Let $\mathcal{B}' = \{k(x), l(x)\} = \{2x^3 + 5x^5, x^3 + 3x^5\}$ be another basis of V , still with $\mathcal{B} = \{f(x), g(x)\} = \{x^3, x^5\}$.

Question: What are the coordinates of $h(x) = 10x^3 - 2x^5$ with respect to \mathcal{B}' now?

Answer:

Well we know that $[k(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $[l(x)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, these are just the coordinates of k , and l with respect to \mathcal{B} .

So now we can construct our P matrix

$$P = \begin{bmatrix} [k(x)]_{\mathcal{B}}, [l(x)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

notice that P 's columns are constructed from $k(x)$ and $l(x)$, expressed in terms of our standard basis \mathcal{B} .

Check:

$$P^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

Then

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ -2 \end{bmatrix} = \begin{bmatrix} 32 \\ -54 \end{bmatrix}$$

This means:

$$h(x) = 32k(x) - 54l(x) = 10x^3 - 2x^5$$

Which is what we expect.

EXAMPLE

Let $V = \mathbb{R}^2$. Standard basis $\mathcal{B} = \{\varepsilon_1, \varepsilon_2\} = \{(1, 0), (0, 1)\}$

$$[(5, 4)]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Fix angle θ , Let

$$\mathcal{B}' = \{(\cos(\theta), \sin(\theta)), (-\sin(\theta), \cos(\theta))\}$$

Question: What is $\begin{bmatrix} 5 \\ 4 \end{bmatrix}_{\mathcal{B}'_{\text{new}}}$?

Answer:

1. Form P

$$[(\cos(\theta), \sin(\theta))]_{\mathcal{B}} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

$$[(-\sin(\theta), \cos(\theta))]_{\mathcal{B}} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

Then

$$P = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Fact:

$$P^{-1} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

so we have

$$\begin{aligned}
[(5, 4)]_{\mathcal{B}'_{\text{new}}} &= P^{-1} \begin{bmatrix} 5 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 5 \cos(\theta) & 4 \sin(\theta) \\ -5 \sin(\theta) & 4 \cos(\theta) \end{bmatrix}
\end{aligned}$$

2 Linear Transformations

Say V, W are both vector spaces over the same field F .

DEFINITION

A **Linear Transformation** $T : V \rightarrow W$ is a function satisfying two rules

1. For all $\alpha, \beta \in V$,

$$T(\alpha + \beta) = T(\alpha) + T(\beta)$$

Note that the first $+$ is addition in V , but the second is addition in W .

2. For all $\alpha \in V$ and $c \in F$,

$$T(c\alpha) = cT(\alpha)$$

The book combines the two definitions above into one, like this,

$$T(c\alpha + \beta) = cT(\alpha) + T(\beta)$$

Let's quickly take some time to understand what V and W are here. Suppose we have a transformation $T : V \rightarrow W$, then V is the **domain** and W is the **codomain**.

Here, T is just a function, which means that it *must* use all of V , but it *does not* have to use all of W . For example, the following is a perfectly valid transformation.

EXAMPLE

Let $T : \mathcal{P}^3 \rightarrow \mathcal{P}^2$ be the transformation that takes all degree 3 polynomials to the space of degree 2 polynomials, with

$$T(f) = \mathbf{0}$$

for all $f \in \mathcal{P}^3$.

It's obvious that there are more degree 2 polynomials in the world than just the $\mathbf{0}$ polynomial. So here, we say that the $\text{Range}(T) = \{\mathbf{0}\}$, and that

$$\text{Range}(T) \subseteq W$$

but maybe we are getting ahead of ourselves.

2.1 Basic Facts

Suppose that $T : V \rightarrow W$ is a linear transformation

1. $T(0) = 0$

Proof:

$$T(0 + 0) = T(0) + T(0) = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

Note: 0 lives in the field, and $\mathbf{0}$ lives in W , the **codomain** of the transformation T .

⟨ Always be aware of where the 0 lives ⟩

2. For all $\{\alpha_1, \dots, \alpha_n\} \subseteq V$, all $\{c_1, \dots, c_n\} \in F$,

$$c_1 T(\alpha_1) + \dots + c_n T(\alpha_n)$$

Proof Easy induction on n , just follows from part (2) of the definition.

2.2 Examples

Let's look at multiple examples of linear transformations to get an idea of how they behave.

EXAMPLE

We already know that each matrix A has an associated linear transformation T_A . Let's look at this in more detail now.

Let $A \in F^{m \times n}$ be an $m \times n$ matrix with entries from a field F .

Then, let $T_A : F^n \rightarrow F^m$ be defined by

$$T_A(\mathbf{x}) = A\mathbf{x}$$

where \mathbf{x} is a vector in F^n .

Let's check that this is indeed a linear transformation.

Chose any $\mathbf{x}, \mathbf{y} \in F^n$, then

1. ⟨ Check that $T_A(\mathbf{x} + \mathbf{y}) = T_A(\mathbf{x}) + T_A(\mathbf{y})$ ⟩

Let $\mathbf{x}, \mathbf{y} \in V$, then

$$T_A(\mathbf{x} + \mathbf{y}) = A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T_A(\mathbf{x}) + T_A(\mathbf{y})$$

so this works as we expect.

2. ⟨ Check that $T_A(c\mathbf{x}) = cT_A(\mathbf{x})$ for $c \in F$. ⟩

let $c \in F$, then we have

$$T_A(cX) = A \cdot (cX) = cAX = cT_A(X)$$

which is also what we expect.

so we have proved that T_A is a linear transformation!

EXAMPLE

Consider \mathcal{P} the set of all polynomials $a_0 + a_1x + \dots + a_nx^n$.

Let's define $D : \mathcal{P} \rightarrow \mathcal{P}$ which takes a function $f \in \mathcal{P}$ to $f' \in \mathcal{P}$, where f' is the *derivative* of f .

$$D(f) = f'$$

Claim:

D is a linear transformation.

Proof:

Take two functions $f, g \in \mathcal{P}$, then by definition of D , we have

$$D(f + g) = (f + g)' = f' + g' = D(f) + D(g)$$

and for $c \in F$,

$$D(cf) = (cf)' = c \cdot f' = cD(f)$$

so the derivative is a linear transformation!

EXAMPLE

Let $C(\mathbb{R})$ be the set of all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$.

Let's define $I : C(\mathbb{R}) \rightarrow C(\mathbb{R})$ which takes a function $f \in C(\mathbb{R})$ to $F \in C(\mathbb{R})$, where F is the *antiderivative* of f .

$$I(f) = \int_0^x f(t)dt$$

⟨ Note that the integral exists because you can always integrate a continuous function. ⟩

The result is also continuous and differentiable by the Fundamental Theorem of Calculus.

$$D(I(f)) = f$$

Is the **Fundamental Theorem of Calculus**.

Therefore $I(f)$ really *is* continuous, $I(f) \in C(\mathbb{R})$.

Claim:

I is a linear transformation.

Proof:

Take two functions $f, g \in \mathcal{P}$, then by definition of I , we have

$$\begin{aligned} I(f + g) &= \int_0^x (f(t) + g(t))dt \\ &= \int_0^x f(t)dt + \int_0^x g(t)dt \\ &= I(f) + I(g) \end{aligned}$$

and

$$I(cf) = \int_0^x cf(t)dt = c \int_0^x f(t)dt = cI(f)$$

so the integral is a linear transformation!

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Recall: A linear transformation $T : V \rightarrow W$ is a function between two vector spaces over the same field F , satisfying

1. For all $\alpha, \beta \in V$,

$$T(\alpha + \beta) = T(\alpha) + T(\beta)$$

Note that the first $+$ is addition in V , but the second is addition in W .

2. For all $\alpha \in V$ and $c \in F$,

$$T(c\alpha) = cT(\alpha)$$

For all $\alpha_1, \dots, \alpha_k \in V$, and $c_1, \dots, c_k \in F$, it breaks nicely into

$$T(c_1\alpha_1 + \dots + c_k\alpha_k) = c_1T(\alpha_1) + \dots + c_kT(\alpha_k)$$

EXAMPLE

$I^* : C(\mathbb{R}) \rightarrow \mathbb{R}$ (all continuous functions from \mathbb{R} to \mathbb{R})

$$I^*(f) = \int_0^1 f(x)dx$$

$$I^*(x^2) = \int_0^1 x^2 dx = \left. \frac{x^3}{3} \right|_0^1 = \frac{1}{3}$$

Note that the output of I^* is just a number here. Additionally, I^* is linear: you can split integrals up for polynomials, and you can take constants outside.

For any V, W , we also have

$$X : V \rightarrow W$$

Is the zero transformation. It takes any $\alpha \in V$ to the 0 of W . We'll use this to prove theorems about linear transformations later.

THEOREM

Let's prove existence and uniqueness of linear transformations.

1. Linear Transformations $T : V \rightarrow W$ are **determined** by their behavior on a basis \mathcal{B} of V . More precisely,

Suppose that $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ is a basis for V and suppose that $T, U : V \rightarrow W$ are both linear transformations (and they agree on a basis), such that

$$T(\alpha_1) = U(\alpha_1), T(\alpha_2) = U(\alpha_2), \dots, T(\alpha_n) = U(\alpha_n)$$

Then $T = U$

2. For **any map** $T_0 : \mathcal{B} \rightarrow W$, there is a unique linear transformation $T : V \rightarrow W$ with $T \supseteq T_0$. In other words,

Let $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ be **any basis** for V and let β_1, \dots, β_n be **any vectors** in W .

Then, there is a **unique** linear transformation $T : V \rightarrow W$ such that

$$T(\alpha_1) = \beta_1, T(\alpha_2) = \beta_2, \dots, T(\alpha_n) = \beta_n$$

PROOF

1. **Uniqueness:** Chose any $\alpha \in V$, since \mathcal{B} is a basis,

⟨ Will show that $T = U \Leftrightarrow$ For any $\alpha \in V$, $T(\alpha) = U(\alpha)$ ⟩

$$\alpha = c_1\alpha_1 + \dots + c_n\alpha_n$$

for some **unique** $c_1, \dots, c_n \in F$.

Since T is a linear transformation,

$$T(\alpha) = c_1T(\alpha_1) + \dots + c_nT(\alpha_n)$$

Likewise with U ,

$$U(\alpha) = c_1U(\alpha_1) + \dots + c_nU(\alpha_n)$$

But, since $T(\alpha_1) = U(\alpha_1), \dots, T(\alpha_n) = U(\alpha_n)$, $T(\alpha) = U(\alpha)$.

⟨ Essentially, if T, U work the same for all α_i , then their sum will obviously be the same, and so they'll give the same result for the same α . ⟩

Note that this theorem *still* works for infinite dimensional vector spaces.

2. **Existence:** Chose any $\alpha \in V$. ⟨ We must define $T(\alpha)$ ⟩

Since \mathcal{B} is a basis, we can write

$$\alpha = c_1\alpha_1 + \dots + c_n\alpha_n$$

which is unique.

Define

$$T(\alpha) := c_1\beta_1 + \dots + c_n\beta_n \in W$$

Check: Is T linear?

Say $\gamma = d_1\alpha_1 + \dots + d_n\alpha_n$, $\delta = e_1\alpha_1 + \dots + e_n\alpha_n$.

In V , we have that $\gamma + \delta = (d_1 + e_1)\alpha_1 + \dots + (d_n + e_n)\alpha_n$.

By our definition of T , we have

$$\begin{aligned} T(\gamma + \delta) &= (d_1 + e_1)\beta_1 + \dots + (d_n + e_n)\beta_n \\ &= (d_1\beta_1 + \dots + d_n\beta_n) + (e_1\beta_1 + \dots + e_n\beta_n) \\ &= T(\gamma) + T(\delta) \end{aligned}$$

Check: $T(c\gamma) = cT(\gamma)$

So such a transformation T exists. Additionally by part (1), it is unique.



Let $T : V \rightarrow W$ be a linear transformation.

DEFINITION

$\text{Range}(T) = \{T(\alpha) : \alpha \in V\} \subseteq W$ is the set of all vectors in W hit by T .

Fact: $\text{Range}(T)$ is a **subspace** of W .

1. 0 is in it. This is because $T(0) = 0$, obviously.

2. **Combinations of α_i are in it**

Say that $\beta_1, \beta_2 \in \text{Range}(T)$. \langle must show that $\beta_1 + \beta_2 \in \text{Range}(T)$ \rangle

Since $\beta_1 \in \text{Range}(T)$, there is some $\alpha_1 \in V$ such that

$$T(\alpha_1) = \beta_1$$

similarly for β_2 . Now $T(\alpha_1 + \alpha_2) = T(\alpha_1) + T(\alpha_2) = \beta_1 + \beta_2$, since T is linear. So $T(\alpha_1 + \alpha_2) = \beta_1 + \beta_2$ so $\beta_1 + \beta_2 \in \text{Range}(T)$ \langle since $\alpha_1, \alpha_2 \in V$ means that $\alpha_1 + \alpha_2 \in V$, because it's a vector space! \rangle

3. **Scaling Works:** Say $\beta \in \text{Range}(T)$, and $c \in F$. Chose $\alpha \in V$ such that $T(\alpha) = \beta$. Then $T(c\alpha) = cT(\alpha) = c\beta$, therefore $c\beta \in \text{Range}(T)$.

In other books this space is also called the **image** of T .

DEFINITION

The **Null Space** of $T : V \rightarrow W$ is the set

$$\text{Null}(T) = \{\alpha \in V | T(\alpha) = \mathbf{0}\} \subseteq V$$

\langle In other words, this is the set of all vectors α in V that, after a transformation T is applied, go to $\mathbf{0}$. Note that $\mathbf{0}$ here is the zero of the vector space $W \subseteq V$. \rangle

This is also sometimes called the **Kernel** of T .

THEOREM

Let $T : V \rightarrow W$ be a linear transformation. $\text{Null}(T)$ is a subspace of V .

PROOF

Let $\alpha, \beta \in \text{Null}(T)$ and $c \in F$. Then,

$$T(c\alpha + \beta) = cT(\alpha) + T(\beta) = c\mathbf{0} + \mathbf{0} = \mathbf{0} \Rightarrow c\alpha + \beta \in \text{Null}(T)$$



It's pretty easy to see from this (and it should make sense) that the Null Space for a transformation T is itself a vector space.

DEFINITION

The **Nullity** of T is the dimension of the Null space of T .

DEFINITION

The **Rank** of T is the dimension of $\text{Range}(T)$. If this is equal to the dimension of W , T is said to have **full rank**.

Note again that this comes back to our definition of W for our transformation T . Earlier, we saw that W was the *codomain* of T . If you think about how functions behave, this is like having a *surjective* function.

EXAMPLE

Let \mathcal{P}_2 be the set of all polynomials of degree 2 or less over a field F . Then, we have $\dim(\mathcal{P}_2) = 3$.

Consider the linear transformation $D : \mathcal{P}_2 \rightarrow \mathcal{P}_2$, the differentiation operator. Then

$$\text{Range}(D) = \text{Span}(\{D(1), D(x), D(x^2)\}) = \text{Span}(\{1, 2x\}) \Rightarrow \text{Rank}(D) = 2$$

In other words, the Range of D is the Span of a basis of \mathcal{P}_2 (in this case $\{1, x, x^2\}$) after being evaluated through D , so $\{1, 2x\}$. So the rank of D here is 2.

For the Null Space of D , we have that

$$\text{Null}(D) = \{c \in F\} \Rightarrow \text{Nullity}(D) = 1$$

The Null Space is the set of all constant functions since those are the function that, on D , go to $\mathbf{0}$.

2.3 The Rank-Nullity Theorem

RANK-NULLITY THEOREM

Let V be a vector space with $\dim V = n$. Let $T : V \rightarrow W$.

$$\text{Rank}(T) + \text{Nullity}(T) = \dim V = n$$

PROOF

First, choose $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ to be a basis for $\text{Null}(T)$. This set is necessarily linearly independent in V . So, we can choose an additional $\{\alpha_{k+1}, \dots, \alpha_n\}$ so that $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis of V .

Certainly, $k \leq n$, since $\text{Null}(T)$ is a subspace of V .

We claim $A = \{T(\alpha_{k+1}), \dots, T(\alpha_n)\}$ is a basis for $\text{Range}(T)$. From this we have our theorem.

Clearly, $A \subseteq \text{Range}(T)$. We also have, that since $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis of V , $\{T(\alpha_i)\}$ spans $\text{Range}(T)$.

However, $T(\alpha_1) = T(\alpha_2) = \dots = T(\alpha_k) = \mathbf{0}$, since they are in the null space, and hence do not contribute to the span. Thus, A spans $\text{Range}(V)$. Now we need only show A is linearly independent. We choose constants such that

$$c_{k+1}T(\alpha_{k+1}) + \dots + c_nT(\alpha_n) = \mathbf{0}$$

Let

$$\alpha^* = c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n \in V$$

We then have

$$T(\alpha^*) = c_{k+1}T(\alpha_{k+1}) + \cdots + c_nT(\alpha_n) = \mathbf{0} \Rightarrow \alpha^* \in \text{Null}(T)$$

So, we then have that, since α^* is in the null space,

$$\alpha^* = d_1\alpha_1 + d_2\alpha_2 + \cdots + d_k\alpha_k = c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n$$

$$d_1\alpha_1 + d_2\alpha_2 + \cdots + d_k\alpha_k - c_{k+1}\alpha_{k+1} - \cdots - c_n\alpha_n = \mathbf{0} \in V$$

But since $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis of V , all the constants are zero, and in particular all of the c_i are zero. So, $\{T(\alpha_{k+1}), \dots, T(\alpha_n)\}$ is linearly independent and is thus a basis of $\text{Range}(T)$.

■

Now that we have the rank-nullity theorem, we can analyze transformations and their matrices.

DEFINITION

Let A be a matrix in $F^{m \times n}$.

The **Column Space** is the vector space spanned by the n columns of A . This is precisely $\text{Range}(T_A)$.

The **Row Space** is the vector space spanned by the m rows of A .

THEOREM

Let A be a matrix, that when row-reduced has n unknowns and r non-zero rows. $\text{Nullity}(T_A) = n - r$

PROOF

This follows from the fact that elementary row operations preserve the row space, and that solving a linear system in r equations with n unknowns will have $n - r$ degrees of freedom.

TODO I guess I can believe this but some more info would be nice.

■

NOTE

Let A be a matrix. Then the following are equal

- The dimension of the row space of A
- The dimension of the column space of A
- The number of nonzero rows in the row-reduced form of A
- $\text{Rank}(T_A)$

This follows immediately from the above and the Rank-Nullity Theorem.

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Suppose that A is an $m \times n$ matrix. Now suppose that we row reduce A , let's call this matrix A^{rr} . Then we have that

$$\text{RowSpace}(A) = \text{RowSpace}(A^{rr})$$

And we know that $\text{Rank}(A)$ is the number of non-zero rows of A^{rr} which we call r .

Moreover, the solution set of the homogeneous system $A\mathbf{x} = \mathbf{0}$ has dimension $n - r$, where n is the number of columns subtract the number of redundant equations.

Now, we know that for a matrix A , there is an associated linear transformation $T_A : F^n \rightarrow F^m$.

Last time, we also saw that

1. $\text{Range}(T_A) = \text{ColSpace}(A)$,
2. $\text{Null}(T_A)$ is the solution set of $A\mathbf{x} = \mathbf{0}$.

Now we can put everything together. Recall the Rank-Nullity theorem, then we have that, for any linear transformation T_A ,

1. $\text{Rank}(T_A) + \text{Nullity}(T_A) = \dim(F^n) = n$
2. $\text{Rank}(T_A) := \dim(\text{Range}(T_A))$
3. $\text{Nullity}(T_A) = \dim(\text{Null}(A)) = n - r$, which is exactly the dimension of the set of all solutions to the homogeneous.
4. Finally we have that

$$\begin{aligned}\text{Rank}(A) &= \dim(\text{RowSpace}(A)) = \dim(\text{ColSpace}(A)) \\ &= \dim(\text{RowSpace}(A^{rr})) \\ &= \text{Rank}(T_A) \\ &= r\end{aligned}$$

Recall also that $\text{Nullity}(T_A) = \dim(\text{Null}(T_A)) = n - r$.

Consider a matrix A where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Then

$$A^{rr} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Is the row reduced matrix.

A basis for the row space of A is

$$\{(1, 0, 1, 1), (0, 1, 1, 1/3)\}$$

but another is

$$\{(1, 2, 3, 4), (1, 0, 1, 1)\}$$

We have $T_A : \mathbb{R}^4 \rightarrow \mathbb{R}^3$, and $\text{Rank}(T_A) = 2$.

Basis for $\text{Range}(T_A)$ equals the basis for $\text{Col Space}(A)$

There are many more linear transformations than the ones given by a matrix, for instance the derivative or integrals.

Let $T : V \rightarrow W$ be a linear transformation. From here there are two questions we can now ask.

1. Is T onto?

It is if and only if $\text{Range}(T) = W$. We saw this earlier. In terms of dimension, this means that $\text{Rank}(T) = \dim(W)$.

Note that here, V, W must be **finite dimensional**.

2. Is T one to one?

This requires some more work.

THEOREM

$T : V \rightarrow W$ is one to one if and only if $\text{Null}(T) = \{\mathbf{0}\}$.

⟨ In other words, the Null space must only contain the zero vector. ⟩

PROOF

Assume that T is one to one. We know that $T(\mathbf{0}_V) = \mathbf{0}_W$. Chose any $\alpha \in \text{Null}(T)$, then $T(\alpha) = \mathbf{0}_W$, by definition of being in the Null Space. Since T is one to one, α must equal $\mathbf{0}_V$.

Now assume that $\text{Null}(T)$ is just $\mathbf{0}_W$. To see that T is one to one, chose any $\alpha, \alpha' \in V$, with $T(\alpha) = T(\alpha')$. Then $T(\alpha - \alpha') = T(\alpha) - T(\alpha')$ by linearity, but then since $\alpha = \alpha'$, $T(\alpha - \alpha') = \mathbf{0}$ so $T(\alpha - \alpha')$ must be in the Null space of T , and since $\text{Null}(T) = \{\mathbf{0}\}$, and $\alpha - \alpha' = \mathbf{0}$, so $\alpha = \alpha'$ and thus T is one to one.



DEFINITION

T is called **non-singular** if T is one to one.

This is just another term for something we already know.

THEOREM

Now suppose that $T : V \rightarrow W$ is a linear transformation with $\dim(V) = \dim(W)$. Then T is one to one if and only if T is onto.

PROOF

By the Rank-Nullity theorem from last time, we have that

$$\text{Rank}(T) + \text{Nullity}(T) = \dim(V)$$

Now, assume that T is one to one, then $\text{Nullity}(T) = 0$, but then $\text{Rank}(T) = \dim(V) = \dim(W)$.

Now conversely, assume that T is onto. Then

$$\text{Rank}(T) = \dim(W) = \dim(V)$$

Therefore $\text{Nullity}(T) = 0$, and so T is one to one.



We are now starting to get a pretty good understanding of linear transformations, but suppose that we now want to combine them.

2.4 Combining Linear Transformations

Say $T : V \rightarrow W$ and $U : W \rightarrow Y$ are linear transformations over F .

⟨ then $U \circ T : V \rightarrow Y$ is a function. ⟩

Check the following:

1. $U \circ T$ is a linear transformation.
 ⟨ You know how to do this, just check that they scale and add as we expect. ⟩
2. If both T and U are one to one, then the composition is also one to one.
3. If both T and U are onto, the composition is also onto.

NOTE

$T \circ U$ would **not** be a linear transformation, assuming that Y and V are not the same vector space.

⟨ Linear transformations don't commute nicely like that. ⟩

Let's now look at T again.

DEFINITION

A linear transformation $T : V \rightarrow W$ is called **invertible** if there is a linear transformation $U : W \rightarrow V$ such that

1. $U \circ T : V \rightarrow V$ is the identity from V . In other words

$$U(T(\alpha)) = \alpha$$

For any $\alpha \in V$.

2. $T \circ U : W \rightarrow W$

$$T(U(\alpha)) = \alpha$$

For any $\alpha \in W$.

NOTE

It might be interesting for you to prove that, if one of the above applies, the other automatically applies as well.

If T is invertible, we call such a U T^{-1} , the inverse transformation of T .

NOTE

Inverse transformations are unique, if they exist.

⟨ We didn't talk about this in class but it *has* to be true. ⟩

Proposition: If $T : V \rightarrow W$ is an *invertible* linear transformation if and only if T is both one to one and onto.

NOTE

If T has an inverse, then it must be the case that $\dim(V) = \dim(W)$.

If this is surprising, just consider that this follows from the fact that T must be both one to one, and onto in order to have an inverse.

Today was exam review. As such, everything for today is written as examples.

What to expect

1. Short answer **True / False**. Then write a sentence explaining your choice, doesn't need to be a proof.
2. Matrix stuff. Row Space, Col Space, Rank of Matrix, Solution set of homogeneous system, etc...
3. Linear Transformations. Polynomials, $\{e^{ix}, e^{-ix}\}$ stuff, etc...
4. Full Proofs of statements. Rather straightforward, just involving linear independence, or spanning, or dimensions, etc...

The exam is 50 minutes, problems *will* be reasonable. At this point, we have definitely covered everything that will be on exam 1.

3 Example Problems

EXAMPLE

Problem 3.1.4

Want: A linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that $T(1, -1, 1) = (1, 0)$, and $T(1, 1, 1) = (0, 1)$

Firstly, *is there one?*

The two vectors passed to T are linearly independent, so **they can be expanded to a basis**. Say for example,

$$\mathcal{B} = \{(1, -1, 1), (1, 1, 1), (0, 0, 1)\}$$

Where \mathcal{B} is a basis of \mathbb{R}^3 . Now, there will be a *unique* linear transformation that will take it to any two points in \mathbb{R}^2 (even if those two points are not linearly independent.)

Fundamental Fact:

If V, W are vector spaces, and $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ is a basis of V , then for any $\beta_1, \dots, \beta_n \subseteq W$, there is *exactly* one linear transformation T for which

$$T(\alpha_1) = \beta_1, \dots, T(\alpha_n) = \beta_n$$

NOTE: β_i do *not* have to be linearly independent! For instance consider the 0 transformation, then $\beta_1 = \dots = \beta_n = \mathbf{0} \in W$.

Question: How do we define such a T ?

Take any $\alpha^* = c_1\alpha_1 + \dots + c_n\alpha_n$. Then by linearity,

$$T(\alpha^*) = c_1\beta_1 + \dots + c_n\beta_n$$

NOTE: This shows that such a T *does* exist, but it's not *constructive*. We don't *actually* know what it is; we only know what properties it has, and that it exists.

NOTE: Before we expanded $\{(1, -1, 1), (1, 1, 1)\}$ to a basis, there were an *infinite* family of linear transformations. This is because we got to choose the last vector of \mathcal{B} , in our case $(0, 0, 1)$. If we have chosen, say, $(0, 1, 0)$ instead, then we would have gotten an entirely different transformation T .

EXAMPLE

Problem 2.4.6a

Question: How do we show that e^{ix} and e^{-ix} are linearly independent?

Let $f_1(x) = 1, f_2(x) = e^{ix}, f_3(x) = e^{-ix}$.

Claim: f_1, f_2, f_3 are *linearly independent*.

Recall what this means. For *any* x ,

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = \mathbf{0}$$

Implies that c_1, c_2, c_3 must all be 0. Where $\mathbf{0}$ is the zero function: $z(x) = 0$ for all x .

$$c_1 + c_2 e^{ix} + c_3 e^{-ix} = \mathbf{0}$$

Now let $x = -100i$, then $e^{i(-100i)} = e^{100}$, then $e^{-i(-100i)} = e^{-100}$, so we have

$$c_1 + c_2 e^{100} + c_3 e^{-100} = \mathbf{0}$$

Therefore, c_1, c_2, c_3 *must* all be 0, since clearly all the functions are positive for $x = -100i$. This shows that the functions are linearly independent.

Since $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = \mathbf{0}$ *has* to work for *any* value of x , if you just find *one* counterexample (like $x = -100i$ above), that shows that they are linearly independent.

EXAMPLE

Question: Suppose that the field F is \mathbb{C} . Is

$$W = \{g_1(x) = 1, g_2(x) = \sin x, g_3(x) = \cos x\}$$

Linearly independent?

Answer: Yes!

Suppose that for some $x = \theta$ we have

$$\begin{aligned} c_1 g_1(x) + c_2 g_2(x) + c_3 g_3(x) &= \mathbf{0} \\ c_1 + c_2 \sin x + c_3 \cos x &= \mathbf{0} \end{aligned}$$

Then

$$c_2 \sin x + c_3 \cos x = -c_1$$

Then look at $\theta + \pi$,

$$c_2 \sin(\theta + \pi) + c_3 \cos(\theta + \pi) = -c_1$$

So then we have the set of equations

$$c_2 \sin \theta + c_3 \cos \theta = c_2 \sin(\theta + \pi) + c_3 \cos(\theta + \pi)$$

But since $\sin \theta = -\sin(\theta + \pi)$ and $\cos \theta = -\cos(\theta + \pi)$, we have

$$\begin{aligned} c_2 \sin \theta + c_3 \cos \theta &= -c_2 \sin \theta - c_3 \cos \theta \\ 2c_2 \sin \theta + 2c_3 \cos \theta &= 0 \end{aligned}$$

But in order for this to be true for all θ , c_1 and c_2 must both be 0.

NOTE: We know that $\sin^2 x + \cos^2 x = 1$, but we're not allowed to square g_2 and g_3 here. That's not a linear operation, so it's not relevant. However, it *is* true that $\{1, \cos^2 x, \sin^2 x\}$ are linearly dependent.

NOTE: $f_2 \in \text{Span}(\{g_1, g_2, g_3\})$.

$$\begin{aligned} e^{-ix} &= \cos(-x) + i \sin(-x) \\ e^{-ix} &= \cos(x) - i \sin(x) \end{aligned}$$

So $f_1 \in \text{Span}(\{1, \sin x, \cos x\})$, but $\dim(W) = 3$, and we already know that f_1, f_2, f_3 are linearly independent from the previous example, so we just found another basis for W

$$\{f_1, f_2, f_3\} = \text{Span}(W)$$

EXAMPLE

Problem 2.4.4d

Let $W = \text{Span}(\{(1, 0, i), (1 + i, 1, -1)\})$.

Question: Is this set linearly independent?

Yes, neither is 0, or a multiple of the other.

Question: What is $\dim(W)$?

2

So a basis could be

$$\mathcal{B} = \{(1, 0, i), (1 + i, 1, -1)\}$$

The vectors themselves.

Question: Let $\beta_1 = (1, 1, 0)$. Is $\beta_1 \in W$?

Let's try to make it.

$$(1, 1, 0) = c_1(1, 0, i) + c_2(1 + i, 1, -1)$$

Need: $1 = c_1 + c_2(i + i)$, and $1 = c_1(0) + c_2(1)$ so $c_2 = 1$, and finally, $0 = c_1(i) + c_2(-1)$.

So $\beta_1 = (-i)(1, 0, -i) + 1(1 + i, 1, -1)$

EXAMPLE

4d on Homework 3

If $\dim(V) = \dim(W)$, and $\mathcal{B} = \{\alpha_1, \dots, \alpha_k\}$ is a basis for V , must

$$T(\alpha_1) + \dots + T(\alpha_k)$$

Be a basis for W ?

Answer: No!

Let T be the 0 transformation, then everything is mapped to 0.

NOTE: Here it's worth reiterating the difference between the *codomain* and the *range* of T .

Suppose that $T : V \rightarrow W$ is a linear transformation that maps vectors from V to W . Then, we say that V is the *domain* of T , and that W is the *codomain* of T .

We know that a linear transformation is just a function, and one thing that we know about functions is that they *must* use their entire domain (so everything from V has to map *somewhere* in W), but they *don't* have to use all of their codomain.

The nuance here is the word "*somewhere*". Suppose that we have the function $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) = 5$. Then the domain of f is \mathbb{R} and so is the codomain. Here, we can see that 5 is *somewhere* in the codomain of f , but it's certainly *not all* of it. Then, we say that the *range* of f is $\{5\}$. This is all just by definition.

Let's look at another example. Suppose that $g : \mathbb{R} \rightarrow \mathbb{Z}$ with $g(x) = \lfloor x \rfloor$ this time. In this case, g *does* use up all of its codomain, since every value in \mathbb{R} will be mapped to a value in \mathbb{Z} .

Linear transformation behave in exactly the same way. When we talk about $\dim(W)$, we're talking about the size of a basis of W , but T makes no promises about mapping elements from V to all of it.

Mon. Feb 27 2023

From this point, the course is going to become much more abstract.

Fix V, W vector spaces over the same field F .

Let $L(V, W)$ consist of all linear transformations $T : V \rightarrow W$.

THEOREM

$L(V, W)$ is a vector space.

PROOF

Say that T, U are each linear transformations. Let $T + U : V \rightarrow W$ be defined by

$$(T + U)(\alpha) = T(\alpha) + U(\alpha)$$

For $T \in L(V, W)$ and $c \in F$. Let $cT : V \rightarrow W$ be the linear transformation

$$(cT)(\alpha) = cT(\alpha)$$

It's easy to check that these linear transformations satisfy the properties of being a vector space.



THEOREM

Suppose that $\dim(V) = n$ and $\dim(W) = m$, then $\dim(L(V, W)) = nm$.

PROOF

Recall that a linear transformation $T : V \rightarrow W$ is determined by what it does to a basis of V .

Chose a basis $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ of V , and $\mathcal{B}' = \{\beta_1, \dots, \beta_m\}$ of W .

For any $1 \leq p \leq m$ and $1 \leq q \leq n$,

Let $E^{pq} : V \rightarrow W$ be determined by

$$E^{pq}(\alpha_i) = \begin{cases} \beta_j & \text{if } i = q \\ 0 & \text{if } i \neq q \end{cases} = \delta_{iq} \cdot \beta_j$$

Where δ_{iq} is the “Kronecker δ function” defined by

$$\delta_{iq} = \begin{cases} 1 & \text{if } i = q \\ 0 & \text{if } i \neq q \end{cases}$$

Claim: $E^{pq} : 1 \leq p \leq m, 1 \leq q \leq n$ is a basis for $L(V, W)$.

Proof: \langle Why does $\{E^{pq}\}$ span $L(V, W)$? \rangle

For $1 \leq p \leq m, 1 \leq q \leq n$, $E^{pq}(\alpha_q) = \beta_p$ but $E^{pq}(\alpha_{q'}) = 0$ for all $q' \neq q$.

Choose any $T \in L(V, W)$, i.e. $T : V \rightarrow W$ is a linear transformation. \langle What does this T do to V ? \rangle

For $1 \leq q \leq n$, say $T(\alpha_q) = A_{1q}\beta_1 + A_{2q}\beta_2 + \dots + A_{mq}\beta_m$, for some A_{1q}, \dots, A_{mq}

\langle Here, we’re building an $m \times n$ matrix! \rangle

Subclaim:

$$T = \sum_{p=1}^m \sum_{q=1}^n A_{pq} E^{pq}$$

\langle The subclaim shows that T is in the span of E^{pq} \rangle

Proof of Subclaim

Let $U = \sum_{p=1}^m \sum_{q=1}^n A_{pq} E^{pq}$. \langle We’re going to show that T , and U do the same thing to every basis element. Linear transformations are equal if and only if they agree on a basis. \rangle

Fix $1 \leq q \leq n$

$$\begin{aligned} U(\alpha_q) &= \sum_{p=1}^m \sum_{q=1}^n A_{pq} E^{pq}(\alpha_q) \\ &= \sum_{p=1}^m A_{pq} \beta_p \\ &= A_{1q}\beta_1 + A_{2q}\beta_2 + \dots + A_{mq}\beta_m \\ &= T(\alpha_q) \end{aligned}$$

So we see that T and U agree on every $\alpha \in \mathcal{B}$, so $T = U$.

⟨ Now we need to show that they are linearly independent. ⟩

Subclaim 2: E^{pq} are linearly independent.

Chose $\{c_{pq}\} \in F$ such that

$$\sum_{p=1}^m \sum_{q=1}^n c_{pq} E^{pq} = 0$$

⟨ Now we must show that all c_{pq} must be 0. ⟩

Fix any $1 \leq q \leq n$. Then

$$\begin{aligned} U(\alpha_q) &= \sum_{p=1}^m \left(\sum_{q=1}^n c_{pq} E^{pq}(\alpha_q) \right) \\ &= \sum_{p=1}^m c_{pq} \beta_p \end{aligned} \quad \text{By cheatsheet =0}$$

Since U is the zero transformation. Thus

$$c_{1q}\beta_1 + c_{2q}\beta_2 + \cdots + c_{mq}\beta_m = 0$$

Since $\{\beta_1, \dots, \beta_m\}$ is a basis for W .

This means that $c_{1q} = c_{2q} = \cdots = c_{mq} = 0$.

This holds for every $1 \leq q \leq n$, therefore all c_{pq} must be 0, so $\{E^{pq}\}$ is linearly independent.

■

Wed. Mar 1 2023

DEFINITION

Suppose that V, W are vector spaces over the same field F . An **isomorphism** is a linear transformation T that has an inverse $U : W \rightarrow V$ satisfying

$$U \circ T = I_V$$

and

$$T \circ U = I_W$$

Write T^{-1} for this U if it exists.

Note that T here is *necessarily* a bijection.

Note that V and W must be over the **same field** for T to be an isomorphism.

DEFINITION

The vector spaces V, W are called **isomorphic** if there exists an *isomorphism* T from V to W .

Note that there may be *many different* isomorphisms.

What does this all mean? Well if V and W are isomorphic, then, even if they are very different, they will behave in very similar ways.

The following hold

- $\dim V = \dim W$
- If $V' \subseteq V$ is a subspace, then $T(V') = W' \subseteq W$ is a subspace of W with $\dim(V') = \dim(W')$. Basically if V and W are isomorphic, then V' and W' are also isomorphic.

Recall: A linear transformation $T : V \rightarrow W$ is an isomorphism if and only if T is both one to one and onto (in other words, if T is a bijection)

Specical Case: If we're lucky and $\dim(V) = \dim(W)$, then there's an easier test. If $T : V \rightarrow W$ is a linear transformation, then the following are equivalent

- T is an isomorphism
- T is one to one
- T is onto

Let's look at examples!

EXAMPLE

Let V be a vector space over F of dimension $\dim(V) = n$. Let $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ be an ordered basis for V . The coordinate transformation

$$C_{\mathcal{B}} : V \rightarrow F^n$$

is an isomorphism, sending $\alpha \mapsto [\alpha]_{\mathcal{B}}$. In other words it translates α to the language of \mathcal{B} .

If $V = \mathcal{P}^2$, $\mathcal{B} = \{1, x, x^2\}$, for any $f \in \mathcal{P}^2$, $f(x) = a_0 + a_1x + a_2x^2$,

$$C_{\mathcal{B}}(f) = [f]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \in F^3$$

is an isomorphism.

Given any $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \in F^3$, Let $\alpha = a_0 + a_1x + a_2x^2 \in \mathcal{P}^2$, then $C_{\mathcal{B}}(\alpha) = [\alpha]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$ is **onto**.

Conclusion:

If V has dimension n , then

$$V \cong F^n$$

and we say that V is *isomorphic* to F^n

This is great! Now we can compare V directly to some F^n .

Corrolary

If V, W are vector spaces over the same field F and $\dim(V) = \dim(W)$, then $V \cong W$.

This is huge! As long as linear transformations have the same dimension, they behave in the same way.

Proof

There exists a $C_B : V \rightarrow F^n$ and $C_{B'} : W \rightarrow F^n$, so there *must* exist a linear transformation $C_{B'}^{-1} \circ C_B$ is a linear transformation going from $V \rightarrow W$, this is an isomorphism!

Check

- The composition of any 2 isomorphisms *is* an isomorphism
- If T is an isomorphism going from V to W , then $T^{-1} : W \rightarrow V$ is an isomorphism

Let's recall some things

For any field F and any $m, n \geq 1$. If $F^{m \times n}$ consists of all $m \times n$ matrices over F , $F^{m \times n}$ is a vector space of dimension mn

$$\dim(F^{m \times n}) = mn$$

After all, there are mn free variables in the basis.

Now, let V, W be vector spaces over F . Let $\dim(V) = nb$ and $\dim(W) = m$. Now let $L(V, W)$ be the set of all linear transformations $T : V \rightarrow W$.

We saw that $L(V, W)$ is a vector space of dimension mn , and that

$$\{E^{pq} : 1 \leq q \leq n, 1 \leq p \leq m\}$$

is a basis for $L(V, W)$. But notice: $L(V, W)$ is a vector space of dimension mn , but so is $F^{m \times n}$. So they must be isomorphic!

$$L(V, W) \cong F^{m \times n}$$

Question: What is a linear transformation giving this isomorphism?

Sure, they're isomorphic, but how do we get from one to the other?

So the input here is just a linear transformation $T : V \rightarrow W$ (an element of $L(V, W)$), and the output is some m by n matrix (an element of $F^{m \times n}$).

Chose a basis $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ for V with $\dim(V) = n$, and a basis $\mathcal{B}' = \{\beta_1, \dots, \beta_m\}$ for W with $\dim(W) = m$. Then there is an isomorphism C_B Taking V to F^n .

But there is also a coordinate isomorphism $C_{B'}$ from W to F^m .

But recall that T goes from V to W , so the diagram commutes.

$$\text{Let } M_{\mathcal{B}'}^{\mathcal{B}}(T) = [[T(\alpha_1)]_{\mathcal{B}'}, [T(\alpha_2)]_{\mathcal{B}'}, \dots, [T(\alpha_n)]_{\mathcal{B}'}] .$$

Then, we propose that, for any $\alpha \in V$,

$$M_{\mathcal{B}'}^{\mathcal{B}}(T) \cdot [\alpha]_{\mathcal{B}} = [T(\alpha)]_{\mathcal{B}'}$$

3.1 Diagram

$$\begin{array}{ccc}
 \alpha \in V & \xrightarrow{C_{\mathcal{B}}} & [\alpha]_{\mathcal{B}} \in F^n \\
 \downarrow T & & \downarrow M_{\mathcal{B}'}^{\mathcal{B}}(T) \\
 T(\alpha) \in W & \xrightarrow{C_{\mathcal{B}'}} & [T(\alpha)]_{\mathcal{B}'} \in F^n
 \end{array}$$

Let's give a concrete example for this.

EXAMPLE

Let $D : \mathcal{P}_2 \rightarrow \mathcal{P}_1$

$$D(a_0 + a_1x + a_2x^2) = a_1 + 2a_2x$$

$\mathcal{B} = \{1, x, x^2\}$ be a basis for \mathcal{P}_2 , and $\mathcal{B}' = \{1, x\}$ be a basis for \mathcal{P}_1 .

Question

What is the matrix $M_{\mathcal{B}'}^{\mathcal{B}}(D)$? Well

$$D(1) = 0, D(x) = 1, D(x^2) = 2x, \text{ then } [D(1)]_{\mathcal{B}'} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, [D(x)]_{\mathcal{B}'} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, [D(x^2)]_{\mathcal{B}'} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

So

$$M_{\mathcal{B}'}^{\mathcal{B}}(D) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

But what does this mean? Well, chose any $f \in \mathcal{P}_2$, say

$$f(x) = 5 + 3x - x^2$$

What are the coordinates of f with respect to \mathcal{B} ?

$$[f]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$$

Then

$$M_{\mathcal{B}'}^{\mathcal{B}}(D) \cdot [f]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

But what does this final vector mean? Well, it's just $[D(f)]_{\mathcal{B}'}$, in other words, it's the derivative of f with respect to the basis \mathcal{B}' !

Really, all we're doing here is moving around in the diagram. All we need to do is to apply T to every α_i , living in V .

Note: Look at Homework number 13 on 3.4.

To show that $\{E^{pq}\}$ span $L(V, W)$, chose $T : V \rightarrow W$. Think about $M_{\mathcal{B}'}^{\mathcal{B}}(E^{pq})$