Contents

Fri. Feb 10 2023

1 Vector Spaces

Suppose that V is a finite dimensional vector space over F, with $\dim(V) = n$.

V may have *many different* bases, we know that they all have the same size n.

Say $\mathcal{B} = \{\alpha_1, ..., \alpha_n\}$ is a basis fix the ordering of $\mathcal{B}$.

Fix the ordering of $\mathcal{B}$.

Theorem

For any $\alpha \in V$, there is a unique n tuple $(x_1, ..., x_n) \in F^n$ such that

$$\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n$$

Proof

Existence is immediate, since $\mathcal{B}$ is a basis, thus $\mathcal{B}$ spans V.

Uniqueness

Say $\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n$ and $\alpha = y_1\alpha_1 + \cdots + y_n\alpha_n$.

Then we have that

$x_1\alpha_1 + \cdots + x_n\alpha_n - y_1\alpha_1 + \cdots + y_n\alpha_n = 0$, so $(x_1 - y_1)\alpha_1 + \cdots + (x_n - y_n)\alpha_n = 0$

But since $\{\alpha_1, ..., \alpha_n\}$ is linearly independent, all coefficients must be 0.

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What this means is that, for a vector space V, there is an associated mapping in F^n. Notice that we know nothing about the vectors α_i.

We define $[\alpha]_{\mathcal{B}}$ to be the *coordinates* of α with respect to $\mathcal{B}$.

Check: The mapping $\alpha \mapsto [\alpha]_{\mathcal{B}} \in F^n$ satisfies

1. One to one-ness
2. Onto-ness
3. "Additive", for any $\alpha, \beta \in V$, if $\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n$ and $\beta = y_1\alpha_1 + \cdots + y_n\alpha_n$. Then

$$[\alpha + \beta]_{\mathcal{B}} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = [\alpha]_{\mathcal{B}} + [\beta]_{\mathcal{B}}$$

4. $[c\alpha]_{\mathcal{B}} + c[\alpha]_{\mathcal{B}}$

There exists an *isomorphism* between V and F^n.

Example

Let $\mathcal{P}$ be the space of al polynomials. Let $f(x) = x^3$, and $g(x) = x^5$. Then, let

$$V = \text{Span}\{f, g\} = \{\text{all } ax^3 + bx^5 : a, b \in F\}$$

then, $\dim(V) = 2$, since f and g are linearly independent.

Typical $h(x) \in V$, say $h(x) = 10x^3 - 2x^5$.

$$[h]_{\mathcal{B}} = \begin{bmatrix} 10 \\ -2 \end{bmatrix}$$

$\langle$ $[h]_{\mathcal{B}}$ is the mapping of h to F^n. **TODO** is this right? $\rangle$

Now let $k(x) = 2x^3 + 4x^5$ and $l(x) = x^3 + 3x^5$. Since k, l are linearly independent, they form another basis of V.

$$\mathcal{B}' = \{k(x), l(x)\}$$

1.1 Change of Basis

Given $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$, and $\mathcal{B}' = \{\alpha_1', ..., \alpha_n'\}$ bases for V.

We want to describe the map going from $[\alpha]_{\mathcal{B}} \mapsto [\alpha]_{\mathcal{B}'}$.

$\langle$ We want to find The $\mathcal{B}$ coordinate of $\alpha \mapsto$ the $\mathcal{B}'$ coordinate of α $\rangle$

Step 1.

Compute the $\mathcal{B}$ coordinate of $\alpha_1', ..., \alpha_n'$, *old* coordinates of the *new* basis elements.

Step 2.

For an $n \times m$ matrix

$$P = \Big[[\alpha_1']_{\mathcal{B}}, \ldots, [\alpha_n']_{\mathcal{B}} \Big]$$

Check: for any $\alpha \in V$

$$[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$$

Ans: This is what we actually want

$$[\alpha]_{\mathcal{B}'} = P^{-1}[\alpha]_{\mathcal{B}}$$

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TODO Missing *some* info

Want: Describe the mapping $T : F^n \to F^n$

$$T([\alpha]_{\mathcal{B}_{\text{old}}}) = [\alpha]_{\mathcal{B}'_{\text{new}}}$$

$\langle$ If we switch the basis for some reason, we want to see what the new coordinates are. $\rangle$

To do this: For each α'_j, compute $[\alpha'_j]_{\mathcal{B}_{\text{old}}}$. Let

$$P = \Big[[\alpha'_1]_{\mathcal{B}_{\text{old}}} \cdots [\alpha'_n]_{\mathcal{B}_{\text{old}}}\Big]$$

be an $n \times n$ matrix.

Claim: For any $\alpha \in V$

$$P \cdot [\alpha]_{\mathcal{B}'_{\text{new}}} = [\alpha]_{\mathcal{B}_{\text{old}}}$$

How?

$$P \cdot [\alpha'_1]_{\mathcal{B}'_{\text{new}}} = P \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [\alpha'_1]_{\mathcal{B}_{\text{old}}}$$

This is the 1^{st} column of P, and similarly for all columns.

Thus: For any $\alpha \in V$,

$$[\alpha]_{\text{new}} = P^{-1} \cdot [\alpha]_{\text{old}}$$

Example

In practice, we have the following.

$V = \text{Span}(\{x^3, x^5\})$ subspace of $\mathcal{P}$, the set of all polynomials. Let $f(x) = x^3, g(x) = x^5, \mathcal{B} = \{x^3, x^5\}$. Let $h(x) = 10x^3 - 2x^5 \in V$.

Question: What are the coordinates of h with respect to $\mathcal{B}$?

Answer:

$$[h]_{\mathcal{B}} = \begin{bmatrix} 10 \\ -2 \end{bmatrix}$$

Let's now see what happens when we create a new basis $\mathcal{B}'$.

Example

Let $k(x) = 2x^3 + 5x^5$, $l(x) = x^3 + 3x^5$.

Let $\mathcal{B}' = \{k(x), l(x)\} = \{2x^3 + 5x^5, x^3 + 3x^5\}$ be another basis of V, still with $\mathcal{B} = \{f(x), g(x)\} = \{x^3, x^5\}$.

Question: What are the coordinates of $h(x) = 10x^3 - 2x^5$ with respect to $\mathcal{B}'$ now?

Answer:

Well we know that $[k(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $[l(x)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, these are just the coordinates of k, and l with respect to $\mathcal{B}$.

So now we can construct our P matrix

$$P = \Big[[k(x)]_{\mathcal{B}}, [l(x)]_{\mathcal{B}}\Big] = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

notice that P's columns are constructed from $k(x)$ and $l(x)$, expressed in terms of our standard basis $\mathcal{B}$.

Check:

$$P^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

Then

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ -2 \end{bmatrix} = \begin{bmatrix} 32 \\ -54 \end{bmatrix}$$

This means:

$$h(x) = 32k(x) - 54l(x) = 10x^3 - 2x^5$$

Which is what we expect.

Example

Let $V = \mathbb{R}^2$. Standard basis $\mathcal{B} = \{\varepsilon_1, \varepsilon_2\} = \{(1,0), (0,1)\}$

$$[(5,4)]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Fix angle θ, Let

$$\mathcal{B}' = \{(\cos(\theta), \sin(\theta)), (-\sin(\theta), \cos(\theta))\}$$

Question: What is $\begin{bmatrix} 5 \\ 4 \end{bmatrix}_{\mathcal{B}'_{\text{new}}}$?

Answer:

1. Form P

$$[(\cos(\theta), \sin(\theta))]_{\mathcal{B}} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

$$[(-\sin(\theta), \cos(\theta))]_{\mathcal{B}} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

Then

$$P = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Fact:

$$P^{-1} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

so we have

$$
\begin{aligned}
[(5,4)]_{\mathcal{B}'_{\text{new}}} =& P^{-1}\begin{bmatrix}5\\4\end{bmatrix} \\
&= \begin{bmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}5\\4\end{bmatrix} \\
&= \begin{bmatrix}5\cos(\theta) & 4\sin(\theta)\\ -5\sin(\theta) & 4\cos(\theta)\end{bmatrix}
\end{aligned}
$$

2 Linear Transformations

Say V, W are both vector spaces over the same field F.

Definition

A **Linear Transformation** $T : V \to W$ is a function satisifying two rules

1. For all $\alpha, \beta \in V$,

$$T(\alpha + \beta) = T(\alpha) + T(\beta)$$

   Note that the first $+$ is addition in V, but the second is addition in W.

2. For all $\alpha \in V$ and $c \in F$,

$$T(c\alpha) = cT(\alpha)$$

The book combines the two definitions above into one, like this,

$$T(c\alpha + \beta) = cT(\alpha) + T(\beta)$$

Let's quickly take some time to understand what V and W are here. Suppose we have a transformation $T : V \to W$, then V is the **domain** and W is the **codomain**.

Here, T is just a function, which means that it *must* use all of T, but it *does not* have to use all of W. For example, the following is a perfectly valid transformation.

Example

Let $T : \mathcal{P}^3 \to \mathcal{P}^2$ be the transformation that takes all degree 3 polynomials to the space of degree 2 polynomials, with

$$T(f) = \mathbf{0}$$

for all $f \in \mathcal{P}^3$.

Its obvious that there are more degree 2 polynomials in the world than just the $\mathbf{0}$ polynomial. So here, we say that the Range$(T) = \{\mathbf{0}\}$, and that

$$\text{Range}(T) \subseteq W$$

but maybe we are getting ahead of ourselves.

2.1 Basic Facts

Suppose that $T : V \to W$ is a linear transformation

1. $T(0) = 0$
   Proof:

   $$T(0+0) = T(0) + T(0) = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

   Note: 0 lives in the field, and $\mathbf{0}$ lives in W, the **codomain** of the transformation T.
   ⟨ Always be aware of where the 0 lives ⟩
2. For all $\{\alpha_1, ..., \alpha_n\} \subseteq V$, all $\{c_1, ..., c_n\} \in F$,

   $$c_1 T(\alpha_1) + \cdots + c_n T(\alpha_n)$$

   Proof Easy induction on n, just follows from part (2) of the definition.

2.2 Examples

Let's look at multiple examples of linear transformations to get an idea of how they behave.

Example

We already know that each matrix A has an associated linear transformation T_A. Let's look at this in more detail now.

Let $A \in F^{m \times n}$ be an $m \times n$ matrix with entries from a field F.

Then, let $T_A : F^n \to F^m$ be defined by

$$T_A(\mathbf{x}) = A\mathbf{x}$$

where $\mathbf{x}$ is a vector in F^n.

Let's check that this is indeed a linear transformation.

Chose any $\mathbf{x}, \mathbf{y} \in F^n$, then

1. ⟨ Check that $T_A(\mathbf{x} + \mathbf{y}) = T_A(\mathbf{x}) + T_A(\mathbf{y})$ ⟩
   Let $\mathbf{x}, \mathbf{y} \in V$, then

   $$T_A(\mathbf{x} + \mathbf{y}) = A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T_A(\mathbf{x}) + T_A(\mathbf{y})$$

   so this works as we expect.
2. ⟨ Check that $T_A(c\mathbf{x}) = cT_A(\mathbf{x})$ for $c \in F$. ⟩
   let $c \in F$, then we have

   $$T_A(cX) = A \cdot (cX) = cAX = cT_A(X)$$

   which is also what we expect.

so we have proved that T_A is a linear transformation!

Example

Consider $\mathcal{P}$ the set of all polynomials $a_0 + a_1 x + \cdots + a_n x^n$.

Let's define $D : \mathcal{P} \to \mathcal{P}$ which takes a function $f \in \mathcal{P}$ to $f' \in \mathcal{P}$, where f' is the *derivative* of f.

$$D(f) = f'$$

Claim:

D is a linear transformation.

Proof:

Take two functions $f, g \in \mathcal{P}$, then by definition of D, we have

$$D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$$

and for $c \in F$,

$$D(cf) = (cf)' = c \cdot f' = cD(f)$$

so the derivative is a linear transformation!

Example

Let $C(\mathbb{R})$ be the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$.

Let's define $I : C(\mathbb{R}) \to C(\mathbb{R})$ which takes a function $f \in C(\mathbb{R})$ to $F \in C(\mathbb{R})$, where F is the *antiderivative* of f.

$$I(f) = \int_0^x f(t)dt$$

$\langle$ Note that the integral exists because you can always integrate a continuous function. $\rangle$

The result is also continuous and differentiable by the Fundamental Theorem of Calculus.

$$D(I(f)) = f$$

Is the **Fundamental Theorem of Calculus**.

Therefore $I(f)$ really *is* continuous, $I(f) \in C(\mathbb{R})$.

Claim:

I is a linear transformation.

Proof:

Take two functions $f, g \in \mathcal{P}$, then by definition of I, we have

$$\begin{aligned} I(f+g) = &\int_0^x (f(t) + g(t))dt \\ = &\int_0^x f(t)dt + \int_0^x g(t)dt \\ = &I(f) + I(g) \end{aligned}$$

and

$$I(cf) = \int_0^x cf(t)dt = c\int_0^x f(t)dt = cI(f)$$

so the integral is a linear transformation!

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Recall: A linear transformation $T : V \to W$ is a function between two vector spaces over the same field F, satisifying

1. For all $\alpha, \beta \in V$,

$$T(\alpha + \beta) = T(\alpha) + T(\beta)$$

   Note that the first + is addition in V, but the second is addition in W.

2. For all $\alpha \in V$ and $c \in F$,

$$T(c\alpha) = cT(\alpha)$$

For all $\alpha_1, ..., \alpha_k \in V$, and $c_1, ..., c_k \in F$, it breaks nicely into

$$T(c_1\alpha_1 + \cdots + c_k\alpha_k) = c_1T(\alpha_1) + \cdots + c_kT(\alpha_k)$$

Example

$I^* : C(\mathbb{R}) \to \mathbb{R}$ (all continuous functions from $\mathbb{R}$ to $\mathbb{R}$)

$$I^*(f) = \int_0^1 f(x)dx$$

$$I^*(x^2) = \int_0^1 x^2 dx = \frac{x^3}{x}\Big|_0^1 = \frac{1}{3}$$

Note that the output of $I*$ is just a number here. Additionally, I^* is linear: you can split integrals up for polynomials, and you can take constants outside.

For any V, W, we also have

$$X : V \to W$$

Is the zero transformation. It takes any $\alpha \in V$ to the 0 of W. We'll use this to prove theorems about linear transformations later.

Theorem

Let's prove existence and uniqueness of linear transformations.

1. Linear Transformations $T : V \to W$ are **determined** by their behavior on a basis $\mathcal{B}$ of V. More precisely,

   Suppose that $\mathcal{B} = \{\alpha_1, ..., \alpha_n\}$ is a basis for V and suppose that $T, U : V \to W$ are both linear transformations (and they agree on a basis), such that

$$T(\alpha_1) = U(\alpha_1), T(\alpha_2) = U(\alpha_2), ..., T(\alpha_n) = U(\alpha_n)$$

   Then $T = U$

2. For **any map** $T_0 : \mathcal{B} \to W$, there s a unique linear transformation $T : V \to W$ with $T \supseteq T_0$. In other words,

   Let $\mathcal{B} = \{\alpha_1, ..., \alpha_n\}$ be **any basis** for V and let $\beta_1, ..., \beta_n$ be **any vectors** in W.

   Then, there is a **unique** linear transformation $T : V \to W$ such that

$$T(\alpha_1) = \beta_1, T(\alpha_2) = \beta_2, ..., T(\alpha_n) = \beta_n$$

Proof

1. **Uniqueness**: Chose any $\alpha \in V$, since $\mathcal{B}$ is a basis,

   $\langle$ Will show that $T = U \Leftrightarrow$ For any $\alpha \in V$, $T(\alpha) = U(\alpha)$ $\rangle$

$$\alpha = c_1\alpha_1 + \cdots + c_n\alpha_n$$

   for some **unique** $c_1, ..., c_n \in F$.

   Since T is a linear transformation,

$$T(\alpha) = c_1 T(\alpha_1) + \cdots + c_n T(\alpha_n)$$

   Likewise with U,

$$U(\alpha) = c_1 U(\alpha_1) + \cdots + c_n U(\alpha_n)$$

   But, since $T(\alpha_1) = U(\alpha_1), ..., T(\alpha_n) = U(\alpha_n)$, $T(\alpha) = U(\alpha)$.

   $\langle$ Essentially, if T, U work the same for all α_i, then their sum will obviously be the same, and so they'll give the same result for the same α. $\rangle$

   Note that this theorem *still* works for infinite dimensional vector spaces.

2. **Existence**: Chose any $\alpha \in V$. $\langle$ We must define $T(\alpha)$ $\rangle$

   Since $\mathcal{B}$ is a basis, we can write

$$\alpha = c_1\alpha_1 + \cdots + c_n\alpha_n$$

   which is unique.

   Define

$$T(\alpha) := c_1\beta_1 + \cdots + c_n\beta_n \in W$$

   Check: Is T linear?

   Say $\gamma = d_1\alpha_1 + \cdots + d_n\alpha_n$, $\delta = e_1\alpha_1 + \cdots + e_n\alpha_n$.

   In V, we have that $\gamma + \delta = (d_1 + e_1)\alpha_1 + \cdots + (d_n + e_n)\alpha_n$.

   By our definition of T, we have

$$\begin{aligned} T(\gamma + \delta) =& (d_1 + e_1)\beta_1 + \cdots + (d_n + e_n)\beta_n \\ =& (d_1\beta_1 + \cdots + d_n\beta_n) + (e_1\beta_1 + \cdots + e_n\beta_n) \\ =& T(\gamma) + T(\delta) \end{aligned}$$

   Check: $T(c\gamma) = cT(\delta)$

   So such a tranformation T exists. Additionally by part (1), it is unique.

■

Let $T : V \to W$ be a linear transformation.

Definition

$\text{Range}(T) = \{T(\alpha) : \alpha \in V\} \subseteq W$ is the set of all vectors in W hit by T.

Fact: Range(T) is a **subspace** of W.

1. 0 is in it. This is because $T(0) = 0$, obviously.
2. **Combinations of α_i are in it**
   Say that $\beta_1, \beta_2 \in \text{Range}(T)$. $\langle$ must show that $\beta_1 + \beta_2 \in \text{Range}(T)$ $\rangle$
   Since $\beta_1 \in \text{Range}(T)$, there is some $\alpha_1 \in V$ such that

   $$T(\alpha_1) = \beta_1$$

   similarly for β_2. Now $T(\alpha_1 + \alpha_2) = T(\alpha_1) + T(\alpha_2) = \beta_1 + \beta_2$, since T is linear. So $T(\alpha_1 + \alpha_2) = \beta_1 + \beta_2$ so $\beta_1 + \beta_2 \in \text{Range}(T)$ $\langle$ since $\alpha_1, \alpha_2 \in V$ means that $\alpha_1 + \alpha_2 \in V$, because it's a vector space! $\rangle$
3. **Scaling Works:** Say $\beta \in \text{Range}(T)$, and $c \in F$. Chose $\alpha \in V$ such that $T(\alpha) = \beta$. Then $T(c\alpha) = cT(\beta)c\beta$, therefore $c\alpha \in \text{Range}(T)$.

In other books this space is also called the **image** of T.

Definition

The **Null Space** of $T : V \to W$ is the set

$$\text{Null}(T) = \{\alpha \in V | T(\alpha) = \mathbf{0}\}$$

$\langle$ In other words, this is the set of all vectors α in V that, after a transformation T is applied, go to $\mathbf{0}$. Note that $\mathbf{0}$ here is the zero of the vector space $W \subseteq V$. $\rangle$

This is also sometimes called the **Kernel** of T.

Theorem

Let $T : V \to W$ be a linear transformation. Null(T) is a subspace of V.

Proof

Let $\alpha, \beta \in \text{Null}(T)$ and $c \in F$. Then,

$$T(c\alpha + \beta) = cT(\alpha) + T(\beta) = c\mathbf{0} + \mathbf{0} = \mathbf{0} \Rightarrow c\alpha + \beta \in \text{Null}(T)$$

■

It's pretty easy to see from this (and it should make sense) that the Null Space for a transformation T is itself a vector space.

Definition

The **Nullity** of T is the dimension of the Null space of T.

Definition

The **Rank** of T is the dimension of Range(T). Is this is equal to the dimension of W, T is said to have **full rank**.

Note again that this comes back to our definition of W for our transformation T. Earlier, we saw that W was the *codomain* of T. If you think about how functions behave, this is like having a *surjective* function.

Example

Let $\mathcal{P}_2$ be the set of all polynomials of degree 2 or less over a field F. Then, we have $\dim(\mathcal{P}_2) = 3$.

Consider the linear transformation $D : \mathcal{P}_2 \to \mathcal{P}_2$, the differentiation operator. Then

$$\text{Range}(D) = \text{Span}(\{D(1), D(x), D(x^2)\}) = \text{Span}(\{1, 2x\}) \Rightarrow \text{Rank}(D) = 2$$

In other words, the Range of D is the Span of a basis of $\mathcal{P}_2$ (in this case $\{1, x, x^2\}$) after being evaluated through D, so $\{1, 2x\}$. So the rank of D here is 2.

For the Null Space of D, we have that

$$\text{Null}(D) = \{c \in F\} \Rightarrow \text{Nullity}(D) = 1$$

The Null Space is the set of all constant functions since those are the function that, on D, go to $\mathbf{0}$.

2.3 The Rank-Nullity Theorem

Rank-Nullity Theorem

Let V be a vector space with $\dim V = n$. Let $T : V \to W$.

$$\text{Rank}(T) + \text{Nullity}(T) = \dim V = n$$

Proof

First, choose $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ to be a basis for Null(T). This set is necessarily linearly independent in V. So, we can choose an additional $\{\alpha_{k+1}, \cdots, \alpha_n\}$ so that $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is a basis of V.

Certainly, $k \leq n$, since Null(T) is a subspace of V.

We claim $A = \{T(\alpha_{k+1}), \cdots, T(\alpha_n)\}$ is a basis for Range(T). From this we have our theorem.

Clearly, $A \subseteq \text{Range}(T)$. We also have, that since $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is a basis of V, $\{T(\alpha_i)\}$ spans Range(T).

However, $T(\alpha_1) = T(\alpha_2) = \cdots = T(\alpha_k) = \mathbf{0}$, since they are in the null space, and hence do not contribute to the span. Thus, A spans Range(V). Now we need only show A is linearly independent. We choose constants such that

$$c_{k+1}T(\alpha_{k+1}) + \cdots + c_nT(\alpha_n) = \mathbf{0}$$

Let

$$\alpha^* = c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n \in V$$

We then have

$$T(\alpha^*) = c_{k+1}T(\alpha_{k+1}) + \cdots + c_nT(\alpha_n) = \mathbf{0} \Rightarrow \alpha^* \in \text{Null}(T)$$

So, we then have that, since α^* is in the null space,

$$\alpha^* = d_1\alpha_1 + d_2\alpha_2 + \cdots + d_k\alpha_k = c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n$$

$$d_1\alpha_1 + d_2\alpha_2 + \cdots + d_k\alpha_k - c_{k+1}\alpha_{k+1} - \cdots - c_n\alpha_n = \mathbf{0} \in V$$

But since $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is a basis of V, all the constants are zero, and in particular all of the c_i are zero. So, $\{T(\alpha_{k+1}), \cdots, T(\alpha_n)\}$ is linearly independent and is thus a basis of Range(T).

■

Now that we have the rank-nullity theorem, we can analyze transformations and their matrices.

Definition

Let A be a matrix in $F^{m\times n}$.

The **Column Space** is the vector space spanned by the n columns of A. This is precisely Range(T_A).

The **Row Space** is the vector space spanned by the m rows of A.

Theorem

Let A be a matrix, that when row-reduced has n unknowns and r non-zero rows. Nullity$(T_A) = n - r$

Proof

This follows from the fact that elementary row operations preserve the row space, and that solving a linear system in r equations with n unknowns will have $n - r$ degrees of freedom.

TODO I guess I can believe this but some more info would be nice.

■

Note

Let A be a matrix. Then the following are equal

- The dimension of the row space of A
- The dimension of the column space of A
- The number of nonzero rows in the row-reduced form of A
- Rank(T_A)

This follows immediately from the above and the Rank-Nullity Theorem.

Mon. Feb 20 2023

Suppose that A is an $m \times n$ matrix. Now suppose that we row reduce A, let's call this matrix A^{rr}. Then we have that

$$\text{RowSpace}(A) = \text{RowSpace}(A^{rr})$$

And we know that $\text{Rank}(A)$ is the number of non-zero rows of A^{rr} which we call r.

Moreover, the solution set of the homogeneous system $A\mathbf{x} = \mathbf{0}$ has dimension $n - r$, where n is the number of columns subtract the number of redundant equations.

Now, we know that for a matrix A, there is an associated linear transformation $T_A : F^n \to F^m$.

Last time, we also saw that

1. $\text{Range}(T_A) = \text{ColSpace}(A)$,
2. $\text{Null}(T_A)$ is the solution set of $A\mathbf{x} = \mathbf{0}$.

Now we can put everything together. Recall the Rank-Nullity theorem, then we have that, for any linear transformation T_A,

1. $\text{Rank}(T_A) + \text{Nullity}(T_A) = \dim(F^n) = n$
2. $\text{Rank}(T_A) := \dim(\text{Range}(T_A))$
3. $\text{Nullity}(T_A) = \dim(\text{Null}(A)) = n - r$, which is exactly the dimension of the set of all solutions to the homogeneous.
4. Finally we have that

$$\begin{aligned} \text{Rank}(A) &= \dim(\text{RowSpace}(A)) = \dim(\text{ColSpace}(A)) \\ &= \dim(\text{RowSpace}(A^{rr})) \\ &= \text{Rank}(T_A) \\ &= r \end{aligned}$$

Recall also that $\text{Nullity}(T_A) = \dim(\text{Null}(T_A)) = n - r$.

Consider a matrix A where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Then

$$A^{rr} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Is the row reduced matrix.

A basis for the row space of A is

$$\{(1, 0, 1, 1), (0, 1, 1, 1/3)\}$$

but another is

$$\{(1, 2, 3, 4), (1, 0, 1, 1)\}$$

We have $T_A : \mathbb{R}^4 \to \mathbb{R}^3$, and $\text{Rank}(T_A) = 2$.

Basis for $\text{Range}(T_A)$ equals the basis for Col Space(A)

There are many more linear transformations than the ones given by a matrix, for instance the derivative or integrals.

Let $T : V \to W$ be a linear transformation. From here there are two questions we can now ask.

1. Is T onto?

   It is if and only if $\text{Range}(T) = W$. We saw this earlier. In terms of dimension, this means that $\text{Rank}(T) = \dim(W)$.

   Note that here, V, W must be **finite dimensional**.

2. Is T one to one?

   This requires some more work.

Theorem

$T : V \to W$ is one to one if and only if $\text{Null}(T) = \{\mathbf{0}\}$.

$\langle$ In other words, the Null space must only contain the zero vector. $\rangle$

Proof

Assume that T is one to one. We know that $T(\mathbf{0}_V) = \mathbf{0}_W$. Chose any $\alpha \in \text{Null}(T)$, then $T(\alpha) = \mathbf{0}_W$, by definition of being in the Null Space. Since T is one to one, α must equal $\mathbf{0}_V$.

Now assume that $\text{Null}(T)$ is just $\mathbf{0}_W$. To see that T is one to one, chose any $\alpha, \alpha' \in V$, with $T(\alpha) = T(\alpha')$. Then $T(\alpha - \alpha') = T(\alpha) - T(\alpha')$ by linearity, but then since $\alpha = \alpha'$, $T(\alpha - \alpha') = \mathbf{0}$ so $T(\alpha - \alpha')$ must be in the Null space of T, and since $\text{Null}(T) = \{\mathbf{0}\}$, and $\alpha - \alpha' = 0$, so $\alpha = \alpha'$ and thus T is one to one.

■

Definition

T is called **non-singular** if T is one to one.

This is just another term for something we already know.

Theorem

Now suppose that $T : V \to W$ is a linear transformation with $\dim(V) = \dim(W)$. Then T is one to one if and only if T is onto.

Proof

By the Rank-Nullity theorem from last time, we have that

$$\text{Rank}(T) + \text{Nullity}(T) = \dim(V)$$

Now, assume that T is one to one, then $\text{Nullity}(T) = 0$, but then $\text{Rank}(T) = \dim(V) = \dim(W)$.

Now conversely, assume that T is onto. Then

$$\mathrm{Rank}(T) = \dim(W) = \dim(V)$$

Therefore $\mathrm{Nullity}(T) = 0$, and so T is one to one.

■

We are now starting to get a pretty good understanding of linear transformations, but suppose that we now want to combine them.

2.4 Combining Linear Transformations

Say $T : V \to W$ and $U : W \to Y$ are linear transformations over F.

⟨ then $U \circ T : V \to Y$ is a function. ⟩

Check the following:

1. $U \circ T$ is a linear transformation.
   ⟨ You know how to do this, just check that they scale and add as we expect. ⟩
2. If both T and U are one to one, then the composition is also one to one.
3. If both T and U are onto, the composition is also onto.

Note

$T \circ U$ would **not** be a linear transformation, assuming that Y and V are not the same vector space.

⟨ Linear transformations don't commute nicely like that. ⟩

Let's now look at T again.

Definition

A linear transformation $T : V \to W$ is called **invertible** if there is a linear transformation $U : W \to V$ such that

1. $U \circ T : V \to V$ is the identity from V. In other words
$$U(T(\alpha)) = \alpha$$
   For any $\alpha \in V$.
2. $T \circ U : W \to W$
$$T(U(\alpha)) = \alpha$$
   For any $\alpha \in W$.

Note

It might be interesting for you to prove that, if one of the above applies, the other automatically applies as well.

If T is invertible, we call such a U T^{-1}, the inverse transformation of T.

> **Note**
>
> Inverse transformations are unique, if they exist.
>
> $\langle$ We didn't talk about this in class but it *has* to be true. $\rangle$

Proposition: If $T : V \to W$ is an *invertible* linear transformation if and only if T is both one to one and onto.

> **Note**
>
> If T has an inverse, then it must be the case that $\dim(V) = \dim(W)$.
>
> If this is surprising, just consider that this follows from the fact that T must be both one to one, and onto in order to have an inverse.