Fri. Feb 10 2023

1 Vector Spaces

Suppose that V is a finite dimensional vector space over F, with $\dim(V) = n$.

V may have *many different* bases, we know that they all have the same size n.

Say $\mathcal{B} = \{\alpha_1, ..., \alpha_n\}$ is a basis fix the ordering of $\mathcal{B}$.

Fix the ordering of $\mathcal{B}$.

Theorem

For any $\alpha \in V$, there is a unique n tuple $(x_1, ..., x_n) \in F^n$ such that

$$\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n$$

Proof

Existence is immediate, since $\mathcal{B}$ is a basis, thus $\mathcal{B}$ spans V.

Uniqueness

Say $\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n$ and $\alpha = y_1\alpha_1 + \cdots + y_n\alpha_n$.

Then we have that

$x_1\alpha_1 + \cdots + x_n\alpha_n - y_1\alpha_1 + \cdots + y_n\alpha_n = 0$, so $(x_1 - y_1)\alpha_1 + \cdots + (x_n - y_n)\alpha_n = 0$

But since $\{\alpha_1, ..., \alpha_n\}$ is linearly independent, all coefficients must be 0.

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What this means is that, for a vector space V, there is an associated mapping in F^n. Notice that we know nothing about the vectors α_i.

Check: The mapping $\alpha \mapsto [\alpha]_{\mathcal{B}} \in F^n$ satisfies

1. One to one-ness
2. Onto-ness
3. "Additive", for any $\alpha, \beta \in V$, if $\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n$ and $\beta = y_1\alpha_1 + \cdots + y_n\alpha_n$. Then

$$[\alpha + \beta]_{\mathcal{B}} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = [\alpha]_{\mathcal{B}} + [\beta]_{\mathcal{B}}$$

4. $[c\alpha]_{\mathcal{B}} + c[\alpha]_{\mathcal{B}}$

There exists an *isomorphism* between V and F^n.

Example

Let $\mathcal{P}$ be the space of al polynomials. Let $f(x) = x^3$, and $g(x) = x^5$. Then, let

$$V = \text{Span}\{f, g\} = \{\text{all } ax^3 + bx^5 : a, b \in F\}$$

then, $\dim(V) = 2$, since f and g are linearly independent.

Typical $h(x) \in V$, say $h(x) = 10x^3 - 2x^5$.

$$[h]_{\mathcal{B}} = \begin{bmatrix} 10 \\ -2 \end{bmatrix}$$

$\langle$ $[h]_{\mathcal{B}}$ is the mapping of h to F^n. **TODO** is this right? $\rangle$

Now let $k(x) = 2x^3 + 4x^5$ and $l(x) = x^3 + 3x^5$. Since k, l are linearly independent, they form another basis of V.

$$\mathcal{B}' = \{k(x), l(x)\}$$

1.1 Change of Basis

Given $\mathcal{B} = \{\alpha_1, ..., \alpha_n\}$, and $\mathcal{B}' = \{\alpha_1', ..., \alpha_n'\}$ bases for V.

We want to describe the map going from $[\alpha]_{\mathcal{B}} \mapsto [\alpha]_{\mathcal{B}'}$.

$\langle$ We want to find The $\mathcal{B}$ coordinate of α $\mapsto$ the $\mathcal{B}'$ coordinate of α $\rangle$

Step 1.

Compute the $\mathcal{B}$ coordinate of $\alpha_1', ..., \alpha_n'$, *old* coordinates of the *new* basis elements.

Step 2.

For an $n \times m$ matrix

$$P = \Big[[\alpha_1']_{\mathcal{B}}, ..., [\alpha_n']_{\mathcal{B}}\Big]$$

Check: for any $\alpha \in V$

$$[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$$

Ans: This is what we actually want

$$[\alpha]_{\mathcal{B}'} = P^{-1}[\alpha]_{\mathcal{B}}$$