This document compiles all homework problems from MATH405 in Spring2023. It should be used as practice, but it's important for the reader to understand the solutions.

The reader is encouraged to attempt the problems before looking at the solutions, obviously.

Contents

1 Homework 1

1.1 Problem 1.3.2

If

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}$$

find all solutions to $AX = 0$ by row-reducing A.

SOLUTION

$$\begin{aligned} A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -1 \\ 1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -1 \\ 0 & -1 & -1 \end{bmatrix} \\ \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 6 & 0 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Since $A \sim I$, the only vector $\mathbf{x}$ that maps to $\mathbf{0}$ is the $\mathbf{0}$ vector itself.

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1.2 Problem 1.3.3

If

$$A = \begin{bmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

find all solutions to $A\mathbf{x} = 2\mathbf{x}$ and all solutions to $A\mathbf{x} = 3\mathbf{x}$.

SOLUTION

Let $\mathbf{x}$ be such that $A\mathbf{x} = 3\mathbf{x}$, then we have that $A\mathbf{x} - 3\mathbf{x} = 0$ and so $(A - 3I)\mathbf{x} = 0$. Solving, we get

$$A - 3I = \begin{bmatrix} 6-3 & -4 & 0 \\ 4 & -2-3 & 0 \\ -1 & 0 & 3-3 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 0 \\ 4 & -5 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 3 & -4 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so we get

$$(A - 3I)\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So we get that $x_1 = x_2 = 0$, and x_3 is free. In other words, $\mathbf{x}$ is of the form

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$$

for $c \in \mathbb{R}$.

Similarly for $A\mathbf{x} = 2\mathbf{x}$,

$$A - 2I = \begin{bmatrix} 6-2 & -4 & 0 \\ 4 & -2-2 & 0 \\ -1 & 0 & 3-2 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 0 \\ 4 & -4 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So we get

$$(A - 2I)\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so we get that $x_1 - x_2 = 0$, and $-x_1 + x_3 = 0$, which entails that $x_1 = x_2 = x_3$. In other words, $\mathbf{x}$ is of the form

$$\mathbf{x} = \begin{bmatrix} c \\ c \\ c \end{bmatrix}$$

for $c \in \mathbb{R}$.

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1.3 Problem 1.5.3

Find two different 2×2 matrices A such that $A^2 = 0$ and $A \neq 0$.

SOLUTION
Consider the matrices

$$A = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}$$

for $c \in \mathbb{R}$. Then,

$$\begin{aligned} A^2 &= \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

by matrix multiplication.

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1.4 Problem 1.5.6

Discover whether

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

is invertible, and find A^{-1} if it exists.

SOLUTION

$$A = \begin{bmatrix}1&2&3&4\\0&2&3&4\\0&0&3&4\\0&0&0&4\end{bmatrix} \sim \begin{bmatrix}1&0&0&0\\0&2&3&4\\0&0&3&4\\0&0&0&4\end{bmatrix} \sim \begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&3&4\\0&0&0&4\end{bmatrix} \sim \begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&3&0\\0&0&0&4\end{bmatrix} \sim \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Since $A \sim I$, A is invertible,

$$\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\0&2&3&4&0&1&0&0\\0&0&3&4&0&0&1&0\\0&0&0&4&0&0&0&1\end{array}\right] \sim \left[\begin{array}{cccc|cccc}1&0&0&0&1&-1&0&0\\0&2&3&4&0&1&0&0\\0&0&3&4&0&0&1&0\\0&0&0&4&0&0&0&1\end{array}\right]$$
$$\sim \left[\begin{array}{cccc|cccc}1&0&0&0&1&-1&0&0\\0&2&0&0&0&1&-1&0\\0&0&3&4&0&0&1&0\\0&0&0&4&0&0&0&1\end{array}\right] \sim \left[\begin{array}{cccc|cccc}1&0&0&0&1&-1&0&0\\0&2&0&0&0&1&-1&0\\0&0&3&0&0&0&1&-1\\0&0&0&4&0&0&0&1\end{array}\right]$$
$$\sim \left[\begin{array}{cccc|cccc}1&0&0&0&1&-1&0&0\\0&1&0&0&0&\frac{1}{2}&-\frac{1}{2}&0\\0&0&1&0&0&0&\frac{1}{3}&-\frac{1}{3}\\0&0&0&1&0&0&0&\frac{1}{4}\end{array}\right]$$

Thus we have

$$A^{-1} = \begin{bmatrix}1&-1&0&0\\0&1/2&-1/2&0\\0&0&1/3&-1/3\\0&0&0&1/4\end{bmatrix}$$

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1.5 Problem 2.1.4

Let V be the set of all pairs (x, y) of real numbers, and F be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$$
$$c(x, y) = (cx, y)$$

Is V, with these operations, a vector space over the field of real numbers?

SOLUTION
Let $c = 0, v = (x_1, y_1)$, and $w = (x_2, y_2)$ with $y_1 \neq y_2$. Then $c\mathbf{v} = (cx_1, y_1) = (0, y_1) = \mathbf{0}$, and $c\mathbf{w} = (cx_2, y_2) = (0, y_2) = \mathbf{0}$. But then $\mathbf{0}$ is not unique, so this is not a vector space.

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1.6 Problem 2.1.5

On R^n, define two operations

$$\begin{aligned} \alpha \oplus \beta =& \alpha - \beta \\ c \cdot \alpha =& - c\alpha \end{aligned}$$

The operations on the right are usual ones. Which of the axioms for a vector space are satisfied by $(R^n, \oplus, \cdot)$?

SOLUTION
Let $\alpha, \beta, \gamma \in V$. Now consider

$$\begin{aligned} &\alpha \oplus (\beta \oplus \gamma) \\ =&\alpha - (\beta - \gamma) \\ =&\alpha - \beta + \gamma \end{aligned}$$

and

$$\begin{aligned} &(\alpha \oplus \beta) \oplus \gamma \\ =&(\alpha - \beta) - \gamma \\ =&\alpha - \beta - \gamma \end{aligned}$$

So V is not associative, and thus it is not a vector space.

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1.7 Problem 2.1.7

Let V be the set of all pairs (x, y) of real numbers, and F be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, 0)$$
$$c(x, y) = (cx, 0)$$

Is V, with these operations, a vector space over the field of real numbers?

SOLUTION
No. Consider the vector $\mathbf{v} = (x, y)$, $\mathbf{w_1} = (0, c_1)$ and $\mathbf{w_2} = (0, c_2)$ for $c_1, c_2 \in F$. Then, by definition of $+$ in V, $\mathbf{v} + \mathbf{w_1} = (x + 0, 0) = \mathbf{v} + \mathbf{w_2}$. Thus $\mathbf{0}$ is not unique, and so V is not a vector space.

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1.8 Problem 2.2.1

Which of the following sets of vectors $\alpha = (a_1, ..., a_n)$ in R^n are subspaces of R^n $(n \geq 3)$?

a) all α such that $a_1 \geq 0$

b) all α such that $a_1 + 3a_2 = a_3$

c) all α such that $a_2 = a_1^2$

d) all α such that $a_1 a_2 = 0$

e) all α such that a_2 is rational

SOLUTION
Let W be the subspace.

a) W is not a subspace. Let $c \in R$ with $c < 0$ and α be such that $a_1 > 0$. Then $c\alpha = (ca_1, ..., ca_n)$ but since $a_1 > 0$ and $c < 0$, $ca_1 < 0$ and thus $c\alpha \notin \{\alpha\}$ but then $\{\alpha\}$ is not closed under scalar multiplication, and so it is not a subspace of R^n.

b) W is a subspace.

(a) $0 \in W$

The $\mathbf{0}$ vector is produced by letting all $a_i = 0$, then $a_3 = 0 + 3(0) = 0 \in W$.

(b) W is closed under addition

Let $\alpha_1 = (a_{11}, a_{12}, ..., a_{1n}), \alpha_2 = (a_{21}, a_{22}, ..., a_{2n}) \in V$, then $\alpha_3 = \alpha_1 + \alpha_2 = (a_{11} + a_{21}, a_{12} + a_{22}, a_{13} + a_{23})$ But $a_{33} = a_{13} + a_{23} = (a_{11} + a_{21}) + 3(a_{12} + a_{22})$ so $\alpha_3 \in W$.

(c) W is closed under scalar multiplication

Let $\alpha = (a_1, a_2, a_3, ..., a_n)$ with $a_3 = a_1 + 3a_2$. Let $c \in R$, then $c\alpha = (ca_1, ca_2, ca_3, ..., ca_n)$ but $ca_3 = c(a_1 + 3a_2) = ca_1 + 3ca_2$ so $c\alpha \in W$

c) W is not a subspace. Recall that squaring is not a linear operation.

d) W is not a subspace. Consider $\alpha = (0,1), \beta = (1,0)$. Then $\alpha + \beta = (1,1) \notin W$ so W is not closed under addition.

e) This is not a subspace. Let $c \in R$ be irrational. Then $c\alpha = (ca_1, ca_2, ..., ca_n)$, but since a_2 is rational and c is irrational, ca_2 is irrational and so $c\alpha \notin \{\alpha\}$. Once again, $\{\alpha\}$ is not closed under scalar multiplication and so it is not a subspace of R^n.

■

1.9 Problem 2.2.2

Let V be the real vector space of all functions f from R into R. Which of the following sets of functions are subspaces of V?

a) all f such that $f(x^2) = f(x)^2$

b) all f such that $f(0) = f(1)$

c) all f such that $f(3) = 1 + f(5)$

d) all f such that $f(-1) = 0$

e) all f which are continuous

SOLUTION

a) Not closed under addition. Consider $f(x) = g(x) = x$, then $(f + g)(x) = f(x) + g(x) = 2x$.

b) This is a subspace.

c) This is not a subspace because it does not contain the $\mathbf{0}$ vector. For instance, suppose that $f(5) = 0$, then $f(3) = 1$. Inversely, if $f(3) = 0$, then $f(5) = -1$, thus it is not a vector space.

d) This is a subspace because f at -1 will scale and add, so it is closed under addition and scalar multiplication. Additionally, the zero function is in f.

e) This is fine. The 0 function is continuous, and the sum of two continuous functions is continuous. Continuous functions can be scaled and remain continuous, thus this is a subspace.

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2 Homework 2

2.1 Problem 1

Suppose that W_1 and W_2 are both subspaces of a vector space V. Let

$$W_1 + W_2 := \{\alpha + \beta : \alpha \in W_1 \text{ and } \beta \in W_2\}$$

and let $W_1 \cap W_2 = \{\alpha \in V : \alpha \in W_1 \text{ and } \alpha \in W_2\}$.

1. Prove that $W_1 + W_2$ is a subspace of V.
2. Prove that $W_1 \cap W_2$ is a subspace of V.
3. Prove that $W_1 \cap W_2$ is a subset (and hence a subspace) of $W_1 + W_2$.

SOLUTION

1. Let α^*, β^* be arbirary vectors in $W_1 + W_2$, with $\alpha^* = \alpha_1 + \beta_1$ and $\beta^* = \alpha_2 + \beta_2$, $\alpha_1, \alpha_2 \in W_1$, and $\beta_1, \beta_2 \in W_2$. Finally, let c be arbitrary in F. Then

$$\begin{aligned} c\alpha^* + \beta^* =& c(\alpha_1 + \beta_1) + (\alpha_2 + \beta_2) \\ =& c\alpha_1 + c\beta_1 + \alpha_2 + \beta_2 \\ =& c\alpha_1 + \alpha_2 + c\beta_1 + \beta_2 \\ =& (c\alpha_1 + \alpha_2) + (c\beta_1 + \beta_2) \end{aligned}$$

but $(c\alpha_1 + \alpha_2) \in W_1$ since W_1 is a subspace and therefore closed. Similarly, $(c\beta_1 + \beta_2) \in W_2$. Thus $(c\alpha_1 + \alpha_2) + (c\beta_1 + \beta_2) \in W_1 + W_2$ and so it is closed under scalar multiplication and vector addition, and so it is a subspace.

2. Let $W = W_1 \cap W_2$. Now let $\alpha, \beta \in W$, and $c \in F$, then by defintion of W, $\alpha, \beta \in W_1, W_2$, but then, $c\alpha + \beta \in W_1$ since W_1 is a vector space, and $c\alpha + \beta \in W_2$ for the same reason. Then $c\alpha + \beta \in W$ so W is closed under scalar multiplication and vector addition, and so it is a subspace.
3. Let $W = W_1 \cap W_2$. Now let $\alpha \in W$, then by definition of W, $\alpha \in W_1$ and $\alpha \in W_2$. Since W_2 is a subspace, $\mathbf{0} \in W_2$, then $\alpha + \mathbf{0} \in W_1 + W_2$, so W is a subset of $W_1 + W_2$.

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2.2 Problem 2.2.7

Let W_1 and W_2 be subspaces of a vector space V such that the set-theoretic union of W_1 and W_2 is also a subspace. Prove that one of the spaces W_i is contained in the other.

SOLUTION

Suppose for the sake of contradiction that neither W_1 nor W_2 is a subset of the other. Let $\alpha \in W_1$, and $\beta \in W_2$ be arbitrary. Then, $\alpha + \beta \in W_1 \cup W_2$, and WLOG, suppose that $\alpha + \beta \in W_1$. Since $\alpha \in W_1$, $(-\alpha) \in W_1$ so $\alpha + (-\alpha) + \beta \in W_1$. Then $\beta \in W_1$, but this is a contradiction, so WLOG, W_1 is contained in W_2.

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2.3 Problem 2.2.8

Let V be the vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$. Let V_e be the subset of even functions $f(-x) = f(x)$. Let V_o be the subset of odd functions $f(-x) = -f(x)$.

1. Prove that V_e and V_o are subspaces.
2. Prove that $V_e + V_o = V$.
3. Prove that $V_e \cap V_o = 0$.

SOLUTION

1. Let $f_1, f_2 \in V_e$ be arbitrary, and $c \in \mathbb{R}$, then

$$\begin{aligned}&(cf_1 + f_2)(-x)\\=&cf_1(-x) + f_2(-x)\\=&cf_1(x) + f_2(x)\\=&(cf_1 + f_2)(x)\end{aligned}$$

so $cf_1(x) + f_2(x) \in V_e$ and therefore V_e is a subspace.

Similarly for V_o, if $g_1, g_2 \in V_o$, then

$$\begin{aligned}
&(cg_1 + g_2)(-x) \\
=&cg_1(-x) + g_2(-x) \\
=&-cg_1(x) - g_2(x) \\
=&-(cg_1(x) + g_2(x)) \\
=&-(cg_1 + g_2)(x)
\end{aligned}$$

$cg_1(x) + g_2(x) \in V_o$ since odd functions are closed under addition and scalar multiplication.

2. Let f be an arbirary function in V, now consider the two functions

$$g(x) = \frac{f(x) + f(-x)}{2}$$
$$h(x) = \frac{f(x) - f(-x)}{2}$$

Notice,

$$g(x) = \frac{f(x) + f(-x)}{2} = \frac{f(-x) + f(x)}{2}$$

so g is even, and thus $g \in V_e$. Additionally,

$$h(x) = \frac{f(x) - f(-x)}{2} = \frac{-f(-x) + f(x)}{2} = \frac{-(f(-x) - f(x))}{2}$$

so h is odd, and thus $h \in V_o$. Finally, we can see that

$$g(x) + h(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = \frac{f(x) + \cancel{f(-x)} + f(x) \cancel{- f(-x)}}{2} = f(x)$$

Thus we can see that $V_e + V_o = V$.

3. Let $f \in V_e$ be an arbitrary even function that isn't $\mathbf{0}$, and let $x \in \mathbb{R}$ be arbitrary. Then by definition of even functions, $f(-x) = -f(x) \neq f(x)$, so f cannot be odd. Therefore, $f \notin V_o$. Thus the only element of $V_e \cap V_o = \mathbf{0}$.

■

2.4 Problem 2.2.9

Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$, and $W_1 \cap W_2 = \{0\}$. Prove that for each vector α in V, there are *unique* vectors α_1 in W_1 and α_2 in W_2 such that $\alpha = \alpha_1 + \alpha_2$.

SOLUTION

Suppose for the sake of contradiction that there is some $\beta_1 \in W_1$ and $\beta_2 \in W_2$. Then we have

$$\begin{aligned}\alpha =&\alpha_1 + \alpha_2 = \beta_1 + \beta_2 \\ =&(\alpha_1 - \beta_1) = (\beta_2 - \alpha_2)\end{aligned}$$

but $(\alpha_1 - \beta_1) \in W_1$ since it is closed, and $(\beta_2 - \alpha_2) \in W_2$ since it is also closed. But since $(\alpha_1 - \beta_1) = (\beta_2 - \alpha_2)$, $(\alpha_1 - \beta_1) \in W_1 \cap W_2$ and $(\beta_2 - \alpha_2) \in W_1 \cap W_2$ which is a contradiction.

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2.5 Problem 2.3.1

Prove that if two vectors are linearly dependent, one of them is a scalar multiple of the other.

SOLUTION

Suppose that $v, w \in V$ are linearly dependent. Then, by definition,

$$c_1 v + c_2 w = 0$$

For $c_1, c_2 \in F$ and, WLOG, assume that $c_1 \neq 0$. Then

$$\begin{aligned}c_1 v + c_2 w &= 0 \\ c_1 v &= -c_2 w \\ v &= -\frac{c_2}{c_1} w\end{aligned}$$

So w is a scalar multiple of v.

■

2.6 Problem 2.3.5

Fund three vectors in $\mathbb{R}^3$ which are linearly dependent, and are such that any two of them are linearly independent.

SOLUTION

$$\begin{bmatrix}1\\0\\1\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}1\\1\\1\end{bmatrix}$$

Work.

■

2.7 Problem 2.3.6

Let V be the vector space of all 2×2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V which has four elements.

SOLUTION
Consider the following basis

$$A = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \; B = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} \; C = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} \; D = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$\{A, B, C, D\}$ are observably linearly independent.

■

2.8 Problem 2.3.7

Let V be the vector space of Exercise 6. Let W_1 be the set of matrices of the form

$$\begin{bmatrix}x & -x\\y & x\end{bmatrix}$$

And let W_2 be the set of matrices of the form

$$\begin{bmatrix}a & b\\-a & c\end{bmatrix}$$

1. Prove that W_1 and W_2 are subspaces of V.
2. Find the dimensions of $W_1, W_2, W_1 + W_2$, and $W_1 \cap W_2$.

SOLUTION

1. Let $M_1, M_2 \in W_1$, and $c \in F$. Then

$$\begin{aligned} & cM_1 + M_2 \\ =& c\begin{bmatrix} x_1 & -x_1 \\ y_1 & x_1 \end{bmatrix} + \begin{bmatrix} x_2 & -x_2 \\ y_2 & x_2 \end{bmatrix} \\ =& \begin{bmatrix} cx_1 + x_2 & -(cx_1 + x_2) \\ cy_1 + y_2 & cx_1 + x_2 \end{bmatrix} \end{aligned}$$

   So $(cM_1 + M_2) \in W_1$ and thus W_1 is a subspace of W. Similarly for W_2, let $N_1, N_2 \in W_2$, and $c \in F$. Then

$$\begin{aligned} & cN_1 + N_2 \\ =& c\begin{bmatrix} a_1 & b_1 \\ -a_1 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ -a_2 & c_2 \end{bmatrix} \\ =& \begin{bmatrix} (ca_1 + a_2) & cb_1 + b_2 \\ -(ca_1 + a_2) & cc_1 + c_2 \end{bmatrix} \end{aligned}$$

   So $(cN_1 + N_2) \in W_2$ and thus W_2 is a subspace of W.
2. (a) $\dim(W_1) = 2$
   (b) $\dim(W_2) = 3$
   (c) $\dim(W_1 + W_2) = 4$
   (d) We know that $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$, so $\dim(W_1 \cap W_2) = 1$.

■

2.9 Problem 2.3.10

Let V be a vector space over the field F. Suppose that there are a finite number of vectors $\alpha_1, ..., \alpha_r$ in V which span V. Prove that V is finite-dimensional.

SOLUTION
If there are a finite number of vectors that spam V, then a finite basis can be constructed for V, and so V is finite-dimensional.

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2.10 Problem 2.3.14

Let V be the set of real numbers. Regard V as a vector space over the field of *rational* numbers, with the usual operations. Prove that this vector space is *not* finite dimensional.

SOLUTION
Let $B = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a linearly independent set with $\alpha_i \in V$. Then $\text{Span}(B)$ is in bijection with $\mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}$. Since the finite cartesian product of countable sets is countable, $|\text{Span}(B)|$ is countable. However since $\mathbb{R}$ is uncountable, B cannot span $\mathbb{R}$.

■

3 Homework 3

3.1 Problem 2.4.1

Show that the vectors

$$\alpha_1 = (1,1,0,0), \qquad \alpha_2 = (0,0,1,1),$$
$$\alpha_3 = (1,0,0,4), \qquad \alpha_4 = (0,0,0,2),$$

Form a basis for $\mathbb{R}^4$. Find the coordinates of each of the standard basis vectors in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

SOLUTION
$(1,0,0,0) = \alpha_3 - 2\alpha_4$, $(0,1,0,0) = \alpha_1 - \alpha_3 + 2\alpha_4$, $(0,0,1,0) = \alpha_2 - \alpha_4/2$, and finally $(0,0,0,1) = \frac{1}{2}\alpha_4$.

Since the standard basis of $\mathbb{R}^4$ can be expressed from $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, they must form a basis of $\mathbb{R}^4$.

■

3.2 Problem 2.4.2

Find the coordinate matrix of the vector $(1,0,1)$ in the basis of $\mathbb{C}^3$ consisting of the vectors

$$(2i,1,0) \qquad (2,-1,1) \qquad (0,1+i,1-i)$$

In that order.

SOLUTION
We have

$$\begin{aligned}
&\begin{bmatrix} 2i & 2 & 0 \\ 1 & -1 & i+1 \\ 0 & 1 & 1-i \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\
&= \left(\frac{1+i}{4}\right) \begin{bmatrix} -2i & -2-2i & -2+2i \\ -1-i & -2+2i & 2+2i \\ i & 2 & 2-2i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ \frac{i}{2} \\ \frac{3}{4} - \frac{i}{4} \end{bmatrix}
\end{aligned}$$

$\blacksquare$

3.3 Problem 2.4.4

Let W be the subspace of $\mathbb{C}^3$ spanned by $\alpha_1 = (1, 0, i)$ and $\alpha_2 = (1 + i, 1, -1)$.

(a) Show that α_1 and α_2 form a basis for W.

(b) Show that the vector $\beta_1 = (1, 1, 0)$ and $\beta_2 = (1, i, 1 + i)$ are in W and form another basis for W.

(c) What are the coordinates of α_1 and α_2 in the ordered basis $\{\beta_1, \beta_2\}$ for W?

SOLUTION

(a)

$$c_1 \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} + c_2 \begin{bmatrix} i+i \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + c_2 i \\ c_2 \\ c_1 i - c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $c_2 = 0$, but then $c_1 i - c_2 = 0$ implies that $c_1 i = 0$, so $c_1 = 0$. Therefore α_1, α_2 are linearly independent and form a basis.

(b) $\beta_1 = \alpha_2 - i\alpha_1$, and $\beta_2 = (2 - i)\alpha_1 + i\alpha_2$

Since β_1, and β_2 can be written as a non-trivial linear combination of α_1 and α_2, they themselves must be linearly independent, and thus form a basis.

(c) We want to solve

$$\begin{bmatrix} 1 & 1 \\ 1 & i \\ 0 & 1+i \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}$$

Here, $c_1 = \frac{1}{2} - \frac{i}{2}, c_2 = \frac{i}{2} + \frac{1}{2}$ work. And

$$\begin{bmatrix} 1 & 1 \\ 1 & i \\ 0 & 1+i \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1+i \\ 1 \\ -1 \end{bmatrix}$$

Here, $c_1 = \frac{3}{2} - \frac{i}{2}, c_2 = \frac{i}{2} - \frac{1}{2}$ work.

■

3.4 Problem 2.4.6

Let V be the vector space over the complex numbers of all functions from $\mathbb{R}$ into $\mathbb{C}$, i.e., the space of all complex-valued functions on the real line. Let $f_1(x) = 1, f_2(x) = e^{ix}, f_3(x) = e^{-ix}$.

(a) Prove that f_1, f_2, and f_3 are linearly independent.

(b) Let $g_1(x) = 1$, $g_2(x) = \cos x$, $g_3(x) = \sin x$. Find an invertible 3×3 matrix P such that

$$g_j = \sum_{i=1}^{3} P_{ij} f_i$$

SOLUTION

(a) consider

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$

For $x = 0, 1, \pi$, we have

$$\begin{aligned} c_1 + c_2 e^{i\cdot 0} + c_3 e^{-i\cdot 0} &= 0 \\ c_1 + c_2 e^{i} + c_3 e^{-i} &= 0 \\ c_1 + c_2 e^{i\pi} + c_3 e^{-i\pi} &= 0 \end{aligned}$$

Which is equivalent to

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i} & e^{-i} \\ 1 & e^{-i\pi} & e^{i\pi} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i} & e^{-i} \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

And we have that

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i} & e^{-i} \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i} & e^{-i} \\ 2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i} & e^{-i} \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ 0 & e^{i} & e^{-i} \\ 1 & 0 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 0 & 1 & 1 \\ 0 & e^{2i} & 1 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1-e^{2i} & 0 \\ 0 & e^{2i} & 1 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & e^{2i} & 1 \\ 0 & 1-e^{2i} & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & e^{2i} & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 0 & e^{2i} & 1 \\ 0 & e^{2i} & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & e^{2i} & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

So $\{f_1, f_2, f_3\}$ are linearly independent.

(b)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -i/2 \\ 0 & 1/2 & i/2 \end{bmatrix}$$

■

3.5 Problem 2.4.7

Let V be the real vector space of all polynomial functions from $\mathbb{R}$ into $\mathbb{R}$ of degree 2 or less, i.e., the space of all functions f of the form

$$f(x) = c_0 + c_1 x + c_2 x^2$$

Let t be a fixed real number and define

$$g_1(x) = 1 \qquad g_2(x) = x + t \qquad g_3(x) = (x+t)^2$$

Prove that $\mathcal{B} = \{g_1, g_2, g_3\}$ is a basis for V. If

$$f(x) = c_0 + c_1 x + c_2 x^2$$

What are the coordinates of f in this ordered basis $\mathcal{B}$?

SOLUTION

Let $\mathcal{B}' = \{1, x, x^2\}$ be the standard basis for all polynomials of degree 2 or less. Then $[g_1]_{\mathcal{B}'} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $[g_2]_{\mathcal{B}'} = \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$, and finally $[g_3]_{\mathcal{B}'} = \begin{bmatrix} t^2 \\ 2t \\ 1 \end{bmatrix}$. Now

$$P = \Big[[g_1]_{\mathcal{B}'}, [g_2]_{\mathcal{B}'}, [g_3]_{\mathcal{B}'}\Big] = \begin{bmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{bmatrix}$$

Since $P \sim I$, we see that all $[g_i]_{\mathcal{B}'}$ are linearly independent and so they form a basis.

To find the coordinates of $f(x)$ in $\mathcal{B}$, simply compute

$$P^{-1} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

■

3.6 Problem 3.1.1

Which of the following functions T from $\mathbb{R}^2$ into $\mathbb{R}^2$ are linear transformations?

(a) $T(x_1, x_2) = (1 + x_1, x_2)$

(b) $T(x_1, x_2) = (x_1, x_2)$

(c) $T(x_1, x_2) = (x_1^2, x_2)$

(d) $T(x_1, x_2) = (\sin x_1, x_2)$

(e) $T(x_1, x_2) = (x_1 - x_2, 0)$

SOLUTION

(a) Not linear. Consider $T(0) + T(0) = (2, 0) \neq 0$

(b) Linear. This is the identity transformation

(c) Not linear. $T(c\alpha + \beta) = ((c\alpha_1 + \beta_1)^2, \alpha_2 + \beta_2) \neq cT(\alpha) + t(\beta)$

(d) Not linear. $\sin(c_1 x_1 + x_2) \neq c_1 \sin x_1 + \sin x_2$

(e) Linear. $T(c\alpha + \beta) = (c\alpha_1 - \beta_1, 0) = (c\alpha_1, 0) + (\beta_1, 0) = cT(\alpha) + T(\beta)$

■

3.7 Problem 3.1.2

Find the range, rank, null space, and nullity for the zero transformation and the identity transformation on a finite-dimensional space V.

SOLUTION

Zero Transformation

1. Range: The 0 vector

2. Rank: 0
3. Null Space: V
4. Nullity: $\dim(V)$

Identity Transformation

1. Range: V
2. Rank: Full, $\dim(V)$
3. Null Space: The 0 vector
4. Nullity: 0. Only the 0 vector

■

3.8 Problem 3.1.3

Describe the range and the null space for differentiation transformation of Example 2. Do the same for the integration transformation of Example 5.

SOLUTION
Differentiation Transformation

1. Range: The set of all polynomials $\mathcal{P}$.
2. Null Space: The null space is all constant functions.

Integration Transformation

1. Range: The set of functions $g(x) = I(f)$ for which $g(0) = 0$, since by definition

$$I(f) = \int_0^x f(t)dt$$

   And $\int_0^0 f(t)dt = 0$ for any function f.
2. Null Space: The null space is only the function for which

$$\int_0^x f(t)dt$$

   For all x, which is the zero function.

■

3.9 Problem 3.1.4

Is there a linear transformation T from $\mathbb{R}^3$ into $\mathbb{R}^2$ such that $T(1,-1,1)=(1,0)$ and $T(1,1,1)=(0,1)$?

SOLUTION

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Works. Notice

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

■

3.10 Problem 3.1.7

Let F be a subfield of the complex numbers and let T be the function from F^3 into F^3 defined by

$$T(x_1,x_2,x_3)=(x_1-x_2+2x_3, 2x_1+x_2, -x_1-2x_2+2x_3)$$

(a) Verify that T is a linear transformation.

(b) If (a,b,c) is a vector in F^3, what are the conditions on $a,b,$ and c that the vector be in the range of T? What is the rank of T?

(c) If (a,b,c) is a vector in F^3, what are the conditions on $a,b,$ and c that (a,b,c) be in the null space of T? What is the nullity of T?

SOLUTION

(a) Let $\alpha,\beta\in V$ be arbitrary, and $c\in F$ be arbitrary. Then

$$
\begin{aligned}
T(c\alpha+\beta) &= \begin{bmatrix} (c\alpha_1+\beta_1)-(c\alpha_2+\beta_2)+2(c\alpha_3+\beta_3) \\ 2(c\alpha_1+\beta_1)+(c\alpha_2+\beta_2) \\ -(c\alpha_1+\beta_1)-2(c\alpha_2+\beta_2)+2(c\alpha_3+\beta_3) \end{bmatrix} \\
&= \begin{bmatrix} c\alpha_1+\beta_1-c\alpha_2-\beta_2+2c\alpha_3+2\beta_3 \\ 2c\alpha_1+2\beta_1+c\alpha_2+\beta_2 \\ -c\alpha_1-\beta_1-2c\alpha_2-2\beta_2+2c\alpha_3+2\beta_3 \end{bmatrix} \\
&= c\begin{bmatrix} \alpha_1-\alpha_2+2\alpha_3 \\ 2\alpha_1+\alpha_2 \\ -\alpha_1-2\alpha_2+2\alpha_3 \end{bmatrix} \begin{bmatrix} \beta_1-\beta_2+2\beta_3 \\ 2\beta_1+\beta_2 \\ -\beta_1-2\beta_2+2\beta_3 \end{bmatrix} \\
&= cT(\alpha)+T(\beta)
\end{aligned}
$$

So T is a linear transformation.

(b) We can express T as the following matrix

$$
\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

So the rank of T is 2.

(c) The conditions for (a, b, c) to be in the null space of T is the following

$$
\begin{aligned}
a &= b-2c \\
b &= -2a \\
c &= b+\frac{1}{2}a
\end{aligned}
$$

■

3.11 Problem 3.1.8

Describe explicitly a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ which has as its range the subspace spanned by $(1, 0, -1)$ and $(1, 2, 2)$.

SOLUTION

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ -1 & 2 & 0 \end{bmatrix}$$

Works, but any arragement of the columns of the matrix above will also work.

■

3.12 Problem 4

Suppose V, W are vector spaces over a field F and $T : V \to W$ is any linear transformation.

(a) Suppose that $\alpha_1, \alpha_2, \alpha_3 \in V$ and $5\alpha_1 + 2\alpha_2 + \alpha_3 = 0$. Prove that $\{T(\alpha_1), T(\alpha_2), T(\alpha_3)\}$ are linearly dependent.

(b) More generally, prove that if $\{\alpha_1, ..., \alpha_k\} \subseteq V$ is any linearly dependent set, then $\{T(\alpha_1), ..., T(\alpha_k)\}$ is linearly dependent in W.

(c) Prove that if $\{\beta_1, ..., \beta_k\} \subseteq W$ are linearly independent and $\alpha_1, ..., \alpha_k$ satisfy $T(\alpha_1) = \beta_1, ..., T(\alpha_k) = \beta_k$, then $\{\alpha_1, ..., \alpha_k\}$ is linearly independent in V.

(d) Is the following statement True or False? If $\dim(V) = \dim(W)$, then for any basis $\mathcal{B} = \{\alpha_1, ..., \alpha_k\}$ of V, the set $\{T(\alpha_1), ..., T(\alpha_k)\}$ is a basis for W. Explain your reasoning.

SOLUTION

(a) $\{T(\alpha_1), T(\alpha_2), T(\alpha_3)\}$ is linearly dependent because $\{\alpha_1, \alpha_2, \alpha_3\}$ are linearly dependent. Simply notice, for example, that α_3 can be produced from $-5\alpha_1 - 2\alpha_2$

$$5\alpha_1 + 2\alpha_2 + \alpha_3 = 0 \Leftrightarrow -5\alpha_1 - 2\alpha_2 = \alpha_3$$

Then, just apply T and get

$$T(5\alpha_1 + 2\alpha_2 + \alpha_3) = 5T(\alpha_1) + 2T(\alpha_2) + T(\alpha_3) = 0$$

By properties of T. But then

$$-5T(\alpha_1) - 2T(\alpha_2) = T(\alpha_3)$$

So $\{T(\alpha_1), T(\alpha_2), T(\alpha_3)\}$ are linearly dependent.

(b) This simply follows from the properties of T. Consider any linearly dependent set $\{\alpha_1, \ldots, \alpha_k\} \subseteq V$. Then by definition of linear dependence

$$c_1\alpha_1 + \cdots + c_k\alpha_k = 0$$

With $c_1, c_2, \ldots, c_k \in F$ not all 0. Now

$$T(c_1\alpha_1 + \cdots + c_k\alpha_k) = T(0) = c_1T(\alpha_1) + \cdots + c_kT(\alpha_k) = 0$$

By properties of T.

(c) This is the contrapositive of (b). For a proof of (b), see (b)

(d) **False**: Consider $T(\alpha) = \mathbf{0}$, then for $\alpha_1 \neq \alpha_2 \in V$, $T(\alpha_1) = T(\alpha_2) = \mathbf{0}$.

■