HOMOLOGY AND COHOMOLOGY EX.2

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Exercise 3.8

Let $f : \Delta^n \to \Delta^0$ be the unique map. $(\Delta^0)_k$ contains only 1 element for any $k \in \mathbb{N}$, namely $(\Delta^0)_k = \{*\} = \{(0 \cdots 0)\}$. Then there is a canonical inclusion $i : \Delta^0 \to \Delta^n$, sending $*$ to the k-simplex $(0 \cdots 0)$ in Δ^n for each k, and f sends everything to $*$.
Obviously,

$$\begin{aligned} f \circ i : \Delta^0 &\to \Delta^0 \\ * &\mapsto * \end{aligned}$$

then $f \circ i = \mathrm{id}_{\Delta^0}$. On the other hand,

$$\begin{aligned} (i \circ f)_k : (\Delta^n)_k &\to (\Delta^n)_k \\ \sigma &\mapsto (0 \cdots 0). \end{aligned}$$

Consider $H : \Delta^n \times \Delta^1 \to \Delta^n$ such that $H_k(\sigma, t)(i) = \sigma(i) \cdot t(i) : [k] \to [n]$. Then this induces an order-preserving map, since if $i \leq j$, then $\sigma(i) \leq \sigma(j)$ and $t(i) \leq t(j)$, therefore $\sigma(i) \cdot t(i) \leq \sigma(j) \cdot t(j)$.
Moreover, if $t = (1 \cdots 1)$, then $H_k(\sigma, (1 \cdots 1))(i) = \sigma(i)$ for each i, this is the same as σ. And if $t = (0 \cdots 0)$, $H_k(\sigma, 0 \cdots 0))(i) = 0$, which induces $(0 \cdots 0) \in (\Delta^n)_k$. Therefore, the following diagram commutes:

$$\begin{tikzcd}
\Delta^n \times \Delta^0 = \Delta^n \times (1 \cdots 1) \arrow[r, equal] \arrow[d, "\delta_1"] & \Delta^n \arrow[dr, "\mathrm{id}_{\Delta^n}"] & \\
\Delta^n \times \Delta^1 \arrow[rr, "H"] & & \Delta^n \\
\Delta^n \times \Delta^0 = \Delta^n \times (0 \cdots 0) \arrow[r, equal] \arrow[u, "\delta_0"'] & \Delta^n \arrow[ur, "i \circ f"] &
\end{tikzcd}$$

□

Exercise 3.10

Let $\varphi \in \text{Hom}(T, A)$, set $a_*(\varphi) = a \circ \varphi$. Need to show a_* is injective. If $a_*(\varphi)(x) = 0$ for every $x \in T$, this means $a(\varphi(x)) = 0 \forall\ x \in T$. However, since a is injective, this is true only when $\phi = 0$ on T. Therefore ker $a_* = \{0\}$, a_* is injective.

Similarly, for $\phi \in \text{Hom}(T, B)$, set $b_*(\phi) = b \circ \phi$. Then $b_* \circ a_*(\varphi)(x) = b \circ a \circ \varphi(x) = 0$, and ker $g_* \subset$ im f_*.

Now consider

$$\mathbb{Z} \to \mathbb{Z}/2 \to 0$$

which is an exact sequence. However,

$$\text{Hom}(\mathbb{Z}/2, \mathbb{Z}) = 0 \to \text{Hom}(\mathbb{Z}/2, \mathbb{Z}/2) \simeq \mathbb{Z}/2$$

is not surjective.

To that $\text{Hom}(T, -)$ is exact when T is a free abelian group with $T = \mathbb{Z}[S]$ for some set S, it remains to check b_* is surjective. Take $s \in S$. For every $\gamma \in \text{Hom}(T, C)$, since b is surjective, there exists $\varphi \in \text{Hom}(T, B)$ such that $b(\phi(s)) = \gamma(s)$, which induces $b_*\phi = \gamma$. Therefore b_* is surjective.

For $\otimes$, consider

$$0 \to A \otimes T \xrightarrow{a\otimes \text{id}} B \otimes T \xrightarrow{b\otimes \text{id}} C \otimes T \to 0$$

then for any $x \in A$, $t \in T$,

$$(b \otimes \text{id}) \circ (a \otimes \text{id})(x \otimes t) = ((b \circ a)(x)) \otimes t = 0 \otimes t = 0,$$

therefore it is a complex.

$b \otimes \text{id}$ is surjective, indeed, for any $z \in C$ and $t \in T$, since b is surjective, $\exists y \in B$ such that $b(y) = z$, therefore $(b \otimes \text{id})(y \otimes t) = z \otimes t$. By construction of $B \otimes T$ and $C \otimes T$, $b \otimes \text{id}$ is also surjective.

However, $a \otimes \text{id}$ is not necessarily injective. Consider

$$0 \to \mathbb{Z}/2 \xrightarrow{i} \mathbb{Z}/3$$

where i is injective. But $\mathbb{Z}/2 \otimes \mathbb{Z}/2 = \mathbb{Z}/2$, $\mathbb{Z}/3 \otimes \mathbb{Z}/2 = 0$, therefore

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \otimes \mathbb{Z}/2 \arrow[r, "i_*"] \arrow[d, equal] & \mathbb{Z}/3 \otimes \mathbb{Z}/2 \arrow[d, equal] \\
& \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

i_* is not injective.

Nevertheless, if T is a free abelian group, $T = \mathbb{Z}[S]$ for a set S, assume that for some $x_i \in A$, $s_i \in S$,

$$(a \otimes \mathrm{id})(\sum_{i=1}^{n} x_i \otimes s_i) = 0$$

this induces

$$0 = \sum_{i=1}^{n} a(x_i) \otimes s_i$$

therefore, for each i, $a(x_i) \otimes s_i = 0$. Either $s_i = 0$ or $a(x_i) = 0$. If $a(x_i) = 0$, since a is injective, this implies $x_i = 0$. Therefore

$$\sum_{i=1}^{n} a(x_i) \otimes s_i = \sum_{i=1}^{n} 0 = 0$$

hence $a \otimes \mathrm{id}$ is injective.

□

Exercise 4.5

Replacing C with $N(\mathbb{Z}[X])$ induces a short exact sequence

$$0 \to H_n(X)/l \to H_n(N(\mathbb{Z}(X))/l) \to (H_{n-1}(X))_l \to 0.$$

Since

$$N_n(\mathbb{Z}[X])/l = (\oplus_{x \in X_n^{\text{non-deg}}} \mathbb{Z})/l \simeq \oplus_{x \in X_n^{\text{non-deg}}} (\mathbb{Z}/l) \simeq N_n((\mathbb{Z}/l)[X])$$

and therefore $H_n(N(\mathbb{Z}(X))/l) = H_n(N(\mathbb{Z}/l)[X]) = H_n(X, \mathbb{Z}/l)$, and this sequence becomes

$$0 \to H_n(X)/l \to H_n(X, \mathbb{Z}/l) \to (H_{n-1}(X))_l \to 0.$$

Inserting $X = S^k$, one has $H_n(S^k)$ is $\mathbb{Z}$ for $n = k$ and 0 otherwise, and this sequence induces

$$0 \to \mathbb{Z}/l \to H_k(S^k, \mathbb{Z}/l) = \mathbb{Z}/l \to 0 \to 0.$$

Now for $X = P^2$, since

$$H_n(P^2, \Lambda) = \begin{cases} \Lambda & n = 0 \\ \Lambda/2 & n = 1 \\ \{x \in \Lambda : 2x = 0\} =: \Lambda' & n = 2 \\ 0 & \text{else} \end{cases}$$

and in particular $H_1(P^2) = \mathbb{Z}/2$, $H_2(P^2) = 0$, there are sequences

$$0 \to 0/l = 0 \to \{x \in \mathbb{Z}/l : 2x = 0\} \to (\mathbb{Z}/2)_l \to 0 \tag{1}$$

$$0 \to \mathbb{Z}/2 \to (\mathbb{Z}/l)/2 \to (\mathbb{Z}/l)/l = \mathbb{Z}/l \to 0 \tag{2}$$

These sequences can be further simplified; WLOG assume $l \geq 0$.
If $l = 0$, $\Lambda' = 0$, $(\mathbb{Z}/2)_l = \mathbb{Z}/2$, $(\mathbb{Z}/l)/2 = \mathbb{Z}/2$, therefore the sequences become

$$0 \to 0 \to 0 \to \mathbb{Z}/2 \to 0$$

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to \mathbb{Z} \to 0$$

If $l = 1$, then $\Lambda' = 0$, $(\mathbb{Z}/2)_l = 0$, $(\mathbb{Z}/l)/2 = 0$. Sequence (1) becomes trivial (with everything being 0), and (2) becomes

$$0 \to \mathbb{Z}/2 \to 0 \to 0 \to 0$$

If $l \geq 2$, $(\mathbb{Z}/l)/2 = \mathbb{Z}/2$. Moreover, if l is even, $\Lambda' = \mathbb{Z}/2$, $(\mathbb{Z}/2)_l = \mathbb{Z}/2$,

$$0 \to 0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0$$

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to \mathbb{Z}/l \to 0$$

If l is odd, then $\Lambda' = 0$, $(\mathbb{Z}/2)_l = 0$, then (1) is trivial since everything is 0, and (2) becomes

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to \mathbb{Z}/l \to 0.$$

$\square$

Exercise 4.6

(1) By definition, $N_n(A) = C_n(A)/D_n(A)$ for each n. Consider p as the quotient map. Then the sequence

$$0 \to D \xrightarrow{i} C \xrightarrow{p} N \to 0$$

is exact, since the inclusion $i_n : D_n(A) = A_n^{\text{deg}} \to C_n(A) = A_n$ is injective, and the quotient map $p_n : C_n(A) \to N_n(A) = C_n(A)/A_n^{\text{deg}}$ is surjective. By snake lemma, this induces a long exact sequence:

$$\cdots \to H_n(D) \xrightarrow{H_n(i)} H_n(C) \xrightarrow{H_n(p)} H_n(N) \to H_{n-1}(D) \to \cdots$$

If D is exact, then $H_n(D) = 0$ for every n. This sequence then becomes

$$\cdots \to 0 \to H_n(C) \xrightarrow{H_n(p)} H_n(N) \to 0 \to \cdots$$

This is equivalent to saying $H_n(p)$ is surjective and bijective for all n. Since it is linear, it is an isomorphism. Therefore, p is a quasi-isomorphism by definition.

(2) For C and C', (by arguments similar to (1)) there is a short exact sequence

$$0 \to C' \to C \to C/C' \to 0$$

and induces a long exact sequence by snake lemma:

$$\cdots \to H_{n+1}(C/C') \to H_n(C') \to H_n(C) \to H_n(C/C') \to H_{n-1}(C') \to H_{n-1}(C) \to \cdots$$

Since C' and C/C' are exact, this sequence can be simplified to

$$\cdots \to 0 \to H_n(C') \to 0 \to 0 \to H_{n-1}(C') \to 0 \to \cdots$$

since this sequence is exact, the only choice for $H_n(C')$ would be $H_n(C') = 0$ for all n. Therefore C' is exact.

(3) First check $(D^{(p)})_n \subset (D^{(p+1)})_n$ for all n and p:

For $n \leq p$, $(D^{(p+1)})_n = (D^{(p)})_n = D_n$, trivial.

For $n = p + 1$,

$$(D^{(p)})_{p+1} = D_{p+1} = A_{p+1}^{\text{deg}},$$

$$(D^{(p+1)})_{p+1} = \sigma_0(C_p) + \cdots + \sigma_{p+1}(C)_p = \sigma_0 A_p + \cdots + \sigma_{p+1} A_p.$$

Then this inclusion is obvious by definition.

For $n > p+1$,

$$(D^{(p)})_n = \sigma_0(C_{n-1}) + \cdots + \sigma_p(C_{n-1}),$$

$$(D^{(p+1)})_n = \sigma_0(C_{n-1}) + \cdots + \sigma_p(C_{n-1}) + \sigma_{p+1}(C_{n-1}),$$

then the inclusion is again obvious. Therefore $(D^{(p)})_n \subset (D^{(p+1)})_n$ for all n and p.

Second, check that $\partial\partial = 0$ for

$$(D^{(p)})_{n+2} \xrightarrow{\partial} (D^{(p)})_{n+1} \xrightarrow{\partial} (D^{(p)})_n.$$

This is immediate from $\partial\partial = 0$, inherited from C.

(4) Notice that $D_n = \sigma_0(C_{n-1}) + \cdots + \sigma_n(C_{n-1})$. Then for $n \geq p$,

$$(D^{(p)})_n/(D^{(p-1)})_n \simeq \sigma_p(C_{n-1}),$$

and for $n < p$,

$$(D^{(p)})_n/(D^{(p-1)})_n = D_n/D_n = 0.$$

Need to find h such the diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & \sigma_p(C_p) \arrow[r, "\partial"] \arrow[d, "\mathrm{id}"] \arrow[dl, "h_{p+1}"'] & \sigma_p(C_{p-1}) \arrow[r, "\partial"] \arrow[d, "\mathrm{id}"] \arrow[dl, "h_p"'] & 0 \arrow[r, "\partial"] \arrow[d, "\mathrm{id}"] \arrow[dl, "h_{p-1}"'] & 0 \arrow[r] \arrow[dl, "h_{p-2}"'] & \cdots \\
\cdots \sigma_p(C_{p+1}) \arrow[r, "\partial"] & \sigma_p(C_p) \arrow[r, "\partial"] & \sigma_p(C_{p-1}) \arrow[r, "\partial"] & 0 \arrow[r] & \cdots &
\end{tikzcd}$$

Let δ_k be the face maps on C_k. It suffices to find h_n for $n > p+2$. With simplicial identities and the quotient, one may observe that

$$\begin{aligned}
\partial\sigma_p(x) &= \sum_{k=0}^{n}(-1)^k \delta_k \sigma_p(x) \\
&= \sum_{k=0}^{p}(-1)^k \delta_k \sigma_p(x) + \sum_{k=p+2}^{n}(-1)^k \sigma_p \delta_{k-1}(x) \\
&= \sum_{k=p+2}^{n}(-1)^k \sigma_p \delta_{k-1}(x)
\end{aligned}$$

furthermore, for $n > p+2$,

$$\begin{aligned}
\partial\sigma_p(\sigma_p(x)) - \sigma_{p-1}\partial(\sigma_p(x)) &= \sum_{k=p+2}^{n+1}(-1)^k\sigma_p\delta_{i-1}\sigma_p(x) - \sigma_p\sum_{i=p+2}^{n}(-1)^k\sigma_p\delta_{k-1}(x) \\
&= \sum_{k=p+2}^{n+1}(-1)^k\sigma_p^2\delta_{k-2}(x) - \sum_{k=p+2}^{n}(-1)^k\sigma_p^2\delta_{k-1}(x) \\
&= \sum_{k=p+1}^{n}(-1)^k\sigma_p^2\delta_{k-1}(x) - \sum_{k=p+2}^{n}(-1)^k\sigma_p^2\delta_{k-1}(x) \\
&= (-1)^{p+1}\sigma_p^2\delta_p(x) \\
&= (-1)^{p+1}\sigma_p(x).
\end{aligned}$$

This induces

$$\begin{aligned}
\mathrm{id} &= (-1)^{p+1}\partial\sigma_p - (-1)^{p+1}\sigma_{p-1}\partial \\
&= \partial((-1)^{p+1}\sigma_p) + (-1)^p\sigma_{p-1}\partial
\end{aligned}$$

Therefore may take $h_n = (-1)^{p+1}\sigma_p$, and $(D^{(p)})_n/(D^{(p-1)})_n$ is null-homotopic via h_n.
(5) Since $(D^{(p)})_n/(D^{(p-1)})_n$ is null-homotopic, it is exact. Also notice that for $p=0$, $D^{(0)}$ is given by $(D^{(0)})_n = \sigma_0(C_{n-1})$, I guess it is exact too but I could not provide a valid proof. Then by (2) and induction on p, $D^(p)$ is exact for all p, and therefore D is exact too.

□

$$\to H_n(X\setminus\{x\}) \to H_n(X) \to H_n(X, X\setminus\{x\}) \xrightarrow{d} H_{n-1}(X\setminus\{x\}) \to$$

in this sequence,

$$H_n(X\setminus\{x\}) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} \text{ if } n=0 \\ 0 \text{ else} \end{cases} \quad \text{and } H_n(X) = \begin{cases} \mathbb{Z} \text{ if } n=0 \\ 0 \text{ else} \end{cases}$$

Since $X\setminus\{x\}$ is a union of two disjoint intervals and X is convex. The non-trivial part of the sequence is then

$$0 \to H_1(X, X\setminus\{x\}) \xrightarrow{d,\text{ inj}} \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z} \xrightarrow{\text{surj}} H_0(X, X\setminus\{x\}) \xrightarrow{d} 0$$