HOMOLOGY AND COHOMOLOGY EX.1

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Exercise 2.4

(a) This is immediate by Yoneda lemma:

$$N([n])_m = (N([n]))[m] = \mathrm{Hom}_{\mathrm{Cat}}([m],[n]) = (\Delta^n)_m.$$

(b) The 2-simplices of $N(C)$ are

$$N(C)_2 = \{g \circ f : X \to Z \mid f : X \to Y,\ g : Y \to Z,\ X,\ Y,\ Z \in \mathrm{Obj}(C)\}$$

(c) To show N is a functor, one needs to show that N (c.1) preserved identity, and (c.2) preserves composition.

Let C be a small category. The n-simplices of $N(C)$ are actually the composition of n morphisms between objects in C, i.e., of the form

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{x} \cdots \xrightarrow{f_n} X_n$$

denote for $(f_1, \dots, f_n)$ for simplicity.

(c.1) N preserves identity:

Let C be a small category, α be an n-simplex of $N(C)$, $\alpha = (f_1, \dots, f_n)$. Then for each $f_i,\ i = 1, \dots, n$, $N(\mathrm{id}_C)(f_i) = (f_i) \in N(C)_1$. Therefore, for α, one has

$$\begin{aligned} N(\mathrm{id}_C)(\alpha) &= N(\mathrm{id}_C)(f_1 \circ \cdots \circ f_n) \\ &= f_1 \circ \cdots \circ f_n = (f_1, \dots, f_n) = \alpha \in N(C)_n \end{aligned}$$

since $N(\mathrm{id}_C)(\alpha)$ and $\mathrm{id}_{N(C)}(\alpha)$ represents the same compositions of maps. Thus $N(\mathrm{id}_C) = \mathrm{id}_{N(C)}$.

(c.2) N preserves composition:

Let $C,\ D,\ E$ be small categories, and $F : C \to D$, $G : D \to E$ functors. Again, let $\alpha = (f_1, \dots, f_n)$ be an n-simplex of $N(C)$. Then $G \circ F : C \to E$.

For a morphism $f : X \to Y$ in C, let $F(f) := N(F)(f)$ (and resp. $G(g) := N(G)(g)$ for g morphism in D), this induces, for a morphism f in C:

1. $N(F)$ sends f to $F(F)$, and therefore $N(G) \circ N(F)$ sends f firstly to $F(f)$ then to $G(F(f))$ a simplex in $N(E)$;

2. $G \circ F$ is the composition of G and F, then $N(G \circ F) = N(G(F))$ sends f to a simplex in $N(E)$, which is $G(F(f))$;

therefore $N(G \circ F)$ and $N(G) \circ N(F)$ acts the same on a morphism f. Now since by assumption $F,\ G$ are functors, they preserve compositions, and one has

$$\begin{aligned}(N(G) \circ N(F))(\alpha) &= (N(G))(F(f_1 \circ \cdots f_n)) \\ &= G(F(f_1) \circ \cdots \circ F(f_n)) \\ &= (G \circ F)(f_1) \circ \cdots \circ (G \circ F)(f_n) \\ &= (G \circ F)(f_1 \circ \cdots \circ f_n) \\ &= N(G \circ F)(\alpha)\end{aligned}$$

Therefore $N(G \circ F) = N(G) \circ N(F)$ for every simplex in C.

Combining (c.1) and (c.2), one has that N is a functor.

(d) Need to show $N : \mathrm{Hom}_{\mathrm{Cat}}(C, D) \to \mathrm{Hom}_{\mathrm{sSet}}(N(C), N(D))$ is bijective.

(d.1) Injectivity:

Let $f,\ g$ be morphisms in C, then $N(f),\ N(g)$ are simplices in $N(C)$. Assume $N(f) = N(g)$, then they acts the same on vertices and edges of $N(C)$, which implies f and g acts the same on C by definition of N.

(d.2) Surjectivity:

Let $F : N(C) \to N(D)$ be a morphism between simplicial sets. This induces maps $F : N(C)_n \to N(D)_n$ for every $n \in \mathbb{N}$.

For $n = 0$, F maps $\mathrm{Obj}(C)$ to $\mathrm{Obj}(D)$. Let X be an object in C, then $F(C) \in \mathrm{Obj}(D)$.

For $n = 1$, consider $X,\ Y \in \mathrm{Obj}(C)$ with a morphism $f : X \to Y$. F induces a morphism $F(f) : F(X) \to F(Y)$ in D. Then $F(f) \in \mathrm{Hom}(N(C), N(D))$. One can define $G : X \to Y$ in this way:

1. for $X \in \mathrm{Obj}(C)$, $G(X) := F(X)$;

2. for morphism $f : X \to Y$ in C, $G(f) := F(f)$.

Claim: (1) G is a functor; (2) $G = N(F)$ for all $n \in \mathcal{N}$.

For (1), G obviously preserves identity. Let $X_0 \xrightarrow{f_1} X_1 \xrightarrow{x} X_2$ be in C. Then $f_1 \circ x$ is a 2-simplex of $N(C)_2$, which induces $G(X_0) \xrightarrow{G(f_1)} G(X_1) \xrightarrow{G(x)} G(X_2)$, and implies G preserves composition.

For (2), let α be an n-simplex of $N(C)_n$. For $n \leq 1$, $G(\alpha) = N(F)(\alpha)$ obviously holds. And by induction, we have that this also holds for $n \geq 2$.

Therefore $G \in \mathrm{Hom}(X, Y)$ and $G = N(F)$, and this proves that N is surjective. $\square$

Exercise 2.7

If $n = 0$, then $\partial\Delta^0 = \emptyset$, then this becomes trivial. Therefore consider $n \geq 1$ only. Let $\delta_k,\ \sigma_k$ be as usual. Let f be an m-simplex in $(\partial\Delta^n)_m$, i.e., $f : [m] \to [n],\ \mathrm{im}\ f \subsetneq [n]$. Define the degeneracy maps $d_k(f) := f \circ \delta_k : [m+1] \to [n]$. Then

$$\mathrm{im}\ d_k(f) = \mathrm{im}\ f \circ \delta_k \subseteq \mathrm{im}\ f \subsetneq [n],$$

therefore $d_k(f) \in (\partial\Delta^n)_{m+1}$.

Similarly, define the face maps $s_k(f) := f \circ \sigma_k : [m-1] \to [n]$, and also $s_k(f) \in (\partial\Delta^n)_{m-1}$.

Using cosimplicial identities, one can easily check the face maps and degeneracy maps in $\partial\Delta^n$ satisfies the simplicial identities:

For $i < j$,

$$d_i d_j(f) = f \circ \delta_j \circ \delta_i = f \circ \delta_i \circ \delta_{j-1} = d_{j-1} d_i(f).$$

For $i > j$,

$$s_i s_j(f) = f \circ \sigma_j \circ \sigma_i = f \circ \sigma_{i-1} \circ \sigma_j = s_j s_{i-1}(f).$$

And

$$d_i s_j(f) = f \circ \sigma_j \circ \delta_i = \begin{cases} f \circ \delta_i \circ \sigma_{j-1} = s_{j-1} d_i(f) & i < j; \\ \mathrm{id}(f) & i = j \text{ or } i = j+1; \\ f \circ \delta_{i-1} \circ \sigma_j = s_j d_{i-1}(f) & i > j+1. \end{cases}$$

Therefore the face maps and degeneracy maps satisfies the simplicial identities; thus $\partial\Delta^n$ is indeed a simplicila set. Especially, the vertices of $\partial\Delta^2$ are $\{0, 1, 2\}$, non-degenerate edges are $\{01, 02, 12\}$.

Now for $S^n := \Delta^n \sqcup_{\partial\Delta^n} \Delta^0$, consider the maps

$$p : \partial\Delta^n \to \Delta^0 = \{\star\} \qquad\qquad i : \partial\Delta^n \to \Delta^n$$
$$\alpha \mapsto \star \qquad\qquad \alpha \mapsto \alpha$$

The 0-simpices $(S^n)_0 = \{0, 1, \cdots, n\} \sqcup_{\{0,1,\cdots,n\}} \{\star\} \cong \{\star\}$ is obviously non-degenerate since there is no degeneracy map that maps to 0-simplices.
Notice that for $m < n$, $(\partial\Delta^n)_m = (\Delta^n)_m$, since an order-preserving $f : [m] \to [n]$ is never surjective. Thus one has

$$(S^n)_m = (\Delta^n)_m \sqcup_{(\partial\Delta^n)_m} (\Delta^0)_m = (\Delta^n)_m \sqcup_{(\Delta^n)_m} (\Delta^0)_m \cong \{\star\}.$$

Moreover, since $0 \cdots 0 \sqcup_{0\cdots 0} 0 \cdots 0 \in (S^n)_m$ is obviously degenerate, we have that every m-simplex is degenerate for $m < n$.
For $m = n$, notice that $(\Delta^n)_n = (\partial\Delta^n)_n \cup \{01 \cdots n\}$, therefore

$$(S^n)_n \cong \{01 \cdots n, 0 \cdots 0\} =: \{\gamma, \star\}.$$

$\star$ is degenerate by previous statement. Left to show γ is non-degenerate. Indeed, by construction, $\gamma = 01 \cdots n \sqcup_{(\partial\Delta^n)_n} 0 \cdots 0$. If γ is degenerate, this means that $01 \cdots n$ is degenerate in $(\Delta^n)_n$, which is not true. Therefore, γ is non-degenerate.
For $m > n$, one has

$$(S^n)_m \cong \{\star\} \cup ((\Delta^n)_m \setminus (\partial\Delta^n)_m).$$

By previous argument, $\star$ is degenerate. For an order-preserving surjective map $f : [m] \to [n]$, f must has duplication, therefore it is degenerate. Therefore, for $m > n$, every m-simplex is degenerate.
Summing up, S^n has only 2 non-degenerate simplices, namely $(0)_{S^1} = \star \in (S^n)_0$, which is in dimension 0; and $\gamma = (01 \cdots n)_{\partial\Delta^n} \in (S^n)_n$, which is in dimension n. $\square$

Exercise 2.8

(1) $(\Delta^1)_2 = \{000, 001, 011, 111\}$ has 4 elements. Therefore X_2 has $4 \times 4 = 16$ elements, namely, X has 16 2-simplices.

If $(f,g) \in X_2$ is degenerate, then by definition, there exists $a,\ b \in X_1$ such that

$$s^1_{k,X}(a,b) = (f,g).$$

By definition, $s^1_{k,X} = s_k \times s_k$, where $s_k : (\Delta^1)_1 \to (\Delta^1)_2$, i.e.,

$$(s_k \times s_k)(a,b) = (s_k(a), s_k(b)) = (f,g) \Leftrightarrow \begin{cases} s_k(a) = f \\ s_k(b) = g \end{cases}$$

Thanks to the fact that the simplicial set Δ^1 is small, I decide to solve this by exhaustion: The degeneracy maps from $(\Delta^1)_1$ to $(\Delta^1)_2$ are s_0 and s_1, and $(\Delta^1)_1 = \{00, 01, 11\}$. Then for the two maps,

$$\begin{aligned} s_0(00) &= 000 \qquad & s_1(00) &= 000 \\ s_0(01) &= 001 \qquad & s_1(01) &= 011 \\ s_0(11) &= 111 \qquad & s_1(11) &= 111 \end{aligned}$$

Therefore X_2 has 2 non-degenerate 2-simplices, namely, $(001, 011)$ and $(011, 001)$.

(2) If X has a non-degenerate 3-simplex, then it is a tetrahedra whose vertices are vertices of X. The vertices of X are $(0,0),\ (0,1),\ (1,0),\ (1,1)$, which cannot form a tetrahedra, since they are all in the same plane.

(3) Consider $k = 4$. Let $(f_4, g_4) \in X_4$, then $f_4,\ g_4 \in (\Delta^1)_4$ are degenerate (since all k-simplices of Δ^1 is degenerate for $k \geq 3$). Then there exists $f_3,\ g_3 \in (\Delta^1)_3$ and degeneracy maps $s^3_m = s_m,\ s^3_n = s_n$ such that

$$(f_4, g_4) = (s_m(f_3), s_n(g_3)).$$

From **(2)** we have $(f_3, g_3) \in X_3$ is degenerate, then there exists $x,\ y \in (\Delta^1)_2$ and degeneracy map $s^2_{l,X} = s^2_l \times s^2_l$ with $s^2_l = s_l$ such that

$$(f_3, g_3) = s^2_{l,X}(x,y) = (s_l(x), s_l(y))$$

therefore

$$(f_4, g_4) = (s_m \circ s_l(x), s_n \circ s_l(y)).$$

Recall that, for $i > j$, the simplicial ideneity gives $s_i s_j = s_j s_{i-1}$. Then,

- if $m > l$, then $s_m s_l = s_l s_{m-1}$;
- if $m \leq l$, then $l+1 > m$, therefore $s_{l+1} s_m = s_m s_l$.

Therefore, $(*)$ $s_m s_l = s_l s_{\bar{m}}$ for some $\bar{m}$; and by similar statement, $s_n s_l = s_l s_{\bar{n}}$ for some $\bar{n}$.

As a consequence,

$$(f_4, g_4) = (s_l s_{\bar{m}}(x), s_l s_{\bar{n}}(y)) = s_l(s_{\bar{m}}(x), s_{\bar{n}}(y)),$$

which proves that every 4-simplex of X is degenerate.

Now consider k-simplices of X with $k \geq 4$. I will use induction on k. Assume that the statement holds for k, then for $k+1$:

Let $(\alpha, \beta) \in X_{k+1}$, then $\alpha,\ \beta \in (\Delta^1)_{k+1}$, which are degenerate. Therefore, there exists $\alpha',\ \beta' \in (\Delta^1)_k,\ s_m,\ s_n$ degeneracy maps on X_k such that

$$(\alpha, \beta) = (s_m(\alpha'), s_n(\beta')). \tag{1}$$

Again, since $(\alpha', \beta') \in X_k$ is degenerate (since all k-simplices of X are degenerate), there exists $(f, g) \in X_{k-1}$ and degenerate map $s_{l, X_{k-1}} = s_l \times s_l$ such that

$$(\alpha', \beta') = s_{l, Xk-1}(f, g) = (s_l(f), s_l(g)). \tag{2}$$

Combining (1) and (2),

$$(\alpha, \beta) = (s_m s_l(f), s_n s_l(g)) \overset{(\bigstar)}{=} (s_l s_{\bar{m}}(x), s_l s_{\bar{n}}(y)) = s_l(s_{\bar{m}}(x), s_{\bar{n}}(y)).$$

This shows that $(k+1)$-simplices of X are all degenerate. Therefore the statement holds for all $k \geq 4$. $\square$

Exercise: Prove that $\Delta^{i-1} \star \Delta^{j-1} \cong \Delta^{i+j-1}$.

Claim: associativity holds for $\star$.

Indeed, let $X,\ Y,\ Z$ be simplicial sets,

$$((X \star Y) \star Z)_e = \bigcup_{\substack{k+l=e \\ k,l \geq 0}} (X \star Y)_k \times Z_l = \bigcup_{\substack{k+l=e \\ k,l \geq 0}} \Big(\bigcup_{\substack{m+n=k \\ m,n \geq 0}} X_m \times Y_n \Big) \times Z_l \Big) = \bigcup_{\substack{l+m+n=e \\ l,m,n \geq 0}} X_m \times Y_n \times Z_l$$

$$(X \star (Y \star Z))_e = \bigcup_{\substack{k+l=e \\ k,l \geq 0}} X_k \times (Y \star Z)_l = \bigcup_{\substack{k+l=e \\ k,l \geq 0}} X_k \times \Big(\bigcup_{\substack{m+n=k \\ m,n \geq 0}} (Y_m \times Z_n) \Big) = \bigcup_{\substack{k+m+l=e \\ k,m,l \geq 0}} X_k \times Y_m \times Z_n$$

Therefore $(X \star Y) \star Z = X \star (Y \star Z)$, and it makes sense to write $X \star Y \star Z$.

Intuitively, if we have $n+1$ copies of Δ^0 $(= \coprod_{n+1 \text{ times}} \Delta^0)$, then we have $n+1$ vertices, which induces Δ^n.

Claim: $(*)$ $\Delta^n \cong \Delta^0 \star \cdots \star \Delta^0$ ($n+1$ times, $n \geq 1$).

For simplicity, denote $\Delta^0 \star \cdots \star \Delta^0$ ($n+1$ times) by $\star^{n+1}\Delta^0$.

Prove by induction on n. For $n = 1$,

$$(\Delta^0 \star \Delta^0)_0 \cong (\Delta^0)_0 \coprod (\Delta^0)_0 = \text{two points} := \{x_0, x_1\} \cong \{0, 1\} = (\Delta^1)_0.$$

Moreover,

$$\begin{aligned}(\Delta^0 \star \Delta^0)_1 &\cong (\Delta^0)_1 \coprod (\Delta^0)_1 \coprod (\Delta^0)_0 \times (\Delta^0)_0 \\ &= \{x_0x_0, x_1x_1, x_0x_1\} \cong \{00, 11, 01\} = (\Delta^1)_1,\end{aligned}$$

Therefore $\Delta^0 \star \Delta^0 \cong \Delta^1$, the statement $(*)$ holds for $n = 1$.

Now assume that $(*)$ holds for n. Consider case $n+1$. WLOG let $(\Delta^n)_0 = \{x_0, \ldots, x_n\} = \{0, 1, \ldots, n\}$, and $(\Delta^0)_0 = \{x_{n+1}\} = \{0\}$. Then for each $k \leq n+1$,

$$(\Delta^n \star \Delta^0)_k = (\Delta^n)_k \coprod (\Delta^0)_k \coprod (\coprod_{i+j=k-1} (\Delta^n)_i \times (\Delta^0)_j),$$

then obviously, for every $k \leq n+1$, $(\Delta^n \star \Delta^0)_k \cong (\Delta^{n+1})_k$ by the canonical map, therefore $\Delta^{n+1} \cong \Delta^n \star \Delta^0 \cong \Delta$, and as a result,

$$\begin{aligned}\Delta^{i-1} \star \Delta^{j-1} &\cong (\star^i \Delta^0) \star (\star^j \Delta^0) \\ &= \star^{i+j} \Delta^0 \\ &\cong \Delta^{i+j-1}.\end{aligned}$$

□

Exercise 2.9

(a) Let $(S^1)_0 = \{0\}$, $(S^1)_1 = \{00, 01\} = \{*, \gamma\}$, and γ is the non-degenerate 1-simplex. Using the notation as in the picture, the only non-degenerate 1-simplex is a, which is the edge of M. Therefore i maps γ to a.

Since i is a map, it is a natural transformation, therefore for $\alpha : [m] \to [n]$, $\alpha^* \circ i_n = i_m \circ \alpha^*$. Especially, i commutes with face maps and degeneracy maps. Then,

$$i(0) = i(d_1(01)) = (d_1 \circ i)(\gamma) = d_1(a) = y.$$

For 1-simplices,

$$i(00) = i(s_0(0)) = s_0(y) = \text{the edge joining } y \text{ and } y$$

such edge does not exist in M; however, notice that $00 = 11$ in $(S^1)_1$, one has

$$i(00) = i(11) = i(s_0(1)) = (s_0 \circ i)(1) = \text{the line joining } x \text{ and } x = d,$$

hence i is well defined for S^1. Therefore i exists and is unique.

(b) Let $p : M \to S^1$ be a map of simplicial sets.

Since $(S^1)_0$ is a single point, $p(x) = p(y) = 0$. This induces

$$p(d) = p(s_0(x)) = s_0(p(x)) = 00 = *$$

Let $p(a) = p(c) = p(b) = \gamma$. This is well-defined since $d_0 \circ p,\ d_1 \circ p,\ p \circ d_0,\ p \circ d_1$ maps everything to $0 \in (S^1)_0$. (Again, thanks to the fact that S^1 and M are not big)By direct check, i.e.,

$$\begin{cases} (p \circ i)(0) = p(y) = 0 \\ (p \circ i)(00) = p(d) = 00 \\ (p \circ i)(01) = p(a) = \gamma = 01 \end{cases} \Rightarrow p \circ i = \mathrm{id}_{S^1}$$

Obviously $p \circ i$ is homotopic to id_{S^1}. On the other hand,

$$\begin{cases} (i \circ p)(x) = (i \circ p)(y) = i(0) = x \\ (i \circ p)(a) = (i \circ p)(b) = (i \circ p)(c) = i(\gamma) = a \\ (i \circ p)(d) = i(00) = d \end{cases}$$

Consider $h : M \times \Delta^1 \to M$ such that

$$h^0(\cdot, 0) = h^0(\cdot, 1) = x$$

$$h^1(a, \cdot) = h^1(b, \cdot) = h^1(c, \cdot) = a,\ h^1(d, \cdot) = d$$

then h is a map and $i \circ p$ is simplicial homotopic to id_M through h. $\square$

If some $e \in M_1 = (\Delta^1)_1 \sqcup_{(\Delta^1)_1 \sqcup (\Delta^1)_1} (M_1')$ is degenerate, then it is in the equivalent class of some degenerate 1-simplex of $(\Delta^1)_1$ or M_1'. Consider only the non-degenerate ones of M_1'.