Frobenius manifolds and flat pencils of metrics

Xueyan Zhang

Abstract

This thesis investigates Frobenius manifolds from a differential geometric perspective. We begin by employing tools from Riemannian geometry to establish the equivalence between Frobenius manifolds and flat pencils of metrics. In this context, we analyze the Levi-Civita connection, the Euler and unity vector fields, and the potential function appearing in the structure equations. Concrete examples are provided to illustrate the emergence of this structure, including explicit computations of the associated metrics and bihamiltonian systems. Finally, we extend the Frobenius manifold framework to supermanifolds and compute the explicit form of the bihamiltonian system in the $(3, 2)$ case.

Contents

Introduction

An informal way to think about a Frobenius manifold is as a Riemannian manifold equipped with additional structure compatible with a Frobenius algebra on each tangent space, namely, for a metric $\langle \, , \, \rangle$,

$$\langle \partial_a \circ \partial_b, \partial_c \rangle = \langle \partial_a, \partial_b \circ \partial_c \rangle.$$

The concept of Frobenius manifolds was introduced by Dubrovin [4] as a geometric framework that connects complex geometry, certain classes of differential equations arising in mathematical physics, and algebraic structures with strong compatibility properties. A main feature of Frobenius manifolds is their equivalence with flat pencils of metrics – pairs of flat (contravariant) metrics whose linear combinations remain flat. These structures are introduced in Section 1, along with the WDVV equations – a family of partial differential equations that encode the associativity condition of the Frobenius algebra structure.

Section 2 follows the main steps of Dubrovin's work [3] to give a detailed geometric proof of the equivalence between Frobenius manifolds and flat pencils of metrics, using tools from Riemannian geometry.

In Section 3, we introduce bihamiltonian systems and their intrinsic link to flat pencils of metrics. These systems often appear in the study of compatible Poisson brackets and play an important role in the geometric interpretation of integrable hierarchies.

Section 4 presents two examples where Frobenius manifold structures and flat pencils of metrics arise naturally from Hurwitz spaces. We describe the geometry of these spaces and explicitly compute the graded homogeneous Poisson bracket of degree 0 and 1 (also called the KdV hierarchies) associated with the cases $n = 1$ and $n = 2$.

Inspired by [12], Section 5 extends the discussion to supermanifolds that incorporate supersymmetry by combining bosonic (even) and fermionic (odd) variables. The symmetry of a metric is replaced with supersymmetry:

$$g_{ab} = (-1)^{|a||b|} g_{ba},$$

where $|a|$ and $|b|$ are the degrees of the coordinates x^a and x^b respectively. We introduce the geometry of Riemannian supermanifolds and construct Frobenius manifold structures and flat pencils of metrics in this setting. As a concrete example, we analyze a $(3, 2)$ supermanifold and compute its associated bihamiltonian structure.

1 Preliminaries

1.1 Flat pencils of metrics

Let M be an n-dimensional manifold. A Riemannian metric on M is known to be a smooth, symmetric, positive-definite 2-tensor. For each $x \in M$, g_x is an inner product on T_xM. In local coordinates $x^1, \ldots, x^n$, g can be written as

$$g = g_{ij} dx^i \otimes dx^j,$$

or

$$ds^2 = g_{ij}(x) dx^i dx^j. \tag{1.1}$$

(Here and below, a summation over repeated indices is assumed.) Similarly, one can define a "metric" on T^*_xM as a (2,0)-tensor:

Definition 1.1. A symmetric bilinear form (,) on T^*M is a *contravariant metric* if it is invertible on an open dense subset $M_0 \subset M$. In local coordinates $x^1, \ldots, x^n$, a contravariant metric is specified as a (2,0)-tensor:

$$(dx^i, dx^j) = g^{ij}(x), \; i, j = 1, \ldots, n. \tag{1.2}$$

g is invertible in the sense that the matrix (g^{ij}) is invertible, that is, on M_0, $(g_{ij}(x)) = (g^{ij}(x))^{-1}$ determines a metric in the usual sense as (1.1) (which, however, may not be positive definite).

This definition naturally indicates that

$$g_{ik} g^{kj} = \delta_i^j,$$

where δ_i^j is the Kronecker delta.

Definition 1.2. The *contravariant Levi-Civita connection* for the metric (,) is determined by a collection of n^3 functions $\Gamma_k^{ij}(x)$ such that on M_0

$$\Gamma_k^{ij}(x) := -g^{is}(x) \Gamma_{sk}^j(x) \tag{1.3}$$

where $\Gamma_{sk}^j(x)$ corresponds to the Levi-Civita connection for the metric (1.2).

Lemma 1.3. The coefficients Γ_{jk}^i of the contravariant Levi-Civita connection are determined uniquely on M_0 from the system of linear equations

$$g^{is} \Gamma_s^{jk} = g^{js} \Gamma_s^{ik} \tag{1.4}$$

$$\Gamma_k^{ij} + \Gamma_k^{ji} = \partial_k g^{ij}, \tag{1.5}$$

where

$$\partial_k = \frac{\partial}{\partial x^k}.$$

Proof. Since Levi-Civita of metric is torsion free, it satisfies $\Gamma_{ts}^k = \Gamma_{st}^k$. From the definition of Γ_k^{ij},

$$g^{is} \Gamma_s^{jk} = -g^{is} g^{jt} \Gamma_{ts}^k,$$

$$g^{js} \Gamma_s^{ik} = -g^{js} g^{it} \Gamma_{ts}^k = -g^{jt} g^{is} \Gamma_{st}^k = g^{is}(-g^{jt} \Gamma_{st}^k) = g^{is} \Gamma_s^{jk},$$

(1.4) follows directly from symmetry condition of Levi-Civita connection, namely $\Gamma_{st}^k = \Gamma_{ts}^k$.

Compatibility with the metric for Levi-Civita connection $\nabla g = 0$ reads

$$0 = \nabla_k g_{ij} = \partial_k g_{ij} - \Gamma_{ki}^m g_{mj} - \Gamma_{kj}^m g_{im} \tag{1.6}$$

for any $i, j, k = 1, \ldots, n$. Thus for fixed i and j,

$$\begin{aligned} 0 &= \partial_k(g_{is}g^{sj}) = g^{sj}\partial_k g_{is} + g_{is}\partial_k g^{sj}, \\ \Rightarrow - g^{sj}\partial_k g_{is} &= g_{is}\partial_k g^{sj}. \end{aligned}$$

Contract by g^{ni},

$$\begin{aligned} RHS &= g^{ni}g_{is}\partial_k g^{sj} = \delta^n_s \partial_k g^{sj} = \partial_k g^{nj}, \\ LHS &= -g^{ni}g^{sj}(\Gamma^m_{ki}g_{ms} + \Gamma^m_{ks}g_{im}) \\ &= (g^{sj}g_{ms})(-g^{ni}\Gamma^m_{ki}) + (g^{ni}g_{im})(-g^{sj}\Gamma^m_{ks}) \\ &= \delta^j_m \Gamma^{nm}_k + \delta^n_m \Gamma^{jm}_k \\ &= \Gamma^{nj}_k + \Gamma^{jn}_k. \end{aligned}$$

Thus (1.5) is proved. Uniqueness follows from the uniqueness and existence of the Levi-Civita connection. □

With a contravariant connection, one can define the operators ∇^u as *covariant derivatives along any 1-form* $u \in T^*M$ with similar properties of a connection. One can associate the operator $\nabla^i := \nabla^{dx^i}$ to the usual covariant derivative by raising the index

$$\nabla^i = g^{is}\nabla_s \tag{1.7}$$

in the sense that

$$\nabla^{dx^i} dx^j = \Gamma^{ij}_k dx^k = g^{is}\Gamma^j_{ks}dx^k = g^{is}\nabla_{dx^s}dx^k.$$

Then for $u = u_i dx^i$ and $v = v_j dx^j$, notice that $dx^i v_j = \partial_i v_j$,

$$\begin{aligned} \nabla^u v &= u_i g^{is}\nabla_s(v_j dx^j) \\ &= u_i(g^{is}dx^s v_k dx_k + v_j \nabla_s dx_j) \\ &= u_i(g^{ij}\partial_j v_k + v_j g^{is}\Gamma^k_{sj}dx^k) \\ &= u_i(g^{ij}\partial_j v_k + v_j \Gamma^{ij}_k)dx^k. \end{aligned}$$

Definition 1.4. A function $f(x)$ is called *flat coordinate* of the contravariant metric $(\ ,\ )$ if the differential $\xi = df$ is covariantly constant w.r.t. the Levi-Civita connection, namely,

$$\nabla f = 0,$$

or equivalently, for $i, j = 1, \ldots, n$,

$$g^{is}\partial_s \xi_j + \Gamma^{is}_j \xi_s = 0,$$

where $\xi_j = \partial_i f$.

Definition 1.5. A contravariant metric is said to be *flat* iff on M_0 there locally exists n independent flat coordinates.

Following the arguments for the usual metric, by choosing a system of flat coordinates, one can reduce the matrix (g^{ij}) to a constant form, and Γ^i_{jk} to zero.

Lemma 1.6. The contravariant metric is flat iff the Riemannian curvature tensor

$$R^{ijk}_l := g^{is}(\partial_s \Gamma^{jk}_l - \partial_l \Gamma^{jk}_s) + \Gamma^{ij}_s \Gamma^{sk}_l - \Gamma^{ik}_s \Gamma^{sj}_l \tag{1.8}$$

identically vanishes.

Proof. Recall that for a connection, the Riemannian curvature is defined as

$$R^l_{ijk} := \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^l_{is}\Gamma^s_{jk} - \Gamma^l_{js}\Gamma^s_{ik}$$

and there exists a system of flat coordinates iff R^l_{ijk} identically vanishes. By (1.3) and (1.5),

$$\begin{aligned}\partial_s \Gamma^{jk}_l &= -\,\partial_s(g^{jn}\Gamma^k_{nl}) \\ &= -\,g^{jn}\partial_s\Gamma^k_{nl} - \Gamma^k_{nl}\partial_s g^{jn} \\ &= -\,g^{jn}\partial_s\Gamma^k_{nl} - \Gamma^k_{nl}(\Gamma^{jn}_s + \Gamma^{nj}_s),\end{aligned}$$

and similarly for $\partial_l \Gamma^{jk}_s$. Therefore, R^{ijk}_l can be rewritten as

$$\begin{aligned}R^{ijk}_l &= g^{is}g^{jn}(\partial_l\Gamma^k_{ns} - \partial_s\Gamma^k_{nl}) + g^{is}\left[(\Gamma^{nj}_l + \Gamma^{jn}_l)\Gamma^k_{ns} - (\Gamma^{jn}_s + \Gamma^{nj}_s)\Gamma^k_{nl}\right] + \Gamma^{ij}_s\Gamma^{sk}_l - \Gamma^{ik}_s\Gamma^{sj}_l \\ &= g^{is}g^{jn}(\partial_l\Gamma^k_{ns} - \partial_s\Gamma^k_{nl}) - \Gamma^{ik}_n(\Gamma^{nj}_l + \Gamma^{jn}_l) - g^{is}(\Gamma^{jn}_s + \Gamma^{nj}_s)\Gamma^k_{nl} + \Gamma^{ij}_s\Gamma^{sk}_l - \Gamma^{ik}_s\Gamma^{sj}_l \\ &= g^{is}g^{jn}(\partial_l\Gamma^k_{ns} - \partial_s\Gamma^k_{nl}) - \Gamma^{ik}_n\Gamma^{jn}_l - g^{is}\Gamma^{jn}_s\Gamma^k_{nl} + (\Gamma^{ij}_s\Gamma^{sk}_l - g^{is}\Gamma^{nj}_s\Gamma^k_{nl}) + (\Gamma^{ik}_n\Gamma^{nj}_l - \Gamma^{ik}_s\Gamma^{sj}_l).\end{aligned}$$

Notice that there are summations over n and s thus these two indices can be changed freely (and so is for the change of n to p that will be used later). Therefore,

$$\Gamma^{ik}_s\Gamma^{sj}_l = \Gamma^{ik}_n\Gamma^{nj}_l,$$

together with (1.3) and (1.4),

$$\Gamma^{ij}_s\Gamma^{sk}_l = -g^{sn}\Gamma^k_{nl}\Gamma^{ij}_s = -g^{si}\Gamma^k_{nl}\Gamma^{nj}_s.$$

Thus R^{ijk}_l can be further simplified:

$$\begin{aligned}R^{ijk}_l &= g^{is}g^{jn}(\partial_l\Gamma^k_{ns} - \partial_s\Gamma^k_{nl}) - \Gamma^{ik}_n\Gamma^{jn}_l - g^{is}\Gamma^{jn}_s\Gamma^k_{nl} \\ &= g^{is}g^{jp}(\partial_l\Gamma^k_{ps} - \partial_s\Gamma^k_{pl}) - g^{is}g^{jp}\Gamma^k_{sn}\Gamma^n_{pl} + g^{is}g^{jp}\Gamma^n_{ps}\Gamma^k_{nl} \\ &= g^{is}g^{jp}R^k_{lsp}.\end{aligned}$$

□

Now we can introduce the definition and homogeneity condition for the flat pencil of metrics.

Definition 1.7. Two contravariant metrics $(\ ,\ )_1$ and $(\ ,\ )_2$ form a *flat pencil* if:

1) The linear combination

$$(\ ,\ )_1 - \lambda(\ ,\ )_2 \tag{1.9}$$

for any λ is a contravariant metric on M.

2) If ${\Gamma_1}^{ij}_k$ and ${\Gamma_2}^{ij}_k$ are the contravariant Levi-Civita connections for these two metrics, then for any λ, the linear combination

$${\Gamma_1}^{ij}_k - \lambda{\Gamma_2}^{ij}_k$$

is the contravariant Levi-Civita connection for the metric (1.9).

3) The metric (1.9) is flat for any λ.

Definition 1.8. The flat pencil is said to be *quasihomogeneous of degree d* if there exists a function τ on M such that the vector fields

$$E_f := \nabla_1\tau \tag{1.10}$$

$$e_f := \nabla_2\tau, \tag{1.11}$$

that means

$$E^i_f = g^{is}_1\partial_s\tau, \quad e^i_f = g^{is}_2\partial_s\tau,$$

that satisfies the following properties for the contravariant metrics $(\ ,\ )_1$ and $(\ ,\ )_2$:

$$[e_f, E_f] = e_f \tag{1.12}$$

$$\mathcal{L}_{E_f}(\ ,\ )_1 = (d-1)(\ ,\ )_1 \tag{1.13}$$

$$\mathcal{L}_{e_f}(\ ,\ )_1 = (\ ,\ )_2 \tag{1.14}$$

$$\mathcal{L}_{e_f}(\ ,\ )_2 = 0. \tag{1.15}$$

Here the lower index "$_f$" is added to E and e to avoid confusion with unity vector and Euler vector field of Frobenius manifold that will be introduced later.

Corollary 1.9. For g_{1ij}, one has

$$\mathcal{L}_{E_f} g_{1ij} = (1-d) g_{1ij}. \tag{1.16}$$

Proof.

$$\mathcal{L}_{E_f}(g_1{}^{ij} g_{1ik}) = \mathcal{L}_{E_f}\delta_k^j = 0.$$

On the other hand, by Leibniz rule for Lie derivative and (1.13),

$$\begin{aligned} 0 = \mathcal{L}_{E_f}(g_1{}^{ij} g_{1ik}) &= g_1{}^{ij}\mathcal{L}_{E_f} g_{1ik} + g_{1ik}\mathcal{L}_{E_f} g_1{}^{ij} \\ &= g_1{}^{ij}\mathcal{L}_{E_f} g_{1ik} + (d-1) g_{1ik} g_1{}^{ij} \\ &= g_1{}^{ij}\mathcal{L}_{E_f} g_{1ik} + (d-1)\delta_k^j. \end{aligned}$$

Thus, contract by g_{1js},

$$\begin{aligned} (1-d)\delta_k^j g_{1js} = g_{1js} g_1{}^{ij}\mathcal{L}_{E_f} g_{1ik} = \delta_s^i \mathcal{L}_{E_f} g_{1ik} \\ \Rightarrow \mathcal{L}_{E_f} g_{1ij} = (1-d) g_{1ij}. \end{aligned}$$

□

1.2 Frobenius manifolds and WDVV equations

1.2.1 WDVV equations

From here on, we will introduce the WDVV equations. A function $F = F(t),\ t = (t^1, \ldots, t^n)$ with its third derivatives

$$c(\partial_\alpha, \partial_\beta, \partial_\gamma) = \partial_\alpha\partial_\beta\partial_\gamma F = c_{\alpha\beta\gamma} \tag{1.17}$$

must satisfy the following properties:

1. Normalization:

$$\eta_{\alpha\beta} := c_{1\alpha\beta}(t) \tag{1.18}$$

where $\eta_{\alpha\beta}$ are constants. Define

$$(\eta^{\alpha\beta}) := (\eta_{\alpha\beta})^{-1}. \tag{1.19}$$

The matrices $(\eta_{\alpha\beta})$ and $(\eta^{\alpha\beta})$ will be used for lowering and raising indices.

2. Associativity. By lifting the index, define a family of functions

$$c_{\alpha\beta}^\gamma(t) := \eta^{\gamma\epsilon} c_{\epsilon\alpha\beta}(t) \tag{1.20}$$

with n-dimensional space and its bases $e_1, \ldots, e_n$. The multiplication

$$e_\alpha \cdot e_\beta := c_{\alpha\beta}^\gamma(t) e_\gamma \tag{1.21}$$

defines a structure of associative algebra A_t.

3. Quasihomogeneity. $F(t)$ is a quasihomogeneous function of its variables:

$$F(c^{d_1} t^1, \ldots, c^{d_n} t^n) = c^{d_F} F(t^1, \ldots, t^n) \tag{1.22}$$

for any nonzero c and for some numbers $d_1, \ldots, d_n, d_F$. Denote $\deg t^\alpha := d_\alpha,\ \deg F := d_F$.

Lemma 1.10. The unity vector field of the algebra A_t is e_1.

Proof. $e_1 \cdot e_\alpha = c_{1\alpha}^\gamma(t) e_\gamma = \eta^{\gamma\varepsilon} c_{\varepsilon 1 \alpha}(t) e_\gamma = \eta^{\gamma\varepsilon}\eta_{\varepsilon\alpha} e_\gamma = \delta_\alpha^\gamma e_\gamma = e_\alpha$. $\square$

For convenience, we introduce the *Euler vector field*

$$E = E^\alpha(t)\partial_\alpha$$

that satisfies

$$\mathcal{L}_E F(t) = E^\alpha \partial_\alpha F(t) = d_F \cdot F(t). \tag{1.23}$$

Apply ∂_c on both sides of (1.22) and taking $c = 1$, one has

$$\begin{aligned} RHS &= d_\alpha c^{d_\alpha - 1} t^\alpha \frac{\partial F(t)}{\partial \alpha}\bigg|_{c=1} = d_\alpha t^\alpha \partial_\alpha F(t), \\ LHS &= d_F c^{d_F - 1} F(t)\big|_{c=1} = d_F \cdot F(t), \end{aligned}$$

thus (1.23) implies

$$E^\alpha(t) = d_\alpha t^\alpha. \tag{1.24}$$

Remark 1. We will consider $F(t)$ up to adding a quadratic function in $t^1, \ldots, t^n$. Thus (1.23) can be modified as

$$\mathcal{L}_E F(t) = d_F F(t) + A_{\alpha\beta} t^\alpha t^\beta + B_\alpha t^\alpha + C. \tag{1.25}$$

Lemma 1.11. For the Euler vector field E and the flat metric $\eta_{\alpha\beta}$,

$$\mathcal{L}_E \eta_{\alpha\beta} = (d_F - d_1)\eta_{\alpha\beta}, \tag{1.26}$$

where $d_F,\ d_1$ are defined in (1.22).

Proof. Differentiate (1.25) w.r.t. $t_1,\ t_\alpha,\ t_\beta$,

$$\begin{aligned} LHS &= \partial_1\partial_\alpha\partial_\beta(\mathcal{L}_E F(t)) \\ &= \partial_1\partial_\alpha\partial_\beta(d_\gamma t^\gamma \partial_\gamma F(t)) \\ &= (d_\alpha + d_\beta + d_1)\eta_{\alpha\beta} \\ &= \mathcal{L}_E F(t) + d_1 \eta_{\alpha\beta} \\ RHS &= \partial_1\partial_\alpha\partial_\beta(d_F F(t) + A_{\alpha\beta} t^\alpha t^\beta + B_\alpha t^\alpha + C) \\ &= d_F \eta_{\alpha\beta} \end{aligned}$$

lemma is proved. $\square$

Corollary 1.12. If $\eta_{11} \neq 0$, then by a linear change of coordinates t^α, the matrix $\eta_{\alpha\beta}$ can be reduced to the antidiagonal form

$$\eta_{\alpha\beta} = \delta_{\alpha+\beta, n+1}. \tag{1.27}$$

In these coordinates,

$$F(t) = \frac{1}{2}(t^1)^2 t^n + \frac{1}{2} t^1 \sum_{\alpha=2}^{n-1} t^\alpha t^{n-\alpha+1} + f(t^2, \ldots, t^n) \tag{1.28}$$

for some function $f(t^2, \ldots, t^n)$.

Proof. Follows from (1.18). $\square$

Remark 2. The degrees $d_1, \ldots, d_n, d_F$ are well defined up to a nonzero factor and unique up to reordering. For convenience the degrees $d_1, \ldots, d_n, d_F$ will be normalized such that

$$d_1 = 1. \tag{1.29}$$

Remark 3. Associativity of the multiplication in the Frobenius algebra induces the WDVV equations:

$$(\partial_\alpha \cdot \partial_\beta) \cdot \partial_\gamma = \partial_\alpha \cdot (\partial_\beta \cdot \partial_\gamma) \tag{1.30}$$

implies that

$$c(\partial_\alpha \cdot \partial_\beta, \partial_\gamma, \partial_\delta) = \langle (\partial_\alpha \cdot \partial_\beta) \cdot \partial_\gamma, \partial_\delta \rangle = \langle \partial_\alpha \cdot (\partial_\beta \cdot \partial_\gamma), \partial_\delta \rangle = c(\partial_\alpha, \partial_\beta \cdot \partial_\gamma, \partial_\delta).$$

And since

$$\begin{aligned} c(\partial_\alpha \cdot \partial_\beta, \partial_\gamma, \partial_\delta) &= \langle (\partial_\alpha \cdot \partial_\beta) \cdot \partial_\gamma, \partial_\delta \rangle \\ &= c_{\alpha\beta\lambda}\eta^{\lambda\mu} \langle \partial_\mu \cdot \partial_\gamma, \partial_\delta \rangle \\ &= c_{\alpha\beta\lambda}\eta^{\lambda\mu} c_{\mu\gamma\delta} \\ &= \partial_\alpha\partial_\beta\partial_\lambda F(t)\eta^{\lambda\mu}\partial_\mu\partial_\gamma\partial_\delta F(t), \end{aligned}$$

similarly,

$$c(\partial_\alpha, \partial_\beta \cdot \partial_\gamma, \partial_\delta) = c_{\beta\gamma\delta}\eta^{\lambda\mu} c_{\mu\lambda\alpha} = \partial_\delta\partial_\beta\partial_\gamma F(t)\eta^{\lambda\mu}\partial_\mu\partial_\gamma\partial_\alpha F(t).$$

These induce the *WDVV associativity equations* for F:

$$\partial_\alpha\partial_\beta\partial_\lambda F(t)\eta^{\lambda\mu}\partial_\mu\partial_\gamma\partial_\delta F(t) = \partial_\delta\partial_\beta\partial_\gamma F(t)\eta^{\lambda\mu}\partial_\mu\partial_\gamma\partial_\alpha F(t),\ \ \alpha, \beta, \gamma, \delta = 1, \ldots, n.$$

A solution to the WDVV equations is called *(primary) free energy*.

Example 1.13. Consider $n = 3$. The function F in (1.28) has the form

$$F(t) = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 + f(t_2, t_3) \tag{1.31}$$

for a function $f(x, y)$. The functions $c_\gamma^{\alpha\beta}(t)$ are calculated through (1.17), (1.20), and by the multiplication law (1.21),

$$\begin{aligned} e_1 e_i &= e_i,\ \ i = 1, 2, 3, \\ e_2^2 &= f_{xxy}e_1 + f_{xxx}e_2 + e_3, \\ e_2 e_3 &= f_{xyy}e_1 + f_{xxy}e_2, \\ e_3^2 &= f_{yyy}e_1 + f_{xyy}e_2, \end{aligned}$$

where the lower indices of f refer to the partial derivatives w.r.t. x or y. Inserting (1.13) into

$$(e_2^2)e_3 = e_2(e_2 e_3),$$

one gets

$$\begin{aligned} LHS &= (f_{xxy}e_1 + f_{xxx}e_2 + e_3)e_3 \\ &= f_{xxy}e_3 + f_{xxx}e_2 e_3 + e_3^2 \\ &= f_{xxx}f_{xyy} + f_{yyy} + (f_{xxx}f_{xxy} + f_{xyy})e_2 + f_{xxy}e_3, \\ RHS &= (f_{xyy}e_1 + f_{xxy}e_2)e_2 \\ &= f_{xyy}e_2 + f_{xxy}(f_{xxy}e_1 + f_{xxx}e_2 + e_3) \\ &= f_{xxy}^2 + (f_{xxx}f_{xyy} + f_{xyy})e_2 + f_{xxy}e_3, \end{aligned}$$

which induces

$$f_{xxy}^2 = f_{yyy} + f_{xxx}f_{xyy}. \tag{1.32}$$

This case will be discussed again for an example of supermanifolds in section 5.

1.2.2 Frobenius manifolds

Another important structure that will be discussed is the Frobenius manifold. In contrast to homogeneity conditions for flat pencils of metrics, the "homogeneity conditions" of Frobenius manifolds are defined through grading operators.

Definition 1.14. A *Frobenius algebra* is a pair $(A, \langle\ ,\ \rangle)$ where A is a commutative associative algebra (over $\mathbb{R}$ or $\mathbb{C}$) with a unity and $\langle\ ,\ \rangle$ stands for a symmetric non-degenerate *invariant* bilinear form on A, that is,

$$\langle ab, c\rangle = \langle a, bc\rangle \tag{1.33}$$

for any arbitrary 3 vectors $a, b, c \in A$.

Definition 1.15. The Frobenius algebra is said to be *graded* if a linear operator $Q : A \to A$ and a number d are defined such that

$$Q(ab) = Q(a)b + aQ(b) \tag{1.34}$$

$$\langle Q(a), b\rangle + \langle a, Q(b)\rangle = d\langle a, b\rangle \tag{1.35}$$

for any $a, b \in A$. The operator Q is then called *grading operator* and the number d is called *charge* of the Frobenius algebra. If Q is diagonalizable, one may assign degrees to the eigenvectors e_α of Q:

$$\deg(e_\alpha) = q_\alpha \text{ if } Q(e_\alpha) = q_\alpha e_\alpha. \tag{1.36}$$

Then the usual property of the degree of the product of homogeneous elements of the algebras holds true:

$$\deg(ab) = \deg a \deg b.$$

Besides, $\langle a, b\rangle$ can be nonzero only if $\deg a + \deg b = d$, where d is the charge:

$$(\deg a + \deg b)\langle a, b\rangle = \langle Q(a), b\rangle + \langle a, Q(b)\rangle = d\langle a, b\rangle.$$

Consider also graded Frobenius algebras $(A, \langle\ ,\ \rangle)$ over graded commutative associative rings R. There are two operators $Q_R : R \to R$ and $Q_A : A \to A$ satisfying the following properties for $\alpha, \beta \in R$ and $a, b \in A$:

$$Q_R(\alpha\beta) = Q_R(\alpha)\beta + \alpha Q_R(\beta) \tag{1.37a}$$

$$Q_A(ab) = Q_A(a)b + aQ_A(b) \tag{1.37b}$$

$$Q_A(\alpha a) = Q_R(\alpha)a + \alpha Q_A(a) \tag{1.37c}$$

$$Q_R\langle a, b\rangle + d\langle a, b\rangle = \langle Q_A(a), b\rangle + \langle a, Q_A(b)\rangle \tag{1.37d}$$

Definition 1.16. A (smooth, analytic) *Frobenius structure* on the manifold M is a structure of Frobenius algebra on the tangent space $T_tM = (A_t, \langle\ ,\ \rangle_t)$ depending (smoothly, analytically) on the point t. This structure must satisfy the following axioms:

FM1. The metric on M included by the invariant bilinear form $\langle\ ,\ \rangle_t$ is flat. Denote ∇ the Levi-Civita connection for the metric (a symmetric, non-degenerate bilinear form on TM) $\langle\ ,\ \rangle_t$, the unity vector field e_F must be covariantly constant, namely,

$$\nabla e_F = 0. \tag{1.38}$$

Notice that the metric may not be positive definite. The flatness of the metric, i.e., vanishing of the Riemannian tensor means that there exists a system of flat coordinates $(t^1, \ldots, t^n)$ such that the matrix $\langle\partial_\alpha, \partial_\beta\rangle$ of the metric in these coordinates becomes constant.

FM2. Let c be the following system of trilinear forms on TM

$$c(u, v, w) := \langle u \cdot v, w\rangle. \tag{1.39}$$

The four-linear form

$$(\nabla_z c)(u,v,w), u,v,w,z \in TM \tag{1.40}$$

must also be symmetric.
FM3. Observe that the space $Vect(M)$ of vector fields on M naturally builds a Frobenius algebra over the algebra $Func(M)$ of (smooth, analytic) functions on M. A linear vector field $E_F \in Vect(M)$ is an *Euler vector field* such that

$$\nabla\nabla E_F = 0. \tag{1.41}$$

The operators

$$\begin{aligned} Q_{Func(M)} &:= E_F \ (= Q_R) \\ Q_{Vect(M)} &:= \mathrm{id} + \mathrm{ad}_{E_F} (= Q_A) \end{aligned} \tag{1.42}$$

introduce in $Vect(M)$ a structure of graded Frobenius algebra of a given charge d over the graded ring $Func(M)$.

We write $c_{\alpha\beta\gamma}$ for $c(\partial_\alpha, \partial_\beta, \partial_\gamma)$, and define

$$c^\gamma_{\alpha\beta}(t) := \eta^{\gamma\epsilon} c_{\varepsilon\alpha\beta}(t).$$

.

FM1 means that, in flat coordinates, the unity vector field is constant. Usually, the flat coordinates are chosen in such a way that

$$e_F = \partial/\partial t^1. \tag{1.43}$$

Lemma 1.17. The axiom **FM3** in flat coordinates $t^1, \ldots, t^n$ can be recast into the following equivalent form:

$$\mathcal{L}_{E_F} c^\gamma_{\alpha\beta} = c^\gamma_{\alpha\beta} \tag{1.44a}$$

$$\mathcal{L}_{E_F} \eta_{\alpha\beta} = (2-d)\eta_{\alpha\beta}. \tag{1.44b}$$

Proof. (1.41) implies that E_F has the form

$$E^i_F = q_i t^i, \tag{1.45}$$

where q_i are constants. Therefore, for ∂_α, $\alpha = 1, \ldots, n$,

$$\mathcal{L}_{E_F}\partial_\alpha = E_F\partial_\alpha - \partial_\alpha E_F = q_\epsilon\partial_\epsilon\partial_\alpha - \partial_\alpha q_\epsilon \partial_\epsilon = 0, \tag{1.46}$$

By (1.37b) and using the operators given as (1.42),

$$\begin{aligned} &\partial_\alpha \cdot \partial_\beta + \mathcal{L}_{E_F}(c^\gamma_{\alpha\beta}\partial_\gamma) = \partial_\alpha \cdot \partial_\beta + \mathcal{L}_{E_F}\partial_\alpha \cdot \partial_\beta + \partial_\alpha \cdot \partial_\beta + \partial_\alpha \cdot \mathcal{L}_{E_F}\partial_\beta \\ \Rightarrow\ & c^\gamma_{\alpha\beta}\mathcal{L}_{E_F}\partial_\gamma + (\mathcal{L}_{E_F} c^\gamma_{\alpha\beta})\partial_\gamma = \partial_\alpha \cdot \partial_\beta = c^\gamma_{\alpha\beta}\partial_\gamma \\ \Rightarrow\ & \mathcal{L}_{E_F} c^\gamma_{\alpha\beta} = c^\gamma_{\alpha\beta} \end{aligned}$$

and (1.44a) is proved.
And (1.37d) gives

$$E_F\langle\partial_\alpha, \partial_\beta\rangle + d\langle\partial_\alpha, \partial_\beta\rangle = \langle\partial_\alpha + \mathcal{L}_{E_F}\partial_\alpha, \partial_\beta\rangle + \langle\partial_\alpha, \partial_\beta + \mathcal{L}_{E_F}\partial_\beta\rangle.$$

by definition of $\eta^{\alpha\beta}$ and (1.46), this becomes

$$\mathcal{L}_{E_F}\eta_{\alpha\beta} = (2-d)\eta_{\alpha\beta}.$$

Thus (1.44b) is proved. $\square$

Lemma 1.18. The function $F(t)$ such that

$$\partial_\alpha \partial_\beta \partial_\gamma F(t) = c(\partial_\alpha, \partial_\beta, \partial_\gamma) = c_{\alpha\beta\gamma}$$

$$\partial_\varepsilon \partial_\alpha \partial_\beta \partial_\gamma F(t) = (\nabla_\varepsilon c)(\partial_\alpha, \partial_\beta, \partial_\gamma)$$

has the following quasihomogeneity equation

$$\mathcal{L}_{E_F} F(t) = (3-d)F(t) + A_{\alpha\beta} t^\alpha t^\beta + B_\alpha t^\alpha + C \tag{1.47}$$

for some constants $A_{\alpha\beta},\ B_\alpha,\ C$.

Proof. From Lemma 1.17,

$$\begin{aligned} \mathcal{L}_{E_F} c_{\alpha\beta\gamma} &= c^\epsilon_{\alpha\beta} \mathcal{L}_{E_F} \eta_{\gamma\epsilon} + \eta_{\gamma\epsilon} \mathcal{L}_{E_F} c^\epsilon_{\alpha\beta} \\ &= (2-d)\eta_{\gamma\epsilon} c^\epsilon_{\alpha\beta} + \eta_{\gamma\epsilon} c^\epsilon_{\alpha\beta} \\ &= (3-d) c_{\alpha\beta\gamma} \\ &= \partial_\alpha \partial_\beta \partial_\gamma ((3-d)F). \end{aligned}$$

Moreover, c as a tensor with components $c_{\alpha\beta\gamma}$ has the following:

$$\begin{aligned} (\mathcal{L}_{E_F} c)_{\alpha\beta\gamma} &= E^\epsilon_F \partial_\epsilon c_{\alpha\beta\gamma} + \partial_\alpha E^\epsilon \cdot c_{\epsilon\beta\gamma} + \partial_\beta E^\epsilon \cdot c_{\alpha\epsilon\gamma} + \partial_\gamma E^\epsilon \cdot c_{\alpha\beta\epsilon} \\ &= \mathcal{L}_{E_F} c_{\alpha\beta\gamma} + (q_\alpha + q_\beta + q_\gamma) c_{\alpha\beta\gamma} \\ &= \partial_\alpha \partial_\beta \partial_\gamma \mathcal{L}_{E_F} F. \end{aligned}$$

Therefore, one has

$$\partial_\alpha \partial_\beta \partial_\gamma \left[\mathcal{L}_{E_F} F - (3-d)F \right] = 0,$$

and this implies (1.47). □

Example 1.19. Consider the space M

$$M = \left\{ \lambda(p) = p^{n+1} + a_n p^{n-1} + \cdots + a_2 p + a_1 | a_1, \ldots, a_n \in \mathbb{C} \right\},$$

and we identify the tangent plane to M with the space of all polynomials of degree less than n. The algebra A_λ on $T_\lambda M$ coincides with

$$A_\lambda = \mathbb{C}[p]/(\lambda'(p)),$$

where

$$\lambda'(p) = \frac{d\lambda(p)}{dp}.$$

The flat coordinate is defined by

$$\langle f, g \rangle_\lambda = \operatorname*{res}_{p=\infty} \frac{f(p)g(p)}{\lambda'(p)}.$$

The unity vector field and the Euler vector field are

$$e = \frac{\partial}{\partial a_1},\ E = \sum_{i=1}^{n} \frac{n-i+1}{n+1} a_i \frac{\partial}{\partial a_i}.$$

In section 4, this example will be discussed in detail with explicit computations for $n = 1$ and $n = 2$.

Lemma 1.20. Any solution of WDVV equations with $d_1 \neq 0$ defined in a domain $t \in M$ determines in this domain the structure of a Frobenius manifold by the formulae

$$\partial_\alpha \cdot \partial_\beta := c^\gamma_{\alpha\beta}(t)\partial_\gamma, \tag{1.48a}$$

$$\langle \partial_\alpha, \partial_\beta \rangle := \eta_{\alpha\beta}, \tag{1.48b}$$

$$e := \partial_1, \tag{1.48c}$$

and the Euler vector field has the form (1.25).

Conversely, locally any Frobenius manifold with the structure (1.48a)–(1.48c)(1.24) corresponds to some solution of WDVV equations.

Proof. Starting from a solution $F(t)$ of the WDVV equations, the metric (1.48b) is obviously flat. Therefore, in these coordinates, the covariant derivatives coincide with partial derivatives. As a result, for the vector field (1.48c), one has

$$\nabla e = 0.$$

Thus **FM1** is satisfied. The trilinear and four-linear forms in **FM2** given as

$$c(\partial_\alpha, \partial_\beta, \partial_\gamma) = \partial_\alpha\partial_\beta\partial_\gamma F(t),$$

$$(\nabla_\alpha c)(\partial_\beta, \partial_\gamma, \partial_\varepsilon) = \partial_\alpha\partial_\beta\partial_\gamma\partial_\varepsilon F(t),$$

, where the four-linear form is obviously symmetric. Thus **FM2** is also satisfied. **FM3** is obvious.

Conversely, consider a Frobenius manifold with locally flat coordinates $t^1, \dots, t^n$ such that the metric $\langle\ ,\ \rangle$ is constant, taking $u = \partial_\alpha, v = \partial_\beta,\ w = \partial_\gamma,\ z = \partial_\varepsilon$, **FM2** reads

$$c(\partial_\alpha, \partial_\beta, \partial_\gamma) = \langle \partial_\alpha \cdot \partial_\beta, \partial_\gamma \rangle,$$

and

$$\partial_\varepsilon c(\partial_\alpha, \partial_\beta, \partial_\gamma)$$

is symmetric. The symmetry implies a function $F(t)$ such that

$$c(\partial_\alpha, \partial_\beta, \partial_\gamma) = \partial_\alpha\partial_\beta\partial_\gamma F(t).$$

Since $\nabla e_F = 0$, one can obtain (1.48c) by a linear transformation of t^1.

We also need to show that $F(t)$ satisfies (1.25). Following Lemma 1.2.2, this is immediate by choosing $d = 3 - d_F$. The lemma is proved. □

2 Equivalence between structures

2.1 From Frobenius Manifolds to Flat Pencils

Definition 2.1. The *intersection form* of the Frobenius manifold is the bilinear form on T^*M

$$i_{E_F}(\omega_1 \cdot \omega_2). \tag{2.1}$$

Here i_{E_F} represents the inner product, namely, for a p-form ω and $X \in TM$,

$$(i_X\omega)(X_1, \ldots, X_{p-1}) = \omega(X, X_1, \ldots, X_{p-1}).$$

Put

$$(\ ,\ )_2 = \langle\ ,\ \rangle$$

which is the bilinear form on the Frobenius manifold, and put

$$(\omega_1, \omega_2) \equiv (\omega_1, \omega_2)_1 = i_{E_F}(\omega_1 \cdot \omega_2). \tag{2.2}$$

Theorem 2.2. The metrics $(\ ,\ )$ and $\langle\ ,\ \rangle$ on a Frobenius manifold form a flat pencil quasihomogeneous of degree d.

To prove the theorem, we need to check that the metrics satisfy the three conditions in Definition 1.7.

Lemma 2.3. In the flat coordinates $t^1, \ldots, t^n$ for $\langle\ ,\ \rangle$, setting

$$dt^\alpha \cdot dt^\beta = c_\gamma^{\alpha\beta}(t) dt^\gamma, \tag{2.3}$$

the components of the bilinear form (2.1) are given by the formula

$$g^{\alpha\beta}(t) = (dt^\alpha, dt^\beta) = E_F^\epsilon(t) c_\epsilon^{\alpha\beta}(t) \tag{2.4}$$

$$= R_\epsilon^\alpha F^{\epsilon\beta}(t) + F^{\alpha\epsilon}(t) R_\epsilon^\beta + A^{\alpha\beta} \tag{2.5}$$

where

$$R_\beta^\alpha(t) = \frac{d-1}{2}\delta_\beta^\alpha + \partial_\beta E_F^\alpha(t) \tag{2.6a}$$

$$c_\gamma^{\alpha\beta}(t) = \eta^{\alpha\epsilon} c_{\epsilon\gamma}^\beta(t) \tag{2.6b}$$

$$F^{\alpha\beta}(t) = \eta^{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu F(t) \tag{2.6c}$$

$$A^{\alpha\beta} = \eta^{\alpha\lambda}\eta^{\beta\mu} A_{\lambda\mu}. \tag{2.6d}$$

Proof. (2.6a) follows directly from (2.3) and (2.2):

$$(dt^\alpha, dt^\beta) \overset{(2.2)}{=} i_{E_F}(dt^\alpha \cdot dt^\beta) \overset{(2.3)}{=} i_{E_F}(c_\gamma^{\alpha\beta} dt^\gamma) = c_\gamma^{\alpha\beta} dt^\gamma (E_F^\epsilon \partial_\epsilon) = E_F^\epsilon(t) c_\epsilon^{\alpha\beta}(t).$$

Differentiate w.r.t. t^μ and t^λ on both sides of (1.47), and contract by $\eta^{\alpha\lambda}\eta^{\beta\mu}$.

$$\begin{aligned}\eta^{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu(\mathcal{L}_{E_F}F) &= (3-d)\eta^{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu F + \eta^{\alpha\lambda}\eta^{\beta\mu}A_{\lambda\mu}\\ &= (3-d)F^{\alpha\beta} + A^{\alpha\beta}.\end{aligned}$$

By (1.41),

$$\begin{aligned}&\eta^{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu(\mathcal{L}_{E_F}F)\\ =&\eta^{\alpha\lambda}\eta^{\beta\mu}(\partial_\lambda\partial_\mu E_F^\epsilon \partial_\epsilon F + \partial_\mu E_F^\epsilon \partial_\lambda\partial_\epsilon F + \partial_\lambda E_F^\epsilon \partial_\mu\partial_\epsilon F + E_F^\epsilon \partial_\lambda\partial_\mu\partial_\epsilon F)\\ =&(\eta^{\beta\mu}\partial_\mu E_F^\epsilon)(\eta^{\alpha\lambda}\partial_\lambda\partial_\epsilon F) + (\eta^{\alpha\lambda}\partial_\lambda E_F^\epsilon)(\eta^{\beta\mu}\partial_\mu\partial_\epsilon F) + E_F^\epsilon c_\epsilon^{\alpha\beta}.\end{aligned}$$

Using the same technique of Corollary 1.9 and by (1.44a), (1.44b), one has

$$\mathcal{L}_{E_F} c_\gamma^{\alpha\beta} = c_\gamma^{\alpha\beta}, \tag{2.7}$$

$$\mathcal{L}_{E_F} \eta^{\alpha\beta} = (d-2)\eta^{\alpha\beta}. \tag{2.8}$$

Thus

$$\begin{aligned}
(d-2)\eta^{\alpha\beta} = \mathcal{L}_{E_F}\eta^{\alpha\beta} &= E_F^\mu \partial_\mu \eta^{\alpha\beta} - \partial_\mu E_F^\alpha \eta^{\mu\beta} - \partial_\mu E_F^\beta \eta^{\alpha\mu} \\
&= -\eta^{\mu\beta}\partial_\mu E_F^\alpha - \eta^{\alpha\mu}\partial_\mu E_F^\beta \\
\Rightarrow (\eta^{\beta\mu}\partial_\mu E_F^\epsilon)(\eta^{\alpha\lambda}\partial_\lambda\partial_\epsilon F) &= ((2-d)\eta^{\beta\epsilon} - \eta^{\mu\epsilon}\partial_\mu E_F^\beta)\eta^{\alpha\lambda}\partial_\lambda\partial_\epsilon F \\
&= (2-d)\eta^{\beta\epsilon}\eta^{\alpha\lambda}\partial_\lambda\partial_\epsilon F - \partial_\mu E_F^\beta(\eta^{\mu\epsilon}\eta^{\alpha\lambda}\partial_\lambda\partial_\epsilon F) \\
&= (2-d)F^{\alpha\beta} - F^{\alpha\epsilon}\partial_\epsilon E_F^\beta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(3-d)F^{\alpha\beta} + A^{\alpha\beta} &= (2-d)F^{\alpha\beta} - F^{\alpha\epsilon}\partial_\epsilon E_F^\beta + (2-d)F^{\alpha\beta} - F^{\beta\epsilon}\partial_\epsilon E_F^\alpha + E_F^\epsilon c_\epsilon^{\alpha\beta} \\
\Rightarrow E_F^\epsilon c_\epsilon^{\alpha\beta} &= (d-1)F^{\alpha\beta} + F^{\alpha\epsilon}\partial_\epsilon E_F^\beta + F^{\beta\epsilon}\partial_\epsilon E_F^\alpha + A^{\alpha\beta} \\
&= \frac{d-1}{2}\delta_\epsilon^\alpha F^{\epsilon\beta} + \frac{d-1}{2}F^{\alpha\epsilon}\delta_\epsilon^\beta + F^{\alpha\epsilon}\partial_\epsilon E_F^\beta + \partial_\epsilon E_F^\alpha \cdot F^{\beta\epsilon} + A^{\alpha\beta} \\
&= R_\epsilon^\alpha F^{\epsilon\beta} + F^{\alpha\epsilon}R_\epsilon^\beta + A^{\alpha\beta}.
\end{aligned}$$

□

Lemma 2.4. $g^{\alpha\beta}(t) - \lambda\eta^{\alpha\beta}$ does not degenerate for any λ.

Proof. By (1.18),

$$c_1^{\alpha\beta}(t) = \eta^{\alpha\epsilon}\eta^{\beta\gamma}c_{1\epsilon\gamma}(t) = \eta^{\alpha\epsilon}\eta^{\beta\gamma}\eta_{\epsilon\gamma} = \delta_\gamma^\alpha\eta^{\beta\gamma} = \eta^{\alpha\beta}.$$

Moreover, using (1.45),

$$E_F^\epsilon(t) = q_\epsilon t^\epsilon.$$

Take e_F as (1.43) and E_F that satisfies (1.12), $q_1 = 1$. Thus, for small $t^2, \ldots, t^n$, the previous lemma implies

$$\begin{aligned}
g^{\alpha\beta}(t) = q_\epsilon t^\epsilon c_\epsilon^{\alpha\beta} &\simeq \eta^{\alpha\beta}t^1 \\
\Rightarrow g^{\alpha\beta}(t) - \lambda\eta^{\alpha\beta} &= \eta^{\alpha\beta}(t^1 - \lambda) + \lambda\eta^{\alpha\beta}
\end{aligned}$$

which does not degenerate in t^1, thus is non-degenerate. □

Lemma 2.5. The contravariant Levi-Civita connection for the metric $(\ ,\ ) - \lambda\langle\ ,\ \rangle$ is

$$\Gamma_\gamma^{\alpha\beta}(t) = c_\gamma^{\alpha\epsilon}(t)R_\epsilon^\beta. \tag{2.9}$$

Proof. By uniqueness of Levi-Civita connection, it suffices to show that $g^{\alpha\beta}(t) - \lambda\eta^{\alpha\beta}$ satisfies (1.4) and (1.5). Let

$$g_\lambda^{\alpha\beta} := g^{\alpha\beta} - \lambda\eta^{\alpha\beta}.$$

From definitions and commutativity of multiplication $\partial_a \cdot \partial_b = \partial_b \cdot \partial_a$, one has

$$\eta^{\mu\epsilon}\Gamma_\epsilon^{\sigma\nu} \overset{(2.9),(2.6b)}{=} \eta^{\mu\epsilon}\eta^{\sigma\omega}c_{\omega\epsilon}^\gamma R_\gamma^\nu \overset{\text{associativity}}{=} \eta^{\sigma\omega}\eta^{\mu\epsilon}c_{\epsilon\omega}^\gamma R_\gamma^\nu = \eta^{\sigma\omega}c_\omega^{\mu\gamma}R_\gamma^\nu = \eta^{\sigma\omega}\Gamma_\omega^{\mu\nu} = \eta^{\sigma\epsilon}\Gamma_\epsilon^{\mu\nu}.$$

On the other hand, (1.30) gives

$$c_{\alpha\beta}^\epsilon c_{\epsilon\gamma}^\lambda \partial_\lambda = c_{\beta\gamma}^\epsilon c_{\epsilon\alpha}^\lambda \partial_\lambda.$$

Contract by $\eta^{\alpha\mu}\eta^{\gamma\sigma}$:

$$c^{\epsilon}_{\alpha\beta}c^{\lambda}_{\epsilon\gamma}\cdot\eta^{\alpha\mu}\eta^{\gamma\sigma}=\underbrace{(\eta^{\alpha\mu}c^{\epsilon}_{\alpha\beta})}_{c^{\mu\epsilon}_{\beta}}\underbrace{(\eta^{\gamma\sigma}c^{\lambda}_{\epsilon\gamma})}_{c^{\sigma\lambda}_{\epsilon}}=(\eta^{\gamma\sigma}c^{\epsilon}_{\beta\gamma})(\eta^{\alpha\mu}c^{\lambda}_{\epsilon\alpha})=c^{\sigma\epsilon}_{\beta}c^{\mu\lambda}_{\epsilon} \tag{2.10}$$

$$\Rightarrow g^{\mu\epsilon}\Gamma^{\sigma\nu}_{\epsilon}\overset{(2.9),(2.4)}{=}E^{\beta}c^{\mu\epsilon}_{\beta}c^{\sigma\lambda}_{\epsilon}R^{\nu}_{\lambda}\overset{(2.10)}{=}E^{\beta}c^{\sigma\epsilon}_{\beta}c^{\mu\lambda}_{\epsilon}R^{\nu}_{\lambda}\overset{(2.9),(2.4)}{=}g^{\sigma\epsilon}\Gamma^{\mu\nu}_{\epsilon}.$$

Therefore, as a linear combination of $g^{\alpha\beta}$ and $\eta^{\alpha\beta}$, g_{λ} and $\Gamma^{\alpha\beta}_{\gamma}$ defined as (2.9) satisfies (1.4).

Also, notice that, by (1.41),

$$\partial_{\gamma}R^{\alpha}_{\beta}=\frac{d-1}{2}\partial_{\gamma}\delta^{\alpha}_{\beta}+\partial_{\gamma}\partial_{\beta}E^{\alpha}=0$$

and

$$\partial_{\gamma}F^{\alpha\beta}(t)=\eta^{\alpha\lambda}\eta^{\beta\mu}\partial_{\gamma}\partial_{\lambda}\partial_{\mu}F(t).=c^{\alpha\beta}_{\gamma}(t)$$

differentiate (2.5) by t^{γ},

$$\partial_{\gamma}g^{\alpha\beta}(t)=F^{\epsilon\beta}\partial_{\gamma}R^{\alpha}_{\epsilon}+R^{\alpha}_{\epsilon}\partial_{\gamma}F^{\epsilon\beta}+F^{\epsilon\alpha}\partial_{\gamma}R^{\beta}_{\epsilon}+R^{\beta}_{\epsilon}\partial_{\gamma}F^{\epsilon\alpha}+\partial_{\gamma}A^{\alpha\beta}=R^{\alpha}_{\epsilon}c^{\epsilon\beta}_{\gamma}+R^{\beta}_{\epsilon}c^{\alpha\epsilon}_{\gamma}$$

since $\eta^{\alpha\beta}$ is constant and $\partial_{\gamma}\eta^{\alpha\beta}=0$, this proves (1.5). □

Lemma 2.6. The curvature of the pencil of the metrics vanishes identically for any λ.

Proof. Observe that

$$\partial_{\epsilon}c^{\beta\gamma}_{\delta}=\eta^{\beta\lambda}\eta^{\gamma\mu}\partial_{\epsilon}\partial_{\lambda}\partial_{\mu}\partial_{\delta}F(t)=\partial_{\delta}c^{\beta\gamma}_{\epsilon}$$

multiply by R^{γ}_{λ}, we have

$$\partial_{\epsilon}\Gamma^{\beta\gamma}_{\delta}=\partial_{\delta}\Gamma^{\beta\gamma}_{\epsilon}. \tag{2.11}$$

For (2.9), by (2.10),

$$\Gamma^{\alpha\beta}_{\epsilon}\Gamma^{\epsilon\gamma}_{\delta}=c^{\alpha\lambda}_{\epsilon}R^{\beta}_{\lambda}c^{\epsilon\mu}_{\delta}R^{\gamma}_{\mu}=c^{\alpha\mu}_{\epsilon}R^{\beta}_{\mu}c^{\epsilon\lambda}_{\delta}R^{\gamma}_{\lambda}=c^{\alpha\lambda}_{\epsilon}R^{\beta}_{\mu}c^{\epsilon\mu}_{\delta}R^{\gamma}_{\lambda}=\Gamma^{\alpha\gamma}_{\epsilon}\Gamma^{\epsilon\beta}_{\delta} \tag{2.12}$$

with (2.11), (2.12), the curvature tensor identically vanishes:

$$R^{\alpha\beta\gamma}_{\delta}=(g^{\alpha\epsilon}-\lambda\eta^{\alpha\epsilon})(\partial_{\epsilon}\Gamma^{\beta\gamma}_{\delta}-\partial_{\delta}\Gamma^{\beta\gamma}_{\epsilon})+\Gamma^{\alpha\beta}_{\epsilon}\Gamma^{\epsilon\gamma}_{\delta}-\Gamma^{\alpha\gamma}_{\epsilon}\Gamma^{\epsilon\beta}_{\delta}=0.$$

□

Lemma 2.7. Set

$$\tau=\eta_{1\alpha}t^{\alpha},$$

then e_f and E_f defined in (1.10) and (1.11) coincides with the unit vector field e_F and the Euler vector field E_F of the Frobenius structure, and satisfies (1.12)-(1.15).

Proof. Choose coordinates that satisfies (1.43), then (1.10) follows since

$$\nabla_2\tau=(\eta^{is}\eta_{1s})\partial_i=\delta^{i}_{1}\partial_i=\partial_1=e_F,$$

and (1.11) follows from (2.1), (1.45) and (1.18):

$$(\nabla_1\tau)_i=g^{is}\eta_{1s}=E^{\epsilon}_{F}c^{is}_{\epsilon}\eta_{1s}=q_{\epsilon}t^{\epsilon}\eta_{1s}\eta^{i\lambda}\eta^{s\mu}c_{\lambda\mu\epsilon}=q_{\epsilon}t^{\epsilon}\delta^{1}_{\mu}\eta^{i\lambda}c_{\lambda\mu\epsilon}=q_{\epsilon}t^{\epsilon}\eta^{i\lambda}\eta_{\lambda\epsilon}=q_{\epsilon}t^{\epsilon}\delta^{i}_{\epsilon}=E^{i}_{f}.$$

Moreover, by choosing coordinates such that $q_1=1$,

$$[e_F,E_F]=e_FE_F-E_Fe_F=e_F.$$

Since $(\ , \ )_1$ is defined as the intersection form (2.1), that is,

$$g^{\alpha\beta} = E^\epsilon c_\epsilon^{\alpha\beta} = E^\epsilon g^{\alpha\gamma} c_{\gamma\epsilon}^\beta,$$

by (2.7) and (2.8),

$$\begin{aligned}
\mathcal{L}_{E_F} g^{\alpha\beta} &= \mathcal{L}_{E_F}(E_F^\epsilon \eta^{\alpha\gamma} c_{\gamma\epsilon}^\beta) \\
&= \eta^{\alpha\gamma} c_{\gamma\epsilon}^\beta \mathcal{L}_{E_F} E_F^\epsilon + E_F^\epsilon c_{\gamma\epsilon}^\beta \mathcal{L}_{E_F} \eta^{\alpha\gamma} + E_F^\epsilon \eta^{\alpha\gamma} \mathcal{L}_{E_F} c_{\gamma\epsilon}^\beta \\
&= 0 + (d-2) E_F^\epsilon c_{\gamma\epsilon}^\beta \eta^{\alpha\gamma} + E_F^\epsilon \eta^{\alpha\gamma} c_{\gamma\epsilon}^\beta \\
&= (d-1) E_F^\epsilon c_\epsilon^{\alpha\beta} \\
&= (d-1) g^{\alpha\beta}.
\end{aligned}$$

Notice that, since $\eta^{\alpha\beta}$ is constant,

$$\mathcal{L}_{e_F} \eta^{\alpha\beta} = 0,$$

and

$$\mathcal{L}_e E_F^\epsilon = \partial_1 (q_\epsilon t^\epsilon) = q_\epsilon \partial_1 t^\epsilon = \delta_\epsilon^1.$$

By (1.18),

$$\mathcal{L}_{e_F} c_{\beta\gamma}^\alpha = \partial_1 c_{\beta\gamma}^\alpha = \eta^{\alpha\epsilon} \partial_1 \partial_\epsilon \partial_\beta \partial_\gamma F = \eta^{\alpha\epsilon} \partial_\beta c_{1\epsilon\gamma} = \eta^{\alpha\epsilon} \partial_\beta \eta_{\epsilon\gamma} = 0$$

Therefore,

$$\begin{aligned}
\mathcal{L}_{e_F} g^{\alpha\beta} &= \mathcal{L}_{e_F}(E_F^\epsilon \eta^{\alpha\gamma} c_{\gamma\epsilon}^\beta) \\
&= \eta^{\alpha\gamma} c_{\gamma\epsilon}^\beta \mathcal{L}_{e_F} E_F^\epsilon + E_F^\epsilon c_{\gamma\epsilon}^\beta \mathcal{L}_{e_F} \eta^{\alpha\gamma} + E_F^\epsilon \eta^{\alpha\gamma} \mathcal{L}_{e_F} c_{\gamma\epsilon}^\beta \\
&= \eta^{\alpha\gamma} c_{\gamma 1}^\beta \\
&= \eta^{\alpha\gamma} \eta^{\beta\epsilon} \eta_{\epsilon\gamma} \\
&= \eta^{\alpha\gamma} \delta_\gamma^\beta = \eta^{\alpha\beta}.
\end{aligned}$$

□

Combining the lemmas, Theorem 2.2 is proved.

2.2 From Flat Pencils to Frobenius Manifolds

Lemma 2.8. The difference between two Christoffel symbols behaves tensorially.

Proof. Claim: the difference between two connections is a tensor. Indeed, let ∇, $\tilde{\nabla}$ be two connections, then

$$(\nabla - \tilde{\nabla})_{fX} Y = f \nabla_X Y - f \tilde{\nabla}_X Y = f(\nabla - \tilde{\nabla})_X Y,$$

$$(\nabla - \tilde{\nabla})_X fY = (XfY + f\nabla_X Y) - (XfY + f\tilde{\nabla}_X Y) = f(\nabla - \tilde{\nabla})_X Y.$$

This is true for both covariant and contravariant connections since they are defined to follow the same properties.

Let Γ_{ij}^k and $\tilde{\Gamma}_{ij}^k$ be Christoffel symbols of the two connections respectively, then

$$(\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k)(f\partial_k) = f(\nabla - \tilde{\nabla})_{\partial_i} \partial_j = f(\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k)\partial_k.$$

□

Lemma 2.9. The functions

$$\Delta^{ijk}(x) = {g_2}^{js} {\Gamma_1}_s^{ik} - {g_1}^{is} {\Gamma_2}_s^{jk}$$

are components of a rank three tensor (namely, a trilinear form on T^*M). Two flat metrics ${g_1}^{ij}$ and ${g_2}^{ij}$ can be simultaneously reduced to constant form iff $\Delta^{ijk} = 0$.

Proof. Lemma 2.8 shows that ${\Gamma_1}^k_{st} - {\Gamma_2}^k_{st}$ is a tensor. Multiply by $g_1^{is} g_2^{jt}$,

$$g_1^{is} g_2^{jt} ({\Gamma_1}^k_{st} - {\Gamma_2}^k_{st}) = {g_2}^{jt} {\Gamma_1}^{ik}_t - {g_1}^{is} {\Gamma_2}^{jk}_s = \Delta^{ijk}.$$

If the two metrics are simultaneously reducible to a constant form, then the Christoffel symbols are identically 0, thus $\Delta^{ijk} = 0$. Conversely, if $\Delta^{ijk} = 0$, the two Christoffel symbols are identical. Thus, if one of the metrics can be reduced to constant form, by choosing flat coordinates for one metric, one of the Christoffel symbols is 0, and the other one must also be 0. □

Consider a (2,1)-tensor

$$\Delta^{ij}_i := g_{2is} \Delta^{sjk} \tag{2.13}$$

and a bilinear operator

$$\begin{gathered} (u, v) \mapsto \Delta(u, v) \\ \Delta(u, v)_k = u_i v_j \Delta^{ij}_k(x) \end{gathered} \tag{2.14}$$

for $u,\ v \in T^*_x M_0$.

Lemma 2.10. For a flat pencil of metrics, the tensor (2.13) satisfies the following properties:

$$(\Delta(u, v), w)_1 = (u, \Delta(w, v))_1 \tag{2.15}$$

$$(\Delta(u, v), w)_2 = (u, \Delta(w, v))_2 \tag{2.16}$$

$$\Delta(\Delta(u, v), w) = \Delta(\Delta(u, w), v) \tag{2.17}$$

$$\nabla^u_2 \Delta(v, w) - \nabla^v_2 \Delta(u, w) = \Delta(\nabla^u_2 v - \nabla^v_2 u, w) \tag{2.18}$$

Proof. Choose a system of flat coordinates $x^1, \ldots, x^n$ for the metric $(\ ,\ )_2$. In these coordinates,

$${\Gamma_2}^{ij}_k = 0,$$

and

$$\Delta^{ij}_k = g_{2is} \Delta^{sjk} = g_{2is} {g_2}^{jt} {\Gamma_1}^{sk}_t = g_{2is} {g_2}^{st} g_{2st} {g_2}^{jt} {\Gamma_1}^{sk}_t = \delta^i_t \delta^j_s {\Gamma_1}^{sk}_t = {\Gamma_1}^{ij}_t. \tag{2.19}$$

By definition of flat pencil, the connection for the linear pencil $(\ ,\ )_1 - \lambda(\ ,\ )_2$ gives

$${\Gamma_1}^{ij}_t - \lambda {\Gamma_2}^{ij}_k = \Delta^{ij}_k,$$

and its metric is given as

$${g_1}^{is} - \lambda {g_2}^{is}.$$

Thus (1.4) and (1.5) are satisfied. Namely for any λ,

$$\begin{gathered} ({g_1}^{is} - \lambda {g_2}^{is}) \Delta^{jk}_s = ({g_1}^{js} - \lambda {g_2}^{js}) \Delta^{ik}_s \\ \Rightarrow ({g_1}^{is} \Delta^{jk}_s - {g_1}^{js} \Delta^{ik}_s) = \lambda ({g_2}^{is} \Delta^{jk}_s - {g_2}^{js} \Delta^{ik}_s). \end{gathered}$$

Both sides of the equation must vanish, thus

$${g_1}^{is} \Delta^{jk}_s = {g_1}^{js} \Delta^{ik}_s \tag{2.20}$$

$${g_2}^{is} \Delta^{jk}_s = {g_2}^{js} \Delta^{ik}_s \tag{2.21}$$

Expanding (2.15), one has

$$\begin{gathered} u_i v_j w_k \Delta^{ij}_t {g_2}^{tk} = u_i v_j w_k \Delta^{kj}_t {g_2}^{it} \\ \Leftrightarrow \Delta^{ij}_t {g_2}^{tk} = \Delta^{kj}_t {g_2}^{it} \end{gathered}$$

which coincides with (2.20). Similarly, (2.16) coincides with (2.21). For (2.17) and (2.18), vanishing of curvature of flat metric gives

$$({g_1}^{is} - \lambda {g_2}^{is})(\partial_s \Delta_l^{jk} - \partial_l \Delta_s^{jk}) - \Delta_s^{ij}\Delta_l^{sk} + \Delta_s^{ik}\Delta_l^{sj} = 0.$$

Vanishing of coefficients of λ and remaining term gives respectively

$$\partial_s \Delta_l^{jk} = \partial_l \Delta_s^{jk}, \tag{2.22}$$

$$\Delta_s^{ij}\Delta_l^{sk} = \Delta_s^{ik}\Delta_l^{sj}. \tag{2.23}$$

and coincides with (2.22) and (2.23) respectively. □

Corollary 2.11. There exists a vector field $f = f^\alpha \partial_\alpha$ such that

$$\Delta^{\alpha\beta\gamma} = \nabla_2^\alpha \nabla_2^j f^\gamma.$$

Proof. (2.22) implies that there exists a tensor $f^{\alpha\beta}$ such that

$$\Delta_\gamma^{\alpha\beta} = \partial_\gamma f^{\alpha\beta},$$

moreover, (2.21) implies that there exists a vector field f such that

$$f^{\alpha\beta} = {g_2}^{\alpha\gamma}\partial_\gamma f^\beta,$$

thus

$$\Delta^{\alpha\beta\gamma} = {g_2}^{\epsilon\alpha}{g_2}^{\beta\alpha}\partial_\epsilon \partial_\alpha f^\gamma = \nabla_2^\alpha \nabla_2^j f^\gamma.$$

□

Lemma 2.12. For a quasihomogeneous flat pencil, the following equations hold true

$$\nabla_2 \nabla_2 \tau = 0 \tag{2.24}$$

$$\nabla_2 \nabla_2 E_f = 0. \tag{2.25}$$

Proof. For a flat pencil for $(\ ,\ )_2$, since ${g_2}^{ij}$ is constant, (1.15) implies

$$0 = \mathcal{L}_{e_f}{g_2}^{ij} = e_f^s \partial_s {g_2}^{ij} - \partial_s e_f^i {g_2}^{sj} - \partial_s e_f^j {g_2}^{is} = -\partial_s e_f^i {g_2}^{sj} - \partial_s e_f^j {g_2}^{is}.$$

Differentiate (1.11) w.r.t. t^s,

$$\begin{aligned} \partial_s e_f^i &= \partial_s {g_2}^{ik}\partial_s \tau + {g_2}^{ik}\partial_s\partial_k \tau = {g_2}^{ik}\partial_s\partial_k\tau \\ \Rightarrow 0 = \mathcal{L}_{e_f}{g_2}^{ij} &= -{g_2}^{sj}{g_2}^{ik}\partial_s\partial_k\tau - {g_2}^{is}{g_2}^{jk}\partial_s\partial_k\tau \\ &= -2{g_2}^{sj}{g_2}^{ik}\partial_s\partial_k\tau \end{aligned}$$

which implies (2.24). By (1.13)-(1.15), for contravariant metrics $(\ ,\ )_1$ and $(\ ,\ )_2$,

$$\begin{aligned} \mathcal{L}_{E_f}(\ ,\ )_2 &= \mathcal{L}_{E_f}(\mathcal{L}_{e_f}(\ ,\ )_1) \\ &= \mathcal{L}_{[E_f,e_f]}(\ ,\ )_1 + \mathcal{L}_{e_f}(\mathcal{L}_{E_f}(\ ,\ )_1) \\ &= \mathcal{L}_{-e_f}(\ ,\ )_1 + (d-1)\mathcal{L}_{e_f}(\ ,\ )_1 \\ &= (d-2)\mathcal{L}_{e_f}(\ ,\ )_1 \\ &= (d-2)(\ ,\ )_2, \end{aligned}$$

this means

$$\mathcal{L}_{E_f}{g_2}^{ij} = (d-2){g_2}^{ij}, \tag{2.26}$$

analogously, differentiating w.r.t. t^s,

$$\partial_s\partial_t E_f^i = 0,$$

hence (2.25) is proved. □

Corollary 2.13. The eigenvalues of the matrix

$$\nabla_{2i} E_f^j(x) \tag{2.27}$$

does not depend on the point of the manifold.

Corollary follows immediately from (2.25) that implies $\nabla_2 E_f$ is constant.

Definition 2.14. A quasihomogeneous flat pencil is said to be *regular* if the (1,1)-tensor

$$\bar{R}_i^j := \frac{d-1}{2}\delta_i^j + \nabla_{2i} E_f^j \tag{2.28}$$

does not degenerate on M.

In a flat coordinates for $(\ ,\ )_2$, $\bar{R}_i^j$ has the following form:

$$\bar{R}_i^j = \frac{d-1}{2}\delta_i^j + \partial_i E_f^j$$

Theorem 2.15. Let M be a manifold carrying a regular quasihomogeneous flat pencil. Denote $M_0 \subset M$ the subset of M where the (contravariant) metric $(\ ,\ )_2$ is invertible. Define the multiplication of 1-forms on M_0 putting

$$u \cdot v := \Delta(u, R^{-1} v). \tag{2.29}$$

Then there exists a unique Frobenius structure on M such that

$$\langle\ ,\ \rangle = (\ ,\ )_2. \tag{2.30}$$

Then the following properties are satisfied:

(1) the multiplication of tangent vectors is $\langle\ ,\ \rangle$-dual to the product (2.29), namely, $\langle\ ,\ \rangle$ is non-degenerate;

(2) the unity vector field e_F and the Euler vector field E_F coincide with the form for e_f and E_f in (1.10) and (1.11) respectively;

(3) the intersection form is equal to $(\ ,\ )_1$.

Choose flat coordinates $t^1, \dots, t^n$ for the metric $(\ ,\ )_2$. Then

$$\eta^{\alpha\beta} := (dt^\alpha, dt^\beta)_2 \tag{2.31}$$

is a constant symmetric invertible matrix. Denote

$$g^{\alpha\beta}(t) := (dt^\alpha, dt^\beta)_1$$

and

$$K_\alpha^\beta := \partial_\alpha E_f^\beta. \tag{2.32}$$

Lemma 2.12 implies the matrix K_α^β is constant. The components of the contravariant Levi-Civita connection for the metric $g^{\alpha\beta}$ in these coordinates is denoted $\Gamma_\gamma^{\alpha\beta}$ $(= {\Gamma_1}_\gamma^{\alpha\beta})$. In this coordinate system, by (2.19),

$$\Delta_\gamma^{\alpha\beta} = \Gamma_\gamma^{\alpha\beta}. \tag{2.33}$$

Lemma 2.16. The vector field e_f is constant in the coordinates t^α. It is an eigenvector of the operator (2.32) with the eigenvalue $1 - d$.

Proof. By (2.24), $\nabla_2\nabla_2\tau = 0$, thus $e_f = \nabla_2\tau$ is constant. Normalize the choice of flat coordinates that satisfies

$$e_f^\alpha = \eta^{\alpha n}, \tag{2.34}$$

thus

$$\tau = t^n + \text{constant}. \tag{2.35}$$

Thus, according to (1.10),

$$E_f^\alpha(t) = g^{\alpha\beta}\partial_\beta\tau = g^{\alpha n}(t) \tag{2.36}$$

for $\alpha = 1, \ldots, n$.

By (1.12),

$$\eta^{\beta n}\partial_\beta = e_f = [e_f, E_f] = (\eta^{\alpha n}\partial_\alpha)(E_f^\beta\partial_\beta) - (E_f^\beta\partial_\beta)(\eta^{\alpha n}\partial_\alpha) = \eta^{\alpha n}\partial_\alpha E_f^\beta\partial_\beta = \eta^{\alpha n}K_\alpha^\beta\partial_\beta,$$

therefore

$$\eta^{\alpha n} = \eta^{\epsilon n}K_\epsilon^\alpha. \tag{2.37}$$

By (2.26) and (2.37)

$$\begin{aligned}(d-2)\eta^{n\alpha} &= \mathcal{L}_{E_f}\eta^{n\alpha} \\ &= E_f^\epsilon\partial_\epsilon\eta^{n\alpha} - \partial_\epsilon E_f^n\eta^{\epsilon\alpha} - \partial_\epsilon E_f^\alpha\eta^{n\epsilon} \\ &= -\eta^{\epsilon\alpha}K_\epsilon^n - \eta^{n\epsilon}K_\epsilon^\alpha \\ &= -\eta^{\epsilon\alpha}K_\epsilon^n - \eta^{\alpha n},\end{aligned}$$

therefore

$$\eta^{\epsilon\alpha}K_\epsilon^n = (1-d)\eta^{\alpha n},$$

contract by $\eta_{n\epsilon}$,

$$\eta_{n\epsilon}\eta^{\epsilon\alpha}K_\epsilon^n = (1-d)\eta^{\alpha n}\eta_{n\epsilon} = (1-d)\delta_\epsilon^\alpha.$$

As a result,

$$K_\epsilon^\alpha = (1-d)\delta_\epsilon^\alpha. \tag{2.38}$$

□

Lemma 2.17. In the coordinates t^α,

$$\Delta_\beta^{\alpha n} = \frac{1-d}{2}\delta_\beta^\alpha, \tag{2.39}$$

$$\Delta_\beta^{n\alpha} = \frac{d-1}{2}\delta_\beta^\alpha + K_\beta^\alpha. \tag{2.40}$$

Proof. By (1.16), Christoffel formula has

$$\begin{aligned}\Gamma_{\alpha\beta}^n &= \frac{1}{2}g^{n\epsilon}(\partial_\alpha g_{\epsilon\beta} + \partial_\beta g_{\alpha\epsilon} - \partial_\epsilon g_{\alpha\beta}) \\ &= -\frac{1}{2}\mathcal{L}_{E_f}g_{\alpha\beta} = \frac{d-1}{2}g_{\alpha\beta},\end{aligned}$$

contract by $g^{\alpha\epsilon}$, by (2.33),

$$\frac{1-d}{2}\delta_\beta^\epsilon = \Gamma_\beta^{\epsilon n} = \Delta_\beta^{\epsilon n}.$$

This proves (2.39).

By (1.5), using (2.39),

$$K_\beta^\alpha = \partial_\beta g^{\alpha n} = \Gamma_\beta^{\alpha n} + \Gamma_\beta^{n\alpha} = \frac{1-d}{2}\delta_\beta^\alpha + \Delta_\beta^{n\alpha},$$

and this proves (2.40). □

Lemma 2.18.

$$\mathcal{L}_{E_f}\Delta^{\alpha\beta}_{\gamma} = (d-1)\Delta^{\alpha\beta}_{\gamma}, \tag{2.41}$$

$$\mathcal{L}_{e_f}\Delta^{\alpha\beta}_{\gamma} = 0. \tag{2.42}$$

Proof. Write ∂_{E_f} for $E^{\epsilon}_{f}\partial_{\epsilon}$ and similarly for ∂_{e_f}. Denote

$$\begin{aligned}\tilde{\Gamma}^{\alpha\beta}_{\gamma} :&= \mathcal{L}_{E_f}\Delta^{\alpha\beta}_{\gamma} + (1-d)\Delta^{\alpha\beta}_{\gamma}\\ &= \partial_{E_f}\Gamma^{\alpha\beta}_{\gamma} - K^{\alpha}_{\epsilon}\Gamma^{\epsilon\beta}_{\gamma} - \Gamma^{\alpha\epsilon}_{\gamma}K^{\beta}_{\epsilon} + K^{\epsilon}_{\gamma}\Gamma^{\alpha\beta}_{\epsilon} + (1-d)\Gamma^{\alpha\beta}_{\gamma}\end{aligned}$$

then

$$\mathcal{L}_{E_f}\Delta^{\alpha\beta}_{\gamma} = (d-1)\Delta^{\alpha\beta}_{\gamma}$$
$$\mathcal{L}_{e_f}\Delta^{\alpha\beta}_{\gamma} = 0.$$

Claim: $\tilde{\Gamma}^{\alpha\beta}_{\gamma}$ satisfies

$$\begin{aligned}\tilde{\Gamma}^{\alpha\beta}_{\gamma} + \tilde{\Gamma}^{\beta\alpha}_{\gamma} &= 0\\ g^{\alpha\epsilon}\tilde{\Gamma}^{\beta\gamma}_{\epsilon} &= g^{\beta\epsilon}\tilde{\Gamma}^{\alpha\gamma}_{\epsilon}\end{aligned} \tag{2.43}$$

Recall that, for contravariant metrics, (1.4) and (1.5) need to be satisfied:

$$\begin{aligned}\Gamma^{\alpha\beta}_{\gamma} + \Gamma^{\beta\alpha}_{\gamma} &= \partial_{\gamma}g^{\alpha\beta}\\ g^{\alpha\epsilon}\Gamma^{\beta\gamma}_{\epsilon} &= g^{\beta\epsilon}\Gamma^{\alpha\gamma}_{\epsilon}\end{aligned} \tag{2.44}$$

differentiate the (1.4) w.r.t. E_f:

$$\begin{aligned}\partial_{E_f}(\Gamma^{\alpha\beta}_{\gamma} + \Gamma^{\beta\alpha}_{\gamma}) = \partial_{E_f}\partial_{\gamma}g^{\alpha\beta} &= E^{\epsilon}_{f}\partial_{\epsilon}\partial_{\gamma}g^{\alpha\beta}\\ &= \partial_{\gamma}(E^{\epsilon}_{f}\partial_{\epsilon}g^{\alpha\beta}) - \partial_{\gamma}E^{\epsilon}_{f}\partial_{\gamma}g^{\alpha\beta}\\ &= \partial_{\gamma}\partial_{E_f}g^{\alpha\beta} - K^{\epsilon}_{\gamma}\partial_{\epsilon}g^{\alpha\beta}\end{aligned}$$

by (1.13),

$$\begin{aligned}(d-1)g^{\alpha\beta} &= \mathcal{L}_{E_f}g^{\alpha\beta}\\ &= \partial_{E_f}g^{\alpha\beta} - \partial_{\epsilon}E^{\alpha}_{f}g^{\epsilon\beta} - \partial_{\epsilon}E^{\beta}_{f}g^{\alpha\epsilon}\\ &= \partial_{E_f}g^{\alpha\beta} - K^{\alpha}_{\epsilon}g^{\epsilon\beta} - K^{\beta}_{\epsilon}g^{\alpha\epsilon}\\ \Rightarrow \partial_{E_f}g^{\alpha\beta} &= (d-1)g^{\alpha\beta} + K^{\alpha}_{\epsilon}g^{\epsilon\beta} + K^{\beta}_{\epsilon}g^{\alpha\epsilon}\end{aligned}$$

(2.25) implies that $\partial_{\gamma}K^{\alpha}_{\beta}$=0, and due to (1.13),

$$\begin{aligned}\partial_{E_f}(\Gamma^{\alpha\beta}_{\gamma} + \Gamma^{\beta\alpha}_{\gamma}) &= (d-1)\partial_{\gamma}g^{\alpha\beta} + \partial_{\gamma}(K^{\alpha}_{\epsilon}g^{\epsilon\beta} + K^{\beta}_{\epsilon}g^{\alpha\epsilon}) - K^{\epsilon}_{\gamma}\partial_{\epsilon}g^{\alpha\beta}\\ &= (d-1)\partial_{\gamma}g^{\alpha\beta} + \partial_{\gamma}K^{\alpha}_{\epsilon}g^{\epsilon\beta} + \partial_{\gamma}K^{\beta}_{\epsilon}g^{\alpha\epsilon} + K^{\alpha}_{\epsilon}\partial_{\gamma}g^{\epsilon\beta} + K^{\beta}_{\epsilon}\partial_{\gamma}g^{\alpha\epsilon} - K^{\epsilon}_{\gamma}\partial_{\epsilon}g^{\alpha\beta}\\ &= (d-1)(\Gamma^{\alpha\beta}_{\gamma} + \Gamma^{\beta\alpha}_{\gamma}) + K^{\alpha}_{\epsilon}(\Gamma^{\epsilon\beta}_{\gamma} + \Gamma^{\beta\epsilon}_{\gamma}) + K^{\beta}_{\epsilon}(\Gamma^{\alpha\epsilon}_{\gamma} + \Gamma^{\epsilon\alpha}_{\gamma}) - K^{\epsilon}_{\gamma}(\Gamma^{\alpha\beta}_{\epsilon} + \Gamma^{\beta\alpha}_{\epsilon})\end{aligned}$$

compare this formula to definition of $\tilde{\Gamma^{\alpha\beta}_{\gamma}}$, the equation implies

$$\tilde{\Gamma}^{\alpha\beta}_{\gamma} = -\tilde{\Gamma}^{\beta\alpha}_{\gamma}.$$

Similarly, differentiate the second equation of (2.44) w.r.t. E_f,

$$\begin{aligned}\partial_{E}(g^{\alpha\epsilon}\Gamma^{\beta\gamma}_{\epsilon}) &= \partial_{E}g^{\beta\epsilon}\Gamma^{\beta\gamma}_{\epsilon} + g^{\alpha\epsilon}\partial_{E}\Gamma^{\beta\gamma}_{\epsilon}\\ &= ((d-1)g^{\alpha\epsilon} + K^{\alpha}_{\mu}g^{\mu\epsilon} + K^{\epsilon}_{\mu}g^{\alpha\mu})\Gamma^{\beta\gamma}_{\epsilon} + g^{\alpha\epsilon}\partial_{E}\Gamma^{\beta\gamma}_{\epsilon}\\ \partial_{E}(g^{\beta\epsilon}\Gamma^{\alpha\gamma}_{\epsilon}) &= ((d-1)g^{\beta\epsilon} + K^{\beta}_{\mu}g^{\mu\epsilon} + K^{\epsilon}_{\mu}g^{\beta\mu})\Gamma^{\alpha\gamma}_{\epsilon} + g^{\beta\epsilon}\partial_{E}\Gamma^{\beta\gamma}_{\epsilon}\\ \Rightarrow 0 &= \partial_{E}(g^{\alpha\epsilon}\Gamma^{\beta\gamma}_{\epsilon} - g^{\beta\epsilon}\Gamma^{\alpha\gamma}_{\epsilon})\\ &= (g^{\alpha\epsilon}\partial_{E}\Gamma^{\beta\gamma}_{\epsilon} - g^{\beta\epsilon}\partial_{E}\Gamma^{\beta\gamma}_{\epsilon}) + (K^{\alpha}_{\mu}\Gamma^{\beta\gamma}_{\epsilon} - K^{\beta}_{\mu}\Gamma^{\alpha\gamma}_{\epsilon})g^{\mu\epsilon} + K^{\epsilon}_{\mu}(g^{\alpha\mu}\Gamma^{\beta\gamma}_{\epsilon} - g^{\beta\mu}\Gamma^{\alpha\gamma}_{\epsilon})\end{aligned}$$

notice that

$$
\begin{aligned}
(K^\alpha_\mu\Gamma^{\beta\gamma}_\epsilon - K^\beta_\mu\Gamma^{\alpha\gamma}_\epsilon)g^{\mu\epsilon} &= K^\alpha_\mu\Gamma^{\beta\gamma}_\epsilon g^{\mu\epsilon} - K^\beta_\mu\Gamma^{\alpha\gamma}_\epsilon g^{\mu\epsilon}\\
&= K^\alpha_\mu g^{\beta\epsilon}\Gamma^{\mu\gamma}_\epsilon - K^\beta_\mu g^{\alpha\epsilon}\Gamma^{\mu\gamma}_\epsilon\\
&= -(K^\beta_\mu g^{\alpha\epsilon} - K^\alpha_\mu g^{\beta\epsilon})\Gamma^{\mu\gamma}_\epsilon
\end{aligned}
$$

thus, by comparing the equations,

$$
\begin{aligned}
g^{\alpha\epsilon}\tilde\Gamma^{\beta\gamma}_\epsilon - g^{\beta\epsilon}\tilde\Gamma^{\alpha\gamma}_\epsilon =&(g^{\alpha\epsilon}\partial_E\Gamma^{\beta\gamma}_\epsilon - g^{\beta\epsilon}\partial_E\Gamma^{\alpha\gamma}_\epsilon) + (1-d)(g^{\alpha\epsilon}\Gamma^{\beta\gamma}_\epsilon - g^{\beta\epsilon}\Gamma^{\alpha\gamma}_\epsilon)\\
&-(K^\beta_\mu g^{\alpha\epsilon} - K^\alpha_\mu g^{\beta\epsilon})\Gamma^{\mu\gamma}_\epsilon - K^\gamma_\mu(g^{\alpha\epsilon}\Gamma^{\beta\mu}_\epsilon - g^{\beta\epsilon}\Gamma^{\alpha\mu}_\epsilon) + K^\mu_\epsilon(g^{\alpha\epsilon}\Gamma^{\beta\gamma}_\mu - g^{\beta\epsilon}\Gamma^{\alpha\gamma}_\mu)\\
=&(g^{\alpha\epsilon}\partial_E\Gamma^{\beta\gamma}_\epsilon - g^{\beta\epsilon}\partial_E\Gamma^{\alpha\gamma}_\epsilon) - (K^\beta_\mu g^{\alpha\epsilon} - K^\alpha_\mu g^{\beta\epsilon})\Gamma^{\mu\gamma}_\epsilon + K^\mu_\epsilon(g^{\alpha\epsilon}\Gamma^{\beta\gamma}_\mu - g^{\beta\epsilon}\Gamma^{\alpha\gamma}_\mu)\\
=&0.
\end{aligned}
$$

By uniqueness of Levi-Civita connection, (2.44) has unique solution for given $g^{\alpha\beta}$, thus the only solution of (2.43) is the trivial solution, namely $\tilde\Gamma^{\alpha\beta}_\gamma = 0$, thus

$$
\begin{aligned}
0 = \tilde\Gamma^{\alpha\beta}_\gamma &= \mathcal{L}_E\Delta^{\alpha\beta}_\gamma + (1-d)\Delta^{\alpha\beta}_\gamma\\
\Rightarrow \mathcal{L}_E\Delta^{\alpha\beta}_\gamma &= (d-1)\Delta^{\alpha\beta}_\gamma.
\end{aligned}
$$

This proves (2.41).
Similarly, since $e^\alpha = \eta^{\alpha n}$ is constant, set

$$
\begin{aligned}
\bar\Gamma^{\alpha\beta}_\gamma :&= \mathcal{L}_e\Delta^{\alpha\beta}_\gamma\\
&\equiv \partial_e\Gamma^{\alpha\beta}_\gamma - \partial_\epsilon e^\alpha\Gamma^{\epsilon\beta}_\gamma - \partial_\epsilon e^\beta\Gamma^{\alpha\epsilon}_\gamma + \partial_\gamma e^\epsilon\Gamma^{\alpha\beta}_\epsilon\\
&= \partial_e\Gamma^{\alpha\beta}_\gamma,
\end{aligned}
$$

then

$$
\bar\Gamma^{\alpha\beta}_\gamma + \bar\Gamma^{\beta\alpha}_\gamma = \partial_e(\Gamma^{\alpha\beta}_\gamma + \Gamma^{\beta\alpha}_\gamma) = \partial_e\partial_\gamma g^{\alpha\beta} = \partial_\gamma\partial_e g^{\alpha\beta}.
$$

(1.14) gives

$$
\begin{aligned}
\eta^{\alpha\beta} &= \mathcal{L}_e g^{\alpha\beta} = \partial_e g^{\alpha\beta}\\
\Rightarrow \bar\Gamma^{\alpha\beta}_\gamma + \bar\Gamma^{\beta\alpha}_\gamma &= \partial_\gamma\eta^{\alpha\beta} = 0.
\end{aligned}
$$

Corollary 2.11 implies that

$$
\Delta^{\alpha\beta\gamma} = \Delta^{\beta\alpha\gamma}.
$$

By definition of Δ, this is equivalent to

$$
\eta^{\alpha\epsilon}\Gamma^{\beta\gamma}_\epsilon = \eta^{\alpha\epsilon}\eta_{\epsilon\theta}\Delta^{\theta\beta\gamma} = \Delta^{\alpha\beta\gamma} = \Delta^{\beta\alpha\gamma} = \eta^{\beta\epsilon}\Gamma^{\alpha\gamma}_\epsilon. \tag{2.45}
$$

Differentiate (1.5) w.r.t. e_f, by (1.14),

$$
\begin{aligned}
\Gamma^{\beta\gamma}_\epsilon\partial_e g^{\alpha\epsilon} + g^{\alpha\epsilon}\partial_e\Gamma^{\beta\gamma}_\epsilon &= \Gamma^{\alpha\gamma}_\epsilon\partial_e g^{\beta\epsilon} + g^{\beta\epsilon}\partial_e\Gamma^{\alpha\gamma}_\epsilon\\
\Rightarrow \eta^{\alpha\epsilon}\Gamma^{\beta\gamma}_\epsilon + g^{\alpha\epsilon}\partial_e\Gamma^{\beta\gamma}_\epsilon &= \eta^{\beta\epsilon}\Gamma^{\alpha\gamma}_\epsilon + g^{\beta\epsilon}\partial_e\Gamma^{\alpha\gamma}_\epsilon.
\end{aligned}
$$

Together with (2.45), this implies

$$
g^{\alpha\epsilon}\bar\Gamma^{\beta\gamma}_\epsilon = g^{\beta\epsilon}\bar\Gamma^{\alpha\gamma}_\epsilon.
$$

Therefore $\bar\Gamma^{\alpha\beta}_\gamma$ satisfies the equations (2.43) by replacing $\tilde\Gamma$ with $\bar\Gamma$. By similar arguments for uniqueness of the Levi-Civita connection, $\bar\Gamma^{\alpha\beta}_\gamma = 0$. □

Corollary 2.19. Let u, v be two 1-forms covariantly constant w.r.t. ∇_2, that is, $\nabla_2 u = \nabla_2 v = 0$. Then the multiplication

$$(u, v) \mapsto \Delta(u, v)$$

on T^*M satisfies the equations

$$\Delta(u, v) + \Delta(v, u) = \mathrm{d}(u, v) \tag{2.46}$$

$$\Delta(\bar{R}(u), v) + \Delta(u, \bar{R}(v)) = \mathrm{d}(u, \bar{R}(v)). \tag{2.47}$$

Proof. Let $u = u_i dt^i$ and $v = v_j dt^j$. By (1.5) and (2.33),

$$\begin{aligned}\Delta(u, v) + \Delta(v, u) &= u_i v_j (\Delta_k^{ij} + \Delta_k^{ji}) dt^k \\ &= u_i v_j \partial_k g^{ij} dt^k.\end{aligned}$$

For (2.47), from (1.16) and (1.10),

$$\begin{aligned}(1-d) g_{ij} &= \mathcal{L}_{E_f} g_{ij} \\ &= \partial_{E_f} g_{ij} + g_{sj} \partial_i E_f^s + g_{is} \partial_j E_f^s \\ &= g^{sk} \partial_k \tau \partial_s g_{ij} + g_{sj} \partial_i (g^{sk} \partial_k \tau) + g_{is} \partial_j (g^{sk} \partial_k \tau)\end{aligned}$$

Notice that

$$\begin{aligned}g_{sj} \partial_i (g^{sk} \partial_k \tau) &= \partial_i (g_{sj} g^{sk} \partial_k \tau) - g^{sk} \partial_k \tau \partial_i g_{sj} \\ &= \partial_i \partial_j \tau - g^{sk} \partial_k \tau \partial_i g_{sj},\end{aligned}$$

and similar for $g_{is} \partial_j (g^{sk} \partial_k \tau)$. Therefore, with (1.6) and (2.19),

$$\begin{aligned}(1-d) g_{ij} &= 2 \partial_i \partial_j \tau + g^{sk} \partial_k \tau \partial_s g_{ij} - g^{sk} \partial_k \tau \partial_i g_{sj} - g^{sk} \partial_k \tau \partial_j g_{is} \\ &= 2 \partial_i \partial_j \tau - 2 {\Gamma_1}_{ij}^s \partial_s \tau.\end{aligned} \tag{2.48}$$

Observe that, with (1.5),

$$\begin{aligned}g_{sj} \nabla_{1i} E_f^s &= g_{sj} (\partial_i E_f^s + {\Gamma_1}_{ik}^s E_f^k) \\ &= g_{sj} (\partial_i (g^{sk} \partial_k \tau) + {\Gamma_1}_{ik}^s g^{kp} \partial_p \tau) \\ &= g_{sj} (g^{sk} \partial_i \partial_k \tau + \partial_i g^{sk} \partial_k \tau + {\Gamma_1}_{ik}^s g^{kp} \partial_p \tau) \\ &= \partial_i \partial_j \tau + g_{sj} ({\Gamma_1}_i^{sk} + {\Gamma_1}_i^{ks}) \partial_k \tau + {\Gamma_1}_{ij}^s \partial_s \tau.\end{aligned}$$

By (1.3),

$$\begin{aligned}g_{sj} {\Gamma_1}_i^{sk} &= -{\Gamma_1}_{ij}^k, \\ g_{sj} {\Gamma_1}_i^{ks} &= -g_{sj} g^{kj} {\Gamma_1}_{ji}^s = -{\Gamma_1}_{ij}^k,\end{aligned}$$

therefore,

$$g_{sj} \nabla_{1i} E_f^s = \partial_i \partial_j \tau - {\Gamma_1}_{ij}^k \partial_k \tau. \tag{2.49}$$

(2.48) and (2.49) imply

$$\begin{aligned}\frac{1-d}{2} g_{ij} &= \nabla_{1i} E_f^j \\ &= \partial_i E^j + {\Gamma_1}_{is}^j E_f^s,\end{aligned}$$

thus

$$-{\Gamma_1}_{is}^j E_f^s = \frac{d-1}{2} \delta_i^j + \partial_i E_f^j = \bar{R}_i^j. \tag{2.50}$$

Writing (2.47) in coordinates:

$$u_iv_j(\frac{d-1}{2}\Delta_k^{ij}+\Delta_k^{is}\partial_s E_f^j+\frac{d-1}{2}\Delta_k^{ij}+\Delta_k^{js}\partial_s E_f^i)=u_iv_j\partial_k(g^{is}\bar{R}_s^j).$$

by (2.50), (2.19), (1.3) and (2.22),

$$\begin{aligned}\partial_k(g^{is}\bar{R}_s^j)=&\partial_k(-g^{is}\Gamma_{sp}^j E_f^p)\\ =&\partial_k(\Delta_p^{ij}E_f^p)\\ =&E_f^p\partial_k\Delta_p^{ij}+\Delta_p^{ij}\partial_k E_f^p\\ =&E_f^p\partial_p\Delta_k^{ij}+\Delta_p^{ij}\partial_k E_f^p,\end{aligned}$$

thus (2.47) is equivalent to

$$\begin{aligned}&(d-1)\Delta_k^{ij}+\Delta_k^{is}\partial_s E_f^j+\Delta_k^{sj}\partial_s E_f^i=E_f^p\partial_p\Delta_k^{ij}+\Delta_p^{ij}\partial_k E_f^p\\ \Leftrightarrow\ &(d-1)\Delta_k^{ij}=\partial_{E_f}\Delta_k^{ij}+\Delta_s^{ij}\partial_k E_f^s-\Delta_k^{is}\partial_s E_f^j-\Delta_k^{sj}\partial_s E_f^i=\mathcal{L}_{E_f}\Delta_k^{ij},\end{aligned}$$

which is just (2.41). □

Fix a point $t\in M_0$, denote

$$V=T_{t_0}^*M.$$

Define the linear operator

$$\Lambda:V\to V,\ \Lambda=\frac{d-2}{2}\mathbf{1}+K.$$

By definition of K and $\bar{R}$,

$$\Lambda=\frac{d-2}{2}\mathbf{1}+K=-\frac{1}{2}\mathbf{1}+\bar{R}.$$

Lemma 2.20. The operator Λ is skew-symmetric w.r.t. $(\ ,\ )_2=\langle\ ,\ \rangle$, namely

$$\langle\Lambda u,v\rangle+\langle u,\Lambda v\rangle=0.$$

Proof. It suffices to show for $u=dt^\alpha$ and $v=dt^\beta$.

$$\begin{aligned}\langle\Lambda dt^\alpha,dt^\beta\rangle+\langle dt^\alpha,\Lambda dt^\beta\rangle&=2\cdot\frac{d-2}{2}\langle dt^\alpha,dt^\beta\rangle+\langle Kdt^\alpha,dt^\beta\rangle+\langle dt^\alpha,Kdt^\beta\rangle\\ &=(d-2)\eta^{\alpha\beta}+\langle K_\epsilon^\alpha dt^\epsilon,dt^\beta\rangle+\langle dt^\alpha,K_\epsilon^\beta dt^\epsilon\rangle\end{aligned}$$

By (2.26),

$$\begin{aligned}(d-2)\eta^{\alpha\beta}&=\mathcal{L}_{E_f}\eta^{\alpha\beta}\\ &=\partial_{E_f}\eta^{\alpha\beta}-\eta^{\epsilon\beta}\partial_\epsilon E_f^\alpha-\eta^{\alpha\epsilon}\partial_\epsilon E_f^\beta.\end{aligned}$$

Since $\eta^{\alpha\beta}$ is constant, and

$$\partial_\epsilon E^\alpha\eta^{\epsilon\beta}=K_\epsilon^\alpha\langle dt^\epsilon,dt^\beta\rangle=\langle K_\epsilon^\alpha dt^\epsilon,dt^\beta\rangle,$$

similarly, $\partial_\epsilon E^\beta\eta^{\alpha\epsilon}=\langle dt^\alpha,K_\epsilon^\beta dt^\epsilon\rangle$. Combining these equations, the lemma is proved. □

Let

$$V=\oplus_\lambda V_\lambda$$

be the root decomposition of the space V w.r.t. the root spaces of the operator Λ, that is, for $v\in V_\mu$, there exists $k\in\mathbb{N}$ (called the *height* of v) such that

$$(\Lambda-\mu\mathbf{1})^k v=0,\ (\Lambda-\mu\mathbf{1})^{k-1}v=0.$$

Then one has the following properties:

Lemma 2.21. The root subspaces V_λ and V_μ are $\langle\ ,\ \rangle$-orthogonal if $\lambda+\mu \neq 0$. The pairing

$$\langle\ ,\ \rangle : V_\lambda \times V_{-\lambda} \to \mathbf{C}$$

does not degenerate.

Proof. Let $u \in V_\lambda,\ v \in V_\mu$ with $\lambda+\mu \neq 0$,

$$\begin{aligned} 0 &= \langle \Lambda u, v\rangle + \langle u, \Lambda v\rangle \\ &= \lambda\langle u, v\rangle + \mu\langle u, v\rangle \\ &= (\lambda+\mu)\langle u, v\rangle \end{aligned}$$

since $\lambda+\mu \neq 0$, this implies $\langle u, v\rangle = 0$. As a result, the pairing does not degenerate. $\square$

Regularity of $\bar{R}$ implies that, for $v \in V_{-\frac{1}{2}}$,

$$-\frac{1}{2}v = \Lambda v = -\frac{1}{2}v + \bar{R}(v)$$

$$\Rightarrow \bar{R}(v) = 0$$

$$\Rightarrow v = 0$$

$$\Rightarrow V_{-\frac{1}{2}} = 0.$$

This also implies that $\frac{d-2}{2} \neq -\frac{1}{2} \Rightarrow d \neq 1$.

Lemma 2.22. The multiplication (2.29) on V is commutative.

Proof. Let $u,\ v$ be 1-forms in a small neighborhood of the point t_0 covariantly constant w.r.t. ∇_2. By (2.47) and (2.29),

$$\begin{aligned} \mathrm{d}(u, v) &= \mathrm{d}(u, \bar{R}(\bar{R}^{-1}(v))) \\ &= \Delta(\bar{R}(u), \bar{R}^{-1}(v)) + \Delta(u, \bar{R}(\bar{R}^{-1}(v))) \\ &= \bar{R}(u)\cdot v + u \cdot \bar{R}(v). \end{aligned} \tag{2.51}$$

It suffices to prove the lemma for $u \in V_\lambda,\ v \in V_\mu$, where $\lambda,\ \mu \neq \frac{1}{2}$ since $V_{-\frac{1}{2}} = 0$. For these u and v,

$$\bar{R}(u) = \left(\frac{1}{2}+\lambda\right)u, \quad \bar{R}(v) = \left(\frac{1}{2}+\mu\right)v. \tag{2.52}$$

By (2.51),

$$\mathrm{d}(u, v) = \left(\frac{1}{2}+\lambda\right)u\cdot v + u\cdot\left(\frac{1}{2}+\mu\right)v = (1+\lambda+\mu)u\cdot v. \tag{2.53}$$

Let $u^{(k)},\ v^{(l)}$ be the adjoint vectors for the eigenvectors of heights k and k respectively, namely,

$$(\Lambda-\lambda\mathbf{1})^k u^{(k)} = 0, \quad (\Lambda-\lambda\mathbf{1})^{k-1}u^{(k)} \neq 0,$$

$$\Rightarrow (\Lambda-\lambda\mathbf{1})^k u^{(k)} = (\Lambda-\lambda\mathbf{1})^{k-1}(\Lambda-\lambda\mathbf{1})u^{(k)}$$

$$\Rightarrow u^{(k-1)} = (\Lambda-\lambda\mathbf{1})u^{(k)} = \Lambda(u^{(k)}) - \lambda u^{(k)}, \quad u^{(-1)} = 0, \quad u^{(1)} = u,$$

thus

$$R(u^{(k)}) = \Lambda(u^{(k)}) + \frac{1}{2}u^{(k)} = u^{(k-1)} + \left(\frac{1}{2}+\lambda\right)u^{(k)}.$$

Similarly,

$$v^{(l-1)} = (\Lambda-\lambda\mathbf{1})v^{(l)} = \Lambda(v^{(l)}) - \lambda v^{(l)}, \quad v^{(-1)} = 0, \quad v^{(1)} = v,$$

$$\bar{R}(v^{(l)}) = \Lambda(v^{(l)}) + \frac{1}{2}v^{(l)} = v^{(l-1)} + \left(\frac{1}{2}+\lambda\right)v^{(l)}.$$

Substitute $u \mapsto u^{(k)},\ v \mapsto v^{(l)}$ in (2.51),

$$\begin{aligned} \mathrm{d}(u^{(k)}, v^{(l)}) &= (u^{(k-1)} + (\frac{1}{2}+\lambda)u^{(k)}) \cdot v^{(l)} + u^{(k)} \cdot (v^{(l-1)} + (\frac{1}{2}+\lambda)v^{(l)}) \\ &= (1+\lambda+\mu)u^{(k)} \cdot v^{(l)} + u^{(k-1)} \cdot v^{(l)} + u^{(k)} \cdot v^{(l-1)}. \end{aligned} \tag{2.54}$$

Case 1: $1+\lambda+\mu \neq 0$. Then [(2.46) implies that

$$\mathrm{d}(u,v) = \mathrm{d}(v,u),$$

thus

$$u \cdot v = \frac{1}{1+\lambda+\mu}\mathrm{d}(u,v) = \frac{1}{1+\lambda+\mu}\mathrm{d}(v,u) = v \cdot u.$$

This proves commutativity for u and v.
Now consider $u^{(k)}$ and $v^{(l)}$. Use induction on the sum of the heights $k+l$. Commutativity holds for $k+l = 0,\ 1,\ 2$. Now assume that commutativity is true for height sum $k+l-1$, then for $k+l$,

$$\begin{aligned} (1+\lambda+\mu)u^{(k)} \cdot v^{(l)} &= \mathrm{d}(u^{(k)}, v^{(l)}) - u^{(k-1)} \cdot v^{(l)} - u^{(k)} \cdot v^{(l-1)} \\ &= \mathrm{d}(v^{(l)}, u^{(k)}) - v^{(l)} \cdot u^{(k-1)} - v^{(l-1)} \cdot u^{(k)} \\ &= (1+\lambda+\mu)v^{(l)} \cdot u^{(k)} \end{aligned}$$

thus commutativity holds for $u^{(k)}$ and $v^{(l)}$.
Case 2: $1+\lambda+\mu = 0$, thus

$$\frac{1}{2}+\lambda = -\left(\frac{1}{2}+\mu\right). \tag{2.55}$$

Then by (2.46) and (2.53),

$$0 = \mathrm{d}(u,v) = \Delta(u,v) + \Delta(v,u). \tag{2.56}$$

Moreover, (2.53) implies

$$\bar{R}^{-1}(u) = \frac{1}{\frac{1}{2}+\lambda}u, \quad \bar{R}^{-1}(v) = \frac{1}{\frac{1}{2}+\mu}v,$$

by [TBF][(2.14) of Dubrovin, (5),(6)],

$$u \cdot v = \Delta(u, \bar{R}^{-1}(v)) = \frac{\Delta(u,v)}{\frac{1}{2}+\mu} = \frac{\Delta(v,u)}{\frac{1}{2}+\lambda} = \Delta(v, \bar{R}^{-1}(u)) = v \cdot u.$$

For the adjoint vectors, notice that

$$\left(\frac{1}{2}+\lambda\right)u^{(k)} = \bar{R}(u^{(k)}) - u^{(k-1)}$$

therefore

$$\begin{aligned} v^{(l)} \cdot u^{(k)} &= \frac{1}{\frac{1}{2}+\lambda}(v^{(l)} \cdot \bar{R}(u^{(k)}) - v^{(l)} \cdot u^{(k-1)}) \\ &= \frac{1}{\frac{1}{2}+\lambda}(\Delta(v^{(l)}, u^{k}) - v^{(l)} \cdot u^{(k-1)}), \end{aligned}$$

similarly,

$$\begin{aligned} u^{(k)} \cdot v^{(l)} &= \frac{1}{\frac{1}{2}+\mu}(\Delta(u^{(k)}, v^{(l)}) - u^{(k)} \cdot v^{(l-1)}) \\ &= -\frac{1}{\frac{1}{2}+\lambda}(\Delta(u^{(k)}, v^{(l)}) - u^{(k)} \cdot v^{(l-1)}). \end{aligned}$$

Thus, by (2.54)

$$
\begin{aligned}
v^{(l)} \cdot u^{(k)} - u^{(k)} \cdot v^{(l)} &= \frac{1}{\frac{1}{2}+\lambda}((\Delta(v^{(l)}, u^{k}) + \Delta(u^{(k)}, v^{(l)})) - (v^{(l)} \cdot u^{(k-1)} + u^{(k)} \cdot v^{(l-1)}) \\
&= \frac{1}{\frac{1}{2}+\lambda}(\mathrm{d}(u^{(k)}, v^{(l)}) - (v^{(l)} \cdot u^{(k-1)} + u^{(k)} \cdot v^{(l-1)})) \\
&= \frac{1}{\frac{1}{2}+\lambda}((u^{(k-1)} \cdot v^{(l)} + u^{(k)} \cdot v^{(l-1)}) - (v^{(l)} \cdot u^{(k-1)} + u^{(k)} \cdot v^{(l-1)})).
\end{aligned}
$$

Again, by induction on $k+l$, commutativity also holds for this case. Lemma is proved. $\square$

We obtain a symmetric multiplication on the cotangent planes T_t^*M which defines $c_\gamma^{\alpha\beta}(t)$:

$$
(dt^\alpha, dt^\beta) \mapsto dt^\alpha \cdot dt^\beta := c_\gamma^{\alpha\beta}(t) dt^\gamma. \tag{2.57}
$$

The 1-form $dt^n = d\tau$ is the unity of this multiplication. Indeed, by (2.39),

$$
\Delta(dt^\alpha, dt^n) = \Delta_\epsilon^{\alpha n} dt^\epsilon = \frac{1-d}{2}\delta_\epsilon^\alpha dt^\epsilon = \frac{1-d}{2} dt^\alpha. \tag{2.58}
$$

By (2.38) and (2.58),

$$
\begin{aligned}
&K_\alpha^n = (1-d)\delta_\alpha^n \\
\Rightarrow &K dt^n = K_\epsilon^n dt^\epsilon = (1-d)\delta_\epsilon^n dt^\epsilon = (1-d) dt^n \\
\Rightarrow &\Lambda dt^n = \frac{d-2}{2} dt^n + K dt^n = (\frac{d-2}{2} + 1 - d) dt^n = -\frac{d}{2} dt^n \\
\Rightarrow &\bar{R}(dt^n) = (\Lambda + \frac{1}{2}) dt^n = \frac{1-d}{2} dt^n
\end{aligned}
$$

$$
\Rightarrow dt^\alpha \cdot dt^n = \Delta(dt^\alpha, \bar{R}^{-1}(dt^n)) = \frac{2}{1-d}\Delta(dt^\alpha, dt^n) = dt^\alpha. \tag{2.59}
$$

Along with (2.44) and commutativity, this implies associativity of this multiplication:

$$
\begin{aligned}
(dt^\alpha \cdot dt^\beta) \cdot dt^\gamma &= \Delta(\Delta(dt^\alpha, \bar{R}^{-1}(t^\beta)), \bar{R}^{-1}(dt^\gamma)) \\
&= \Delta(\Delta(dt^\alpha, \bar{R}^{-1}(t^\gamma)), \bar{R}^{-1}(dt^\beta)) \\
&= (dt^\alpha \cdot dt^\gamma) \cdot dt^\beta,
\end{aligned} \tag{2.60}
$$

thus

$$
(dt^\alpha \cdot dt^\beta) \cdot dt^\gamma = (dt^\alpha \cdot dt^\gamma) \cdot dt^\beta = (dt^\gamma \cdot dt^\alpha) \cdot dt^\beta = (dt^\gamma \cdot dt^\beta) \cdot dt^\alpha = dt^\alpha \cdot (dt^\gamma \cdot dt^\beta).
$$

On T_tM, by duality, one has

$$
\partial_\alpha \cdot \partial_\beta = c_{\alpha\beta}^\gamma(t)\partial_\gamma,
$$

where

$$
c_{\alpha\beta}^\gamma(t) = \eta_{\alpha\epsilon} c_\beta^{\epsilon\gamma}(t).
$$

Lemma 2.23. This multiplication along with $\langle\ ,\ \rangle$ satisfies the axioms for Frobenius structure.

Proof. Need to prove that this structure sasfied **FM1, FM2, FM3** and $(\ ,\ )_1$ coincides with the intersection form.

(i) **FM1**: The vector defined as (2.34) is the unity of this multiplication. Indeed, (2.59) implies

$$
c_\gamma^{\alpha n} = \delta_\gamma^\alpha,
$$

thus

$$e_f \cdot \partial_\alpha = \eta^{\epsilon n}\partial_\epsilon \cdot \partial_\alpha = \eta^{\epsilon n}\eta_{\epsilon\mu}c_\alpha^{\gamma\mu}\partial_\gamma = c_\alpha^{\gamma n}\partial_\gamma = \partial_\alpha,$$

Thus **FM1** is satisfied.

(ii) **FM2**: Need to show symmetry in $\alpha,\ \beta,\ \gamma$ for

$$\langle \partial_\alpha \cdot \partial_\beta, \partial_\gamma \rangle,$$

and symmetry in $\alpha,\ \beta,\ \gamma,\ \delta$ for

$$\partial_\delta \langle \partial_\alpha \cdot \partial_\beta, \partial_\gamma \rangle.$$

Commutativity of the product implies symmetry for the α and β. Moreover, by the algebra structure and linearity of $\Delta(\ ,\ )$,

$$(\Delta(dt^\alpha, dt^\beta), dt^\gamma)_2 = ((dt^\alpha \cdot \bar{R}(dt^\beta)), dt^\gamma)_2 = \bar{R}(c_\epsilon^{\alpha\beta}dt^\epsilon, dt^\gamma) = \bar{R}c_\epsilon^{\alpha\beta}\eta^{\epsilon\gamma},$$

and similar for $(dt^\alpha, \Delta(dt^\gamma, dt^\beta))_2 = (\Delta(dt^\gamma, dt^\beta), dt^\alpha)_2$. Thus (2.16) implies

$$c_\epsilon^{\alpha\beta}\eta^{\epsilon\gamma} = c_\epsilon^{\gamma\beta}\eta^{\epsilon\alpha}.$$

Therefore,

$$\langle \partial_\alpha \cdot \partial_\beta, \partial_\gamma \rangle = \langle c_{\alpha\beta}^\epsilon \partial_\epsilon, \partial_\gamma \rangle = c_{\alpha\beta}^\epsilon \eta_{\epsilon\gamma} = c_\beta^{\mu\epsilon}\eta_{\mu\alpha}\eta_{\epsilon\gamma} = c_{\gamma\beta}^\mu \eta_{\mu\alpha} = \langle \partial_\gamma \cdot \partial_\beta, \partial_\alpha \rangle.$$

Thus symmetry for α and γ holds. As a result,

$$\langle \partial_\alpha \cdot \partial_\beta, \partial_\gamma \rangle = \langle \partial_\beta \cdot \partial_\alpha, \partial_\gamma \rangle = \langle \partial_\gamma \cdot \partial_\alpha, \partial_\beta \rangle = \langle \partial_\alpha \cdot \partial_\gamma, \partial_\beta \rangle$$

symmetry of β and γ is granted. Thus symmetry w.r.t. $\alpha,\ \beta,\ \gamma$ is proved.

To prove symmetry for $\partial_\delta \langle \partial_\alpha \cdot \partial_\beta, \partial_\gamma \rangle$, by commutativity of the multiplication and symmetry for $\langle \partial_\alpha \cdot \partial_\beta, \partial_\gamma \rangle$, it suffices to show symmetry of δ and any one of the other components. This symmetry follows from (2.18). For $dt^\alpha,\ dt^\beta,\ dt^\gamma$, (2.18) implies

$$\nabla_2^\alpha(dt^\beta \cdot dt^\gamma) - \nabla_2^\beta(dt^\alpha \cdot dt^\gamma) = (\nabla_2^\alpha dt^\beta - \nabla_2^\beta dt^\alpha) \cdot dt^\gamma = 0$$
$$\Rightarrow \nabla^\alpha c_\epsilon^{\beta\gamma} = \nabla^\beta c_\epsilon^{\alpha\gamma},$$

by (1.7),

$$\begin{aligned}\partial_\delta \langle \partial_\alpha \cdot \partial_\beta, \partial_\gamma \rangle &= \nabla_\delta(c_{\alpha\beta}^\epsilon \eta_{\epsilon\gamma}) = \eta_{\epsilon\gamma}\nabla_\delta c_{\alpha\beta}^\epsilon = \eta_{\epsilon\gamma}\eta_{\delta i}\eta_{\alpha j}\nabla^i c_\beta^{\epsilon j} \\ &= \eta_{\epsilon\gamma}\eta_{\delta i}\eta_{\alpha j}\nabla^j c_\beta^{\epsilon i} = \eta_{\epsilon\gamma}\nabla_\alpha c_{\beta\delta}^\epsilon = \partial_\alpha \langle \partial_\delta \cdot \partial_\beta, \partial_\gamma \rangle.\end{aligned}$$

(iii) **FM3**: To prove **FM3** is satisfied, it suffices to prove (1.41), (1.44a), (1.44b). (1.41) immediately follows from (2.25). Recall the definition of K, (2.32) gives

$$\begin{aligned}\mathcal{L}_{E_f} K_\alpha^\beta &= E_f^\epsilon \partial_\epsilon K_\beta^\alpha + \partial_\epsilon E_f^\beta K_\alpha^\epsilon - \partial_\alpha E_f^\epsilon K_\epsilon^\beta \\ &= E_f^\epsilon \partial_\epsilon \partial_\beta E_f^\alpha + \partial_\epsilon E_f^\alpha \partial_\beta E_f^\epsilon - \partial_\epsilon E_f^\beta \partial_\epsilon E_f^\alpha \\ &= 0.\end{aligned} \tag{2.61}$$

By the definitions,

$$c_\gamma^{\alpha\beta}dt^\gamma = dt^\alpha \cdot dt^\beta = \Delta(dt^\alpha, \bar{R}^{-1}dt^\beta) = (\bar{R}^{-1}dt^\beta)_\epsilon \Delta_\gamma^{\epsilon\alpha}dt^\gamma = (\bar{R}^{-1})_\epsilon^\beta \Delta_\gamma^{\epsilon\alpha}dt^\gamma,$$

thus

$$c_\gamma^{\alpha\beta} = (\bar{R}^{-1})_\epsilon^\beta \Delta_\gamma^{\epsilon\alpha}. \tag{2.62}$$

Contract by $\bar{R}_\beta^\mu$,

$$\bar{R}_\beta^\mu c_\gamma^{\alpha\beta} = \bar{R}_\beta^\mu (\bar{R}^{-1})_\epsilon^\beta \Delta_\gamma^{\epsilon\alpha} = \delta_\epsilon^\mu \Delta_\gamma^{\epsilon\alpha} = \Delta_\gamma^{\mu\alpha},$$

thus, by (2.61),

$$\begin{aligned}\mathcal{L}_{E_f}(\bar{R}^\mu_\beta c^{\alpha\beta}_\gamma) &= \bar{R}^\mu_\beta \mathcal{L}_{E_f} c^{\alpha\beta}_\gamma + c^{\alpha\beta}_\gamma \mathcal{L}_{E_f} \bar{R}^\mu_\beta \\ &= \bar{R}^\mu_\beta \mathcal{L}_{E_f} c^{\alpha\beta}_\gamma + c^{\alpha\beta}_\gamma \mathcal{L}_{E_f} (\frac{d-1}{2}\delta^\mu_\beta + K^\mu_\beta) \\ &= \bar{R}^\mu_\beta \mathcal{L}_{E_f} c^{\alpha\beta}_\gamma,\end{aligned}$$

and by (2.41) and (2.62),

$$\mathcal{L}_{E_f}\Delta^{\mu\alpha}_\gamma = (d-1)\Delta^{\mu\alpha}_\gamma = (d-1)\bar{R}^\mu_\beta c^{\alpha\beta}_\gamma.$$

Combining these equations,

$$\mathcal{L}_{E_f} c^{\alpha\beta}_\gamma = (d-1)c^{\alpha\beta}_\gamma.$$

Lower the index α, by (2.26), and using the same technique for Lemma 1.17,

$$\mathcal{L}_{E_f} c^\beta_{\alpha\gamma} = \mathcal{L}_{E_f}(\eta_{\alpha\epsilon} c^{\beta\epsilon}_\gamma) = \eta_{\alpha\epsilon}\mathcal{L}_{E_f} c^{\beta\epsilon}_\gamma + c^{\beta\epsilon}_\gamma \mathcal{L}_{E_f}\eta_{\alpha\epsilon} = (d-1)\eta_{\alpha\epsilon}c^{\beta\epsilon}_\gamma + (2-d)\eta_{\alpha\epsilon}c^{\beta\epsilon}_\gamma = \eta_{\alpha\epsilon}c^{\beta\epsilon}_\gamma = c^\beta_{\alpha_\gamma}.$$

Hence **FM3** is proved.

(iv) It remains to show $g^{\alpha\beta}$ coincides with the intersection form. It suffices to show $g^{\alpha\beta}$ coincides with (2.5). This follows from (2.51) and (2.46):

$$dt^\alpha \cdot \bar{R}dt^\beta + \bar{R}dt^\alpha \cdot dt^\beta = \mathrm{d}(u,v) = \partial_\gamma g^{\alpha\beta} dt^\gamma.$$

Since

$$dt^\alpha \cdot \bar{R}dt^\beta + \bar{R}dt^\alpha \cdot dt^\beta = dt^\alpha \cdot \bar{R}^\beta_\epsilon dt^\epsilon + \bar{R}^\alpha_\epsilon dt^\epsilon \cdot dt^\beta = \bar{R}^\alpha_\epsilon c^{\epsilon\beta}_\mu dt^\mu + c^{\alpha\epsilon}_\mu \bar{R}^\beta_\epsilon dt^\mu,$$

this implies

$$\bar{R}^\alpha_\epsilon c^{\epsilon\beta}_\gamma + c^{\alpha\epsilon}_\gamma \bar{R}^\beta_\epsilon = \partial_\gamma g^{\alpha\beta}. \tag{2.63}$$

Set $F(t)$ be such that

$$\partial_\alpha\partial_\beta\partial_\gamma F(t) = \eta_{\alpha\epsilon}\eta_{\beta\mu}c^{\epsilon\mu}_\gamma(t)$$

equivalently,

$$\eta^{\alpha\epsilon}\eta^{\beta\mu}\partial_\alpha\partial_\beta\partial_\gamma F(t) = c^{\epsilon\mu}_\gamma(t).$$

By (2.25),

$$\partial_\gamma \bar{R}^\alpha_\beta = \partial_\gamma\partial_\beta E^\alpha = 0,$$

thus

$$\partial_\gamma(\bar{R}^\alpha_\epsilon c^{\epsilon\beta}_\gamma + c^{\alpha\epsilon}_\gamma \bar{R}^\beta_\epsilon) = \partial_\gamma(c^{\epsilon\beta}_\gamma + c^{\alpha\epsilon}_\gamma). \tag{2.64}$$

(2.63) and (2.64) implies $c^{\epsilon\beta}_\gamma + c^{\alpha\epsilon}_\gamma$ and $g^{\alpha\beta}$ differs by a constant matrix. □

Remark 4. Regularity assumption of non-degenerateness of the operator R can be relaxed. Take $d=1$. Then R is always degenerate. Take τ as (2.35), then by (2.38),

$$R(d\tau) = R^n_j dt^j = K^n_j dt^i = (1-d)dt^n = 0. \tag{2.65}$$

Notice that regularity is required only to prove commutativity. Thus to show (2.15) holds for $d=1$, it suffices to show that commutativity holds for $v \in V_{-\frac{1}{2}}$. By the definition (2.29) of the multiplication, it suffices to show $R^{-1}(0) = V_{-\frac{1}{2}}$ is 1-dimensional. Indeed, reversing the process (2.65), the only 1-form in $V_{-\frac{1}{2}}$ is $d\tau$ up to a scalar multiplication. Moreover, by (2.40),

$$\Delta(d\tau, v) = v^j\Delta^{nj})kdx^k = v^j(\frac{d-1}{2}\delta^j_k + K^j_k)dx^k = v^j d^j_k dx^k = R(v),$$

therefore

$$d\tau \cdot v = \Delta(d\tau, R^{-1}v) = R(R^{-1}v) = v,$$

and $d\tau$ is the left unity of the multiplication. Defining $v \cdot d\tau = v$, we obtain the Frobenius structure.

3 Flat Pencils and Bihamiltonian Structures on Loop Spaces

We define the loop space $L(M)$ of all smooth maps

$$S^1 \to M.$$

In a local coordinate system $x^1, \ldots, x^n$, any such map is given by a 2π-periodic smooth vector-function $(x^1(s), \ldots, x^n(s))$. Consider the $local functionals$

$$I[x] = \frac{1}{2\pi} \int_0^{2\pi} P\left(x; \dot{x}, \ddot{x}, \ldots, x^{(m)}\right) ds \tag{3.1}$$

$$x = (x^1, \ldots, x^n),\ \dot{x} = (\dot{x}^1, \ldots, \dot{x}^n) = \frac{dx}{ds},\ \ldots$$

as "functions" on the loop space, where $P\left(x; \dot{x}, \ddot{x}, \ldots, x^{(m)}\right)$ is a polynomial in $\dot{x}, \ddot{x}, \ldots$ with coefficients in smooth functions of x. The nonnegative integer m may depend on the functional.

Definition 3.1. Introduce the δ-function on the circle defined by the identity

$$\frac{1}{2\pi} \int_0^{2\pi} f(s)\delta(s) ds = f(0)$$

for any smooth 2π-periodic function $f(s)$. The derivative of δ-function are defined by the equations

$$\frac{1}{2\pi} \int_0^{2\pi} f(s_2)\delta^{(k)}(s_1 - s_2) ds_2 = f^{(k)}(s_1).$$

Definition 3.2. Consider the *local invariant Poisson bracket* defined as

$$\left\{x^i(s_1), x^j(s_2)\right\} = \sum_{k=0}^{N} a_k^{ij} \left(x(s_1); \dot{x}(s_1), \ddot{x}(s_1), \ldots, x^{(m_k)}(s_1)\right) \delta^{(k)}(s_1 - s_2). \tag{3.2}$$

The variational derivatives

$$I[x + \delta x] - I[x] = \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta I}{\delta x^i(s)} \delta x^i(s) ds + O(||\delta x||^2). \tag{3.3}$$

For the local functionals of the form 3.1, the variational derivatives can be obtained by applying the Euler-Lagrange operator:

$$\frac{\delta I}{\delta x^i(s)} = \frac{\partial P}{\partial x^i} - \frac{d}{ds}\frac{\partial P}{\partial \dot{x}^i} + \frac{d^2}{ds^2}\frac{\partial P}{\partial \ddot{x}^i} - \cdots \tag{3.4}$$

For $I_1,\ I_2 \in L(M)$, consider the local translation Poisson bracket

$$\{I_1, I_2\} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{\delta I_1}{\delta x^i(s_1)} \left\{x^i(s_1), x^j(s_2)\right\} \frac{\delta I_2}{\delta x^j(s_2)} ds_1 ds_2. \tag{3.5}$$

Let

$$A^{ij} := \sum_{k=0}^{N} a_k^{ij} \left(x(s_1); \dot{x}(s_1), \ddot{x}(s_1), \ldots, x^{(m_k)}(s_1)\right) \frac{d^k}{ds^k}. \tag{3.6}$$

Write $a_k^{ij}(s_1)$ for $a_k^{ij}\left(x(s_1);\dot{x}(s_1),\ddot{x}(s_1),\ldots,x^{(m_k)}(s_1)\right)$ for simplicity. Then (3.5) can be transformed into:

$$\begin{aligned}\{I_1,I_2\} &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{\delta I_1}{\delta x^i(s_1)}\sum_{k=1}^N a_k^{ij}(s_1)\frac{\delta I_2}{\delta x^j(s_2)}\delta^{(k)}(s_1-s_2)ds_1ds_2\\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\delta I_1}{\delta x^i(s_1)}\sum_{k=1}^N a_k^{ij}(s_1)\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{\delta I_2}{\delta x^j(s_2)}\delta^{(k)}(s_1-s_2)ds_2\right]ds_1\\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\delta I_1}{\delta x^i(s_1)}\sum_{k=1}^N a_k^{ij}(s_1)\frac{d^k}{dx^k}\frac{\delta I_2}{\delta x^j(s_2)}ds_1\\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\delta I_1}{\delta x^i(s_1)}A_k^{ij}\frac{\delta I_2}{\delta x^j(s_2)}ds_1.\end{aligned}\tag{3.7}$$

The Poisson bracket needs to be independent on the choice of the local coordinates on M, thus the coefficients a_k^{ij} must transform in an appropriate way. Consider the changes of coordinates $y = y(x)$. The transformation law of the coefficients is determined by the Leibniz identity for (3.2):

$$\{y^p(s_1),y^q(s_2)\} = \frac{\partial y^p}{\partial x^i}(s_1)\frac{\partial y^q}{\partial x^j}(s_2)\left\{x^i(s_1),x^j(s_2)\right\}. \tag{3.8}$$

Moreover, the derivative of δ-function gives

$$f(s_2)\delta^{(k)}(s_1-s_2) = \sum_{l=0}^k \binom{k}{l} f^{(l)}(s_1)\delta^{(k-1)}(s_1-s_2). \tag{3.9}$$

Definition 3.3. Assign *degrees* to the derivatives as

$$\deg\frac{d^k x^i}{ds^k} = k,\ \ k = 1,2,\ldots \tag{3.10}$$

and put

$$\deg f(u) = 0 \tag{3.11}$$

for any function independent of the derivatives. We say that the bilinear operation (3.2) (or (3.7)) is *graded homogeneous of the degree D* if the coefficients are graded homogeneous polynomials in the derivatives of the degree

$$\deg a_k^{ij}(x;\dot{x},\ddot{x},\ldots,x^{(m)}) = D-k,\ \ k = 0,1,\ldots \tag{3.12}$$

Obviously, the order of (3.2) cannot be greater than D.

Lemma 3.4. The degree D does not depend on the choice of local coordinates $x^1,\ldots,x^n$.

Lemma 3.4 follows immediately from (3.8) and (3.9).

Example 3.5. The graded homogeneous Poisson bracket of degree 0 has the form

$$\left\{x^i(s_1),x^j(s_2)\right\} = h^{ij}(x)\delta(s_1-s_2) \tag{3.13}$$

for some $h^{ij}(x)$ a Poisson bracket on the manifold M,

Example 3.6. The graded homogeneous Poisson bracket of degree 1 has the form

$$\left\{x^i(s_1),x^j(s_2)\right\} = \bar{g}^{ij}(x(s_1))\dot{\delta}(s_1-s_2) + \bar{\Gamma}_k^{ij}(x)\dot{x}^k\delta(s_1-s_2), \tag{3.14}$$

where the coefficients $\bar{g}^{ij}(x)$ and $\bar{\Gamma}_k^{ij}$ are some functions on M depending on the choice of local coordinates, and this class of Poisson brackets is called *Poisson brackets of hydrodynamics type.*

Theorem 3.7. The graded homogeneous non-degenerate Poisson brackets of the degree 1 on the loop space $L(M)$ are in 1-to-1 correspondence with flat contravariant metrics $g^{ij}(x)$ on M. The coefficients $\bar{\Gamma}^{ij}_k$ in (3.14) coincide with the Levi-Civita connection of this metric.

Following Theorem 3.7, the Poisson bracket (3.14) can be rewritten in the following constant form:

$$\left\{t^\alpha(s_1), t^\beta(s_2)\right\} = \eta^{\alpha\beta}\dot{\delta}(s_1 - s_2), \tag{3.15}$$

where $\eta^{\alpha\beta}$ are constants, and are the entries of the matrix of the metric in the flat coordinates t.

Definition 3.8. Two Poisson brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ on the same space are called *compatible* if the bilinear combination

$$\{\ ,\ \}_1 - \lambda\{\ ,\ \}_2 \tag{3.16}$$

is a Poisson bracket for any λ.

Theorem 3.9. Two graded homogeneous non-degenerate Poisson brackets of the degree 1 on the loop space $L(M)$ are compatible iff the correspondent flat metrics form a flat pencil.

Proof. The linear combination (3.16) of two Poisson brackets of the form (3.14) reads

$$\begin{aligned} &\left\{x^i(s_1), x^j(s_2)\right\}_1 - \lambda\left\{x^i(s_1), x^j(s_2)\right\}_2 \\ &= \left[g_1^{ij}(x(s_1)) - \lambda g_2^{ij}(x(s_1))\right]\dot{\delta}(s_1 - s_2) + \left[{\Gamma_1}^{ij}_k(x) - \lambda{\Gamma_2}^{ij}_k(x)\right]\dot{x}^k\delta(s_1 - s_2), \end{aligned} \tag{3.17}$$

and then the theorem immediately follows from Theorem 3.7. □

As a result of the theorems, (3.13) and (3.14) associates a flat pencil of metric $g^{\alpha\beta} - \lambda\eta^{\alpha\beta}$ with flat coordinates $\{t^a\}$ to a family of Poisson brackets

$$\begin{aligned} &\left\{t^\alpha(s_1), t^\beta(s_2)\right\}_1 = \eta^{\alpha\beta}\dot{\delta}(s_1 - s_2), \\ &\left\{t^\alpha(s_1), t^\beta(s_2)\right\}_2 = g^{\alpha\beta}(t(s_1))\dot{\delta}(s_1 - s_2) + \Gamma^{\alpha\beta}_\gamma(t)\dot{t}^\gamma\delta(s_1 - s_2), \end{aligned}$$

which are called the *KdV hierarchies* of the pencil.

Remark 5. Observe that, for $d \neq 1$, the variable

$$T(s) := \frac{2}{1-d}\tau(s) \tag{3.18}$$

where the flat coordinates τ was defined in (2.35) has the Poisson bracket with itself has the form

$$\{T(s_1), T(s_2)\}_1 = [T(s_1) + T(s_2)]\dot{\delta}(s_1 - s_2), \tag{3.19}$$

and

$$\{t^\alpha(s_1), T(s_2)\}_1 = \frac{2}{1-d}E^\alpha(t(s_1))\dot{\delta}(s_1 - s_2) + \dot{t}^\alpha\delta(s_1 - s_2). \tag{3.20}$$

(3.20) follows immediately from (2.36) and (2.39).

For (3.19), since

$$\{T(s_1), T(s_2)\}_1 = \left(\frac{2}{1-d}\right)^2\left[g_1^{nn}(t(s_1))\dot{\delta}(s_1 - s_2) + {\Gamma_1}^{nn}_k\dot{t}^k(s_1)\delta(s_1 - s_2)\right].$$

Due to (2.39) and (3.9),

$$\begin{aligned} \left(\frac{2}{1-d}\right)^2{\Gamma_1}^{nn}_k\dot{t}^n(s_1)\delta(s_1 - s_2) &= \left(\frac{2}{1-d}\right)^2 \cdot \frac{1-d}{2}\delta^n_k\dot{t}^k(s_1)\delta(s_1 - s_2) \\ &= \frac{2}{1-d}\dot{t}^n(s_1)\delta(s_1 - s_2) \\ &= \frac{2}{1-d}(t^n(s_2)\dot{\delta}(s_1 - s_2) - t^n(s_1)\dot{\delta}(s_1 - s_2)) \\ &= [T(s_2) - T(s_1)]\,\dot{\delta}(s_1 - s_2). \end{aligned} \tag{3.21}$$

Since $\nabla\nabla E_f = 0$, one can assume $E_f^n = \alpha t^n$ for some constant α. By (2.38),

$$(1-d)\delta_k^n = \partial_k E_f^n \;\Rightarrow\; \alpha = 1-d.$$

By (2.36),

$$g_1{}^{nn} = E_f^n = 1-d.$$

Thus

$$\left(\frac{2}{1-d}\right)^2 g_1^{nn}(t(s_1))\dot{\delta}(s_1-s_2) = 2\cdot\frac{2}{1-d}t(s_1))\dot{\delta}(s_1-s_2) = 2T(s_1). \tag{3.22}$$

Combining (3.21) and (3.22), (3.19) is proved.

4 Differential Geometry of Hurwitz Spaces

4.1 Geometry of Hurwitz spaces

A Riemann surface is a connected manifold of dimension 1. Hurwitz spaces are moduli spaces of pairs (C, λ) where C is a smooth algebraic curve of genus g and λ is a meromorphic function on C of degree $n+1$. A Hurwitz space requires a moduli space $M = M_{g;n_0,\ldots,n_m}$ of dimension

$$n = 2g + n_0 + \cdots + n_m + 2m$$

of sets

$$(C; \infty_0, \ldots, \infty_m; \lambda) \in M_{g;n_0,\ldots,n_m},$$

where C is a Riemannian surface with marked points $\infty_0, \ldots, \infty_m$, and a marked meromorphic function

$$\lambda : C \to CP^1 \tag{4.1}$$

such that

$$\lambda^{-1}(\infty) = \infty_0 \cup \cdots \cup \infty_m,$$

and has degree $n_i + 1$ near the point ∞_i.

Denote the critical values of λ

$$u^j = \lambda(P_j),\ d\lambda|_{P_j} = 0,\ j = 1, \ldots, n$$

where $u^i \neq u^j$ for $i \neq j$. A Hurwitz space also requires a one-dimensional affine group act that

$$(C; \infty_0, \ldots, \infty_m; \lambda; \ldots) \mapsto (C; \infty_0, \ldots, \infty_m; a\lambda + b; \ldots) \tag{4.2}$$

$$u^i \mapsto au^i + b,\ i = 1, \ldots, n. \tag{4.3}$$

To describe the Frobenius structure, set

$$\partial_i := \frac{\partial}{\partial u^i},$$

and define the multiplication

$$\partial_i \cdot \partial_j = \delta_{ij} \partial_i, \tag{4.4}$$

where δ_{ij} denotes the Kronecker delta. The unity vector e and the Euler vector E are chosen to be

$$e = \sum_{i=1}^{n} \partial_i, \quad E = \sum_{i=1}^{n} u^i \partial_i. \tag{4.5}$$

A one-form Ω on the manifold with a Frobenius algebra structure in the tangent planes is called *admissible* if the invariant inner product

$$\langle a, b \rangle_\Omega := \Omega(a \cdot b) \tag{4.6}$$

determines a Frobenius structure on the manifold. Any quadratic differential Q on C holomorphic for $|\lambda| < \infty$ determines a one-form

$$\Omega_Q := \sum_{i=1}^{n} du^i \operatorname*{res}_{P_i} \frac{Q}{d\lambda}. \tag{4.7}$$

The covering $\hat{M} = \hat{M}_{g;n_0,\cdots,n_m}$ consists of the sets

$$(C; \lambda; k_0, \ldots, k_;; a_1, \ldots, a_g; b_1, \ldots, b_g) \in \hat{M}_{g;n_0,\ldots,n_m}$$

with a marked symplectic basis $a_1, \dots, a_g, b_1, \dots, b_g \in H_1(C, \mathbb{Z})$ and marked branches $k_0, \dots, k_m$ of roots of λ near $\infty_0, \dots, \infty_m$ of the orders $n_0+1, \dots, n_m+1$ resp.,

$$k_i^{n_i+1}(P) = \lambda(P), P \text{ near } \infty_i. \tag{4.8}$$

The admissible quadratic differentials on the Hurwitz space are constructed as squares

$$Q = \phi^2$$

of certain differentials ϕ on C, called *primary differentials.* The list of primary differentials are given below:

Type 1.

$$\phi = \phi_{t^{i;\alpha}} := -\frac{1}{\alpha} dk_i^\alpha + \text{regular terms} \quad \text{near } \infty_i, \ i = 0, \dots, m, \ \alpha = 1, \dots, n_i,$$

$$\oint_{a_j} \phi_{t^{i;\alpha}} = 0;$$

Type 2.

$$\phi = \sum_{i=1}^m \delta_i \phi_{v^i} \text{ for } i = 1, \dots, m,$$

$$\phi_{v^i} := -d\lambda + \text{regular terms} \quad \text{near } \infty_i,$$

$$\oint_{a_j} \phi_{v^i} = 0;$$

Type 3.

$$\phi = \sum_{i=1}^m \alpha_i \phi_{w^i},$$

$$\oint_{a_j} \phi = 0;$$

Type 4.

$$\phi = \sum_{i=1}^g \beta_i \phi_{r^i},$$

$$\phi_{r^i}(P + b_j) - \phi_{r^i}(P) = -\delta_{ij} d\lambda,$$

$$\oint_{a_j} \phi_{r^i} = 0;$$

Type 5.

$$\phi = \sum_{i=1}^g \gamma_i \phi_{s^i},$$

$$\oint_{a_j} \phi_{s^i} = \delta_{ij}.$$

For any two tangent fields ∂', ∂'', the metric corresponding to the one-form Ω_{ϕ^2} then has the form

$$ds_{\phi^2} = \langle \partial', \partial'' \rangle := \Omega_{\phi^2}(\partial' \cdot \partial''). \tag{4.9}$$

Theorem 4.1. For any primary differential ϕ among the list, the multiplication (4.4), the unity and Euler vector field (4.5) and the one-form Ω_{ϕ^2} determine a structure of Frobenius manifold on $\hat{M}$. The corresponding flat coordinates t^A, $A = 1, \ldots, A$ consists of the five parts

$$t^A = (t^{i;\alpha}, i = 0, \ldots, m, \alpha = 1, \ldots, n_i; p^i, q^i, i = 1, \ldots, m; r^i, s^i, i = 1, \ldots, g) \tag{4.10}$$

where

$$t^{i;\alpha} = \operatorname*{res}_{\infty_i} k_i^{-\alpha} p d\lambda, \; i = 0, \ldots, m, \; \alpha = 1, \ldots, n_i; \tag{4.11a}$$

$$p^i = \text{v.p.} \int_{\infty_0}^{\infty_i} dp, \; i = 1, \ldots, m; \tag{4.11b}$$

$$q^i = -\operatorname*{res}_{\infty_i} \lambda dp, \; i = 1, \ldots, m; \tag{4.11c}$$

$$r^i = \oint_{b^i} dp; \tag{4.11d}$$

$$s^i = -\frac{1}{2\pi i} \oint_{a_i} \lambda dp, \; i = 1, \ldots, g. \tag{4.11e}$$

The metric (4.9) in these coordinates has the form

$$\eta_{t^{i;\alpha} t^{j;\beta}} = \frac{1}{n_i + 1} \delta_{ij} \delta_{\alpha+\beta, n_i+1}, \tag{4.12a}$$

$$\eta_{v^i w^j} = \frac{1}{n_i + 1} \delta_{ij}, \tag{4.12b}$$

$$\eta_{r^i s^j} = \frac{1}{2\pi i} \delta_{ij}, \tag{4.12c}$$

other components of η vanish.

The metric $\langle \ , \ \rangle$ is then given as

$$\langle \partial_{t^A}, \partial_{t^B} \rangle_\phi = \sum_{|\lambda| < \infty} \operatorname*{res}_{d\lambda = 0} \frac{\phi_A \phi_B}{d\lambda}, \tag{4.13}$$

or equivalently, together with the trilinear form (1.17) and the metric $(\ , \ )$, for tangent vector fields ∂', ∂'', ∂''',

$$\langle \partial', \partial'' \rangle = -\sum_{|\lambda| < \infty} \operatorname*{res}_{d\lambda = 0} \frac{\partial'(\lambda dp) \partial''(\lambda dp)}{d\lambda} \tag{4.14}$$

$$(\partial', \partial'') = -\sum_{|\lambda| < \infty} \operatorname*{res}_{d\lambda = 0} \frac{\partial'(\log \lambda dp) \partial''(\log \lambda dp)}{d \log \lambda} \tag{4.15}$$

$$c\,(\partial', \partial'', \partial''') = -\sum_{|\lambda| < \infty} \operatorname*{res}_{d\lambda = 0} \frac{\partial'(\lambda dp) \partial''(\lambda dp) \partial'''(\lambda dp)}{dp \; d\lambda} \tag{4.16}$$

where

$$d\lambda = \frac{\partial \lambda}{\partial p}, \; d \log \lambda = \frac{\partial \log \lambda}{\partial p} dp.$$

We will focus on the Hurwitz space $M_{0;n}$, that is, the case $g = 0$, $m = 0$, $n_0 = n$ of the Hurwitz space which consists of all the polynomials of the form

$$\lambda(p) = p^{n+1} + a_n p^{n-1} + \ldots + a_1, \; a_1, \ldots, a_n \in \mathbb{C}, \tag{4.17}$$

where the coefficient of p^n is zero by a translation of p and the coefficient of p^{n+1} is 1 by a rescaling of p. The affine transformations $\lambda \mapsto a\lambda + b$ acts on (4.17) as

$$\begin{aligned} p &\mapsto a^{\frac{1}{n+1}} p, \\ a_i &\mapsto a_i a^{\frac{n-i+2}{n+1}} \text{ for } i > i, \\ a_1 &\mapsto a_1 a + b. \end{aligned}$$

This can be verified through simple calculations. After the affine transformation, λ becomes

$$\begin{aligned} &\left(a^{\frac{1}{n+1}} p\right)^{n+1} + \sum_{i=2}^{n} a_i a^{\frac{n-i+2}{n+1}} \left(a^{\frac{1}{n+1}} p\right)^{i-1} + a_1 a + b \\ =& a p^{n+1} + \sum_{i=2}^{n} a_i a p^{i-1} + a_1 a + b \\ =& a\lambda(p) + b. \end{aligned}$$

Since $a\lambda(p) + b$ and $\lambda(p)$ share the same critical points, (4.3) is also satisfied. In this case, λ has n critical values, which are $u^1, \dots, u^n$.

The flat coordinates of $M_{0;n}$ do not vanish only for the form (4.11a), namely

$$t^\alpha = t^{0;\alpha}, \ \alpha = 1, \dots, n.$$

The metric is thus

$$ds^2 = \sum_{\alpha=1}^{n} dt^\alpha dt^{n+1-\alpha}.$$

We also need to construct the unity vector field and the Euler vector field [6],[1].

Theorem 4.2. The flat coordinates satisfy

$$p(k) = k - \frac{1}{n+1}\left(\frac{t^n}{k} + \frac{t^{n-1}}{k^2} + \cdots + \frac{t^2}{k^{n-1}} + \frac{t^1}{k^n}\right) + O\left(\frac{1}{k^{n+1}}\right) \tag{4.18}$$

where k is defined as (4.8), namely,

$$p(k)^{n+1} + a_2 p(k)^{n-1} + \cdots + a_n p(k) + a_{n+1} = k^{n+1}.$$

Lemma 4.3. $t^1, \dots, t^n$ defined in (4.18) has the form

$$t^\alpha = -\operatorname*{res}_{\lambda=\infty} \left(p(\lambda)\lambda^{\frac{-\alpha}{n+1}} d\lambda\right). \tag{4.19}$$

Proof. Since $k = \lambda^{\frac{1}{n+1}}$,

$$\begin{aligned} p(\lambda)\lambda^{\frac{-\alpha}{n+1}} d\lambda &= \left(k - \frac{1}{n+1}\left(\frac{t^n}{k} + \frac{t^{n-1}}{k^2} + \cdots + \frac{t^2}{k^{n-1}} + \frac{t^1}{k^n}\right) + O\left(\frac{1}{t^{n+1}}\right)\right) k^{-\alpha}(n+1)k^n dk \\ &= (n+1)k^{n+1-\alpha} - \sum_{\beta=1}^{n} \frac{t^\beta}{k^{\alpha+1-\beta}} + O\left(\frac{1}{k^{\alpha+1}}\right) dk, \end{aligned}$$

thus

$$\begin{aligned} -\operatorname*{res}_{\lambda=\infty} \left(p(\lambda)\lambda^{\frac{-\alpha}{n+1}} d\lambda\right) &= \sum_{\beta=1}^{n} t^\beta \operatorname*{res}_{k=\infty} \frac{1}{k^{\alpha+1-\beta}} \\ &= t^\alpha. \end{aligned}$$

□

Lemma 4.4.
$$\frac{n+1}{n+1-\alpha}\operatorname*{res}_{p=\infty}(\lambda^{\frac{n+1-\alpha}{n+1}}(p)dp) = -\operatorname*{res}_{\lambda=\infty}(p\lambda^{\frac{-\alpha}{n+1}}d\lambda).$$

Proof. Using integration by parts,
$$\frac{n+1}{n+1-\alpha}\int \lambda^{\frac{n+1-\alpha}{n+1}}(p)dp = \frac{n+1}{n+1-\alpha}p\lambda^{\frac{n+1-\alpha}{n+1}} - \int p\lambda^{\frac{-\alpha}{n+1}}d\lambda. \tag{4.20}$$
Notice that
$$\operatorname*{res}_{\infty} p\lambda^{\frac{n+1-\alpha}{n+1}}(p) = 0,$$
lemma is proved. □

Theorem 4.2 is hence proved. This implies that
$$t^i = -a_{i+1} + B_i(a_{i+2}, \dots, a_n),\ 0 \le i \le n-1.$$
for some function B_i.

Section 2.2 of [1] explained that for any polynomial $\lambda \in \mathbb{C}[z_1, \dots, z_k]$ can be written as
$$\lambda = \sum_{j=1}^{n} \lambda(P_j)\partial_j = E,$$
therefore λ defined by (4.9) itself is an Euler vector field in this sense.

Observe that E can be expressed in the coordinates $(a_1, \dots, a_n)$:
$$\lambda(p) - \frac{p}{n+1}\lambda(p) = \sum_{j=1}^{n} \frac{n+2-j}{n+1}a_j z^{j-1},$$
in other words,
$$\begin{aligned} E &= \sum_{j=1}^{n} \frac{n+2-j}{n+1}a_{j+1}\frac{\partial}{\partial a_{j+1}} \\ &= \sum_{j=0}^{n-1} \frac{n+1-j}{n+1}a_j\frac{\partial}{\partial a_j}. \end{aligned}$$
Assume $\deg a_i = n-i+2$, so that $\deg \lambda = n+1$, $\deg B_i = n-i+1$ and λ and B_i are homogeneous. Then
$$\begin{aligned} E \cdot t^i &= \sum_{j=1}^{n} \frac{n+2-j}{n+1}a_j\frac{\partial t^i}{\partial a_j} \\ &= -\frac{n+2-i}{n+1}a_i + \sum_{j=i+1}^{n} \frac{n+2-j}{n+1}a_j\frac{\partial B_i}{\partial a_j} \\ &= \frac{n+2-i}{n+1}t^i - \left(\frac{n+2-i}{n+1}B_i - \sum_{j=i+2}^{n} \frac{n+2-j}{n+1}a_j\frac{\partial B_i}{\partial a_j}\right) \\ &= \frac{n+2-i}{n+1}t^i \end{aligned}$$
by homogeneity of B_i. Therefore, in flat coordinates,
$$E = \sum_{j=1}^{n} \frac{n+2-j}{n+1}t^j\frac{\partial}{\partial t^j}.$$

Similarly, the unit vector field can be written as

$$e = \frac{\partial}{\partial a_1},$$

thus, via direct calculation using

$$t^\alpha = \frac{n+1}{n+1-\alpha} \operatorname*{res}_{p=\infty} (\lambda^{\frac{n+1-\alpha}{n+1}}(p)dp),$$

one can find that

$$e \cdot t^\alpha = \frac{\partial t^\alpha}{\partial a_1} = \delta_{1\alpha},$$

therefore

$$e = \frac{\partial}{\partial t^1}$$

in flat coordinates.

In the following content we will see two concrete examples for $n = 1$ and $n = 2$.

4.2 Example of n=1

For $n = 1$, one has

$$\lambda(p) = p^2 + a,$$

The flat coordinates are given by (4.11a) as

$$t^{0;1} = -2a,$$

one may take by normalization

$$t^1 = \frac{a}{\sqrt{2}}.$$

Then, w.r.t. this coordinate, by (4.14),

$$\eta_{11} = \langle \partial_1, \partial_1 \rangle = 1.$$

For readability, we can lower the indices of t^i:

$$t_j = \eta_{ji} t^i,$$

thus

$$t_1 = \eta_{11} t^1 = t^1 = \frac{a}{\sqrt{2}}$$

and the trilinear form (1.17) is given by (4.16) as

$$\langle \partial_1, \partial_1, \partial_1 \rangle = 2\sqrt{2},$$

and this implies the free energy F of the corresponding Frobenius manifold

$$F = \frac{\sqrt{2}}{3} t_1^3,$$

and the Euler vector field is

$$E = t_1 \partial t_1.$$

By (2.4) and raising the indices ((2.6b), (1.20)), one gets the contravariant metrics g:

$$g^{11} = 2\sqrt{2}\ t_1,$$

$$g_{11} = \frac{1}{2\sqrt{2}t_1}.$$

Recall that the Christoffel symbol corresponding to the Levi-Civita connection can be calculated as

$$\Gamma^i_{kl} = \frac{1}{2} g^{im}(\partial_l g_{mk} + \partial_k g_{ml} - \partial_m g_{kl}) \tag{4.21}$$

thus

$$\Gamma^1_{11} = -\frac{1}{2t_1},$$

and by raising the indices (1.3),

$$\Gamma^{11}_1 = \sqrt{2}.$$

Thus one can also find the Poisson brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ of (3.14):

$$\left\{t^1(s_1), t^1(s_2)\right\}_1 = \sqrt{2}(2\ t^1\dot{\delta}(s_1 - s_2) + \dot{t^1}\delta(s_1 - s_2)),$$

$$\left\{t^1(s_1), t^1(s_2)\right\}_2 = \dot{\delta}(s_1 - s_2).$$

4.3 Example of n=2

For n=2,

$$\lambda = p^3 + ap^2 + b,$$

the flat coordinates given by (4.11a) are

$$t^{0;1} = -6b, \quad t^{0;2} = -2a.$$

which can be normalized

$$t^1 = \frac{1}{3}a, \quad t^2 = b.$$

One has for the metric

$$\langle \partial_1, \partial_2 \rangle = 1$$

and vanishes elsewhere. Thus the metric $\langle\ ,\ \rangle$ is

$$\left(\eta^{\alpha\beta}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Again, for readability, one can lower the indices,

$$t_1 = t^2 = b, \quad t_2 = t^1 = \frac{1}{3}a.$$

For the trilinear form,

$$\langle \partial_1, \partial_1, \partial_1 \rangle = 3a,$$
$$\langle \partial_1, \partial_2, \partial_2 \rangle = 1,$$

which induces the free energy

$$F = \frac{1}{2}t_1^2 t_2 + \frac{a}{2}t_2^3$$

which has the quasihomogeneity degrees

$$\deg t_1 = \deg t_2 = 1,\ \deg F = 3.$$

The Euler vector field is therefore

$$E = t_1\partial_1 + t_2\partial_2.$$

From the intersection form, one may obtain the metric g

$$\left(g^{\alpha\beta}\right) = \begin{pmatrix} 3at_2 & t_1 \\ t_1 & t_2 \end{pmatrix}.$$

The contravariant Levi-Civita connection is given as

$$\begin{aligned}
\Gamma_1^{11} &= 0, & \Gamma_1^{12} &= \frac{1}{2}, & \Gamma_1^{21} &= \frac{1}{2}, & \Gamma_1^{22} &= 0, \\
\Gamma_2^{11} &= \frac{3a}{2}, & \Gamma_2^{12} &= 0, & \Gamma_2^{21} &= 0, & \Gamma_2^{22} &= \frac{1}{2}.
\end{aligned}$$

The KdV hierarchies are hence:

$$\begin{aligned}
&\left\{t^1(s_1), t^1(s_2)\right\}_1 = 3at^1\dot{\delta}(s_1 - s_2) + \frac{3a}{2}\dot{t}^2\delta(s_1 - s_2), \\
&\left\{t^1(s_1), t^2(s_2)\right\}_1 = t^2(s_1)\dot{\delta}(s_1 - s_2) + \frac{1}{2}\dot{t}^1\delta(s_1 - s_2), \\
&\left\{t^2(s_1), t^1(s_2)\right\}_1 = t^2(s_1)\dot{\delta}(s_1 - s_2) + \frac{1}{2}\dot{t}^1\delta(s_1 - s_2), \\
&\left\{t^2(s_1), t^2(s_2)\right\}_1 = t^1(s_1)\dot{\delta}(s_1 - s_2) + \frac{1}{2}\dot{t}^2\delta(s_1 - s_2), \\
&\left\{t^1(s_1), t^1(s_2)\right\}_2 = 0, \\
&\left\{t^1(s_1), t^2(s_2)\right\}_2 = \left\{t^2(s_1), t^1(s_2)\right\}_2 = \dot{\delta}(s_1 - s_2), \\
&\left\{t^2(s_1), t^2(s_2)\right\}_2 = 0.
\end{aligned}$$

5 Frobenius manifold structure for a supermanifold

This section aims to discuss an example of commutativity replaced with super-commutativity. We will endow a supermanifold with a Frobenius structure. Throughout this section, homogeneity will be considered under the principles of supermanifold theory.

The homogeneity is understood with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading, corresponding to parity of functions and tensors, where even elements have degree 0, and odd elements have degree 1 (note that this is not the same as the quasihomogeneity degree defined by (1.22)). In the following content, an even (resp. odd) function always refers to a homogeneous function of even (resp. odd) degree. All geometric objects in this section (metrics, connections, curvature, etc), if not specified, are understood in the supergeometric sense. Some propositions are introduced in Appendix A.

Given local coordinates $\{x^a\}$, each x^a is assigned with a $\mathbb{Z}/2\mathbb{Z}$-degree $|x^a|$. For simplicity, $(-1)^{|x^a||x^b|}$ will be written in the form as $(-1)^{ab}$ when no confusion arises. For the coordinates, one has supercommutativity:

$$x^a x^b = (-1)^{ab} x^b x^a. \tag{5.1}$$

A (p,q) supermanifold is often described as a manifold with p bosonic coordinates and q fermionic coordinates, i.e., $x_1,\ldots,x_p,\xi_1,\ldots,\xi_q$, where $|x_i|=0,\ |\xi_j|=1$. This structure is locally isomorphic to $\mathbb{C}^\infty(\mathbb{R}^p)\times\Lambda(\xi_1,\ldots,\xi_q)$. The supersymmetry can be interpreted as

$$x_i x_j = x_j x_i,\ x_i\xi_j = \xi_j x_i,\ \xi_i\xi_j = -\xi_j\xi_i. \tag{5.2}$$

One may notice that taking $i=j$ in the third equation implies

$$\xi_j^2 = 0,$$

therefore a homogeneous function on a supermanifold is at most linear w.r.t. ξ_j. In this section, we will focus on the case for a $(p,2)$ manifold.

5.1 Contravariant connections on supermanifold

On a supermanifold, define the *contravariant Levi-Civita connection* with respect to a bilinear form g as

$$\begin{aligned}\Gamma^{ij}_k(x) :&= -g^{is}(x)g^{jt}(x)\Gamma^l_{ks}(x)g_{lt}(x)\\ &= -(-1)^j g^{is}\Gamma^j_{ks}.\end{aligned} \tag{5.3}$$

Analogous to Lemma 1.3, the functions Γ^{ij}_k have the following propositions:

Lemma 5.1. The coefficients $\Gamma^{ij}_k(x)$ of the contravariant Levi-Civita connection on a supermanifold are determined uniquely from the system of linear equations

$$g^{is}\Gamma^{jk}_s = (-1)^{ij}g^{js}\Gamma^{ik}_s, \tag{5.4}$$

$$\partial_k g^{ij} = (-1)^j\Gamma^{ij}_k + (-1)^{ij+j}\Gamma^{ji}_k. \tag{5.5}$$

Proof. For (5.4), applying (5.3) directly:

$$\begin{aligned}g^{is}\Gamma^{jk}_s &= g^{is}g^{jx}g^{ky}\Gamma^z_{sx}g_{zy},\\ g^{js}\Gamma^{ik}_s &= g^{js}g^{ix}g^{ky}\Gamma^z_{sx}g_{zy}\\ &= (-1)^{sx}g^{is}g^{jx}g^{ky}\Gamma^z_{sx}g_{zy}.\end{aligned}$$

Observe that, due to (A.5),

$$(-1)^{sx}g^{is}g^{jx}g^{ky}\Gamma^z_{sx}g_{zy} = \sum_{\substack{|s|=|i|\\|x|=|j|}}(-1)^{sx}g^{is}g^{jx}g^{ky}\Gamma^z_{sx}g_{zy} = (-1)^{ij}g^{is}g^{jx}g^{ky}\Gamma^z_{sx}g_{zy}. \tag{5.6}$$

As a result,

$$g^{js}\Gamma^{ik}_s = (-1)^{ij} g^{is} g^{jx} g^{ky} \Gamma^z_{sx} g_{zy} = (-1)^{ij} g^{is} \Gamma^{jk}_s.$$

To show (5.5), since for all $i,\ t,\ k = 1, \dots, p+q$, applying the same technique as (5.6),

$$\begin{aligned} 0 = \partial_k (g_{ij} g^{jt}) &\overset{(A.1)}{=} \partial_k g_{ij} \cdot g^{jt} + (-1)^{k(i+j)} g_{ij} \partial_k g^{jt} \\ &\overset{(A.9)}{=} \underbrace{\Gamma^s_{ki} g_{sj} g^{jt}}_{\Gamma^s_{ki}\delta^t_s \ = \Gamma^t_{ki}} + (-1)^{ij} \Gamma^s_{kj} g_{si} g^{jt} + g_{ij} \partial_k g^{jt} \end{aligned}$$

contract by g_{tn}:

$$\begin{aligned} 0 =& \Gamma^t_{ki} g_{tn} + (-1)^{ij} \Gamma^s_{kj} g_{si} g^{jt} g_{tn} + g_{ij} \partial_k g^{jt} g_{tn} \\ =& \Gamma^t_{ki} g_{tn} + (-1)^{in} \Gamma^s_{kn} g_{si} + g_{ij} \partial_k g^{jt} g_{tn} \end{aligned}$$

contract by $g^{ai} g^{bn}$

$$\begin{aligned} 0 =& g^{ai} g^{bn} \Gamma^t_{ki} g_{tn} + (-1)^{in} g^{ai} g^{bn} \Gamma^s_{kn} g_{si} + g^{ai} g^{bn} g_{ij} \partial_k g^{jt} g_{tn} \\ =& -\Gamma^{ab}_k + (-1)^{an+a} g_{si} g^{ia} g^{bn} \Gamma^s_{kn} + (-1)^b g^{ai} g_{ij} g_{tn} g^{nb} \partial_k g^{jt} \\ =& -\Gamma^{ab}_k + (-1)^{an} (-1)^a b^{bn} \Gamma^a_{kn} + (-1)^b \partial_k g^{ab} \\ =& -\Gamma^{ab}_k - (-1)^{ab} \Gamma^{ba}_k + (-1)^b \partial_k g^{ab}. \end{aligned}$$

As a result, we get (5.5).

□

5.2 WDVV systems and Frobenius supermanifold

We endow a supermanifold M with a tensor η of rank 2 and a tensor A of rank 3. The components of the tensors are then η_{ab} and A^c_{ab} respectively. The degrees of the tensors are taken as

$$|\eta_{ab}| = |x^a| + |x^b|, \ |A^c_{ab}| = |x^a| + |x^b| + |x^c|.$$

In this way, the tensors are homogeneous and even, in the sense that

$$\eta_{ab} dx^a \otimes dx^b$$

has degree 0.

We require that η_{ab} satisfies

$$\eta_{ab} = (-1)^{ab} \eta_{ba},$$

which defines a pairing

$$\langle \partial_a, \partial_b \rangle := \eta_{ab}. \tag{5.7}$$

We assume η_{ab} is non-degenerate so that the inverse $(\eta^{ab}) = (\eta_{ab})^{-1}$ exists. We will also assume that η_{ab} are constant. The pairing is thus a metric. Furthermore, since η is constant, it is a flat metric in a flat coordinate system.

The tensor A^c_{ab} will be used for constructing a structure of a $C^\infty(M)$-algebra on TM with a multiplication $\circ$:

$$\partial_a \circ \partial_b := A^c_{ab} \partial_c.$$

The multiplication $\circ$ must satisfy the following.

1. (Super)commutativity:

$$A^c_{ba} = (-1)^{ab} A^c_{ab}. \tag{5.8}$$

2. Associativity:

$$A^e_{ab}A^d_{ec} = (-1)^{a(b+c)}A^e_{bc}A^d_{ea}, \tag{5.9a}$$

$$\partial_d A^c_{ab} = (-1)^{ad}\partial_a A^c_{db}. \tag{5.9b}$$

Notice that (5.9a) is equivalent to

$$\partial_a \circ (\partial_b \circ \partial_c) = (\partial_a \circ \partial_b) \circ \partial_c.$$

3. The multiplication admits a Frobenius algebra, i.e.,

$$\langle \partial_a \circ \partial_b, \partial_c \rangle = \langle \partial_a, \partial_b \circ \partial_c \rangle. \tag{5.10}$$

Setting

$$A^c_{ab} := A_{abe}g^{ec},$$

we naturally have $A_{abc} = A^e_{ab}g_{ec}$. (5.10) reads

$$A_{abc} = (-1)^{a(b+c)}A_{bca}. \tag{5.11}$$

4. Existence of identity: an even coordinate vector field ∂_0 satisfying

$$A^c_{0b} = \delta^c_b.$$

Or equivalently,

$$A_{0bc} = \eta_{bc}. \tag{5.12}$$

5. Existence of potential: an even (local) function Φ on M is called a *potential* if

$$A_{abc} = \partial_a\partial_b\partial_c\Phi.$$

It is easy to check that (5.7), (5.8), (5.9b), (5.11) follow from this definition. In particular, (5.9a) gives the WDVV equations on the supermanifold:

$$\partial_a\partial_b\partial_e\Phi\eta^{ef}\partial_f\partial_c\partial_d\Phi = (-1)^{a(b+c)}\partial_b\partial_c\partial_e\Phi\eta^{ef}\partial_f\partial_a\partial_d\Phi. \tag{5.13}$$

6. Potential should be quasihomogeneous: Φ on a (p,q) supermanifold, there exists $d_1, \dots, d_{m+n}$ such that for any nonzero c

$$c^{d_F}\Phi(x_1, \dots, x_p, \xi_1, \dots, \xi_q) = \Phi(c^{d_1}x_1, \dots, c^{d_p}x_p, c^{d_{p+1}}\xi_1, \dots, c^{d_{p+q}}\xi_q).$$

We will still consider the quasihomogeneity up to adding of a quadratic function. Again, quasihomogeneity introduces the Euler vector field

$$E := d_a x_a \partial_a + d_\alpha \xi_\alpha \partial_\alpha, \tag{5.14}$$

and the unity vector field

$$e := \frac{\partial}{\partial x_1}. \tag{5.15}$$

Since x_1 is even, the unity vector field e is also even. Moreover, since for any a and α, $|x_a\partial_a| = |\xi_\alpha\partial_\alpha| = 0$, the Euler vector field E is homogeneous and even. Consequently, the homogeneity conditions (1.44a), (1.44b) are preserved.

Similar to the classical case discussed in Section 1.2, the triple (M, η, A) induces a Frobenius supermanifold structure. It is therefore natural to investigate whether a Frobenius supermanifold implies a flat pencil of metrics on the supermanifold.

On the Frobenius supermanifold, setting

$$dx^a \cdot dx^b := A^{ab}_c dx^c,$$

where

$$A_c^{ab} := \eta^{ia}\eta^{jb}A_{cij} = (-1)^b\eta^{ai}A_{ci}^b,$$

the equivalence between a Frobenius manifold and a flat pencil of metrics follows analogously from Section 2. Then one may derive the bilinear form g:

$$g^{ab} := i_E(dx^a \cdot dx^b) = E^s A_s^{ab} = E^s\eta^{ai}\eta^{bj}A_{sij}. \tag{5.16}$$

We have the analogy for the super-setting:

Lemma 5.2. For $|a| = |b|$, g^{ab} satisfies

$$g^{ab} = R_s^a\Phi^{sb} + (-1)^{ab}\Phi^{as}R_s^b + A^{ba}$$

where

$$R_b^a = \frac{d-1}{2}\delta_b^a + \partial_b E^a,$$
$$\Phi^{ab} = \eta^{ai}\eta^{bj}\partial_i\partial_j\Phi,$$
$$A^{ab} = \eta^{ai}\eta^{bj}A_{ij}.$$

Proof. Since e and E are even, (2.8) is preserved. As a result, analogous to the classical case,

$$(\eta^{ai}\partial_i E^s)(\eta^{bj}\partial_s\partial_j F) = (2-d)F^{ab} - \partial_i E^a \cdot F^{ib},$$
$$(\eta^{bj}\partial_j E^s)(\eta^{ai}\partial_s\partial_i F) = (2-d)F^{ba} - \partial_j E^b \cdot F^{ja},$$

and

$$\begin{aligned}(3-d)F^{ab} &= (-1)^{ab}(\eta^{bj}\partial_j E^s)(\eta^{ai}\partial_s\partial_i F) + (\eta^{ai}\partial_i E^s)(\eta^{bj}\partial_s\partial_j F) + E^s\eta^{ai}\eta^{bj}A_{sij}\\ &= (4-2d)F^{ab} - \partial_i E^a \cdot F^{ib} - (-1)^{ab}\partial_j E^b \cdot F^{ja} + E^s\eta^{ai}\eta^{bj}A_{sij}\end{aligned}$$

which implies

$$\begin{aligned}g^{ab} = E^s\eta^{ai}\eta^{bj}A_{sij} &= (d-1)F^{ab} + \partial_i E^a \cdot F^{ib} + (-1)^{ab}\partial_j E^b \cdot F^{ja} + A^{ba}\\ &= \left[\frac{d-1}{2}F^{ab} + \partial_i E^a \cdot F^{ib}\right] + (-1)^{ab}\left[\frac{d-1}{2}F^{ba} + \partial_j E^b \cdot F^{ja}\right] + A^{ba}\\ &= R_s^a\Phi^{sb} + (-1)^{ab}\Phi^{as}R_s^b + A^{ba}.\end{aligned}$$

□

Lemma 5.3. The contravariant Levi-Civita connection for $g - \lambda\eta$ is

$$\Gamma_c^{ab} = (-1)^b R_s^b A_c^{as}. \tag{5.18}$$

Proof. It suffices to show (5.18) satisfies (5.1). Observe that (5.18) implies $\Gamma_c^{ab} = \gamma(b)A_c^{ab}$ for some constant γ determined by the index b, thus (5.4) follows from

$$\eta^{as}\Gamma_s^{bc} = \gamma(c)\eta^{as}\eta^{bi}\eta^{cj}A_{sij} = (-1)^{ab}\gamma(c)\eta^{bi}\eta^{as}\eta^{cj}A_{isj} = (-1)^{ab}\eta^{bi}\Gamma_i^{ac},$$

$$g^{as}\Gamma_s^{bc} = (-1)^{s+c}\gamma(c)\eta^{ai}\eta^{bj}A_{ti}^s A_{sj}^c \overset{(5.9a)}{=} (-1)^{s+c+ab}\gamma(c)\eta^{ai}\eta^{bj}A_{tj}^s A_{si}^c = (-1)^{ab}g^{bs}\Gamma_s^{ac}.$$

(5.5) follows immediately from Lemma 5.2. □

From the above, one may deduce a flat pencil formed by the flat metric η and the bilinear form g given as (5.16), thus inducing the KdV hierarchies.

5.3 Example of the (3,2) supermanifold

For the $(3,2)$ case, where there are 3 even variables x_1, x_2, x_3 and two odd variables ξ_1, ξ_2, consider the matrix η_{ab}

$$\eta = \left(\begin{array}{ccc|cc} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & -\alpha & 0 \end{array}\right).$$

Since the potential Φ is even, it has the form

$$\Phi = F_1(x_1, x_2, x_3) + F_2(x_1, x_2, x_3)\xi_1\xi_2. \tag{5.20}$$

According to (5.12), Φ has the form

$$\Phi = G(x_2, x_3) + F(x_2, x_3)\xi_1\xi_2 + \frac{1}{2}x_1^2 x_2 + \frac{1}{2}x_1 x_2^2 - \alpha x_1\xi_1\xi_2 + c_1 x_1^2 + c_2 x_1 x_2 + c_3 x_1 x_3 + c_4 x_1$$

for some functions $G(x, y)$ and $F(x, y)$.

Lemma 5.4. The WDVV equations (5.13) for Φ reads

$$-F_x^2 + 2\alpha F_y = 0, \tag{5.21}$$
$$-\alpha F_x G_{xyy} + \alpha^2 G_{yyy} - F_y^2 = 0, \tag{5.22}$$
$$-\alpha F_x G_{xxy} + \alpha^2 G_{xyy} - F_x F_y = 0, \tag{5.23}$$
$$-\alpha F_x G_{xxx} + \alpha^2 G_{xxy} - F_x^2 - \alpha F_y = 0. \tag{5.24}$$

As a result of the above differential equations,

$$-G_{xyy}G_{xxx} + G_{xxy}^2 - G_{yyy} = 0. \tag{5.25}$$

Proof. Here, an example of obtaining (5.24) will be demonstrated, and (5.21), (5.22), (5.23) can be derived in the same way.

Taking $(a, b, c, d) = (1, 1, 3, 4)$ in the WDVV equation, one has

$$\begin{aligned} 0 &= F_{xxy}\xi_1\xi_2 + G_{xxy} - \frac{1}{\alpha^2}F_x^2 - \frac{1}{\alpha}F_y + \frac{1}{\alpha}F_{xx}^2\xi_1\xi_2 - \frac{1}{\alpha}F_x F_{xxx}\xi_1\xi_2 - \frac{1}{\alpha}F_x G_{xxx} \\ &= (F_{xxy} + \frac{1}{\alpha}F_{xx}^2 - \frac{1}{\alpha}F_x F_{xxx})\xi_1\xi_2 - \frac{1}{\alpha^2}F_x^2 + G_{xxy} - \frac{1}{\alpha}F_y - \frac{1}{\alpha}F_x G_{xxx}. \end{aligned}$$

Since the functions F and G don't include ξ_i, this implies

$$\begin{aligned} &-\frac{1}{\alpha^2}F_x^2 + G_{xxy} - \frac{1}{\alpha}F_y - \frac{1}{\alpha}F_x G_{xxx} = 0 \\ \Leftrightarrow &-\alpha F_x G_{xxx} + \alpha^2 G_{xxy} - F_x^2 - \alpha F_y = 0, \end{aligned}$$

which gives (5.24).

From (5.21),

$$F_x^2 = 2\alpha F_y, \tag{5.26}$$

(5.22) and (5.24) gives respectively

$$\begin{aligned} \alpha F_x G_{xyy} &= \alpha^2 G_{yyy} - F_y^2 \\ \alpha F_x G_{xxx} &= \alpha^2 G_{xxy} - 3\alpha F_y. \end{aligned}$$

Multiply on both sides

$$\underbrace{\alpha^2 F_x^2}_{2\alpha^3 F_y} G_{xxx} G_{xyy} = \alpha^4 G_{yyy} G_{xxy} - 3\alpha^3 F_y G_{yyy} - \alpha^2 F_y^2 G_{xxy} + 3\alpha F_y^3 \tag{a}$$

(5.23) gives

$$\alpha^2 F_x G_{xyy} = \alpha^3 G_{yyy} - \alpha F_y^2,$$

and (5.24) can be rewritten as

$$\begin{aligned}
\underbrace{\alpha F_x^2}_{2\alpha^2 F_y} G_{xxy} &= \alpha^2 F_x G_{xyy} - F_x^2 F_y \\
&= \alpha^3 G_{yyy} - 3\alpha F_y^2 \\
\Rightarrow 2\alpha F_y G_{xxy} &= \alpha^2 G_{yyy} - 3F_y^2.
\end{aligned}$$

Therefore (a) can be reformulated

$$\begin{aligned}
2\alpha^3 F_y G_{xxx} G_{xyy} &= \alpha^2 G_{xxy}(2\alpha F_y G_{xxy} + 3F_y^2) - 3\alpha^3 F_y G_{yyy} - \alpha^2 F_y^2 G_{xxy} + 3\alpha F_y^3 \\
&= 2\alpha^3 F_y G_{xxy}^2 + 2\alpha^2 F_y^2 G_{xxy} - 3\alpha^3 F_y G_{yyy} + 3\alpha F_y^3 \\
&= 2\alpha^3 F_y G_{xxy}^2 + \alpha F_y(\alpha^2 G_{yyy} - 3F_y^2) - 3\alpha^3 F_y G_{yyy} + 3\alpha F_y^3 \\
&= 2\alpha^3 F_y G_{xxy}^2 - 2\alpha^3 F_y G_{yyy} \\
\Rightarrow G_{xxx} G_{xyy} &= G_{xxy}^2 - G_{yyy},
\end{aligned}$$

and we have (5.25). □

(5.25) – (5.24) are solved as[12]: for an arbitrary function $f(s)$,

$$\begin{aligned}
x &= -\frac{f'(s)}{\alpha} y + s \\
F &= -\frac{f'(s)^2}{2\alpha} y + f(s).
\end{aligned} \tag{5.27}$$

For an arbitrary function $g_1(s)$, $g_2(s)$ and $g_3(s)$ satisfy the following equations

$$\begin{aligned}
g_2''(s) &= -2g_1'' \cdot \frac{f''}{\alpha} - g_1' \frac{f'''}{\alpha} + \frac{3}{2} \frac{(f')^2}{\alpha^2} \\
g_3'(s) &= -g_2' \frac{f''}{\alpha} - g_1' \frac{(f'')^2}{\alpha^2} - \frac{(f')^3}{2\alpha^3}
\end{aligned}$$

the function G is then

$$G(x, y) = \frac{1}{8} \frac{f'(y)^4}{\alpha^4} + g_3(y) x^2 + g_2(y) x + g_1(y).$$

Example 5.5. Choose $f(s) = \frac{\alpha}{2} s^2$ in (5.27) and $g_1(s) = -\frac{1}{8} s^4$. Then

$$F(x, y) = \frac{\alpha x^2}{2(1-y)},\ G(x, y) = \frac{x^4}{8(y-1)},$$

Φ has

$$\deg x_1 = 1,\ \deg x_2 = \frac{1}{2},\ \deg x_3 = 0,\ \deg \xi_1 + \deg \xi_2 = 1,$$
$$\deg \Phi = \deg G = 2,\ \deg F = 1.$$

The degrees induce the Euler vector field

$$E = x_1\frac{\partial}{\partial x_1} + \frac{1}{2}x_2\frac{\partial}{\partial x_2} + \theta\xi_1\frac{\partial}{\partial \xi_1} + (1-\theta)\xi_2\frac{\partial}{\partial \xi_2},\ \theta\in[0,1].$$

The metric η implies a system of coordinates $\{x^a\}$:

$$x^1 = x_3,\ x^2 = x_2,\ x^3 = x_1,\ \xi^1 = \alpha\xi_2,\ \xi^2 = -\alpha\xi_1.$$

For simplicity, we denote $\lambda := \frac{1}{1-x_3} = \frac{1}{1-x^1}$. The components of the bilinear form g are given by (5.16) as

$$\begin{aligned}
&g^{11} = x_2^2\lambda^3(-\frac{1}{2}x_2^2 + 2\alpha\xi_1\xi_2), && g^{12} = g^{21} = 2x_2\lambda^2(-\frac{3}{2}x_2^2 + \alpha(1+4\theta)\xi_1\xi_2),\\
&g^{13} = g^{31} = x_1, && g^{14} = g^{41} = -2\alpha x_2^2\lambda^2((\theta+1)\xi_1 + (\theta-1)\xi_2),\\
&g^{15} = g^{51} = 2(2-\theta)\lambda^2 x_2^2\xi_2, && g^{22} = x_1 + \lambda(-3x_2^2 + \alpha\xi_1\xi_2),\\
&g^{23} = g^{32} = \frac{1}{2}x_2, && g^{24} = g^{42} = (\frac{1}{2}+\theta)\lambda x_2\xi_1,\\
&g^{25} = g^{52} = 2(3-2\theta)\lambda x_2, && g^{33} = g^{44} = g^{55} = 0,\\
&g^{34} = g^{43} = -\theta\xi_1, && g^{35} = g^{53} = (\theta-1)\xi_2,\\
&g^{45} = -g^{54} = 2\alpha(3-2\theta)\lambda x_2.
\end{aligned}$$

With (5.3), we have the KdV hierarchies, which are given in Appendix B.

Appendix A Geometry of supermanifolds

Since the symmetry condition has changed from the content, it is necessary to pose the definition of a metric on supermanifolds, as well as the connection and derivation properties. We adopt the notions as defined in [7]. The definitions are reproduced below.

Definition A.1. The *tangent space* T_qM of M at p is the space of homogeneous *derivations* φ such that for homogeneous $f,\ g$,

$$\varphi(fg) = \varphi(f)g(p) + (-1)^{|\varphi||f|}f(p)\varphi(g). \tag{A.1}$$

$TM,\ T_q^*M, T^*M, \mathrm{Vec}(M)$ are defined similarly to the classical case, with whose elements must be homogeneous.

Remark 6. The coordinate system $\{x^a\}$ defines a basis $\partial_a = \partial/\partial x^a$ of vector fields on TM and a basis dx^a of 1-forms on T^*M. The vector field ∂_a and 1-form dx^a have the same parity as x^a.

Definition A.2. [11] Let $M,\ N$ be modules over a commutative algebra A. A mapping $B : M \times N \to A$ is called a *bilinear form* if it is additive in both arguments, and for $m \in M,\ n \in N,\ a \in A$,

$$B(ma, n) = B(m, an), \tag{A.2a}$$

$$B(m, na) = B(m, n)a. \tag{A.2b}$$

Moreover,

$$|B(m, n)| = |B| + |m| + |n|. \tag{A.3}$$

Definition A.3. A *Riemannian metric* on the supermanifold M is an even K-bilinear form (K is $\mathbb{R}$ or $\mathbb{C}$) $\langle\ ,\ \rangle$ on TM such that for any $q \in M$ and homogeneous $X, Y \in T_qM$,

$$\langle X, Y\rangle = (-1)^{|X||Y|}\langle Y, X\rangle. \tag{A.4}$$

Since K is considered as purely even, the bilinear form requires

$$\langle X, Y\rangle = 0 \text{ if } |X| \neq |Y|, \tag{A.5}$$

as a result, the matrix (g_{ab}) has the form

$$g = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right),$$

and g_{ab} is either 0 or an even function. Therefore, for any function f, one always has

$$g_{ab}f = fg_{ab}. \tag{A.6}$$

Remark 7. Since a metric has even degree, bilinearity of a metric $\langle\ ,\ \rangle$ on a supermanifold reads

$$\langle fX, Y\rangle = f\langle X, Y\rangle, \tag{A.7a}$$

$$\langle X, fY\rangle = (-1)^{|X||f|}f\langle X, Y\rangle. \tag{A.7b}$$

Definition A.4. The bracket $[\ ,\ ]$ defines a bilinear form $[\ ,\ ] : \mathrm{Vec}(M) \times \mathrm{Vec}(M) \to \mathrm{Vec}(M)$ for homogeneous vector fields $X,\ Y$:

$$[X, Y]f := X(Yf) - (-1)^{|X||Y|}Y(Xf).$$

Definition A.5. [2] The *inner product* of a vector field $X \in TM$ and a p-form ω is defined as

$$i_X\omega(X_1, \ldots, X_{p-1}) := (-1)^{|X||\omega|}\omega(X, X_1, \ldots, X_{p-1}).$$

Definition A.6. [2] For $X \in TM$, the *Lie derivative* w.r.t. X is

$$\mathcal{L}_X := i_X \circ d + d \circ i_X,$$

where d is the exterior differential as in the usual sense. The explicit expression of the Lie derivative for a p-form ω is given as

$$\begin{aligned}\mathcal{L}_X \omega(X_1, \ldots, X_p) =& (-1)^{|X| \sum_{i=1}^{p} |x_i|} X(\omega(X_1, \ldots, X_p)) \\ &- (-1)^{|X||\omega|} \sum_{i=1}^{p} (-1)^{|X| \sum_{j<i} |X_j|} \omega(X_1, \ldots, [X, X_i], \ldots, X_p).\end{aligned}$$

Definition A.7. A (*linear*) *connection* for the supermanifold M is a bilinear map $\nabla : \mathrm{Vec}(M) \times \mathrm{Vec}(M) \to \mathrm{Vec}(M)$ such that for homogeneous vector fields X, Y and a smooth function f, the following properties hold:

(a) $\nabla_{fX} Y = f \nabla_X Y$,

(b) $\nabla_X (fY) = X(f)Y + (-1)^{|X||f|} f \nabla_X Y$.

Moreover,

$$|\nabla_X Y| = |X| + |Y|.$$

The *Christoffel symbols* Γ^k_{ij} are defined in the same way as in the classical case, that is, for a system of coordinates $\{x^a\}$,

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k.$$

And the degrees of the Christoffel symbols are given as

$$|\Gamma^k_{ij}| = |x^i| + |x^j| + |x^k|.$$

Definition A.8. The *torsion* T of a connection ∇ of a supermanifold is defined as

$$T(X, Y) := \nabla_X Y - (-1)^{|X||Y|} \nabla_Y X - [X, Y].$$

Theorem-Definition A.9. On a supermanifold M with a Riemannian metric $\langle\ ,\ \rangle$ defined as Definition A.3, there exists a unique connection that has zero torsion and is compatible with the metric, i.e.,

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + (-1)^{|X||Y|} \langle Y, \nabla_X Z\rangle.$$

Such a connection is called the *Levi-Civita connection.*

Remark 8. On a supermanifold, having zero torsion and compatibility with the metric g read respectively

$$\Gamma^k_{ij} = (-1)^{ij} \Gamma^k_{ji}, \tag{A.8}$$

$$\partial_k g_{ij} = \Gamma^s_{ki} g_{sj} + (-1)^{ij} \Gamma^s_{kj} g_{si}. \tag{A.9}$$

Definition A.10. The *Riemannian curvature tensor* is given as

$$R(X, Y)Z := [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z.$$

The above definition induces the Riemannian curvature tensor on a supermanifold

$$R^l_{ijk} := \partial_i \Gamma^l_{jk} - (-1)^{ij} \partial_j \Gamma^l_{ik} + (-1)^{i(j+k)} \Gamma^s_{jk} \Gamma^l_{si} - (-1)^{jk} \Gamma^s_{ik} \Gamma^l_{sj}$$

The definition of *flatness* is inherited from Section 1.1. Analogous to the classical case, flat coordinates exist iff R^l_{ijk} identically vanishes.

Appendix B KdV hierarchy for the example

The KdV hierarchy of Example 5.5 is

$$
\begin{aligned}
\{x^1(s_1),x^1(s_2)\}_1 =& -x^{2^2}\lambda^3(\frac{1}{2}x^{2^2}-2\alpha\xi^1\xi^2)\dot\delta(s_1-s_2)+x^2\lambda^2\left[\lambda(\frac{2}{\alpha}\xi^1\xi^2-x^{2^2})\dot x^2\right.\\
&\left.+3x^2\lambda^2(\frac{x^{2^2}}{4}+\frac{1}{\alpha}\xi^1\xi^2)\dot x^3\right]\delta(s_1-s_2)\\
\{x^1(s_1),x^2(s_2)\}_1 =& 2x^2\lambda^2\left(-\frac{3}{2}x^{2^2}+\alpha(1+4\theta)\xi_1\xi_2\right)\dot\delta(s_1-s_2)+\frac{1}{2}\lambda^2\left[(\frac{1}{\alpha}\xi^1\xi^2-\frac{3}{2}x^{2^2})\dot x^2\right.\\
&\left.+(\frac{2}{\alpha}\xi^1\xi^2-x^{2^2})\lambda x^2\dot x^3\right]\delta(s_1-s_2)\\
\{x^1(s_1),x^3(s_2)\}_1 =& x^3\dot\delta(s_1-s_2)\\
\{x^1(s_1),\xi^1(s_2)\}_1 =& -2x^{2^2}\lambda^2\left((\theta-1)\xi^1-(\theta+1)\xi^2\right)\dot\delta(s_1-s_2)-\left[\theta\lambda^2x^2(\xi_1\dot x^2+x^2\lambda)\xi_1\dot x^3+\frac{\alpha}{2}x^2\dot\xi^1\right]\delta(s_1-s_2)\\
\{x^1(s_1),\xi^2(s_2)\}_1 =& \frac{2}{\alpha}(2-\theta)\lambda^2(x^2)^2\xi^1\dot\delta(s_1-s_2)+\frac{(1-\theta)}{\alpha}x^{2^2}\lambda^2\left[\xi^1\dot x^2+x^2\lambda\xi^1\dot x^3+\frac{\alpha}{2}x^2\dot\xi^2\right]\delta(s_1-s_2)\\
\{x^2(s_1),x^1(s_2)\}_1 =& 2x_2\lambda^2(-\frac{3}{2}x_2^2+\alpha(1+4\theta)\xi_1\xi_2)\dot\delta(s_1-s_2)+\lambda^2\left[(\xi^1\xi^2+\alpha)\dot x^2+x^2\lambda(2\xi^1\xi^2-x^{2^2})\dot x^3\right]\delta(s_1-s_2)\\
\{x^2(s_1),x^2(s_2)\}_1 =& \left(x_1+\lambda(-3x_2^2+\alpha\xi_1\xi_2)\right)\dot\delta(s_1-s_2)+\frac{1}{2}\left[\dot x^1-3x^2\lambda\dot x^2+(\lambda^2\xi^1\xi^2-\frac{3}{2}x^{2^2}\lambda^2)\dot x^3\right]\delta(x_1-x_2)\\
\{x^2(s_1),x^3(s_2)\}_1 =& \frac{1}{2}x^2\dot\delta(s_1-s_2)\\
\{x^2(s_1),\xi^1(s_2)\}_1 =& -\frac{1}{\alpha}(\frac{1}{2}+\theta)\lambda x^2\xi^2\dot\delta(s_1-s_2)+\frac{\theta}{\alpha}\lambda\left[\xi^1\dot x^2+x\lambda\dot x^3+\alpha x\dot\xi^1\right]\delta(s_1-s_2)\\
\{x^2(s_1),\xi^2(s_2)\}_1 =& 2(3-2\theta)\lambda x^2\dot\delta(s_1-s_2)-\frac{\theta-1}{\alpha}\lambda\left[\xi^1\dot x^2+x^2\lambda\dot x^3+\alpha x^2\dot\xi^2\right]\delta(s_1-s_2)\\
\{x^3(s_1),x^1(s_2)\}_1 =& x^3\dot\delta(s_1-s_2)+\dot x^1\delta(s_1-s_2)\\
\{x^3(s_2),x^2(s_2)\}_1 =& \frac{1}{2}x_2\dot\delta(s_1-s_2)+\frac{1}{2}\dot x^2\delta(s_1-s_2)\\
\{x^3(s_2),x^3(s_2)\}_1 =& 0\\
\{x^3(s_2),\xi^1(s_2)\}_1 =& \frac{\theta}{\alpha}\xi^2\dot\delta(s_1-s_2)-\theta\delta(s_1-s_2)\\
\{x^3(s_2),\xi^2(s_2)\}_1 =& \frac{\theta-1}{\alpha}\xi^1\dot\delta(s_1-s_2)-(1-\theta)\dot\xi^2\delta(s_1-s_2)\\
\{\xi^1(s_1),x^1(s_2)\}_1 =& -2x^{2^2}\lambda^2\left((\theta-1)\xi^1-(\theta+1)\xi^2\right)\dot\delta(s_1-s_2)-\frac{1}{\alpha}x\lambda^2\left[\xi^2\dot x^2+x\lambda\xi^2\dot x^3-\frac{\alpha}{2}x\dot\xi^1\right]\delta(s_1-s_2)\\
\{\xi^1(s_1),x^2(s_2)\}_1 =& -\frac{1}{\alpha}(\frac{1}{2}+\theta)\lambda x^2\xi^2\dot\delta(s_1-s_2)-\frac{1}{2\alpha}\lambda\left[\xi^2\dot x^2+x\lambda\xi^2\dot x^3-\alpha x\dot\xi^1\right]\delta(s_1-s_2)\\
\{\xi^1(s_1),x^3(s_2)\}_1 =& -\frac{\theta}{\alpha}\xi^2\dot\delta(s_1-s_2)\\
\{\xi^1(s_1),\xi^1(s_2)\}_1 =& 0\\
\{\xi^1(s_1),\xi^2(s_2)\}_1 =& 2\alpha(3-2\theta)\lambda x^2\dot\delta(s_1-s_2)-\frac{1-\theta}{\alpha}\left[-\dot x^1+x^2\lambda\dot x^2+\frac{1}{2}x^{2^2}\lambda^2\dot x^3\right]\delta(s_1-s_2)\\
\{\xi^2(s_1),x^1(s_2)\}_1 =& \frac{2}{\alpha}(2-\theta)\lambda^2(x^2)^2\xi^1\dot\delta(s_1-s_2)+\frac{1}{\alpha}x^2\lambda^2\left[\xi^1\dot x^2+x^2\lambda\dot x^3+\frac{\alpha}{2}x^2\dot\xi^2\right]\dot{(}s_1-s_2)\\
\{\xi^2(s_1),x^2(s_2)\}_1 =& 2(3-2\theta)\lambda x^2\dot\delta(s_1-s_2)+\frac{1}{2\alpha}\lambda\left[\xi^1\dot x^2+x^2\lambda\xi^1\dot x^3+\alpha x^2\dot\xi^2\right]\delta(s_1-s_2)
\end{aligned}
$$

$$
\begin{aligned}
\left\{\xi^2(s_1), x^3(s_2)\right\}_1 =& \frac{\theta-1}{\alpha}\xi^1\dot{\delta}(s_1-s_2) \\
\left\{\xi^2(s_1), \xi^1(s_2)\right\}_1 =& 2\alpha(2\theta-3)\lambda x^2\dot{\delta}(s_1-s_2) - \frac{\lambda}{\alpha}\left[\dot{x}^1 - x^2\lambda\dot{x}^2 - \frac{1}{2}{x^2}^2\lambda^2\dot{x}^3\right]\delta(s_1-s_2) \\
\left\{\xi^2(s_1), \xi^2(s_2)\right\}_1 =& 0.
\end{aligned}
$$

$$
\begin{aligned}
\left\{x^1(s_1), x^3(s_2)\right\}_2 =& \left\{x^3(s_1), x^1(s_2)\right\}_2 = \left\{x^2(s_1), x^2(s_2)\right\}_2 = \delta(s_1-s_2), \\
\left\{\xi^1(s_1), \xi^2(s_2)\right\}_2 =& -\frac{1}{\alpha}\delta(s_1-s_2), \\
\left\{\xi^2(s_1), \xi^1(s_2)\right\}_2 =& \frac{1}{\alpha}\delta(s_1-s_2),
\end{aligned}
$$

and all other brackets vanish.

References

[1] Michèle Audin. "An introduction to Frobenius manifolds, moduli spaces of stable maps and quantum cohomology". working paper or preprint. May 1997. URL: https://hal.science/hal-00129568.

[2] C. Bartocci, U. Bruzzo, and D. Hernández-Ruipérez. *The Geometry of Supermanifolds*. Mathematics and Its Applications. Springer Netherlands, 1991. ISBN: 9780792314400. URL: https://books.google.it/books?id=9gY0PZ6ghlIC.

[3] Boris Dubrovin. *Flat pencils of metrics and Frobenius manifolds*. 1998. arXiv: math/9803106 [math.DG]. URL: https://arxiv.org/abs/math/9803106.

[4] Boris Dubrovin. *Geometry of 2d topological field theories*. 1994. arXiv: hep-th/9407018 [hep-th]. URL: https://arxiv.org/abs/hep-th/9407018.

[5] S. Duplij. *Concise Encyclopedia of Supersymmetry: And Noncommutative Structures in Mathematics and Physics*. Concise Encyclopedia of Supersymmetry. Springer Netherlands, 2005. ISBN: 9781402013386. URL: https://books.google.it/books?id=YAP8oVQ4Q9AC.

[6] Guilherme F. Almeida. "The differential geometry of the orbit space of extended affine Jacobi group An". In: *Journal of Geometry and Physics* 171 (2022), p. 104409. ISSN: 0393-0440. DOI: https://doi.org/10.1016/j.geomphys.2021.104409. URL: https://www.sciencedirect.com/science/article/pii/S0393044021002552.

[7] Oliver Goertsches. *Riemannian Supergeometry*. 2006. arXiv: math/0604143 [math.DG]. URL: https://arxiv.org/abs/math/0604143.

[8] M. Kontsevich and Yu. Manin. "Gromov-Witten classes, quantum cohomology, and enumerative geometry". In: *Communications in Mathematical Physics* 164.3 (Aug. 1994), pp. 525–562. ISSN: 1432-0916. DOI: 10.1007/bf02101490. URL: http://dx.doi.org/10.1007/BF02101490.

[9] J.M. Lee. *Introduction to Riemannian Manifolds*. Graduate Texts in Mathematics. Springer International Publishing, 2019. ISBN: 9783319917542. URL: https://books.google.it/books?id=UIPltQEACAAJ.

[10] J.M. Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics. Springer, 2003. ISBN: 9780387954486. URL: https://books.google.it/books?id=eqfgZtjQceYC.

[11] Leites and D A. "INTRODUCTION TO THE THEORY OF SUPERMANIFOLDS". In: *Russian Mathematical Surveys* 35.1 (1980).

[12] Ruud Martini and P.H.M. Kersten. "WDVV-equations and their solution for a (3,2) and (3,4) supermanifold; the isotropic and nonisotropic case". Undefined. In: *Electronic Proceedings of the EMIS*. Feb. 1998.