Well-Posedness for Moreau's Sweeping Process

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Theorem. Let Y be Hilbert space, $C_* \subset Y$ be nonempty, closed and convex, $l : [0,T] \to Y$ be Lipschitz with constant L. Set $C(t) = C_* - l(t)$. Let $u_0 \in C(0)$. Consider the sweeping problem:

$$(SP)\begin{cases} \dot{u}(t) \in -N_{C(t)}(u(t)) \\ u(0) = u_0 \end{cases}$$

(SP) admits one and only one Caratheodory solution $u : [0,T] \to Y$. This solutionis Lipschitz with constant L and depends continuously on the initial condition u_0.

Lemma. (Bochner integration theory) Let (X, Σ, μ) be a measure space and let B be a Banach space. A **simple function** is

$$s(x) = \sum_{i=1}^{n} b_i \cdot 1_{E_i}(x).$$

Define $\int_x s(x) d\mu := \sum_{i=1}^{n} \mu(E_i) b_i \in B$. A measurable function f is said to be **Bochner integral** provided there exists a sequence of simple functions $\{s_k\}$ such that

$$\lim_{k \to +\infty} \int_x \|f(x) - s_k(x)\| d\mu = 0.$$

In this case, the limit $\lim\limits_{k \to +\infty} \int_x s_x d\mu$ exists, and by definition, it is equal to $\int_x f d\mu$.

Lemma. (Mazur) Let E be a Banach space, and let $\{x_n\} \subset E$ be such that $x_n \rightharpoonup x$ (weakly in E). Then there exists sequence $\{y_n\} \subset E$ suth that

$$y_n \in co(\bigcup_{i=n}^{\infty} \{x_i\})$$

$$y_n \to x \text{ (strongly in } E)$$

i.e., $\forall n,\ \exists m_n \in \mathbb{N},\ \lambda_n^i \in [0,1],\ i = 1, \cdots, m_n,\ \sum_{i=0}^{m_n} \lambda_n^i = 1$, such that

$$y_n := \sum_{i=0}^{m_n} \lambda_n^i x_{n+i} \to x \text{ as } n \to \infty.$$

Proof of the theorem. The proof is in three steps:

1. Construction of u
2. Show that u is a solution of (SP)
3. Uniqueness and continuous dependence on u_0

Step 1. Construction of u: via a discretization method called "catching up algorithm"(a kind of Euler polygonal).

Let $t_n^i = T\dfrac{i}{2^n},\ i = 0, 1, \cdots, 2^n$. Define:

$u_n^0 := u_0 \forall n$

$u_n^1 := \pi_{C(t_n^1)}(u_n^0)$ (exist and is unique by convexity and closedness)

$u_n(t) := u_0 + \dfrac{2^n}{T} t(u_n^1 - u_n^0), t \in [0, t_n^1]$ (observe that $u_n(0) = u_0, u_n(t_n^1) = u_n^1$)

Recursively, define

$$u_n^i := \pi_{C(t_n^i)}(u^{i-1}), i = 1, \cdots, 2^n$$
$$u_n(t) := u_n^{i-1} + (t - t_n^{i-1})\frac{2^n}{T}(u_n^i - u_n^{i-1}), t \in [t_n^{i-1}, t_n^i]$$

Then $u_n(\cdot)$ has following **properties**:

(a) $u_n(t_n^i) \in C(t_n^i), i = 1, \cdots, 2^n$.

(b) $\dot{u}_n(t) = \dfrac{2^n}{T}(u_n^i) - u_n^{i-1}\ \forall t \in (t_n^{i-1}, t_n^i)$.

(c) $\dot{u}_n(t) \in -N_{C(t_n^i)}(u_n(t_n^i))\ \forall t \in (t_n^{i-1}, t_n^i), i = 1, \cdots, 2^n$.

(d) $\{u_n\}$ is Lipschitz uniformly with respect to n, with constant L.

(e) $d(u_n(t), C(t_n^i)) \leq \left\| u_n(t) - u_n^i \right\| = \left\| u_n(t) - u_n(t_n^i) \right\| \leq L(t - t_n^i) \leq L\dfrac{T}{2^n}\ \forall t \in [t_n^{i-1}, t_n^i]$.

(a),(b),(e) follows from construction and direct computation.

Explanation of (c): $u_n^i = \pi_{C(t_n^i)(u_n^{i-1})}$ is the minimizer of $\dfrac{1}{2}\left\| u - u_n^{i-1} \right\|^2$ over $u \in C(t_n^i)$. Since the norm is lower semicontinuous, proper and convex with respect to u, it follows from the necessary optimality condition that

$$0 \in u_n^i - u_n^{i-1} + N_{C(t_n^i)}(u_n^i)$$

$$\Rightarrow \dot{u}_n(t) = \frac{2^n}{T} \in -\frac{2^n}{T} N_{C(t_n^i)}(u_n^i) = -N_{C(t_n^i)}(u_n^i).$$

Proof of (d):

$$\|\dot{u}_n(t)\| = \frac{2^n}{T}(u_n^i - u_n^{i-1}) = \frac{2^n}{T} d((u_n^{i-1}), C(t_n^i))$$

Since $u_n^{i-1} \in C(t_n^{i-1}) = C_* - l(t_n^{i-1})$, i.e., $\exists v_n^{i-1} \in C_*$ such that $u_n^{i-1} = v_n^{i-1} - l(t^{i-1_n})$, thus

$$\begin{aligned} d(u_n^{i-1}, C(t_n^i)) &\leq \left\| u_n^{i-1} - (v_n^{i-1} - l(t_n^i)) \right\| \\ &= \left\| (v_n^{i-1} - l(t_n^{i-1})) - (v_n^{i-1} - l(t_n^i)) \right\| \\ &= \left\| l(t_n^{i-1}) - l(t_n^i) \right\| \\ &\leq L \left\| t_n^{i-1} - t_n^i \right\| \end{aligned}$$

Thus

$$\|\dot{u}_n(t)\| \leq \frac{2^n}{T} L(t_n^i - t_n^{i-1}) = L.$$

I now prove that $\|u_n\|$ is Cauchy in $\mathcal{C}^0([0,T]; Y)$. Define $\tau_n(t) := t_n^i$ for $t \in [t_n^{i-1}, t_n^i)$, then obviously, $0 \leq \tau_n(t) - t \leq \dfrac{T}{2^n}$ for any t. Therefore $\tau_n(t) \rightrightarrows t$ on $[0,T]$, and property (c) can be rewritten as:

$$\dot{u}_n(t) \in -N_{C(\tau_n(t))}(u_n(\tau_n(t))) \text{ a.e. } t \in [0,T]$$

(1.1) $\quad$ i.e., $\langle -\dot{u}_n(t), u - u_n(\tau_n(t)) \rangle \leq 0\ \forall u \in C(\tau_n(t))$ a.e. $t \in [0,T]$

Set $u_n^m(t) := \pi_{C(\tau_n(t))}(u_m(\tau_n(t)))$ $(\in C(\tau_n(t)))$. And now look into equation (1.1):

$$\begin{aligned}
&\langle -\dot{u}_n(t), u_m(\tau_m(t)) - u_n(\tau_n(t))\rangle \\
=&\langle -\dot{u}_n(t), u_m(t) - u_n(\tau_n(t))\rangle + \langle -u_n(t), u_m(\tau_m(t)) - u_n^m(t)\rangle \\
\leq&|\langle \dot{u}_n(t), u_n^m(t) - u_n(\tau_n(t)) - u_n^m(t)\rangle| \\
\leq&\|\dot{u}_n(t)\|\|u_m(\tau_n(t)) - u_n^m(t)\| \\
\leq&L \cdot d(u_m(\tau_m(t)), C(\tau_n(t))) \\
\leq&L \cdot \|l(\tau_m(t)) - l(\tau_n(t))\| \\
\leq&L^2 \cdot |\tau_m(t) - \tau_n(t)|.
\end{aligned}$$

Since $l(\cdot)$ is Lipschitzwith constant L,

$$\langle -\dot{u}_n(t), u_m(\tau_m(t)) - u_n(\tau_n(t))\rangle \leq L^2\left(\frac{T}{2^m} + \frac{T}{2^n}\right) \tag{2.1}$$

Interchanging m and n,

$$\langle -\dot{u}_m(t), u_n(\tau_n(t)) - u_m(\tau_m(t))\rangle \leq L^2\left(\frac{T}{2^n} + \frac{T}{2^m}\right) \tag{2.2}$$

(2.1)+(2.2) gives:

$$\langle \dot{u}_n(t) - \dot{u}_m(t), u_n(\tau_n(t)) - u_m(\tau_m(t))\rangle \leq L^2 T(\frac{1}{2^{m-1}} + \frac{1}{2^{n-1}})$$

$$\begin{aligned}
\text{LHS of } (2.1) + (2.2) =&\langle \dot{u}_n(t) - \dot{u}_m(t), u_n(t) - u_m(t)\rangle \\
&+ \langle \dot{u}_n(t) - \dot{u}_m(t), u_n(\tau_n(t)) - u_n(t)\rangle + \langle \dot{u}_n(t) - \dot{u}_m(t), u_m(t) - u_m(\tau_n(t))\rangle \\
\geq&\langle \dot{u}_n(t) - \dot{u}_m(t), u_n(t) - u_m(t)\rangle \\
&- \|\dot{u}_n(t) - \dot{u}_m(t)\|(\|u_m(\tau_n(t)) - u_m(t)\| + \|u_n(\tau_n(t)) - u_n(t)\|) \\
\geq&\langle \dot{u}_n(t) - \dot{u}_m(t), u_n(t) - u_m(t)\rangle - L(L \cdot \frac{T}{2^m} + L \cdot \frac{T}{2^n})
\end{aligned}$$

Therefore,

$$\langle \dot{u}_n(t) - \dot{u}_m(t), u_n(t) - u_m(t)\rangle \leq L^2 T(\frac{1}{2^{m-2}} + \frac{1}{2^{n-2}})$$

i.e.,

$$\frac{d}{dt}\frac{1}{2}\|u_n(t) - u_m(t)\|^2 \leq L^2 T(\frac{1}{2^{m-2}} + \frac{1}{2^{n-2}})$$

$$\Rightarrow \|u_n(t) - u_m(t)\|^2 \leq L^2 T(\frac{1}{2^{m-2}} + \frac{1}{2^{n-2}})\ \forall m, n \in \mathbb{N} \text{ in } [0,T]$$

Therefore $\{u_n\}$ is Cauchy, and it converges to some $u : [0,T] \to Y$ that is Lipschitzwith constant L.

Step 2. Show that u is a solution of (SP): need information on $\dot{u}$. Since $L^2(0,T;Y)$ is reflexive (proved with Bochner integration theory), $\{\dot{u}_n\}$ bounded in $L^2(0,T;Y)$ $\Rightarrow$up to a subsequence, $\{\dot{u}_n\}$ converges weakly to some $v \in L^2(0,T;Y)$, i.e.,

$$u_n \to u \text{ in } [0,T],$$
$$\dot{u}_n \rightharpoonup v \text{ in } L^2(0,T;Y).$$

Claim: u is differentiable a.e. and $u(t) = u_0 + \int_0^t v(s)ds\ \forall t \in [0,T]$, so that $\dot{u} = v$ a.e. in $[0,T]$. Indeed, for $\{\dot{u}_n\}$, one has

$$u_n(t) = x_0 + \int_0^t \dot{u}_n(s)ds$$

weak convergence of $\dot{u}_n(t)$ to v in $L^2(0,T;Y)$ means:

$$\forall w \in L^2(0,T;Y),\ \int_0^T \langle w(s), \dot{u}_n(s)\rangle ds \to \int_0^T \langle w(s), v(s)\rangle ds \text{ as } n \to +\infty.$$

Fix $w \in Y,\ t \in [0,T]$. Set $w(s) := w|_{[0,t)}(s) \in L^2(0,T;Y)$. Then

$$\int_0^T \langle ws, \dot{u}_n(s)\rangle ds = \int_0^T \langle w, \dot{u}_n(s)\rangle ds = \langle w, \int_0^T \dot{u}_n(s)ds\rangle = \langle w, u_n(t) - u_0\rangle$$

Letting $n \to \infty$, by weak convergence, the above equations become

$$\int_0^T \langle w, v(s)\rangle ds = \langle w, \int_0^T v(s)ds\rangle = \langle w, u(t) - u_0\rangle,$$

which means $\int_0^T v(s)ds = u(t) - u_0$. In particular, $\dot{u}_t \rightharpoonup \dot{u}$ in $L^2(0,T;Y)$. By Mazur's lemma, up to a subsequence, $\exists N_n \in \mathbb{N},\ \lambda_n^i,\ i = 1, \cdots, N_n$, with $\lambda_n^i \in [0,1], \sum_{i=1}^{N_n} \lambda_n^i = 1$, such that

$$\sum_{i=1}^{N_n} \lambda_n^i \dot{u}_n(t) \to \dot{u}(t) \text{ a.e. } t \in [0,T] \text{ as } n \to \infty.$$

Recall property (c):$\forall t \in [0,T],\ \forall u \in C_*$,

$$\langle -\dot{u}_i(t), u - l(\tau_i(t)) - u_i(\tau_i(t))\rangle \leq 0 \tag{2.3}$$

For each i, multiply by λ_n^i and sum up the N_n inequalities, one has

$$\sum_{i=1}^{N_n} \lambda_n^i \langle -\dot{u}_i(t), u - l(\tau_i(t)) - u_i(\tau_i(t)) \rangle \leq 0. \tag{3.0}$$

Since

$$\sum_{i=1}^{N_n} \lambda_n^i \langle -\dot{u}_i(t), u \rangle = \langle \sum_{i=1}^{N_n} -\dot{u}_i(t), u \rangle \to \langle -\dot{u}(t), u \rangle \tag{3.1}$$

$$\begin{aligned} \sum_{i=1}^{N_n} \lambda_n^i \langle -\dot{u}_i(t), -l(\tau_i(t)) \rangle &= \sum_{i=1}^{N_n} \lambda_n^i \langle \dot{u}_i(t), l(t) \rangle \\ &= \langle \sum_{i=1}^{N_n} \lambda_n^i \dot{u}_i(t), l(t) \rangle \to \langle \dot{u}(t), l(t) \rangle = \langle -\dot{u}(t), -l(t) \rangle \end{aligned} \tag{3.2}$$

$$\begin{aligned} \sum_{i=1}^{N_n} \lambda_n^i \langle -\dot{u}_i(t), -u_i(\tau_i(t)) \rangle &= \sum_{i=1}^{N_n} \langle \dot{u}_i(t), u_i(t) \rangle \\ &= \sum_{i=1}^{N_n} \frac{d}{dt} \frac{1}{2} \|u_i(t)\|^2 \\ &\geq \frac{d}{dt} \frac{1}{2} \left\| \sum_{i=1}^{N_n} u_i(t) \right\|^2 \to \frac{d}{dt} \frac{1}{2} \|u(t)\|^2 = \langle -\dot{u}(t), -u(t) \rangle \end{aligned} \tag{3.3}$$

Summing up (3.1), (3.2), (3.3), one gets that, $\forall u \in C_*$, a.e. $t \in [0, T]$,

$$\langle -\dot{u}(t), -l(t) \rangle \leq \sum_{i=1}^{N_n} \lambda_n^i \langle -\dot{u}_i(t), u - l(\tau_i(t)) - u_i(\tau_i(t)) \rangle \leq 0,$$

i.e.,

$$\dot{u}(t) \in -N_{C(t)}(u(t)).$$

Step 3. Uniqueness and continuous dependence on u_0. Assume $u_1(\cdot)$, $u_2(\cdot)$ are solutions of (SP), $u_i(0) = u_i^0$ for $i = 1, 2$. Then, for a.e. $t \in [0, T]$,

$$\langle -\dot{u}_1(t), u_2(t) - u_1(t) \rangle \leq 0$$

$$\langle -\dot{u}_2(t), u_1(t) - u_2(t) \rangle \leq 0$$

Summing up:

$$0 \geq \langle \dot{u}_2(t) - \dot{u}_1(t), u_2(t) - u_1(t) \rangle = \frac{d}{dt} \|u_2(t) - u_1(t)\|^2$$

Integrate on both sides:

$$\|u_2(t) - u_1(t)\|^2 \leq \left\| u_2^0 - u_1^0 \right\|^2,$$

which implies uniqueness and continuous dependence on initial condition.□

$A = \sum_k \bigtimes_{i \in I} U_{k,i}$, with $U_{k,i} \in \mathcal{E}_i$