Frobenius manifolds and flat pencils of metrics

Xueyan Zhang

In Section 1 and Section 2, we will introduce the flat pencils of metrics and Frobenius manifolds. On a n-dimensional manifold M, for a metric g_{ij} and its inverse (called contravariant metric) g^{ij}, the contravariant Levi-Civita connection is defined as

$$\Gamma_k^{ij} := -g^{is}\Gamma_{sk}^j.$$

A contravariant metric is said to be falt iff there locally exists n independent functions that are covariantly constant w.r.t. the contravariant Levi-Civita connection. Two contravariant metrics g and η, together with their contravariant Levi-Civita connections ${\Gamma_1}_k^{ij}$ and ${\Gamma_2}_k^{ij}$ resp., form a flat pencil of metrics if the linear combinations of the metrics $g - \lambda\eta$ are still a flat metric.

An informal way to think about a Frobenius manifold is as a Riemannian manifold equipped with additional structure compatible with a Frobenius algebra on each tangent space, namely, for a metric $\langle\ ,\ \rangle$,

$$\langle \partial_a \circ \partial_b, \partial_c \rangle = \langle \partial_a, \partial_b \circ \partial_c \rangle.$$

In Section 3, we introduce bihamiltonian systems and their link to Frobenius manifolds. The corresponding flat pencil of metrics of a Frobenius manifold carries a family of Poisson brackets, called KdV hierarchies, given as

$$\begin{aligned}
\left\{t^\alpha(s_1), t^\beta(s_2)\right\}_1 &= \eta^{\alpha\beta}\dot{\delta}(s_1 - s_2),\\
\left\{t^\alpha(s_1), t^\beta(s_2)\right\}_2 &= g^{\alpha\beta}(t(s_1)\dot{\delta}(s_1 - s_2)) + \Gamma_\gamma^{\alpha\beta}(t)\dot{t}^\gamma\delta(s_1 - s_2).
\end{aligned}$$

Section 4 presents two examples where Frobenius manifolds arise naturally from Hurwitz spaces $M_{0;n}$, which consist of all polynomials of the form

$$\lambda(p) = p^{n+1} + a_n p^{n-1} + \cdots + a_1,\ a_1, \ldots, a_n \in \mathbb{C}.$$

We describe the geometry of the Hurwitz space $M_{0;n}$, and explicitly compute the flat pencils of metrics and KdV hierarchies for cases $n = 1$ and $n = 2$.

Section 5 extends the discussion to supermanifolds that incorporate supersymmetry by combining bosonic (even) and fermionic (odd) variables. The symmetry of a metric is replaced with supersymmetry:

$$g_{ab} = (-1)^{|a||b|}g_{ba},$$

where $|a|$ and $|b|$ are the degrees of the coordinates x^a and x^b respectively. We introduce the geometry of Riemannian supermanifolds and try to construct Frobenius supermanifold structures and flat pencils of metrics in this setting. As a concrete example, we analyze a $(3, 2)$ supermanifold and try to find its associated KdV hierarchies.