Distinguished Triangles and Triangulated Categories

Definition 1. A functor $f : \mathcal{C} \to \mathbf{Set}$ is called *representable* if $\exists X \in \mathrm{Ob}(\mathcal{C})$ such that F is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(X, \cdot)$.
X is unique up to isomorphism, called *representative* of f.

Definition 2. An *additive category* is a category $\mathcal{C}$ such that

(i) $\forall X, Y \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has a structure of additive (i.e. abelian) group, and composition law is bilinear.

(ii) $\exists 0 \in \mathrm{Ob}(\mathcal{C})$ such that $\mathrm{Hom}_{\mathcal{C}}(0, 0) = 0$.

(iii) $\forall X, Y \in \mathrm{Ob}(\mathcal{C})$, the functor

$$W \mapsto \mathrm{Hom}_{\mathcal{C}}(X, W) \times \mathrm{Hom}_{\mathcal{C}}(Y, W)$$

is representable. The representative is denoted as $X \oplus Y$.

(iv) $\forall X, Y \in \mathrm{Ob}(\mathcal{C})$, the functor

$$W \mapsto \mathrm{Hom}_{\mathcal{C}}(W, X) \times \mathrm{Hom}_{\mathcal{C}}(W, Y)$$

is representable. The representative is denoted as $X \times Y$.

(iii) and (iv) are equivalent under the conditions (i) and (ii), and $X \oplus Y \simeq X \times Y$.

Definition 3. A functor of additive categories $F : \mathcal{C} \to \mathcal{C}'$ is *additive* if $\forall X, Y \in \mathrm{Ob}(\mathcal{C})$,

$$F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$$

is a group homomorphism.

Definition 4. ($\mathrm{Ker}\, f$, $\mathrm{Coker}\, f$, $\mathrm{Im}\, f$, $\mathrm{Coim}\, f$ for $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.)

(i) If

$$\mathrm{Ker}(\mathrm{Hom}(\cdot, f)) : Z \mapsto \mathrm{Ker}(\mathrm{Hom}(Z, f)) = \{u \in \mathrm{Hom}(Z, X) : f \circ u = 0\}$$

is representable, its representative is called *kernel* of f, denoted $\mathrm{Ker}\, f$.

(ii) If

$$\mathrm{Ker}(\mathrm{Hom}(f, \cdot)) : Z \mapsto \mathrm{Ker}(\mathrm{Hom}(f, Z)) = \{u \in \mathrm{Hom}(Y, Z) : u \circ f = 0\}$$

is representable, its representative is called *cokernel* of f, denoted $\mathrm{Coker}\, f$.

The kernel and cokernel satisfies the *universal properties* as follows:
Assume f has a kernel, then there exists a morphism of functors

$$\beta : \mathrm{Hom}_{\mathcal{C}}(\cdot, \mathrm{Ker}\ f) \to \mathrm{Hom}_{\mathcal{C}}(\cdot, X)$$

and $\beta(\mathrm{id}_{\mathrm{Ker}\ f})$ defines a morphism $\alpha : \mathrm{Ker}\ f \to X$ that satisfies the universal property: for any morphism $e : W \to X$ such that $f \circ e = 0$, there exists unique h such that the following diagram commutes:

$$\begin{tikzcd}
W \arrow[r, "e"] \arrow[dr, dotted, "\exists ! h"'] & X \arrow[r, "f"] & Y \\
& \mathrm{Ker}\ f \arrow[u, "\alpha"'] \arrow[ur, dashed, "\text{0 by definition of Ker}"'] &
\end{tikzcd}$$

Now assume f has a cokernel. $\exists \delta : \mathrm{Hom}_{\mathcal{C}}(\mathrm{Coker}\ f, \cdot) \to \mathrm{Hom}_{\mathcal{C}}(Y, \cdot)$, and $\delta(\mathrm{id}_{\mathrm{Coker}\ f}) =: \gamma : Y \to \mathrm{Coker}\ f$ such that $\forall g$ with $g \circ f = 0$, $\exists ! u$ that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, dashed, "\text{0 by definition of Coker}"'] & Y \arrow[r, "g"] & Z \\
& \mathrm{Ker}\ f \arrow[u, "\gamma"'] \arrow[ur, dotted, "\exists ! u"'] &
\end{tikzcd}$$

Definition 5. (continued) $\mathrm{Coim}\ f := \mathrm{Coker}(\mathrm{Ker}\ f \xrightarrow{\alpha} X)$,
$\mathrm{Im}\ f := \mathrm{Ker}(Y \xrightarrow{\gamma} \mathrm{Coker}\ f)$.

Universal properties gives

$$\begin{tikzcd}
\mathrm{Ker}\ f \arrow[r, "\alpha"] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "\gamma"] & \mathrm{Coker}\ f \\
& \mathrm{Coim}\ f \arrow[r, dotted] & \mathrm{Im}\ f \arrow[u] &
\end{tikzcd}$$

the dotted arrow induces a natural map.

Definition 6. Let $\mathcal{C}$ be an additive category. $\mathcal{C}$ is an *abelian category* if

(i) for any morphism $f : X \to Y$, $\mathrm{Ker}\ f$ and $\mathrm{Coker}\ f$ exist.

(ii) $\mathrm{Coim}\ f \xrightarrow{\sim} \mathrm{Im}\ f$. (isomorphic)

From now on, if not specified, $\mathcal{C}$ is always an abelian category.

Definition 7. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is *exact* if

(i) $g \circ f = 0$

(ii) $\mathrm{Im}\ f \xrightarrow{\sim} \mathrm{Ker}\ g$.

Definition 8. An additive functor F is said to be *left (resp. right) exact* if for an exact sequence

$$0 \to X' \to X \to X''$$

$$(\text{resp. } X' \to X \to X'' \to 0)$$

the sequence

$$0 \to F(X') \to F(X) \to F(X'')$$

$$(\text{resp. } F(X') \to F(X) \to F(X'') \to 0)$$

is also exact. F is *exact* if it is left and right exact.

Convention. If $0 \to X \xrightarrow{f} Y \to Z \to 0$ is exact, then one may write $Z = Y/X$. Moreover, $\operatorname{Coker} f = Y/\operatorname{Im} f$.

Definition 9. *Complex* X in $\mathcal{C}$: $\{X^n, d_X^n\}_{n\in\mathcal{Z}}$. Definition seen in class.
Morphism of complexes $f : X \to Y$ is a sequence $\{f^n : X^n \to Y^n\}$ such that

$$\begin{tikzcd} X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"'] & X^{n+1} \arrow[d, "f^{n+1}"] \\ Y^n \arrow[r, "d_Y^n"'] & Y^{n+1} \end{tikzcd}$$

commutes.

Definition 10. $C(\mathcal{C})$:=category of complexes in $\mathcal{C}$.
Complex $X[k]$ is defined as

$$\begin{cases} X[k]^n = X^{n+k} \\ d_X^n[k] = (-1)^k d_X^{n+k} \end{cases}$$

and for $f : X \to Y$, $f[k] : X[k] \to Y[k]$ is such that $f[k]^n = f^{n+k}$.

Definition 11. A morphism $f : X \to Y$ is *homotopic to 0* if for any n there exists $s^n : X^n \to Y^{n-1}$ such that

$$f^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n.$$

$$\begin{tikzcd} & X^n \arrow[r, "d_X^n"] \arrow[dl, "s^n"'] \arrow[d, "f^n"] & X^{n+1} \arrow[dl, "s^{n+1}"] \\ Y^{n-1} \arrow[r, "d_Y^{n-1}"'] & Y^n & \end{tikzcd}$$

f is *homotopic to* g if $f - g$ is homotopic to 0.

Definition 12. $Ht(X, Y) := \left\{f \in \operatorname{Hom}_{C(\mathcal{C})}(X, Y) : f \text{ homotopic to } 0\right\}.$

We may observe that the composition has

$$\begin{array}{c} Ht(X,Y) \times \mathrm{Hom}_{C(\mathcal{C})}(Y,Z) \\ \mathrm{Hom}_{C(\mathcal{C})}(X,Y) \times Ht(Y,Z) \end{array} \to Ht(X,Z)$$

Definition 13. We define a category $K(\mathcal{C})$:

$$\begin{cases} \mathrm{Ob}(K(\mathcal{C})) = \mathrm{Ob}(C(\mathcal{C})) \\ \mathrm{Hom}_{K(\mathcal{C})}(X,Y) = \mathrm{Hom}_{C(\mathcal{C})}(X,Y)/Ht(X,Y) \end{cases}$$

Definition 14. Let $X \in \mathrm{Ob}(C(\mathcal{C}))$. Define

$$\begin{aligned} Z^k(X) &:= \mathrm{Ker}\ d_X^k, \\ B^k(X) &:= \mathrm{Im}\ d_X^{k-1}, \\ H^k(X) &:= Z^k(X)/B^k(X). \end{aligned}$$

Proposition 15. Let

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

be an exact sequence. Then there is a long exact sequence:

$$\cdots \to H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(X) \to \cdots$$

Proof is given in class.

Definition 16. Let $f \in \mathrm{Hom}_{C(\mathcal{C})}(X,Y)$. Then the *mapping cone* $M(f)$ is a complex

$$\begin{cases} M(f)^n = X^{n+1} \oplus Y^n \\ d^n_{M(f)} = \begin{pmatrix} d^n_{X[1]} & 0 \\ f^{n+1} & d^n_Y \end{pmatrix} \end{cases}$$

We define two morphisms $\alpha(f) : Y \to M(f)$, $\beta(f) : M(f) \to X[1]$:

$$\alpha(f)^n = \begin{pmatrix} 0 \\ \mathrm{id}_{Y^n} \end{pmatrix}, \quad \beta(f)^n = (\mathrm{id}_{X^{n+1}}, 0).$$

Lemma 17. For any $f : X \to Y$, there exsits an isomorphism ϕ in $K(\mathcal{C})$ and the following diagram commutes in $K(\mathcal{C})$:

$$\begin{tikzcd} Y \arrow[r, "\alpha(f)"] \arrow[d, "\mathrm{id}_Y"'] & M(f) \arrow[r, "\beta(f)"] \arrow[d, "\mathrm{id}_{M(f)}"'] & X[1] \arrow[r, "-f[1]"] \arrow[d, shift left, "\exists\phi"] & Y[1] \arrow[d, "\mathrm{id}_{Y[1]}"] \\ Y \arrow[r, "\alpha(f)"'] & M(f) \arrow[r, "\alpha(\alpha(f))"'] & M(\alpha(f)) \arrow[r, "\beta(\alpha(f))"'] \arrow[u, shift left, "\psi"] & Y[1] \end{tikzcd}$$

Proof. $M(\alpha(f)) = Y^{n+1} \oplus X^{n+1} \oplus Y^n$. Let

$$\phi^n = \begin{pmatrix} -f^{n+1} \\ \mathrm{id}_{X^{n+1}} \\ 0 \end{pmatrix}, \quad \psi^n = (0, \mathrm{id}_{X^{n+1}}, 0)\,.$$

We need to prove

(i) ϕ, ψ are morphisms of complexes.

(ii) $\psi \circ \phi = \mathrm{id}_{X[1]}$.

(iii) $\phi \circ \psi$ is homotopic to $\mathrm{id}_{M(\alpha(f))}$.

(iv) $\psi \circ \alpha(\alpha(f)) = \beta(f)$.

(v) $\beta(\alpha(f)) \circ \phi = -f[1]$.

(i), (ii), (iv), (v) are easy to see. For (iii), take

$$s^n = \begin{pmatrix} 0 & 0 & \mathrm{id}_{Y^n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathrm{id}_{M(\alpha(f))^n} - \phi^n \circ \psi^n = s^{n+1} \circ d^n_{M(\alpha(f))} + d^{n-1}_{M(\alpha(f))} \circ s^n$$

which finishes the proof. □

Definition 18. A *triangle* is $K(\mathcal{C})$ is a sequence

$$X \to Y \to Z \to X[1].$$

A *distinguished triangle* is a triangle isomorphic to

$$X' \to Y' \to M(f) \to X'[1].$$

Proposition 19. The distinguished triangle satisfies the following:

(TR 0) A triangle isomorphic to a distinguished triangle is a distinguished triangle.

(TR 1) $\forall X \in \mathrm{Ob}(K(\mathcal{C}))$, $X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$ is a distinguished triangle.

(TR 2) Any morphism $f : X \to Y$ can be embedded in a distinguished triangle.

(TR 3) $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle iff $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is a distinguished triangle.

(TR 4) If the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"'] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

commutes, then there exists w such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"'] & Z \arrow[r] \arrow[d, "\exists w"] & X[1] \arrow[d, "u[1]"] \\ X' \arrow[r, "f'"] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

commutes.

(TR 5) (Octahedral axiom) If there are distinguished triangles

$$X \xrightarrow{f} Y \to Z' \to X[1]$$
$$Y \xrightarrow{g} Z \to X' \to Y[1]$$
$$X \xrightarrow{g \circ f} Z \to Y' \to X[1]$$

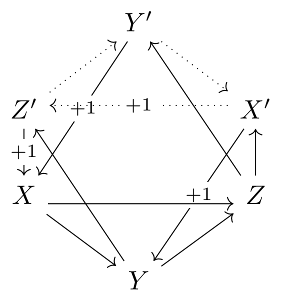
then there exists a distinguished triangle

$$Z' \to Y' \to X' \to Z'[1]$$

such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] & Y \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X[1] \arrow[d, "\mathrm{id}_{X[1]}"'] \\ X \arrow[r, "g \circ f"] \arrow[d] & Z \arrow[r] \arrow[d, "\mathrm{id}_Z"'] & Y' \arrow[r] \arrow[d] & X[1] \arrow[d] \\ Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "\mathrm{id}_{X'}"'] & Y[1] \arrow[d] \\ Z' \arrow[r] & Y' \arrow[r] & X' \arrow[r] & Z'[1] \end{tikzcd}$$

(TR 5) may be interpreted as follows:



Proof. Since (TR 0), (TR 2) are obvious and (TR 5) is not used in later content of this seminar, we will only prove (TR 1), (TR 3), (TR 4). Starting with (TR 3), take $Z = M(f)$. Previous lemma gives

$$\begin{array}{ccccccccc}
X & \longrightarrow & Y & \longrightarrow & M(f) & \longrightarrow & X[1] & \longrightarrow & Y[1] \\
{\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \exists\phi\ iso} & & \downarrow{\scriptstyle \mathrm{id}} \\
X & \longrightarrow & Y & \longrightarrow & M(f) & \longrightarrow & M(\alpha(f)) & \longrightarrow & Y[1]
\end{array}$$

For (TR 1), consider $f : 0 \to X$. Then $M(f)^n = 0^{n+1} \oplus X^n = X^n$. By (TR 3) there is a distinguished triangle

$$\begin{array}{ccccccc}
X & \xrightarrow{\alpha(f)=\mathrm{id}_X} & M(f) & \longrightarrow & 0[1] & \longrightarrow & X[1] \\
 & & \Downarrow & & \Downarrow & & \\
 & & X & & 0 & &
\end{array}$$

and this gives (TR 1).

For (TR 4), by definition of $K(\mathcal{C})$, $v^n \circ f^n$ is homotopic to ${f'}^n \circ u^n$, thus there exists $s^n : X^n \to Y^n$ such that

$$v^n \circ f^n - {f'}^n \circ u^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n.$$

Choosing

$$w^n = \begin{pmatrix} u^{n+1} & 0 \\ s^{n+1} & v^n \end{pmatrix}$$

finishes the proof. $\square$

Definition 20. (Generalization) Let $T : \mathcal{C} \to \mathcal{C}$ be an automorphism. A *triangle* is a sequence

$$X \to Y \to Z \to T(X).$$

A *triangulated category* $\mathcal{C}$ concists of

$$\begin{cases} \text{additive category } \mathcal{C} \text{ with automorphism } T : \mathcal{C} \to \mathcal{C} \\ \text{family of triangles called “distinguished triangle”} \end{cases}$$

and (TR 0)–(TR 5) are satisfied by replacing $\cdot[1]$ by $T(\cdot)$.

A functor of triangulated categories $F : \mathcal{C} \to \mathcal{C}'$ is such that $F \circ T \simeq T' \circ F$, and F sends distinguished triangle in $\mathcal{C}$ into distinguished triangle in $\mathcal{C}'$.

Definition 21. A functor F is a *cohomological functor* if it for any distinguished triangle $X \to Y \to Z \to T(X)$, the sequence

$$F(X) \to F(Y) \to F(Z)$$

is exact.

Proposition 22. The distinguished triangle and cohomologcal functor satisfies the following propositions:

(i) If $X \xrightarrow{f} Y \xrightarrow{g} Z \to T(X)$ is a distinguished triangle, then $g \circ f = 0$.

(ii) For $W \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(W, \cdot)$, $\mathrm{Hom}_{\mathcal{C}}(\cdot, W)$ are cohomological functors.

Proof. By (TR 1), there is a diagram of distinguished triangles

$$\text{(TR 1)} \qquad \begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, dotted, "\exists\phi"] & T(X) \arrow[d, "\mathrm{id}"] \\ X \arrow[r] & Y \arrow[r, "g"] & Z \arrow[r] & T(X) \end{tikzcd}$$

a morphism at the dotted arrow that makes the whole diagram commutes exists by (TR 4). Therefore the middle square gives $g \circ f = 0$. Hence (i) is proved.
For (ii), consider $\mathrm{Hom}_{\mathcal{C}}(W, \cdot)$. Assume $X \xrightarrow{f} Y \xrightarrow{g} Z \to T(X)$ is a distinguished triangle. Here we want to show

$$\mathrm{Hom}_{\mathcal{C}}(W, X) \xrightarrow{f\circ} \mathrm{Hom}_{\mathcal{C}}(W, X) \xrightarrow{g\circ} \mathrm{Hom}_{\mathcal{C}}(W, Z)$$

exact, i.e., (a) $g \circ f \circ \cdot = 0$, (b) $\mathrm{Ker}(g \circ \cdot) \simeq \mathrm{Im}(f \circ \cdot)$. (a) follows directly from (i). (i) also implies that $\mathrm{Ker}(g \circ \cdot) \supseteq \mathrm{Im}(f \circ \cdot)$. It suffices to show $\subseteq$. Let $\phi \in \mathrm{Hom}_{\mathcal{C}}(W, Y)$ such that $g \circ \phi = 0$. Then we have the following commutative diagram

$$\begin{tikzcd} \text{(TR 0)} & W \arrow[r] \arrow[d, dotted, "\exists\psi"] & W \arrow[r] \arrow[d, "\phi"] & 0 \arrow[r] \arrow[d] & T(W) \arrow[r] \arrow[d, dotted, "\exists T(\psi)"] & T(W) \arrow[d, "T(\phi)"] \\ \text{hypothesis} & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & T(X) \arrow[r] & T(Y) \end{tikzcd}$$

$T(\psi)$ exists by (TR 4) and ψ exists since T is an automorphism. Thus $\mathrm{Hom}_{\mathcal{C}}(W, \cdot)$ is cohomogical, $\mathrm{Hom}_{\mathcal{C}}(\cdot, W)$ is also cohomological by reversing the vertical arrows. □

Corollary 23. Consider the commutative diagram of two distinguished triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\phi"] & Y \arrow[r] \arrow[d, "\psi"] & Z \arrow[r] \arrow[d, "\theta"] & T(X) \arrow[d, "T(\phi)"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & T(X') \end{tikzcd}$$

If ϕ and ψ are isomorphisms, then θ is also isomorphism.

Proof. Since $\mathrm{Hom}(W, \cdot)$ is cohomological, the rows of the following diagram are exact:

$$\begin{tikzcd} \mathrm{Hom}(W, X) \arrow[r] \arrow[d, "\phi\circ"'] & \mathrm{Hom}(W, Y) \arrow[r] \arrow[d, "\psi\circ"'] & \mathrm{Hom}(W, Z) \arrow[r] \arrow[d, "\theta\circ"'] & \mathrm{Hom}(W, T(X)) \arrow[r] \arrow[d, "T(\phi)\circ"'] & \mathrm{Hom}(W, T(Y)) \arrow[d, "T(\psi)"] \\ \mathrm{Hom}(W, X') \arrow[r] & \mathrm{Hom}(W, Y') \arrow[r] & \mathrm{Hom}(W, Z') \arrow[r] & \mathrm{Hom}(W, T(X')) \arrow[r] & \mathrm{Hom}(W, T(Y')) \end{tikzcd}$$

then $\theta \circ \cdot$ is isomorphism by 5-lemma. Thus θ is an isomorphism. □

Proposition 24. Let $\mathcal{C}$ be an abelian category. Then $H^0(\cdot) : K(\mathcal{C}) \to \mathcal{C}$ is a cohomologcal functor.

Proof. Consider the distinguished triangle $Y \to M(f) \to X[1] \xrightarrow{-f[1]} Y[1]$.
Claim: $0 \to Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1] \to 0$ is exact.
We need to check:

(a) $\beta(f) \circ \alpha(f) = 0$: follows by definition of $\alpha(f)$ and $\beta(f)$.

(b) $\mathrm{Im}(\alpha(f)) \simeq \mathrm{Ker}(\beta(f)^n)$: follows since $\mathrm{Im}(\alpha(f)) \simeq Y^n$ and $\mathrm{Ker}(\beta(f)^n) \simeq Y^n$.

Therefore there exists a long exact sequence by Proposition 15

$$0 \to \boxed{H^0(Y) \to H^0(M(f)) \to H^0(X[1])} \to H^1(Y) \to \cdots$$

the boxed part gives the wanted result. $\square$