Assignment 1

HW-1

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Discrete Structures.

1. <u>Contrapositive Proofs:</u>

a) $\forall n \in \mathbb{Z},\ 7 \nmid n^2 \Rightarrow 7 \nmid n$.

<u>Contrapositive statement:</u> $\forall n \in \mathbb{Z},\ 7 \mid n \Rightarrow 7 \mid n^2$

<u>Proof:</u> Let $n = 7k$, where $k \in \mathbb{Z}$

$\Rightarrow n^2 = (7k)^2 = 49k^2$

$\Rightarrow 7 \mid n^2$

Hence, Proved.

b) $m, n \in \mathbb{Z}^+$, such that $mn = 100$

$\Rightarrow m \leq 10$ or $n \leq 10$

Contrapositive statement: $m > 10$ & $n > 10$

$\Rightarrow mn \neq 100$, where $m, n \in \mathbb{Z}^+$.

<u>Proof:</u> Let $m = 10 + a$ and $n = 10 + b$,

where $a, b \in \mathbb{N}$

$$mn = (10+a)(10+b)$$

$$= 100 + 10(a+b) + ab$$

$10(a+b) + ab \in \mathbb{N}$ since $a, b \in \mathbb{N}$.

$\Rightarrow \quad mn = 100 + t$, where $t \in \mathbb{N}$

$\Rightarrow \quad mn \neq 100$

Hence, Proved.

c) $x \in \mathbb{R}$, such that $x \in (0,1)$, then $x > x^2$.

Contrapositive statement: $x^2 > x \Rightarrow x \notin (0,1)$, ~~the~~

where $x \in \mathbb{R}$.

<u>Proof</u>: <u>Case 1</u>: $x > 0$

$x^2 > x \Rightarrow x > 1$

<u>Case 2</u>: $x < 0$

$x^2 > x \Rightarrow x < 1$

~~$\Rightarrow$ For $x^2 > x$, $x \in ((0,\infty) \cap (1,\infty)) \cup ((-\infty,0) \cup (-\infty,1))$~~

~~$\Rightarrow x \in -\infty, 0$~~

~~$\Rightarrow$~~

$\Rightarrow$ For $x^2 > x$, $x \in ((-\infty,0) \cap (-\infty,1)) \cup ((0,\infty) \cap (1,\infty))$

$\Rightarrow \quad x \in (-\infty, 0) \cup (1, \infty)$

$\Rightarrow \quad x \in \mathbb{R} \setminus (0,1)$

$\Rightarrow \quad$ ~~$x \notin 0,10$~~ $x \notin (0,1)$

Hence, Proved.

2. Equivalence Proofs:

1. $x, y \in \mathbb{R}$. To Prove: $|x+y| = |x| + |y| \Leftrightarrow xy \geq 0$

Proof: LHS $\Rightarrow$ RHS:

Let $a, b \in \mathbb{R}^+ \cup \{0\}$

Case 1: $x = a$, $y = b$

$|x+y| = \cancel{x} = \cancel{x+y}\ a+b$

$|x| + |y| = \cancel{x+y}\ a+b \Rightarrow |x+y| = |x| + |y|$

$xy = ab \geq 0$

Case 2: $x = -a$, $y = -b$

$|x+y| = |-(a+b)| = a+b$

$|x| + |y| = a + b$

$\Rightarrow |x+y| = |x| + |y|$

$xy = (-a)(-b) = ab \geq 0$

Case 3: $x = -a$, $y = b$

$|x+y| = |b-a|$

$|x| + |y| = a+b$

$|x+y| \neq |x| + |y|$

$xy = -ab \leq 0$

Case 4: $x = a, \quad y = -b$

$|x+y| = |a-b|$

$|x|+|y| = a+b$

$|x+y| \neq |x|+|y|$

$xy = -ab \leq 0$

Hence, LHS $\Rightarrow$ RHS $\left(|x+y| = |x|+|y| \Rightarrow xy \geq 0 \right)$

RHS $\Rightarrow$ LHS:

$xy \geq 0$

Case 1: $x \geq 0, \quad y \geq 0$

$|x+y| = x+y$

$|x|+|y| = x+y = |x+y|$

Case 2: $x \leq 0, \quad y \leq 0$

$|x+y| = -(x+y)$

$|x|+|y| = -x-y = |x+y|$

$\therefore \quad xy \geq 0 \Rightarrow |x+y| = |x|+|y|$

Hence, Proved.

2. $\forall m, n \in \mathbb{Z}$, $m \mid n$ and $n \mid m \Leftrightarrow m = n$

Proof: LHS $\Rightarrow$ RHS:

$m \mid n \Rightarrow m \leq n$

$n \mid m \Rightarrow n \leq m$

$m \leq n$ & $n \leq m \Rightarrow m = n$

$\therefore$ $m \mid n$ & $n \mid m \Rightarrow m = n$

RHS $\Rightarrow$ LHS:

$m = n \Rightarrow m = n \times 1 \Rightarrow n \mid m$

$n = m \Rightarrow n = m \times 1 \Rightarrow m \mid n$

$\therefore$ $m = n \Rightarrow m \mid n$ & $n \mid m$

Hence, Proved.

3. Contradiction Proofs:

1. $A = \left\{\frac{n-1}{n} : n \in \mathbb{Z}^+\right\}$ does not have a ~~maximum~~ largest value.

Contradiction: $A : \left\{\frac{n-1}{n} : n \in \mathbb{Z}^+\right\}$ has a ~~maximum~~ largest value.

Proof: Let A have a largest value a

$\cancel{a = \frac{k-1}{k}}$ where $a = \frac{k-1}{k}$ for some $k \in \mathbb{Z}^+$

Consider $\frac{(k+1)-1}{k+1} = \frac{k}{k+1} > \frac{k-1}{k}$

$\cancel{\Rightarrow \exists}$ $\Rightarrow$ For any k, $(k+1)$ will provide a higher value.

$\Rightarrow$ A does not have a maximum value.

$\Rightarrow$ The assumption was wrong.

Hence proved.

2. Mean of four distinct numbers is $n \in \mathbb{Z}$, then at least one of the four integers is greater than $n+1$.

Contradiction: None of the four integers is greater than $n+1$.

Proof: Consider four distinct integers a, b, c & d.

Consider, $a, b, c, d \leq n+1$.

For highest value of mean with given conditions the four integers are:

$$n-2, \; n-1, \; n, \; n+1.$$

$$\text{mean} = \frac{n-2+n-1+n+n+1}{4} = n - \frac{1}{2} < n$$

Hence, a mean of n cannot be achieved given that none of the four distinct integers is greater than $n+1$.

$\Rightarrow$ The Assumption was wrong

$\Rightarrow$ ~~Mea~~ At least one of the four numbers is greater than $n+1$.

Hence, Proved.

3. $\nexists\, r \in \mathbb{Q}$, such that $2^r = 3$

Contradiction: $\exists\, r \in \mathbb{Q}$, such that $2^r = 3$

Proof: let $r = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ & $q \neq 0$

$$2^{\frac{p}{q}} = 3$$

$$\Rightarrow 2^p = 3^q$$

But 2^p is even and 3^q is odd

$$\Rightarrow 2^p \neq 3^q$$

$\Rightarrow$ The assumption is wrong

$\Rightarrow \nexists\, r \in \mathbb{Q}$, such that $2^r = 3$

Hence, Proved.

4. Existence Proofs:

1. Out of 367 people, at least two of them have the same birthdays.

~~Contradic~~

Proof: Assume no two of them have the same birthdays

Number of unique possible birthdays : 366

Let the first 366 people uniquely have a different birthday out of the 366 possible birthdays.

The 367^{th} person's birthday will also have to be one of these 366 days.

$\Rightarrow$ 367^{th} person will be sharing his/her birthday with ~~someone~~ someone else.

$\Rightarrow$ Two people have the same birthday

$\Rightarrow$ The assumption was wrong

$\Rightarrow$ At least two of the 367 will have the same birthday.

2. $\{b_1, b_2, b_3 \cdots b_n\} \in \mathbb{Z}$, such that $\Sigma b_k^2 < n$.

<u>To Prove</u>: at least one of the integers in the set is 0.

<u>Proof</u>: Assume $b_1, b_2 \cdots b_n \neq 0$ (none of the integers is 0)

$\Rightarrow b_k^2 \geq 1 \quad \forall k \in \{1, 2, 3 \cdots n\}$

$\cancel{\Rightarrow \sum_{k=1}^{n} b_k^2}$

$\Rightarrow b_1^2 + b_2^2 + \dots + b_n^2 \geq 1 + 1 + 1 \dots$ n times

$\Rightarrow \sum_{k=1}^{n} b_k^2 \geq n$

But $\sum_{k=1}^{n} b_k^2 < n$

$\Rightarrow$ Our assumption was wrong

$\Rightarrow$ At least one of the integers is zero.