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Assignment – 2

Discrete Structures (MA5.101)

1) Proof: $\forall a \in \mathbb{N}$, we can write a as $k \times n + r$, where $k \in \mathbb{N}$ and $0 \leq r < n$

Here r is the remainder we obtain when we divide a by n.

r can take n different values $\rightarrow 0, 1, 2 \ldots n-1$.

If two numbers leave the same remainder when divided by n, their difference is a multiple of n.

Proof: let $a = kn + r$, $b = ln + r$

$a - b = (k - l)n$, which is divisible by n.

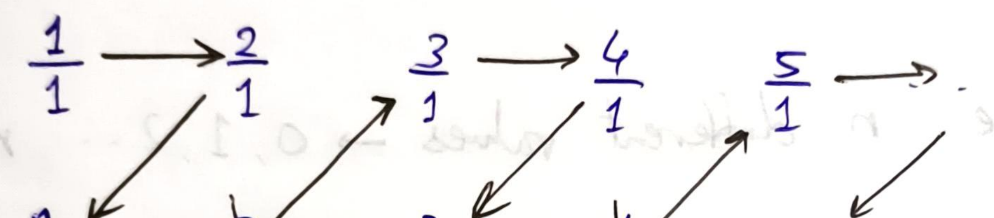
Assume n different numbers ~~take the~~ have the n different remainders. If we ~~take~~ consider another number, it will have the same remainder as one of the previous n numbers, by pigeon-hole principle.

$\Rightarrow$ Its difference with this number will be divisible by n. $\square$

8) Consider the set of all positive rational numbers,

$$\mathbb{Q}^+ := \left\{ \frac{p}{q} \,\middle/\, p, q \in \mathbb{N} \right\}$$

We can take all these numbers in an infinitely extending two dimensional grid, where the numerator increases as we move to the right and denominator increases as we move down.



If we count as shown by the black arrows, we will clearly count every positive rational number. We can map each rational number to a natural number.

Hence, we can say that the set of positive rational numbers is countable.

$\square$

10) $f: X \to Y$, $A, B \subseteq X$

a) Let $y \in f(A \cup B)$, where $y \in Y$ and $y = f(x)$, where $x \in X$

$\Leftrightarrow x \in A \cup B$

$\Leftrightarrow x \in A$ or $x \in B$

$\Leftrightarrow y \in f(A)$ or $y \in f(B)$

$\Leftrightarrow y \in f(A) \cup f(B)$

Hence, $f(A \cup B) = f(A) \cup f(B)$

$\square$

b) For some $x \in X$ & $y \in Y$, let $f(x) = y$

Let $y \in f(A \cap B)$

$\Leftrightarrow x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$

$\Rightarrow f(x) \in f(A)$ and $f(x) \in f(B)$

$\Leftrightarrow y \in f(A) \cap f(B)$

$\Rightarrow$ Hence, $f(A \cap B) \subseteq f(A) \cap f(B)$ $\square$

c) for some $x \in X$ and $y \in Y$, let $f(x) = y$.

let $y \in f(A-B)$

$\Leftrightarrow x \in (A-B)$

$\Leftrightarrow x \in A \cap B'$ (By definition)

$\Leftrightarrow x \in A$ and not in B

$\Leftrightarrow f(x) \in f(A)$ and $f(x) \notin f(B)$ [$\because f$ is injective]

$\Leftrightarrow y \in f(A) \cap f(B)'$

$\Leftrightarrow y \in f(A) - f(B)$

Hence, $f(A-B) = f(A) - f(B)$, if f is injective.

$\square$

11) $f: X \to Y$, $S \subseteq Y$. $f^{-1}(S) = x \in X \mid f(x) \in S$, $A, B \subseteq Y$

a) for some $x \in X$ and $y \in S$, let $f(x) = y$

~~let y ∈~~ let $x \in f^{-1}(A \cup B)$

$\Leftrightarrow y \in A \cup B$ $\Leftrightarrow y \in A$ or $y \in B$

$\Leftrightarrow f^{-1}(y) \in f^{-1}(A)$ or $f^{-1}(y) \in f^{-1}(B)$

$\Leftrightarrow x \in f^{-1}(A) \cup f^{-1}(B)$

Hence, $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

$\square$

b). for some $x \in \cancel{X}$ and $y \in S$, let $f(x) = y$

~~Since~~ Since all inverse functions are ~~bijective~~ injective, f^{-1} is ~~bijective.~~ injective.

Let $x \in f^{-1}(A \cap B) \Leftrightarrow y \in A \cap B$

$\Leftrightarrow y \in A$ and $y \in B$ ~~$\Leftrightarrow f^{-1}(y) \in f^{-1}(A)$~~

$\Leftrightarrow f^{-1}(y) \in f^{-1}(A)$ and $f^{-1}(y) \in f^{-1}(B)$ [$\because f^{-1}$ is injective]

$\Leftrightarrow x \in f^{-1}(A) \cap f^{-1}(B)$

Hence, $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

$\square$

c) For some $x \in X$ and $y \in S$, let $f(x) = y$

Since all inverse functions are injective, f^{-1} is injective.

Let $x \in f^{-1}(A - B) \Leftrightarrow y \in A - B$

$\Leftrightarrow y \in A \cap B'$ (By definition of set difference)

$\Leftrightarrow y \in A$ and $y \notin B$

$\Leftrightarrow x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$ [$\because f^{-1}$ is injective]

$\Leftrightarrow x \in f^{-1}(A) \cap f^{-1}(B)' \Leftrightarrow x \in f^{-1}(A) - f^{-1}(B)$

Hence, $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$

$\square$

12) a) $f: A \to B$ is strictly increasing

$\Leftrightarrow \forall \; a_1, a_2 \in A$

$a_1 > a_2 \Leftrightarrow f(a_1) > f(a_2)$

Let for any $x, y \in A$,

$f(x) = f(y)$ *

~~Let $x > y$,~~ Assume $x > y$,

Since f is strictly increasing,

$\Rightarrow f(x) > f(y)$

This is a contradiction.

Hence $x \not> y$ — ①

Assume $x < y$,

Since f is strictly increasing

$\Rightarrow f(x) < f(y)$

this contradicts * our statement

$\Rightarrow$ Hence $x \not< y$ — ②

from ① and ②,

$x = y$

$f(x) = f(y) \Rightarrow x = y$

Hence, f is injective.

$\square$

b) $f: A \to B$ is strictly decreasing

$\Leftrightarrow \forall a_1, a_2 \in A$

$a_1 < a_2 \Leftrightarrow f(a_1) > f(a_2)$

let for any $x, y \in A$,

$f(x) = f(y)$

Assume $x > y$,

$\Rightarrow f(x) < f(y)$ [strictly decreasing]

This is a contradiction.

Hence, $x \ngtr y$ — ①

Assume $x < y$,

$\Rightarrow f(x) > f(y)$

This is a contradiction

Hence, $x \nless y$ — ②

From ① and ②,

$x = y$

$f(x) = f(y) \Rightarrow x = y$

Hence f is injective.

$\square$

13) a) $f(x) = \frac{x}{1+x^2}$

Domain $\equiv (-\infty, \infty) = R$

Range $\equiv \left[-\frac{1}{2}, \frac{1}{2}\right]$

b) $f(x) = 2 - |x-5|$

Domain $\equiv (-\infty, \infty) = \mathbb{R}$

Since $|x-5|$'s minimum value is 2,

Range $\equiv (-\infty, 2)$

c) $f(x) = \frac{x^2}{1+x^2}$

Domain $\equiv (-\infty, \infty) = \mathbb{R}$

Range $\equiv [0, 1)$

d) $f(x) = \frac{x^2-3x+2}{x-2}$

$f(x) = \frac{(x-1)(x-2)}{x-2} = x-1$

Domain $\equiv \cancel{(-\infty, \infty)} = \mathbb{R} - \{2\}$

Range $\equiv (-\infty, \infty) = \mathbb{R} - \{1\}$

e) $f(x) = \log_2 (x - x^2)$

$x - x^2 > 0$

$\Rightarrow x > x^2$

$\Rightarrow x \in (0, 1) \rightarrow$ Domain

Domain $\equiv (0, 1)$

Range $\equiv [-2, 0)$

14). $f : \mathbb{R} \rightarrow \mathbb{R}, \; f(x) = |x - 1|$

$g : \mathbb{Z} \rightarrow \mathbb{R}, \; g(x) = \frac{2x}{x^2 - 3}$

$h : \mathbb{R} \rightarrow \mathbb{R}, \; h(x) = 2x + 1$

a) $f\left(\frac{2}{3}\right) = \left|\frac{2}{3} - 1\right| = \left|-\frac{1}{3}\right| = \frac{1}{3}$

b) $g \circ h\left(\frac{1}{2}\right)$

$h\left(\frac{1}{2}\right) = 2 \times \frac{1}{2} + 1 = 2$

$g(2) = \frac{4}{4-3} = 4 \Rightarrow g \circ h\left(\frac{1}{2}\right) = 4$

c) $f \circ f(-2)$

$f(-2) = |-2-1| = |-3| = 3$

$f(3) = |3-1| = |2| = 2$

$\Rightarrow f \circ f(-2) = 2$

d) $f \circ h(x)$

$h(x) = 2x+1$

$f(h(x)) = |2x+1-1| = |2x|$

$\Rightarrow f \circ h(x) = |2x|$

e) $f \circ g(x)$

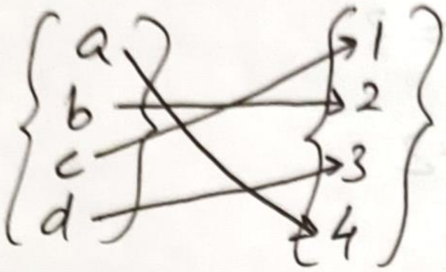
$g(x) = \frac{2x}{x^2-3}$

$f(g(x)) = \left|\frac{2x}{x^2-3} - 1\right| = \left|\frac{2x - x^2+3}{x^2-3}\right| = \left|\frac{x^2-2x-3}{x^2-3}\right|$

$= \left|\frac{(x-3)(x+1)}{x^2-3}\right|$

(Please turn Over)

15) a) $f: \{a, b, c, d\} \to \{1, 2, 3, 4\}$



Since each element in $\{a, b, c, d\}$ is mapped uniquely to each image, f is injective

Since each element in $\{1, 2, 3, 4\}$ has a pre-image, f is also surjective.

Hence, f is both injective and surjective.

b) $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{x^2+1}{x^2+2}$$

~~Since f(x) is a strictly increasing fn~~

$\forall\ a \in \mathbb{R} - \{0\}$, we have $-a$, where $f(a) = f(-a)$. Hence, $f(x)$ is not injective.

The range of f is $\cancel{0, 1}$ $\left[\frac{1}{2}, 1\right)$ ~~and~~ $\neq \mathbb{R}$

Hence $f(x)$ is not surjective.

Hence, $f(x)$ is neither injective nor surjective

c) $f: \mathbb{Z} \to \mathbb{Z}$,

$f(n) = \lfloor \frac{n}{2} \rfloor$

$\forall k \in \mathbb{Z}$, $f(2k) = f(2k+1) = k$

$\Rightarrow$ f is not injective.

All the integers in the codomain have a pre-image we can find by simply multiplying the image by 2.

$\Rightarrow$ f is surjective

Hence, f is only surjective.

d) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$

f is an exponential function. Hence, it is injective.

Range of f is $\mathbb{R}^+$, hence, it is not surjective

Hence f is only injective.

e) $f: \mathbb{R} \to \mathbb{R}^+$, $f(x) = e^x$

f is an exponential function. Hence, it is injective.

Range of f is $\mathbb{R}^+$ and hence, it is surjective.

Hence, f is injective and surjective.

16) a) $f: \mathbb{N} \to \mathbb{N}$, $f(x) = x^2$

No, f is not invertible

Since $(\sqrt{x})$ is not defined $\forall x \in \mathbb{N}$

$\sqrt{x^2}$ is defined and hence, f is ~~right~~ left invertible

b) $f: \mathbb{Z} \to \mathbb{Z}$

if $n \equiv 0 \pmod 3$,

$f(n) = n + 4 \equiv 1 \pmod 3$

if $n \equiv 1 \pmod 3$,

$f(n) = -n-3 = -3k-1-3 = 3t-4 \equiv 2 \pmod 3$

if $n \equiv 2 \pmod 3$,

$f(n) = n+1 = 3k+3 \equiv 0 \pmod 3$

Hence, f is invertible and its inverse is:

$$f^{-1}(x) = \begin{cases} x-1 & \text{if } x \equiv 0 \pmod 3 \\ x-4 & \text{if } x \equiv 1 \pmod 3 \\ -x-3 & \text{if } x \equiv 2 \pmod 3 \end{cases}$$

c) $f : [0,\infty) \to (-\infty,\infty)$

f is not invertible as $f(0) = f(1) = 0$

and is hence not injective.

It is hence not left or right invertible either.

17). a) If 1 is not in the range, each of the five elements in A have 2 choices (2 or 3) to be mapped to

$= 2^5 = 32$

b) $\underbrace{{}^5C_2}_{\text{to pick 2 pre-images}} \times \underbrace{2^3}_{\text{2 choices for other numbers.}} = 10 \times 8 = 80$

c) $3^4 = 81$ as each of the other elements have 3 choices.

d) We cannot have any one-one function as $|A| > |B|$ (By pigeonhole principle)

$\Rightarrow = 0$

e) All functions are many-one $= 3^5 = 243$

As each number in A has 3 choices.

6) ~~Let~~ If we pick k, we cannot pick $2k$

$\forall k \in \{1, 2, 3 \dots n\}$

Hence the only way to pick n numbers without one of them being a multiple of another is to pick $\{n+1, n+2, n+3 \dots 2n\}$. We still have have to now pick one number from $\{1, 2 \dots n\}$.

But if we pick k from this set, $2k$ is already in the set $\{n+1, n+2 \dots 2n\}$. Hence, we cannot pick $n+1$ such numbers not exceeding $2n$.

$\square$

5) For any two numbers comprised of only 1's, their difference will only have 0's and 1's as its digits.

For any n, consider the ~~first n+1 elements of the~~ set: $\{1, 11, 111, 1111 \dots 11 \cdots n+1 \text{ times}\}$

Each of these numbers can possibly leave remainders $0, 1, 2 \dots n-1$, when divided by n.

These are n different ~~remaindd~~ remainders. But we have n+1 numbers. Hence at least two of these numbers will have the same remainder when divided by 'n'.

⇒ Their difference will be a multiple of 'n'.

$\square$

3). Let us consider the contradiction to the given statement: Given a room of 6 people, there need not be 3 people who know each other ~~and there~~ or ~~need not be~~ 3 people who do not know each other.

~~This~~ let us take the maximum cases: ~~2 people know each~~ ~~Case 1:~~ 2 people know each other and 2 people do not know each other. The remaining two people either know each other or do not know each other, which contradicts our statement.

Hence, 3 people either know each other or 3 people do not know each other.

7) $a \Rightarrow b$

A is countable $\Leftrightarrow$ there exists a bijection from A to ~~S⊆N~~ ~~where S⊆N ⇒ there exists a bijection fr~~ S, where

$S \subseteq \mathbb{N}$.

$\Rightarrow$ there exists a bijection from S to A (we can consider the inverse)

$\Rightarrow$ there exists a surjection from $\mathbb{N}$ to A (since $S \subseteq \mathbb{N}$)

$b \Rightarrow c$

There exists a surjection from $\mathbb{N}$ to A

~~⇒~~ Consider f, which is such a surjection.

Consider any one random preimage for every element in A.

Consider a function which maps each of these elements of A

to an element in $\mathbb{N}$.

$\Rightarrow$ There exists an injection from A to $\mathbb{N}$.

$c \Rightarrow a$

There exists an injection from A to $\mathbb{N}$

$\Rightarrow$ ~~There exists a bijecti~~ Consider the subset of $\mathbb{N}$, which are images of the bijection from A

$\Rightarrow$ There exists a bijection from A to a subset of $\mathbb{N}$.

$\Rightarrow$ A is countable. $\square$

2) Let us divide our square into 25 smaller squares. The side length of each of these squares is $\frac{1}{5}$, which makes the diagonal $\frac{\sqrt{2}}{5}$

$\frac{\sqrt{2}}{5} < \frac{2}{7}$ and hence, the diameter of required circle > diagonal length of each square.

$\Rightarrow$ We can always find a circle which completely contains any square.

Out of the 51 points, let 2 points be in each of these 25 squares. This is the most spaced out we can place the points. the last point has to be placed in one of the pre-occupied squares. Consider the circle containing this square with three points. this circle hence contains 3 points.

If the points are placed in any different arrangement, they will be placed closer and hence, we can definitely find a circle containing 3 of them.

$\square$

4) Each of the 5 computers has access to each of the 5 routers. For this, we will need $5 \times 5 = 25$ connections.

The remaining computers can be connected to ~~each~~ one of the pre-existing computers and hence we will require 5 more connections. Hence, we need a total of $25 + 5 = 30$ connections.