International Institute of Information Technology Hyderabad

Discrete Structures (MA5.101)

Total Marks: 100

Instructions: Submit preferably handwritten scanned pdf file in the moodle under Assignments directory.

1. Given a set of $n+1$ distinct numbers for $n > 1$, using pigeon hole principle, prove that there exist a pair of elements whose difference is divisible by n. [2.5]
2. Given 51 points in a square of size 1 square unit, show that there exists a circle of radius $1/7$ that would contain 3 of these points. [2.5]
3. Given a room of 6 people, show that there must be either 3 people who know each other, or 3 people who does not know each other. [2.5]
4. What is the least number of computers required to connect 10 computers to 5 routers, in order to guarantee that 5 computers can directly access 5 routers? [2.5]
5. Show that for every integer n, there is a multiple of n that has only 0s and 1s in its decimal expansion [2.5]
6. Show that among any $n+1$ positive integers not exceeding $2n$, there must be an integer that divides one of the other integers [2.5]
7. Let A be a non-empty set. Show that the following are equivalent.
   (a) A is countable.
   (b) There exists a surjection $f : N \to A$
   (c) There exists an injection $g : A \to N$ [10]
8. Prove that the set of all positive rational numbers is countable. [5]
9. If A and B are countable sets then show that $A \times B$ is countable. [2.5]
10. For $f : X \to Y$ and $A, B \subseteq X$, prove (or disprove) that
    (a) $f(A \cup B) = f(A) \cup f(B)$.

(b) $f(A \cap B) \subseteq f(A) \cap f(B)$.

(c) $f(A - B) \subseteq f(A) - f(B)$, if f is injective. [7.5]

11. For $f : X \to Y$, let S $\subseteq$ Y. Define $f^{-1}(S) = x \in X | f(x) \in S$. Let $A, B \subseteq Y$. Prove (or disprove) that

(a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(b) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(c) $f^{-1}(A - B) = f^{-1}(A) - f^{-1}B$. [7.5]

12. Show that a strictly increasing or strictly decreasing function is injective. [2.5]

13. Define the range and domain of the following functions defined over R:

(a) $f(x) = \frac{x}{1+x^2}$

(b) $f(x) = 2 - |x - 5|$

(c) $f(x) = \frac{x^2}{1+x^2}$

(d) $f(x) = \frac{x^2-3x+2}{x-2}$

(e) $f(x) = \log_2(x - x^2)$ [10]

14. Three functions f, g, h are defined as below,

$f : R \to R,\ f(x) = |x - 1|$

$g : Z \to R,\ g(x) = \frac{2x}{x^2-3}$

$h : R \to R,\ h(x) = 2x + 1$

For each of the following, either evaluate the expression or explain why it is not defined.

(a) $f(\frac{2}{3})$

(b) $g \circ h(\frac{1}{2})$

(c) $f \circ f(-2)$

(d) $f \circ h(x)$

(e) $f \circ g(x)$ [5]

15. Determine which of the following functions are injective, surjective, or both.

(a) $f : \{a, b, c, d\} \to \{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$

(b) $f : R \to R,\ f(x) = \frac{x^2+1}{x^2+2}$

(c) $f : Z \to Z,\ f(n) = [\frac{n}{2}]$

(d) $f : R \to R,\ f(x) = e^x$

(e) $f : R \to R^+,\ f(x) = e^x$ [10]

16. Are the following functions invertible? If yes, find the inverse of the function. If not, check if they are left invertible or right invertable.

(a) $f : N \to N$ defined by $f(x) = x^2$

(b) $f : Z \to Z$ defined by

$$f(x) = \begin{cases} n+4 & \text{if n} \equiv 0 \text{ (mod 3)} \\ -n-3 & \text{if n} \equiv 1 \text{ (mod 3)} \\ n+1 & \text{if n} \equiv 2 \text{ (mod 3)} \end{cases}$$

(c) $f : [0, \infty) \to (-\infty, \infty)$ defined by

$$f(x) = \begin{cases} ln(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

[15]

17. Let $f : A \to B$ where $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3\}$ be two sets then answer the following questions

    Note : Answer and a one line explanation for each part is enough.

    (a) In how many functions, does the range not contain 1?

    (b) In how many functions, does 1 in B have exactly 2 pre-images in A?

    (c) In how many functions, will $f(1) = 1$?

    (d) How many functions would be one-one?

    (e) How many functions would be many-one?

[10]

All the best!!!