Discrete Structures

Permutations and Combinations

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IIIT, Hyderabad

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Outline

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  - proofs such as Pigeon Hole Principle (PHP)

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- It is dates back to the Upper Paleolithic period of human history, and is approximately 20,000 years old.
- The bone is 10 cm long and contains a series of notches, which many scientists believe were used for counting.

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- Sumer was a region of ancient Mesopotamia in the Middle East

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How many numbers are there between m and n, $m < n$?

Answer

$m - n + 1$

How do we count?

Question-1

How many numbers between 33 and 67 are divisible by 4?

Outline

Sum Rule: Statement and Example

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Sum Rule Example

How many of the Pizza or Burgers places are there?

Answer

There are 7 Pizzas and there are 5 Burgers, hence, by sum rule, we have $7 + 5 = 12$

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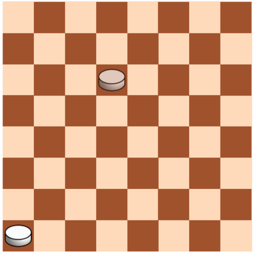

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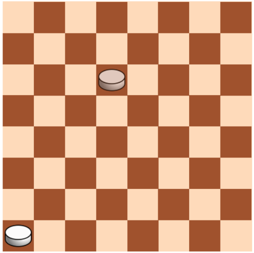

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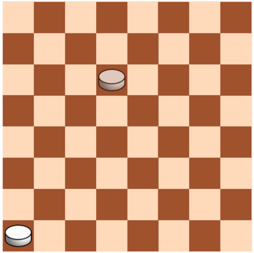

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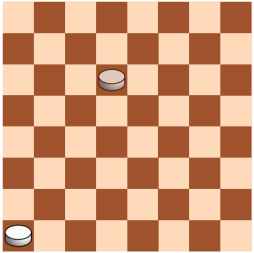

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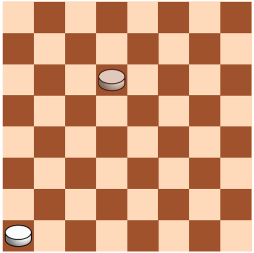

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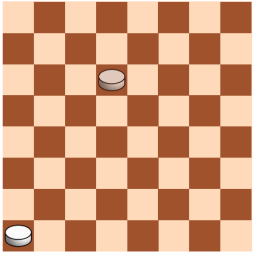

- There are two types of moves: move right and move up
- To get to the column 4, we need 3 moves to the right
- To get to the row 6, we need 5 moves up
- Applying sum rule: In total, we need $3+5=8$ moves

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Caution with Sum Rule

In the rule of sum, no object should belong to both types!

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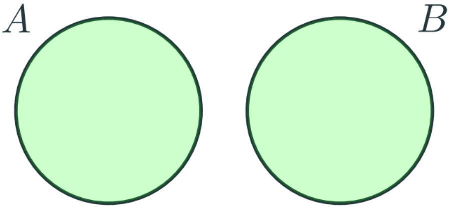

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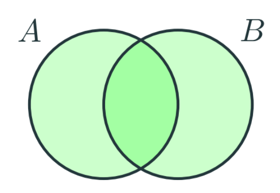

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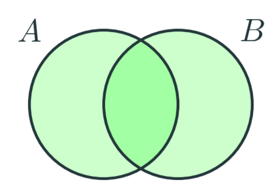

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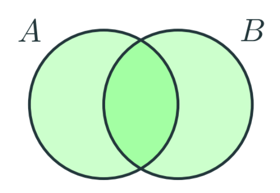

- If we consider $|A| + |B|$ as in sum rule, then we will be wrong
- We will count elements that belong to both A and B twice
- $|A \cup B| = |A| + |B| - |A \cap B|$ (Inclusion-Exclusion Principle)

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- Let us count all the numbers from 1 to 10:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10$$

- Here 6 is divisible both by 2 and by 3. Hence, rule of sum can't be applied!

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Number of Paths

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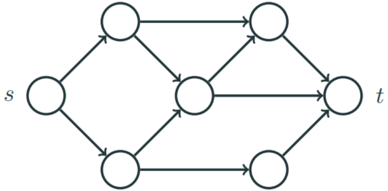

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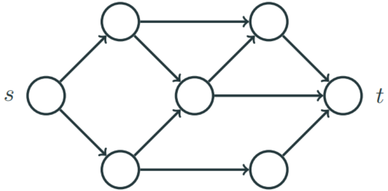

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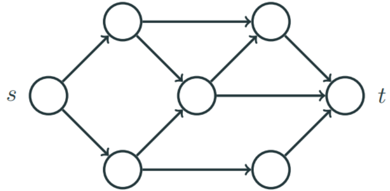

- counting can be done recursively
- for each node count the number of paths from s to this node
  - sum rule will be used

Outline

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Rule of Product Using Sets

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If there is a finite set A and a finite set B, then there are $|A| \times |B|$ pairs of objects, the first from A, and the second from B

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- Can you now answer the question above?

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- thus, the answer is a product of n by itself k times, that is n^k

Number of Number Plates

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Question

How many vehicles are there?

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- If we fix 5 in one place, then there are $5 \times 5 \times 5 = 125$ sequences
- There are 4 ways to arrange 5 among 4 places
- Hence, there are $4 \times 125 = 500$ four digit numbers below 10,000 with exactly one 5

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$$\begin{array}{ccccc} 1 & 2 & 3 & \dots & k \\ * & * & * & \cdots & * \\ n & n-1 & n-2 & & \end{array}$$

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- Hence there are

$$n \times (n-k) \times \cdots (n-k+1)$$

$k-$permutations, which is $n!/(n-k)!$

Permutation Examples

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Question

In how many ways we can arrange n different books in n different bins on shelf?

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Answer

Hint: Use previous result with $k = n$.

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Above Question Reformulated

We are essentially asking: What is the number of ways of choosing 3 elements out of a set containing 5 elements?

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- There are five choices of the first friend,

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- But each group of three friends is counted $3! = 6$ times, that is, a group $\{a, b, c\}$ is counted as $abc, acb, bac, bca, cab, cba$
- Thus, we need to divide by $3!$, the answer is $(5 \times 4 \times 3)/3! = 10$

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- But each group of three friends is counted $3! = 6$ times, that is, a group $\{a, b, c\}$ is counted as $abc, acb, bac, bca, cab, cba$
- Thus, we need to divide by $3!$, the answer is $(5 \times 4 \times 3)/3! = 10$
- Since ordering does not matter, we call them combinations!

Answer to Previous Question...

Answer

- There are five choices of the first friend, four choices of the second friend, and three choices of the third friend
- How many choices are there for choosing 3 friends, assuming ordering?
- In total, there are $5 \times 4 \times 3 = 60$ choices assuming ordering
- But each group of three friends is counted $3! = 6$ times, that is, a group $\{a, b, c\}$ is counted as $abc, acb, bac, bca, cab, cba$
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- We define combinations in next slide...

Combinations: $k-$combination

Definition of $k-$combination

For a set S, a $k-$combination is a subset of S of size k

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Pronounced:"n choose k". Proof by example!

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Number of $k-$combinations

The number of $k-$combinations of an n element set is given by

$$\frac{n!}{(n-k)!}$$

Derive formula for n choose k...

$\binom{5}{3}$

$3!$

abc	abd	abe	acd	ace	ade	bcd	bce	bde	cde
acb	adb	aeb	adc	aec	aed	bdc	bec	bed	ced
bac	bad	bae	cad	cae	dae	cbd	cbe	dbe	dce
bca	bda	bea	cda	cea	dea	cdb	ceb	deb	dec
cba	dba	eba	dca	eca	eda	dcb	ecb	edb	edc
cab	dab	eab	dac	eac	ead	dbc	ebc	ebd	ecd

$$3!\binom{5}{3} = \frac{5!}{(5-3)!}$$

Outline

Pascal triangle...

Pascal triangle...

Question

There are n students. What is the number of ways of forming a team of k students out of them?

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A result...

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof of previous identity...

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- Fix one of the students, call him Ramesh
- Then there are two types of teams:

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Prove the following $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

- Fix one of the students, call him Ramesh
- Then there are two types of teams:
  - Teams without Ramesh: $\binom{n-1}{k-1}$

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Theorem

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- Then there are two types of teams:
  - Teams without Ramesh: $\binom{n-1}{k-1}$
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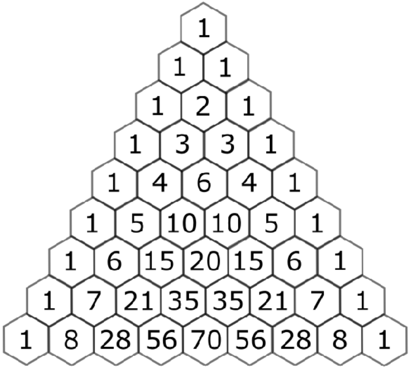
  - Apply sum rule to conclude the proof

Hence recursion for n choose k is...

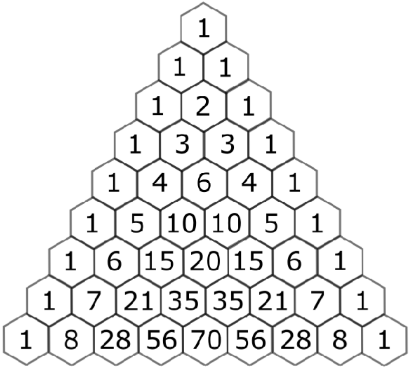
$$\binom{n}{k} = \binom{n-2}{k-2} + \binom{n-2}{k-1} + \binom{n-2}{k-1} + \binom{n-2}{k} = \cdots$$

Pascal’s Triangle

Pascal's Triangle



Pascal's Triangle



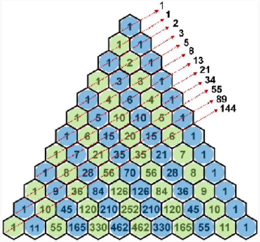
Quiz

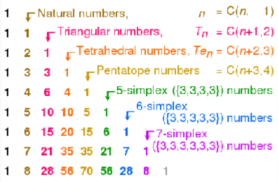
Do you know how to grow Pascal's triangle? What is the rule?

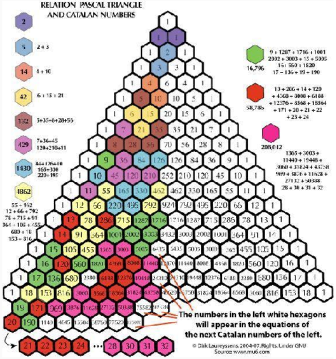
Pascal’s Triangle and Many Relations...

Pascal's Triangle and Many Relations...

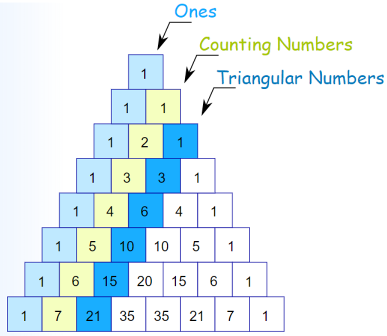
$$(a+b)^0 = 1$$
$$(a+b)^1 = 1a \quad 1b$$
$$(a+b)^2 = 1a^2 \quad 2ab \quad 1b^2$$
$$(a+b)^3 = 1a^3 \quad 3a^2b \quad 3ab^2 \quad 1b^3$$
$$(a+b)^4 = 1a^4 \quad 4a^3b \quad 6a^2b^2 \quad 4ab^3 \quad 1b^4$$
$$(a+b)^5 = 1a^5 \quad 5a^4b \quad 10a^3b^2 \quad 10a^2b^3 \quad 5ab^4 \quad 1b^5$$
$$(a+b)^6 = 1a^6 \quad 6a^5b \quad 15a^4b^2 \quad 20a^3b^3 \quad 15a^2b^4 \quad 6ab^5 \quad 1b^6$$
$$(a+b)^7 = 1a^7 \quad 7a^6b \quad 21a^5b^2 \quad 35a^4b^3 \quad 35a^3b^4 \quad 21a^2b^5 \quad 7ab^6 \quad 1b^7$$
$$(a+b)^8 = 1a^8 \quad 8a^7b \quad 28a^6b^2 \quad 56a^5b^3 \quad 70a^4b^4 \quad 56a^3b^5 \quad 28a^2b^6 \quad 8ab^7 \quad 1b^8$$



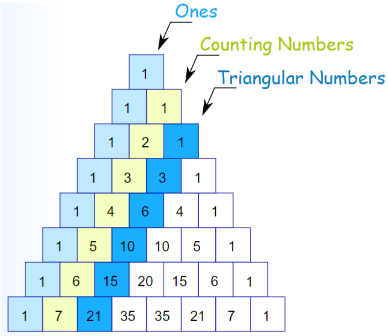


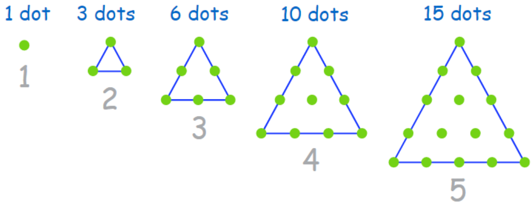


Pascal's Triangle and Triangular Numbers



Pascal's Triangle and Triangular Numbers

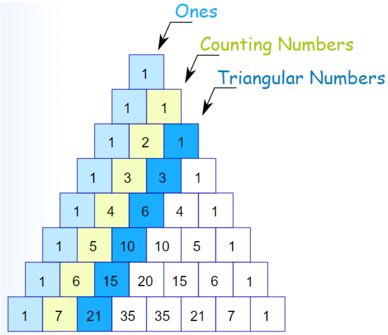


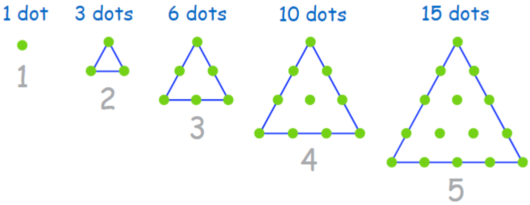


- Triangular numbers are the number of dots

- If we look at the indicated colors, we obtain counting numbers, triangular numbers, etc

Pascal's Triangle and Triangular Numbers





- Triangular numbers are the number of dots
- Add one more row and dots to get next triangular number

- If we look at the indicated colors, we obtain counting numbers, triangular numbers, etc

Pascal's Triangle: Horizontal Sums and Exponents of 11

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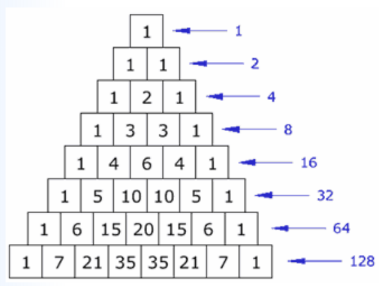
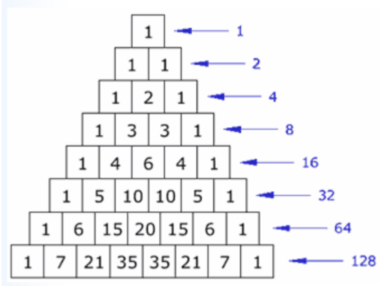
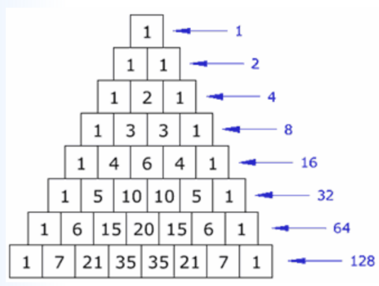


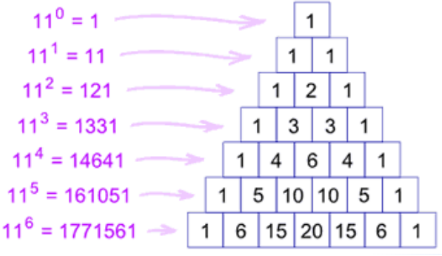
Pascal's Triangle: Horizontal Sums and Exponents of 11



- The horizontal sums are 2^i, i is the ith row

Pascal's Triangle: Horizontal Sums and Exponents of 11





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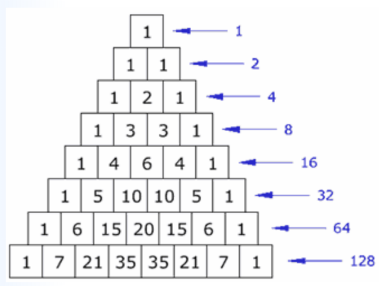
Pascal's Triangle: Horizontal Sums and Exponents of 11



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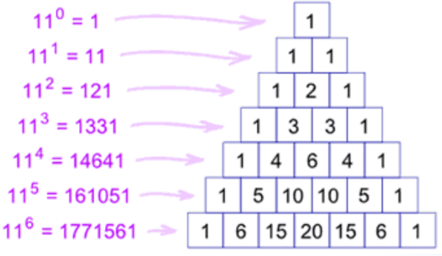
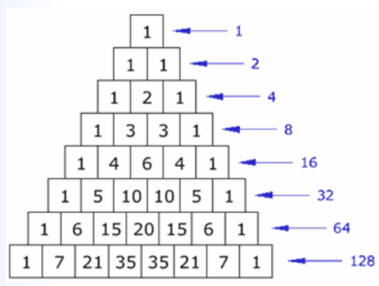


- The row entries are digits of powers of 11

Pascal's Triangle: Horizontal Sums and Exponents of 11



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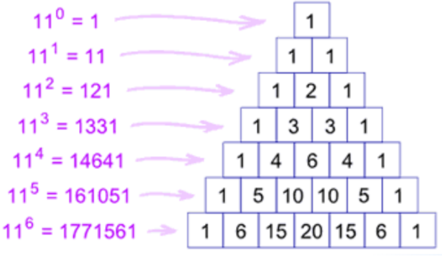
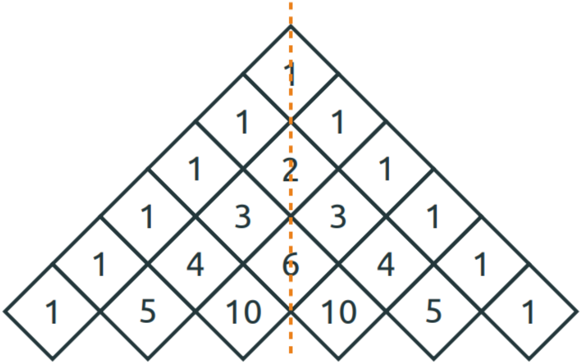


- The row entries are digits of powers of 11
- The entries of the ith row are digits of 11^i

Pascal’s Triangle and Symmetry

Pascal's Triangle and Symmetry



$$\binom{n}{k} = \binom{n}{n-k}$$

Proof of symmetry...

Theorem

Prove that

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- $\binom{n}{k}$ is the number of ways of selecting a team of size k out of n students

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Prove that

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- $\binom{n}{n-k}$ is the number of ways of selecting a team of size $n-k$ out of n students

Proof of symmetry...

Theorem

Prove that

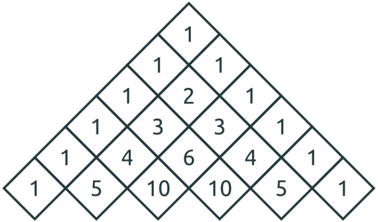
$$\binom{n}{k} = \binom{n}{n-k}$$

- $\binom{n}{k}$ is the number of ways of selecting a team of size k out of n students
- $\binom{n}{n-k}$ is the number of ways of selecting a team of size $n-k$ out of n students

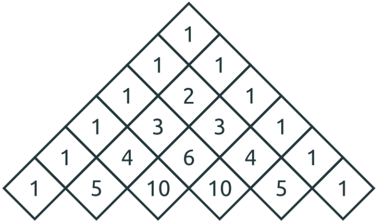
Answer

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!}$$

Row Sums of Pascal's Triangle...

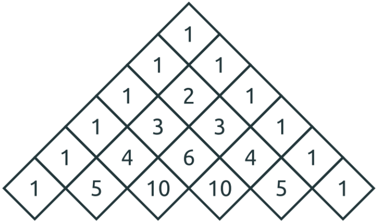


Row Sums of Pascal's Triangle...



$$\begin{array}{c} 1 \\ 1 + 1 \\ 1 + 2 + 1 \\ 1 + 3 + 3 + 1 \\ 1 + 4 + 6 + 4 + 1 \\ 1 + 5 + 10 + 10 + 5 + 1 \end{array}$$

Row Sums of Pascal's Triangle...



$$
\begin{array}{c}
1 \\
1 + 1 \\
1 + 2 + 1 \\
1 + 3 + 3 + 1 \\
1 + 4 + 6 + 4 + 1 \\
1 + 5 + 10 + 10 + 5 + 1
\end{array}
$$

Theorem

The sum of all the numbers in the n-th row of Pascal's triangle is equal to 2^n :

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Theorem (Prove this...)

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- The base case (0-th row) holds

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- The base case (0-th row) holds
- We'll show that the sum of each row is twice the sum of the previous row
- $\binom{n}{k}$ is the number of k-subsets of a set of size n
- The sum $\binom{n}{k}$ for all k (from 0 to n) is the number of all subsets of an n element set; this is 2^n by the product rule (how?)

Alternating Row Sum in Pascal

Alternating Row Sum in Pascal

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & - & 1 & & & & & = 0\\
 & & & & 1 & - & 2 & + & 1 & & & & = 0\\
 & & & 1 & - & 3 & + & 3 & - & 1 & & & = 0\\
 & & 1 & - & 4 & + & 6 & - & 4 & + & 1 & & = 0\\
1 & - & 5 & + & 10 & - & 10 & + & 5 & - & 1 & & = 0
\end{array}
$$

Alternating Row Sum in Pascal

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & - & 1 & & & & & = 0 \\
 & & & & 1 & - & 2 & + & 1 & & & & = 0 \\
 & & & 1 & - & 3 & + & 3 & - & 1 & & & = 0 \\
 & & 1 & - & 4 & + & 6 & - & 4 & + & 1 & & = 0 \\
1 & - & 5 & + & 10 & - & 10 & + & 5 & - & 1 & & = 0
\end{array}
$$

Theorem

For $n > 0$, $\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

Alternating Row Sum in Pascal

$$
\begin{array}{ccccccccccccccc}
 & & & & & & 1 & & & & & & & & \\
 & & & & & 1 & - & 1 & & & & & & & = 0 \\
 & & & & 1 & - & 2 & + & 1 & & & & & & = 0 \\
 & & & 1 & - & 3 & + & 3 & - & 1 & & & & & = 0 \\
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\end{array}
$$

Theorem

For $n > 0$, $\quad \sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$

- Hint: Number of odd size subsets = Number of even size subsets

Counting Problems...

Counting Problems...

Question

What is the number of 5-card hands dealt off of a standard 52-card deck?

Counting Problems...

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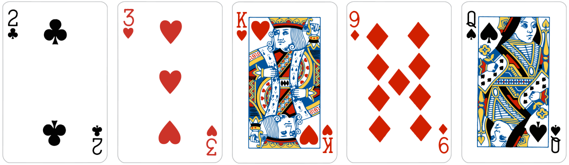

Counting Problems...

Question

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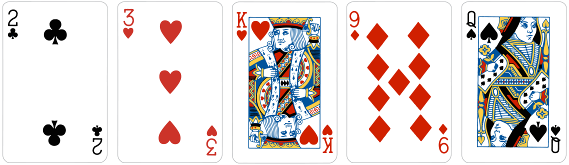

Answer

$$\binom{52}{5} = \frac{52!}{5!47!} = \frac{52 \times 52 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2598960$$

Counting Problems ...

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What is the number of 5-card hands with two hearts and three spades?

Counting Problems ...

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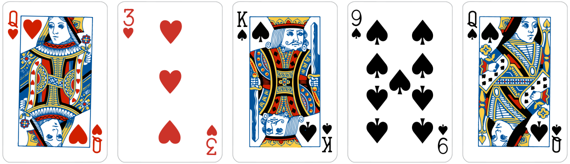
What is the number of 5-card hands with two hearts and three spades?



Counting Problems ...

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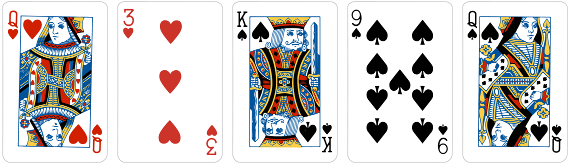
What is the number of 5-card hands with two hearts and three spades?



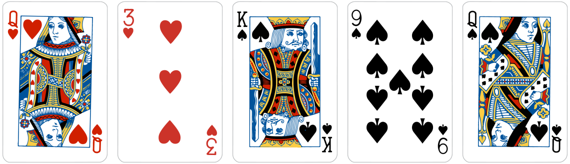
Answer

- Number of ways of picking 2 hearts from 13 hearts

Counting Problems ...

Question

What is the number of 5-card hands with two hearts and three spades?



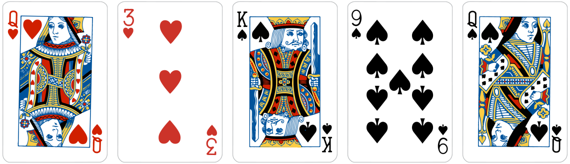
Answer

- Number of ways of picking 2 hearts from 13 hearts
- Number of ways of picking 3 spades from 13 spades

Counting Problems ...

Question

What is the number of 5-card hands with two hearts and three spades?



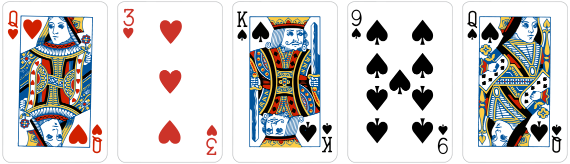
Answer

- Number of ways of picking 2 hearts from 13 hearts
- Number of ways of picking 3 spades from 13 spades
- Now apply product rule!

Counting Problems ...

Question

What is the number of 5-card hands with two hearts and three spades?



Answer

- Number of ways of picking 2 hearts from 13 hearts
- Number of ways of picking 3 spades from 13 spades
- Now apply product rule!
- The answer is: $\binom{13}{2}\binom{13}{2} = 22308$

Counting Problems...

Question

What is the number of non-negative integers with at most four digits at least one of which is equal to 7?

Counting Problems...

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- Total number of 4 digit numbers $= 10^4$
- Total number of 4 digit number that does not contain $7 = 9^4$

Counting Problems...

Question

What is the number of non-negative integers with at most four digits at least one of which is equal to 7?

- Total number of 4 digit numbers $= 10^4$
- Total number of 4 digit number that does not contain $7 = 9^4$
- Hence, the answer is $10^4 - 9^4 = 3439$

Counting Problems...

Counting Problems...

Question

What is the number of non-negative integers with at most four digits whose digits are increasing?

Counting Problems...

Question

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Counting Problems...

Question

What is the number of non-negative integers with at most four digits whose digits are increasing?

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- Hence, it is nothing but picking 4 different digits out of 10 digits!

Counting Problems...

Question

What is the number of non-negative integers with at most four digits whose digits are increasing?

- We can choose 4 different digits and we can rearrange them in increasing order
- Hence, it is nothing but picking 4 different digits out of 10 digits!
- Hence, the answer is $\binom{10}{4} = 210$

Counting Problems...

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A piece can move one step up or one step to the right. What is the number of ways of getting from the cell [0, 0] (bottom left corner) to the cell [5, 3]?

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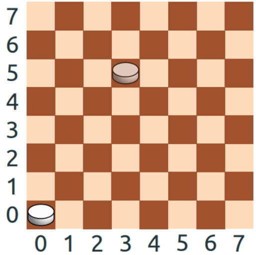

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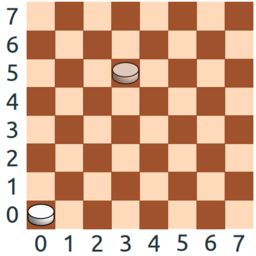

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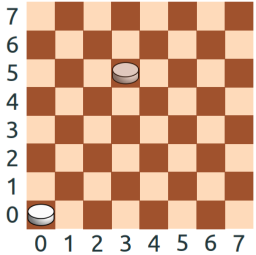

- We want to go to the cell [5,3]. How many ways we can go?
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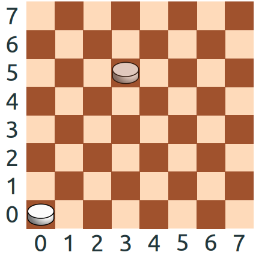

- We want to go to the cell [5,3]. How many ways we can go?
- Any path to [5,3] must involves 3 moves right and 5 moves up!
- Hence, answer is $\binom{8}{3} = 56$

Answer using Pascal's triangle...

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It is now only a matter of filling the (5,3) cell...

Combinations with or without repetitions...

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So far we have considered selections of k items out of n possible options. Consider $n = 2$ and $n = 3$ options: a, b, c

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So far we have considered selections of k items out of n possible options. The formulas we have derived are the following:

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- Let us try to find out...

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Question

So, what could be your answer?

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We have an unlimited supply of tomatoes, cucumbers, and onions. We want to make a salad out of 4 units among these three ingredients (we do not have to use all ingredients). How many different salads we can make?

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- We pick 4 items out of 3 options with repetitions
- Order does not matter; Still do not know how to count
- We will list all possible salads, then count them
- But we want to do it wisely!

Solution continued...

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Our goal: To pick 4 items out of 3 salads (Onions, Bell Peppers, Cucumber)

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- There are 15 possible combinations. Do we see any structure?

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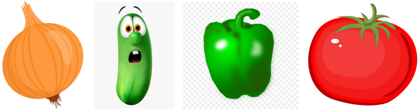

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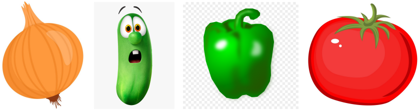

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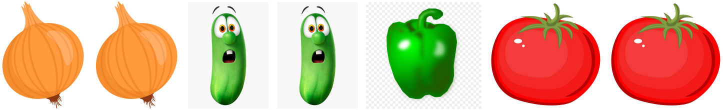

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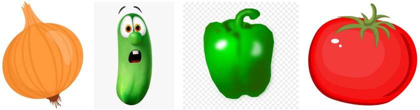

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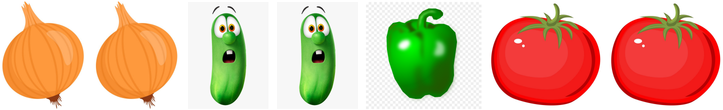

- Do you already see a way to find all possible combinations?

Combinations with Repetitions...

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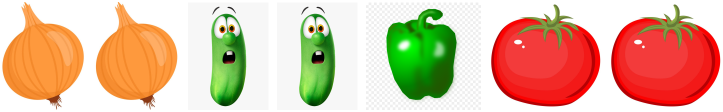

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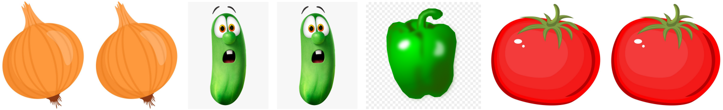

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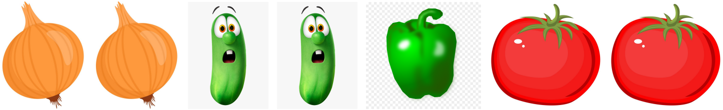

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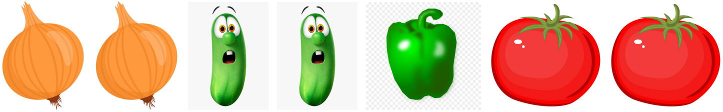

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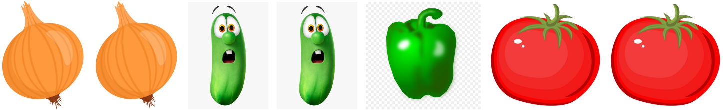

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- The number of ways we can put the delimiters determine the number of combinations

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Recall, we want to put delimiters...

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- Total number of objects (7 items + 3 delimiters) is 10

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- Form the example above, we need 3=(4-1) delimiters!
- Total number of objects (7 items + 3 delimiters) is 10
- The problem now reduces to arranging 3 delimiters among 10 items! Voila!

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The formula for number of combinations of size k of n objects with repetitions is

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Outline

Distributing Assignments Among People

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Suppose there are 4 people and 9 different assignments. Each person should receive one assignment. Assignments for different people should be different. How many ways are there to do it?

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Quiz

Where does this problem fit in our combination and permutation table?

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- Here assignments and people are different: so a ordered case
- No repetitions: same assignment is not given to two persons
- It is a case of ordered without repetitions, i.e., $k-$permutations

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- The answer is $\left(\dfrac{9!}{(9-4)!}\right) = 9 \times 8 \times 7 \times 6 = 3024$

Distributing Assignments Among People: Twist

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Problem with a twist!

There are 4 people and 9 different assignments. We need to distribute all assignments among people. No assignment should be assigned to two people. Every person can be given arbitrary number of assignments from 0 to 9. How many ways are there to do it?

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Quiz

Which category this problem belongs to? 

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- Each person receives several assignments; look at one by one
- First person can be assigned arbitrary number of assignments
  - It is a case of counting all possible subsets of assignments
- Same assignment can't be given to two persons, the number of subsets of assignments for second person depends on what we chose for first person! How to attack this problem?

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Options	4	4	4	4	4	4	4	4	4

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Options	4	4	4	4	4	4	4	4	4

- There are $4^9 = 262144$ choices! This was a case of Tuples!

Quiz-1

https://tinyurl.com/y2g93ofb

Distributing Candies Among Kids

Distributing Candies Among Kids

Problem

There are 15 identical candies. How many ways are there to distribute them among 7 kids?

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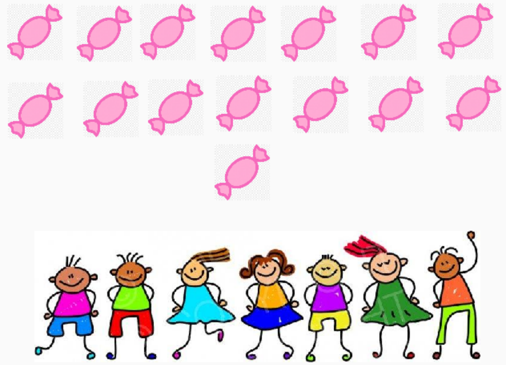

Distributing Candies Among Kids

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Analyzing the problem, and fitting it into the known case

- When distributing candies, we pick one of 7 kids

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- Since the candies to be distributed are identical, ordering doesn't matter

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- Since all the candies are same, repetitions are allowed
- Since the candies to be distributed are identical, ordering doesn't matter
- This is a case of combinations with repetitions

Distributing Candies Among Kids: Solution Finally!

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Analyzing the problem, and fitting it into the known case

- Number of combinations = size of combination
- Number of options = Number of kids
- The answer is $\binom{15+(7-1)}{(7-1)} = \binom{21}{3} = 54264$

Fair Distributions..

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A problem with previous distribution...

No kid will like not having even a single candy. Infact, a kid would like to have it all! How can we change the problem?

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A problem with previous distribution...

We want that each kid receives atleast one candy!

Let us change our previous problem so that each kid is ensured atleast one candy!

Distributing Candies with Fair Distribution...

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- Does our previous approach work here? What should you do?

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Quiz

- Does our previous approach work here? What should you do?
- Can we reduce this problem to previous problem? If yes, how?

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- Give kids atleast one candy. Then we are left with 15-7=8 candies...

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- Now we can distribute these 8 candies as in previous problem!
- It becomes a problem of Combinations with Repetitions
  - #combinations = 8;   #options = 7
- Answer $= \binom{8+(7-1)}{(7-1)} = \binom{14}{6} = 3003$

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Question

Assume that 15 identical candies are distributed among 7 kids. How many ways are there that will leave some kids without any candy?

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Hint

Use the answer to previous two problems!

Numbers with Fixed Sum of Digits

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- But what next? How many options for the 2nd digit?

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Understanding the problem...

- It is clear that we have nine options for the 1st digit
- But what next? How many options for the 2nd digit?
- It is already tricky...

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- $\#\text{combinations} = 9; \quad \#\text{options} = 4$

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- Multiple ones will go to at least one of the four places, and get added
- This belongs to the category: "unordered with repetitions"
- $\#\text{combinations} = 9; \quad \#\text{options} = 4$
- Hence, answer is $\binom{9+(4-1)}{(4-1)} = \binom{12}{3} = 220$

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- Hence, answer is $\binom{10+(4-1)}{(4-1)} = \binom{13}{3} = 286$

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- Hence, answer is $\binom{10+(4-1)}{(4-1)} = \binom{13}{3} = 286$
- But this is wrong! Why? The answer is off by 4!

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- When we assign all 10 ones to one position, but digits can be only upto 9

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- We need to substract the additional cases! How?

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- Hence, final correct answer = 286-4 = 282

Numbers with Non-increasing Digits Problem

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- If we try to count options for each position and apply the product rule, there are problems
- 10 options for the first position, but for the second the number of options depends on the first number
- Usual approach doesn't work; try from another angle!

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- We pick digits from 0 to 9 to be in our number

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- We pick digits from 0 to 9 to be in our number
- Once we picked four digits, our number is uniquely determined
- Order of picks does not matter; repetitions are allowed
- $\#\text{combinations} = 4; \quad \#\text{options} = 10$
- The answer is $\binom{4+(10-1)}{(10-1)} = \binom{13}{4} = 715$

Group Work

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Fact

Group learning is a wonderful way to learn and share knowledge.

Splitting into Working Groups

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- This problems is more tricky; There are several ways to solve it
- But we need to combine several ideas
- How do we approach this problem?

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- For the second group we have 10 people left, there are $\binom{10}{2}$ options, so on

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- So, overall we have the following options

$$\binom{12}{2} \times \binom{10}{2} \times \binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2}$$

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- Is this correct answer?

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- Is this correct answer? No! Why?

- Let us number the students...

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- According to our enumeration, we could have the following

$$\{3,7\},\{1,5\},\{6,9\},\{11,2\},\{8,12\},\{4,10\}$$

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- That is, ordering across groups also does not matter!

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- That is, ordering across groups also does not matter!
- We had 6! redundant splitting, which we need to divide by
- We have counted each assignment 6! times, hence, after dividing, the answer is

$$\binom{12}{2} \times \binom{10}{2} \times \binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2} \times \frac{1}{6!} = \frac{12!}{6! \times 2^6} = 10395$$