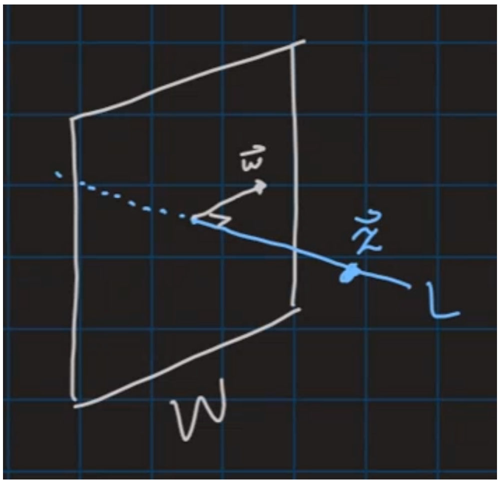
Last lecture we started an example about orthogonality with a plane and a vector, here's that example...

Example

let W be a plane through the origin in $\mathbb{R}^3$, and L be the line through the origin and orthogonal to W.

If $\vec{z}$ and $\vec{w}$ are nonzero, $\vec{z}$ is on L, and $\vec{w}$ is in W and $\vec{z} \cdot \vec{w} = 0$.



L consists of all vectors orthogonal to W, and W consists of all vectors orthogonal to L.

We can describe their relationship with the notation

$$L = W^{\perp} \quad \text{or} \quad W = L^{\perp}$$

which reads "L is W perpendicular" or "W perp" for short.

If W is a subspace of $\mathbb{R}^n$, then

1. a vector $\vec{x}$ is in $W^{\perp}$ if, and only if, it is orthogonal to every vector in a set that spans W.
2. $W^{\perp}$ is a subspace of $\mathbb{R}^n$

This leads to a theorem...

Theorem 1 *Let $A \in \mathbb{R}^{m \times n}$. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T.*

$$(Row\ A)^{\perp} = Nul\ A \qquad (Col\ A)^{\perp} = Nul\ A^T$$

Dillhoff goes over a paper that's relevant but won't be on homework or exams so I won't go over it here. Here's a link to the lecture.

1 Orthogonal Basis

What is notable about the standard basis vs. one that is skewed?

A set of vectors $\{\vec{u_1}, \cdots, \vec{u_n}\}$ is said to be an orthogonal set if each pair of distinct is orthogonal to each other.

$$\vec{u_i} \cdot \vec{u_j} = 0 \qquad i \neq j$$

Example

Take the standard basis

$$\vec{e_1} = (1, 0, 0) \quad \vec{e_2} = (0, 1, 0) \quad \vec{e_3} = (0, 0, 1)$$

We can verify that each vector is orthogonal to the others. These are pretty simple so I'll just show one:

$$\vec{e_1} \cdot \vec{e_2} = (1)(0) + (0)(1) + (0)(0) = 0$$

A lot of examples are just like this. To verify, we would just need to check the dot product of each pair of vectors.

Theorem 2 *If $S = \{\vec{u_1}, \cdots, \vec{u_n}\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and is a basis for the subspace spanned by S.*

An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

Theorem 3 *Let $\{\vec{u_1}, \cdots, \vec{u_n}\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\vec{y} \in W$, the weights of the linear combination*

$$\vec{y} = c_1\vec{u} + \cdots + c_n\vec{u_n}$$

are given by

$$c_j = \frac{\vec{y} \cdot \vec{u_j}}{\vec{u_j} \cdot \vec{u_j}} \qquad (j = 1, \cdots, n)$$

Example

Let

$$S = \begin{bmatrix} \vec{u_1} & \vec{u_2} & \vec{u_3} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

be an orthogonal basis. Express $\vec{y} = (2, 3, 19)$ as a linear combination of the vectors of S. We could use the matrix equation to solve for this but there's another way. We can use the theorem above to get

$$\begin{array}{ccc} \vec{y} \cdot \vec{u_1} = 24 & \vec{y} \cdot \vec{u_2} = 18 & \vec{y} \cdot \vec{u_3} = -16 \\ \vec{u_1} \cdot \vec{u_1} = 3 & \vec{u_2} \cdot \vec{u_2} = 6 & \vec{u_3} \cdot \vec{u_3} = 2 \end{array}$$

So we could express $\vec{y}$ as

$$\vec{y} = \frac{\vec{y} \cdot \vec{u_1}}{\vec{u_1} \cdot \vec{u_1}}\vec{u_1} + \frac{\vec{y} \cdot \vec{u_2}}{\vec{u_2} \cdot \vec{u_2}}\vec{u_2} + \frac{\vec{y} \cdot \vec{u_3}}{\vec{u_3} \cdot \vec{u_3}}\vec{u_3}$$

$$= 8\vec{u_1} + 3\vec{u_2} - 8\vec{u_3}$$

So $\vec{x} = (8, 3, -8)$.

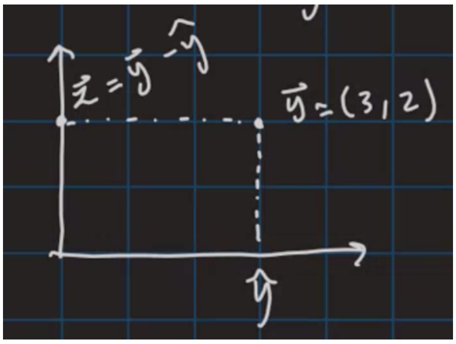
2 Orthogonal Projection

Let's say we want to decompose a vector $\vec{y}$ into the sum of 2 vectors

1. a multiple of $\vec{u}$
2. one orthogonal to $\vec{u}$

$$\vec{y} = \hat{y} + \vec{z}$$

Where $\hat{y} = \alpha\vec{u}$ for some $\alpha \in \mathbb{R}$ and $\vec{z}$ is orthogonal to $\vec{u}$.



Note: Dillhoff didn't write it but we can call the horizontal axis $\vec{u}$.

Let $\vec{z} = \vec{y} - \alpha\vec{u}$. Then $\vec{y} - \hat{y}$ is orthogonal to $\vec{u}$ if, and only if,

$$\vec{0} = (\vec{y} - \alpha\vec{u}) \cdot \vec{u} = \vec{y} \cdot \vec{u} - (\alpha\vec{u}) \cdot \vec{u} = \vec{y} \cdot \vec{u} - \alpha(\vec{u} \cdot \vec{u})$$

$\vec{y} = \hat{y} + \vec{z}$ is satisfied with $\vec{z}$ orthogonal to $\vec{u}$ if, and only if

$$\alpha = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \quad \text{and} \quad \hat{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\,\vec{u}$$

We call $\hat{y}$ the **orthogonal projection** of $\vec{y}$ onto $\vec{u}$.

$$\hat{y} = \text{proj}_L\vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\,\vec{u}$$

This can allow us to compute the closest point to a given subspace $\vec{u}$.

Visualization