CSE 3380 - Solutions of Non-Homogeneous Systems and Matrix Operations
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1 Review

We left off last time with parametric vector form. For example, given

$$10x_1 - 3x_2 - 2x_3 = 0$$

We solve for x_1

$$x_1 = .3x_2 + .2x_3$$

We can write the result in parametric vector form, writing all the free variables (x_2, x_3) as coefficients of vectors.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} .2x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$\underbrace{x_2 \begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix}}_{\text{parametric vector form}}$$

In general,

$$\vec{x} = s\vec{u} + t\vec{v} \qquad (s, t \in \mathbb{R})$$

2 Solutions of Non-Homogeneous Systems

So $A\vec{x} = \vec{b}$, where $\vec{b} \neq \vec{0}$

2.1 Example

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & 8 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

First, throw it into an augmented matrix,

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & 8 & -4 \end{bmatrix} \begin{matrix} \sim \\ +R1 \\ -2\cdot R2 \end{matrix} \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{bmatrix} \begin{matrix} \\ \sim \\ +3\cdot R1 \end{matrix} \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \sim \\ \cdot\frac{1}{3} \\ \\ \end{matrix}$$

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} -5\cdot R2 \\ \sim \\ \\ \end{matrix} \begin{bmatrix} 3 & 0 & -4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \cdot\frac{1}{3} \\ \sim \\ \\ \end{matrix} \begin{bmatrix} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The solution set can be written as

$$\begin{cases} x_1 = \frac{4}{3}x_3 - 1 \\ x_2 = 2 \\ x_3 \text{ is free} \end{cases}$$

Or, it can be written in parametric vector form,

$$\vec{x} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\vec{p}} + x_3 \underbrace{\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}}$$

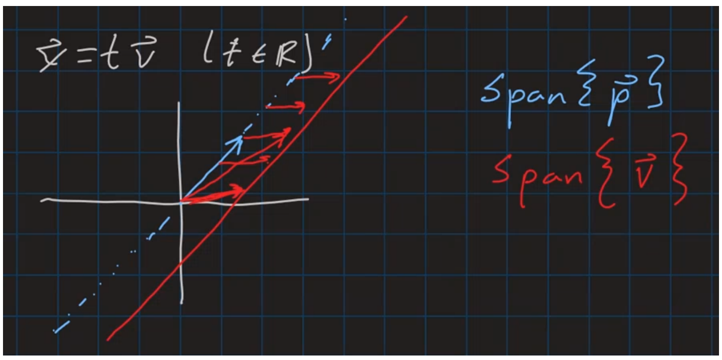
Generally,

$$\vec{x} = \vec{p} + x_3\vec{v}$$

For a non-homogeneous system like the one above, the solution will be of the form $\vec{x} = \vec{p} + t\vec{v}$ where t is a scalar. For a homogeneous system, we'll get $\vec{x} = t\vec{v}$ where t is a scalar.

Note: Translation

For the following screenshot, we have $\vec{x} = s\vec{p} + \vec{v}$. (Ignore the $\vec{x} = t\vec{v}$ that he wrote in the corner).

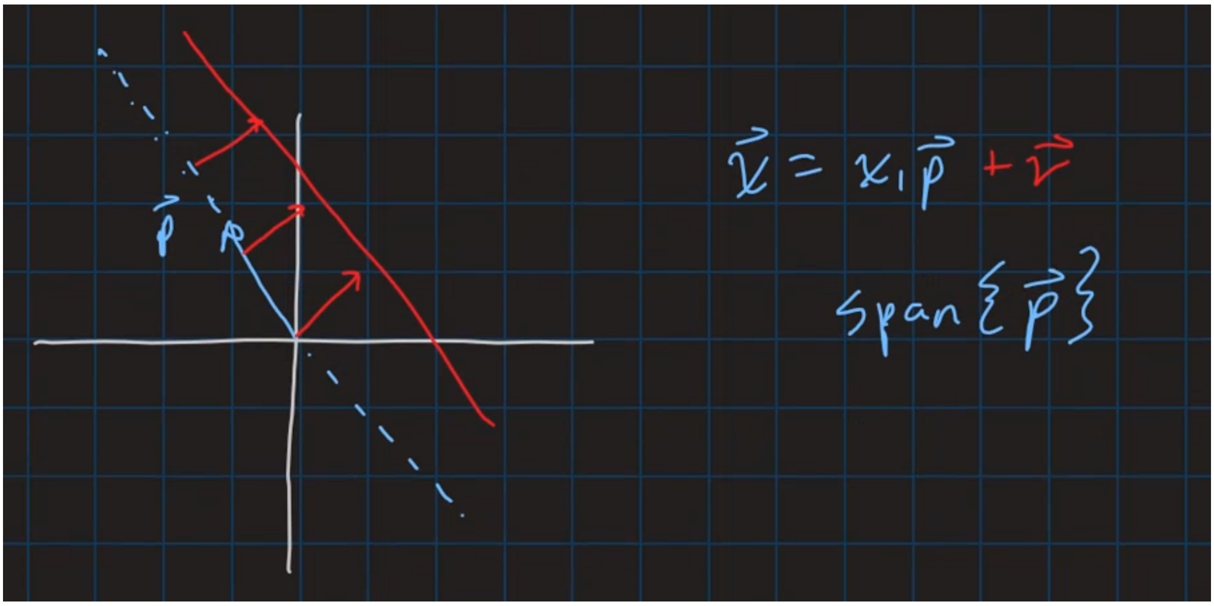


The Span$\{\vec{p}\}$ is the line formed by scaling the $\vec{p}$ vector (blue line). By adding the vector $\vec{v}$, we **translate** the line. Because $\vec{v}$ is not scaled, it forms a line, not a plane.

Theorem 1 *Suppose $A\vec{x} = \vec{b}$ is consistent for some $\vec{b}$, and let $\vec{p}$ be a solution. The solution set of $A\vec{x} = \vec{b}$ is the set of all vectors of the form*

$$\vec{w} = \vec{p} + \vec{V_n}$$

where $\vec{V_n}$ is any solution of the homogeneous system $A\vec{x} = \vec{0}$.



Example

He gives a matrix, but I won't show the steps here because it's not relevant. Long story short, there's 6 variables and we end up with this solution set

$$\begin{cases} x_1 = -5x_2 - 8x_4 - x_5 \\ x_3 = 7x_4 - 4x_5 \\ x_6 = 0 \\ x_2, x_4, x_5 \text{ are free} \end{cases}$$

We'll convert to parametric form,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8 \\ 0 \\ 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

3 Matrix Operations

The general form of an $m \times n$ (row $\times$ column) matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

$\vec{a}_i$ would be a column vector.

a_{ij} would be a scalar.

Main Diagonal

a_{11} to a_{ij} to a_{mn} is the main diagonal, entries of a_{ij} where $i = j$.

Diagonal Matrix

A square matrix $(n \times n)$ whose non-diagonal entries are 0.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Zero Matrix

$a_{ij} = 0$ for all i, j.

Equality

Two matrices A, B are equal if they are the same size and have the same entries

Sums and Scalar Multiples

This is an extension of vector sums and scalars.

Sums

If $A \in \mathbb{R}^{\mathrm{m \times n}}$, $B \in \mathbb{R}^{\mathrm{m \times n}}$, then $A + B$ is an $m \times n$ matrix whole columns are the sums of the corresponding columns from A and B.

Example

$$A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$$

We can't add $A + C$ or $B + C$ because of the size.

Scalar Multiples

$$r \in \mathbb{R}$$
$$rA = \begin{bmatrix} r\vec{a}_1 & r\vec{a}_2 & \cdots & r\vec{a}_n \end{bmatrix}$$

Properties of Matrix Operations

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$

$$A + B = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} + \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} (a_1 + b_1) & (a_2 + b_2) & \cdots & (a_n b_n) \end{bmatrix}$$