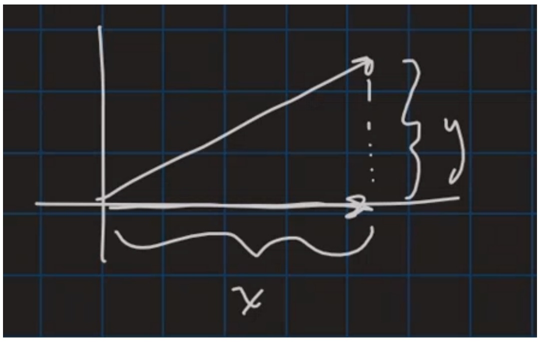
1 Projection (Cont.)

A projection is a vector that is projected down into a lower dimension.

For a 2D vector, we could just remove the y value, leaving us with a 1D line segment along the x-axis



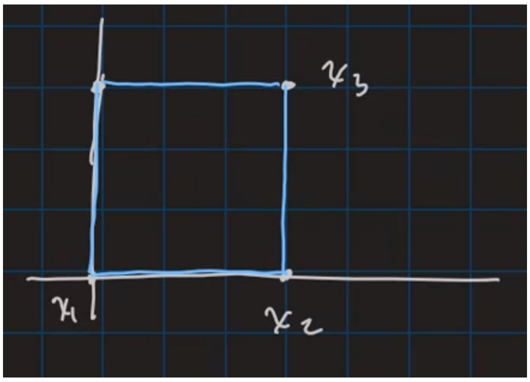
This is common in computer graphics, where we may want to project a 3D image onto a 2D plane, like we're rendering a frame.

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We can use A to project a 3D vector onto a 2D plane in $\mathbb{R}^3$ (not $\mathbb{R}^2$).

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

Example: Shearing

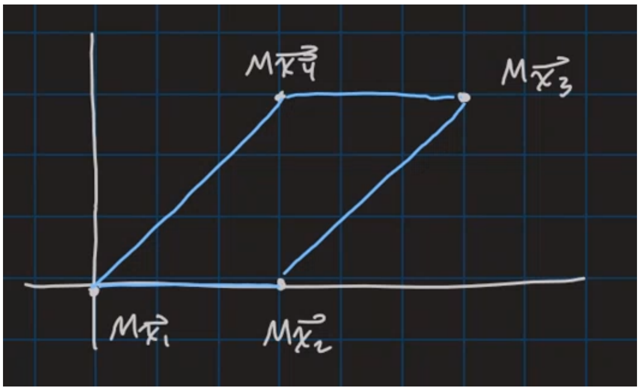
We're going to make a square out of 4 points.



We'll have a transformation matrix $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. We're going to calculate $M\vec{x_1}$, $M\vec{x_2}$, $\cdots$

$$M\vec{x_1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$M\vec{x_2} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$
$$M\vec{x_3} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$
$$M\vec{x_4} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Now we have new values for x_i, which we can plot.



Linear Transformations: What Makes it Linear?

Let $A \in \mathbb{R}^{m \times n}$ matrix.

These properties are useful

- $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$
- $A(c\vec{u}) = cA\vec{u}$

A transformation T is **linear** if

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}$, $\vec{v}$ in the domain of T.
2. $T(c\vec{u}) = cT(\vec{u})$ for all c and $\vec{u}$ in the domain of T.

$\therefore$ every matrix transformation is a linear transformation. (the reverse is not true)

If T is a linear transformation

1. $T(\vec{0}) = \vec{0}$
2. $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$

From (2), we can generalize to the following

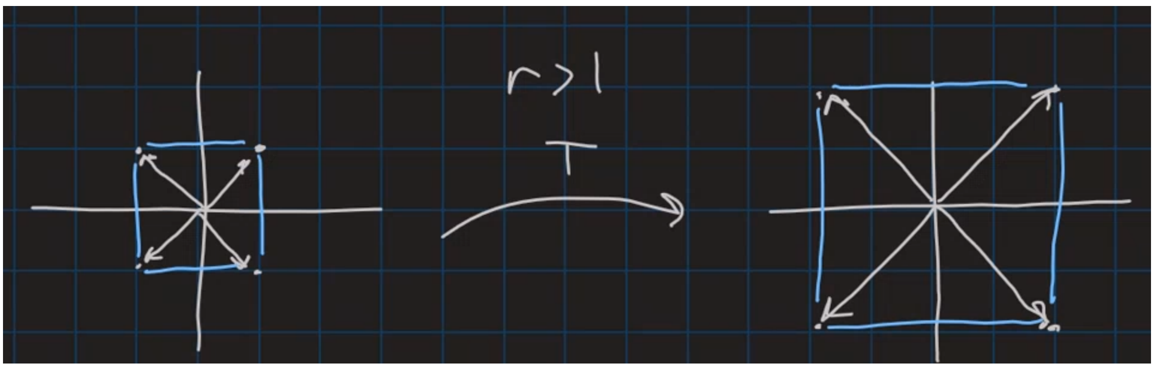
$$T(c_1\vec{v_1} + \cdots + c_p\vec{v_p}) = c_1T(\vec{v_1}) + \cdots + c_pT(\vec{v_p})$$

Example: Contraction and Dilation

Define a transform $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{x}) = r\vec{x}$ where $r \in \mathbb{R}$.

When $0 \leq r < 1$, T is called a **contraction**

When $r > 1$, T is called a **dilation**



Dilation

Show that T is a linear transformation.

Let $\vec{u}$, $\vec{v} \in \mathbb{R}^2$ and $c, d \in \mathbb{R}$, then by the definition of this transformation

$$T(c\vec{u} + d\vec{v}) = r(c\vec{u} + d\vec{v})$$

$$= rc\vec{u} + rd\vec{v}$$

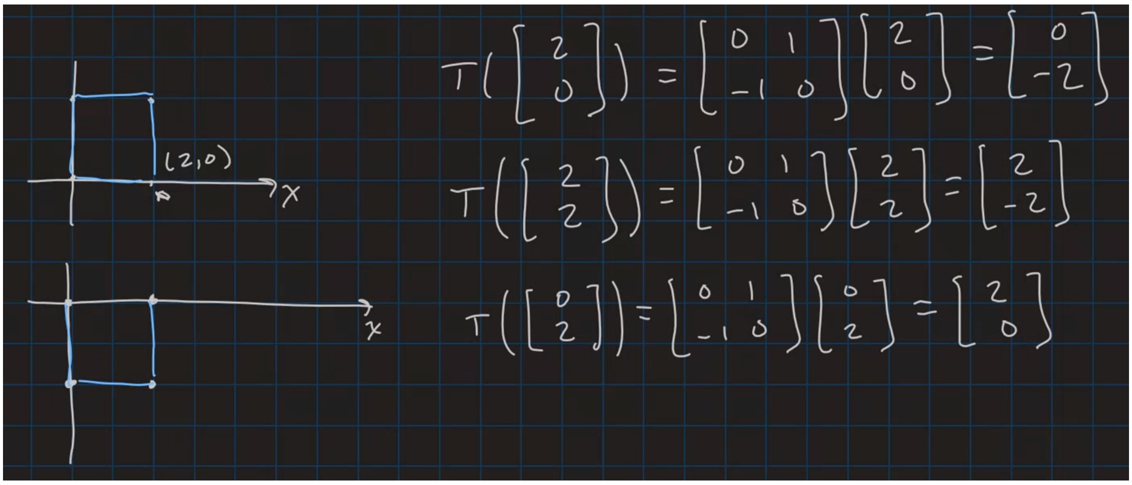
$$c(r\vec{u}) + d(r\vec{v})$$

$$cT(\vec{u}) + dT(\vec{v})$$

Example: Rotation by 90°

$T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$

Find the images of the vectors $\vec{v_i}$ (shown in image) under T.



Example: Dimensionality Reduction

It's important to try to reduce high dimension data to lower dimensions for efficiency and space in computer graphics.

Given a 2×3 vector of weights

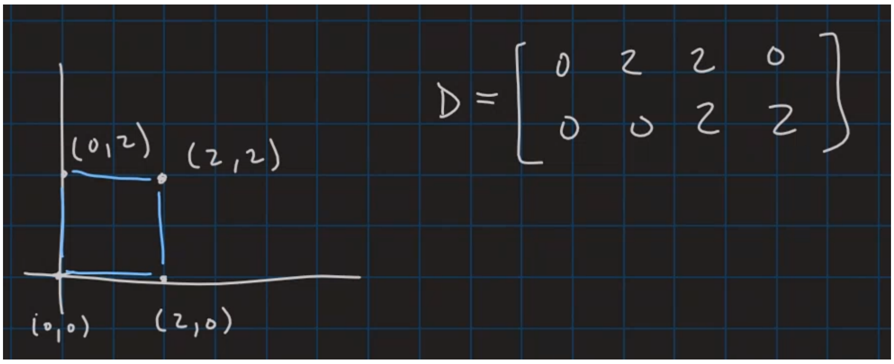
$$\begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

In this case, don't think about $\vec{x}$ as a vector in space, but a set of features. For example, x_1 could be the color of a shape, and x_2 is the size and x_3 is the shape itself.

If x_1 is the shape color and all shapes are blue, we could set $w_{11} = 0$ and $w_{21} = 0$. This would set all x_1 to 0. Since all shapes in our set are blue, we don't need to store this because it's a given.

2 Computer Graphics

When representing something like a shape in 2D space, we can use a matrix of points.



There are 3 common transformations

1. Scaling — making objects bigger or smaller
2. Rotation — changing the orientation of objects
3. Translation — moving objects

Scaling

In 2D, a matrix that scales a point (or vertex) by some values

$$S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

Side note: in computer graphics, objects are typically normalized.

Rotation Matrices

We can rotate something counter clockwise with

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

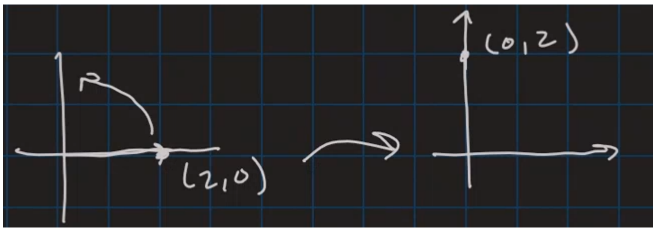
Example

What translation matrix would be need to rotate something by 90° ($\frac{\pi}{2}$) counter clockwise

$$R\left(\frac{\pi}{2}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Applying that to some arbitrary x and y

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$



We'll expand on this in the next lecture (L21).