1 Review: Linear Combination

Let $\vec{x_1}$, $\vec{x_2}$, $\cdots$, $\vec{x_k}$ be vectors in a linear space K and α_1, α_2, $\cdots$, α_k be numbers in K.

$$\vec{y} = \alpha_1\vec{x_1} + \cdots + \alpha_k\vec{x_k}$$

If $\vec{y} = \vec{0}$, then the vectors $\vec{x_i}$ are linearly independent.

Special Cases

- A set with 1 vector $\vec{v}$ is linearly independent if, and only if, $\vec{v} \neq \vec{0}$.
- A set with 2 vectors in linearly independent if, and only if, one vector is not a multiple of the other.
- A set containing $\vec{0}$ is linearly dependent.

Theorem 1 *An indexed set $\{\vec{v_1}, \cdots, \vec{v_p}\}$ of two or more vectors with $\vec{v_1} \neq \vec{0}$ is linearly dependent if, and only if, some $\vec{v_j}$ (with $j > 1$) is a linear combination of the preceding vectors $\vec{v_1}$, $\cdots$, $\vec{v_{j-1}}$.*

Example

Let

$$f_1(t) = 3$$
$$f_2(t) = \frac{t}{2}$$
$$f_3(t) = t + 9$$

is $\{f_1, f_2, f_3\}$ linearly independent?

We can write $f_3(t)$ as a linear combination of the previous two vectors (functions)

$$f_3(t) = 2f_2(t) + 3f_1(t)$$

So *no*, the set is not linearly independent.

Example

The set $\{\sin t, \cos t\}$ is linearly independent in $C[0, 1]$ – the space of all continuous functions.

There is no such scalar c such that

$$\cos t = c \cdot \sin t \qquad \forall\, t \in [0, 1]$$

This make sense if you think about it graphically. There is no c that you can scale $\sin t$ by to get $\cos t$. The peaks and valleys are always in the wrong place. Because we can't reach one as a linear combination of the other, they are linearly independent.

2 Basis

Let H be some subspace of a vector space V. A set of vectors B in V is a **basis** for H if

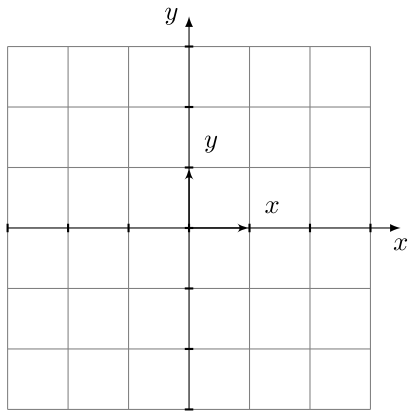
(i) B is a linearly independent set

(ii) The subspace spanned by B coincides with H

$$H = \text{Span}\{B\}$$

A basis we all already know is the standard coordinate system, or Cartesian coordinate system, with x, y or $\hat{i}$, $\hat{j}$

Cartesian Coordinate Plane



$$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Any other point in this space is a linear combination of x and y.

Relation to Invertible Matrices

If $A \in \mathbb{R}^{n \times n}$ and is invertible, then cols(A) form a basis for $\mathbb{R}^n$. They are linearly independent by IMT (invertible matrix theorem).

Let $I_n \in \mathbb{R}^{n \times n}$ be the $n \times n$ identity matrix and $\vec{e}_1, \cdots, \vec{e}_n$ be the cols(I_n). Then $\{\vec{e_1}, \cdots, \vec{e_n}\}$ is called the **standard basis** for $\mathbb{R}^n$.

Example

Let $\vec{v_1} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$

Is $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ a basis for $\mathbb{R}^3$?

We can use row reduction to get to an identity matrix. If we reach I_3, then it is a basis.

$$\begin{bmatrix} 0 & 1 & 4 \\ 1 & 3 & 3 \\ 3 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 4 \\ 3 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

I've skipped a few steps, but we can actually stop here. We can see that we have n (3) pivot positions, which means this system is consistent and I_3 can be reached, so the vectors $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ do form a basis for $\mathbb{R}^3$.

Example

This is a popular example. I don't really understand it.

Let $S = \{1, t, t^2, \cdots, t^k\}$. Verify that S is a basis for $\mathbb{P}^k$ (space of polynomial functions).

By definition, S spans $\mathbb{P}^k$. We need to show that S is linearly independent. Show that the coefficients $C_0, \cdots, C_k$ satisfy

$$C_0 1 + C_1 t + C_2 t^2 + \cdots + C_k t^k = \vec{0}(t)$$

Where $\vec{0}(t)$ is called the **zero polynomial**.

The only polynomial with more 0 values than its degree is the zero polynomial. The eq. above holds for t only if $C_0 = \cdots = C_k = 0$. By definition of linear independence, S is linearly independent and is a basis for $\mathbb{P}^k$.

Theorem 2 (Spanning Set Theorem) *Let $S = \{\vec{v_1}, \cdots, \vec{v_p}\}$ be a set in a linear space V, and $H = Span\{\vec{v_1}, \cdots, \vec{v_p}\}$.*

(a) if one of the vectors in S is a linear combination of the others, then the set formed from removing the vector still spans H.

(b) If $H \neq \{\vec{0}\}$, some subset of S is a basis for H. (S is a basis for it's own Span)

Example

Let $\vec{v_1} = \begin{bmatrix} 3 \\ -4 \\ -4 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 1 \\ 7 \\ -4 \end{bmatrix}$ and $H = \text{Span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$.

We're going to skip some steps here. $\vec{v_3} = 2\vec{v_1} - \vec{v_2}$. Show that $\text{Span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}\} = \text{Span}\{\vec{v_1}, \vec{v_2}\}$, then find a basis for H.

Every vector in $\text{Span}\{\vec{v_1}, \vec{v_2}\}$ belongs in H. Because $\vec{v_3}$ is a linear combination of the first two, we can exclude it from the span and still have the same span.

We can say that

$$\begin{aligned} \vec{x} &= c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} \\ &= c_1\vec{v_1} + c_2\vec{v_2} + c_3(2\vec{v_1} - \vec{v_2}) \\ &= (c_1 + 2c_3)\vec{v_1} + (c_2 - c_3)\vec{v_2} \end{aligned}$$

This is just a linear combination of $\vec{v_1}$ and $\vec{v_2}$. Thus, $\vec{x} \in \text{Span}\{\vec{v_1}, \vec{v_2}\}$ as is every vector in H.