Remember from last lecture's notes:

A vector space is a nonempty set V over a field of objects, called vectors, on which two operations are defined: (1) addition and (2) scalar multiplication.

1 Subspaces

Just as a subset is a smaller "slice" of a set (eg. $A \subset B$), a subspace is a smaller slice of a vector space. Suppose a set of elements L of a vector space K has the following properties:

a) if $\vec{x} \in L$, $\vec{y} \in L$, then $\vec{x} + \vec{y} \in L$

b) if $\vec{x} \in L$ and λ is an element of the field K, then $\lambda\vec{x} \in L$

We need to verify these two properties over the subspace just as we did for vector spaces

Verify the Zero Vector of L

let $\vec{x}$ be any element of L. Then $\lambda\vec{x} \in L$ for every $\lambda \in K$ (property b)

We choose

$$\lambda = 0 \implies \lambda\vec{x} = \vec{0} \in L$$

Verify Negative Element

Let $\lambda = -1$, then $(-1)\vec{x} = -\vec{x}$.

Let $\vec{y} = (-1) \cdot \vec{x}$.

$\vec{x} + \vec{y} = 1 \cdot \vec{x} + (-1) \cdot \vec{x} = (1 - 1) \cdot \vec{x} = 0$.

So by property a, $\vec{x} + \vec{y} \in L$, therefore $-\vec{x}$ is the negative element of $\vec{x}$

In general, every set $L \subset K$ that satisfies properties (a) and (b) is called a **linear subset** (or **linear subspace**)

a) if $\vec{x} \in L$, $\vec{y} \in L$, then $\vec{x} + \vec{y} \in L$

b) if $\vec{x} \in L$ and λ is an element of the field K, then $\lambda\vec{x} \in L$

Trivial Examples

The simplest possible linear subspace is

$$L = \left\{\vec{0}\right\}$$

which is the **zero subspace**. Here is another trivial subspace

$$K \subseteq K$$

Geometric Interpretation of a Subspace

Let V_3 be the 3D space. We might say that all vectors parallel to a plane or a line form a subspace. Take the 3 dimensional space of x_1, x_2, and x_3. We could say that every point on the plane of x_1 and x_2 is a subspace of V_3. Or we could say that every point on the line of x_3 is a subspace of V_3. Our subspaces must contain the zero vector $\vec{0}$.

If these subspaces did not go through the origin, they would not be subspaces.

Is $\mathbb{R}^2$ a subset of $\mathbb{R}^3$?

In $\mathbb{R}^2$,

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

which means we have a structure (a vector of 2 elements) such that x and y are both real numbers. For $\mathbb{R}^3$

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

These may look similar at first, but there's no way we could go from an element in $\mathbb{R}^2$ to an element in $\mathbb{R}^3$, because the $\mathbb{R}^2$ element has no concept of z. We can try to zero out an element of $\mathbb{R}^3$ to emulate $\mathbb{R}^2$, but that doesn't quite work:

$$\left\{ \begin{bmatrix} 0 \\ y \\ z \end{bmatrix} : y, z \in \mathbb{R} \right\} \neq \mathbb{R}^2$$

We can make a subset of $\mathbb{R}^3$ that looks like $\mathbb{R}^2$ (a plane), but it is not.

A Subspace Spanned by a Set

Let V be a vector space and $H = \text{Span}\{\vec{v_1}, \vec{v_2}\}$. Show that $H \subset V$.

The first thing we'll ask is if the zero vector is in this space. We know that if we add 2 vectors of a span, the sum will also be in that span. We can show this with

$$\vec{0} = 0\vec{v_1} + 0\vec{v_2}$$

To prove addition, we'll create 4 arbitrary coefficients s_1, s_2, t_1, and t_2. We'll use these to create 2 new vectors within the span from $\vec{v_1}$ and $\vec{v_2}$. We need to show that a linear combination of these new vectors are in H, $\vec{u} + \vec{w} \in H$

$$\vec{u} = s_1\vec{v_1} + s_2\vec{v_2} \qquad \vec{w} = t_1\vec{v_1} + t_2\vec{v_2}$$

$$\vec{u} + \vec{w} = (s_1\vec{v_1} + s_2\vec{v_2}) + (t_1\vec{v_1} + t_2\vec{v_2})$$

$$= (s_1 + t_1)\vec{v_1} + (s_2 + t_2)\vec{v_2}$$

To prove scalar multiplication, we'll use a similar process.

$$c\vec{u} = c(s_1\vec{v_1} + s_2\vec{v_2}) = (cs_1)\vec{v_1} + (cs_2)\vec{v_2}$$

And we arrive at virtually the same result as the step before, so H is closed under addition and scalar multiplication.

Theorem 1 *If $\vec{V_1}$, ..., $\vec{V_p}$ are in a vector space V, then Span$\left\{\vec{V_1}, \cdots, \vec{V_p}\right\}$ is a subspace of V. Span$\left\{\vec{V_1}, \cdots, \vec{V_p}\right\}$ is the subspace spanned (or generated) by $\left\{\vec{V_1}, \cdots, \vec{V_p}\right\}$.*

Example

Let H be the set of all vectors of the form $(a-3b,\ b-a,\ a,\ b)$

$$H = \{(a-3b,\ b-a,\ a,\ b) : a, b \in \mathbb{R}\}$$

Show that $H \subset \mathbb{R}^4$.

Writing this as a vector will help us

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v_1}} + b \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v_2}}$$

By writing this is parametric vector form, we've basically written the definition of a Span is (a linear combination of 2 vectors). These two vectors create a plane in $\mathbb{R}^4$. $H = \text{Span}\{\vec{v_1}, \vec{v_2}\}$ and $H \subset \mathbb{R}^4$ by Theorem 1.

Example

For what values of h will $\vec{y}$ be in the subspace of $\mathbb{R}^3$ spanned by the vectors $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$ if the following

$$\vec{v_1} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} \quad \vec{v_3} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

Another way of phrasing this question is "for what values of h is $\vec{y}$ within the span of these vectors?" We can write this as a linear system of equations.

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$$

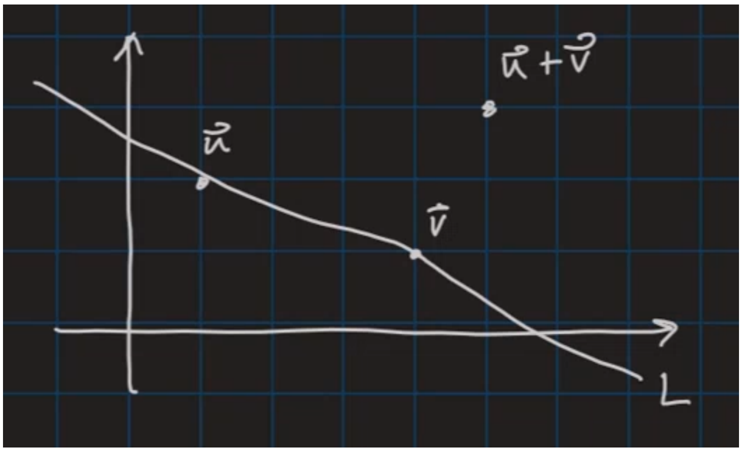
$$0 = h - 5 \implies h = 5$$

So the system is consistent if, and only if, $h - 5 = 0$. Therefore $\vec{y} \in \text{Span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ if, and only if, $h = 5$.

Example (4.1 #4)

Construct a geometric figure that shows why a line in $\mathbb{R}^2$ *not* through the origin is not closed under addition.

Remember that "closed under addition" means that we can take any two elements from the set (2 vectors from the origin to a point in the line, in this case) and add them together to get another element of the set. We can draw some arbitrary line that doesn't cross through the origin, and add 2 arbitrary vectors from that line to get a vector that is not in the line.



2 Null Space

Consider the following linear system

$$\begin{aligned} x_1 - 3x_2 - 2x_3 &= 0 \\ -5x_1 + 9x_2 + x_3 &= 0 \end{aligned}$$

The solution set $\vec{x}$ satisfying $A\vec{x} = 0$ is called the **null space** of A. Remember that it is only when we have a free variable that we get a non trivial solution to $A\vec{x} = 0$. Because there are 2 equations and 3 variables, if this system is consistent then it will have a free variable, and therefore a nontrivial solution.

The **null space** of an $n \times n$ matrix A, written as Nul A, is the set of all solutions of the homogenous equation $A\vec{x} = \vec{0}$.

$$\text{Nul } A = \left\{\vec{x} : \vec{x} \in \mathrm{\mathbb{R}^n \text{ and } A\tilde{x} = \tilde{0}}\right\}$$

Example

The examples given in the book are very simple. They ask to verify that a vector belongs to the null space of a matrix A. You just have to verify that $A\vec{u} = \vec{0}$, which we've done before.