CSE 3380 - Pivots and Row Reduction Algorithm
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1 Review

Last lecture (notes here [1]) we covered two types of matrix forms:

1. Row Echelon form
   $\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 1 & 5 \end{bmatrix}$
   (This may not have a solution, just an example)
2. Reduced Row Echelon Form (RREF)
   $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
   This is also called an identity matrix.

2 Pivots

Theorem 1 *any nonzero matrix may be* ***row reduced*** *into more than one matrix in echelon form. The reduced echelon form obtained from a matrix is unique.*

The leading value in each row is called a ***pivot****. It's position is called a* ***pivot position****. In RREF, pivots will always be 1.*

Definition 1 *A* ***pivot position*** *in a matrix A is a location in that matrix A that corresponds to a leading 1 in the RREF of it. A* ***pivot column*** *is a column that contains a pivot position.*

A **pivot** is a nonzero number in a pivot position that is used as needed to create zeroes via row operations.

Example

The bold column/values are pivots.

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \begin{matrix} \rightarrow \\ -2 \cdot eq.1 \\ +eq.1 \end{matrix} \begin{bmatrix} \mathbf{2} & 1 & 1 & 5 \\ \mathbf{0} & -8 & -2 & -12 \\ \mathbf{0} & 8 & 3 & 14 \end{bmatrix} \begin{matrix} \\ \rightarrow \\ +eq.2 \end{matrix} \begin{bmatrix} \mathbf{2} & 1 & 1 & 5 \\ 0 & \mathbf{-8} & -2 & -12 \\ 0 & 0 & \mathbf{1} & 2 \end{bmatrix}$$

[1] `CSE3380/CSE_3380__L2_Elementary_Operations_and_Row_Echelon_Form.pdf` on my notes site.

3 Row Reduction Algorithm

He says there are 4 steps to get to echelon form, then a 5th to get to RREF, but he doesn't name them. Here's an example with explanations.

Given this matrix, find a solution set.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

First, there's a 0 in a potential pivot position (row 1, column 1). We'll swap eq.1 with eq.3. Then, we'll subtract eq.1 from eq.2 to zero out the first column in eq.2.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \begin{matrix} \\ -eq.1 \\ \rightarrow \end{matrix} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \begin{matrix} \\ \rightarrow \\ -3/2 \cdot eq.2 \end{matrix}$$

Next, we'll subtract $-3/2\cdot$ eq.2 from eq.3 to zero out eq.3 in column 2. After that, it's in row echelon form. This is the final step in the "forward pass". Then we can start the "backward pass" to zero out everything else and create a matrix in RREF.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{matrix} -6 \cdot eq.3 \\ -2 \cdot eq.3 \quad \div 2 \\ \rightarrow \end{matrix} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & 9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{matrix} +9 \cdot eq.2 \\ \rightarrow \\ \end{matrix}$$

We can create more zeroes and reduce as much as possible.

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{matrix} \div 3 \\ \rightarrow \\ \end{matrix} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

From this reduced form, we can find the following equations. We can't find a solution since we have 5 unknowns and only 3 equations.

$$\begin{Bmatrix} x_1 - 2x_3 + 3x_4 = -2 \\ x_2 - 2x_3 + 2x_4 = -7 \\ x_5 = 4 \end{Bmatrix}$$

4 Solution Sets and Free Variables

Systems can have 0, 1, many, or ∞ solutions.

We may reduce a matrix and get something like this:

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Writing out the equations, we get

$$\left\{\begin{matrix} x_1 - 5x_3 = 1 \\ x_2 + x_3 = 4 \\ 0 = 0 \end{matrix}\right\}$$

Because of the last equation, x_3 is called a **free variable**. The other variables are called **basic variables**. With a free variable, we can choose any value. This system has infinitely many solutions. We can solve the first two equations in terms of x_3 then choose an arbitrary value for x_3 to give them values. We'll choose $x_3 = 0$ for simplicity.

We write the **solution set** as follows:

$$\left\{\begin{matrix} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{matrix}\right\}$$

Choosing $x_3 = 0$ gives us one possible solution, $x_1 = 1$, $x_2 = 4$, $x_3 = 0$.