CSE 2315 - Predicate Logic
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Luke Sweeney
UT Arlington

1 Variables and Statements

A **variable** is a symbol that stands for an individual in a collection or set. For example, x could stand for one of the days of the week. We may let $x =$ Monday or $x =$ Tuesday.

A sentence containing a variable may have no truth value, and is then called an **incomplete statement**. An incomplete statement is about the individuals in a definite domain or set. When we replace the variable by the name of a specific individual in the set we may obtain a statement (with a truth value) about that individual. An example of an incomplete statement is "x has 30 days". If we substituted x with a month, it becomes a complete statement.

2 Quantifiers and Predicates

Statements can have a limiting *qualifier*, which says how many items the statement pertains to. Quantifiers can be "for some" or "for all/every", indicating how many objects have a certain property.

Quantifiers:

1. Universal ($\forall$) - everything in the domain of interpretation
   - "For all"
   - "given any"
   - "for each"
   - "for every"
2. Existential ($\exists$) - one, some, or a few things in the domain of interpretation.
   - "for some"
   - "there exists"
   - "there is a"
   - "for at least one"

A **predicate** is the verbal statement that describes the property of a variable. Usually represented by the letter P, the notation $P(x)$ is used to represent some unspecified property of predicate that x may have.

- e.g. $P(x) = x$ has 30 days
- $P(\text{April}) =$ April has 30 days (this is true)

- $P(\text{January})$ = January has 30 days (this is false)

We can combine quantifiers and predicates to get complete statements of the form $(\forall x)P(x)$ or $(\exists x)P(x)$. The collection of objects from which x may be drawn is called the domain of interpretation.

Example of the truth value of expressions formed using quantifiers and predicates

TRUE - $\forall(x)P(x)$ where x is all the months and $P(x) = x$ has less than 32 days.

FALSE - $\forall(x)P(x)$ where the domain of interpretation is the collection of all flowers and $P(x) = x$ is yellow.

TRUE - $\forall(x)P(x)$ where $P(x)$ is the property that x is a plant, and the domain of interpretation is the collection of all flowers.

FALSE - $\forall(x)P(x)$ where $P(x) = x$ is positive, and the domain of interpretation consists of all integers.

FALSE - Can you find one interpretation in which both $\forall(x)P(x)$ is true and $(\exists x)P(x)$ is false?

YES? - Can you find one interpretation in which both $(\exists x)P(x)$ is true and $(\forall x)P(x)$ is false? I think he said the wrong thing here.

Predicates involving properties of single variables are **unary** predicates. You can have binary, ternary, and n-ary predicates.
$(\forall x)(\exists y)Q(x, y)$ is a binary predicate. It reads that for all things x, there exists a y such that x is in the Q relation with y.

Example

(This example is kind of weird, I don't think it's a good example).

$(\forall x)(\exists y)(\exists z)\,[Q_1(x, y) \wedge Q_2(x, z)]$. This statement has 2 properties (Q_1 and Q_2). It reads: for every x, there exists one, some, or a few y and one, some, or a few z such that x and y are in the Q_1 relation and x and z are in the Q_2 relation.

$(\forall x)$: All males

$(\exists y)$: People named Carl

$(\exists z)$: People who are 6'8"

$Q_1(x, y)$: A male who has the name Carl

$Q_2(x, z)$: A male who is 6'8"

Interpretation

An **interpretation** for an expression involving predicates consists of the following:

- A collection of objects, called the domain of interpretation, which must include at least one object.
- An assignment of a property of the objects in the domain to each predicate in the expression.
- An assignment of a particular object in the domain to each constant symbol in the expression.

Predicate wffs can be built similar to propositional wffs using logical connectives with predicates and quantifiers.

Examples of predicate wffs:

- $(\forall x)[P(x) \to Q(x)]$ - For all elements in the domain of x, if x has the property of P then x also has the property of Q
- $(\forall x)\,[(\exists y)\,[P(x,y) \vee Q(x,y)] \to R(x)]$
- $S(x,y) \wedge R(x,y)$

Brackets are used to identify the scope of the variable.

$$(\forall x)\,[(\exists y)\,[P(x,y) \vee Q(x,y)] \to R(x)]$$

$(\forall x)$ applies to the whole statement, while $(\exists y)$ applies to $P(x,y) \vee Q(x,y)$.

Free Variables In the expression

$$(\forall x)\,[P(x,y) \to (\exists y)Q(x,y)]$$

the variable y in $P(x,y)$ is not within the scope of a y quantifier, hence y is called a free variable. Such expressions might not have a truth value at all.

$$(\exists x)\,[A(x) \wedge (\forall y)\,[B(x,y) \to C(y)]]$$

$A(x)$ is "$x > 0$", $B(x,y)$ is "$x > y$" and $C(y)$ is "$y \leq 0$" where the domain of x is positive integers and the domain of y is all integers.

$\forall$ and $\exists$ with implications

Important Almost always, $\exists$ goes with $\wedge$ (conjunction) and $\forall$ goes with $\to$ (implication).

More Examples of Predicates

- $D(x) = x$ is a dog, $R(x) = x$ is a rabbit, $C(x, y) = x$ chases y.

  All dogs chase rabbits $\Leftrightarrow$ For anything, if it is a dog, then for any other thing, if it is a rabbit, then the dog chases it.

$$(\forall x)\,[D(x) \to (\forall y)\,[R(y) \to C(x, y)]]$$

- Some dogs chase all rabbits $\Leftrightarrow$ There is something that is a dog and for any other thing, if that thing is a rabbit, then the dog chases it.

$$(\exists x)\,[D(x) \wedge (\forall y)\,[R(y) \to C(x, y)]]$$

- Only dogs chase rabbits $\Leftrightarrow$ For any two things, if one is a rabbit and the other chases it, then the other is a dog.

$$(\forall x)(\forall y)\,[R(y) \wedge C(x, y) \to D(x)]$$

3 Negation of Statements

Quantifiers should also be negated alongside predicates. $A(x) = x$ is fun

$$(\forall x)A(x) \Leftrightarrow \text{everything is fun}$$
$$[(\forall x)A(x)]' \Leftrightarrow (\exists x)A(x)' \Leftrightarrow \text{Some things are not fun}$$

Class Exercises

Given the following predicates, write a wff to express the statement.

$S(x) = x$ is a student

$I(x) = x$ is intelligent

$M(x) = x$ likes music

- All students are intelligent $\Leftrightarrow (\forall x)(S(x) \to I(x))$
- Some intelligent students like music $\Leftrightarrow (\exists x)\,[S(x) \wedge I(x) \wedge M(x)]$
- Everyone who likes music is a stupid student $\Leftrightarrow (\forall x)\,[M(x) \to S(x) \wedge I(x)']$

4 Validity

With expressions using predicates and quantifiers, validity takes the place of true or false. A valid expression is analogous to a tautology of propositional logic. A predicate wff is valid if it is true in all possible interpretations. A valid predicate wff is intrinsically true.

	Propositional Wffs	Predicate Wffs
Truth values	True or false – depends on the truth value of statement letters	True, false or neither (if the wff has a free variable)
Intrinsic truth	Tautology – true for all truth values of its statements	Valid wff – true for all interpretations
Methodology	Truth table to determine if it is a tautology	No algorithm to determine validity

Examples

- $(\forall x)P(x) \to (\exists x)P(x)$ is valid because if every element in the domain of interpretation has a property, then at least one must have the property.
- $(\forall x)P(x) \to P(a)$ is valid. a is a constant, meaning one specific example of the domain. Same reason as above.
- $(\exists x)P(x) \to (\forall x)P(x)$ is not valid, because you're going from the specific to the general.
- $(\forall x)\left[P(x) \vee Q(x)\right] \to (\forall x)P(x) \vee (\forall x)Q(x)$ is not valid. Take the domain of all integers. The hypothesis says that all integers are even or odd or both, while the conclusion says that all integers are even or all integers are odd.

What is the truth value of the following statements where the domain of interpretation consists of all the integers?

- $(\exists y)(\forall x)(x + y = 0)$

  False. This says that there is one specific integer y that satisfies the equation for all integers x.
- $(\exists y)(\exists x)(x^2 = y)$

  True. Take any number and square it, and there exists one number that is equal to that square.
- $(\forall x)[x < 0 \to (\exists y)(y > 0 \wedge x + y = 0)]$