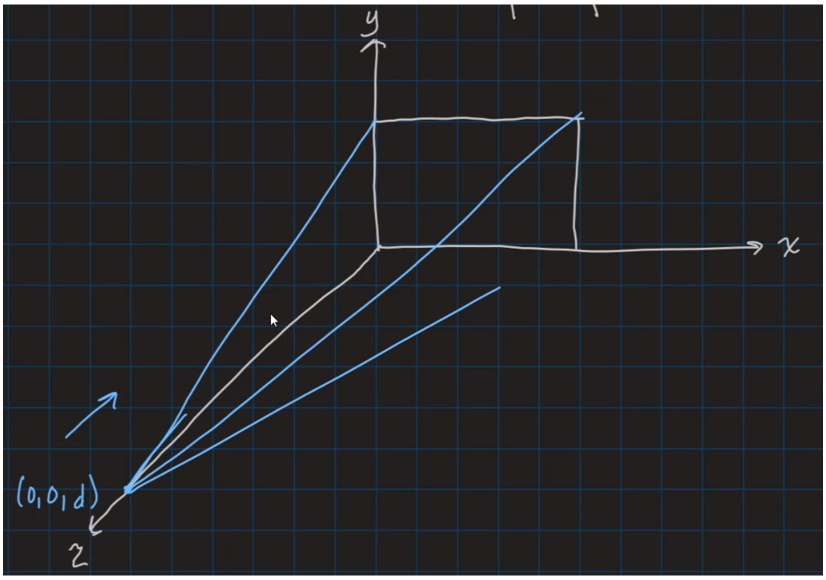
1 Projective Geometry, Cont.

Last lecture we continued talking about translation, rotation, and scaling of objects in both 2D and 3D spaces. Homogeneous coordinates can also help us with perspective.

Perspective Projections

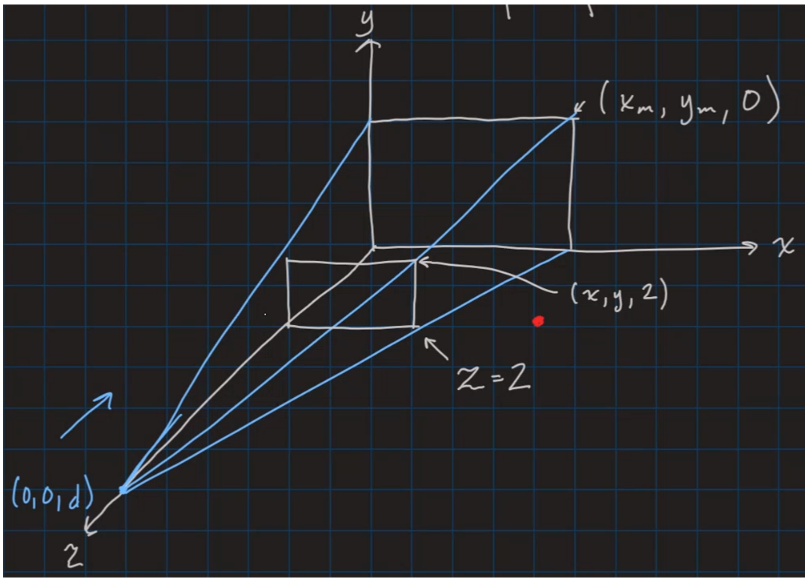
How do we simulate perspective?

In this illustration, we project points onto a plane at $z = 0$ with a camera model along the z axis at $z = d$. The camera is looking down towards the image plane. We'll define a "frustum" which connects from our camera to the corners of our plane.



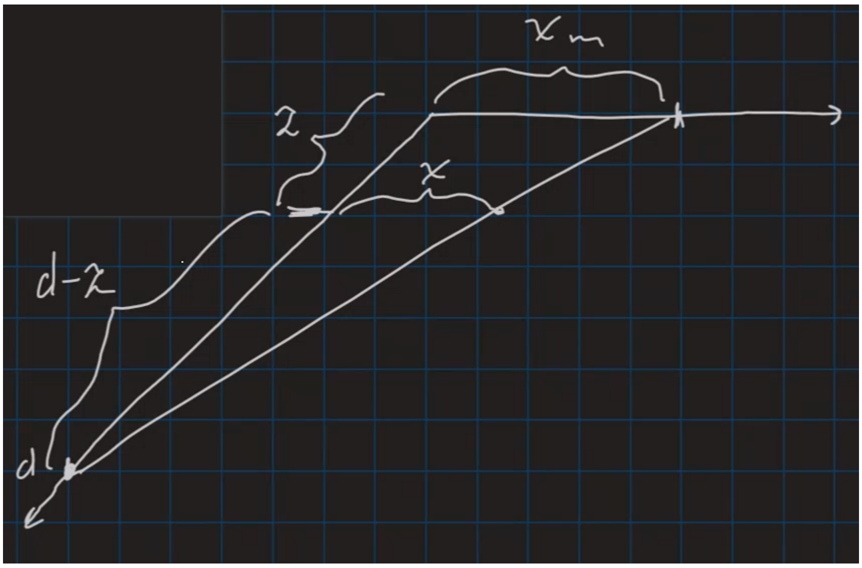
Everything within this frustum volume will appear in our image plane. Usually we would define a z point (near d, a little closer to the origin) that would be our 'clipping point". Anything between the clipping point and camera would be omitted. We won't bother with that here, just thought it worth mentioning.

We'll define a plane in the frustum at $z = 2$. This is where we'll project our objects. We'll call one of the corners $(x, y, 2)$. We'll also call one corner of the image plane $(x_m, y_m, 0)$.



We're going to derive a projection matrix that maps all the 3D points within the volume of the frustum onto the image plane. It will help us to draw just the x and z axes.

We'll note the distance (width) of the image plane x_m, the distance d where the camera is, the projection plane at $d - z$ (here it's really $d - 2$ because we've defined $z = 2$, but generally $d - z$) and width x of the projection plane.



Note that there are 2 triangles here: one from the camera to the x line (at distance z) and another from the camera to the x axis.

Using some basic geometry, we get

$$\frac{x_m}{d} = \frac{x}{d - z}$$
$$\implies x_m = \frac{d \cdot x}{d - z} = \frac{x}{1 - z/d}$$
$$\implies y_m = \frac{y}{1 - z/d}$$

How do we create a transformation matrix for this?

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/d & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ 1 - z/d \end{bmatrix}$$

When we set up our 3D homogeneous coordinates, we said

$$x = \frac{X}{W}$$

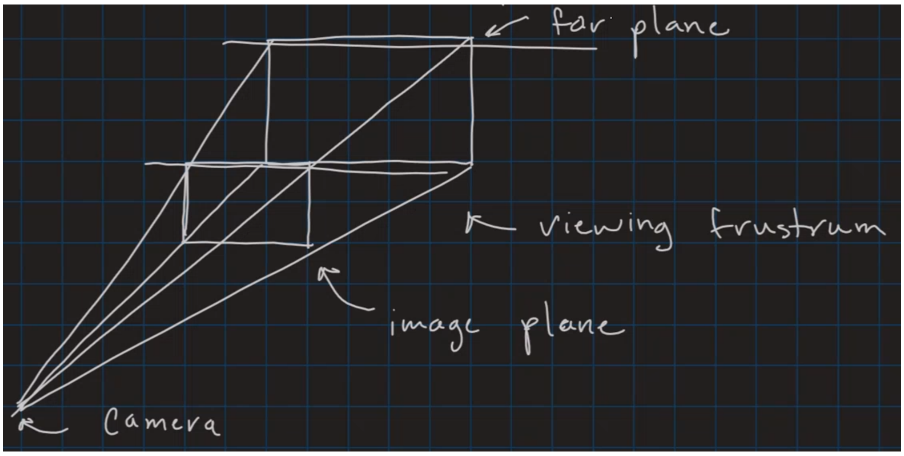
and the same for y and z where W is the last (fourth) coordinate. Now we can use that to get

$$x_m = \frac{x}{1 - z/d} \qquad y_m = \frac{y}{1 - z/d}$$

We can use this transform to project any point in this space. I think what's happening here is that we're taking the image plane and scaling it up, but I'm not really sure how it works. He didn't go over an example.

Terminology

The actual terminology is this:



We have the camera, image plane, far plane, and viewing frustum. Note: the shape is a "frustum" but he's saying "frustrum". Not sure which is correct.

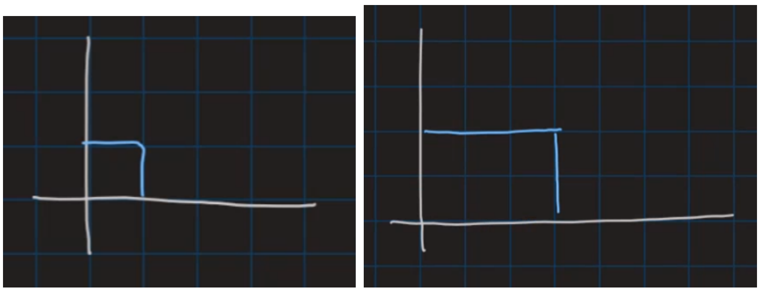
2 The Determinant, continued

The determinant can tell us how a volume or area changes when we apply a linear transformation.

Let's start with a 2D unit square. We'll use the transformation

$$T = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Which will scale the square by 3 units on the x axis and 2 units on the y axis.



How has the area changed? The area of the old square was 1, and the new area is $3 \times 2 = 6$. The determinant of the transformation is $\det T = (2)(3) - (0)(0) = 6$. So the determinant is the factor by which the area changed.

Another way to think about this is a rectangle with width s_x and height s_y, defined by two points A

$$A = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

$$\det A = (s_x)(s_y) - (0)(0) = (\text{width})(\text{height})$$

Theorem 1 *If $A \in \mathbb{R}^{2\times 2}$, the area of the parallelogram determined by the columns of A is $|\det A|$.*

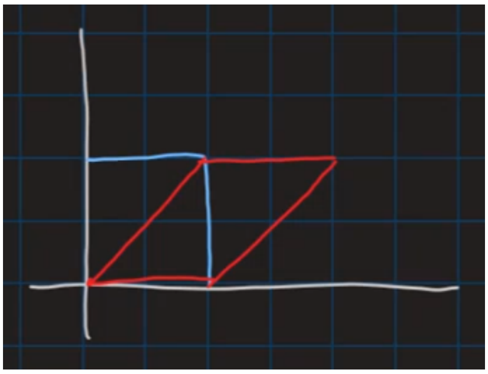
If $A \in \mathbb{R}^{3\times 3}$, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

What about Shearing?

We know of a shearing matrix

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and we can apply that to a 2×2 rectangle (blue) to get the shorn rectangle (red)



The area doesn't change, which we can verify by finding the determinant $\det T = 1$

What about Translation?

Using a translation matrix

$$T = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$$

We know that for a triangular matrix, the determinant is the product of the values on the main diagonal. So for any h and k, translation will not affect the area.

Theorem 2 *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transform determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then*

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$$

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transform determined by a 3×3 matrix A. If S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}$$

This theorem can be applied to any arbitrary shape in $\mathbb{R}^2$ and $\mathbb{R}^3$. We can find how the volume changes even when we don't have tools to find the volume. When we would normally need an intergral, we can use the determinant instead.