CSE 3380 - Into to Linear Algebra
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1 Introduction to Linear Algebra

The oldest known linear algebra problem is a good example of what kind of problem can be solved with linear algebra.

There were two fields, totalling 1800 ft^2, and they produced wheat at the following rates

field 1 $= \frac{2}{3}$ bushel per yrd^2

field 2 $= \frac{1}{2}$ bushel per yrd^2

In total, 1100 bushels of wheat were produced per year. How much did each field produce?

From this information, we can create two equations which form a **system of equations**

$$x_1 + x_2 = 1800$$
$$\frac{2}{3}x_1 + \frac{1}{2}x_2 = 1100$$

2 Form

Linear equations typically have the form

$$a_1x_1 + a_2x_2 + \cdots + a_n + x_n = b$$

x_i = basic variables
a_i = coefficients or *weights*

Linear System

A linear system is a set of equations that use the same variables but different coefficients.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b_1$$
$$a_{n+1}x_1 + a_{n+2}x_2 + \cdots + a_mx_n = b_2$$

Notice that there are always n variables and m coefficients; there could be different coefficients for each term, but the same variables are used.

Solution

A solution is any set of numbers (s_1, s_2, ... s_n) that makes each equation of a linear system true. If a system has ≥ 1 solution, it is **consistent**, otherwise it is inconsistent.

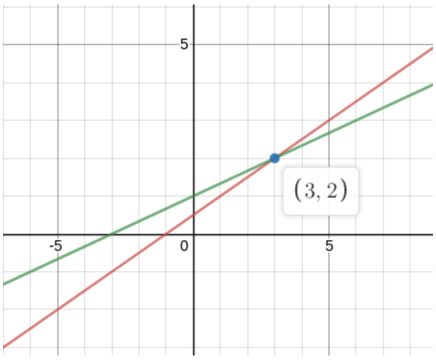
Example

Given the system

$x_1 - 2x_2 = -1$

$-x_1 + 3x_2 = 3$

we can find a solution graphically given the two lines the equations form.



Both lines intersect at $(3, 2)$. Plugging in $x_1 = 3$ and $x_2 = 2$ yields two true equations, so this system is solved.

3 Matrices

We can represent systems of equations more easily through matrices. A matrix contains only the coefficients of a system, not the variables. Here is a system of equations and the corresponding matrix.

$a_1x_1 + a_2x_2 + a_3x_3 = b_1$

$a_4x_1 + a_5x_2 + a_6x_3 = b_2$

$a_7x_1 + a_8x_2 + a_9x_3 = b_3$

$$\begin{bmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3 \\ \mathrm{a}_4 & \mathrm{a}_5 & \mathrm{a}_6 \\ \mathrm{a}_7 & \mathrm{a}_8 & \mathrm{a}_9 \end{bmatrix}$$

Types of Matrices

- **Coefficient Matrix**
  $$\begin{bmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3 \\ \mathrm{a}_4 & \mathrm{a}_5 & \mathrm{a}_6 \\ \mathrm{a}_7 & \mathrm{a}_8 & \mathrm{a}_9 \end{bmatrix}$$
- **Augmented Matrix**
  $$\begin{bmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3 & \mathrm{b}_1 \\ \mathrm{a}_4 & \mathrm{a}_5 & \mathrm{a}_6 & \mathrm{b}_2 \\ \mathrm{a}_7 & \mathrm{a}_8 & \mathrm{a}_9 & \mathrm{b}_3 \end{bmatrix}$$
  Augmented matrices hold the value of the equation in the last column.

- **Identity Matrix** (more on this later)

  $\begin{bmatrix} 1 & 0 & 0 & \mathrm{b}_1 \\ 0 & 1 & 0 & \mathrm{b}_2 \\ 0 & 0 & 1 & \mathrm{b}_3 \end{bmatrix}$

  Notice the 1's on the diagonal and 0's everywhere else. This will cancel all variables except one for each equation. This one yields

$$x_1 = b_1$$
$$x_2 = b_2$$
$$x_3 = b_3$$

Matrices are generally size $m \times n$, where m is the number of rows and n is the number of columns.

Solving a System

Our goal when solving a system is to replace it with a system that is easier to solve, until we get to one with known variables. The identity matrix above is the easiest system possible.

Example

Solve the following system

$$x_1 - 2x_2 + x_3 = 0$$
$$(0) + 2x_2 - 8x_3 = 8$$
$$5x_1 + (0) - 5x_3 = 10$$

We can create a matrix and then take a few steps on that matrix to simplify it.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} 1. \rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix} 2. \rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{bmatrix} 3. \rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & -30 & 30 \end{bmatrix}$$
$$4. \rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} 5. \rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} 6. \rightarrow \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} 7. \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Steps

1. eq.3 -= 5 $\times$ eq.1
2. eq.2 = eq.2 / 2
3. eq.3 -= 10 $\times$ eq.2
4. eq.3 = eq.3 / 30
5. eq.2 += 4$\times$ eq.3
6. eq.1 -= eq.3
7. eq.1 += 2$\times$eq.2

Now we've created an identity matrix. This is the simplest possible form.

Check the Solution

We can get values for our variables and plug those into the original equations to verify the results.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$x_1 = 1$

$x_2 = 0$

$x_3 = -1$

✓ $x_1 - 2x_2 + x_3 = 1 - 2(0) + (-1) = 0$

✓ $(0) + 2x_2 - 8x_3 = 0 + 2(0) - 8(-1) = 8$

✓ $5x_1 + (0) - 5x_3 = 5(1) + (0) - 5(-1) = 10$