CSE 2315 - Statements, Tautologies, and Contradictions
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Luke Sweeney
UT Arlington

1 Formal Logic

Logic is a foundation for an organized and careful method thinking that characterizes reasoned activity. It is also the study of reasoning; specifically concerned with whether something is true or false.

Formal logic focuses on the relationship between statements as opposed to the content of any particular statement.

Applications of formal logic in computer science:

1. Prolog programming language
2. Circuit logic - logic governing computer circuitry

2 Statements

A **statement** or **proposition** is *a sentence that is either true or false, but not both.* The truth value of a statement is T (1) or F (0).

Which of these are statements? (he doesn't actually tell us though)

- All mathematicians wear sandals
- 5 is greater than -2
- Where do you live?
- You are a cool person
- Anyone who wears sandals is an algebraist

An example of statement logic:

- All mathematicians wear sandals
- Anyone who wears sandals is an algebraist
- Therefore, all mathematicians are algebraists

Logic is of no help determining the truth of any one of these statements. But if you assume the first two are true, then the third must be true.

We can use logic like this to prove theorems and to prove that programs do what they are supposed to do.

We can use letters like A, B, C, and D to refer to statements. We can also use *connectives* like $\vee, \wedge, \rightarrow, \leftrightarrow$ to relate statements.

- $\wedge$ represents *and*

- $\vee$ represents *or*
- $\to$ represents implication (if this, then that)
- $\leftrightarrow$ represents equivalence

A statement form or propositional form is an expression made up of statement variables (A, B, C, etc.) and logical connectives ($\wedge$, $\vee$, etc) that becomes a statement when actual statements are substituted for the component statement variables.
Example:

- $(A \wedge A') \to (B \wedge C')$

3 Connectives

1. **Conjunction** ($\wedge$)
   - If A and B are statement variables, the conjunction of A and B is $A \wedge B$, which is read "A and B"
   - $A \wedge B$ is true when both A and B are true
   - $A \wedge B$ is false when at least one of A and B is false
2. **Disjunctive** ($\vee$)
   - The disjunctive of A and B is $A \vee B$, which is read "A or B"
   - $A \vee B$ is true when at least one of A or B is true
   - $A \vee B$ is false only when both A and B are false
3. **Implication** ($\to$)
   - $A \to B$ reads "if A then B"
   - A *implies* the truth of B
   - A is the hypothesis/antecedent statement and B is the conclusion/consequent statement
   - An implication can be **vacuously true** (truth by default) if A if false. If A is false, the statement $A \to B$ is true
   - Example: If Ms. X passes the exam, then she will get the job.
     - Here, B is "she will get the job" and A is "Ms. X passes the exam.
     - The statement states that Ms. X will get the job **if** a certain condition (passing the exam) is met
     - It says nothing about what will happen if the condition is not met.
4. **Negation** ($'$) or **complement**
   - A' reads "not A", which is the opposite of A.
   - $A'' = A$, $A''' = A'$

- Be careful about negating statements. You have to complement both the statement and the operator, ie $\wedge$ would become $\vee$.

5. **Equivalence ($\leftrightarrow$)**
   - "A if, and only if, B" is denoted by $A \leftrightarrow B$.
   - $A \leftrightarrow B$ is true if both A and B have the same truth values.
   - It is false if A and B have opposite truth values
   - $A \leftrightarrow B$ is a short form for $(A \to B) \wedge (B \to A)$
   - This is like $\wedge$ but it works if both are false or both are true. The values must be equal

3.1 Equivalences

Representation of an implication as Or

- Let A' be "you do your homework" and B be "you will flunk".
- The given statement "Either you do your homework or you will flunk" is $A' \vee B$.
- In **if-then** form, $A \to B$ means that "if you do not do your homework, then you will flunk," where A (which is equivalent to A'' if "you do not do your homework".

$(A \to B) \leftrightarrow (A' \vee B)$

Truth table for an implication

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

Truth table for an equivalent statement

A	A'	B	$(A' \vee B)$
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T

Representation of $\leftrightarrow$ as two implications

$(A \to B) \wedge (B \to A)$ is the same as $A \leftrightarrow B$

A	B	$A \to B$	$B \to A$	$(A \to B) \wedge (B \to A)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

4 Well formed formulas (wff)

Combining letters, connectives, and parentheses can generate an expression which is meaningful, called a **wff**. It's basically syntax. If the syntax is incorrect, it's not a wff.

To reduce the number of parentheses, an order is stipulated in which the connectives can be applied, called the order of precendence:

1. Connectives within the innermost parentheses first and then progress outward
2. Negation (')
3. Conjunctions ($\wedge$) and disjunctions ($\vee$)
4. Implications ($\rightarrow$)
5. Equivalence ($\leftrightarrow$)

Example

The truth table for the wff $A \vee B' \rightarrow (A \vee B)'$ is shown below. The main connective according to the rules is implication.

A	B	B'	$A \vee B'$	$A \vee B$	$(A \vee B)'$	$A \vee B' \rightarrow (A \vee B)'$
T	T	F	T	T	F	F
T	F	T	T	T	F	F
F	T	F	F	T	F	T
F	F	T	T	F	T	T

5 Tautology and Contradiction

Letters like P, Q, R, S etc. are used for representing wffs. $[(A \vee B) \wedge C'] \rightarrow A' \vee C$ can be represented by $P \rightarrow Q$ where P is $[(A \vee B) \wedge C']$ and Q is $A' \vee C$.

Tautologies are wffs that are intrinsically true, ie. no matter what the truth value of the statements are that comprise the wff, it is always true. For example, $(A \vee A')$. "It will rain today or it will not rain today" is always true.

Contradictions will always be false no matter the values of the statements, eg. $A \wedge A'$

Two statements are called logically equivalent if, and only if, they have identical truth values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms P and Q is denoted by writing $P \Leftrightarrow Q$ or $P \equiv Q$

Commutative	$A \vee B \Leftrightarrow B \vee A$	$A \wedge B \Leftrightarrow A \wedge A$
Associative	$(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$	$(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$
Distributive	$A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$	$A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$
Identity	$A \vee 0 \Leftrightarrow A$	$A \wedge 1 \Leftrightarrow A$
Complement	$A \vee A' \Leftrightarrow 1$	$A \wedge A' \Leftrightarrow 0$

6 De Morgan's Laws

1. $(A \vee B)' \Leftrightarrow A' \wedge B'$
2. $(A \wedge B)' \Leftrightarrow A' \vee B'$

7 Algorithms

An **algorithm** is a set of instructions that can be mechanically executed in a finite amount of time in order to solve a problem unambiguously.

Algorithms are usually represented by pseudocode.