CSE 3380 - Block Matrices
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Luke Sweeney
UT Arlington
Professor Dillhoff

1 Review

Last lecture we finished with matrix inverses. We went over an algorithm to find an inverse of a matrix, where we set up an augmented matrix like so

$$\begin{bmatrix} A & I \end{bmatrix}$$

and then performed elementary row operations on that matrix until we had

$$\begin{bmatrix} I & A^{-1} \end{bmatrix}$$

that is, using the same operations that it takes to turn the matrix A into an identity matrix will also turn the identity matrix I into A^{-1} (A inverse).

2 Invertable Matrix Theorem

Let $A \in \mathbb{R}^{n \times n}$. The following statements are logically equivalent, meaning **all true** or **all false**:

1. A is an invertable matrix.
2. A is row equivalent to the $n \times n$ identity matrix I_n.
3. A has n pivot positions.
4. The equation $A\vec{x} = \vec{0}$ has only the trivial solution.
5. The columns of A form a linearly independent set.
6. $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$.
7. The columns of A span $\mathbb{R}^n$.
8. There is an $n \times n$ matrix C such that $CA = I$.
9. There is an $n \times n$ matrix D such that $AD = I$.
10. A^T is an invertable matrix.

The negation of any of these statements applies to every singular $n \times n$ matrix. For example, if we say that A^T is not invertable, then we know that all the statements are false for A.

Corollary for Luke for later: $A\vec{x} = \vec{0}$ can only have a non-trivial solution if there is a free variable. Put that in the review.

Example

Find the inverse of the following matrix

$$\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix} \underset{+3\cdot R1}{\sim} \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & -9 & -12 \end{bmatrix} \underset{+3\cdot R2}{\sim} \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix does not have n pivot positions. Thus, it is not invertable.

3 Block (Partition) Matrices

Before, we viewed a matrix as a list of column vectors. We can also partition a large matrix into smaller matrices.

$$A = \begin{bmatrix} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ -8 & -6 & 3 & 1 & 7 & -4 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|cc|c} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array}\right]$$

Here, we've created 6 submatrices, which we can index with A_{11}, A_{12}, etc. Note that we use the capital letter A to denote the submatrix, not a_{11} which would denote a value in the matrix A

This is a 3×6 matrix written as a 2×3 **block matrix**.

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

The matrices A_{ij} are called submatrices

Block Matrix Applications

When are block matrices useful? Whenever we have limited resources. Computers frequency split large matrices up into submatrices to process them concurrently.

- Basic Linear Algebra Subprograms (BLAS)
- Graph theory
- Partial differentiation
- ATLAS (based on BLAS, but more efficient)

Addition and Scalar Multiplication

- Addition between block matrices A and B is valid as long as their partitions are the same.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$A + B = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{bmatrix}$$

  Each submatrix being added must be the same size.

- Scalar multiplication is applied block by block.

Multiplication of Block Matrices

The same row-column rule still applies. If we have two matrices, the inner sizes much match ($n \times m$ and $m \times p$, m must match).

$$A = \left[\begin{array}{rr|rr} 1 & 4 & -9 & -1 \\ -3 & 7 & 3 & -7 \\ -1 & 4 & -1 & 6 \end{array}\right], \qquad B = \left[\begin{array}{rr} -3 & 1 \\ -5 & 1 \\ \hline -3 & -2 \\ -2 & -2 \end{array}\right]$$

$$A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$AB = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_1B_1 & A_2B_2 \end{bmatrix}$$

$$A_1B_1 = \begin{bmatrix} 1 & 4 \\ -3 & 7 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} -23 & 5 \\ -26 & 4 \\ -17 & 3 \end{bmatrix}$$

$$A_2B_2 = \begin{bmatrix} -9 & -1 \\ 3 & -7 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 29 & 20 \\ 5 & 8 \\ -9 & -10 \end{bmatrix}$$

$$AB = \begin{bmatrix} -23 & 5 \\ -26 & 4 \\ -17 & 3 \end{bmatrix} + \begin{bmatrix} 29 & 20 \\ 5 & 8 \\ -9 & -10 \end{bmatrix} = \begin{bmatrix} 6 & 25 \\ -21 & 12 \\ -26 & -7 \end{bmatrix}$$

Generalize the Product AB

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 6 & 1 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

Verify that $AB = \text{col}_1(A)\,\text{row}_1(B) + \text{col}_2(A)\,\text{row}_2(B) + \text{col}_3(A)\,\text{row}_3(B)$

- $\text{col}_1(A)\,\text{row}_1(B) = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 6a & 6b \end{bmatrix}$
- $\text{col}_2(A)\,\text{row}_2(B) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} -2c & -2d \\ c & d \end{bmatrix}$
- $\text{col}_3(A)\,\text{row}_3(B) = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \begin{bmatrix} e & f \end{bmatrix} = \begin{bmatrix} 3e & 3f \\ -3e & -4f \end{bmatrix}$

$$\sum_{k=1}^{3} = \text{col}_k(A)\,\text{row}_k(B) = \begin{bmatrix} 2a - 2c + 3e & 2a - 2d + 3f \\ 6a + c - 4e & 6b + d - 4f \end{bmatrix}$$

Theorem 10

If $A \in \mathbb{R}^{\mathrm{m \times n}}$ and $B \in \mathbb{R}^{\mathrm{n \times p}}$, then

$$AB = \begin{bmatrix} \text{col}_1(A) & \cdots & \text{col}_n(A) \end{bmatrix} \begin{bmatrix} \text{row}_1(B) \\ \vdots \\ \text{row}_n(B) \end{bmatrix}$$

$$= \text{col}_1(A)\,\text{row}_1(B) + \cdots + \text{col}_n(A)\,\text{row}_n(B)$$

(I conceptually understand this but I'm not really sure why he's going over this)