1 Dimensions of Vector Spaces

We already have a pretty good concept of dimensions already, but we'll go through some formal definitions here.

The dimension of a vector space is based on the linearly independent set that makes up it's basis. For example, in $\mathbb{R}^3$, there are 3 basis vectors, so there are 3 dimensions.

Assume $\{\vec{x}, \vec{y}, \vec{z}\}$ is a basis for V. What can we say about $\{\vec{x}, \vec{y}, \vec{z}, \vec{w}\}$? We know that if we add a vector to our basis, it must be linearly dependent; $\vec{w}$ is a linear combination of the others.

Theorem 1 *If a vector space V has a basis $B = \left\{\vec{b_1}, \cdots, \vec{b_n}\right\}$, then any set in V containing more than n vectors must be linearly dependent.*

Theorem 2 *If a vector space V has a basis of n vectors, then V is **finite-dimensional**, and the dimension, $\dim V$, is the number of vectors in a basis V.*

The dimension of a zero vector space $\left\{\vec{0}\right\}$ is defined to be 0.

If V is not spanned by a finite set, V is infinite-dimensional.

Some Common Vector Space Dimensions

$\dim \mathbb{R}^n = n$

$\dim \mathbb{P}_n = n + 1$

Example

Let the x–y plane in $\mathbb{R}^3$ be defined as $P = \text{Span}\,\{\vec{x}, \vec{y}\}$. The basis of P is the set $\vec{x}, \vec{y}$.

$$\dim P = 2$$

Example

Let W be the set of all vectors of the form

$$\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix}$$

The set of vectors that span W can be written using parametric form.

What is the dimension of W?

We should put this in parametric vector form, then set up a matrix and do Gaussian elimination.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

We can tell from here that the maximum dimension possible is 3 because there are 3 columns. But it could be lower. Let's reduce

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dim W = 3$$

Subspace of a Finite-Dimensional Space

Theorem 3 *Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded into a basis for H.*

H is also finite-dimensional and $\dim H \leq \dim V$.

Theorem 4 (The Basis Theorem) *Let V be a p–dimensional vector space, $p \geq 1$.*

Any linearly independent set of exactly p elements in V is automatically a basis for V.

2 Column and Null Space

The **rank** of an $m \times n$ matrix A is the dimension of the column space.

The **nullity** of A is the dimension of the null space.

Theorem 5 (The Rank Theorem)

$$\textit{rank } A + \textit{nullity } A = \#\textit{ of columns in } A$$

Example

$$A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine the dimensions of Nul A, Col A.

To find rank A, we count the vectors in the column space, which are all the columns with pivot positions. rank $A = 3$.

For the nullity, there are a few ways we can find it. We know that rank A+nullity $A = \#$ of columns in A, so $6 - 3 = 3$. We can also count the free variables to find the null space (I think), which is also 3.

Example

If a 6×3 matrix A has rank 3, find $nullity A$ and rank A^T.

We know that there are 3 columns, so by the rank theorem, nullity $A = 0$.

For rank A^T, imagine transposing A. We would have 3 rows and 6 columns. However, we know that there are only 3 basic variables, so rank $A^T = 3$.