1 Null Space

We talked about this last lecture, but a null space is defined as

$$\text{Nul } A = \left\{\vec{x} : \vec{x} \in \mathbb{R}^n \text{ and } A\vec{x} = \vec{0}\right\}$$

Theorem 1 *The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to $A\vec{x} = \vec{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.*

Example

Let H be the set of all vectors in $\mathbb{R}^4$ whose coordinates a, b, c, d satisfy the following equations

$$\begin{aligned} a - 2b + 5c &= d \\ c - a &= b \end{aligned}$$

Show that $H \subset \mathbb{R}^4$.

We'll rewrite this as a system of homogeneous equations

$$\begin{aligned} a - 2b + 5c - d &= 0 \\ c - a - b &= 0 \end{aligned}$$

By the theorem above (the second statement, specifically), $H \subset \mathbb{R}^4$.

Let's revisit the definition above of null space:

$$\text{Nul } A = \left\{\vec{x} : \vec{x} \in \mathbb{R}^n \text{ and } A\vec{x} = \vec{0}\right\}$$

This is not an explicit definition of a set of vectors. This is more like a test that we can run over a set to determine if it's a null space. It's a description of what a set needs to be to be a null space. Let's look at some explicit definitions of null space. To get an explicit definition, we need only solve $A\vec{x} = \vec{0}$.

Example

Find the null space of this system

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

First, we'll solve $\begin{bmatrix} A & \vec{0} \end{bmatrix}$. This yields

$$\begin{aligned} x_1 - 2x_2 - x_4 + 3x_5 &= 5 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

The number of equations and the $0 = 0$ tells us that we have free variables. Our solution set is

$$\begin{cases} x_1 = 2x_2 + x_4 - 3x_5 \\ x_3 = -2x_4 + 2x_5 \\ x_2, x_4, x_5 \text{ are free} \end{cases}$$

We can write this in parametric vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_2\vec{u} + x_4\vec{v} + x_5\vec{w}$$

This description describes all vectors in Nul A.

The spanning set $\{\vec{u}, \vec{v}, \vec{w}\}$ is automatically linearly independent.

When Nul A contains any nonzero vectors, the number of vectors in Span{Nul A} is the number of free variables in $A\vec{x} = \vec{0}$

2 Column Space

The **column space** of a matrix $A \in \mathbb{R}^{m \times n}$, written Col A, is the set of all linear combinations of the columns of A. If $A = [\vec{a_1} \ \cdots \ \vec{a_n}]$, then

$$\text{Col } A = \text{Span}\{\vec{a_1}, \cdots, \vec{a_n}\}$$

Theorem 2 *The column space of an $m \times n$ matrix A is the subspace of $\mathbb{R}^m$.*

Implicit description:

$$\text{Col } A = \left\{\vec{b} : \vec{b} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n\right\}$$

Example

Given

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

Find a matrix such that $W = \text{Col } A$.

We can once again we can use parametric vector form.

$$W = \left\{ a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

$$= \text{Span} \left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Thus, if $A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$, then $W = \text{Col A}$.

Theorem 3 *Columns of A span $\mathbb{R}^m$ if, and only if, $A\vec{x} = \vec{b}$ has a solution for each $\vec{b}$.*

The column space of $A \in \mathbb{R}^{m \times n}$ is all $\mathbb{R}^m$ if, and only if $A\vec{x} = \vec{b}$ has a solution for each $\vec{b} \in \mathbb{R}^m$

3 Row Space

The set of all linear combinations of the row vectors is called the **row space** of A, denoted Row A.

$$\text{Col } A^T = \text{Row } A$$

Example

$$A = \begin{bmatrix} 3 & 4 & 8 & 0 & 1 \\ 1 & 6 & 0 & 9 & 5 \\ 0 & 2 & 0 & 2 & 2 \\ 2 & 3 & 7 & 5 & 1 \end{bmatrix}$$

(a) If Col A is a subspace of $\mathbb{R}^k$, what is k?

Each column has 4 entries, so Col $A \subset \mathbb{R}^k$

(b) If $A\vec{x}$ is defined and $A \in R^{4 \times 5}$, $\vec{x}$ must have what size?

$\vec{x} \in \mathbb{R}^5$, meaning $\vec{x}$ must have 5 entries. Nul $A \subset \mathbb{R}^5$.

(c) Are there any nonzero vectors in Col A or Nul A?

- For Col A, this is easy. We can see that there are no columns of A that are all zero.
- For Nul A - first solve $\begin{bmatrix} A & \vec{0} \end{bmatrix}$

$$\begin{bmatrix} A & \vec{0} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

To cut to the chase, the solution $\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

How do we test if a given vector is in Col A?

(a) to check if $\vec{u}$ is in Col A, solve $A\vec{x} = \vec{u}$, or just check if the system is consistent.

(b) to check if vector $\vec{v} \in \mathbb{R}^5$ is in Nul A, solve $A\vec{v} = \vec{0}$