CSE 3380 - Properties of the Determinant
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This lecture is especially full of theorems. The theorem numbers don't mean anything.

1 More on the Determinant

We left off last time with a definition of the determinant:

Definition 1 *For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form*

$$\pm a_{1j} \det A_{1j}$$

with the sign alternating, where the entries $a_{11}, \cdots, a_{1n}$ are from the first row of A.

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{1j} \det A_{1j}$$

Sidenote: Remember submatrix notation A_{ij} means that you take a matrix A and delete row i, column j and take what remains. You'll see it in the following example.

Example

Let's apply that to a matrix

$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{bmatrix}$

$$\det A = 1 \det \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix} - 5 \det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \det \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}$$

Remember that to find the determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we use $ad - bc$

$$= 1(1-2) - 5(0-0) + 0 = -2$$

Alternate Notation

We can use vertical pipes to represent the determinant

$$\det A = |A|$$
$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Theorem 1 *The determinant of an $n \times m$ matrix A can be computed by a* ***cofactor expansion*** *across any row or down any column.*

The cofactor expansion is

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Across row i

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

Across column j:

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Cofactor Expansion Example

Calculate determinant A across the third row

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

We'll use the theorem above, substituting 3 for i in the row example

$$\det A = a_{31}C_{31} + a_{32}C_{32} + a_{3n}C_{3n}$$

$$\det A = (-1)^{3+1}a_{31} \det A_{31} + (-1)^{3+2}a_{32} \det A_{32} + (-1)^{3+3}a_{33} \det A_{33}$$

$$= 0 \begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$$

$$2 \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = 2((1)(-1) - (0)(2)) = -2$$

Strategy Note: Here, it's easiest to choose row 3 to find the determinant because it has the most amount of zeroes. Those zeroes are the a_{31} and a_{33} from above. If we choose as many 0 factors as possible, many of our calculations will be simplified.

For larger matrices, this formula works recursively. The determinant of a 5×5 matrix would be based on the determinants of the 4×4 submatrices, and so forth.

Theorem 2 *If some matrix A is a triangular matrix, then* $\det A$ *is the product of the entries on the main diagonal.*

There's a strong relationship between the determinant and elementary row operations

Theorem 3 *Let* $A \in \mathbb{R}^{n \times n}$

1. *If a multiple of one row of A is added to another row to produce a matrix B, then* $\det A = \det B$
2. *If two rows are interchanged to produce B, then* $\det B = -\det A$
3. *If one row of A is multiplied to produce B, then* $\det B = k \det A$

Example

Find the determinant of the following matrix

$$A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$$

We'll start doing row operations to reduce the matrix

$$\begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{bmatrix}$$

Now this matrix is in upper triangular form. We can use theorem 2 from above to find the determinant. Because we interchanged two rows in step 2, $\det B = -\det A$

$$\det A = (1)(3)(-5) = 15$$

Theorem 3, part 2 can be generalized as

$$\det A = (-1)^r \det U$$

where r is the number of row exchanges

$$U = \begin{bmatrix} u_{11} & * & * & * \\ 0 & u_{22} & * & * \\ 0 & 0 & u_{33} & * \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & u_{nn} \end{bmatrix}$$

Even more generally, we can say

$$\det A = \begin{cases} (-1)^r(\text{product of pivots}) & \text{when A is invertible} \\ 0 & \text{when A is not invertible} \end{cases}$$

Theorem 4 *A square matrix A is invertible if, and only if,* $\det A \neq 0$

Corollaries

- $\det A = 0$ when the columns of A are linearly dependent
- $\det A = 0$ when the row of A are linearly dependent
- Rows of A are columns of A^T - linearly dependent columns of A^T make A^T singular. When A^T is singular, so is A.

Sidenote: Matrices with many zeroes are called **sparse**. There's an entire sub field of linear algebra that deals with how to represent sparse matrices in memory without wasting memory on the zeroes.

2 Transpose and Column Operations

Theorem 4 establishes that ops. on columns of A^T are analogous to ops. on rows of A.

Theorem 5 *If* $A \in \mathbb{R}^{n \times n}$*, then* $\det A^T = \det A$

Theorem 6 (Multiplicative Property) *If A and B are $n \times n$, then*

$$\det AB = (\det A)(\det B)$$