CSE 3380 - Block Matrices and the Determinant
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Luke Sweeney
UT Arlington
Professor Dillhoff

1 More on Block Matrices

Inverse of a Block Matrix

Given a 2×2 invertible block matrix A

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

Lets say that $A_{11} \in \mathbb{R}^{\mathrm{p} \times \mathrm{p}}$ and $A_{22} \in \mathbb{R}^{\mathrm{q} \times \mathrm{q}}$ (both square matrices). Our goal is to find A^{-1}.

We know that if A is invertible, we know that there is a multiplication with some matrix B such that $B = A^{-1}$ and

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}$$

We'll write out the row-column products (just how you would multiply two matrices)

$$A_{11}B_{11} + A_{12}B_{21} = I_p \tag{1}$$
$$A_{11}B_{12} + A_{12}B_{22} = 0 \tag{2}$$
$$A_{22}B_{21} = 0 \tag{3}$$
$$A_{22}B_{22} = I_q \tag{4}$$

Now we'll solve for each piece of B using the equations above.

- Equation (4) implies that $B_{22} = A_{22}^{-1}$.
- Take equation (3) and apply A_{22}^{-1}

$$\begin{aligned} A_{22}B_{21} &= 0 \\ A_{22}^{-1}A_{22}B_{21} &= 0 \cdot A_{22}^{-1} \\ B_{21} &= 0 \end{aligned}$$

- Now that we know B_{21} and it's in equation (1), we'll solve equation (1). We'll also use the same process as for equation (4)

$$A_{11}B_{11} + A_{12}B_{21} = I_p$$

$$A_{11}B_{11} = I_p$$
$$B_{11} = A_{11}^{-1}$$

- We can rewrite equation (2) and begin eliminating things

$$A_{11}B_{12} = -A_{12}B_{22}$$
$$A_{11}^{-1}A_{11}B_{12} = -A_{11}^{-1}A_{12}B_{22}$$
$$B_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}$$

Now that we know each piece of B, we can write A^{-1}

$$A^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$$

Exams and assignments will probably not have this in them, as there are easier ways. Section 2.4, problems 9 and 10 have some problems over this.

Special Case: Block Diagonal Matrix

A **block diagonal** is a partition matrix with zero blocks off the main diagonal.

This matrix is invertible if, and only if, each block on the diagonal is invertible.

2 The Determinant

The determinant has a long history. Before there was the concept of a matrix, the determinant was used to determine (heh) if a linear system had a solution.

The determinant is 0 only when the column vectors are linearly dependent.

$n \times n$ Determinants

A 2×2 matrix is invertible if, and only if, the determinant of that matrix is not 0,

$$\det A \neq 0$$

The determinant of a 2×2 matrix is

$$\det A = ad - bc = a_{11}a_{22} - a_{12}a_{21}$$

Let's expand to 3×3 matrices. Let $A \in \mathbb{R}^{3\times 3}$ and A is invertible.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(Dillhoff goes through a very long process with a dickload of writing that gets A into triangular form. I'm going to skip all of that and write the conclusion.)

The determinant Δ of a general 3×3 matrix is

$$\Delta = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Or, in more compact form,

$$\Delta = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

Where the matrices A_{11}, A_{12}, and A_{13} are obtained as follows:

For some A_{ij}, delete row i and column j, then select the resulting submatrix.

In general, for an $n \times n$ determinant, the definition relies on the determinants of $(n-1) \times (n-1)$ submatrices.

Definition 1 *For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form*

$$\pm a_{1j} \det A_{1j}$$

with the sign alternating, where the entries a_{11}, a_{12}, ..., a_{1n} are from the first row of A.

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{1j} \det A_{1j}$$