1 Length

The length of a vector can be very useful in all sorts of applications. Vector length can be found with the Pythagorean theorem.

The **length** (or **norm**) of a vector $\vec{v}$ is the nonnegative scalar $||\vec{v}||$ defined by

$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{\vec{v_1}^2 + \vec{v_2}^2 + \cdots + \vec{v_n}^2}$$

and

$$||\vec{v}||^2 = \vec{v} \cdot \vec{v}$$

Given the scalar c, the norm of $c\vec{v}$ is

$$||c\vec{v}|| = |c| \; ||\vec{v}||$$

A vector whose length is 1 is called a **unit vector**. We can **normalize** a vector (make it the unit vector). Normalization is defined as

$$\vec{u} = \frac{\vec{v}}{||\vec{v}||}$$

$\vec{u}$ has length 1, but has the same direction as $\vec{v}$.

Example

Let $\vec{v} = (3, -1, -2, 1)$. Find a unit vector $\vec{w}$ in the same direction as $\vec{v}$.

The length of $\vec{v}$ is

$$||\vec{v}||^2 = \vec{v} \cdot \vec{v} = (3)^2 + (-1)^2 + (-2)^2 + (1)^2 = 15$$

$$||\vec{v}|| = \sqrt{15}$$

$$\vec{w} = \frac{\vec{v}}{||\vec{v}||} = \frac{1}{\sqrt{15}}\vec{v} = \frac{1}{\sqrt{15}} \begin{bmatrix} 3 \\ -1 \\ -2 \\ 1 \end{bmatrix}$$

Example

Take a basis B and normalize it.

$$B = \left\{ \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

We do the same for each vector.

$$||\vec{b_1}||^2 = (-1)^2 + (1)^2 + (2)^2 = 9 \qquad ||\vec{b_1}|| = \sqrt{9} = 3$$

$$||\vec{b_2}|| = \sqrt{14} \qquad ||\vec{b_3}|| = \sqrt{6}$$

Now we can normalize each which I won't show. Just multiply each vector by $\frac{1}{||\vec{b_i}||}$

2 Distances in $\mathbb{R}^n$

Euclidean distance is for $\mathbb{R}^2$, we can extend this to $\mathbb{R}^n$.

Given $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, the distance between then is

$$\text{dist}(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}||$$

Example

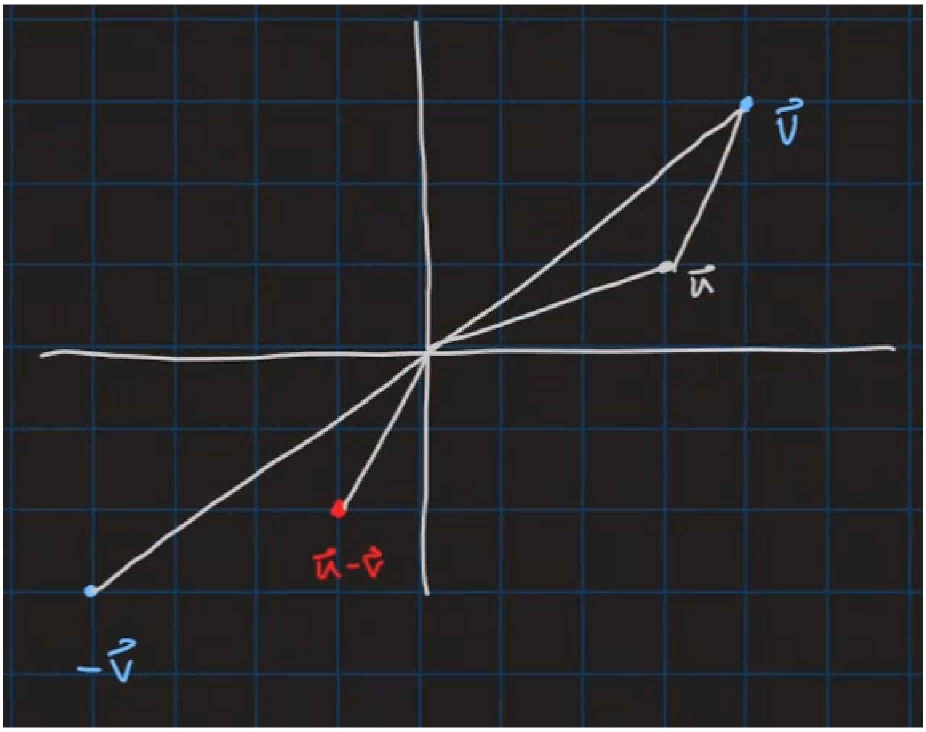
Compute the distance between $\vec{u} = (3, 1)$ and $\vec{v} = (4, 3)$

First we find $\vec{u} - \vec{v}$

$$\vec{u} - \vec{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Then we can take the norm

$$||\vec{u} - \vec{v}|| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$$



This graph shows the distance between $\vec{u}$ and $\vec{v}$

In general, for $\vec{u}, \vec{v} \in \mathbb{R}^n$, we can write

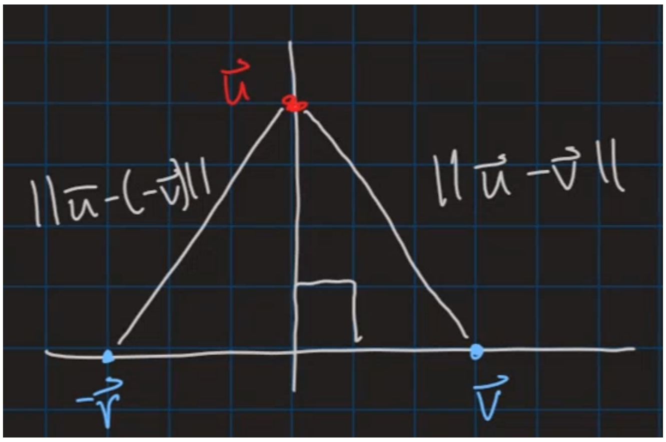
$$\text{dist}(\vec{u}, \vec{v}) = \sqrt{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})}$$

$$= \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}$$

3 Orthogonal Vectors

Perpendicularity, as described in $\mathbb{R}^2$, has an analogue in $\mathbb{R}^n$. Observe some interesting properties of lines that are geometrically perpendicular.

Let $\vec{v} = (3, 0)$ and $\vec{u} = (0, 4)$.



Let's verify by comparing $[\text{dist}(\vec{u}, \vec{v})]^2$ and $[\text{dist}(\vec{u}, \vec{-v})]^2$.

$$[\text{dist}(\vec{u}, \vec{v})]^2 = ||\vec{u} - (-\vec{v})||^2 = ||\vec{u} + \vec{v}||^2$$

$$= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$$

$$= \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}$$

$$= ||\vec{u}||^2 + ||\vec{v}||^2 + 2\vec{u} \cdot \vec{v}$$

Now let's do the other one

$$[\text{dist}(\vec{u}, \vec{v})]^2$$

$$\cdots = ||\vec{u}||^2 + ||\vec{v}||^2 - 2\vec{u} + \vec{v}$$

The lines are perpendicular when

$$-2\vec{u} \cdot \vec{v} = 2\vec{u} \cdot \vec{v}$$

which is only equal if $\vec{u} \cdot \vec{v} = 0$.

Definition 1 *Two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal to each other is $\vec{u} \cdot \vec{v} = \vec{0}$.*
$\vec{0}$ is orthogonal to every vector.

Theorem 1 (Pythagorean Theorem) *Two vectors are orthogonal if, and only if,*

$$||\vec{u} + \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2$$

Sometimes you'll see $||\vec{a} - \vec{b}||_1$ with a little one. This is the L1–distance. There's also L2–distance and so on.

4 Orthogonal Complements

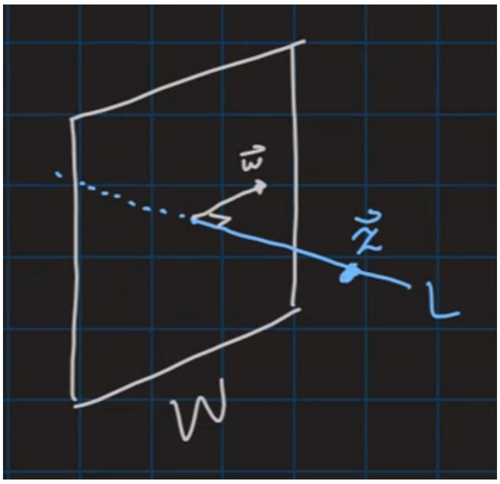
Consider a vector in $\mathbb{R}^3$. How many vectors are orthogonal to it?

An infinite number. The set of (infinite) vectors orthogonal to another vector (in $\mathbb{R}^3$) span a plane. Therefore, to specify a plane, we need only describe a single vector.

Example

let W be a plane through the origin in $\mathbb{R}^3$, and L be the line through the origin and orthogonal to W.

If $\vec{z}$ and $\vec{w}$ are nonzero, $\vec{z}$ is on L, and $\vec{w}$ is in W and $\vec{z} \cdot \vec{w} = 0$.



L consists of all vectors orthogonal to W, and W consists of all vectors orthogonal to L.

We'll pick up with this example next lecture.