We left off part way through an example. Here it is

Example

Diagonalize

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \qquad A = PDP^{-1}$$

Where the columns of P are the eigenvectors of A.

1. Find the eigenvalues of A

$$\det(A - \lambda I) = 0$$

$$\implies -\lambda^3 - 3\lambda^2 + 4$$

$$-(\lambda - 1)(\lambda - 2)^2$$

The eigenvalues are $\lambda = 1$, $\lambda = -2$

2. Find the eigenvectors of A.

Basis for $\lambda = 1$

$$\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Basis for $\lambda = -2$

$$\vec{v_2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v_3} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

3. Now that we have the eigenvectors,

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

4. Now we construct D with the corresponding eigenvalues, according to the columns of eigenvectors in P

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Theorem 1 *An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.*

Consider $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$. When matrices are in upper triangular form like this, the eigenvalues are just the values on the diagonal. $\lambda = 5$, $\lambda = 0$, $\lambda = -2$. The theorem above is sufficient but not necessary.

There's still a way to find P in $A = PDP^{-1}$ even if the eigenvalues are not distinct.

Theorem 2 *Let $A \in \mathbb{R}^{n \times m}$, whose distinct eigenvalues are $\lambda_1, \cdots, \lambda_p$.*

a. *for λ_k, $1 \leq k \leq p$, the dimension of eigenspace for λ_k is $\leq$ the multiplicity of λ_k*

1. *A is diagonalizable if, and only if, the sum of dimensions of eigenspaces equals n. This happens when*

    (i) *the characteristic polynomial factors completely into linear factors*

    (ii) *the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k*

2. *If A is diagonalizable and β_k is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in sets $\beta_1, \cdots, \beta_p$ form an eigenbasius for $\mathbb{R}^n$.*

At this point, this material won't be tested on so I'm going to call it. It's 12:28am. The final is in 452 minutes. Maybe I'll pass, God willing.