CSE 3380 - Matrix Multiplication
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Last lecture we went over matrix addition and scaling which is very similar to vector addition and scaling. See Lecture 8 notes.

1 Matrix Multiplication

Given a matrix A and vector $\vec{x}$, we know $A\vec{x}$ = some vector $\vec{b}$. We can think of $A\vec{x}$ as a vector directly. Multiplying another matrix B, we think of $AB\vec{x}$ as $B \times (A\vec{x})$.

Let $A \in \mathbb{R}^{\mathrm{m}\times\mathrm{n}}$, $B \in \mathbb{R}^{\mathrm{n}\times\mathrm{p}}$, $\vec{x} \in \mathbb{R}^{\mathrm{p}}$.

Restating linear combinations,

$$B\vec{x} = x_1\vec{b}_1 + \cdots + x_p\vec{b}_p$$

now adding A as a factor

$$A(B\vec{x}) = A(x_1\vec{b}_1) + \cdots + A(x_p\vec{b}_p)$$

Definition 1 *If A is $m \times n$ and B is $n \times p$, the product AB is the $m \times p$ matrix whose columns are $A\vec{b}_1, \cdots, A\vec{b}_p$*

$$AB = A\begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p\end{bmatrix}$$

$$AB = \begin{bmatrix}A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p\end{bmatrix}$$

Example

$A = \underbrace{\begin{bmatrix}2 & 3\\ 1 & -5\end{bmatrix}}_{(2\times2)}$ $\qquad B = \underbrace{\begin{bmatrix}4 & 3 & 6\\ 1 & -2 & 3\end{bmatrix}}_{(2\times3)}$

Dimensions: $(2 \times 2)(2 \times 3) = (2 \times 3)$

(The inner dimensions must match, and the resulting dimensions are the outer dimensions.)

$$AB = \begin{bmatrix}\begin{bmatrix}2 & 3\\ 1 & -5\end{bmatrix}\begin{bmatrix}4\\ 1\end{bmatrix} & \begin{bmatrix}2 & 3\\ 1 & -5\end{bmatrix}\begin{bmatrix}3\\ -2\end{bmatrix} & \begin{bmatrix}2 & 3\\ 1 & -5\end{bmatrix}\begin{bmatrix}6\\ 3\end{bmatrix}\end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}}_{(2\times 3)}$$

Size Example

If $A \in \mathbb{R}^{\mathrm{m\times n}}$, $B \in \mathbb{R}^{\mathrm{n\times p}}$

What is the size of AB and BA?

1. $AB = (m \times n)(n \times p) = (m \times p)$

   Because the inner sizes match, it's not undefined. The resulting dimensions are the outer dimensions.

2. $BA = (n \times p)(m \times n) =$ not defined

   The inner dimensions don't match, so it doesn't exist.

2 Row Column Rule for Matrix Multiplication

For matrix multiplication, if AB is defined, then the entry in row i, column j, $(AB)_{ij}$ is the sum of the products of the corresponding entries from row i of A and column j of B.

If $(AB)_{ij}$ denotes the $(i, j)^{th}$ entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

or, more compactly

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Example

$AB = \begin{bmatrix} \mathbf{2} & \mathbf{3} \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & \mathbf{6} \\ 1 & -2 & \mathbf{3} \end{bmatrix}$

Find $(AB)_{13}$ and $(AB)_{22}$.

Each of these is really long to write out with proper notation, but they're really easy to think about. For the first matrix (A), just traverse across the specified row, and for the second matrix (B) traverse down the proper column. I've bolded the matrix entries above used to find $(AB)_{13}$.

$$i = 1 \text{ (Row 1 of } A) \qquad j = 3 \text{ (column 3 of } B)$$

$$(AB)_{13} = \sum_{k=1}^{2} a_{1k}b_{k3} = a_{11}b_{13} + a_{12}b_{23} = (2)(6) + (3)(3) = 21$$

For $(AB)_{22}$, the entries we use are in bold here:

$AB = \begin{bmatrix} 2 & 3 \\ \mathbf{1} & \mathbf{-5} \end{bmatrix} \begin{bmatrix} 4 & \mathbf{3} & 6 \\ 1 & \mathbf{-2} & 3 \end{bmatrix}$

$$(AB)_{22} = (1)(3) + (-5)(-2) = 13$$

Example

$$A = \underbrace{\begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix}}_{(4\times 3)} \qquad B = \underbrace{\begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}}_{(3\times 2)} \qquad (AB) \in \mathbb{R}^{4\times 2}$$

Find the second row of (AB).

We need to find $(AB)_{21}$ and $(AB)_{22}$. We'll multiply row 2 of A by column 1 of B, then column 2 of B:

1. $(AB)_{21} = (-1)(4) + (3)(7) + (-4)(3) = 5$
2. $(AB)_{22} = (-1)(-6) + (3)(1) + (-4)(2) = 1$

so

$$AB = \begin{bmatrix} \cdots & \cdots \\ 5 & 1 \\ \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$$

3 Properties of Matrix Multiplication

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$ Left distributive law
3. $(B + C)A = BA + CA$ Right distributive law
4. $r(AB) = (rA)B = A(rB)$
5. $I_m A = A = AI_n$
   - I_m and I_n are identity matrices if $A \in \mathbb{R}^{m\times n}$