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CSE 3380 - Linear Combinations
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1 Vectors in $\mathbb{R}^n$

$\mathbb{R}^n$ is the collection of all lists (**ordered** n-tuples) of all numbers.

Note: the zero vector $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

Algebraic Properties of Vectors

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3. $\vec{u} + 0 = 0 + \vec{u} = \vec{u}$
4. $\vec{u} + (-\vec{u}) = 0$
5. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
6. $(c + d)\vec{v} = c\vec{v} + d\vec{v}$
7. $c(d\vec{v}) = (cd)\vec{v}$
8. $1\vec{u} = \vec{u}$

2 Linear Combination

Definition 1 *given vectors* $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_p}$ *in* $\mathbb{R}^n$ *and scalars* $c_i \in \mathbb{R}$, *the vector* $\vec{y}$ *defined by*

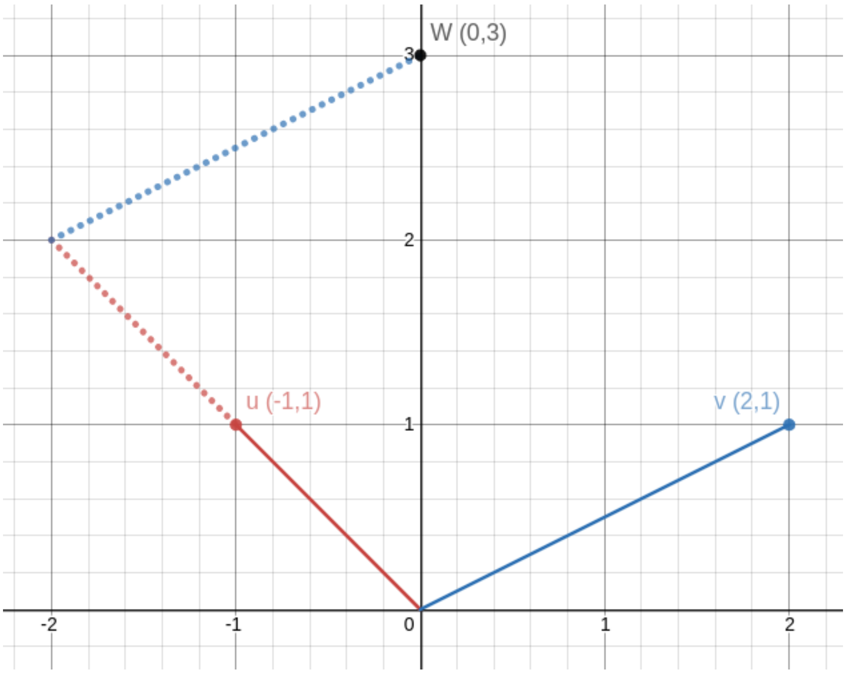
$$\vec{y} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_p\vec{v_p}$$

is called a ***linear combination*** *of* $\vec{v_1}, \cdots, \vec{v_p}$ *with weights* $c_1, \cdots, c_p$.

Example

Let $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

The point $\vec{w} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ can be represented as a linear combination of these two vectors.



Example (3D)

Let $\vec{a_1} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\vec{a_2} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$

Can we write $\vec{b}$ as a linear combination of a_1 and a_2, so that

$$x_1\vec{a_1} + x_2\vec{a_2} = \vec{b}$$

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$$\begin{cases} x_1 + 2x_2 = 7 \\ -2x_1 + 5x_2 = 4 \\ -5x_1 + 6x_2 = -3 \end{cases}$$

We can convert this system to an augmented matrix and solve for a solution.

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \begin{matrix} \rightarrow \\ +2 \cdot eq.1 \\ +5 \cdot eq.1 \end{matrix} \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \begin{matrix} \rightarrow \\ \cdot \frac{1}{9} \\ \ \end{matrix} \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{bmatrix} \begin{matrix} -2 \cdot eq.2 \\ \rightarrow \\ -16 \cdot eq.2 \end{matrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

It may seem like there are infinite solutions ($0 = 0$ implies a free variable), but remember there are only 2 unknowns. This system tells us that we have a single solution, $x_1 = 3$, $x_2 = 2$.

So, $3 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$

Definition 2 *a vector equation* $x_1\vec{a_1} + \cdots + x_n\vec{a_n} = \vec{b}$ *has the same solution set as the linear system whose augmented matrix is*

$$\begin{bmatrix} \vec{a_1} & \vec{a_2} & \cdots & \vec{a_n} & \vec{b} \end{bmatrix}$$

Definition 3 *if* $\vec{v_1}, \cdots, \vec{v_p}$ *are in* $\mathbb{R}^n$*, then the set of all linear combinations of the vectors in denoted by Span*$\{\vec{v_1}, \cdots, \vec{v_p}\}$ *and is called the subset of* $\mathbb{R}^n$ *spanned by* $\vec{v_1}, \cdots, \vec{v_p}$.

Span$\{\vec{v_1}, \cdots, \vec{v_p}\}$ *is the collection of all vectors that can be written in the form*

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n}$$

Example

Let $\vec{a_1} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\vec{a_2} = \begin{bmatrix} 5 \\ -13 \\ 3 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$

Then, Span$\{\vec{a_1}, \vec{a_2}\}$ is a plane through the origin in $\mathbb{R}^3$. Is $\vec{b}$ in that plane?

We can form a linear system and then an augmented matrix, which we can solve for x, y, and z, the coefficients of the vectors needed for $\vec{b}$ to be in the plane.

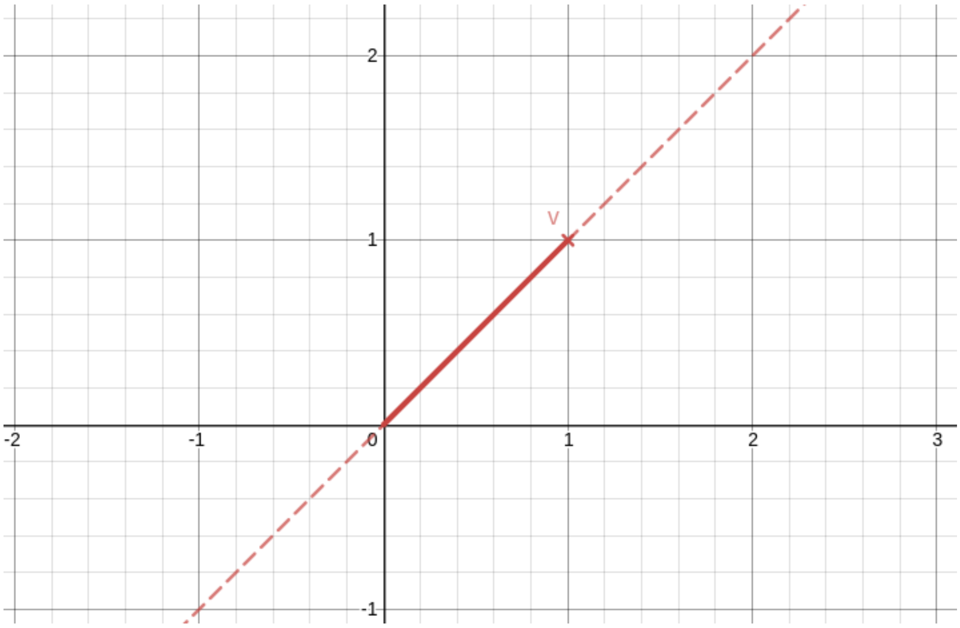
I'll omit some steps because we already know how to solve linear systems.

$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 \\ 0 & -3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

This last matrix contains a contradiction, $0 = 2$. This tells us the system is inconsistent, so $\vec{b}$ is not in the plane.

2.1 Geometric Description of Span$\{\vec{v}\}$ and Span$\{\vec{u}, \vec{v}\}$

For Span$\{\vec{v}\}$, the span is all vectors formed by scaling $\vec{v}$ by any scalar. That's all we can do. This is one dimension.



For Span$\{\vec{u}, \vec{v}\}$, the span contains every point in $\mathbb{R}^2$ space. Here's two vectors and an example of them combining to reach a third arbitrary point.