This is the exam 2 review, where we'll go over a few problems that will be on the exam. The first section is Dillhoff's review sheet.

1 Topics

1. Vector Spaces
   (a) Lectures (and Luke's Notes)
      - March 1, 2021
      - March 3, 2021
      - March 5, 2021
      - April 2, 2021

   (b) Outcomes
      - Identify is a set is a vector space
      - Determine if a set is a subspace of another set
      - Be able to state both the implicit and explicit description of the null space
      - State the dimension of a vector space
      - Determine the rank and nullity of a matrix

   (c) Assignments
      - Assignment 6
      - Assignment 8
2. Linear Independence and Basis
   (a) Lectures
      - March 10
      - March 12

   (b) Outcomes
      - Determine if a set of vectors in linearly independent.
      - Identify if a set of vectors forms a basis of a vector space
      - Explain how a set of vectors could span a space, but not be the basis of it
      - Determine the bases for the null space and column space of a matrix

   (c) Assignments
      - Assignment 6
3. Linear Transformations and Computer Graphics
   (a) Lectures
      - March 12
      - March 22
      - March 24
      - March 26

   (b) Outcomes

- Know the definition of a linear transformation
- Identify the difference between an affine transformation and a linear transformation
- Apply basic linear transformations (rotation and scaling)
- Apply affine transformations (translation)
- Derive a projection matrix given a plane and camera coordinates

(c) Assignments

- Assingment 7

4. Determinants and Volume

(a) Lectures

- March 26
- March 29

(b) Outcomes

- Understand what the determinant of a linear transformation says about the space it is applied to
- Calculate the volume of an object transformed by some linear transformation.

5. Coordinate systems, Change of Basis

(a) Lectures

- March 29
- March 31
- $\cdots$

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2 Coordinate Systems, Change of Basis

Remember that a vector is a combination of it's weights times the basis vectors

$$\vec{y} = c_1\vec{v_1} + \cdots + c_n\vec{v_n}$$

An example in the standard basis for the coordinate (3, 2) is

$$\vec{y} = 3\hat{\imath} + 2\hat{\jmath}$$

Where (3, 2) are the coordinates (weights) and $\hat{\imath}$ and $\hat{\jmath}$ for the basis $B = \left\{\hat{\imath}, \hat{\jmath}\right\}$. This holds for all bases and coordinates. This leads us to...

Change of Basis

In general, for a basis $B = \left\{\vec{b_1}, \cdots, \vec{b_n}\right\}$ with a change-of-basis matrix $P_B = \begin{bmatrix}\vec{b_1} & \cdots & \vec{b_n}\end{bmatrix}$ and vector $\vec{x}$

$$\vec{x} = P_B \ [\vec{x}]_B$$

$$\therefore \quad [\vec{x}]_B = P_B^{-1} \ \vec{x}$$

If you remember that $\vec{x}$ is implicity the same as $[\vec{x}]_\varepsilon$ (where ε is the standard basis $\left\{\hat{\imath}, \hat{\jmath}\right\}$) then you get

$$P_\varepsilon \ [\vec{x}]_\varepsilon = P_B \ [\vec{x}]_B$$

3 Vector Spaces

Dimension

The dimension of a matrix is the amount of pivot columns.

Example

What is the dimension of the following set of vectors?

$$\begin{bmatrix}1\\0\\-3\\2\end{bmatrix}, \begin{bmatrix}0\\1\\2\\-3\end{bmatrix}, \begin{bmatrix}-3\\-4\\1\\6\end{bmatrix}, \begin{bmatrix}1\\-3\\-8\\7\end{bmatrix}, \begin{bmatrix}2\\1\\-6\\9\end{bmatrix}$$

We can throw these into a matrix and get the reduced row echelon form

$$\begin{bmatrix}1 & 0 & -3 & 1 & 2\\0 & 1 & -4 & -3 & 1\\-3 & 2 & 1 & -8 & -6\\2 & -3 & 6 & 7 & 9\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -3 & 0 & 4\\0 & 1 & -4 & 0 & -5\\0 & 0 & 0 & 1 & -2\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

There are 3 pivot columns in that matrix, forming the set

$$\left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \right\}$$

The dimension is 3 because there are 3 pivot columns, $\dim A = 3$

Null Space, Basis of Null space

The null space of a system is set of all vectors $\vec{x}$ that satisfy

$$A\vec{x} = \vec{0}$$

To find the null space of A (Nul A) we *could* create an augmented matrix $\begin{bmatrix} A & \vec{0} \end{bmatrix}$ to solve for $\vec{x}$ but that's kinda boring because the $\vec{0}$ on the end never changes. Instead we'll just reduce A and pretend the column of zeroes is there.

I'll steal the matrix from above, but generally given an RREF matrix like this

$$A = \begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For this matrix, there are a few ways to express the null space of A. We could solve for the basic variables in terms of the free variables and write them like this

$$\begin{aligned} x_1 &= 3x_3 - 4x_5 \\ x_2 &= 4x_3 + 5x_5 \\ x_4 &= 2x_5 \end{aligned}$$

We could also use parametric form which is more useful when finding the basis of the null space

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3x_3 - 4x_5 \\ 4x_3 + 5x_5 \\ x_3 \\ 2x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 5 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

So the basis of the null space is the set

$$B = \left\{ \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Column Space

To find the column space of A, we want the columns of A that correspond to pivot columns in RREF(A). I'll use the same set of vectors as in the dimension example above. We got to RREF(A) and found that columns 1, 2, and 4 were pivot columns, so the column space would be

$$\text{Col } A = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix} \right\}$$

Remember that # of columns = rank $A +$ nullity A. There were 5 columns in the matrix; there are 3 in the column space and 2 in the null space.

Row Space

We can find Row A by finding Col A^T. There will always be the same number of rows in the row space as columns in the column space.

4 Linear Transformations

Basic Transformations

- **Scaling** by a factor of s

$$S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

- **Rotating** by θ
  - Counterclockwise

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

  - Clockwise

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

- **Translation** to $x + h$ and $y + k$

$$T = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$$

Composite transforms

Composite transforms are pretty easy. You'd take multiple transformation matrices like the ones above and multiply then together. You can combine them and it's the same affect. The order might matter depending on what you're doing. Remember that you should write your matrices out **right to left** in the order you want them to be applied. For example

$$T = T_3 T_2 T_1 \, \vec{x}$$

T_1 would be applied first, if that makes sense.

Rotating around a point

If you just apply a rotation matrix to a matrix or vector, it will rotate around the origin. This may not be what you want. If you want to rotate around any arbitrary point (x_1, y_1), then you should translate the matrix or vector until that point (x_1, y_1) is on the origin, then rotate, then translate back. Look in L21 notes (March 24) for an example.