CSE 3380 - More Matrix Inverses
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1 Review

We left off last class (Lecture 10) talking about the inverse and transpositions of matrices, as well as the determinant.

Just remember that

$$A\vec{x} = \vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

2 Continuing Matrix Inverses

Example

$$\begin{cases} 3x_1 + 4x_2 = 3 \\ 5x_1 + 6x_2 = 7 \end{cases}$$

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

If this has a solution, we can write it in the form $\vec{x} = A^{-1}\,\vec{b}$.

First, we calculate A^{-1}. We know that A is invertible because $\det A \neq 0$

$$A^{-1} = \frac{1}{18-20}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$

$$A^{-1}\,\vec{b} = \vec{x} = \begin{bmatrix} -3 & 2 \\ 5/3 & -3/2 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Theorem 6

This applies to all square matrices.

a. if A is invertible, then A^{-1} is also invertible, and

$$(A^{-1})^{-1} = A$$

b. if A and B are both invertible, then so if AB, and AB^{-1} is the product of the inverses of A and B, but reversed

$$(AB)^{-1} = (B^{-1})(A^{-1})$$

c. if A is invertible, then so is A^T and the inverse of A^T is the transpose of the inverse

$$(A^T)^{-1} = (A^{-1})^T$$

3 Elementary Matrices

An **elementary matrix** is one obtained by performing a single elementary row operation on an identity matrix.

For example,

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{-4 \cdot R1}{\sim} = E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

Theorem 1 *A $n \times n$ matrix is invertible if, and only if, A is row equivalent to I_n, and in this case any sequence of row operations that reduces A to I_n also transforms I_n into A^{-1}.*

This basically means that we can start with A and I_n, perform the same operations to both until A is an identity matrix, and I_n will be A^{-1}

Example

Find the inverse of

$$A = \begin{bmatrix} -3 & 1 & -8 \\ 4 & 4 & -4 \\ 6 & 7 & -9 \end{bmatrix}$$

So far, we've used the determinant and the formula from last class to find the inverses of square 2×2 matrices. This obviously won't work here.

Let's use the idea above. We'll tack on an identity matrix, then do some row operations to get $A \to I$. This will look like a single matrix, just remember it's two smashed together.

$$\begin{bmatrix} -3 & 1 & -8 & 1 & 0 & 0 \\ 4 & 4 & -4 & 0 & 1 & 0 \\ 6 & 7 & -9 & 0 & 0 & 1 \end{bmatrix} \overset{\text{swap R2}}{\sim} \begin{bmatrix} 4 & 4 & -4 & 0 & 1 & 0 \\ -3 & 1 & -8 & 1 & 0 & 0 \\ 6 & 7 & -9 & 0 & 0 & 1 \end{bmatrix} \overset{\cdot \frac{1}{4}}{\sim}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 1/4 & 0 \\ -3 & 1 & -8 & 1 & 0 & 0 \\ 6 & 7 & -9 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \sim \\ +3 \cdot R1 \\ -6 \cdot R1 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 0 & 1/4 & 0 \\ 0 & 4 & -11 & 1 & 3/4 & 0 \\ 0 & 1 & -3 & 0 & -3/2 & 1 \end{bmatrix} \underset{\text{swap R2}}{\sim}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 1/4 & 0 \\ 0 & 1 & -3 & 0 & -3/2 & 1 \\ 0 & 4 & -11 & 1 & 3/4 & 0 \end{bmatrix} \begin{array}{c} \\ \sim \\ -4 \cdot R2 \end{array} \begin{bmatrix} 1 & 1 & -1 & 0 & 1/4 & 0 \\ 0 & 1 & -3 & 0 & -3/2 & 1 \\ 0 & 0 & 1 & 1 & 27/4 & -4 \end{bmatrix} \begin{array}{c} +R3 \\ 3 \cdot R3 \\ \sim \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 28/4 & -4 \\ 0 & 1 & 0 & 3 & 75/4 & -11 \\ 0 & 0 & 1 & 1 & 27/4 & -4 \end{bmatrix} \begin{array}{c} -R2 \\ \sim \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & -2 & -47/4 & 7 \\ 0 & 1 & 0 & 3 & 75/4 & -11 \\ 0 & 0 & 1 & 1 & 27/4 & -4 \end{bmatrix}$$

We can see that the left 3 columns (what used to be A) have become the identity matrix I, which means the right 3 columns are A^{-1}

$$A^{-1} = \begin{bmatrix} -2 & -47/4 & 7 \\ 3 & 75/4 & -11 \\ 1 & 27/4 & -4 \end{bmatrix}$$