CSE 3380 - Solution Sets and Vectors
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1 Solution Sets

We left off last lecture with **free variables**. If a system contains a free variable and is consistent, then it has infinitely many solutions.

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases}$$

Here, we obtained a RREF matrix, wrote out the equations, then wrote each variable (x_1, x_2) in terms of the free variables (x_3).

Free Variables Example

Here's an example with 2 free variables.

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

We know this system is consistent because it's in echelon form and there is no $0 = b$, where b is a non-zero number.

We'll do the backwards pass to get the matrix to RREF.

$$\begin{matrix} +2 \cdot eq.3 \\ +eq.3 \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \begin{matrix} \\ \cdot\frac{1}{2} \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \begin{matrix} -2 \cdot eq.2 \\ \rightarrow \\ \end{matrix} \begin{bmatrix} \mathbf{1} & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & \mathbf{1} & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 7 \end{bmatrix}$$

This is now in RREF, with all pivot positions in bold. We have 2 free variables x_2 and x_4. We can write out our equations and solve in terms of the free variables.

$$\begin{cases} x_1 + 6x_2 + 3x_4 = 0 \\ x_3 - 4x_4 = 5 \\ x_5 = 7 \end{cases}$$

(Because we've reduced to RREF, only one basic variable will appear in each equation. This prevents us from having to back substitute.)

Once we solve, we get our solution set with ∞ solutions.

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

Parametric Description of Solution Sets

- When a system has free variables, the solution set has many possible parameters descriptions.
- As a convention, we'll always write equations in terms of other variables.
- A system that is inconsistent may have free variables, but the solution set itself would be empty.

Existence and Uniqueness of a Solution

Given a matrix in echelon form,

1. If there is no $0 = b$, where b is a non-zero number, it is implied that the system is consistent.
2. If the system contains free variables, there is not a unique solution; there are infinitely many solutions.
3. If all basic variables are defined (and there are no free variables), there exists a unique solution.

For example, given this matrix in echelon form,

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

There are no contradictions ($0 = b$), and there are free variables. This means there are infinitely many solutions.

Theorem 1 *A linear system is consistent if, and only if, the right most column of the augmented matrix is* ***not*** *a pivot column.*

2 Vectors

Given a general matrix

$$M = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

A column vector $\vec{v} = \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix}$ or row vector $\vec{v} = \begin{bmatrix} a_{11} \\ \vdots \\ a_{1m} \end{bmatrix}$

Vectors in $\mathbb{R}^2$

A vector in $\mathbb{R}^2$ space (2D space) is the smallest form of a vector.

Vectors are **ordered** sets of numbers.

Operations

Addition

$$\begin{bmatrix}-2\\4\end{bmatrix} + \begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}-2+1\\4+3\end{bmatrix} = \begin{bmatrix}-1\\7\end{bmatrix}$$

Scaling (multiplication)

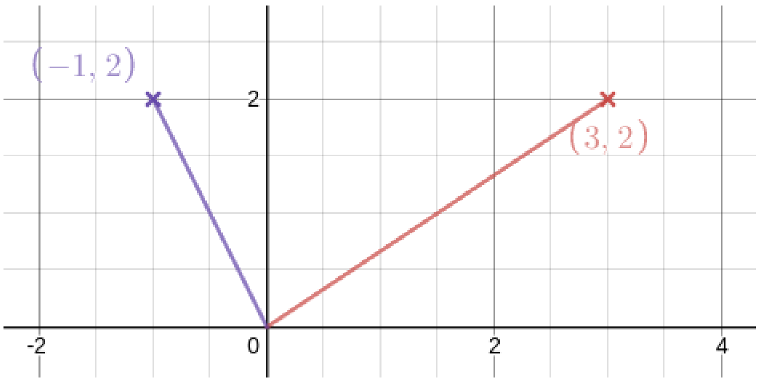
$$\vec{u} = \begin{bmatrix}2\\4\end{bmatrix}$$

$$c\vec{u} = \begin{bmatrix}2c\\4c\end{bmatrix}$$

Geometric Description of a Vector

Vectors in 2D space is pretty basic, we did this in physics. Vectors are typically represented with arrows.

The vectors $\vec{v} = \begin{bmatrix}3\\2\end{bmatrix}$ and $\vec{u} = \begin{bmatrix}-1\\2\end{bmatrix}$ are shown.



Rule: if $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ space are represented as points in the plane, then $\vec{u}+\vec{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\vec{0}$, $\vec{u}$, and $\vec{v}$.

This is the "tip to tail" rule.