CSE 3380 - The Matrix Equation
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1 The Matrix Equation

Previously, we've been using

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Definition 1 *If A is an $m \times n$ matrix with columns a_1, a_2, ..., a_n and if $\vec{x}$ is in $\mathbb{R}^n$, then the product of A and $\vec{x}$, denoted by $A\vec{x}$, is the linear combination of the columns of A and the weights of $\vec{x}$.*

$$A\vec{x} = \begin{bmatrix} \vec{a_1} & \vec{a_2} & \cdots & \vec{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1\vec{a_1} + x_2\vec{a_2} + \cdots + x_n\vec{a_n}$$

Columns of A (n) must be equal to the number of weights in $\vec{x}$. Otherwise, the product is undefined.

Example

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Example

For $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3} \in \mathbb{R}^m$, write the linear combination

$$3\vec{v_1} - 5\vec{v_2} + 7\vec{v_3}$$

as a linear matrix times a vector.

The weights $\vec{x} = \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}$

Note: we can write an inline vector line this: $\vec{w} = \begin{pmatrix} 3 & 5 & -7 \end{pmatrix}$. This is the same as $\vec{x}$ above.

$$3\vec{v_1} - 5\vec{v_2} + 7\vec{v_3} = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix} = A\vec{x}$$

Example

Write the following system as a linear matrix times a vector:

$\begin{cases} x_1 + 2x_2 - x_3 = 4 \\ -5x_2 + 3x_3 = 1 \end{cases}$

We can write the weights $\vec{x}$ times each column of the system's matrix

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Then pull out the x's into an $\vec{x}$ vector.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

This is in the form $A\vec{x} = \vec{b}$

Theorem 1 *if A is an $m \times n$ matrix, with columns $\vec{a_1}$, $\vec{a_2}$, ..., $\vec{a_n}$, and if $\vec{b} \in \mathbb{R}^m$, the matrix equation*

$$A\vec{x} = \vec{b}$$

has the same solution set as the vector equation

$$x_1\vec{a_1} + \cdots + x_n\vec{a_n} = \vec{b}$$

which also has the same solution set as the linear system

$$\begin{bmatrix} \vec{a_1} & \vec{a_2} & \cdots & \vec{a_n} & \vec{b} \end{bmatrix}$$

2 Existence of Solutions

The equation $A\vec{x} = \vec{b}$ has a solution if, and only if, $\vec{b}$ is a linear combination of the columns of A.

Example

Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}, \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

is $A\vec{x} = \vec{b}$ consistent for all possible values of b_1, b_2, b_3?

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}$$

We can write the last equation out

$$0 = b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1)$$

This is not consistent for **every** set of b values (for example, $b_1 = 0$, $b_2 = 0$, $b_3 = 1$), but there do exist solutions. The answer to the question is **no**, this system is not consistent for every set of b values.

Span$\left\{\vec{b_1}, \vec{b_2}\right\}$ forms a plane in $\mathbb{R}^3$. Any value of b_3 chosen that fits on that plane will be a solution, but values exist outside of that plane.

Theorem 2 *Let A be an $m \times n$ matrix, the following statements are logically equivalent:*

1. *For each $\vec{b} \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution*
2. *Each $\vec{b} \in \mathbb{R}^m$ is a linear combination of cols(A)*
3. *cols(A) $\in$ Span $\{\mathbb{R}^m\}$*
4. *A has a pivot in every row*

Proof

(You don't need to know this for the homework/exams, but it's here nonetheless.)

Focus on 1 and 4.

Let U be an echelon form of A. Given $\vec{b} \in \mathbb{R}^m$, we can row reduce the augmented matrix $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ to $\begin{bmatrix} U & \vec{d} \end{bmatrix}$ for some $\vec{d} \in \mathbb{R}^m$.

If (4) is true, each row of U contains a pivot position, and there can be no pivot in the augmented column (remember that the system is inconsistent if there is a pivot position in the augmented column).

So $A\vec{x} = \vec{b}$ has a solution, for any $\vec{b}$ and (1) is true.

If (4) is false, the last row if U is all zeroes. Let $\vec{d}$ be any vector with a 1 in it's last position. Then $\begin{bmatrix} U & \vec{d} \end{bmatrix}$ is inconsistent, so $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ is also inconsistent.

Compute $A\vec{x} = \vec{b}$

Given some $A = \begin{bmatrix} a_1 1 & a_1 2 & a_1 3 \\ a_2 1 & a_2 2 & a_2 3 \\ a_3 1 & a_3 2 & a_3 3 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, compute $A\vec{x}$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} a_{11}x_1 & a_{12}x_2 & a_{13}x_3 \\ a_{21}x_1 & a_{22}x_2 & a_{23}x_3 \\ a_{31}x_1 & a_{32}x_2 & a_{33}x_3 \end{bmatrix}$$

$$b_i = \sum_j (a_{ij} x_j)$$