CSE 3380 - Matrix Powers, Inverse, and Transposition
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Last lecture we went over matrix multiplication. We can quickly summarize what we went over with

$$(AB)_{ij} = \sum_k a_{ik}\, b_{kj}$$

1 Matrix Multiplication Differences

Matrix multiplication has a few differences than regular algebraic multiplication

1. Generally, $AB \neq BA$. There are values where $AB = BA$ is true, but for many values this is not true.
2. $AB = BC$ does not imply $A = C$. Again, there are values where this is true, but it is not always true.
3. If $AB = 0$, it does not imply that A or B are 0.

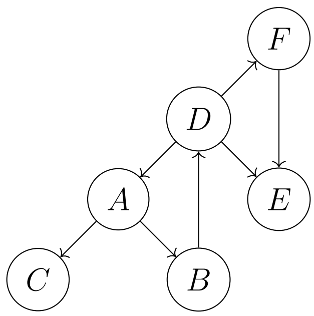
2 Powers of a Matrix

If $A \in \mathbb{R}^{n \times n}$ and if k is a positive integer, then

$$A^k = \underbrace{A \cdot A \cdots A}_{k}$$

A^0 is interpreted as the identity matrix.

Application - Graph Steps



The graph shown is a directed graph with 6 nodes. We can generate what's called an **adjacency matrix**, which denotes the paths from one node to all others.

We'll create a new $n \times n$ matrix. Each row will correspond to which nodes can be reached by a certain node in one step; the adjacent nodes.

For example, node A can reach nodes C and B in one step, but no others. Because of this, node A's row in the adjacency matrix will look like

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The 1's corresponding to columns 2 and 3, for nodes B and C. We can write out the entire adjacency matrix,

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

A^k will tell us which nodes we can reach in k steps. Lets calculate A^2 to find the nodes reachable from any other node in 2 steps.

$$A^2 = AA = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

As an example, taking the second row of this matrix $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ tells us that node B can reach nodes A, E, and F in exactly 2 steps. This is evident from the graph.

3 Transposition

The **transpose** of a matrix A, denoted by A^T, is a matrix whose columns are formed by the rows of A.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

For square matrices, the values just flip over the main diagonal

$$\begin{bmatrix} a & & b \\ & \ddots & \\ c & & d \end{bmatrix}$$

This is related to the determinant that we'll talk about later. Just keep it in the back of your brain.

For non square matrices, just transpose the rows to columns and columns to rows. The dimensions will swap, so double check those; ie. $(3 \times 2) \implies (2 \times 3)$.

Properties of Transpose

a. $(A^T)^T = A$

b. $(A+B)^T = A^T + B^T$

c. $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

4 Matrix Inverse

In algebra, we have a reciprocal, or multiplicative inverse, such that

$$a \cdot a^{-1} = a\,\frac{1}{a} = 1$$

in linear algebra, we have the **inverse**.

We want to find some matrix C such that

$$CA = I \qquad \text{and} \qquad AC = I$$

We denote the inverse of A is A^{-1}

$$A \cdot A^{-1} = I \qquad \text{and} \qquad A^{-1} \cdot A = I$$

Example

Show that $C = A^{-1}$, the inverse of A.

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & -1 \\ -1 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 0 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So $C = A^{-1}$

The Determinant

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $ad - bc \neq 0$, then A is invertable, and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is <u>not</u> invertable.

The denominator $ad - bc$ is called the **determinant**, denoted by

$$\det A = ad - bc$$

Example

Find the inverse of $A = \begin{bmatrix} 3 & 4 \\ -6 & -9 \end{bmatrix}$

First we find the determinant

$$\det A = (3)(-9) - (4)(-6) = -3$$

Then we can plug all those values into the equation given above

$$A^{-1} = \frac{1}{-3} \begin{bmatrix} -9 & -4 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4/3 \\ -2 & -1 \end{bmatrix}$$

Theorem 1 *If A is an invertable $n \times n$ matrix, for each $\vec{b} \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has the unique solution*

$$\vec{x} = A^{-1}\vec{b}$$

This is analogous to the following:

$$\text{If } ab = c, \text{ then } a = b^{-1}c$$