Foresight

Extra credit homework questions 1 and 2:

1. Find an orthogonal basis for the given set of vectors using the Gram-Schmidt process.

$$\dots$$

2. Orthogonalize the given set of vectors using the Gram-Schmidt process.

$$\dots$$

Keep those in mind when listening to this lecture.

We left off talking about orthogonal projections. Long story short, when writing projections by hand, we'll generally want to use

$$\vec{y} = \sum_i \frac{\vec{y} \cdot \vec{u_i}}{\vec{u_i} \cdot \vec{u_i}} \vec{u_i}$$

Dillhoff goes through an example that he says is silly and contrived so I won't include it here. It shouldn't be on the homework or exams. It goes through a very long process, which can all be boiled down to $||\vec{x} - \vec{x}|| = 0$ if the vectors are the same.

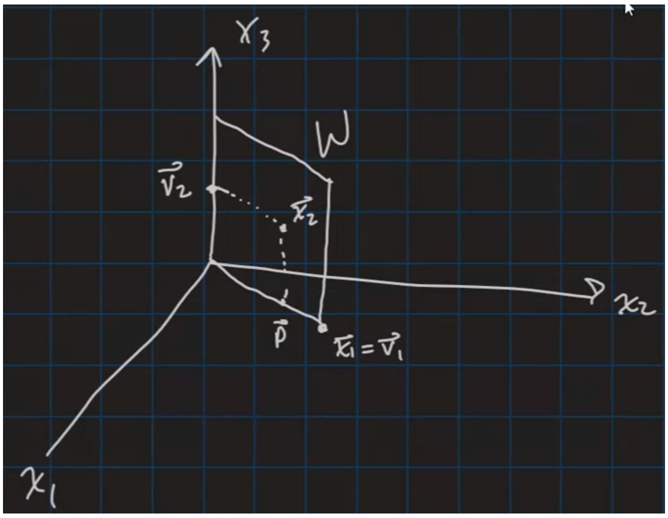
1 The Gram–Schmidt Process

Given a nonzero subspace of $\mathbb{R}^n$, we can produce an orthogonal basis for it through the Gram–Schmidt process.

Let $W = \text{Span}\{\vec{x_1}, \vec{x_2}\}$, where $\vec{x_1} = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\vec{x_2} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Construct an orthogonal basis for W. We want to find two vectors $\{\vec{v_1}, \vec{v_2}\}$ that form an orthogonal basis. We know that both x vectors are already in W by definition. We can choose an arbitrary vector (the first one, in this case) and then find another vector that is orthogonal to it to get an orthogonal basis.

1. Pick $\vec{v_1} = \vec{x_1}$
2. $\vec{x_2} - \vec{p} = \vec{v_2}$ will be orthogonal to $\vec{v_1}$ for some vector $\vec{p}$.



Here, we see the original vectors $\vec{x_1}$ and $\vec{x_2}$ which span W. Then we see the projection of $\vec{x_2}$ onto $\vec{x_1}$ at the vector $\vec{p}$. $\vec{x_2} - \vec{p}$ will give us the vector $\vec{v_2}$, which is orthogonal to $\vec{v_1}$, forming a basis.

What is $\vec{p}$?

$$\vec{p} = proj_{\vec{x_1}} \vec{x_2} = \frac{\vec{x_2} \cdot \vec{x_1}}{\vec{x_1} \cdot \vec{x_1}} \vec{x_1} = \frac{15}{45} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

So to get $\vec{v_2}$, we use

$$\vec{v_2} = \vec{x_2} - \vec{p} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

We our orthogonal basis is

$$B = \left\{ \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$$

Theorem 1 (The Gram–Schmidt Process) *Given a basis $\{\vec{x_1}, \cdots, \vec{x_n}\}$ for a nonzero subspace W of $\mathbb{R}^n$, define*

$$\vec{v_1} = \vec{x_1}$$

$$\vec{v_2} = \vec{x_2} - \frac{\vec{x_2} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1}$$

$$\vec{v_3} = \vec{x_3} - \frac{\vec{x_3} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1} - \frac{\vec{x_3} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}} \vec{v_2}$$

$$\vdots$$

$$\vec{v_n} = \vec{x_n} - \frac{\vec{x_n} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1} - \frac{\vec{x_n} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}} \vec{v_2} - \cdots - \frac{\vec{x_n} \cdot \vec{v_{n-1}}}{\vec{v_{n-1}} \cdot \vec{v_{n-1}}} \vec{v_{n-1}}$$

Then $\{\vec{v_1}, \cdots, \vec{v_n}\}$ is an orthogonal basis for W. Additionally,

$$\mathit{Span}\{\vec{x_1}, \cdots, \vec{x_k}\} = \mathit{Span}\{\vec{v_1}, \cdots, \vec{v_k}\} \text{ for } 1 \le k \le n$$

Example

Let $\vec{x_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{x_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

$W = \text{Span}\{\vec{x_1}, \vec{x_2}, \vec{x_3}\}$. Find an orthogonal basis.

1. $\vec{v_1} = \vec{x_1} \qquad (W_1 = \text{Span}\{\vec{v_1}\})$
2. $\vec{v_2} = \vec{x_2} - proj_{W_1}\vec{x_2}$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

2b. Multiplication by this vector with fractions may be messy. We can optionally scale the vector with no change to the basis because they're orthogonal. We'll multiply by 4 and make $\vec{v_2} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and let $W_2 = \text{Span}\{\vec{v_1}, \vec{v_2}\}$

3. $\vec{v_3} = \vec{x_3} - proj_{W_2}\vec{x_3}$

$$proj_{W_2}\vec{x_3} = \frac{\vec{x_3} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1} + \frac{\vec{x_3} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}} \vec{v_2} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$\vec{v_3} = \vec{x_3} - \begin{bmatrix} 0 \\ 2/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Next lecture we'll finish up the Gram–Schmidt Process and go over QR Factorization.