1 More on Eigenvectors and Eigenvalues

Last lecture we talked about eigenvectors and values. For a transformation A, there could be vectors that satisfy

$$A\vec{v} = \lambda\vec{v} \qquad \lambda \in \mathbb{R}$$

$\vec{v}$ is called the **eigenvector** and λ is called the **eigenvalue**.

We left off talking about rotation transformations. in 2D, there are no eigenvectors for a rotation, but in 3D the eigenvector will be the axis of rotation.

Theorem 1 *The eigenvalues of a triangular matrix are entries on its main diagonal.*

To prove this, let A be an upper triangular 3×3 matrix. Then

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}$$

When λ is equal to any of the values on the main diagonal, it would set one of the pivot values to 0. Remember that λ is an eigenvalue if, and only if, $(A - \lambda I)\vec{x} = \vec{0}$ has a nontrivial solution. Having a free variable gives us nontrivial solutions.

This only happens if some $a_{ii} = \lambda$.

What does a 0 eigenvalue represent? $\lambda = 0$ implies that the equation

$$A\vec{x} = 0\vec{x}$$

has a nontrivial solution. This happens when A is not invertible.

Theorem 2 *If $\vec{v_1}, \cdots, \vec{v_r}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \cdots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\vec{v_1}, \cdots, \vec{v_r}\}$ is linearly independent.*

Given a square matrix A, when or how can we find eigenvalues?

Example

Given $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$, find all λ such that $(A - \lambda I)\vec{x} = \vec{0}$ has a nontrivial solution.

We only need to compute when $\det(A - \lambda I) = 0$.

$$\det \begin{bmatrix} 3 - \lambda & 1 \\ 0 & 2 - \lambda \end{bmatrix} = 0$$

Looking at the values on the diagonal, the determinant would be 0 if $\lambda = 2$ or 3. We can write $ad - bc = 0$ to find values for λ

$$(3 - \lambda)(2 - \lambda) = 0$$

This scalar equation $\det(A - \lambda I) = 0$ is called the **characteristic equation**.

Similarity

If $A, B \in \mathbb{R}^{n\times n}$, then A is similar to B if there is an invertible P such that $P^{-1}AP = B$ or $A = PBP^{-1}$.

If we let $Q = P^{-1}$, then $Q^{-1}BQ = A$. So B is similar to A.

The process of changing A into $P^{-1}AP$ is a **similarity transform**.

Theorem 3 *If $A, B \in \mathbb{R}^{n\times n}$ are similar, then they have the same characteristic polynomial equation and the same eigenvalues.*

2 Diagonalization

$A = PDP^{-1}$ is a factorization, where D is a diagonal matrix.

Example

If $D \in \mathbb{R}^{2\times 2}$ is diagonal, D^n is easy to compute.

$$D^k = \begin{bmatrix} d_{11}^k & 0 \\ 0 & d_{22}^k \end{bmatrix}$$

A^k will be easy to compute.

Example 2

Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$, and $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

Find a formula for A^k.

First we want to find P^{-1}

$$P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$A^2 = (PDP^{-1})(PDP^{-1})$$

$$= PD(P^{-1}P)DP^{-1}$$

$$= PDDP^{-1}$$

$$= PD^2P^{-1}$$

Generalizing for A^k

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

A square matrix is diagonalizable if it is similar to a diagonal matrix... if $A = PDP^{-1}$ for some invertible matrix P and a diagonal matrix D.

Theorem 4 (The Diagonalization Theorem) *$A \in \mathbb{R}^{n\times n}$ is diagonalizable if, and only if, A has n linearly independent eigenvectors.*

$A = PDP^{-1}$, with diagonal D, if and only if the columns of P are n linearly independent eigenvectors of A.

In this case, the diagonal entries of D are the corresponding eigenvalues.

This is saying that A is diagonalizable if, and only if, it has n eigenvectors that form a basis for $\mathbb{R}^n$.

Example

Diagonalize

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

We want to factorize this as $A = PDP^{-1}$.

1. Find the eigenvalues of A.

$$\det(A - \lambda I) = 0$$

$$\implies -\lambda^3 - 3\lambda^2 + 4$$
$$-(\lambda - 1)(\lambda + 2)^2$$

so the eigenvalues are $\lambda = 1$, $\lambda = -2$

2. Find the eigenvectors of A.

Basis for $\lambda = 1$: $\vec{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

for $\lambda = -2$: $\vec{v_2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\vec{v_3} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

> He doesn't have enough time to finish this example so we're picking it up next lecture.