1 Vector Spaces

A **vector space** is a nonempty set V *over a field* of objects, called vectors, on which two operations are defined: (1) addition and (2) scalar multiplication.

These operations are subject to the following axioms and hold for all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ and scalar values c and d:

1. addition, $\vec{u} + \vec{v}$, is closed under V
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
4. There exists some $\vec{0}$ such that $\vec{u} + \vec{0} = \vec{u}$
5. For each $\vec{u}$, there exists some $-\vec{u}$ such that $\vec{u} + (-\vec{u}) = \vec{0}$
6. $c\vec{u} \in V$
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
8. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
9. $c(d\vec{u}) = (cd)\vec{u}$
10. $1\vec{u} = \vec{u}$

Being "**closed**" in a set just means that an operation between 2 elements of a set will produce something that is also in the set.

Properties derived from these axioms:

1. $\vec{0}$ is unique (there's only 1 zero vector)
2. $-\vec{u}$ is unique for each $\vec{u}$ in V
3. $c\vec{u} = \vec{0}$ if $c = 0$
4. $c\vec{0} = \vec{0}$
5. $-\vec{u} = (-1)\vec{u}$

Dillhoff says that the main axioms we need here are **addition**, **scalar multiplication**, and showing that **the zero vector exists**. Most of these axioms are self evident. If we can show that addition and scalar multiplication are closed, and that the zero vector exists, then we'll be in good shape.

This is not in the book, Dillhoff thinks it's good for us to know.

A **number field** is any set of k objects which also have some axioms like the ones above, where things are closed under addition and scalar multiplication. If you add any two items from the field, the sum is also in the field. The same is true for multiplication. Things like integers, rational numbers, complex numbers, etc. are all examples of number fields.

Example

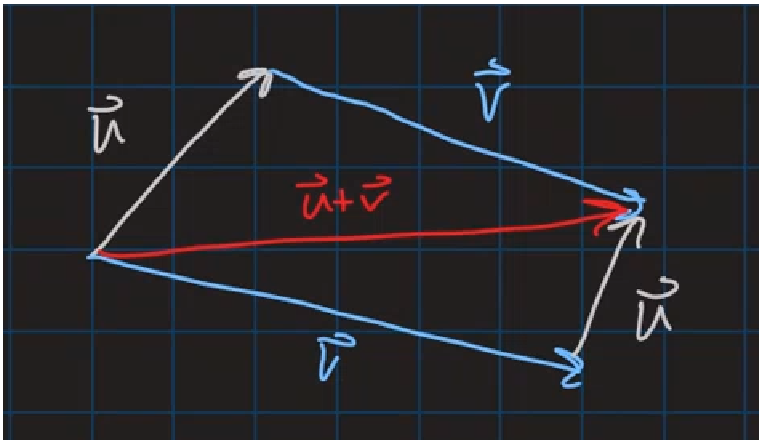
Let V_3 be the set of arrows in 3D.

We want to show that V_3 is a vector space, so we need to show that

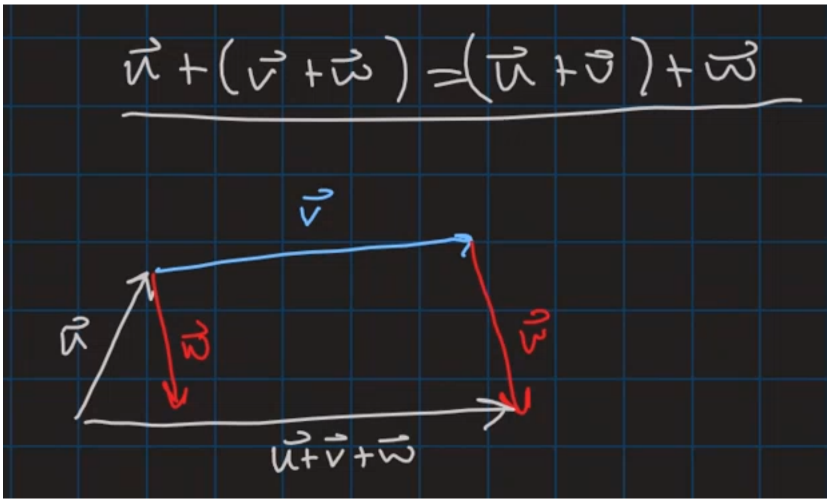
- two vectors are closed under addition
- two vectors are closed under scalar multiplication
- $\vec{0}$ is just an arrow with no length and arbitrary direction.

Addition

Let some $\vec{u}$ and some $\vec{v}$ like so. If we can draw the result of $\vec{u}+\vec{v}$, then it's in the vector space (for this particular problem). We can just show this geometrically.

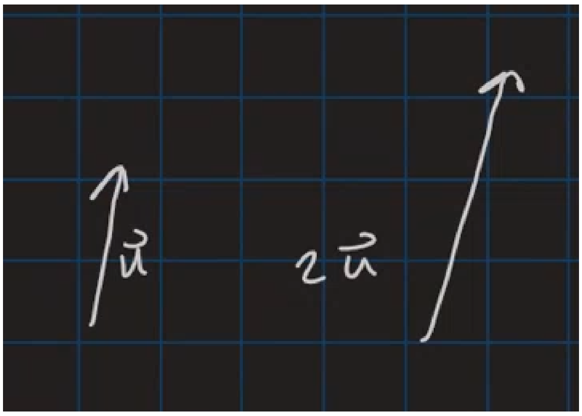


We can also extend this to the property $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$



Scalar Multiplication

We can show that the product of a vector and scalar are closed in the vector space in the same way. Let's just multiply a vector by a scalar.



Example

This is an example that you would see very often when doing any soft of sampling work, like discrete time signals or sampling audio.

Let S be the space of all doubly infinite sequences of numbers.

$$\{y_k\} = (\cdots, y_{-2}, y_{-1}, y_0, y_1, y_2, \cdots)$$

We're going from $-\infty$ to $+\infty$. In the example of time, we can't really say where the beginning or end are because time can be infinitely subdivided.

$\{Z_k\}$ is another element in S

$\{y_k\} + \{Z_k\}$ is the sequence $\{y_k + Z_k\}$ formed by adding corresponding terms.

For scalar multiplication, $c\{y_k\} = \{cy_k\}$

Example

For $n \geq 0$, the set P_n of polynomials of degree n consists of the following

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n$$

Where a_i are the coefficients. a_i and t are both real numbers.

We already know that real numbers are a vector space.

- We can add two different polynomials in terms of t to show that they can be added

$$(p + q)(t) = p(t) + q(t) = (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n$$

- If all coefficients $a_i = 0$, then p is the zero polynomial.
- $(cp)(t) = cp(t) = ca_0 + (ca_1)t + \cdots + (ca_n)t^n$

Example: Real Value Functions

Let V be the set of all real-valued functions on a set $\mathbb{D}$.

- **Addition**

  $f + g$ is the function whose value at $t \in \mathbb{D}$ is $f(t) + g(t)$.

$$f(x) = x^2 + 1 \qquad g(x) = x^3$$

$$(f + g)(t) = x^3 + x^2 + 1$$

- **Scalar multiplication**

  cf is the function whose value at $t \in \mathbb{D}$ is $c \cdot f(t)$

$$(2g)(t) = 2 \cdot g(t) = 2x^3$$

- **Zero Vector**

  $\vec{0}$ is a function that is identically 0, $f(t) = 0 \quad \forall t$

Example (2 in 2.4)

Let W be the union of the first and third quadrants in the xy-plane

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$$

a) if $\vec{u} \in W$ and c is any scalar, is $c\vec{u} \in W$?

$$c \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix} \implies (cx)(cy) = c^2(xy) \geq 0$$

Therefore, $c\vec{u} \in W$

b) Find vector $\vec{u}$ and $\vec{v} \in W$ such that $\vec{u} + \vec{v} \notin W$

We can find a specific counter example. We know that if an element is in the set, then the multiplication of the elements is positive. That is evident from the definition of W, but also makes sense if you consider the restriction of the first and third quadrants. Either both x and y are positive (first quadrant) or both are negative (third quadrant).

$$\vec{u} = \begin{bmatrix} -2 \\ -6 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\vec{u} + \vec{v} = \begin{bmatrix} -2 \\ -6 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \notin W$$

Therefore, addition is not closed $\implies$ W **is not a vector space**. (You can think about this graphically. Try to find two vectors, one in quad 1 and the other in quad 3, that sum to a new vector in quads 2 or 4).