CSE 3380 - Elementary Operations and Row Echelon Form
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1 Review of Previous Example

We ended last class with an example of solving a matrix using elementary operations. Here's that example again, briefly.

$$\begin{matrix} +x_1 & -2x_2 & +x_3 & =0 \\ & +2x_2 & -8x_3 & =8 \\ +5x_1 & & -5x_3 & =+10 \end{matrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \underset{-5 \cdot eq.1}{\rightarrow} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix} 1/2 \cdot eq.2 \rightarrow$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{bmatrix} \underset{-10 \cdot eq.2}{\rightarrow} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{bmatrix} \underset{1/30 \cdot eq.3}{\rightarrow *} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{matrix} -1 \cdot eq.3 \\ +4 \cdot eq.3 \rightarrow \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \overset{+2 \cdot eq.2}{\rightarrow} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Now it's an identity matrix and we have a set of solutions $(x_1, x_2, x_3) = (1, 0, -1)$.

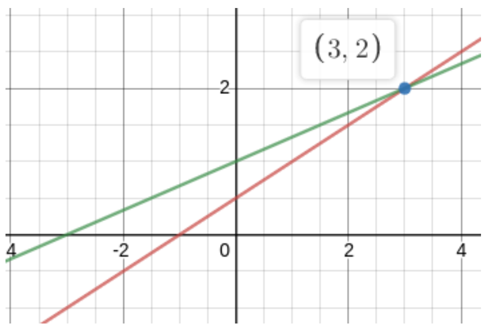
Up until the $\rightarrow *$ step was the "forward pass", getting the augmented matrix to be a triangular form. Then came the "backwards pass", getting the triangular form to an identity matrix.

2 What do these operations look like Geometrically?

Let's take a simple system

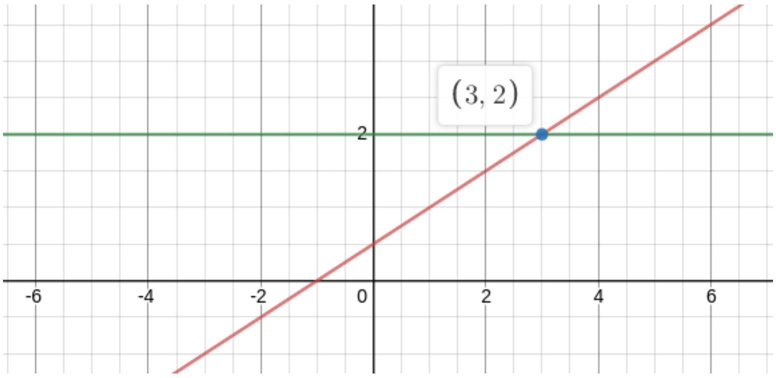
$$\begin{matrix} +x_1 & -2x_2 & =-1 \\ -x_1 & +3x_2 & =+3 \end{matrix} \rightarrow \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \end{bmatrix}$$

We can graph the two lines that these lines form.



From the graph we can already see the solution $(3, 2)$. Let's use elementary operations like we did in the first example to see how the lines change.

$$\begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \end{bmatrix} \underset{+eq1}{\rightarrow} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$



When we changed the second equation, the graph changed but the point of intersection (the solution) did not. As long as we perform valid operations on a system, the solution will remain the same.

3 Elementary Row Operations

There are 3 elementary row operations

1. Replacement

   Replacing an equation with another, or a scaled version of another.

2. Exchange

   Replacing a row with another row.

3. Scaling

   Multiplying every term in a row by the same factor.

Definition: Two matrices are **row equivalent** if there is a sequence of elementary row operations that transforms one into the other. In the above examples, every matrix produced through a step is row equivalent with the others.

(He doesn't really explain uniqueness here)

Example 1

Given the following system, determine if the system is consistent, and if so, find the solution.

$$\begin{Bmatrix} & +x_2 & -4x_3 & = 8 \\ +2x_1 & -3x_2 & +2x_3 & = 1 \\ +4x_1 & -8x_2 & +12x_3 & = 1 \end{Bmatrix}$$

First, we'll put it in augmented form then start performing steps to bring it to an identity matrix.

$$\begin{bmatrix}0 & 1 & -4 & 8\\ 2 & -3 & 2 & 1\\ 4 & -8 & 12 & 1\end{bmatrix} \begin{matrix}\text{swap eq. 2}\\ \text{swap eq. 1}\\ \rightarrow\end{matrix} \begin{bmatrix}2 & -3 & 2 & 1\\ 0 & 1 & -4 & 8\\ 4 & -8 & 12 & 1\end{bmatrix} \begin{matrix}\\ \rightarrow\\ -2\cdot eq.1\end{matrix} \begin{bmatrix}2 & -3 & 2 & 1\\ 0 & 1 & -4 & 8\\ 0 & -2 & 8 & -1\end{bmatrix}$$

$$\begin{matrix}\\ \rightarrow\\ +2\cdot eq.2\end{matrix} \begin{bmatrix}2 & -3 & 2 & 1\\ 0 & 1 & -4 & 8\\ 0 & 0 & 0 & 15\end{bmatrix}$$

And here we can determine that the system is inconsistent because of eq. 3, which states $0 = 15$.

Example 2

Given the system, find it's solutions, if any.

$$\begin{Bmatrix} +2x_1 & +4x_2 & = -4\\ +5x_1 & +7x_2 & = 11 \end{Bmatrix}$$

We'll just use elementary row operations:

$$\begin{bmatrix}2 & 4 & -4\\ 5 & 7 & 11\end{bmatrix} \begin{matrix}1/2\cdot eq.1\\ \rightarrow\end{matrix} \begin{bmatrix}1 & 2 & -2\\ 5 & 7 & 11\end{bmatrix} \begin{matrix}\rightarrow\\ -5\cdot eq.1\end{matrix} \begin{bmatrix}1 & 2 & -2\\ 0 & -3 & 21\end{bmatrix} \begin{matrix}\rightarrow\\ -1/3\cdot eq.3\end{matrix} \begin{bmatrix}1 & 2 & -2\\ 0 & 1 & -7\end{bmatrix} \begin{matrix}-2\cdot eq.2\\ \rightarrow\end{matrix} \begin{bmatrix}1 & 0 & 12\\ 0 & 1 & -7\end{bmatrix}$$

Here we get an identity matrix, with $x_1 = 12$, $x_2 = -7$.

4 Echelon Form

Definition: a rectangular matrix is in **echelon form** (or **row echelon form**) if it has 3 properties:

1. all nonzero rows are above any rows of all zeroes.
2. each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. all entries in a column below a leading entry are zeroes.

For example, the following matrix is in row echelon form

$$\begin{bmatrix}2 & -3 & 2 & 1\\ 0 & 1 & -4 & 8\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Additionally, if the following properties are satisfied, the matrix is said to be in **reduced row echelon form** (RREF)

4. the leading entry in each nenzero row is 1.
5. each leading entry is the only nonzero value in it's column.

$$\begin{bmatrix}1 & 0 & 0 & 29\\ 0 & 1 & 0 & 16\\ 0 & 0 & 1 & 3\end{bmatrix}$$