CSE 3380 - Row Vector Rule and Parametric Vector Descriptions
February 6, 2021
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1 Row Vector Rule

Definition 1 *If $A\vec{x}$ is defined, then the i^{th} entry in $A\vec{x}$ is the sum of the products of the corresponding entries from row i of A and from $\vec{x}$.*

$$A\vec{x} = \vec{b} \qquad\qquad b_i = \sum_j A_{ij}x_j$$

Remember that the product $A\vec{x}$ is only defined when the sizes match.

Example

$$\begin{bmatrix} 3 & -2 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \begin{matrix} b_1 \\ b_2 \end{matrix}$$

$$b_1 = (A_{11})(x_1) + (A_{12})(x_2) = (3)(1) + (-2)(0) = 3$$
$$b_2 = (A_{21})(x_1) + (A_{22})(x_2) = (6)(1) + (1)(0) = 6$$

Here, we're moving from column to column in A, row to row in $\vec{x}$.

Note: in this example, $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a "binary mask". This one selects the first column of A.

Identity Matrix Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (1\cdot x) & + & (0\cdot y) & + & (0\cdot z) \\ (0\cdot x) & + & (1\cdot y) & + & (0\cdot z) \\ (0\cdot x) & + & (0\cdot y) & + & (1\cdot z) \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

2 Properties of Vector-Matrix product $A\vec{x}$

(from the book, section 1.4, theorem 5)

Theorem 1 *If A is $m \times n$, and $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then*

a. $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$

b. $A(c\vec{u}) = c(A\vec{u})$

Dillhoff goes over a proof but we don't need to know it.

3 Homogeneous Linear Systems

Definition 2 *A linear system is called homogeneous if it can be written in the form*

$$A\vec{x} = \vec{0}$$

$$A \in \mathbb{R}^{\mathrm{m\times n}}$$
$$\vec{x} \in \mathbb{R}^{\mathrm{n}}$$
$$\vec{0} \in \mathbb{R}^{\mathrm{m}}$$

In other words, a system is homogeneous if all the constant bits are 0. Homogeneous systems are always consistent.

$A\vec{x} = \vec{0}$ always has at least 1 solution... the trivial solution: $\vec{x} = \vec{0}$. We're usually interested in finding any non-trivial solutions.

Does there exist a non-trivial solution?

Since $A\vec{x} = \vec{0}$ does not have a pivot in the rightmost column of the augmented matrix, it is always consistent. (And because of the trivial solution).

Theorem 2 *The system only has a non-trivial solution when the system has at least one free variable.*

Example

$$\begin{cases} 3x_1 + 5x_2 - 4x_3 = 0 \\ -3x_1 - 2x_2 + 4x_3 = 0 \\ 6x_1 + x_2 - 8x_3 = 0 \end{cases}$$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \begin{matrix} \\ +R1 \\ -2\cdot R1 \end{matrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \sim \\ +3\cdot R2 \end{matrix} \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here we can see that x_3 is free, so there exists a non-trivial solution.

$$\sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{cases} x_1 = \frac{4}{3}x_3 \\ x_2 = 0 \\ x_3 \text{ is free} \end{cases}$$

We can rewrite our solution set like this

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3\vec{v}$$

Example

Given a single equation

$$10x_1 - 3x_2 - 2x_3 = 0$$

Let's write it in terms of x_1

$$x_1 = .3x_2 + .2x_3 \qquad \Longrightarrow x_2 \text{ and } x_3 \text{ are free}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} .2x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}} + x_3 \underbrace{\begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}}$$

The solution is Span$\{\vec{u}, \vec{v}\}$. The trivial solution still exists.

The form we just used is the...

4 Parametric Vector Form

$$10x_1 - 3x_2 - 2x_3 = 0$$

is an implicit description of a plane.

The form

$$\vec{x} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

is called the parametric vector form, and explicit description of the same thing.