1 Linear Independence and Bases

Basis for Nul A

When a null space contains non-zero vectors, it contains the same amount of vectors as free variables. This set is linearly independent with free variables being the weights, this is a basis for Nul A.

Basis for Col A

Given this matrix in echelon form, we know that each non pivot column is a linear combination of the pivot columns. For example, $\vec{b_2} = 4\vec{b_1}$, $\vec{b_4} = 2\vec{b_1} - \vec{b_3}$

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using theorem 5, $\{\vec{b_1}, \vec{b_3}, \vec{b_5}\}$ Span Col A. None of these vectors is a linear combination, so this is our basis.

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

S is a basis for A. We've just chosen each column that contains a pivot position.

Theorem 1 *The pivot columns of a matrix A form the basis for Col A.*

Theorem 2 *If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the non-zero rows of B form a basis for both A and B.*

Example

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By the theorem above, the first 3 rows of B form a basis the row space of A.

Basis Summary

- The most efficient way to describe a space.
- Every vector in the set is necessary: adding a vector will make it not linearly independent, and removing a vector will prevent it from describing a space.

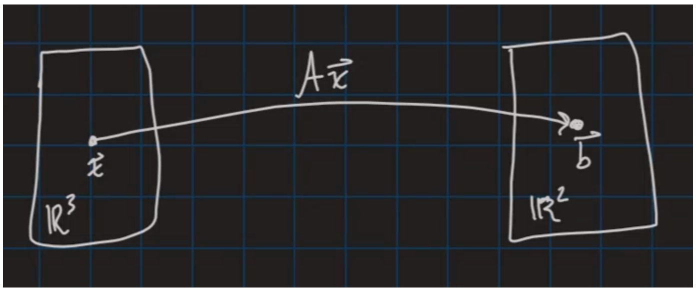
2 Linear Transformations

We'll start with a question: What is the matrix A in the matrix equation $A\vec{x} = \vec{b}$ doing?

A linear combination of cols(A) and the values of $\vec{x}$?

Let's view A as a transformation that **transforms** a vector $\vec{x}$ to some other vector.

Take a 2×3 matrix A and 3×1 vector $\vec{x}$. $A\vec{x} = \vec{b}$ is the equation that transforms $\vec{x}$ from 3D to 2D.



We can view solving $A\vec{x} = \vec{b}$ as finding all vectors in $\mathbb{R}^3$ that are transformed to $\mathbb{R}^2$.

a tranform is also called

- a **function** from one set to another
- a **mapping** between spaces/sets

Definition 1 *In general, a* ***transformation*** *T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\vec{x} \in \mathbb{R}^n$ a vector $T(\vec{x}) \in \mathbb{R}^m$.*

The set $\mathbb{R}^n$ is called the ***domain*** *of T and $\mathbb{R}^m$ is the* ***codomain****.*

Sometimes we'll see the notation

$T : \mathbb{R}^n \to \mathbb{R}^m$

This says that the domain of the transform T is $\mathbb{R}^n$, and the codomain is $\mathbb{R}^m$.

The vector $T(\vec{x})$ is called the **image** of $\vec{x}$. The set of all images is called the **range** of T.

Matrix Transformations

Let $A \in \mathbb{R}^{m \times n}$. The domain of T is $\mathbb{R}^n$ – the number of columns. The codomain of T is $\mathbb{R}^m$ – the number of rows.

$$\vec{x} \mapsto A\vec{x}$$

Example

Let

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \qquad \vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

Define a transformation $T : \mathbb{R}^n \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$ such that

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

a) find $T(\vec{u})$, the image of $\vec{u}$ under T.

$$T(\vec{u}) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

b) Solve $T(\vec{x}) = \vec{b}$ for $\vec{x}$.

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix} \implies \vec{x} = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$$

Another way to say this is that the image of $\vec{x}$ under T is $\vec{b}$.

c) Is there more than one $\vec{x}$ whose image under T is $\vec{b}$?

There is not more than 1 $\vec{x}$ whose image under T is $\vec{b}$. This is asking whether or not $A\vec{x} = \vec{b}$ has a unique solution.

d) Determine if $\vec{c}$ is in the range of T.

It is if it's the image of some $\vec{x} \in \mathbb{R}^n$, or if $\vec{c} = T(\vec{x})$.

The above is analogous to the following question: is the system $A\vec{x} = \vec{c}$ consistent?

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$$

$$0 \neq -35 \quad \therefore \ \vec{c} \text{ is not in the range of } T$$

Projection Matrices

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The transform $\vec{x} \mapsto A\vec{x}$ projects 3D points into x_1–x_2 plane. This is not necessarily a transform from $\mathbb{R}^3$ to $\mathbb{R}^2$.

If we set up the transformation,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

This will project onto the x_1–x_2 plane, but it will still be in $\mathbb{R}^3$.