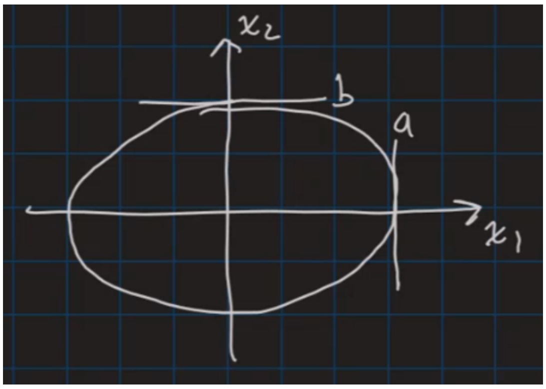
1 Determinants and Volumes, cont.

We left off last lecture talking about approximating areas and volumes with small squares or cubes. It's the classic example of how an integral gives the area under a curve.

Example: Ellipse

Let a and b be positive numbers. Find the area of the region E bounded by the ellipse whose equation is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$



We know that an area of a circle is given by πr^2. The area of a unit disk D $(r = 1)$ is $\pi(1)^2 = \pi$. We will define a transform T which maps the disk D (unit circle) to the ellipse E.

This transform needs to stretch the circle in the x_1 direction by a units and in the x_2 direction by b units.

$$T = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Now we map $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ to $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ by $\vec{x} = T\vec{u}$.

This implies that $u_1 = \frac{x_1}{a}$ and $u_2 = \frac{x_2}{b}$.

We can conclude that

$$\{\text{area of } E\} = \{\text{area of } T(D)\}$$

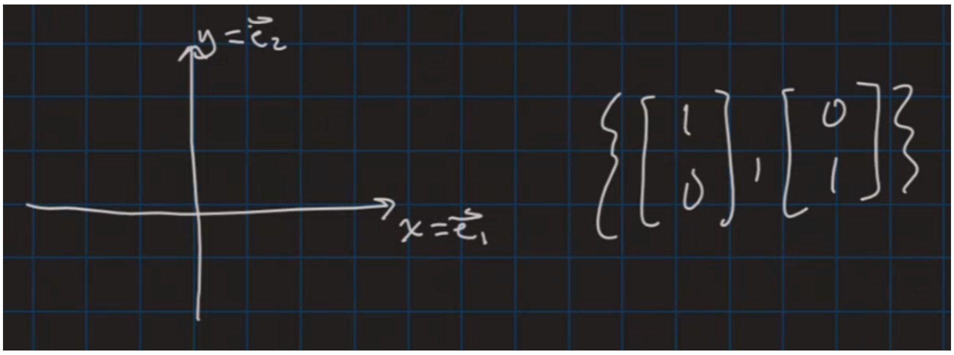
$$= |\det T| \cdot \{\text{area of D}\}$$

$$= |\det T| \cdot \pi = ab \cdot \pi$$

2 Coordinate Systems

Coordinate systems are just a formal description of representing points in a given vector space. They are defined as basis vectors.

Consider Cartesian coordinates



Cartesian coordinates have x on the horizontal axis and y on the vertical, or sometimes $\vec{e_1}$ and $\vec{e_2}$ or $\hat{i}$ and $\hat{j}$.

Because these are basis vectors, we can reach any point in the vector space with

$$\underbrace{c_1}_{\text{coordinate}} \cdot \vec{e_1} + \underbrace{c_2}_{\text{coordinate}} \cdot \vec{e_2}$$

Theorem 1 (Unique Representation Theorem) *Let $B = \left\{\vec{b_1}, \cdots, \vec{b_n}\right\}$ be a basis for a vector space V. Then for each $\vec{x} \in V$, there exists a unique set of scalars $c_1, \cdots, c_n$ such that*

$$\vec{x} = c_1\vec{b_1} + \cdots + c_n\vec{b_n}$$

Basically this means that for every vector in the vector space, there are coordinates to get there when multiplied by the basis vectors. Think of it like how $3\hat{i} + 5\hat{j}$ is just the vector to $(3, 5)$. 3 and 5 are c_1 and c_2, and $\hat{i}$ and $\hat{j}$ are the basis vectors $\vec{b_1}$ and $\vec{b_2}$.

Definition 1 (Coordinates) *If $B = \left\{\vec{b_1}, \cdots, \vec{b_n}\right\}$ is a basis for a vector space V and $\vec{x} \in V$, then the coordinates of $\vec{x}$ relative to B are the weights (coefficients) $c_1, \cdots, c_n$.*

If $c_1, \cdots, c_n$ are B-coordinates of $\vec{x}$, then a vector in $\mathbb{R}^n$ is

$$[\vec{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

This is the coordinate vector relative to B.

There is a transform $\vec{x} \mapsto [\vec{x}]_B$ which is the coordinate mapping.

The standard basis $\varepsilon = \{\vec{e_1}, \cdots, \vec{e_n}\}$. n is implied by content.

Unless otherwise specified, $\vec{x} = [\vec{x}]_\varepsilon$

Converting Between Coordinate Systems

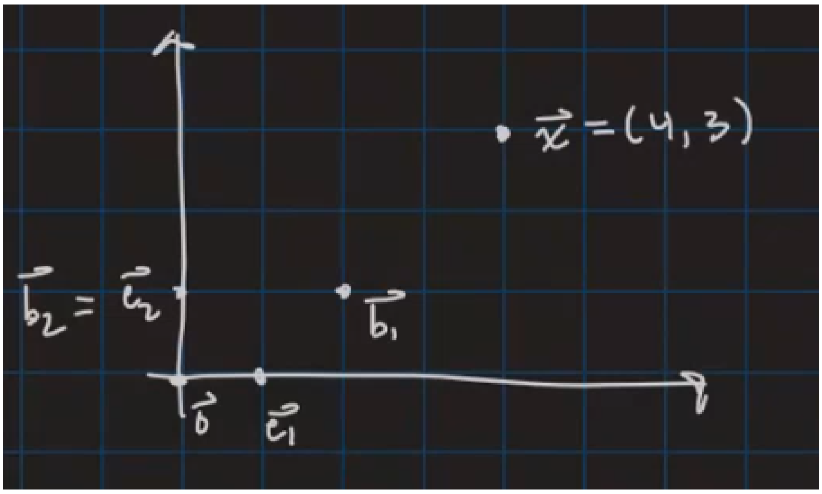
Example

Let ε be the standard basis in $\mathbb{R}^2$ and B be another coordinate system defined as

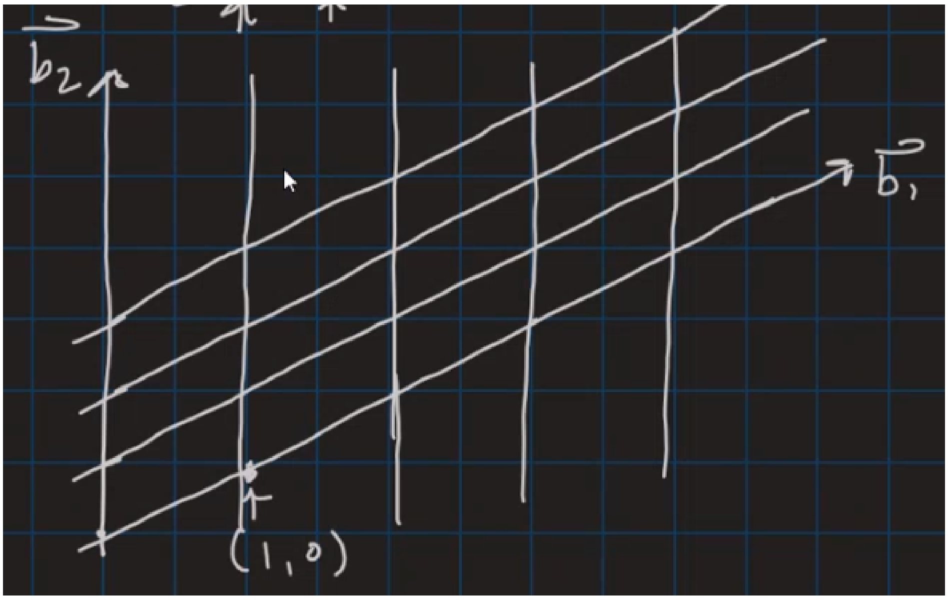
$$B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Let's look at a vector $\vec{x} = (4, 3)$ with respect to both coordinate systems.

This graph shows the vectors plotted with respect to ε:

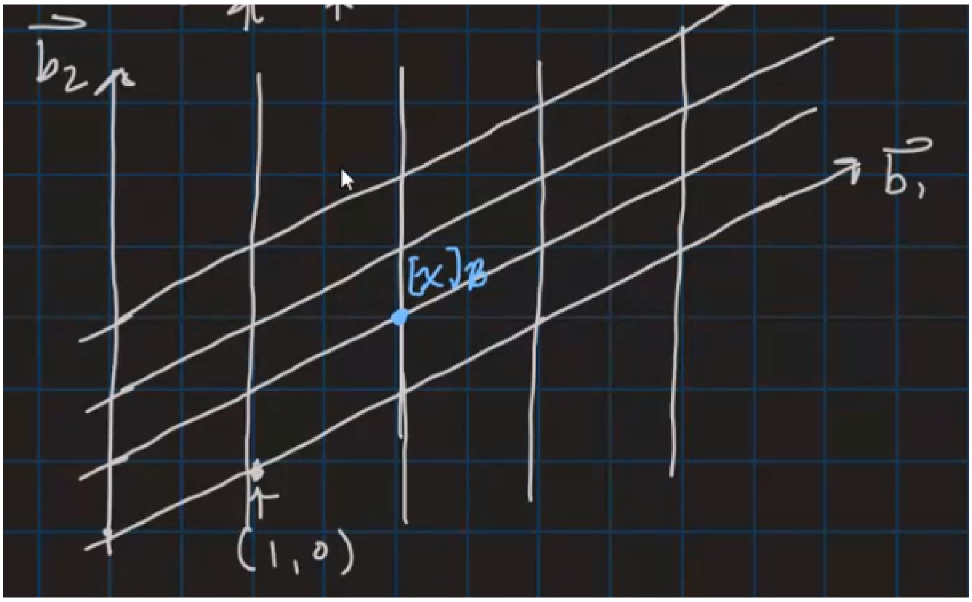


If we look at in with respect to B, we get:



Notice that the blue grid lines in the back are consistent. In the second graph, we're drawing a vertical line at every increment of the $\vec{b_1}$ vector. This is equivalent to (2, 1) in ε.

If we plot $\vec{x}_B$, we get:



$$\vec{x} = [\vec{x}]_\varepsilon = (4, 3)$$
$$[\vec{x}]_B = (2, 1)$$

It's easy to see how these spaces align because both grids are present: the white drawn grid is B while the blue grid is ε. But we need a way to calculate this.

Example

Let $\vec{b_1} = \begin{bmatrix}1\\2\end{bmatrix}$, $\vec{b_2} = \begin{bmatrix}1\\-1\end{bmatrix}$, $\vec{x} = \begin{bmatrix}2\\3\end{bmatrix}$, and $B = \left\{\vec{b_1}, \vec{b_2}\right\}$.

Find $[\vec{x}]_B$.

Let c_1, c_2 be coordinates of B such that

$$\vec{x} = c_1\vec{b_1} + c_2\vec{b_2}$$

$$\begin{bmatrix}1 & 1\\2 & -1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$$

We'll set up the augmented matrix $[A \;\; \vec{x}]$.

$$\begin{bmatrix}A & \vec{x}\end{bmatrix} = \begin{bmatrix}c_1\\c_2\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 2\\2 & -1 & 3\end{bmatrix} \sim \text{skipping steps...} \sim \begin{bmatrix}1 & 0 & 5/3\\0 & 1 & 1/3\end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix}5/3\\1/3\end{bmatrix}$$

We call $P_B = \begin{bmatrix}1 & 1\\2 & 1\end{bmatrix}$ the **change-of-coordinates matrix**. Thus, $\vec{x} = P_B\,[\vec{x}]_B$.

Recall from the IMT that if P_B form a basis in $\mathbb{R}^N$, P_B is invertible, so

$$P_B^{-1} \cdot \vec{x} = [\vec{x}]_B$$

Future Luke here: In summary, just remember:

$$B \cdot [\vec{x}]_B = \vec{x}$$

$$B^{-1} \cdot \vec{x} = [\vec{x}]_B$$