2.01

The Strong & Weak forms of 1D linear eliptic PDEs.

Given u_0, t, $f(x)$ and the constitutive relation

$$\sigma = E u_{,x}$$

STRONG FORM:

Find u such that

$$\frac{d\sigma}{dx} + f = 0 \text{ in } (0, L)$$

$$u(0) = u_0$$

$$\sigma(L) = t$$

WEAK FORM:

Find $u \in \mathcal{S} = \{u \mid u(0) = u_0\}$

such that

$$\forall\, w \in \mathcal{V} = \{w \mid w(0) = 0\}$$

$$\int_0^L w_{,x}\, \sigma A\, dx = \int_0^L w f A\, dx + w(L) t A$$

RECALL: these forms are completely equivalent.

- Approximations of the Strong Form result in Finite Difference Methods
- Approximations of the Weak Form result in the Finite Element Method!

These are infinite-dimensional function spaces.

$$u \in \mathcal{S} \; ; \; w \in \mathcal{V}$$

i.e. in the case of polynomials, we consider polynomials of all orders. From the constant term all the way up to x^n where n may be ∞.

Polynomial case is denoted as:

$$\text{If } \mathcal{S}, \mathcal{V} \in P^n(x), \quad n = 0, 1, 2 \ldots$$

Polynomials of order n.

Construct approximations in finite-dimensional function spaces.

e.g: $P^n(x)$, $n = 0, 1$

Restrict the solution space & weighting function space.

Find $\underbrace{u^h(x)}_{}$ $\in \mathcal{S}^h \subset \mathcal{S}$ subset

denotation
for approximation
of u (u"sup"h)

$$\mathcal{S}^h = \{ u^h \in H^1(0,L) \mid u^h(0) = u_0 \}$$

such that $\forall\ w^h \in \mathcal{V}^h \subset \mathcal{V}$

$$\mathcal{V}^h = \{ w^h \in H^1(0,L) \mid w^h(0) = 0 \}$$

$$\int_0^L w^h_{,x}\, \sigma^h A\, dx = \int_0^L w^h f A\, dx + w^h(L)\, t A$$

Finite-Dimensional Weak Form.

"Galerkin Weak Form"

The Finite-Dimensional Weak Form is NOT equivalent to the Strong Form — in general.

Recall the infinite-dimensional weak form.

Find $u \in \mathcal{S}$ Such that $\forall\ w \in \mathcal{V}$

$$\int_0^L w_{,x}\, \sigma A\, dx = \int_0^L w f A\, dx + w(L)\, t A$$

In proving this is equivalent, to the Strong Form we relied on $\underline{\underline{\forall}}\ w \in \mathcal{V}$.

Now: $u^h \in \mathcal{S}^h$, $w^h \in \mathcal{V}^h \subset \mathcal{V}$, so we lose the arguement of equivalence.

02.02

Function Spaces — Hilbert spaces. {Functional Analysis}

Recall $u^h \in \mathcal{S}^h = \{u^h \in H^1(0,L) \mid u^h(0) = u_0\}$

Consider a function $v : (0,L) \longmapsto \mathbb{R}$

Define the function v to be an L^2-function.
if

$$\int_0^L v^2 \, dx < \infty \quad \text{then } v \in L^2(0,L)$$

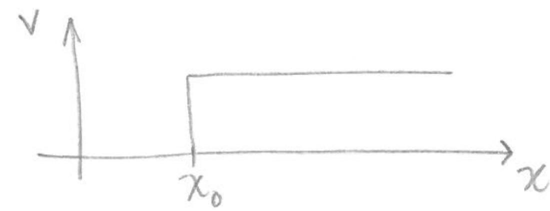
↳ i.e. the function is bounded on the interval.
↳ i.e the function v is L^2 if it is square integrable.

e.g. $v(x) = $ constant

$v(x) = \sum_{k=0}^{n} a_k x^k$ { a polynomial

$v(x) = H(x - x_0)$

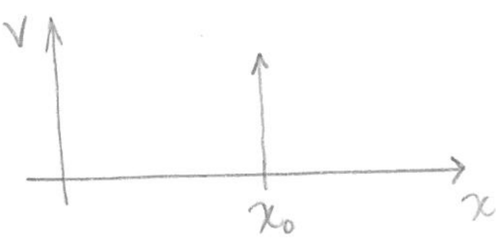
heaviside function located at x_0.



$\Bigg\}\ L^2$

$v(x) = \delta(x - x_0) \notin L^2(0,L)$

Delta function



One can in general define L^p functions, where $p \in \mathbb{R}$

How about control over the derivatives of v? (Regularity)

$v(x) \in H^1(0,L)$ if

$$\int_0^L \left[v^2 + L^2 (v_{,x})^2 \right] dx < \infty$$

to ensure correct units.

<u>Remark</u>: L^2 has been introduced for dimensional purposes.

In general use $m(0,L)^{1/d} = L$; d: spatial dimension

measure of the domain. $d=1$ for 1dimen

2.03

In $\mathbb{R}^3$, $d=3$

$m(\Omega)^{1/d} = m(\Omega)^{1/3} \approx$ "length."

$v \in H^1(0,L)$ if $\displaystyle\int_0^L \left[v^2 + \underbrace{(m(0,L))^2 v_{,x}^2}_{\text{we are controlling } v \text{ \& its first derivative.}} \right] dx < \infty$

Now: $u^h \in \mathcal{S}^h = \{ u^h \in H^1(0,L) \mid u^h(0) = u_0 \}$

$\therefore$ u^h is bounded and so is $u^h_{,x}$.

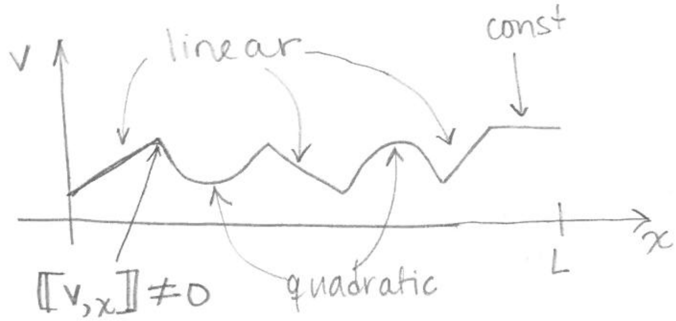
square integrable.

e.g. $H^1(0,L)$ functions

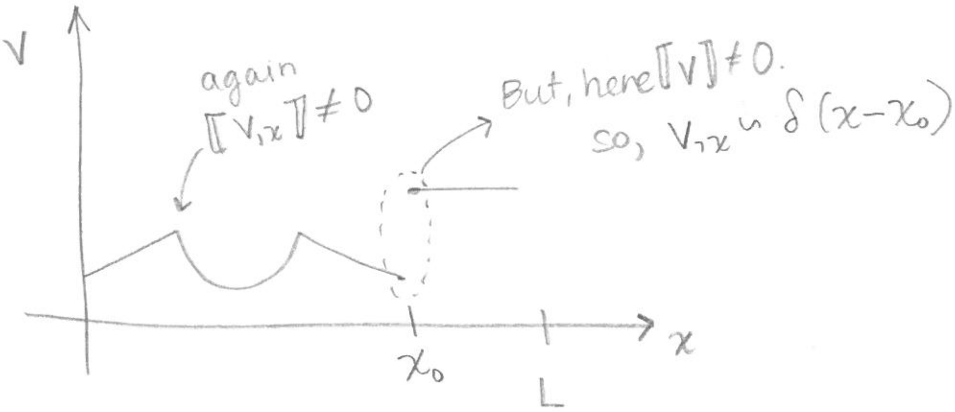
$V(x) =$ constant

$V(x) = \sum_{k=0}^{n} a_k x^k$

$V(x) =$ (see sketch)



But what about this?



so, $v \notin H^1(0,L)$

$\boxed{2.04}$ control over a funct. → Bounded

control over its derivative → Regularity

The Finite Element Method for linear, elliptic PDEs in 1D.

Recall: the Galerkin (Finite-Dimensional) Weak Form.

Find $u^h \in \mathcal{S}^h = \{u^h \in H^1(0,L) \mid u(0) = u_0\}$

often called "trial function"

such that $\forall\, w^h \in \mathcal{V}^h = \{w^h \in H^1(0,L) \mid w^h(0) = 0\}$

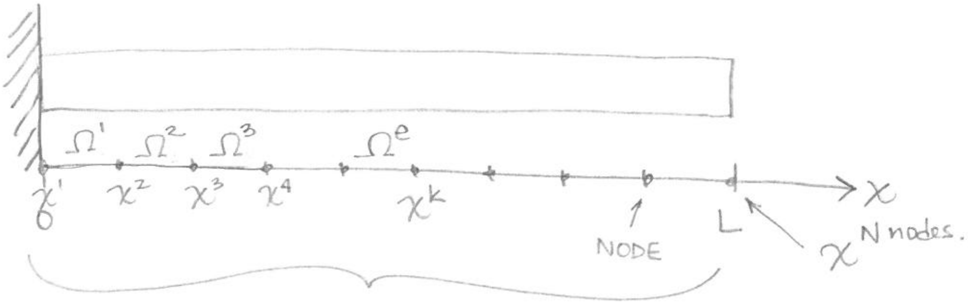
$$\int_0^L w^h_{,x}\, \sigma^h A\, dx = \int_0^L w^h f A\, dx + w^h(L)\, t A.$$

$\sigma^h = E u^h_{,x}$; $f(x)$ ← this is given, so no need to approximate. } NOT a finite dimension.

This also holds for t, but only in 1D case.

How do we obtain u^h, w^h? Alternately, $\mathcal{S}^h, \mathcal{V}^h$?

⇒ Partition $(0,L)$ into "finite elements" which are disjoint subdomains of $(0,L)$.



$\Omega = (0, L)$

↑
general domain

Partitioned Ω into subdomains Ω^e; Ω^e is an open subdomai

$$\overline{\Omega} = \overbrace{\bigcup_{e=1}^{n_{element}}}^{\text{union}} \Omega^e \qquad \Omega^e = (x^e, x^{e+1})$$

$\overline{\Omega}$ is the closure of Ω

$$\overline{\Omega} = \Omega \cup \underbrace{\partial\Omega}_{\text{boundary points of } \Omega}$$

x^e: nodes of the partition.

Ω^e: an element.

The Galerkin Weak Form:

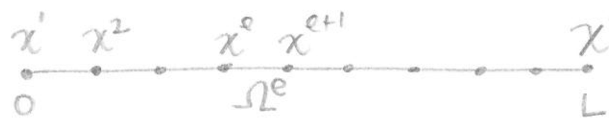
$$\int_\Omega w^h_{,x} \sigma^h A\, dx = \int_\Omega w^h f A\, dx + w^h(L) t A$$

$$\Rightarrow \sum_{e=1}^{n_{el}} \int_{\Omega^e} w^h_{,x} \sigma^h A\, dx = \sum_{e=1}^{n_{el}} \int_{\Omega^e} w^h f A\, dx + w^h(L) t A$$

Represent S^h and V^h over each Ω^e

Recall the Finite Element Partition.

$$\Omega^e = (x^e, x^{e+1})$$



$$\Omega = (0, L)$$

$[N = n_{el} + 1]$

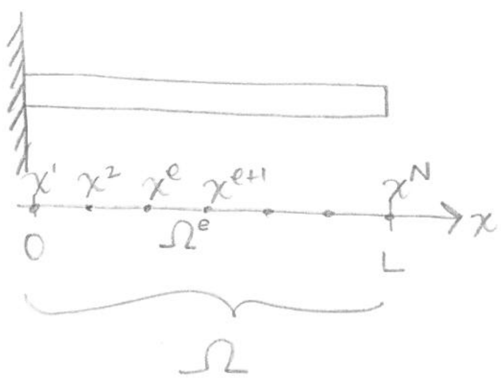
Ω contains all nodal points $\{x^2, x^3 \ldots x^{N-1}\}$

$$\Omega \neq \bigcup_{e=1}^{n_{el}} \Omega^e$$

Ω^e does not contain its nodal points.

$$\bar{\Omega} = \overline{\bigcup_{e=1}^{n_{el}} \Omega^e}$$ } this contains x^1 AND x^N!

$\boxed{2.05}$



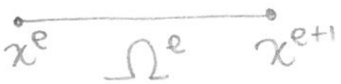
$N_{nodes} = n_{el} + 1$

Galerkin Weak Form

$$\sum_{e=1}^{n_{el}} \int_{\Omega^e} w^h_{,x} \sigma^h A \, dx = \sum_{e=1}^{n_{el}} \int_{\Omega^e} w^h f A \, dx + w^h(L) t \cdot A.$$

Representation of u^h and $w^h(x)$

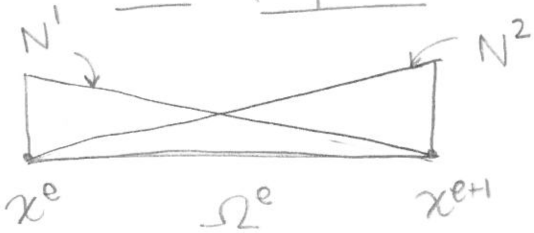
Need to represent u^h and $w^h(x)$ over Ω^e; $e = 1, \ldots, n_{el}$



Define local basis functions on Ω^e — finite number of basis functions — over Ω^e and therefore over Ω.

Two basis functions over Ω^e — polynomials, complete basis

$\Rightarrow$ linear polynomials on Ω^e



$N^1(x)$, $N^2(x)$ number of nodes in the element.

$$u_e^h(x) = \sum_{A=1}^{N_{ne}} N^A(x)\, d_e^A$$

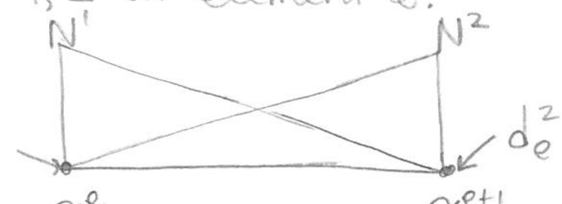
degree of freedom.

$$= N^1(x)\, d_e^1 + N^2(x)\, d_e^2$$

$N^A(x)$: basis function $A = 1, 2$

d_e^A : degree of freedom $A = 1, 2$ on element e.

$\boxed{2.06}$ $$u_e^h(x) = \sum_{A=1}^{N_{ne}} N^A(x)\, d_e^A$$



$N^A(x)$: nodal basis functions

d_e^A : nodal degrees of freedom.

Similarly for $w_e^h(x)$:

$$w_e^h(x) = \sum_{A=1}^{N_{ne}} \underbrace{N^A(x)}_{\text{basis is same}} \; \underbrace{c_e^A}_{\text{degree of freedom is different.}}$$

$\hookrightarrow$ Bubnov – Galerkin Method

$\hookrightarrow$ same basis function for u_e^h and w_e^h.

Petrov – Galerkin is when bases are different.

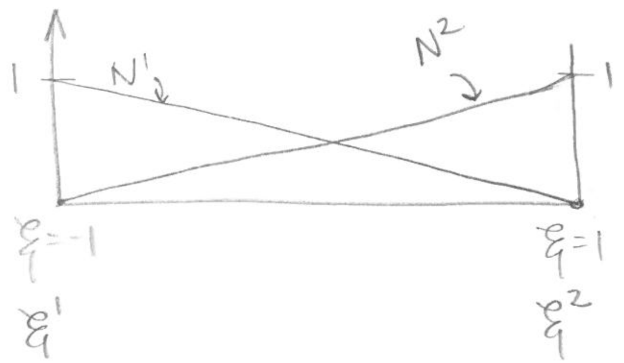
Physical Domain

Bi-unit domain (aka mathematical domain)

$$N^1(x) = N^1(\underbrace{x(\xi)}_{\text{due to the mapping}}) = N^1(\xi)$$

$$N^2(x) = N^2(x(\xi)) = N^2(\xi)$$

$$N^1(\xi) = \frac{1-\xi}{2}, \quad N^2(\xi) = \frac{1+\xi}{2}$$



$N^1(-1) = 1$

$N^1(1) = 0$

$N^2(-1) = 0$

$N^2(1) = 1$

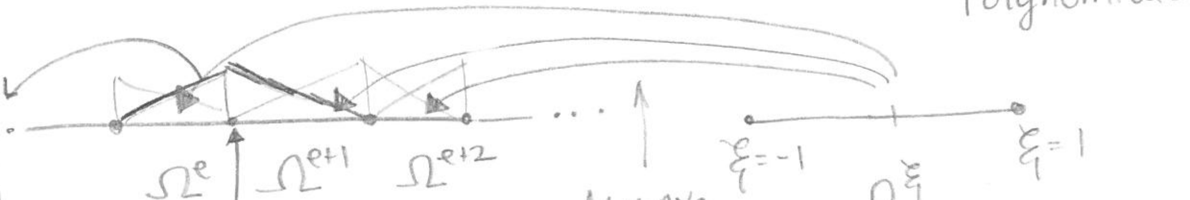
$$N^A(\xi^B) = \underbrace{\delta_{AB}}_{\text{Kronecher delta}} = \begin{cases} 1 & \text{if } A = B \\ 0 & \text{if } A \neq B \end{cases}$$

2.07 — Also, $N^1(\xi) + N^2(\xi) = \frac{1-\xi}{2} + \frac{1+\xi}{2} = 1$

— Generalize-able to higher-order polynomials — Legrange Polynomials

Compact Support of Global Basis Functions

—



basis function for global node e+1. is non-zero only in elements adjoining the node (Ω^e & Ω^{e+1})

x^{e+1} } global node number e+1. also local node number 2 for Ω^e. local node number 1 for Ω^{e+1}.

ALWAYS mapped from the same "parent" domain in the bi-unit domain.

$\boxed{2.08}$ <u>Remark</u>: The local definition of basis functions leads to global basis functions associated with each node, x^e with compact support in Ω^{e-1} and Ω^e.

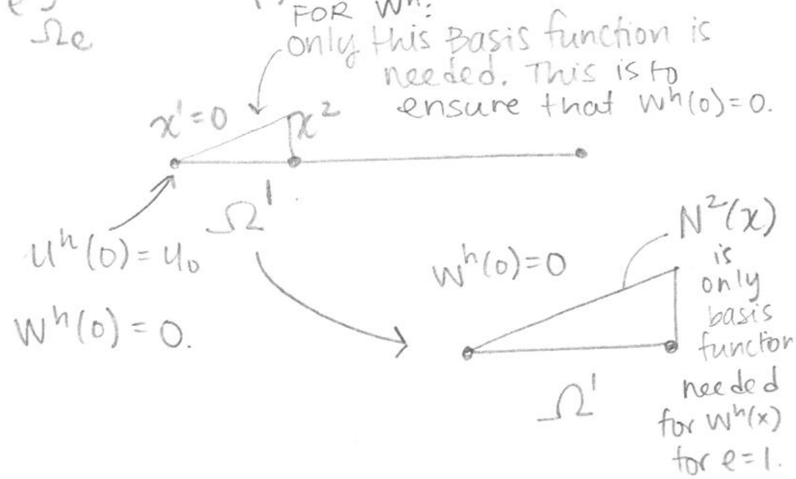
<u>Recall</u>:

$$\sum_e \int_{\Omega_e} \underbrace{w^h_{,x}}_{\text{gradient}} \underbrace{\sigma^h}_{E u^h_{,x}} A\, dx = \sum_e \int_{\Omega_e} w^h f(x(\xi)) A\, dx + w^h(L) t A$$

gradient

another gradient.

<u>NOTE</u>: In Ω^e, for $e=1$



However, for $e=2,\dots,n_{el}$

$$w^h_e(x) = \sum_{A=1}^{N_{n_e}} N^A(x) c^A_e$$

$$w^h_{e=1}(x) = N^2(x) c^2_e$$

Need to compute $w^h_{,x}$ and $\underbrace{u^h_{,x}}_{\substack{\text{needed for} \\ \sigma^h = E u^h_{,x}}}$

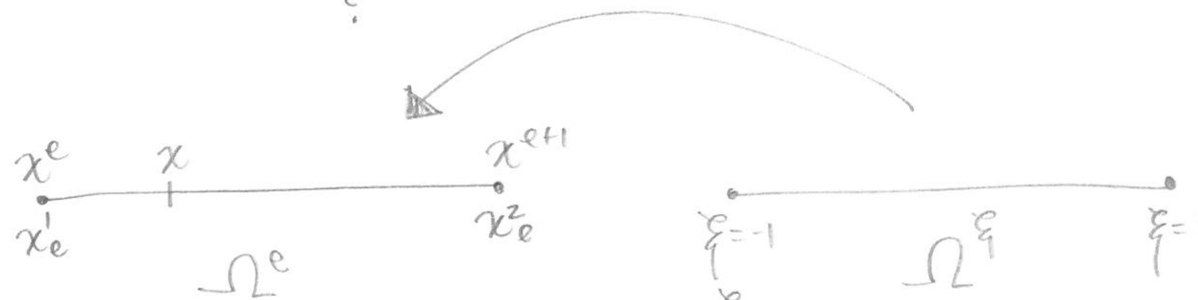
$$u^h_{e,x}(x) = \sum_{A=1}^{N_{n_{el}}} N^A_{,x}(\xi)\, d^A_e \quad ; \quad w^h_{e,x}(\xi) = \sum_{A=1}^{N_{n_e}} N^A_{,x}(\xi)\, c^A_e$$

d.o.f. has no dependence on ξ or x.

CHAIN RULE

$$= \sum_{A=1}^{N_{n_e}} N^A_{,\xi}\, \xi_{,x}\, d^A_e \qquad\qquad = \sum_{A=1}^{N_{n_e}} N^A_{,\xi}\, \xi_{,x}\, c^A_e$$

$\boxed{2.09}$ $\quad u^h_{,x} = \sum_{A=1}^{N_{ne}} N^A_{,\xi}\, \underbrace{\xi_{,x}}_{?}\, d^A_e \quad ; \quad w^h_{,x} = \sum_{A=1}^{N_{ne}} N^A_{,\xi}\, \xi_{,x}\, c^A_e$



Define the mapping to Ω^e from Ω^ξ

$$x(\xi) = \sum_{A=1}^{N_{ne}} \left(N^A(\xi) \right) x^A_e$$

$\{x^1_e, x^2_e\} = \{x^e, x^{e+1}\}$

local nodes $\quad$ global nodes.

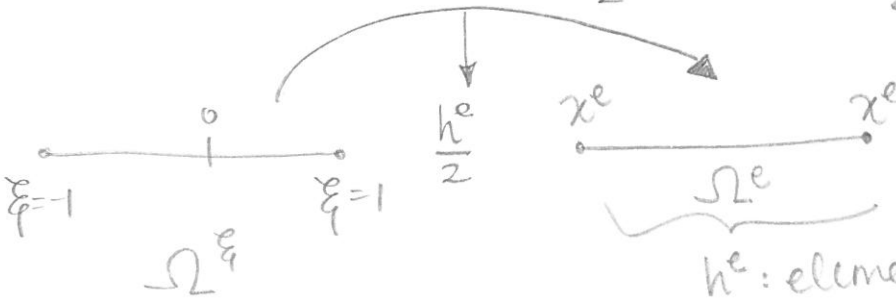
use same basis functions as for representing u^h and w^h. ← ISOPARAMETRIC FORMULATION

$$x_{,\xi}\Big|_e = \sum_{A=1}^{N_{ne}} N^A_{,\xi}\, x^A_e$$

$$= N^1_{,\xi}\, \underbrace{x^1_e}_{\to\, x^e} + N^2_{,\xi}\, \underbrace{x^2_e}_{\to\, x^{e+1}}$$

$$= \frac{d}{d\xi}\left(\frac{1-\xi}{2}\right) x^1_e + \frac{d}{d\xi}\left(\frac{1+\xi}{2}\right) x^2_e$$

$$= \frac{x^2_e - x^1_e}{2} = \frac{x^{e+1} - x^e}{2} = \frac{h^e}{2}$$



h^e: element length.

NON UNIFORM DISCRETIZATION

THIS ALLOWS US TO HAVE ELEMENTS OF UNEQUAL LENGTH.

$\boxed{2.10}$ The isoparametric mapping is invertible.

$$\Rightarrow \quad \xi_{,x} = \frac{1}{x_{,\xi}}$$

$$\Rightarrow \quad u^h_{,x} = \sum_{A=1}^{N_{ne}} N^A_{,\xi} \cdot \underbrace{\xi_{,x}}_{2/h^e} d^A_e$$

$$w^h_{,x} = \sum_{A=1}^{N_{ne}} N^A_{,\xi} \cdot \underbrace{\xi_{,x}}_{2/h^e} c^A_e$$

Consider the integrals:

$$\int_{\Omega^e} w^h_{,x} \sigma^h A \, dx = \int_{\Omega^e} w^h_{,x} E u^h_{,x} A \, dx$$

$$= \int_{\Omega^e} \overbrace{\left(\sum_A N^A_{,\xi} \frac{2}{h^e} c^A_e \right)}^{w^h_{,x}} EA \overbrace{\left(\sum_B N^B_{,\xi} \frac{2}{h^e} d^B_e \right)}^{u^h_{,x}} \underbrace{dx}_{\frac{dx}{d\xi} d\xi}$$

$$\frac{dx}{d\xi} = \frac{h^e}{2}$$

AND

$$\int_{\Omega^e} w^h f(x(\xi)) A \, dx = \int_{\Omega^e} \left(\sum_A N^A c^A_e \right) f(x(\xi)) A \underbrace{dx}_{\frac{dx}{d\xi} d\xi}$$

$$\frac{dx}{d\xi} = \frac{h^e}{2}$$

$$\Rightarrow \int_{\Omega^e} w^h_{,x} EA \, u^h_{,x} \, dx = \int_{\Omega^\xi} \left(\sum_A N^A_{,\xi} \frac{2}{h^e} c^A_e \right) EA \left(\sum_B N^B_{\xi} \frac{2}{\cancel{h^e}} d^B_e \right) \frac{\cancel{h^e}}{\cancel{2}} d\xi$$

$$\Rightarrow \int_{\Omega^e} w^h f A \, dx = \int_{\Omega^\xi} \left(\sum_A N^A c^A_e \right) f(\xi) A \left(\frac{h^e}{2} \right) d\xi$$