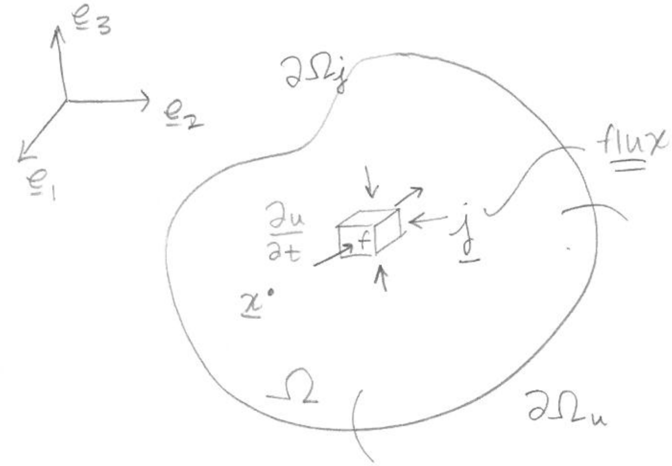
$\boxed{11.01}$ Linear Parabolic PDE in a scalar variable in 3D

→ unsteady heat conduction & mass diffusion in 3D.
↳ (time dependent)

STRONG FORM



Given u_g, $\overset{\text{flux}}{j_n}$, $\overset{\text{source}}{f}$, constitutive relation: $-K_{ij}u_{,j} = j_i$, ρ

Find $u(\underline{x},t)$ such that:

$$\rho\frac{\partial u}{\partial t} = -j_{i,i} + f$$

in $\Omega \times \underbrace{[0,T]}_{\substack{\text{PDE HOLDS} \\ \text{ALSO OVER} \\ \text{THIS TIME} \\ \text{DOMAIN.}}}$

Initial condition:

$u(\underline{x},0) = u_0(\underline{x})$

B.C.'s:

$u = u_g$ on $\partial\Omega_u$

$-j_i n_i = j_n$ on $\partial\Omega_j$

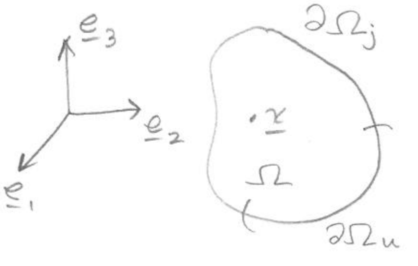
Remarks:

(1) Heat Conduction: ρ is specific heat per unit volume.

(2) Mass Diffusion: PDE is based on the principle of conservatio & $\rho = 1$.

$\boxed{11.02}$ The Strong Form for linear, parabolic pde's in 3D

(scalar unknow



given u_g, j_n, f, $j_i = -K_{ij}u_{,j}$, ρ

find u such that,

$$\rho\frac{\partial u}{\partial t} = -j_{i,i} + f \quad \text{in } \Omega \times [0,T]$$

B.C.'s $\quad u = u_g$ on $\partial\Omega_u$

$-j_i n_i = j_n$ on $\partial\Omega_j$

Initial Condition: $u(\underline{x},0) = u_0(\underline{x})$

Infinite-Dimensional Weak Form:

Consider $w \in \mathcal{V} = \{ w \mid w = 0 \text{ on } \partial\Omega_u \}$

$$\int_\Omega w\rho \frac{\partial u}{\partial t} dV = \underbrace{\int_\Omega \left(-w j_{i,i} + wf\right) dV}_{\text{rewrite using integration by parts.}}$$

$$\Rightarrow \int_\Omega w\rho \frac{\partial u}{\partial t} dV = \int_\Omega w_{,i} j_i \, dV + \int_\Omega wf \, dV - \underbrace{\int_{\partial\Omega} w j_i n_i \, dS}_{= -\int_{\partial\Omega_u} \overset{0}{\cancel{w}} j_i n_i \, dS - \int_{\partial\Omega_j} w \overbrace{\tilde{j}_i n_i}^{= -j_n} dS}$$

$$\Rightarrow \int_\Omega w\rho \frac{\partial u}{\partial t} dV = \int_\Omega w_{,i} \overset{-K_{ij} u_{,j}}{\cancel{j_i}} dV + \int_\Omega wf \, dV + \int_{\partial\Omega_j} w j_n \, dS$$

$$\Rightarrow \int_\Omega w\rho \frac{\partial u}{\partial t} dV + \int_\Omega w_{,i} K_{ij} u_{,j} \, dV = \int_\Omega wf \, dV + \int_{\partial\Omega_j} w j_n \, dS$$

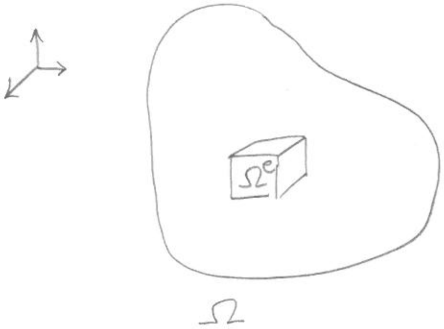
Finite-Dimensional Weak Form:

Find $u^h \in \mathcal{S}^h \subset \mathcal{S}$; $\mathcal{S}^h = \{ u^h \in H^1(\Omega) \mid u^h = u_g \text{ on } \partial\Omega_u \}$

such that $\forall \, w^h \in \mathcal{V}^h \subset \mathcal{V}$; $\mathcal{V}^h = \{ w^h \in H^1(\Omega) \mid w^h = 0 \text{ on } \partial\Omega_u \}$

$$\int_\Omega w^h \rho \frac{\partial u^h}{\partial t} dV + \int_\Omega w^h_{,i} K_{ij} u^h_{,j} \, dV = \int_\Omega w^h f \, dV + \int_{\partial\Omega_j} w^h j_n \, dS$$

Partition $\bar{\Omega} = \overline{\bigcup_e \Omega^e}$



$\boxed{11.03}$ **Basis Functions:**

$$u_e^h(\underline{x},t) = \sum_{A=1}^{n_{ne}} \underbrace{N^A(\underline{x}(\underline{\xi}))}_{\text{spatial discretization}} d_e^A(t)$$

↳ Often called a semi-discrete FE formulation.

$$w_e^h(\underline{x}) = \sum_{A=1}^{n_{ne}} N^A(\underline{x}(\underline{\xi}))\, c_e^A$$

$$\int_\Omega w^h_{,i} K_{ij} u^h_{,j}\, dV = \underline{c}^T \underline{K}\, \underbrace{\underline{d}(t)}_{\text{only difference from before!}} \; ; \quad \int_\Omega w^h f\, dV + \int_{\partial\Omega_j} w^h j_n\, dS = \underline{c}^T \underbrace{\underline{F}}_{\text{same as before.}}$$

Remark: Can have $u_g = u_g(t)$, $j_n = j_n(t)$, $f = f(\underline{x},t)$

So, $\underline{F} = \underline{F}(t)$.

$\boxed{11.04}$ Consider the time-dependent term:

$$\left(\frac{d(d_e^A)}{dt}\right)$$

$$\int_\Omega w^h \rho \frac{\partial u^h}{\partial t} dV \quad \ldots \quad \text{use} \quad \left.\frac{\partial u^h}{\partial t}\right|_e = \sum_A N^A \dot{d}_e^A$$

$$\int_\Omega w^h \rho \frac{\partial u^h}{\partial t} dV = \sum_e \int_{\Omega^e} \left(\sum_A N^A c_e^A\right) \rho \left(\sum_B N^B \dot{d}_e^B\right) dV$$

$$= \sum_e \sum_{A,B} c_e^A \underbrace{\left(\int_{\Omega^e} N^A \rho N^B dV\right)}_{\text{Yields a scalar for each element}} \dot{d}_e^B$$

M_e^{AB} } The assembly of all of the elemental contributions

Use: $$\underline{c}_e = \begin{Bmatrix} c_e^1 \\ \vdots \\ c_e^{n_{ne}} \end{Bmatrix},$$

$$\dot{\underline{d}}_e = \begin{Bmatrix} \dot{d}_e^1 \\ \vdots \\ \dot{d}_e^{n_{ne}} \end{Bmatrix},$$

$$\Rightarrow \int_{\Omega} w^h \rho \frac{\partial u^h}{\partial t}\, dV = \sum_e \underline{c}_e^T \, \underline{\bar{M}}_e \, \dot{\underline{\bar{d}}}_e \Big\}$$

NOTE: The "bar" over $\underline{d}_e$ and $\underline{M}_e$ indicates application of Dirichlet BC's.

often called the "element mass matrix"

Remark: ①For a general element Ω^e such that $\partial\Omega_e \cap \partial\Omega_{\bar{u}} =$

$$\mathrm{DIM}[\underline{c}_e] = n_{ne}$$

$$\Rightarrow \underline{\bar{M}}_e = \begin{bmatrix} M_e^{11} & \cdots & & M_e^{1n_{ne}} \\ \vdots & \ddots & M_e^{AB} & \vdots \\ \vdots & M_e^{BA} & \ddots & \vdots \\ M_e^{n_{ne}1} & \cdots & & M_e^{n_{ne}n_{ne}} \end{bmatrix}; \quad M_e^{AB} = M_e^{BA}$$

$$\underline{\bar{M}}_e = \underline{\bar{M}}_e^T \Big\} \text{ symmetry.}$$

$\underline{\bar{M}}_e$ is positive definite: $\underline{c}_e^T \underline{\bar{M}}_e \underline{c}_e \geq 0 \quad \forall\, \underline{c}_e \in \mathbb{R}^{n_{ne}}$

$\underline{c}$ can be any vector

$= 0$ only if $\underline{c}_e = \underline{0}$

② This is the constitent element mass matrix.

↳ A lumped element mass matrix is:

$$\breve{\underline{M}}_e = \begin{bmatrix} \sum_B M_e^{1B} & 0 & 0 & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ 0 & & \ddots & & \vdots \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & \sum_B M_e^{n_{ne}B} \end{bmatrix}$$

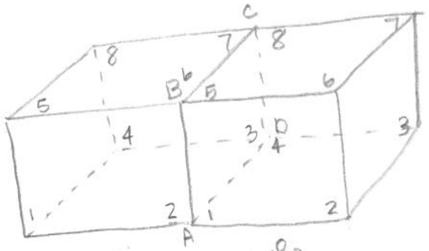
$$\breve{\underline{M}}_e^{AB} = \begin{cases} \sum_C \underline{M}_e^{AC} & \text{if } A = B \\ 0 & \text{if } A \neq B \end{cases}$$

11.05 We have: $$\int_\Omega w^h \rho \frac{\partial u^h}{\partial t}\, dV = \sum_e \underline{c}_e^T \underline{\bar M}_e \underline{\dot d}_e = \underline{c}^T \underline{\bar M} \underline{\dot d}$$

$$\underline{c} = \begin{Bmatrix} c^1 \\ c^2 \\ \vdots \\ c^{n_n - N_D} \end{Bmatrix} \;;\quad \underline{\dot d} = \begin{Bmatrix} \dot d^1 \\ \dot d^2 \\ \vdots \\ \dot d^{n_n} \end{Bmatrix} \;;\ \text{where } \underline{\bar M} = \underset{e}{\mathrm{A}}\, \underline{\bar M}_e$$

← from Dirichlet Boundary conditions.

global dof number



Ω^{e_1} Ω^{e_2}

NODE	Ω^{e_1}	Ω^{e_2}
A	2	1
B	6	5
C	7	8
D	3	4

$$\underline{\bar M} = \begin{array}{c|cccc|c} & \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \\ \hline & \cdots + M_{e_1}^{22} + M_{e_2}^{11} + \cdots & \cdots + M_{e_1}^{26} + M_{e_2}^{15} + \cdots & & & \mathrm{A} \\ & \cdots + M_{e_1}^{62} + M_{e_2}^{51} + \cdots & & & & \mathrm{B} \\ & & & \cdots + M_{e_1}^{77} + M_{e_2}^{88} + \cdots & & \mathrm{C} \\ & & & & & \mathrm{D} \end{array}$$

11.06 Dirichlet Boundary Conditions:

$$\underbrace{\int_\Omega w^h \rho \frac{\partial u^h}{\partial t}\, dV}_{\underline{c}^T \underline{\bar M} \underline{\dot d}} + \underbrace{\int_\Omega w^h_{,i} K_{ij} u^h_{,j}\, dV}_{\underline{c}^T \underline{K} \underline{d}} = \underbrace{\int_\Omega w^h f\, dV + \int_{\partial\Omega_j} w^h j_n\, dS}_{\underline{c}^T \underline{F}}$$

$\underline{K}$ ← SQUARE

$\underline{F}$ ← has dirichlet b.c.'s from non-time derivative integrals.

This form follows if the Dirichlet Boundary Conditions from the integrals without time derivatives have already been accounted for in the matrix-vector form.

$$\underline{c}^T \underline{\bar M} \underline{\dot d} = \langle c^1 \ \cdots\ c^{n_{ne} - N_D} \rangle \begin{bmatrix} \{\underline{\bar M}^{\bar A}\} & \{\underline{\bar M}^{\bar B}\} \end{bmatrix} \begin{Bmatrix} \dot d^1 \\ \dot d^{\bar A} \\ \dot d^{\bar B} \\ \dot d^{n_{ne}} \end{Bmatrix}$$

← Dirichlet B.C.'s on dot $\bar A$ & $\bar B$

Remark: $\dot{d}^{\bar{A}}$ & $\dot{d}^{\bar{B}}$ drive the initial and boundary value problem via time-dependent Dirichlet Boundary conditions.

$$\Rightarrow \quad \underline{c}^T \underline{M}\,\dot{\underline{d}} + \underline{c}^T \underline{K}\,\underline{d} = \underline{c}^T \Big(\underbrace{\underline{F} - \dot{d}^{\bar{A}}\,\underline{\bar{M}}^{\bar{A}} - \dot{d}^{\bar{B}}\,\underline{\bar{M}}^{\bar{B}}}_{\text{redefine as } \underline{F}} \Big)$$

$$\underline{c}^T \left(\underline{M}\dot{\underline{d}} + \underline{K}\underline{d} - \underline{F} \right) = 0 \qquad \forall\, \underline{c} \in \mathbb{R}^{n_{n_e} - N_D}$$

$$\Rightarrow \quad \underline{M}\,\dot{\underline{d}} + \underline{K}\,\underline{d} = \underline{F} \quad \Big\} \text{ Semi-discrete matrix-vector problem}$$

11.07 The Matrix-Vector Equation:

$$\underline{M}\dot{\underline{d}} + \underline{K}\underline{d} = \underline{F}$$ — First-order ordinary differential equation in $\underline{d} \in \mathbb{R}^{n_{df}}$

$$\underline{d}(0) = \underbrace{\begin{Bmatrix} u_0(\underline{x}^A) \\ \vdots \\ u_0(\underline{x}^{n_{df}} \end{Bmatrix}}_{\underline{d}_0}$$

Globally Lumped Mass Matrix: $\underline{M}_\ell$ can be defined

indicates "lumped"

$$M_\ell^{AB} = \begin{cases} \sum_C M^{AC} & \text{if } A = B \\ 0 & \text{if } A \neq B. \end{cases}$$

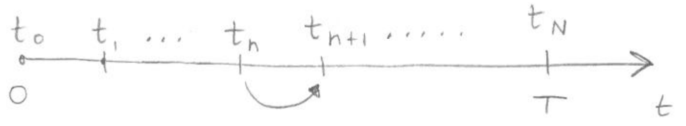
Time Discretization: Finite Difference

↳ Remark:
- Space-time finite element methods do exist.
- Integration over Ω & $[0,T]$.
- Accuracy wrt time is of higher order than with Finite Difference.

Divide $[0,T]$ into sub-intervals: $\underbrace{[t_0,t_1],\ [t_1,t_2],\ \dots,\ [t_{N-1},t_N]}_{N \text{ sub-intervals}}$ such that $t_0 = 0$ & $t_N = T$.

Consider an interval $[t_n, t_{n+1}]$; $n \in [0, N-1]$

↳ Time-stepping: Assumes we know the solution at t_n, then find at t_{n+1}.



Notation: $\underline{d}(t_n)$: the time-exact solution at $t = t_n$

$$\underline{M}\,\dot{\underline{d}} + \underline{K}\,\underline{d} = \underline{F}\,; \quad \underline{d}(0) = d_0$$

$\underline{d}_n$: the algorithmic solution (obtained by a method to integrate the time discretized ODE).

$$\underline{M}\,\dot{\underline{d}}(t_n) + \underline{K}\,\underline{d}(t_n) = \underline{F}(t_n)$$

velocity of the solution.

$$\underbrace{\underline{M}\,\underline{v}_n + \underline{K}\,\underline{d}_n = \underline{F}_n}_{\text{Time-discretized ODE at } t_n} \rightarrow \underline{v}_n \text{ is a time-discretized approximation of } \dot{\underline{d}}$$

$\boxed{11.08}$ At $t = t_{n+1}$: $\underline{M}\,\underline{v}_{n+1} + \underline{K}\,\underline{d}_{n+1} = \underline{F}_{n+1}$

Integration Algorithms: Euler Family for First-Order ODEs.

Consider: $\dot{y} = f(y)$

Algorithm:
$$\frac{y_{n+1} - y_n}{\Delta t} = f(y_{n+\alpha})$$

where: $\Delta t = (t_{n+1} - t_n)$. the time step.

$\alpha \in [0, 1]$

$\alpha = 0$: Forward Euler

$\alpha = 1$: Backward Euler.

$\alpha = 1/2$: Midpoint Rule, Crank-Nicholson Method.

Rewritten as: $y_{n+1} = y_n + \Delta t \cdot f(y_{n+\alpha})$

↳ $y_{n+\alpha} = \alpha y_{n+1} + (1-\alpha) y_n$

$\Rightarrow$ Approximates the time derivative as a linearly varying quantity over the time interval.

$\boxed{11.09}$

Time-Discretized ODE

$$\underline{M}\,\underline{v}_{n+1} + \underline{K}\,\underline{d}_{n+1} = \underline{F}_{n+1}$$

$$\left.\begin{aligned} \underline{d}_{n+1} &= \underline{d}_n + \Delta t\,\underline{v}_{n+\alpha} \\ \underline{v}_{n+\alpha} &= \alpha\,\underline{v}_{n+1} + (1-\alpha)\,\underline{v}_n \end{aligned}\right\} \text{Euler Family}$$

given $\underline{d}_0$

v- Method: First note:

$$\underline{d}_{n+1} = \underline{d}_n + \Delta t\left[\alpha\,\underline{v}_{n+1} + (1-\alpha)\,\underline{v}_n\right]$$

$$= \underbrace{\underline{d}_n + (1-\alpha)\Delta t\,\underline{v}_n}_{\tilde{\underline{d}}_{n+1} \leftarrow \text{prediction}} + \underbrace{\alpha\Delta t\,\underline{v}_{n+1}}_{\text{corrector.}}$$

$$\Rightarrow \underline{d}_{n+1} = \tilde{\underline{d}}_{n+1} + \alpha\Delta t\,\underline{v}_{n+1} \quad \Big\} \text{ Predictor-corrector Method.}$$

$$\underline{M}\,\underline{v}_{n+1} + \underline{K}\,\underline{d}_{n+1} = \underline{F}_{n+1}$$

Substitute predictor-corrector form for $\underline{d}_{n+1}$

$$\underline{M}\,\underline{v}_{n+1} + \underline{K}\left[\tilde{\underline{d}}_{n+1} + \alpha\Delta t\,\underline{v}_{n+1}\right] = \underline{F}_{n+1}$$

$$\Rightarrow \left(\underline{M} + \alpha\Delta t\,\underline{K}\right)\underline{v}_{n+1} = \underline{F}_{n+1} - \underline{K}\,\tilde{\underline{d}}_{n+1}$$

$$\underline{v}_{n+1} = \left(\underline{M} + \alpha\Delta t\,\underline{K}\right)^{-1}\left(\underline{F}_{n+1} - \underline{K}\,\tilde{\underline{d}}_{n+1}\right)$$

Remarks: ① If $\underline{M} = \underline{M}_\ell$ & $\alpha = 0$ (Fwd Euler)

$$\underline{v}_{n+1} = \underline{M}_\ell^{-1}\left(\underline{F}_{n+1} - \underline{K}\,\tilde{\underline{d}}_{n+1}\right)$$

② If $\alpha \neq 0 \rightarrow$ Implicit method.

d-method : $\underline{d}_{n+1} = \tilde{\underline{d}}_{n+1} + \alpha \Delta t \, \underline{v}_{n+1}$

$$\Rightarrow \underline{v}_{n+1} = \frac{\left(\underline{d}_{n+1} - \tilde{\underline{d}}_{n+1}\right)}{\alpha \Delta t} \quad , (\alpha \neq 0)$$

↳ Substitute into discretized form of ODE:

$$\underline{M} \frac{(\underline{d}_{n+1} - \tilde{\underline{d}}_{n+1})}{\alpha \Delta t} + \underline{K} \, \underline{d}_{n+1} = \underline{F}_{n+1}$$

$$(\underline{M} + \alpha \Delta t \, \underline{K}) \, \underline{d}_{n+1} = \alpha \Delta t \, \underline{F}_{n+1} + \underline{M} \, \tilde{\underline{d}}_{n+1}$$

$$\underline{d}_{n+1} = (\underline{M} + \alpha \Delta t \, \underline{K})^{-1} \left[\alpha \Delta t \, \underline{F}_{n+1} + \underset{\uparrow}{\underline{M}} \, \tilde{\underline{d}}_{n+1} \right]$$

if $\underline{M}_{\ell}$ is used, there are fewer operations to form the RHS.

$\boxed{11.10}$ Analysis: Modal decomposition.

$$\underline{M} \frac{(\underline{d}_{n+1} - \underline{d}_n)}{\Delta t} + \underline{K} \, \underline{d}_{n+\alpha} = \underline{F}_{n+\alpha}$$

→ Analyze the stability and consistency of the time-integration algorithms.

Consider the homogeneous ODE:

$$\underline{M} \dot{\underline{d}} + \underline{K} \, \underline{d} = \underline{0} \quad ; \quad \underline{d}(0) = \underline{d}_0$$

$$\underline{M} \left(\frac{\underline{d}_{n+1} - \underline{d}_n}{\Delta t} \right) + \underline{K} \, \underline{d}_{n+\alpha} = \underline{0} \, , \quad \underline{d}_0 \text{ given.}$$

where $\underline{d}_{n+\alpha} = \alpha \, \underline{d}_{n+1} + (1-\alpha) \, \underline{d}_n$

Modal Decomposition

Invoke the related Generalized Eigen value Problem:

$$\underline{M}\,\underline{\Phi} = \lambda\,\underline{K}\,\underline{\Phi}$$

$$\Rightarrow \quad \underline{K}\,\underline{\Phi} = \lambda\,\underline{M}\,\underline{\Phi}\,, \qquad \underline{\Phi} \in \mathbb{R}^{n_{df}}$$

Remark: A standard eigen value problem is

$$\underline{K}\,\underline{\Phi} = \lambda \overset{\underline{I}}{\curvearrowleft} \underline{\Phi}$$

Let $\underline{\Phi}_m$, $m = 1, 2, \cdots, n_{df}$ be eigen vectors

λ_m : the corresponding eigen value.

The eigenvectors and eigen values satisfy:

$$\underline{K}\,\underline{\Phi}_m = \lambda_m\,\underline{M}\,\underline{\Phi}_m\,; \qquad m = 1, \ldots, n_{df}$$

The $\{\underline{\Phi}_m\}_{m=1,\cdots,n_{df}}$ can be orthonormalized to a set $\{\underline{\Psi}_m\}_{m=1,\ldots n_{df}}$

Such that

$$\underline{\Psi}_m \cdot \underline{M}\,\underline{\Psi}_k = \overset{\text{Kronecker Delta}}{\overset{\swarrow}{\delta_{mk}}}$$

$\{\underline{\Psi}_m\}_{m=1,\ldots,n_{df}}$ are $\underline{M}$- orthonormal

$\boxed{11.11}$ Orthonormalization can be constructed by the Gram-Schmidt Method.

$$\underbrace{\{\underline{\Phi}_m\}_{m=1,\dots,n_{df}}}_{\substack{\text{linearly independent}\\ \text{(because } \underline{K} \,\&\, \underline{M} \text{ are}\\ \text{positive definite)}}} \xrightarrow{\text{G-S process}} \underbrace{\{\underline{\Psi}_m\}_{m=1,\dots,n_{df}}}_{\underline{M}\text{-orthonormal}}$$

$$\underbrace{\lambda_k \underline{\Psi}_m \cdot (\underline{M}\,\underline{\Psi}_k)}_{\lambda_k \delta_{mk}} = \underbrace{\underline{\Psi}_m \cdot (\underline{K}\,\underline{\Psi}_k)}$$

$\{\underline{\Psi}_m\}_{m=1,\dots,n_{df}}$ forms a basis in $\mathbb{R}^{n_{df}}$

Any vector, say $\underline{d} = \sum_{m=1}^{n_{df}} d_m \underline{\Psi}_m$ expansion of $\underline{d}$ in the basis.

To get the coefficients, d_m:

$$\underline{\Psi}_k \cdot \underline{M}\;\underline{d} = \underline{\Psi}_k \cdot \sum_{m=1}^{n_{df}} d_m \underline{M}\,\underline{\Psi}_m$$

$$= \sum_{m=1}^{n_{df}} d_m \underbrace{\underline{\Psi}_k \cdot \underline{M}\,\underline{\Psi}_m}_{\delta_{km}}$$

$$\underline{\Psi}_k \cdot (\underline{M}\,\underline{d}) = d_k$$

$\underline{d} = \sum_{m=1}^{n_{df}} d_m \underline{\Psi}_m$ —— modal decomposition of $\underline{d}$

$\underline{\Psi}_m$: m^{th}- mode

d_m: modal coeffecient of $\underline{d}$; $d_m = \underline{\Psi}_m \cdot (\underline{M}\,\underline{d})$

$\boxed{11.12}$ Analysis of time integration algorithms for linear parabolic systems.

Generalized Eigen value Problem:

$$\underline{K}\,\underline{\Psi}_m = \lambda_m \underline{M}\,\underline{\Psi}_m$$

$$\underline{\Psi}_\ell \cdot \underline{M}\,\underline{\Psi}_m = \delta_{\ell m} \Rightarrow \underline{\Psi}_\ell \cdot \underline{K}\,\underline{\Psi}_m = \lambda_m \delta_{\ell m}$$

(orthonormal)

Expansion in $\{\underline{\Psi}_m\}_{m=1,\dots,n_{df}}$

$$\underbrace{\underline{d} = \sum_m d_m \underline{\Psi}_m}_{\text{modal decomposition of } \underline{d}} , \qquad \underbrace{d_m = \underline{\Psi}_m \cdot \underline{M}\,\underline{d}}_{\text{modal coefficients}}$$

Modal decomposition of the time-exact ODE.

$$\underline{M}\,\dot{\underline{d}} + \underline{K}\,\underline{d} = \underline{0} \qquad \underline{d}(0) = \underline{d}_0$$

$$\left.\underline{d}(t) = \sum_m^{n_{df}} d^m_{(t)} \underline{\Psi}^m \right\} \rightarrow \{\underline{\Psi}_m\}_{m=1,\dots,n_{df}} \text{ are fixed in time}$$

$\because$ $\underline{K}$ & $\underline{M}$ are also fixed in time.

$$\Rightarrow \dot{\underline{d}} = \sum_m^{n_{df}} \dot{d}^m_{(t)} \underline{\Psi}^m$$

Substituting,

$$\hookrightarrow \underline{M} \cdot \sum_m^{n_{df}} \dot{d}^m \underline{\Psi}^m + \underline{K} \sum_m^{n_{df}} d^m \underline{\Psi}^m = \underline{0}$$

$$\rightarrow \sum_m^{n_{df}} \dot{d}^m \underline{M}\,\underline{\Psi}^m + \sum_m^{n_{df}} d^m \underline{K}\,\underline{\Psi}^m = \underline{0}$$

$$\underline{\Psi}^\ell \cdot \sum_m^{n_{df}} \dot{d}^m \underline{M}\,\underline{\Psi}^m + \underline{\Psi}^\ell \cdot \sum_m^{n_{df}} d^m \underline{K}\,\underline{\Psi}^m = \overset{\text{now a scalar!}}{0}$$

$\boxed{11.13}$

$$\sum_m \dot{d}^m \underbrace{\underline{\Psi}^l \cdot \underline{M}\,\underline{\Psi}^m}_{\delta_{lm}} + \sum_m d^m \underbrace{\underline{\Psi}^l \cdot \underline{K}\,\underline{\Psi}^m}_{\lambda^m \delta_{lm}} = 0$$

$$\Rightarrow \dot{d}^l + \lambda^l d^l = 0, \quad \forall \quad l = 1, 2, \dots n_{df}$$

"single degree of freedom modal equation"

Do same for time-discrete, homogeneous ODE:

$$\underline{M}\frac{(\underline{d}_{n+1} - \underline{d}_n)}{\Delta t} + \underline{K}\,\underbrace{\underline{d}_{n+\alpha}} = \underline{0}$$

$$\Rightarrow \underline{M}\,(\underline{d}_{n+1} - \underline{d}_n) + \Delta t\,\underline{K}\,\overbrace{\left[\alpha\,\underline{d}_{n+1} + (1-\alpha)\,\underline{d}_n\right]} = \underline{0}$$

Modal decompositions:

$$\underline{d}_{n+1} = \sum_m d^m_{n+1}\,\underline{\Psi}^m \quad ; \quad \underline{d}_n = \sum_m d^m_n\,\underline{\Psi}^m$$

$$\underline{\Psi}^l \cdot \left(\underline{M} + \alpha\Delta t\,\underline{K}\right) \sum_m d^m_{n+1}\,\underline{\Psi}^m - \underline{\Psi}^l \cdot \left(\underline{M} - (1-\alpha)\,\Delta t\,\underline{K}\right) \sum_m d^m_n\,\underline{\Psi}^m = \underline{0}$$

$$\Rightarrow \sum_m d^m_{n+1}\Big(\underbrace{\underline{\Psi}^l \cdot \underline{M}\,\underline{\Psi}^m}_{\delta_{lm}} + \alpha\Delta t\,\underbrace{\underline{\Psi}^l \cdot \underline{K}\,\underline{\Psi}^m}_{\lambda^m \delta_{lm}}\Big) -$$

$$\sum_m d^m_n\Big(\underbrace{\underline{\Psi}^l \cdot \underline{M}\,\underline{\Psi}^m}_{\delta_{lm}} - (1-\alpha)\,\Delta t\,\underbrace{\underline{\Psi}^l \cdot \underline{K}\,\underline{\Psi}^m}_{\lambda^m \delta_{lm}}\Big) = 0$$

$$\Rightarrow \underbrace{d^l_{n+1}\left(1 + \alpha\Delta t\,\lambda^m\right) - d^l_n\left(1 - \Delta t\,(1-\alpha)\,\lambda^m\right) = 0} \quad \forall\ l = 1, \dots, n_{ds}$$

single degree of freedom modal equation
for the time-discrete problem.

$\boxed{11.14}$ SINGLE DOF MODAL EQUATIONS

Time-Exact Case: $\dot{d}^{l} + \lambda^{l} d^{l} = 0$

initial condition: $d^{l}(0) = \underline{\Psi}^{l} \cdot \underline{M}\, \underline{d}(0)$

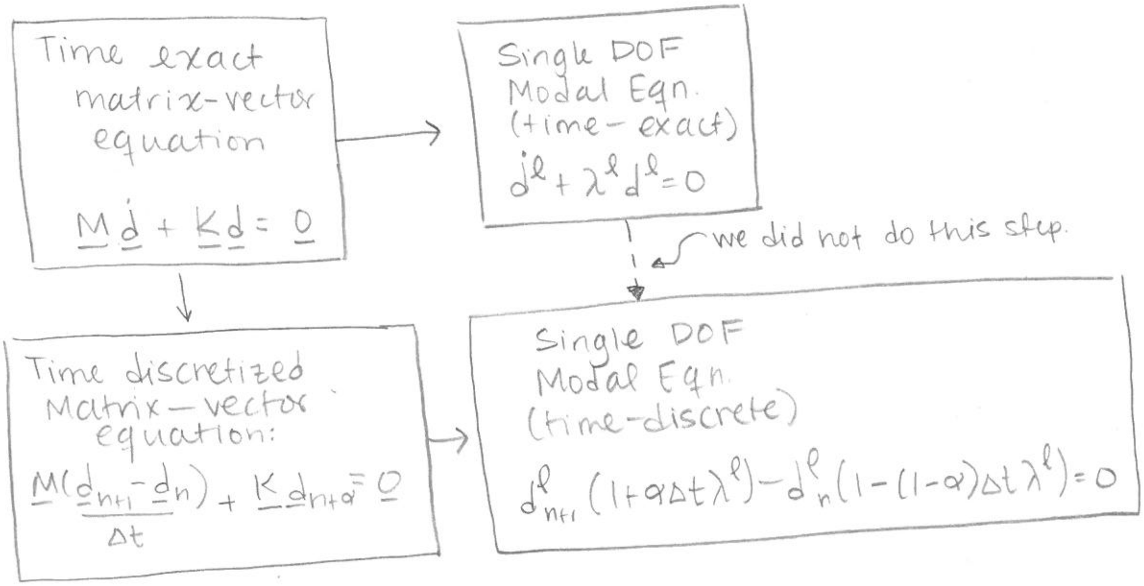
$= \underline{\Psi}^{l} \cdot \underline{M}\, \underline{d}_0$

$= d^{l}_{0}$

Time- Discrete Case: $d^{l}_{n+1}\left(1+\alpha \Delta t \lambda^{l}\right) - d^{l}_{n}\left(1-(1-\alpha)\Delta t \lambda^{l}\right) = 0$

given: d^{l}_{0}

Remark:



$\boxed{11.15}$ Stability: The time-exact case:

$$\dot{d} + \lambda^{h} d = 0$$

λ^{h} is the eigenvalue of a mode that is obtained after spacial discretization.

Exact Solution: $d(t) = d_0 \exp(-\lambda^{h} t)$

$\lambda^{h} \geqslant 0 \quad \because\ \underline{M}$ is positive definite, $\underline{K}$ is positive semi- definite.

$d(t_n+1) \leqslant d(t_n) \qquad (\because\ t_{n+1} > t_n)$

$\frac{d(t_n+1)}{d(t_n)} \leqslant 1 \quad (\text{if } d(t_n) \neq 0)$

Monotonically Decreasing $d(t)$

Time Discrete Equation:

$$d_{n+1}\left(1+\alpha\Delta t\,\lambda^h\right) = d_n\left(1-(1-\alpha)\Delta t\,\lambda^h\right)$$

$$\frac{d_{n+1}}{d_n} = \underbrace{\frac{\left(1-(1-\alpha)\Delta t\,\lambda^h\right)}{\left(1+\alpha\Delta t\,\lambda^h\right)}}$$

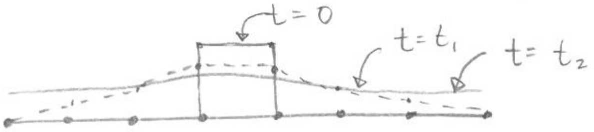
A: amplification factor.

$A(\alpha, \Delta t, \lambda^h)$

Magnitude required since A may change signs.

Stability: $|A| \leq 1$

QUESTION: Decay of MODAL coefficients:



$\underline{d} \in \mathbb{R}^{n_{df}}$, $\underline{d} = \begin{Bmatrix} d^1(t) \\ d^2(t) \\ \vdots \\ d^{n_{df}}(t) \end{Bmatrix}$ nodal degrees of freedom

components of this are what are plotted above.

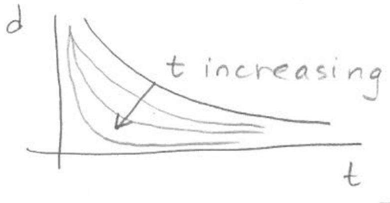
Modal Decomposition

$$\underline{d}(t) = \sum_{m=1}^{n_{df}} \underline{\Psi}^m d^m(t)$$

$\underline{\Psi} \in \mathbb{R}^{n_{df}}$

effectively a shape function. & gives $\underline{d}(t)$ varying amplitudes.

$$d^m(t) = d_0^m \exp(-\lambda^m - t)$$



∴ mode shapes $\underline{\Psi}$ decay in time.

$\boxed{11.16}$ **Amplification Factor:**

$$\frac{d_{n+1}}{d_n} = A = \frac{1-(1-\alpha)\Delta t\,\lambda^h}{1+\alpha\Delta t\lambda^h} \quad ; \quad \alpha \in [0, 1]$$

$$\Delta t > 0$$

$$\lambda^h \geqslant 0$$

Linear Stability Condition: $|A| \leqslant 1 \}$ ↳ guarantees decaying solution.

$$\Rightarrow \quad -1 \leqslant A \leqslant 1$$

$$-1 \leqslant \frac{(1-(1-\alpha)\Delta t\lambda^h)}{1+\alpha\Delta t\lambda^h} \leqslant 1$$

$$\Rightarrow \quad -(1+\alpha\Delta t\lambda^h) \leqslant (1-(1-\alpha)\Delta t\lambda^h) \leqslant (1+\alpha\Delta t\lambda^h)$$

Consider: $\cancel{1}-(1-\alpha)\Delta t\lambda^h \leqslant (\cancel{1}+\alpha\Delta t\lambda^h)$

$$(\cancel{\alpha}+(1-\cancel{\alpha}))\Delta t\lambda^h \geqslant 0$$

$$\Delta t\lambda^h \geqslant 0$$ satisfied. ✓ $\forall\, \alpha \in [0, 1]$

consider: $-(1+\alpha\Delta t\lambda^h) \leqslant 1-(1-\alpha)\Delta t\lambda^h$

$$\Rightarrow \quad 1-(1-\alpha)\Delta t\lambda^h + (1+\alpha\Delta t\lambda^h) \geqslant 0$$

$$\Rightarrow \quad 2+(2\alpha-1)\Delta t\lambda^h \geqslant 0$$

$$\Rightarrow \quad 2 \geqslant (1-2\alpha)\Delta t\lambda^h$$

Case 1: $\alpha \geqslant 1/2$

$$\Rightarrow (1-2\alpha)\Delta t\lambda^h \leqslant 0$$

$\Rightarrow 2 \geqslant (1-2\alpha)\Delta t\lambda^h$ holds unconditionally. $\forall\, \Delta t > 0$

Unconditional Stability if $\alpha \geqslant 1/2$

$\alpha = 1/2$ Crank-Nicholson or Midpoint Rule.

$\alpha = 1$ Backward Euler.

Case 2: $\alpha \in [0, \frac{1}{2})$

$$0 \leqslant \alpha < 1/2$$

$$2 \geq \underbrace{(1-2\alpha)}_{\text{always positive.}} \Delta t \lambda^h \quad , \quad 0 \leq \alpha < 1/2$$

$$\Delta t \leq \frac{2}{(1-2\alpha)\lambda^h} \Big\}$$ conditional stability — conditional upon Δt.

↳ $\alpha = 0$ FORWARD EULER

$O(\lambda^h)$ depends on eigenvalue problem

↑ order of λ^h

$$\underline{K}\,\underline{\Psi} = \lambda^h \underline{M}\,\underline{\Psi}$$

$$O(\lambda^h) \sim O(\underline{M}^{-1}\underline{K})$$

Recall: $$M_e^{AB} \sim \int_{\Omega^e} N^A N^B \, dV \sim 1$$

$$K_e^{AB} \sim \int_{\Omega^e} \overset{\frac{1}{h}}{N^A_{,x_i}} \overset{\frac{1}{h}}{N^B_{,x_j}} K_{ij} \, dV \sim h^{-2}$$

} order $\frac{1}{h}$

$$O(\lambda^h) \sim h^{-2}$$

$$\Rightarrow \Delta t \leq \frac{2}{(1-2\alpha)\underset{h^{-2}}{\lambda^h}}$$

Remark: Restriction on Δt gets more stringent as h decreases.

$\boxed{11.17}$ Stability:

$$\alpha \begin{cases} \geq \frac{1}{2}; & \text{unconditionally stable, any } \Delta t > 0 \\ < \frac{1}{2}; & \text{conditionally stable; } \underbrace{\Delta t \leq \frac{2}{(1-2\alpha)\lambda^h}}_{\text{must hold for all modes.}} \end{cases}$$

$$\Rightarrow \Delta t \leq \frac{2}{(1-2\alpha)\underset{h^{-2}}{\lambda^h_{max}}}$$

$$\Delta t_{max} \sim h^2$$

↳ as we refine the mesh, we must use smaller &

Recall the amplification factor:

$$A = \frac{1-(1-\alpha)\Delta t \lambda^h}{1+\alpha \Delta t \lambda^h}$$

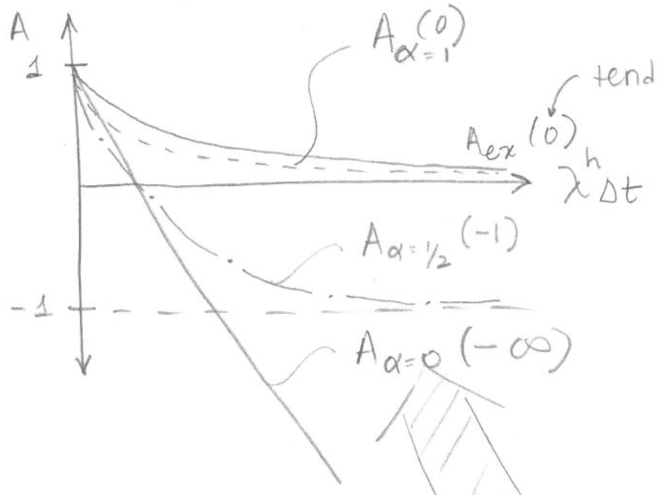
$$d_{n+1} = A\, d_n$$

Behavior of high-order modes:

$\underline{\Psi}_m$: $m = 1, \ldots, n_{df}$

$\lambda^h \leftarrow$ large values here create "higher order"

$\lambda^h \Rightarrow \lambda^h \Delta t$



Exact eqn: $d(t) = d_0 \underbrace{\exp(-\lambda^h t)}_{A_{ex} = e^{-\lambda^h \Delta t}}$

$$A = \frac{1-(1-\alpha)\Delta t \lambda^h}{1+\alpha \Delta t \lambda^h} = \frac{\frac{1}{\lambda^h \Delta t} - (1-\alpha)}{\frac{1}{\lambda^h \Delta t} + \alpha}$$

$\boxed{11.18}$ Remarks: — High-Order MODES —

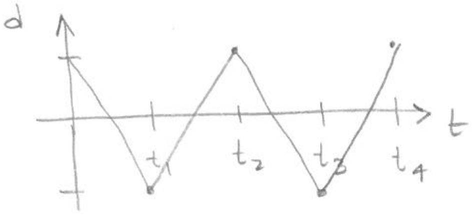
α	$\lim_{\lambda^h \Delta t \to \infty} A$
1	0
1/2	−1
0	$-\infty$

(1) Backward Euler ($\alpha = 1$) dissipates high order modes.
↳ "Numerical Dissipation"
↳ Similar to time-exact eqn.

(2) Forward Euler ($\alpha = 0$) has unbounded A.

(3) Mid-point Rule ($\alpha = 1/2$) has $A \to -1$

$d_{n+1} = -d_n$ } leads to oscillatory behavior.



} Time average eliminates the effect of the oscillations. (usually done in post processing)

Consistency:

reintroduce the Forcing Term

$$\frac{d_{n+1}-d_n}{\Delta t} + \lambda^h d_{n+\alpha} = F_{n+\alpha}\,; \qquad F^{\ell}_{n+\alpha} = \underline{\psi}^{\ell} \cdot \underline{F}_{n+\alpha}$$

ℓ: modal number.

$$d_{n+1} - d_n + \lambda^h \Delta t \left(\alpha d_{n+1} + (1-\alpha) d_n\right) = \Delta t\, F_{n+\alpha}$$

$$d_{n+1}\left(1+\alpha \Delta t \lambda^h\right) - \left(1-(1-\alpha)\right)\Delta t \lambda^h d_n - \Delta t\, F_{n+\alpha} = 0$$

$$\left. d_{n+1} - A d_n - \underbrace{\frac{\Delta t\, F_{n+\alpha}}{1+\alpha \Delta t \lambda^h}}_{L_n} = 0 \right\} \text{Time discrete Modal equation w/ Forcing.}$$

$d(t_{n+1})$: time-exact mode corresponding to λ^h.

a nonzero factor depending on t_n.

$$d(t_{n+1}) - A\, d(t_n) - L_n = \Delta t\, \tilde{\tau}(t_n) \leftarrow \text{consistency condition}$$

Consistency Method:

$$\tau(t_n) \leq c \Delta t^k , \quad k > 0$$

$$\lim_{\Delta t \to 0} \left[d(t_{n+1}) - A d(t_n) - L_n = \Delta t\, c\, \Delta t^k \right] \to 0$$

In the limit $\Delta t \to 0$, the time-discrete equation admits the exact solution.

$$\tau(t_n) \leqslant \underset{\substack{\uparrow \\ \text{constant}}}{c}\, \Delta t^{k} \leftarrow \text{order of accuracy}$$

$$k = \begin{cases} 2, & \alpha = 1/2\,;\ \text{Midpoint} \\ 1, & \text{otherwise; Backward/Forward} \end{cases}$$

$\boxed{11.19}$ Convergence of the time-discrete solution.

Error: $\underline{d}_{n+1} - \underline{d}(t_{n+1}) =: \underline{e}_{n+1} \in \mathbb{R}^{n_{df}}$

Modal Decomposition of $\underline{e}_{n+1}$:

$$\underline{e}_{n+1} = \sum_m e^m_{n+1} \, \underline{\Psi}^m$$

Convergence: $\lim\limits_{(n+1)\to\infty} \left(\underline{e}_{n+1} \cdot \underline{M} \, \underline{e}_{n+1} \right) = 0$

We put $\underline{M}$ here because $\underline{M}$ is positive definite, so $\underline{e}_{n+1} \cdot \underline{M} \, \underline{e}_{n+1} = 0$ iff $\underline{e}_{n+1} = 0$.

But note: $\underline{e}_{n+1} \cdot \underline{M} \, \underline{e}_{n+1} =$

$$\left(\sum_m e^m_{n+1} \underline{\Psi}^m \right) \cdot \underline{M} \left(\sum_\ell e^\ell_{n+1} \underline{\Psi}^\ell \right)$$

$$= \sum_{m,\ell} e^m_{n+1} \underbrace{\left(\underline{\Psi}^m \cdot \underline{M} \, \underline{\Psi}^\ell \right)}_{\delta_{m\ell}} e^\ell_{n+1}$$

$$\Rightarrow \underbrace{\underline{e}_{n+1} \cdot \underline{M} \, \underline{e}_{n+1}}_{\to 0} = \sum_m e^m_{n+1} e^m_{n+1} = \sum_m \big(\underbrace{e^m_{n+1}}_{\text{This then must} \to 0 \text{ also.}} \big)^2$$

$$\Rightarrow \lim_{(n+1)\to\infty} \left(\underline{e}_{n+1} \cdot \underline{M} \, \underline{e}_{n+1} \right) = \lim_{n+1\to\infty} \sum_m \left(e^m_{n+1} \right)^2$$

$$\Rightarrow \lim_{(n+1)\to\infty} \left(\underline{e}_{n+1} \cdot \underline{M} \, \underline{e}_{n+1} \right) = 0 \text{ iff } \lim_{n+1\to\infty} e^m_{n+1} = 0$$

Study convergence of Modal Coefficients, e_{n+1}

Consider: $d_{n+1} - A d_n - L_n = 0$

$\ominus \; \underline{d(t_{n+1}) - A d(t_n) - L_n = \Delta t \, \tau(t_n)}$ } subtract these.

$$\Rightarrow e_{n+1} - A e_n = -\Delta t \, \tau(t_n)$$

$$\Rightarrow e_{n+1} = A e_n - \Delta t \, \tau(t_n)$$

using recursion.

$$\Rightarrow e_n = A e_{n-1} - \Delta t \, \tau(t_{n-1})$$

$$\Rightarrow e_{n+1} = A^2 e_{n-1} - A^0 \Delta t \, \tau(t_n) - A^1 \Delta t \, \tau(t_{n-1})$$

$$\Rightarrow e_{n-1} = A e_{n-2} - \Delta t \, \tau(t_{n-2})$$

$$\Rightarrow e_{n+1} = A^3 e_{n-2} - A^0 \Delta t \, \tau(t_n) - A^1 \Delta t \, \tau(t_{n-1}) - A^2 \Delta t \, \tau(t_{n-3})$$

$\vdots$

∅ since initial condition must be satisfied.

$$\Rightarrow e_{n+1} = A^{n+1} \cancel{e_0} - \sum_{i=0}^{n} A^i \Delta t \, \tau(t_{n-i})$$

$$\Rightarrow e_{n+1} = - \sum_{i=0}^{n} A^i \Delta t \, \tau(t_{n-i})$$

$$\Rightarrow |e_{n+1}| = \left| \sum_{i=0}^{n} A^i \Delta t \, \tau(t_{n-i}) \right|$$

$$\leq \sum_{i=0}^{n} \left| A^i \Delta t \, \tau(t_{n-i}) \right|$$

given by (Triangle Inequality)

$$\leq \sum_{i=0}^{n} \underbrace{|A^i|}_{\text{Property of our method}} \cdot \left| \Delta t \, \tau(t_{n-i}) \right|$$

given by (Cauchy-Schwartz Inequality)

"stability"

$$\leq \sum_{i=0}^{n} 1 \cdot \left| \Delta t \cdot \tau(t_{n-i}) \right|$$

due to consistency.

$$\leq \sum_{i=0}^{n} \Delta t \, \tau_{\max_i}(t_{n-i}) \rightarrow$$

$$\leq \Delta t \sum_{i=0}^{n} |c \Delta t^k|$$

$$\leq t_{n+1} \cdot |c \Delta t^k|$$

But $\lim_{\Delta t \to 0} \left(c \Delta t^{k} \right) = 0$ for $k > 0$. (consistency)

$\Rightarrow \lim_{\Delta t \to 0} \left| e_{n+1} \right| \leqslant 0$ } guarantees Convergence.

Remark: (1) Consistency & Stability IMPLIES convergence.

↳ "Lax Theorem"

(2)

α	Name	Stability	Order of Accuracy	High order Modes
0	Forward Euler	conditional	1	$\lim_{\lambda^h \Delta t \to \infty} A \to -\infty$
$1/2$	Midpoint Rule	unconditional	2	$\lim_{\lambda^h \Delta t \to \infty} A = -1$
1	Backward Euler	unconditional	1	$\lim_{\lambda^h \Delta t \to \infty} A = 0$