The finite element method for problems in physics

Problem set 1

Problem 1.

Consider the following differential equation of elastostatics, in strong form: Find u satisfying

$$(E\,A\,u_{,x})_{,x} + f = 0, \quad \text{in } (0, L),$$

where f is a constant, and s.t. the following sets of boundary condiditions are satisfied:

(a) $u(0) = g_1$, $u(L) = g_2$

(b) $E\,A\,u_{,x} = h_1$ at $x = 0$, $E\,A\,u_{,x} = h_2$ at $x = L$, where $h_1 = h_2 + f\,L$

Comment on peculiarities, if any, of the solutions obtained to (a) or (b).

Problem 2.

For parts (a) through (c), consider the following boundary value problem (BVP) in its strong and weak formulations:

(**S**) On the unit interval $\Omega = (0, 1)$ and $\bar{\Omega} = [0, 1]$, given $f : \Omega \to \mathbb{R}$ and constants A, B, find $u : \bar{\Omega} \to \mathbb{R}$ such that,

$$u_{,xxxx} = f \text{ on } \Omega$$

$$u(1) = 0 \qquad u_{,x}(1) = 0$$

$$u_{,xx}(0) = A \qquad u_{,xxx}(0) = B$$

(**W**) $\mathcal{S} = \mathcal{V} = \{w | w \in H^2(\Omega),\ w(1) = w_{,x}(1) = 0\}$, given $f : \Omega \to \mathbb{R}$, and constants A, B, find $u \in \mathcal{S}$ such that $\forall\, w \in \mathcal{V}$,

$$\int_0^1 w_{,xx} u_{,xx}\, dx = \int_0^1 w f\, dx - w_{,x}(0)A + w(0)B$$

Also observe that $w \in H^2(\Omega)$ (i.e, $\int_0^1 |w|^2\, dx + \int_0^1 |w_{,x}|^2\, dx + \int_0^1 |w_{,xx}|^2\, dx < \infty$), unlike the problem in class where $w \in H^1(\Omega)$.

Prove the following propositions,

a) Let u be a solution to (**S**), then u is also a solution of (**W**).

b) Let u be a solution to (**W**), then u is also a solution of (**S**).

Also,

c) Would you call the boundary condition $u_{,x}(1) = 0$ as essential or natural?. Give reasons.