## $\boxed{3.01}$ The Matrix-Vector Weak Form.

Recall:

$$\int_{\Omega^e} w^h_{,x}\, \sigma^h A \, dx$$

for $e=1$:

$$= \int_{\Omega^e} \underbrace{\left(N^2_{,\xi} \cdot \xi_{,x} c^2_e\right)}_{w^h_{,x}} EA \underbrace{\left(\sum_B N^B_{,\xi}\, \xi_{,x}\, d^B_e\right)}_{u^h_{,x}} dx$$

This leaves out $N^1$ (the first basis function due to the dirichlet boundary condition).

$$= \int_{\Omega^\xi} \left(N^2_{,\xi} \frac{2}{h^e} c^2_e\right) EA \left(\sum_B N^B_{,\xi} \cancel{\frac{2}{h^e}} d^B_e\right) \cancel{\frac{dx}{d\xi}}^{\frac{h^e}{2}} d\xi$$

also for $e=1$:

$$\int_{\Omega^e} w^h f A \, dx = \int_{\Omega^\xi} \left(N^2 c^2_e\right) f A \frac{h^e}{2} d\xi$$

Consider for a general element $\Omega^e$:

$$\int_{\Omega^\xi} \left(\sum_A N^A_{,\xi} c^A_e\right) EA \left(\sum_B N^B_{,\xi} d^B_e\right) \frac{2}{h^e} d\xi = \sum_{A,B} c^A_e d^B_e \int_{\Omega^\xi} N^A_{,\xi} \frac{2EA}{h^e} N^B_{,\xi} d\xi$$

d.o.f's are independent of $\xi$.

$$\int_{\Omega^\xi} \left(\sum_A N^A c^A_e\right) f A \frac{h^e}{2} d\xi = \sum_A c^A_e \int_{\Omega^\xi} N^A f A \frac{h^e}{2} d\xi$$

still independent of $\xi$.

Because $N_1$ isn't used for the first element

For $e=1$, there is no sum over $A$. Instead use $A=2$.

$\because$ $w^h(0)=0 \Rightarrow w^h_{e=1} = N^2 c^2_e$.

⇒ Use matrix-vector product to eliminate the sums over A & B.

$n_{el} = 2$ ← because linear.  ?★ "rho vector"

$$\sum_{A,B=1}^{n_{el}=2} c_e^A \left( \int_{\Omega^\xi} N^A_{,\xi} \frac{2EA}{h^e} N^B_{,\xi} \, d\xi \right) d_e^B = \langle c_e^1 \, c_e^2 \rangle \underbrace{\frac{2EA}{h^e}} \int_{\Omega^\xi} \begin{bmatrix} N^1_{,\xi} N^1_{,\xi} & N^1_{,\xi} N^2_{,\xi} \\ N^2_{,\xi} N^1_{,\xi} & N^2_{,\xi} N^2_{,\xi} \end{bmatrix} d\xi \cdot \begin{Bmatrix} d_e^1 \\ d_e^2 \end{Bmatrix}$$

assuming the simple case where these are constant.

NOTE: This holds if E & A are uniform over $\Omega^e$.

Recall: $N^1(\xi) = \frac{1-\xi}{2}$ & $N^2(\xi) = \frac{1+\xi}{2}$

$\Rightarrow N^1_{,\xi} = -\frac{1}{2}$ & $N^2_{,\xi} = \frac{1}{2}$

$$= \langle c_e^1 \, c_e^2 \rangle \frac{2EA}{h^e} \left. \begin{bmatrix} \frac{1}{4}\xi & -\frac{1}{4}\xi \\ -\frac{1}{4}\xi & \frac{1}{4}\xi \end{bmatrix} \right|_{-1}^{1} \cdot \begin{Bmatrix} d_e^1 \\ d_e^2 \end{Bmatrix}$$


$$= \langle c_e^1 \, c_e^2 \rangle \frac{\cancel{2}EA}{h^e} \begin{bmatrix} \frac{1}{\cancel{2}} & -\frac{1}{\cancel{2}} \\ -\frac{1}{\cancel{2}} & \frac{1}{\cancel{2}} \end{bmatrix} \cdot \begin{Bmatrix} d_e^1 \\ d_e^2 \end{Bmatrix}$$

$$= \langle c_e^1 \, c_e^2 \rangle \frac{EA}{h^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} d_e^1 \\ d_e^2 \end{Bmatrix}$$

THIS COMES FROM THIS: $\int_{\Omega^e} w^h_{,x} \sigma^h A \, dx$

In a related manner:

$$\sum_A c_e^A \int_{\Omega^\xi} N^A \underbrace{fA}_{\text{if uniform over } \Omega^e} \frac{h^e}{2} \, d\xi = \langle c_e^1 \, c_e^2 \rangle \frac{fAh^e}{2} \int_{-1}^{1} \begin{Bmatrix} N^1 \\ N^2 \end{Bmatrix} d\xi$$

$N^1 = \frac{1-\xi}{2}$, $N^2 = \frac{1+\xi}{2}$

$$= \langle c_e^1 \, c_e^2 \rangle \frac{fAh^e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

COMES FROM: $\int_{\Omega^e} w^h f A \, dx$

3.03 NOTATION: $\because$ is "since" or "because"

Note that for $\underline{\underline{e=1}}$

$$\Rightarrow \int_{\Omega^e} w^h_{,x}\, \sigma^h A\, dx = c_e^2 \frac{EA}{h^e} \langle -1 \quad 1 \rangle \begin{Bmatrix} d_e^1 \\ d_e^2 \end{Bmatrix}$$

$$\Rightarrow \int_{\Omega^e} w^h f A\, dx = c_e^2 \frac{fAh^e}{2} \cdot 1$$

Recall: the Finite Dimensional Weak Form

$$\underbrace{\sum_e \int_{\Omega^e} w^h_{,x}\, \sigma^h A\, dx}_{①} = \underbrace{\sum_e \int_{\Omega^e} w^h f A\, dx}_{②} + \underbrace{w^h(L)\, t A}_{③}$$

only comes up at the last element since this represents the Neumann condition

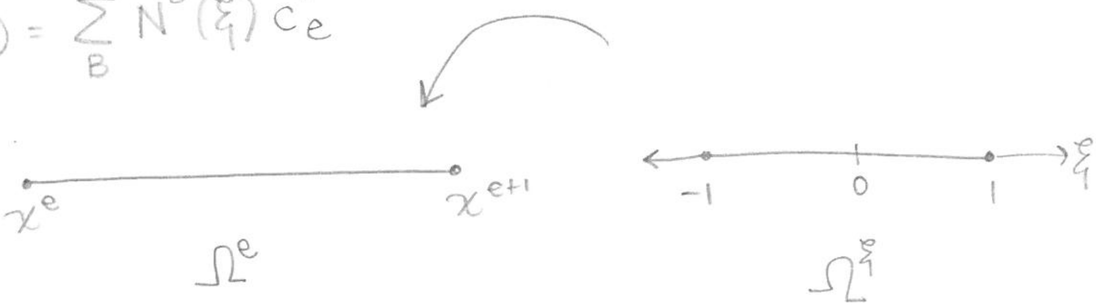
$$① \Rightarrow \underbrace{c_1^2 \frac{EA}{h^1} \langle -1 \quad 1 \rangle \begin{Bmatrix} d_1^1 \\ d_1^2 \end{Bmatrix}}_{\text{contribution from element 1}} + \underbrace{\sum_{e=2}^{n_{el}} \langle c_e^1 \; c_e^2 \rangle \frac{EA}{h^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} d_e^1 \\ d_e^2 \end{Bmatrix}}_{\text{contribution from the rest of the elements!}} =$$

$$② \quad c_1^2 f \frac{Ah^1}{2} + \sum_{e=2}^{n_{el}} \langle c_e^1 c_e^2 \rangle f \frac{Ah^e}{2} \cdot \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \underbrace{c_{n_{el}}^2}_{w^h(L)} \cdot t \cdot A$$

Remark:

$$u_e^h(x) = \sum_A N^A(\xi)\, d_e^A \qquad \text{where } x = f(\xi)$$

$$w_e^h(x) = \sum_B N^B(\xi)\, c_e^B$$



evaluating at the nodes.

$$u^h(x=x^e) = u^h(x(\xi=-1)) = \sum_A \underbrace{N^A(\xi^1)}_{\delta_{A1}} d_e^A = d_e^1$$

The degrees of freedom @ the node

$$u^h(x=x^{e+1}) = u^h(x(\xi=1)) = \sum_A N^A(\xi^2)\, d_e^A = \sum_A \delta_{A2}\, d_e^A = d_e^2$$

Similarly: $w^h(x^e) = c_e^1$ & $w^h(x^{e+1}) = c_e^2$

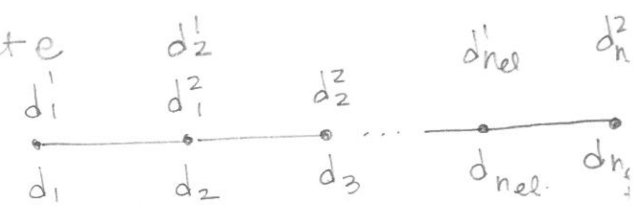
The Kronecker-delta property of the basis functions ensures that the nodal degrees of freedom of the solution field are indeed its values at the nodes. → NOT GENERALLY TRUE, but true for these basis vectors.

NOTE: The following map holds between local and global node numbers or degrees of freedom.

global degree of freedom $e+A-1$

$$d_e^A = d_{e+A-1}$$

local degree of freedom A in element e




Check: $d_1^1 = d_{1+1-1} = d_1$

$d_1^2 = d_{1+2-1} = d_2$

$d_{nel}^1 = d_{1+nel-1} = d_{nel}$ ✓

$d_{nel}^2 = d_{2+nel-1} = d_{nel+1}$

Similarly: $c_e^A = c_{e+A-1}$

$\boxed{3.05}$ Matrix-Vector Weak Form in terms of global matrices & vectors

$$c_1^2 \frac{EA}{h^1} \langle -1 \quad 1 \rangle \begin{Bmatrix} d_1^1 \\ d_1^2 \end{Bmatrix} + \sum_{e=2}^{n_{el}} \langle c_e^1 \; c_e^2 \rangle \overbrace{\frac{EA}{h^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}^{\text{element stiffness matrix, } \underline{K}_e} \begin{Bmatrix} d_e^1 \\ d_e^2 \end{Bmatrix}$$

$$= c_1^2 \frac{fAh^1}{2} + \sum_{e=2}^{n_{el}} \langle c_e^1 \; c_e^2 \rangle \underbrace{\begin{Bmatrix} fAh^e/2 \\ fAh^e/2 \end{Bmatrix}}_{\text{Element Force vector, } \underline{F}_e} + c_{n_{el}}^2 \cdot t \cdot A$$

Finite Element Assembly: (we now use global nodal notation)

$$\langle c_2 \; c_3 \cdots c_{n_{el}} \; c_{n_{el}+1} \rangle \; EA \begin{bmatrix} -\frac{1}{h^1} & \frac{1}{h^1}+\frac{1}{h^2} & -\frac{1}{h^2} & & & \\ & -\frac{1}{h^2} & \frac{1}{h^2}+\frac{1}{h^3} & -\frac{1}{h^3} & & \\ & & -\frac{1}{h^3} & \frac{1}{h^3}+\frac{1}{h^4} & & \\ & & & \ddots & & \\ & & & & \frac{1}{h^{n_{el}-1}}+\frac{1}{h^{n_{el}}} & -\frac{1}{h^{n_{el}}} \\ & & & & -\frac{1}{h^{n_{el}}} & \frac{1}{h^{n_{el}}} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{n_{el}} \\ d_{n_{el}+1} \end{Bmatrix}$$

$$= \underset{e=1}{\overset{n_{el}}{\mathrm{A}}} \langle c_e^1 \; c_e^2 \rangle \underline{K}_e \begin{Bmatrix} d_e^1 \\ d_e^2 \end{Bmatrix}$$

assembling

$\boxed{3.06}$

$$\langle c_2 \; c_3 \cdots c_{n_{el}} \; c_{n_{el}+1} \rangle \frac{fA}{2} \begin{Bmatrix} h^1+h^2 \\ h^2+h^3 \\ h^3+h^4 \\ \vdots \\ h^{n_{el}-1}+h^{n_{el}} \\ h^{n_{el}} \end{Bmatrix} = \underset{e=1}{\overset{n_{el}}{\mathrm{A}}} \langle c_e^1 \; c_e^2 \rangle \begin{Bmatrix} \frac{fAh^e}{2} \\ \frac{fAh^e}{2} \end{Bmatrix}$$

$$\langle c_2 \; c_3 \cdots c_{n_{el}} \; c_{n_{el}+1} \rangle \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ tA \end{Bmatrix} = c_{n_{el}}^2 \cdot t \cdot A$$

$$\langle c_2 \ c_3 \dots c_{n_{el}} \ c_{n_{el}+1} \rangle EA \begin{bmatrix} -\frac{1}{h^1} & \frac{1}{h^1}+\frac{1}{h^2} & -\frac{1}{h^2} & & & \\ & -\frac{1}{h^2} & \frac{1}{h^2}+\frac{1}{h^3} & -\frac{1}{h^3} & & \\ & & -\frac{1}{h^3} & \frac{1}{h^3}+\dots & & \\ & & & \ddots & & \\ & & & & \frac{1}{h^{n_{el}-1}}+\frac{1}{h^{n_{el}}} & -\frac{1}{h^{n_{el}}} \\ & & & & -\frac{1}{h^{n_{el}}} & \frac{1}{h^{n_{el}}} \end{bmatrix} \cdot \begin{Bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n_{el}} \\ d_{n_{el}+1} \end{Bmatrix} =$$

$$\langle c_2 \ c_3 \dots c_{n_{el}} \ c_{n_{el}+1} \rangle \frac{fA}{2} \begin{Bmatrix} h^1+h^2 \\ h^2+h^3 \\ \vdots \\ h^{n_{el}-1}+h^{n_{el}} \\ h^{n_{el}+1} \end{Bmatrix} + \langle c_2 \ c_3 \dots c_{n_{el}} \ c_{n_{el}+1} \rangle \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ tA \end{Bmatrix}$$

## $\boxed{3.07}$ The Finite Element Equations in Matrix-Vector Form.

Recall:

$$\int_\Omega w^h_{,x} \sigma^h A \, dx = \langle c_2 \ c_3 \dots c_{n_{el}+1} \rangle EA \begin{bmatrix} -\frac{1}{h_1} & \frac{1}{h_1}+\frac{1}{h_2} & -\frac{1}{h_2} & & & \\ & -\frac{1}{h_2} & \frac{1}{h_2}+\frac{1}{h_3} & -\frac{1}{h_3} & & \\ & & -\frac{1}{h_3} & \frac{1}{h_3}+\dots & & \\ & & & \ddots & & \\ & & & & \frac{1}{h_{n_{el}-1}}+\frac{1}{h_{n_{el}}} & -\frac{1}{h_{n_{el}}} \\ & & & & -\frac{1}{h_{n_{el}}} & \frac{1}{h_{n_{el}}} \end{bmatrix} \cdot \begin{Bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n_{el}+1} \end{Bmatrix}$$

$$\int_{\Omega} w^h f A\,dx = \langle c_2\ c_3 \cdots c_{n_{el}+1} \rangle fA \begin{Bmatrix} h_1 + h_2 \\ h_2 + h_3 \\ h_3 + \cdots \\ \vdots \\ h_{n_{el}-1} + h_{n_{el}} \\ h_{n_{el}} \end{Bmatrix}$$

NOTE: Using $h_e$ instead of $h^e$. (oops!)

Consider $h^e$ constant for all e.

$$w^h(L)\,tA = \langle c_2\ c_3 \cdots c_{n_{el}+1} \rangle \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ tA \end{Bmatrix}$$

NOTE:

$$\underbrace{\langle c_2\ c_3 \cdots c_{n_{el}+1} \rangle}_{\text{length } n_{el}} \underbrace{\frac{EA}{h^e}}_{\text{constant!}} \left[ \underbrace{\begin{Bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}}_{\text{dimension: } n_{el} \times n_{el}+1} \begin{matrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{matrix} \right] \underbrace{\begin{Bmatrix} \cancel{d_1} \\ d_2 \\ \vdots \\ d_{n_{el}+1} \end{Bmatrix}}_{\text{length } n_{el}+1}$$

$d_1 \to u_0$ because of B.C ✱

$$= \underbrace{\langle c_2\ c_3 \cdots c_{n_{el}+1} \rangle}_{\text{length: } n_{el}} fA \frac{h^e}{2} \underbrace{\begin{Bmatrix} 2 \\ 2 \\ \vdots \\ 1 \end{Bmatrix}}_{\text{length } n_{el}} + \langle c_2\ c_3 \cdots c_{n_{el}+1} \rangle \underbrace{\begin{Bmatrix} 0 \\ 0 \\ \vdots \\ tA \end{Bmatrix}}_{\text{length: } n_{el}}$$

✱ $d_1 = u_0$

$x' = 0$

$\Omega_1$

$$\underbrace{\langle c_2\ c_3 \cdots c_{n_{el}} \rangle}_{\underline{c}^T \text{ since it is a row vector}} \overbrace{\frac{EA}{h^e} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}}^{(n_{el}-1)\times(n_{el}-1) \leftarrow \text{dimension of the matrix } \underline{K}} \begin{Bmatrix} d_2 \\ d_3 \\ \vdots \\ d_{n_{el}+1} \end{Bmatrix} \leftarrow \underline{d} \quad =$$

$$\underset{\underline{c}^T}{\underset{\uparrow}{\langle c_2\ c_3 \cdots c_{n_{el}} \rangle}} \underbrace{\left( \frac{fAh^e}{2} \begin{Bmatrix} 2 \\ \vdots \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \vdots \\ tA \end{Bmatrix} + \frac{EA}{h^e} \begin{Bmatrix} u_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \right)}_{\underline{F}}$$

$\boxed{3.08}$ $\quad \underline{c}^T \underline{K}\, \underline{d} = \underline{c}^T \underline{F} \leftarrow$ Matrix-Vector Weak Form

Recall: $w_e^h = \sum_A N^A c_e^A$

Also: Find $u^h \in \mathcal{S}^h$ s.t. $\forall\, w^h \in \mathcal{V}^h$

$$\int_\Omega w^h_{,x} \sigma^h A\, dx = \int_\Omega w^h f A\, dx + w^h(L)\, t A$$

$$\Longleftrightarrow$$

$$\underline{c}^T \underline{K}\, \underline{d} = \underline{c}^T \underline{F} \qquad \forall\, \underline{c} \in \mathbb{R}^{n_{el}}$$

$\Rightarrow \underline{K}\,\underline{d} = \underline{F}$ } Final Form of Finite Element Equations.

Remarks: (1) $\underline{K}$: Symmetric, positive definite, with banded, tridiagonal structure

symmetric w.r.t. $w^h$ & $u^h$.

symmetry from $\int_\Omega w^h_{,x} \underbrace{E u^h_{,x}}_{\sigma^h} A\, dx$

(2) $\underline{K}$: "Stiffness" matrix

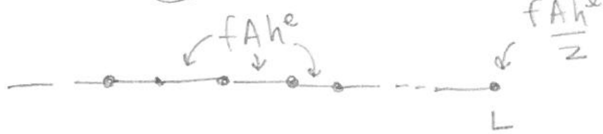
positive definiteness from $E > 0$.

(3) Tridiagonal from single derivative on $w^h$ & $u^h$ & linear basis funct.

(4) $$\underline{F} = \frac{fAh^e}{2}\begin{Bmatrix}2\\ \vdots\\ 1\end{Bmatrix} + \begin{Bmatrix}0\\ \vdots\\ tA\end{Bmatrix} + \underbrace{\frac{EA}{h^e}\begin{Bmatrix}u_0\\ 0\\ 0\\ \vdots\\ 0\end{Bmatrix}}_{\text{"Dirichlet" driven load} \; =0 \text{ in the considered case.}}$$

Dirichlet B.C.

traction (Neumann b.c.)




$$\underline{K}\,\underline{d} = \underline{F}$$

$$\underline{d} = \underline{K}^{-1}\underline{F} \Rightarrow u^h_e = \sum_A N^A d^A_e$$

NOTE

- A matrix $\underline{K}$ is symmetric if $\underline{K} = \underline{K}^T$

$$\Rightarrow \underline{c}^T \underline{K}\,\underline{d} = \underline{d}^T \underline{K}\,\underline{c}$$

- A matrix $\underline{K}$ is positive definite if

$$\underline{K} : n \times n$$

and $\forall\, \underline{d} \in \mathbb{R}^n$, $\underline{d}^T \underline{K}\,\underline{d} \geq 0$.

In particular, $\underline{d}^T \underline{K}\,\underline{d} > 0$ if $\underline{d} \neq \underline{0}$.

$$\Rightarrow \underline{d}^T \underline{K}\,\underline{d} = 0 \text{ iff } \underline{d} = \underline{0}.$$