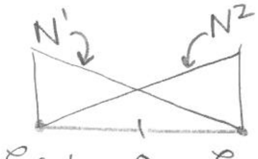
8.01 A unified view of basis functions in 1 – 3 dimensions.

Lagrange Polynomials:

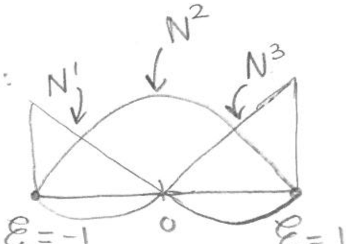
Recall 1D: • Linears:



$$\check{N}^1(\xi_1) = \frac{1-\xi_1}{2}$$

$$\check{N}^2(\xi_1) = \frac{1+\xi_1}{2}$$

• Quadratics:



$$\check{N}^1(\xi_1) = \frac{(1-\xi_1)\xi_1}{2}$$

$$\check{N}^2(\xi_1) = (1-\xi_1)^2$$

$$\check{N}^3(\xi_1) = \frac{(1+\xi_1)\xi_1}{2}$$

Lagrange Polynomials of order k:

$k = \text{d.o.f} - 1$

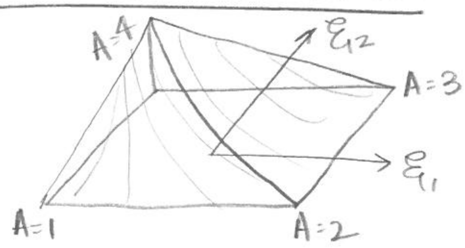
$k = n_{n_1} - 1$

← number of nodes in 1D.

$$N^A(\xi_1) = \prod_{\substack{B=1 \\ B\neq A}}^{n_{n_1}} \frac{(\xi_1 - \xi_1^B)}{(\xi_1^A - \xi_1^B)}$$

Tensor Product Basis Functions in 2-D:

Recall bi-linears.



Shown: N^4

$n_{ne} = \text{d.o.f} = 4$

$= (n_{n_1})^2$ for

$n_{n_1} = 2$ ← linear case.

$$N^A(\xi_1,\xi_2) = \underbrace{\check{N}^B(\xi_1)\cdot\check{N}^C(\xi_2)}_{\text{tensor product formula.}} \quad \text{for} \quad B,C = 1,\dots,n_{n_1}; \quad A = 1,\dots,n_{ne}$$

$N^1(\xi_1,\xi_2) = \check{N}^1(\xi_1)\cdot\check{N}^1(\xi_2)$

$N^2(\xi_1,\xi_2) = \check{N}^2(\xi_1)\cdot\check{N}^1(\xi_2)$

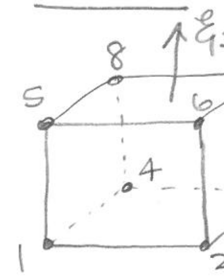
$\vdots$

$N^4(\xi_1,\xi_2) = \check{N}^1(\xi_1)\cdot\check{N}^2(\xi_2)$

8.02 Likewise, in 3D:

$$n_{ne} = (n_{n1})^3 = (2)^3 = 8$$

Trilinears:

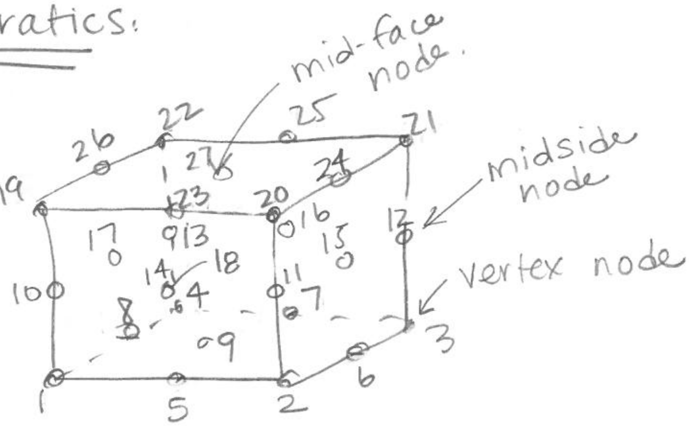


$$N^A(\xi_1, \xi_2, \xi_3) = \tilde{N}^B(\xi_1)\tilde{N}^C(\xi_2)\tilde{N}^D(\xi_3)$$

where $B, C, D \in \{1, \ldots, n_{n1D}\}$

$A \in \{1, \ldots, n_{ne}\}$

Triquadratics:



$$N^5(\xi_1, \xi_2, \xi_3) = \tilde{N}^2(\xi_1) \cdot \tilde{N}^1(\xi_2) \cdot \tilde{N}^1(\xi_3)$$

8.03 Numerical Integration in 1 thru 3 dimensions.

Recall 1D:

$$\int_{-1}^{1} g(\xi_1)\, d\xi_1 = \sum_{l=1}^{n_{int}} g(\xi_1^l)\, w_l$$

quadrature points; weight.

Gaussian Quadrature:

$n_{int} = 1$ $\quad \xi_1^1 = 0, \; w_1 = 2$

$\vdots$

$n_{int} = 3$ $\quad \xi_1^1 = -\sqrt{\frac{3}{5}} \quad w_1 = \frac{5}{9}$

$\xi_1^2 = 0 \quad w_2 = \frac{8}{9}$

$\xi_1^3 = \sqrt{\frac{3}{5}} \quad w_3 = \frac{5}{9}$

} n_{int} point rule integrals $p^{2n_{int}-1}$ exactly

2D: Need to integrate $\int_{-1}^{1}\int_{-1}^{1} g(\xi_1,\xi_2)\,d\xi_1\,d\xi_2$

Numerical integration: $= \int_{-1}^{1}\left(\sum_{l_1=1}^{n_{int}^{1}} g(\xi_1^{l_1},\xi_2)\cdot w_{l_1}\right)d\xi_2$ ← # of integration points

$= \sum_{l_2=1}^{n_{int}^{2}} \sum_{l_1=1}^{n_{int}^{1}} g(\xi_1^{l_1},\xi_2^{l_2})\cdot w_{l_1}\cdot w_{l_2}$

Remark: Can use different # of integration points along ξ_1 & ξ_2

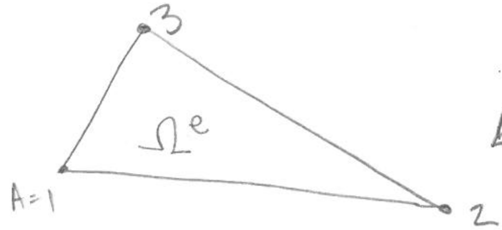
Use Gaussian Quadrature points & weights along each direction.

Sum of weights = 4.

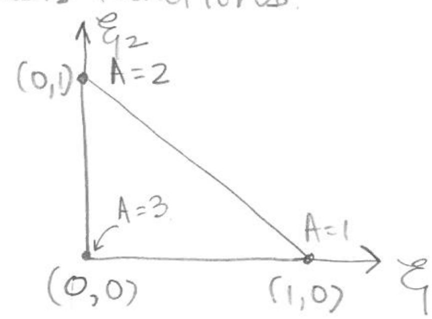
3D: $\int_{\xi_3=-1}^{1}\int_{\xi_2=-1}^{1}\int_{\xi_1=-1}^{1} g(\xi_1,\xi_2,\xi_3)\,d\xi_1\,d\xi_2\,d\xi_3 = \sum_{l_3=1}^{n_{int}^{3}}\sum_{l_2=1}^{n_{int}^{2}}\sum_{l_1=1}^{n_{int}^{1}} g(\xi_1^{l_1},\xi_2^{l_2},\xi_3^{l_3})\cdot w_{l_1}\cdot w_{l_2}\cdot w_{l_3}$

8.04 Simpler Elements: 2D & 3D

Triangular Elements – linear basis functions.



$n_{n_e} = 3$



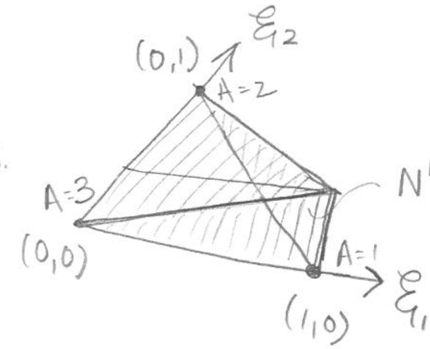
Also define $\xi_3 := 1 - \xi_1 - \xi_2$

"unit" domain (not bi-unit!)

8.05

$$N^1(\xi_1,\xi_2,\xi_3) = \xi_1$$
$$N^2(\xi_1,\xi_2,\xi_3) = \xi_2$$
$$N^3(\xi_1,\xi_2,\xi_3) = \xi_3$$

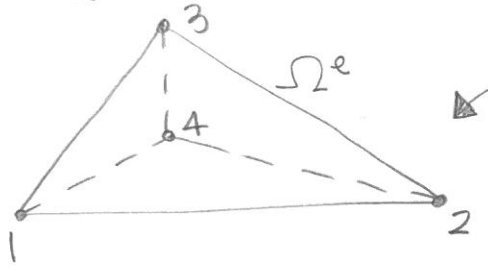
linear basis functions.

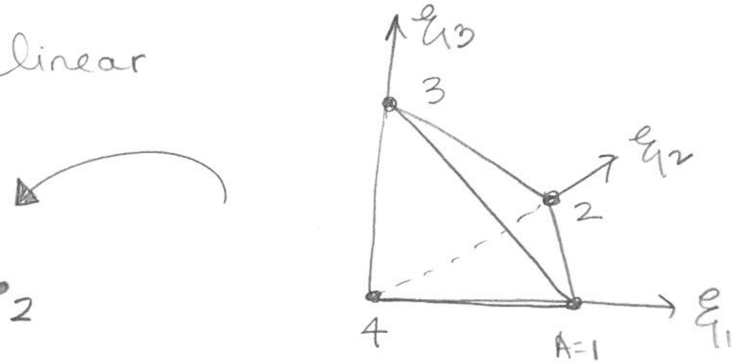


Simplest elements in general are simpler than quadralaterals and hexahedra.

Simularly for N^2, N^3

3D: Tetrahedra – linear





$$\xi_4 = 1 - \xi_1 - \xi_2 - \xi_3$$

$$N^1(\xi_1,\xi_2,\xi_3,\xi_4) = \xi_1$$
$$\vdots$$
$$N^4(\xi_1,\xi_2,\xi_3,\xi_4) = \xi_4$$

Remark: Linear simpler elements lead to constant gradiants:

$$N^A_{,\xi}$$

Can define higher-order tetrahedron

- Numerical integration rules: (Gaussian Quadrature no longer optimal).