4.01 Pure Dirichlet Problems

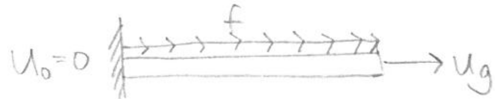
Find $u^h \in \mathcal{S}^h \subset \mathcal{S} = \{u \mid u(0) = u_0,\ u(L) = u_g\}$

$$\Rightarrow \mathcal{S}^h = \{u^h \in H^1(\Omega) \mid u^h(0) = u_0,\ u^h(L) = u_g\}$$

Such that $\forall\ w^h \in \mathcal{V}^h \subset \mathcal{V} = \{w \mid w(0) = 0,\ w(L) = 0\}$

$$\Rightarrow \mathcal{V}^h = \{w^h \in H^1(\Omega) \mid w^h(0) = 0,\ w^h(L) = 0\}$$

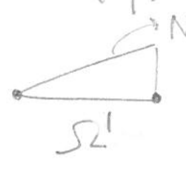
$$\int_\Omega w^h_{,x}\, \sigma^h A\, dx = \int_\Omega w^h f A\, dx$$

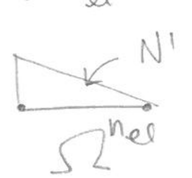


physical picture

Basis Functions: linear polynomials over Ω^e

$$w^h_{e=1} = N^2(\xi)\, c^2_{e=1} \qquad w^h_{e=n_{el}} = N^1(\xi)\, c^1_{e=n_{el}}$$





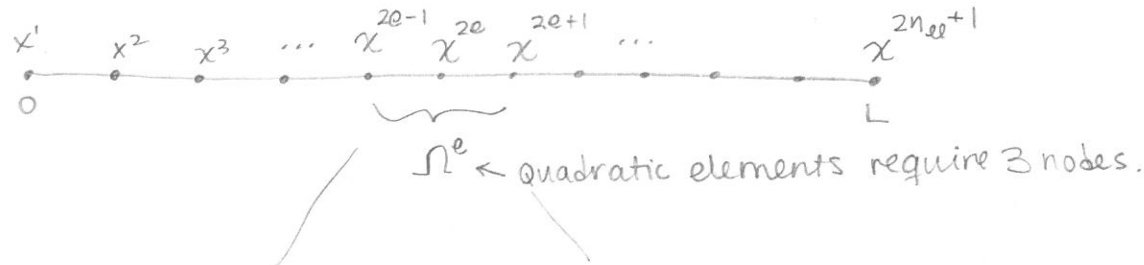
$$\int_\Omega w^h_{,x}\, \sigma^h A\, dx = c^2_1 \frac{EA}{h^e} \langle -1 \quad 1 \rangle \begin{Bmatrix} d^1_1 \\ d^2_1 \end{Bmatrix} + \sum_{e=2}^{n_{el}-1} \langle c^1_e\, c^2_e \rangle \frac{EA}{h^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} d^1_e \\ d^2_e \end{Bmatrix}$$

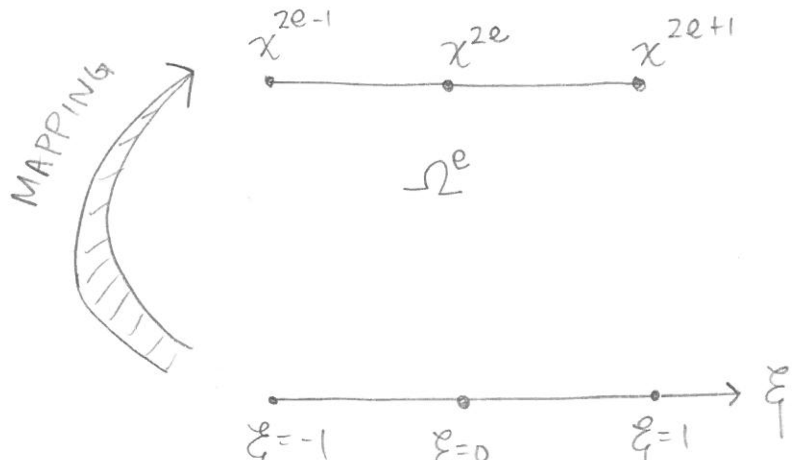
$$+\ c^1_{n_{el}} \frac{EA}{h^e} \langle 1 \ -1 \rangle \begin{Bmatrix} d^1_{n_{el}} \\ d^2_{n_{el}} \end{Bmatrix}$$

$$\int_{\Omega} w^h f A\,dx = c_1^2 f \frac{Ah^e}{2} + \sum_{e=2}^{n_{el}-1} \langle c_e^1\ c_e^2 \rangle f A \frac{h^e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + c_{n_{el}}^1 f \frac{Ah^e}{2}$$

$\boxed{4.03}$ Higher-Order Polynomial Basis functions

→ Quadratic basis functions





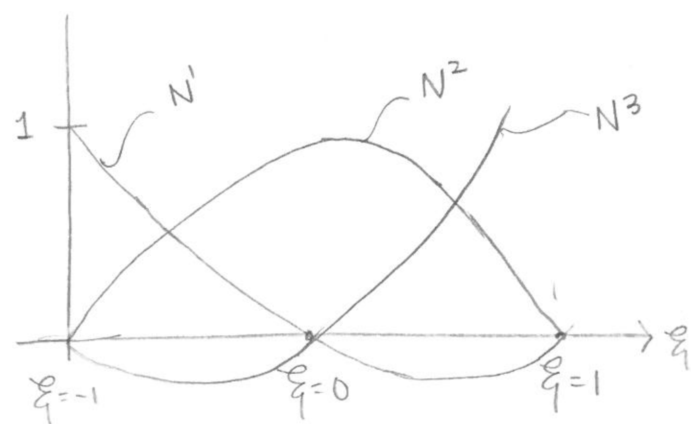
$$u_e^h = \sum_{A=1}^{N_{n_e}=3} N_A(x(\xi))\, d_e^A \quad , \quad w_e^h = \sum_{B=1}^{N_{n_{el}}=3} N_B(x(\xi))\, c_e^B$$

↑ basis function.

$$N^1(\xi) = -\xi\frac{(1-\xi)}{2}$$

$$N^2(\xi) = 1-\xi^2$$

$$N^3(\xi) = \xi\frac{(1+\xi)}{2}$$



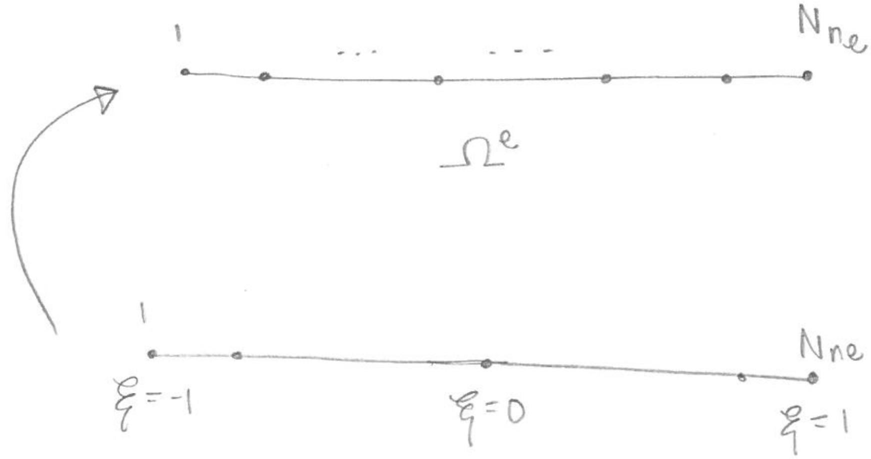
If $\begin{Bmatrix} \xi^1 = -1 \\ \xi^2 = 0 \\ \xi^3 = 1 \end{Bmatrix}$ then $N^A(\xi^B) = \delta_{AB}$ Kronecker delta property

and $\sum_A N^A(\xi) = 1$

Higher-order basis functions:

→ These are generated by a formula for Lagrange polynomial

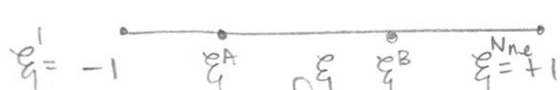
$N_{n_e} - 1$ order polynomials.



$$N^A(\xi) = \frac{\prod_{\substack{B=1 \\ B \neq A}}^{N_{ne}} (\xi - \xi^B)}{\prod_{\substack{B=1 \\ B \neq A}}^{N_{ne}} (\xi^A - \xi^B)} \qquad A = 1, \ldots, N_{ne}$$

product.

Check for linear basis functions:



Linears:

$$N^1(\xi) = \frac{\prod_{\substack{B=1 \\ B \neq 1}}^{2} (\xi - \xi^B)}{\prod_{\substack{B=1 \\ B \neq 1}}^{2} (\xi^1 - \xi^B)} = \frac{(\xi - 1)}{-1-1} = \left(\frac{1-\xi}{2}\right) ✓$$

$\boxed{4.04}$ ★ Lagrange Polynomials satisfy the Kronecker Delta Property. ★

$$N^A(\xi) = \frac{\prod_{\substack{B=1 \\ B\neq A}}^{N_{ne}} (\xi - \xi^B)}{\prod_{\substack{B=1 \\ B\neq A}}^{N_{ne}} (\xi^A - \xi^B)}$$

holds for $A = 1, \dots, N_{ne}$

$$N^A(\xi^A) = \frac{\prod_{\substack{B=1 \\ B\neq A}}^{N_{ne}} (\xi^A - \xi^B)}{\prod_{\substack{B=1 \\ B\neq A}}^{N_{ne}} (\xi^A - \xi^B)} = 1$$

Kronecker delta property!

$$N^A(\xi^C) = \frac{\prod_{\substack{B=1 \\ B\neq A}}^{N_{ne}} (\xi^C - \xi^B)}{\prod_{\substack{B=1 \\ B\neq A}}^{N_{ne}} (\xi^A - \xi^B)}$$

This term is 0 for $B=C$ & since it is a product, the whole product is 0.

Check that $\sum_A N^A(\xi) = 1$ ← Exercise.

Develop the Finite Element Formulation with quadratic basis functions.

$$\underbrace{\sum_{e=1}^{n_{el}} \int_{\Omega^e} w^h_{,x} \cancel{\sigma^h}^{\,E u^h_{,x}} A \, dx}_{\text{let's focus on this term}} = \sum_{e=1}^{n_{el}} \int_{\Omega^e} w^h f A \, dx + \underbrace{w^h(L) \cdot \bar{t} A}_{\text{Dirichlet-Neumann Problem.}}$$

Consider:

$$\int_{\Omega^e} w^h_{,x}\, E\, u^h_{,x}\, A\, dx = \int_{\Omega^e} \underbrace{\left(\sum_{A=1}^{3} N^A_{,\xi}\cdot\xi_{,x}\, c^A_e\right) EA \left(\sum_{B=1}^{3} N^B_{,\xi}\cdot\xi_{,x}\cdot d^B_e\right)}_{\hat{3}}\, dx$$

Quadratic element!

Recall: $u^h_{,x} = \sum_{A=1}^{3} N^A_{,x}\, d^A_e\,;$ $\qquad w^h_{,x} = \sum_{A=1}^{3} N^A_{,x}\, c^A_e$

use the chain rule!

$$N^A_{,x} = N^A_{,\xi}\cdot \xi_{,x}$$

$$N^1_{,\xi} = \frac{d}{d\xi}(N^1) = \frac{d}{d\xi}\left(-\frac{1}{2}\xi(1-\xi)\right) = -\frac{1}{2}+\xi$$

$$N^2_{,\xi} = \frac{d}{d\xi}\left(1-\xi^2\right) = -2\xi$$

$$N^3_{,\xi} = \frac{d}{d\xi}\left(\frac{1}{2}(\xi+1)\xi\right) = \xi+\frac{1}{2}$$

$\boxed{4.05}$ To compute $\xi_{,x}$ — use the invertibility of the mapping

$$x_e(\xi) = \sum_{A=1}^{3} N^A(\xi)\, x^A_e \longrightarrow \text{isoparametric mapping.}$$

nodal coordinates in physical domain

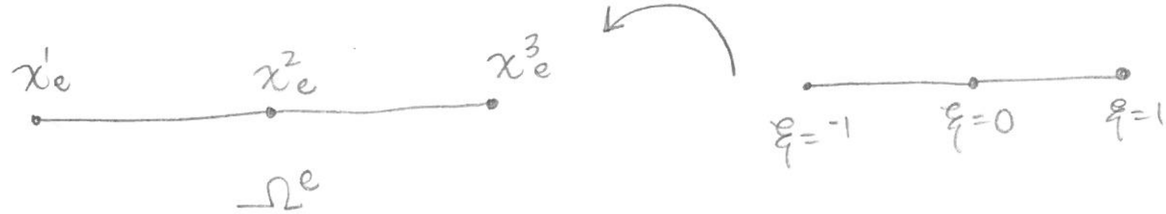
$$\Rightarrow x_{,\xi} = \sum_{A=1}^{3} N^A_{,\xi}\cdot x^A_e$$

already found this!

$$= N^1_{,\xi}\cdot x^1_e + N^2_{,\xi}\cdot x^2_e + N^3_{,\xi}\, x^3_e$$

$$= \frac{1}{2}(-1+2\xi)\cdot x^1_e - 2\xi\, x^2_e + \frac{1}{2}(1+2\xi)\cdot x^3_e$$

$$= \left\{\frac{x^3_e - x^1_e}{2}\right\} + \xi\left(x^3_e - 2x^2_e + x^1_e\right)$$



x_e^2 : midside node

$$\Rightarrow x_e^2 = \frac{x_e^3 + x_e^1}{2}$$

$$\Rightarrow x_{,\xi} = \underbrace{\frac{x_e^3 - x_e^1}{2}}_{\frac{h^e}{2}} + \xi \underbrace{\left(x_e^1 - 2x_e^2 + x_e^3\right)}_{=0}$$

$\frac{h^e}{2}$ ← length of the element!

Invertibility of the map

$$\Rightarrow \xi_{,x} = \frac{1}{x_{,\xi}} = \frac{2}{h^e}$$

$x_{,\xi} = \frac{h^e}{2}$: tangent of the geometric mapping is constant.

— Affine mapping

$$\Rightarrow \int_{\Omega^e} w^h_{,x} EA\, u^h_{,x}\, dx = \int_{\Omega^e} \left(\sum_A N^A_{,\xi} \left(\frac{2}{h^e}\right) \cdot c_e^A\right) EA \left(\sum_B N^B_{,\xi} \left(\frac{2}{h^e}\right) \cdot d_e^B\right) dx$$

$\boxed{4.06}$ Recall: $\displaystyle\int_{\Omega^e} w^h_{,x} EA\, u^h_{,x}\, dx = \int_{\Omega^e} \left(\sum_A N^A_{,\xi} \frac{2}{h^e} c^A_e\right) EA \left(\sum_B N^B_{,\xi} \cancel{\frac{2}{h^e}} d^B_e\right) dx$

Using "Matrix-vector notation":

$$dx = \frac{dx}{d\xi} d\xi = \cancel{\frac{h_e}{2}}$$

$$= \frac{2EA}{h^e} \sum_{A,B} c^A_e \left(\int_{\Omega^\xi} N^A_{,\xi} N^B_{,\xi}\, d\xi \right) d^B_e$$

Symmetric

$$= \langle c^1_e \;\; c^2_e \;\; c^3_e \rangle \frac{2EA}{h^e} \begin{bmatrix} \int_{-1}^{1} N^1_{,\xi} N^1_{,\xi} d\xi & \int_{-1}^{1} N^1_{,\xi} N^2_{,\xi} d\xi & \int_{-1}^{1} N^1_{,\xi} N^3_{,\xi} d\xi \\ \int_{-1}^{1} N^2_{,\xi} N^1_{,\xi} d\xi & \int_{-1}^{1} N^2_{,\xi} N^2_{,\xi} d\xi & \int_{-1}^{1} N^3_{,\xi} N^2_{,\xi} d\xi \\ \int_{-1}^{1} N^3_{,\xi} N^1_{,\xi} d\xi & \int_{-1}^{1} N^3_{,\xi} N^2_{,\xi} d\xi & \int_{-1}^{1} N^3_{,\xi} N^3_{,\xi} d\xi \end{bmatrix} \begin{Bmatrix} d^1_e \\ d^2_e \\ d^3_e \end{Bmatrix}$$

Calculation of integrals: only even terms survive.

$$\int_{-1}^{1} N^1_{,\xi} N^1_{,\xi}\, d\xi = \frac{1}{4}\int_{-1}^{1} (2\xi-1)^2 d\xi = \frac{1}{4}\int_{-1}^{1} (4\xi^2 - 4\xi + 1) d\xi = \frac{1}{4}\left[\frac{4}{3}\xi^3 + \xi\right]_{-1}^{1}$$

$$= \frac{1}{4}\left\{\frac{4}{3}(1) + 1 - \left(-\frac{4}{3} - 1\right)\right\}$$

$$= \left\{\frac{14}{3}\right\}\frac{1}{4} = \frac{7}{6}$$

$$\int_{-1}^{1} N^1_{,\xi} N^2_{,\xi}\, d\xi = \int_{-1}^{1} \frac{1}{2}(2\xi-1)(-2\xi)\, d\xi$$

$$= \frac{1}{2}\int_{-1}^{1} (2\xi - 4\xi^2) d\xi = \frac{-1}{2}\left[4\xi^3/3\right]_{-1}^{1} = -4/3$$

$$\int_{-1}^{1} N^1_{,\xi} N^3_{,\xi}\, d\xi = \int_{-1}^{1} \frac{1}{2}(2\xi-1)\left(\frac{1}{2}\right)(2\xi+1)\, d\xi = \frac{1}{4}\int_{-1}^{1} (4\xi^2 - 1) d\xi = \frac{1}{4}\left(\frac{4\xi^3}{3} - \xi\right)\Bigg|_{-1}^{1}$$

$$= \frac{1}{4}\left(\frac{8}{3}\right) = \frac{2}{6}$$

$$\int_{-1}^{1} N^2_{,\xi} N^2_{,\xi}\, d\xi = \int_{-1}^{1} (-2\xi)^2 d\xi = \left.\frac{4\xi^3}{3}\right|_{-1}^{1} = \frac{8}{3}$$

$$\int_{-1}^{1} N^2_{,\xi} N^3_{,\xi}\, d\xi = \int_{-1}^{1} \tfrac{1}{2}(-2\xi)(2\xi+1)\, d\xi = \tfrac{1}{2}\int_{-1}^{1} (-4\xi^2 - 2\xi)\, d\xi = \left.\tfrac{1}{2}\left[-\frac{4\xi^3}{3}\right]\right|_{-1}^{1} = -\frac{4}{3}$$

$$\int_{-1}^{1} N^3_{,\xi} N^3_{,\xi}\, d\xi = \int_{-1}^{1} \tfrac{1}{4}(2\xi+1)^2 d\xi = \int_{-1}^{1} \tfrac{1}{4}(4\xi^2 + 4\xi + 1)\, d\xi = \left.\tfrac{1}{4}\left[\frac{4\xi^3}{3} + \xi\right]\right|_{-1}^{1}$$

$$= \tfrac{1}{4}\left[\frac{8}{3} + 2\right] = \frac{7}{6}$$

<u>Collecting results:</u>

$$\int_{\Omega^e} w^h_{,x}\, \sigma^h A\, dx = \langle c^1_e\ c^2_e\ c^3_e \rangle \underbrace{\frac{2EA}{h^e} \begin{bmatrix} 7/6 & -4/3 & 1/6 \\ -4/3 & 8/3 & -4/3 \\ 1/6 & -4/3 & 7/6 \end{bmatrix}}_{\underline{K}_e:\ \text{element stiffness matrix.}} \begin{Bmatrix} d^1_e \\ d^2_e \\ d^3_e \end{Bmatrix}$$

$\boxed{4.07}$ Next, consider: $\displaystyle\int_{\Omega^e} w^h f A\, dx = \int_{\Omega^e} \left(\sum_A N^A c^A_e\right) f A\, dx = \sum_{A=1}^{3} c^A_e \int_{\Omega^\xi} N^A f A \left(\frac{h^e}{2}\right) d\xi$

Consider f, A are uniform over Ω^e.

$$= \sum_{A=1}^{3} c^A_e \cdot \frac{f A h^e}{2} \int_{-1}^{1} N^A\, d\xi$$

Using Matrix Vector Notation:

$$= \langle c^1_e\ c^2_e\ c^3_e \rangle \frac{f A h^e}{2} \begin{bmatrix} \int_{-1}^{1} N^1 d\xi \\ \int_{-1}^{1} N^2 d\xi \\ \int_{-1}^{1} N^3 d\xi \end{bmatrix}$$

$$\begin{Bmatrix} \int_{-1}^{1} \tfrac{1}{2}\xi(\xi-1)\, d\xi \\ \int_{-1}^{1} (1-\xi^2)\, d\xi \\ \int_{-1}^{1} \tfrac{1}{2}(\xi)(\xi+1)\, d\xi \end{Bmatrix}$$

$$= \langle c_e^1 \; c_e^2 \; c_e^3 \rangle \frac{fAh^e}{2} \begin{Bmatrix} \frac{1}{2}\xi^3/3 \Big|_{-1}^{1} \\ (\xi - \xi^3/3) \Big|_{-1}^{1} \\ \frac{1}{2}\xi^3/3 \Big|_{-1}^{1} \end{Bmatrix}$$

$$= \langle c_e^1 \; c_e^2 \; c_e^3 \rangle \frac{fAh^e}{2} \begin{Bmatrix} 1/3 \\ 4/3 \\ 1/3 \end{Bmatrix}$$

Assembly of global matrix vector equations.

$$\sum_e \int_{\Omega^e} w^h_{,x} \sigma^h A \, dx = \sum_e \int_{\Omega^e} w^h f A \, dx + w^h(L) t A$$

Recall: For Dirichlet-Neumann Problem $u^h(0) = u_g = 0$

$$\Rightarrow w^h(0) = 0.$$

$\Rightarrow$ For element one $(e=1)$; $w^h_{e=1} = \sum_{A=2}^{3} N^A c_e^A = N^2 c^{2e} + N^3 c^{2e+1}$; $e=1$

Global equations for the matrix vector weak form:

$$\underbrace{\langle c^2 \; c^3 \rangle \frac{2EA}{h^1} \begin{bmatrix} -4/3 & 8/3 & -4/3 \\ 1/6 & -4/3 & 7/6 \end{bmatrix} \begin{Bmatrix} d^1 \\ d^2 \\ d^3 \end{Bmatrix}}_{1^{st} \text{ element}} + \underbrace{\sum_{e=2}^{n_{el}} \langle c^{2e-1} \; c^{2e} \; c^{2e+1} \rangle \frac{2EA}{h^e} \cdot \begin{bmatrix} 7/6 & -4/3 & 1/6 \\ -4/3 & 8/3 & -4/3 \\ 1/6 & -4/3 & 7/6 \end{bmatrix} \begin{Bmatrix} d^{2e-1} \\ d^{2e} \\ d^{2e+1} \end{Bmatrix}}_{\text{all the rest.}}$$

$$= \langle c^2 \; c^3 \rangle \frac{fAh^1}{2} \begin{Bmatrix} 4/3 \\ 1/3 \end{Bmatrix} + \sum_{e=2}^{n_{el}} \langle c^{2e-1} \; c^{2e} \; c^{2e+1} \rangle \cdot \frac{fAh^e}{2} \begin{Bmatrix} 1/3 \\ 4/3 \\ 1/3 \end{Bmatrix} + c^{2n_{el}+1} \cdot tA$$

$\boxed{4.08}$

$$\Rightarrow \langle c^2\ c^3\ c^4\ c^5\ c^6 \dots c^{2n_{el}-1}\ c^{2n_{el}}\ c^{2n_{el}+1}\rangle \frac{2EA}{h} \cdot$$

length $2n^{el}$

if $h^e = h\ \forall e$

$2n^{el} \times (2n^{el}+1)$

$$\begin{bmatrix} -4/3 & 8/3 & -4/3 & & & & & & \\ 1/6 & -4/3 & (7/6+7/6) & -4/3 & 1/6 & \cdots & & & \\ & & -4/3 & 8/3 & -4/3 \cdots & & & & \\ & & 1/6 & -4/3 & 7/6+\cdots & & & & \\ & & & & & & & & \\ & & & & & & \cdots+7/6 & -4/3 & 1/6 \\ & & & & & & -4/3 & 8/3 & -4/3 \\ & & & & & & 1/6 & -4/3 & 7/6 \end{bmatrix} \begin{Bmatrix} d^1 \\ d^2 \\ d^3 \\ d^4 \\ d^5 \\ \vdots \\ d^{2n_{el}-1} \\ d^{2n_{el}} \\ d^{2n_{el}+1} \end{Bmatrix}$$

$$= \langle c^2\ c^3\ c^4\ c^5 \dots c^{2n_{e}-1}\ c^{2n_{el}}\ c^{2n_{el}+1}\rangle \left(\left(\frac{fAh}{2}\right) \begin{Bmatrix} 4/3 \\ 1/3+1/3 \\ 4/3 \\ 1/3+\cdots \\ \vdots \\ \cdots+1/3 \\ 4/3 \\ 1/3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ tA \end{Bmatrix} \right)$$

length $2n^{el}$

length $2n^{el}$

known dirichlet B.C.

$= u_0$

$\boxed{4.09}$ The global matrix-vector equations

$$\langle c^2\ c^3 \dots c^{2n_{el}+1}\rangle \frac{2EA}{h} \begin{bmatrix} -4/3 & 8/3 & -4/3 & & & & & & \\ 1/6 & -4/3 & 7/3 & -4/3 & 1/6 & & & & \\ 0 & & -4/3 & 8/3 & -4/3 & & & & \\ 0 & & 1/6 & -4/3 & 7/3 \cdots & & & & \\ 0 & & & & \vdots & & & & \\ \vdots & & & & & & 7/3 & -4/3 & 1/6 \\ & & & & & & -4/3 & 8/3 & -4/3 \\ 0 & & & & & & 1/6 & -4/3 & 7/6 \end{bmatrix} \begin{Bmatrix} d^1 \\ d^2 \\ d^3 \\ \vdots \\ \vdots \\ d^{2n_{el}+1} \end{Bmatrix}$$

includes components from same degree of freedom, but different elements.

$$= \langle c^2\ c^3 \dots c^{2n_{el}+1}\rangle \left(\frac{fAh}{2} \begin{Bmatrix} 4/3 \\ 2/3 \\ 4/3 \\ \vdots \\ 2/3 \\ 4/3 \\ 1/3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ tA \end{Bmatrix} \right)$$

let's move the dirichlet b.c. term to the right hand side.

$$\Rightarrow \underbrace{\langle c^2\ c^3 \dots c^{2nel+1} \rangle}_{\underline{c}^T} \underbrace{\frac{2EA}{h} \begin{bmatrix} 8/3 & -4/3 & & & & & & \\ -4/3 & 7/3 & -4/3 & 1/6 & & & & \\ & -4/3 & 8/3 & -4/3 & & & & \\ & 1/6 & -4/3 & 7/3 & -4/3 & 1/6 & & \\ & & & & \ddots & & & \\ & & & & & 7/3 & -4/3 & 1/6 \\ & & & & & -4/3 & 8/3 & -4/3 \\ & & & & & 1/6 & -4/3 & 7/6 \end{bmatrix}}_{\underline{K}} \underbrace{\begin{Bmatrix} d^2 \\ d^3 \\ \vdots \\ d^{2nel+1} \end{Bmatrix}}_{\underline{d}}$$

NOW THIS IS SQUARE!

"bandwidth"

mid-node

end node

$$= \underbrace{\langle c^2\ c^3 \dots c^{2nel+1} \rangle}_{\underline{c}^T} \underbrace{\left(\frac{fAh}{2} \begin{Bmatrix} 4/3 \\ 2/3 \\ 4/3 \\ \vdots \\ 1/3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ tA \end{Bmatrix} - \frac{2EA}{h} \underbrace{d^1}_{u_0} \begin{Bmatrix} -4/3 \\ 1/6 \\ 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \right)}_{\underline{F}}$$

So, $\boxed{\underline{c}^T \underline{K}\, \underline{d} = \underline{c}^T \underline{F}}$

Remarks:

(1) $\underline{K}$ matrix components are different from linear case.

(2) Bandwidth of $\underline{K}$ matrix is larger (5) do to use of Quadratic Basis Functions.

(3) Mid-side nodes have a larger contribution to $\underline{F}$ vector due to the use of Quadratic Basis Functions.

$\boxed{4.10}$ Consider the Pure Dirichlet Problem.

Find $u^h \in \mathcal{S}^h \subset \mathcal{S}$, $\mathcal{S}^h = \{ u^h \in H^1(\Omega) \mid u^h(0) = u_0, u^h(L) = u$

such that $\forall\ w^h \in \mathcal{V}^h \subset \mathcal{V}$, $\mathcal{V}^h = \{ w^1 \in H^1(\Omega) \mid w^h(0) = 0, w^h(L) =$

$$\int_\Omega w^h_{,x} \sigma^h A\, dx = \int_\Omega w^h f A\, dx$$

This term gone since it is Pure Dirichlet.

<u>Matrix-Vector weak form — Global</u>

$$\underbrace{\langle c^2\ c^3 \ldots c^{2n_{el}-1}\ c^{2n_{el}} \rangle}_{(2^{n_{el}}-1)} \frac{2EA}{h} \underbrace{\begin{bmatrix} -4/3 & 8/3 & -4/3 & & & & & \\ 1/6 & -4/3 & 7/3 & -4/3 & 1/6 & & & \\ & & -4/3 & 8/3 & -4/3 & 1/6 & & \\ & & 1/6 & -4/3 & 7/3 & -4/3 & 1/6 & \\ & & & & & \ddots & & \\ & & & & & 7/3 & -4/3 & 1/6 \\ & & & & & -4/3 & 8/3 & -4/3 \end{bmatrix}}_{(2^{n_{el}}-1)\times(2^{n_{el}}+1)} \underbrace{\begin{Bmatrix} \overset{u_0}{d^1} \\ d^2 \\ \vdots \\ \\ \underset{u_L}{d^{2n_{el}+1}} \end{Bmatrix}}_{(2^{n_{el}}+1)}$$

$$= \underbrace{\langle c^2\ c^3 \ldots c^{2n_{el}-1}\ c^{2n_{el}} \rangle}_{(2^{n_{el}}-1)} \left(\frac{fAh}{2} \underbrace{\begin{Bmatrix} 4/3 \\ 2/3 \\ 4/3 \\ \vdots \\ \vdots \\ 4/3 \end{Bmatrix}}_{(2^{n_{el}}-1)} \right)$$

$$\Rightarrow \underbrace{\langle c^2\ c^3 \ldots c^{2n_{el}-1}\ c^{2n_{el}} \rangle}_{\underline{C}^T} \underbrace{\frac{2EA}{h} \begin{bmatrix} 8/3 & -4/3 & & & & \\ -4/3 & 7/3 & -4/3 & 1/6 & & \\ & -4/3 & 8/3 & -4/3 & 1/6 & \\ & & & & \ddots & \\ & & & & 7/3 & -4/3 \\ & & & & -4/3 & 8/3 \end{bmatrix}}_{\underline{K}\ \text{square}\ (2^{n_{el}}-1)\times(2^{n_{el}}-1)} \overbrace{\begin{Bmatrix} d^2 \\ \vdots \\ \vdots \\ d^{2n_{el}} \end{Bmatrix}}^{\underline{d}} =$$

$$\underbrace{\langle c^2\ c^3 \ldots c^{2n_{el}} \rangle}_{\underline{C}^T} \underbrace{\left(\frac{fAh}{2} \begin{Bmatrix} 4/3 \\ 2/3 \\ \vdots \\ 4/3 \end{Bmatrix} - \frac{2EA}{h} \overset{u_0}{d^1} \begin{Bmatrix} -4/3 \\ 1/6 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} - \frac{2EA}{h} \underset{u_L}{d^{2n_{el}+1}} \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ 1/6 \\ -4/3 \end{Bmatrix} \right)}_{\underline{F}}$$

$\underline{c}^T \underline{K}\, \underline{d} = \underline{c}^T \underline{F}$ for Dirichlet (Pure) Problem.

$$c^T(\underline{K}\underline{d} - \underline{F}) = 0, \quad \forall\, \underline{c} \in \mathbb{R}^{2n_{el}} \quad \left(\text{Dirichlet - Neuman Problem}\right)$$

-or-

$$\forall\, \underline{c} \in \mathbb{R}^{2n_{el}-1} \quad \left(\text{Dirichlet (Pure) Problem}\right)$$

$$\Rightarrow \underline{K}\underline{d} - \underline{F} = 0$$

$\underline{K}\underline{d} = \underline{F}$ ← standard form for linecr problems

$\underline{d} = \underline{K}^{-1}\underline{F}$ (definite solution of nodal problem.)

4.11 Numerical Integration

Needed if $E, f, \ldots$ are complicated functions of x. or if complicated basis functions are used.

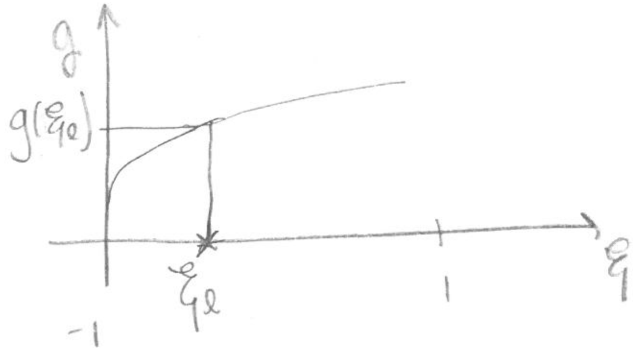
— Gaussian Quadrature, optimal for polynomials—exactly integrate polynomials of certain orders.

Consider,
$$\int_{-1}^{1} g(\xi)\,d\xi = \sum_{l=1}^{n_{int}} g(\xi_l)\cdot w_l \quad \Big\} \text{ Quadrature Rule.}$$

n_{int} ← # of integration points.

ξ_l: an integration point.

w_l: weight ascribed to the integration point.



$$w_l: \quad \sum_{l=1}^{n_{int}} w_l = 2$$

Because if $g(\xi) = \text{const.}$

$$\int_{-1}^{1} g(\xi)\,d\xi = 2(\text{const.})$$

$$= \sum_{l} \overbrace{g(\xi_l)}^{\text{const}}\, w_l = 2(\text{const.})$$

∴ we can integrate constants exactly.

$n_{int} = 1, \quad \xi_{q_1} = 0, \quad w_1 = 2$

$n_{int} = 2, \quad \xi_{q_1} = -1/\sqrt{3}, \quad w_1 = 1$

$\quad\quad\quad\quad \xi_{q_2} = 1/\sqrt{3}, \quad w_2 = 1$

$n_{int} = 3, \quad \xi_{q_1} = -\sqrt{3/5} \quad w_1 = 5/9$

$\quad\quad\quad\quad \xi_{q_2} = 0 \quad w_2 = 8/9$

$\quad\quad\quad\quad \xi_{q_3} = \sqrt{3/5} \quad w_3 = 5/9$

Rules from Gaussian Quadrature.

Gaussian Quadrature Rule with n_{int} points exactly integrates polynomials of order $\leq (2n_{int} - 1)$

$n_{int} = 1$ ——— linears

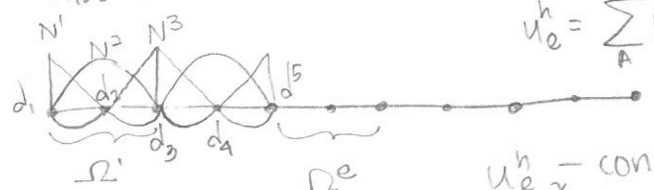
$n_{int} = 2$ ——— cubics

$n_{int} = 3$ ——— pentics

$\vdots$

$\boxed{5.01}$ **Norms**: Consider the finite-dimensional trial solution, u^h

— Also called the "Finite Element Solution"



$u^h_e = \sum_A N^A d^A_e$ — continuous!

↳ $C^0(\Omega)$

mathematic way to say funct. is continuous in the domain Ω

$u^h_{e,x}$ — continuous in Ω^e

$C^0(\Omega^e)$

continuous over each element (but not over whole domain due to the edges) $u^h_{e,x}$ — not in $C^0(\Omega)$

The Lagrange Polynomial Basis Functions have been constructed to be only $C^0(\Omega)$, not $C^n(\Omega)$ for $n > 0$.

n: # of derivatives.