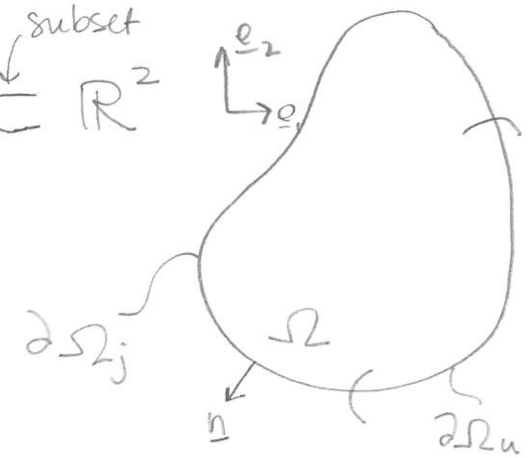
$\boxed{9.01}$ Two-Dimensional linear, elliptic PDEs in a Scalar Variable.

Given: u_g, j_n, f, the constitutive relation $+j_i = -K_{ij}\, u_{,j}$

$(i, j = 1, 2)$

Strong Form: Find u such that

$$-j_{i,i} = \underbrace{f}_{-\bar{f}} \quad \text{in } \Omega \overset{\text{subset}}{\subset} \mathbb{R}^2$$



With boundary conditions:

$$u = u_g \text{ on } \partial\Omega_u$$

$$-j_i n_i = j_n \text{ on } \partial\Omega_j$$

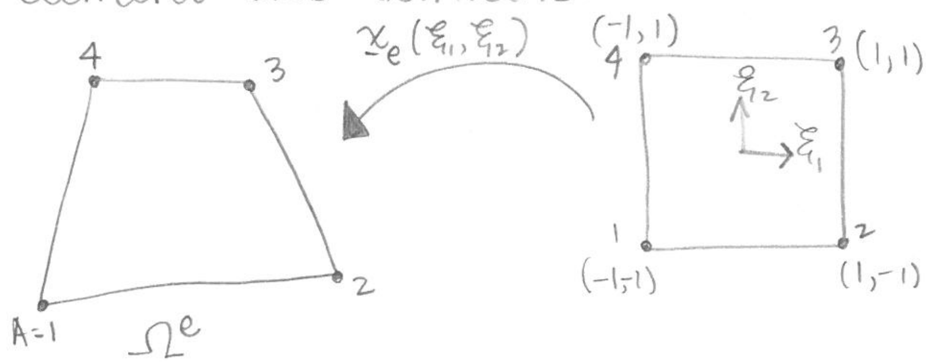
(Finite dimensional) weak form:

Find $u^h \in S^h \subset S$; $S^h = \{u^h \in H^1(\Omega) \mid u = u_g \text{ on } \partial\Omega_u\}$

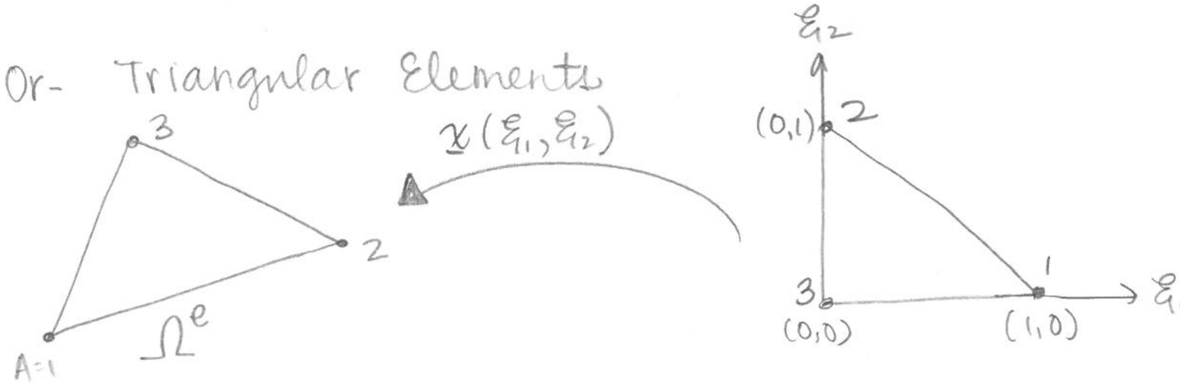
such that $\forall\, w^h \in V^h \subset V$; $V^h = \{w^h \in H^1(\Omega) \mid w^h = 0 \text{ on } \partial\Omega_u\}$

$$-\int_\Omega w^h_{,i}\, \overset{-K_{ij} u^h_{,j}}{\cancel{j^h_i}}\, dA = \int_\Omega w^h \bar{f}\, dA + \int_{\partial\Omega_j} w^h j_n\, dS \qquad i, j, i = 1, 2$$

Element sub-domains:



-Or- Triangular Elements



Basis Functions:

Bilinear Lagrange polynomials: $N^A(\xi_1, \xi_2)$, $A = 1, \ldots, n_{ne} = 4$

linear polynomials: $N^A(\xi_1, \xi_2, \underbrace{\xi_3}_{=1-\xi_1-\xi_2})$, $A = 1, \ldots, n_{ne} = 3$

$$u_e^h = \sum_{A=1}^{n_{ne}} N^A \cdot d_e^A, \quad w_e^h = \sum_{A=1}^{n_{ne}} N^A \cdot c_e^A$$

$\boxed{9.02}$ Gradients:

$$u^h_{,i} = \sum_{A=1}^{n_{ne}} \underbrace{N^A_{,\xi_1} \frac{\partial \xi_1}{\partial x_i} + N^A_{,\xi_2} \frac{\partial \xi_2}{\partial x_i}}_{\substack{\text{Rewrite } N^A(\xi_1, \xi_2, \xi_3) \\ \text{using } \xi_3 = 1 - \xi_1 - \xi_2 \text{ for} \\ \text{triangles.}}}, \quad i = 1, 2$$

How do we compute $\frac{\partial \xi_1}{\partial x_i}$, $\frac{\partial \xi_2}{\partial x_i}$?

Use isoparametric mapping:

use in $u^h_{,i} = \sum_A \frac{\partial N^A}{\partial \xi_I} \frac{\partial \xi_I}{\partial x_i} d_e^A$

$$x_{i_e}(\xi_1, \xi_2) = \sum_{A=1}^{n_{ne}} N^A \cdot x^A_{i_e}$$

$$\Rightarrow \frac{\partial x_i}{\partial \xi_I} = \sum_A N^A_{,\xi_I} x^A_{i_e} \qquad i = 1,2 \quad I = 1,2$$

$$\underline{J}(\underline{\xi}) = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix} \Rightarrow \underline{J}^{-1}(\underline{\xi}) = \begin{bmatrix} \frac{\partial \xi_1}{\partial x_1} & \frac{\partial \xi_1}{\partial x_2} \\ \frac{\partial \xi_2}{\partial x_1} & \frac{\partial \xi_2}{\partial x_2} \end{bmatrix}$$

Compute Element Integrals.

$$-\int_{\Omega} w^h_{,i}\, j^h_i \, dA = -\sum_e \int_{\Omega^e} w^h_{,i}\, j^h_i \, dA$$

$$-\int_{\Omega^e} w^h_{,i}\, j^h_i \, dA = -\sum_{A,B}^{n_{ne}} c_e^A \underbrace{\left(\int_{\xi_2=-1}^{1} \int_{\xi_1=-1}^{1} N^A_{,i} (-K_{ij}) N^B_{,j} \det(\underline{J})\, d\xi_1 d\xi_2 \right)}_{\text{use Numerical Quadrature: } \overline{K}_e^{AB}} d_e^B$$

$i,j = 1,2$

$\overline{K}_e^{AB}$ ← conductivity or diffusivity matrix

$$\int_{\Omega} w^h \bar{f}\, dA = \sum_e \int_{\Omega^e} w^h \bar{f}\, dA$$

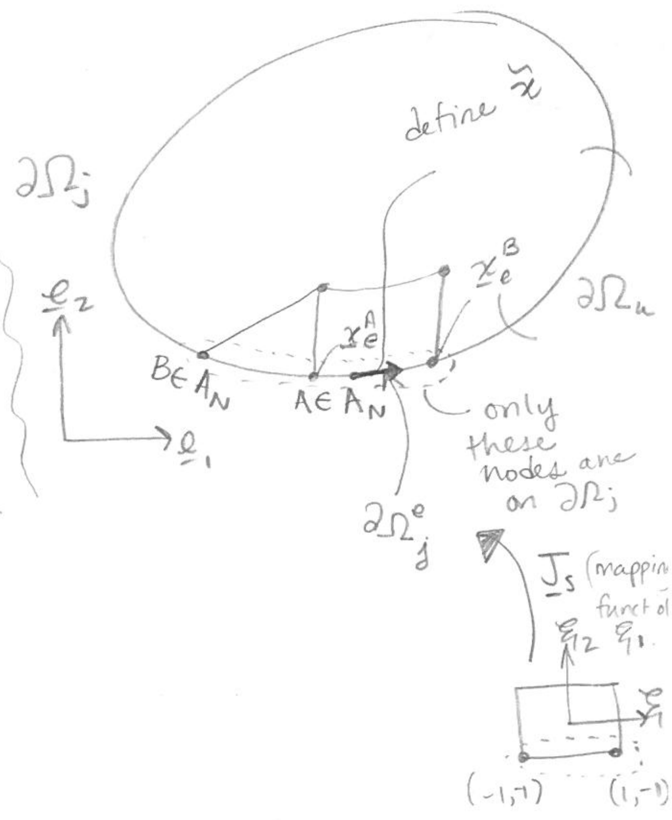
$$\int_{\Omega^e} w^h \bar{f}\, dA = \sum_A c_e^A \underbrace{\left(\int_{\xi_2=-1}^{1} \int_{\xi_1=-1}^{1} N^A \bar{f} \det(\underline{J})\, d\xi_1 d\xi_2 \right)}_{F_e^{\text{internal}_A}}$$

$\boxed{9.03}$ The boundary Integral

$$\int_{\partial\Omega_j} w^h j_n \underbrace{ds}_{\text{elemental curve}} = \sum_{e \in \mathcal{E}_N} \int_{\partial\Omega_j^e} w^h j_n \, ds$$

$$\int_{\partial\Omega_j^e} w^h j_n \, ds = \sum_A c_e^A \underbrace{\int_{\partial\Omega_j^e} N^A j_n \cancel{\det} \underline{J}_s \cdot d\xi_1}_{F_e^{jA}}$$

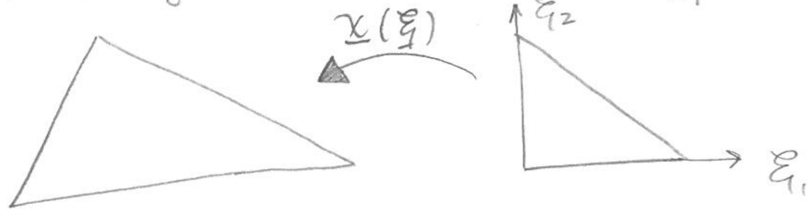
$\frac{\partial \tilde{x}}{\partial \xi_1}$



Similarly for triangular elements!

Remark:

Linear Triangles the matrix $\underline{J}(\underline{\xi}) = \begin{bmatrix} x_{1,\xi_1} & x_{1,\xi_2} \\ x_{2,\xi_1} & x_{2,\xi_2} \end{bmatrix}$ = con



due to linearity of the map. (it's not bi-linear!)

$$\det \underline{J}(\underline{\xi}) = 2\, m(\Omega^e)$$

Assembly:

$$\sum_e \underbrace{\left(\sum_{A,B} c_e^A \bar{K}^{AB} d_e^B\right)} = \sum_e \underbrace{\sum_A c_e^A \bar{F}_e^{int\,A}} + \sum_{e\in\mathcal{E}_N} \underbrace{\sum_{A\in A_N} c_e^A F_e^{jA}}$$

$$\underline{c}_e^T \underbrace{\underline{\bar{K}}_e}_{\text{rectangular due to dirichlet b.c.s}} \underline{\bar{d}}_e = \underline{c}_e^T \underline{\bar{F}}_e^{int} + \underline{c}_e^T \underline{F}_e^j$$

rectangular due to dirichlet b.c.s

$\boxed{9.04}$

Recall that $\underline{\bar{K}}_e$

# of degrees of freedom w/dirichlet b.c.s

no. of nodes in problem

$$(n_{sd} \times n_{np} - N_D) \times (n_{sd} \times n_{np})$$

$n_{sd} \to 2$

dimensions of $\underline{\bar{K}}$

$$\mathop{\Large A}_e \underline{\bar{K}}_e = \underline{\bar{K}}$$

$$\mathop{\Large A}_e \underline{\bar{F}}_e^{int} = \underline{\bar{F}}$$

$$\mathop{\Large A}_{e\in\mathcal{E}_N} \underline{F}_e^j = \underline{F}^j$$

$$\underline{\bar{K}} = \underbrace{\begin{bmatrix} \left\{ \underline{\bar{K}}^{\bar{A}} \right\} & \left\{ \underline{\bar{K}}^{\bar{B}} \right\} \end{bmatrix}}_{(n_{sd}\times n_{np})} \Bigg\} \begin{matrix} n_{sd}\times n_{ne} \\ -N_D \end{matrix}$$

$$\cdot \begin{Bmatrix} d^{\bar{A}} \leftarrow \bar{A} \\ d^{\bar{B}} \leftarrow \bar{B} \end{Bmatrix}$$

known

known

$$\Rightarrow \quad \underline{C}^T \underset{\uparrow}{\underline{K}} \, \underline{d} = \underline{C}^T \Big(\underbrace{\underline{F}^{int} + \underline{F}^{j} - d^{\bar{A}} \underline{\bar{K}}^{A} \; d^{\bar{B}} \underline{\bar{K}}^{\bar{B}}}_{\underline{F}} \Big)$$

$$(n_{sd} \times n_{np} - N_D)^2$$

$$\underline{F}$$
$$(n_{sd} \times n_{np} - N_D)$$

$$\forall \, \underline{c} \in \mathbb{R}^{n_{sd} \times n_{np} - N_D}$$

$$\boxed{\underline{K} \, \underline{d} = \underline{F}}$$