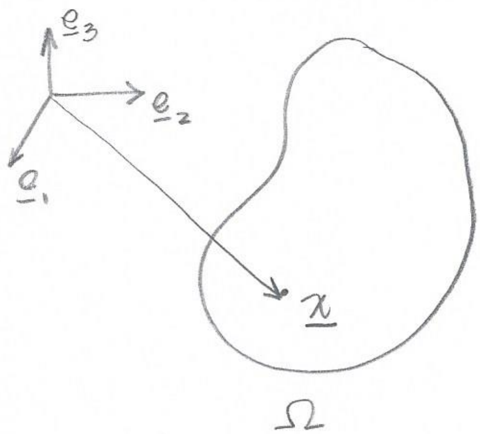
$\boxed{10.01}$ Linear elliptic PDE in 3D for a vector variable.

↳ Linearized Elasticity in 3D



Given: $\underbrace{u_i^g,\ \bar{t}_i,\ f_i}_{\text{coordinate notation}} \rightarrow \underbrace{(\underline{u}^g, \underline{\bar{t}}, \underline{f} \in \mathbb{R}^3)}_{\text{direct notation}}$

and the constitutive relation:

$$\sigma_{ij} = \mathbb{C}_{ijkl}\, \varepsilon_{kl} \rightarrow (\underline{\sigma} = \underline{\mathbb{C}} : \underline{\varepsilon})$$

Cauchy stress tensor

Fourth order elasticity tensor

infinitesimal strain tensor

contraction symbol

$\boxed{10.02}$

and the kinematic relation

$$\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)$$

in direct notation

$$\underline{\varepsilon} = \text{sym}\, \underline{\nabla}\, \underline{u}$$

Find $u_i \rightarrow (\underline{u} \in \mathbb{R}^3)$ such that

$$\underbrace{\sigma_{ij,j}}_{\text{divergence of stress}} + f_i = 0 \quad \text{in } \Omega \quad \Big\}\ \text{Quasi-Static stress Equilibrium Relation.}$$

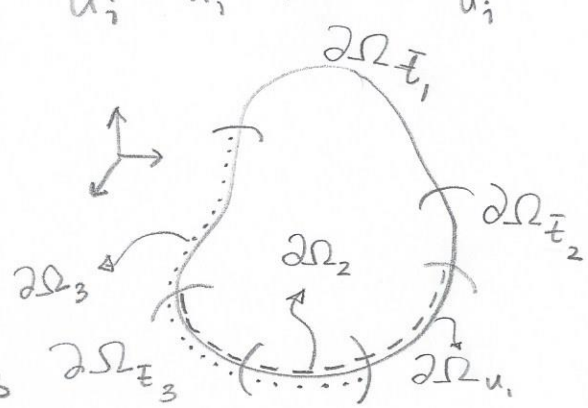
Boundary Conditions: $u_i = u_i^g$ on $\partial\Omega_{u_i}$

NOTE: $i \,\&\, j = 1,2,3$

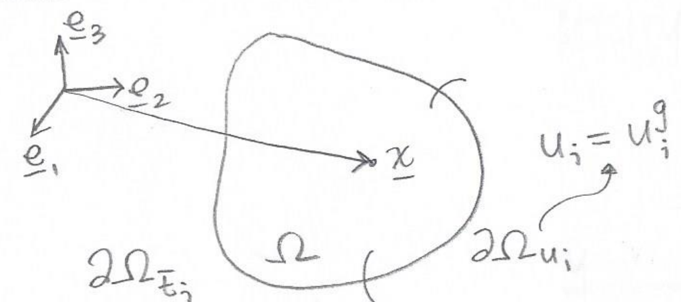
$\sigma_{ij} n_j = \bar{t}_i$ on $\partial\Omega_{t_i}$

$\partial\Omega = \partial\Omega_{u_i} \cup \partial\Omega_{\bar{t}_i}$,

$\partial\Omega_{u_i} \cap \partial\Omega_{\bar{t}_i} = \emptyset$ for $i = 1,2,3$



$\boxed{10.03}$ Linearized elasticity in 3D. — Direct Notation.



$$\partial\Omega = \partial\Omega_{u_i} \cup \partial\Omega_{t_i}$$

$$\partial\Omega_{u_i} \cap \partial\Omega_{t_i} = \emptyset$$

$$(\underline{\mathbb{C}} : \underline{\varepsilon})_{ij} = \mathbb{C}_{ijkl}\,\varepsilon_{kl}$$

Given $u_i^g,\ t_i,\ \underline{f},\ \underline{\sigma} = \underbrace{\underline{\mathbb{C}} : \underline{\varepsilon}}\ ,\ \underline{\varepsilon} = \text{sym}(\underline{\nabla}\,\underline{u})$

Find $\underline{u}$ such that $\underline{\nabla} \cdot \underline{\sigma} + \underline{f} = \underline{0}$ in Ω

$u_i = u_i^g$ on $\partial\Omega_{u_i}$

$(\underline{\sigma}\,\underline{n})_i = t_i$ on $\partial\Omega_{t_i}$

NOTE: cannot be writen in direct notation

also writen as: $\text{div}(\underline{\sigma})$ -or- $\frac{\partial}{\partial \underline{x}} \cdot \underline{\sigma}$

Constitutive Relations:

$$\sigma_{ij} = \mathbb{C}_{ijkl}\,\varepsilon_{kl}$$

$\mathbb{C}_{ijkl}$: forth-order elasticity tensor
constant w.r.t $\underline{\varepsilon}$ for linearized case.
↳ i.e. relation between $\underline{\sigma}$ & $\underline{\varepsilon}$ is linear.

$\underline{\mathbb{C}}$: major symmetry
↳ $\mathbb{C}_{ijkl} = \mathbb{C}_{klij}$
Follows from the fact that $\exists$ a function ("there exists")

$$\Psi : \underbrace{S(3)}_{\text{symmetric 2nd order tensors}} \mapsto \mathbb{R}^+$$

$\Psi(\underline{\varepsilon})$: strain energy density function

Can show that $\mathbb{C}_{ijkl} = \dfrac{\partial^2 \psi}{\partial \varepsilon_{ij}\, \partial \varepsilon_{kl}}$

$\because$ $\underline{\mathbb{C}}$: constant wrt $\underline{\varepsilon}$ $\Rightarrow$ $\psi(\underline{\varepsilon})$: Quadratic.

$$\Rightarrow \mathbb{C}_{ijkl} = \underbrace{\frac{\partial^2 \psi}{\partial \varepsilon_{ij}\, \partial \varepsilon_{kl}} = \frac{\partial^2 \psi}{\partial \varepsilon_{kl}\, \partial \varepsilon_{ij}}}_{\psi \text{ is "smooth" wrt } \underline{\varepsilon}} = \mathbb{C}_{klij}$$

$\mathbb{C}$ has minor symmetries:

(1) $\mathbb{C}_{ijkl} = \mathbb{C}_{jikl}$

Follows because the cauchy stress is symmetric

$\sigma_{ij} = \sigma_{ji}$ — balance of angular momentum

$$\sigma_{ij} = \mathbb{C}_{ijkl}\, \varepsilon_{kl} = \sigma_{ji} = \mathbb{C}_{jikl}\, \varepsilon_{kl}$$

free indicies.

contracted out

(2) $\mathbb{C}_{ijkl} = \mathbb{C}_{ijlk}$

Follows because $\varepsilon_{kl} = \frac{1}{2}\left(u_{k,l} + u_{l,k}\right)$

$$= \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)$$

$\hookrightarrow \varepsilon_{kl} = \varepsilon_{lk}$

$$\sigma_{ij} = \mathbb{C}_{ijkl}\, \varepsilon_{kl} = \underbrace{\mathbb{C}_{ijlk}\, \varepsilon_{lk}}_{\mathbb{C}_{ijlk}\, \varepsilon_{kl}}$$

because l, k are dummy indicies & are being contracted out.

10.04

Constitutive relations, continued

- $\underline{\mathbb{C}}$ is positive definite:

$$\forall \underbrace{\underline{\Theta}}_{\text{general tensor}} \in \underbrace{GL(3)}_{\text{general linear transformations in } \mathbb{R}^3}$$

$$\underline{\Theta} : \underline{\mathbb{C}} : \underline{\Theta} \geq \underset{\text{scalar}}{0} \qquad \left(\Theta_{ij}\, \mathbb{C}_{ijkl}\, \Theta_{kl} \geq 0 \right)$$

"if and only if"

$$\underline{\Theta} : \underline{\mathbb{C}} : \underline{\Theta} = 0 \quad \text{iff} \quad \underline{\Theta} = \underline{\emptyset}$$

i.e. fracture or shear bands

⇒ Linearized Elasticity Theory has no material instabilities

- $\underline{\mathbb{C}}$ for materials that are isotropic

Coordinate Notation:

$$\mathbb{C}_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + 2\mu \cdot \underbrace{\tfrac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)}_{}$$

Kronecker deltas also writen as $\mathbb{1}_{ij}$

$\mathbb{I}_{ijkl}$: fourth-order symmetric identity tensor.

$$\mathbb{I}_{ijkl}\, \Theta_{kl} = \tfrac{1}{2}\left(\Theta_{ij} + \Theta_{ji}\right)$$

Direct Notation:

$$\underline{\mathbb{C}} = \lambda\, \underline{\mathbb{1}} \otimes \underline{\mathbb{1}} + 2\mu\, \underline{\mathbb{I}}$$

($\otimes$:) Diadic Product -or- Tensor Product.

λ, μ: Lamé constants

If E: Young's Modulus Then, $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$

ν : Poisson Ratio

K : Bulk Modulus

$$\mu = \frac{E}{2(1+\nu)} \quad \text{shear Modulus}$$

$$K = \frac{E}{3(1-2\nu)}$$

$-1 < \nu < 1/2$

elastic incompressibility — means we only have real wave speeds.

- Positive Definiteness $\Rightarrow$ $\lambda + 2\mu > 0$, $\mu > 0$

$$C_{longitudinal} = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad C_{shear} = \sqrt{\frac{\mu}{\rho}}$$

wave speed

Propagating Longitude & Shear Waves in the Elastic Material.

$\boxed{10.05}$ Weak form of linearized elasticity

Using Coordinate Notation:

Given u_i^g, t_i, f_i, the constitutive relation: $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$
and the kinematic relation: $\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial u_l} + \frac{\partial u_l}{\partial u_k}\right)$

find $u_i \in \mathcal{S} = \{ u_i \mid u_i = u_i^g \text{ on } \partial\Omega_{u_i} \}$ ← $i = 1,2,3$

such that $\forall$ $w_i \in \mathcal{V} = \{ w_i \mid w_i = 0 \text{ on } \partial\Omega_{u_i} \}$

$$\int_\Omega w_{i,j}\sigma_{ij}\, dV = \int_\Omega w_i f_i \, dV + \sum_{i=1}^{n_{sd}} \int_{\partial\Omega_{t_i}} w_i t_i \, dS$$

→ no implied sum here.

Strong Form implies the weak Form and vice versa.

Given u_i^g, t_i, f_i, constitutive relation $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$
the kinematic relation $\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)$
find u_i such that

$$\sigma_{ij,j} + f_i = 0 \quad \text{in } \Omega$$

BC's:

$$u_i = u_i^g \quad \text{on } \partial\Omega_{u_i}$$

$$\sigma_{ij} n_j = t_i \quad \text{on } \partial\Omega_{t_i}$$

Consider $w_i \in \mathcal{V} = \{ w_i \mid w_i = 0 \text{ on } \partial\Omega_{u_i} \}$

Multiply the PDE by w_i and integrate over Ω.

$$\int_\Omega w_i \underbrace{\sigma_{ij,j}}_{\substack{\text{use} \\ \text{integration} \\ \text{by parts.}}} \, dV + \int_\Omega w_i f_i \, dV = 0$$

NOTE: Here, sums are implied.

Integrate by parts:

$$\Rightarrow \int_\Omega \underbrace{(w_i \sigma_{ij})_{,j}}_{\nabla \cdot (\underline{w} \cdot \underline{\sigma})} \, dV - \int_\Omega w_{i,j} \sigma_{ij} \, dV + \int_\Omega w_i f_i \, dV = 0$$

$\downarrow$ Divergence Theorem

$$\Rightarrow \int_{\partial\Omega} w_i \sigma_{ij} n_j \, dS - \int_\Omega w_{i,j} \sigma_{ij} \, dV + \int_\Omega w_i f_i \, dV = 0$$

$\boxed{10.05q}$ How are the sums computed over $i = 1, 2, 3$?

Strong Form: $\sigma_{ij,j} + f_i = 0 \qquad i = 1,2,3$

$$\Rightarrow \left.\begin{aligned} \sigma_{1j,j} + f_1 &= 0 \\ \sigma_{2j,j} + f_2 &= 0 \\ \sigma_{3j,j} + f_3 &= 0 \end{aligned}\right\}$$

sum implied on j for all 3 equations.

Weak Form: $w_i \sigma_{ij,j} + w_i f_i = 0 \}$ sum on i & j.

collapses 3 equations to just one.

$$\left.\begin{aligned} & w_1 \sigma_{1j,j} + w_2 \sigma_{2j,j} + w_3 \sigma_{3j,j} \\ & + w_1 f_1 + w_2 f_2 + w_3 f_3 = 0 \end{aligned}\right\} \text{ sum over } j.$$

$\boxed{10.06}$ $$\int_{\Omega} w_{i,j}\,\sigma_{ij}\,dV = \int_{\Omega} w_i f_i\,dV + \int_{\partial\Omega} w_i \sigma_{ij} n_j\,dS$$

use $\partial\Omega = \partial\Omega_{u_i} \cup \partial\Omega_{\bar{t}_i}$ $\quad i = 1,2,3$

$$\int_{\Omega} w_{i,j}\sigma_{ij}\,dV = \int_{\Omega} w_i f_i\,dV + \sum_{i=1}^{n_{sd}} \int_{\underset{\text{fixed}}{\partial\Omega_{u_i}} \cup \underset{\text{fixed}}{\partial\Omega_{\bar{t}_i}}} \underset{\text{fixed}}{w_i}\,(\underset{\text{fixed}}{\sigma_{ij} n_j})\,dS$$

Now, Invoke boundary conditions on w_i & $\sigma_{ij} n_j$

$$\int_{\Omega} w_{i,j}\sigma_{ij}\,dV = \int_{\Omega} w_i f_i\,dV + \sum_{i=1}^{n_{sd}}\left[\int_{\partial\Omega_{u_i}} \cancel{w_i}^{=0 \text{ on } \partial\Omega_{u_i}}(\sigma_{ij}n_j)\,dS + \int_{\partial\Omega_{\bar{t}_i}} w_i\,\underbrace{(\sigma_{ij}n_j)}_{=\bar{t}_i \text{ on } \partial\Omega_{t_i}}\,dS\right]$$

$$\int_{\Omega} w_{i,j}\sigma_{ij}\,dV = \int_{\Omega} w_i f_i\,dV + \sum_{i=1}^{n_{sd}} \int_{\partial\Omega_{\bar{t}_i}} w_i \bar{t}_i\,dS$$

__Remark:__ The weak form can be obtained as the Euler-Lagrange conditions of a variational principle on extremization of a free energy function in 3D.

<u>This is the infinite-dimensional weak form.</u>

↳ The finite-dimensional weak form:

Given u_i^g, $\bar{t}_i$, f_i, $\underbrace{\sigma_{ij} = C_{ijkl}\varepsilon_{kl}}_{\text{constitutive relation}}$ & $\underbrace{\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)}_{\text{kinematic relation}}$

Find u_i^h (approximations!) $\in \mathcal{S}^h \subset \mathcal{S}$; $\mathcal{S}^h = \left\{ u_i^h \in H^1(\Omega) \,/\, u_i^h = u_i^g \text{ on } \partial\Omega_{u_i} \right\}$

such that $\forall\, w_i^h \in \mathcal{V}^h \subset \mathcal{V}$; $\mathcal{V}^h = \left\{ w_i^h \in H^1(\Omega) \,|\, w_i^h = 0 \text{ on } \partial\Omega_{t_i} \right\}$

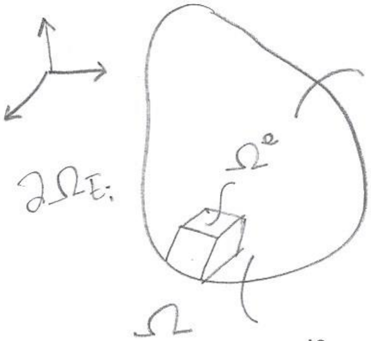
$$\int_\Omega w^h_{i,j}\, \cancelto{= \mathbb{C}_{ijkl}\varepsilon^h_{kl}}{\sigma^h_{ij}}\, dV = \int_\Omega w_i^h f_i \, dV + \sum_{i=1}^{n_{sd}} \int_{\partial\Omega_{t_i}} w_i^h \bar{t}_i \, dS$$

NOTE: $\varepsilon^h_{kl} = \frac{1}{2}\left(\frac{\partial u^h_k}{\partial x_l} + \frac{\partial u^h_l}{\partial x_k} \right)$

$\boxed{10.07}$ The Finite - Dimensional Weak Form & Basis Functions

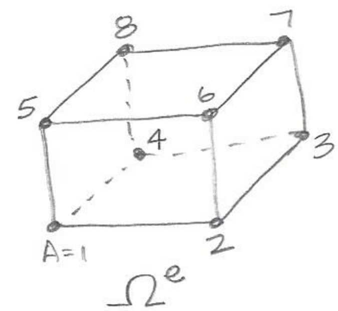
$$\int_\Omega w^h_{i,j}\, \sigma^h_{ij}\, dV = \int_\Omega w_i^h f_i \, dV + \sum_{i=1}^{n_{sd}} \int_{\partial\Omega_{t_i}} w_i^h \bar{t}_i \, dS$$

Basis Functions



$\Omega = \overline{\bigcup_e \Omega^e}$

$\Omega^{e_1} \cap \Omega^{e_2} = \emptyset$



N^A: basis funct. @ local Node A.

$$u^h_{i_e} = \sum_{A=1}^{n_{ne}} N^A d^A_{i_e} \qquad i = 1, \dots, n_{sd}$$

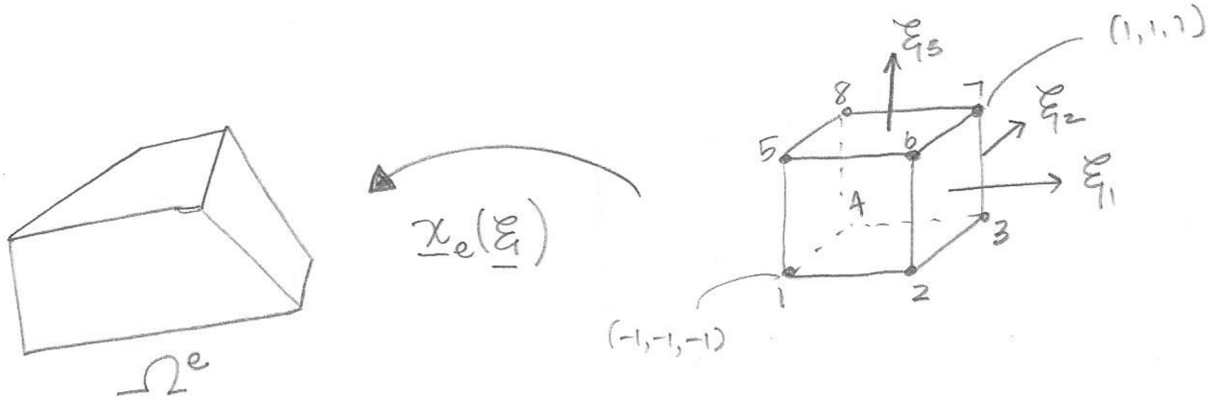
Number of degrees of freedom in Ω^e is $(n_{ne} \times n_{sd})$

$\underline{u}^h_e = \sum_A N^A \underline{d}^A_e$ } direct notation. $\underline{u}^h_e, \underline{d}^A_e \in \mathbb{R}^3$

↳ Vector d.o.f at each node.

$w^h_{i_e} = \sum_A N^A c^A_{i_e}$; $i = 1, \dots, n_{sd}$

$\underline{w}^h_e = \sum_A N^A \underline{c}^A_e$; $\underline{w}^h_e, \underline{c}^A_e \in \mathbb{R}^3$



$$N^A(\xi_1, \xi_2, \xi_3) = \underbrace{\check{N}^{\bar{A}}(\xi_1) \cdot \check{N}^{\bar{B}}(\xi_2) \cdot \check{N}^{\bar{C}}(\xi_3)}$$

Each of these is a 1D Lagrange Polynomial

$\boxed{10.08}$ **Isoparametric Mapping For Geometry**

$$x_{i_e}(\underline{\xi}) = \sum_A N^A(\underline{\xi})\, x^A_{i_e}$$

Gradients:

$$w^h_{e_{i,j}} = \sum_A N^A_{,j}\, c^A_{ie} \;; \quad u^h_{e_{i,j}} = \sum_A N^A_{,j}\, d^A_{ie}$$

Recall: $\displaystyle\int_\Omega w^h_{i,j}\, \sigma^h_{ij}\, dV$

$$\mathbb{C}_{ijkl}\, \varepsilon^h_{kl} = \mathbb{C}_{ijkl}\, u_{k,l}$$

$$\varepsilon^h_{kl} = \underbrace{\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)}$$

$\therefore$ minor symmetry:

$$\mathbb{C}_{ijkl} = \mathbb{C}_{ijlk}$$

$$N^A_{,j} = N^A_{,\xi_I} \cdot \frac{\partial \xi_I}{\partial x_j} \;; \; j = 1,2,3$$

Use Jacobian of the Mapping

$$\underline{J}(\underline{\xi}) = \begin{bmatrix} x_{1,\xi_1} & x_{1,\xi_2} & x_{1,\xi_3} \\ x_{2,\xi_1} & x_{2,\xi_2} & x_{2,\xi_3} \\ x_{3,\xi_1} & x_{3,\xi_2} & x_{3,\xi_3} \end{bmatrix}$$

$$\underline{J}^{-1}(\underline{\xi}) = \begin{bmatrix} \xi_{1,x_1} & \cdots & \xi_{1,x_3} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \xi_{3,x_3} \end{bmatrix}$$

$\boxed{10.09}$ The element integrals.

Finite-Dimensional Weak Form:

$$\sum_e \int_{\Omega_e} w^h_{i,j} \underbrace{\cancel{\sigma^h_{ij}}}_{\mathbb{C}_{ijkl}\, u_{k,l}} dV = \sum_e \int_{\Omega^e} w^h_i f_i \, dV + \sum_{i=1}^{n_{sd}} \sum_{\underbrace{e \in \mathcal{E}_{N_i}}_{\text{elements coinciding w/ Neuman boundary}}} \int_{\partial\Omega^e_{\bar{t}_i}} w^h_i \bar{t}_i \, dS$$

Consider

$$\int_{\Omega_e} \underbrace{w^h_{i,j}}\, \mathbb{C}_{ijkl} \underbrace{u_{k,l}}\, dV$$

$$= \int_{\Omega^e} \left(\sum_A N^A_{,j} c^A_{i_e} \right) \mathbb{C}_{ijkl} \left(\sum_B N^B_{,l} d^B_{k_e} \right) dV$$

$$= \sum_{A,B=1}^{n_{ne}} c^A_{i_e} \left(\int_{\Omega_e} N^A_{,j} \, \mathbb{C}_{ijkl} \, N^B_{,l} \, dV \right) d^B_{k_e}$$

$$= \sum_{A,B} c^A_{i_e} \left(\int_{\Omega_\xi} N^A_{,j} \, \mathbb{C}_{ijkl} \, N^B_{,l} \det[J(\xi)] \, dV_\xi \right) d^B_{k_e}$$

$\int_{-1}\int_{-1}\int_{-1}$ $\qquad dV_\xi = d\xi_1 \, d\xi_2 \, d\xi_3$

using Gaussian Quadrature. $\qquad$ Integral is K^{AB}_{ik}

Numerical Quadrature (or Integration):

$$= \left. \sum_{A,B} c^A_{i_e} K^{AB}_{ik} d^B_{k_e} \right\} \text{— (Scalar)} \qquad \underbrace{i,k = 1,2,3}_{\text{sum is implied.}}$$

10.10

$$\sum_{A,B} \underline{c}_e^{A^T} \underline{K}^{AB} \underline{d}_e^{B} \quad ; \quad \underline{c}_e^A,\ \underline{d}_e^B \in \mathbb{R}^3$$

$$\underline{K}^{AB} \in GL(3)$$

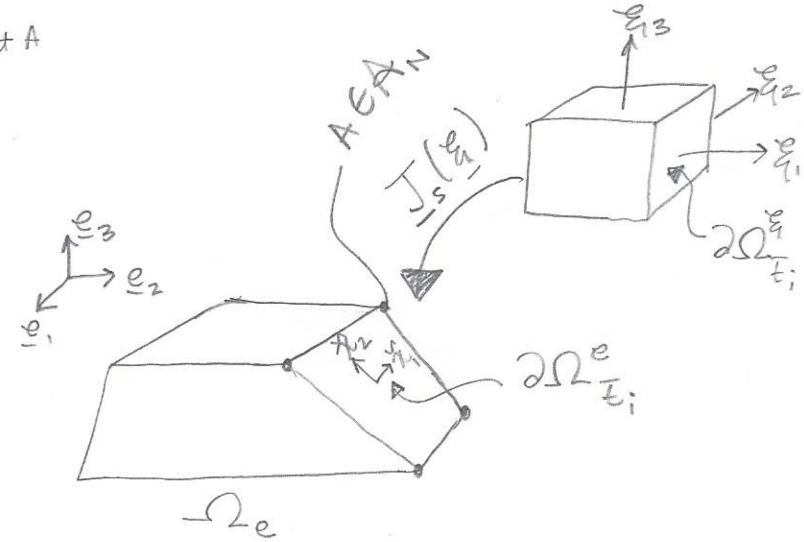
Transpose

Next, Consider $\displaystyle\int_{\Omega_e} w_i^h f_i \, dV = \int_{\Omega_e} \Big(\sum_A N^A c_{i_e}^A\Big) \cdot f_i \, dV$

$$= \sum_A c_{i_e}^A \underbrace{\int_{\Omega_\xi} N^A f_i \det[J(\xi)] \, dV_\xi}_{F_i^{int_A}}$$

$$= \sum_A \underline{c}_e^{A^T} \underline{F}^{int\,A}$$

10.11 <u>The Traction Integral:</u>



$$\sum_{i=1}^{n_{sd}} \sum_{e \in \xi_{N_i}} \underbrace{\int_{\partial\Omega^e_{\bar{t}_i}} w_i^h \bar{t}_i \, dS}$$

$$\int_{\partial\Omega^e_{\bar{t}_i}} w_i^h \bar{t}_i \, dS = \int_{\partial\Omega^e_{\bar{t}_i}} \Big(\sum_{A \in A_N} N^A c_{i_e}^A\Big) \bar{t}_i \, dS = \sum_{A \in A_N} c_{i_e}^A \int_{\partial\Omega^e_{\bar{t}_i}} N^A \bar{t}_i \, dS$$

$$= \sum_{A \in A_N} c_{i_e}^A \underbrace{\int_{\partial\Omega^{\xi}_{\bar{t}_i}} N^A \bar{t}_i \det[\underline{J}_s] \, \overset{d\xi_2 d\xi_3}{dS_\xi}} \quad \text{where } \underline{J}_s \in \overset{(2\times 2)\text{ matrix}}{GL(2)}$$

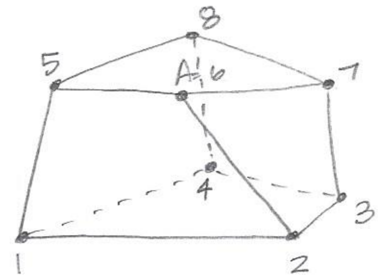
$$= \sum_{A \in A_N} c_{i_e}^A F_i^{\bar{t}A}$$

$$= \sum_A c_{i_e}^A F_i^{\bar{t}_A} \quad ; \quad F_i^{\bar{t}_A} = \begin{cases} \bar{F}_i^{\bar{t}_A} & \text{if } A \in \mathcal{A}_N \\ 0 & \text{otherwise.} \end{cases}$$

The finite-dimensional weak form:

$$\sum_e \sum_{A,B} \underline{c}_e^{A\,T} \underline{K}_e^{AB} \underline{d}_e^B = \sum_e \sum_A \underline{c}_e^{A\,T} \underline{F}_e^{\text{int}\,A} + \sum_{i=1}^{n_{sd}} \sum_{e \in \mathcal{E}_{N_i}} \sum_A c_{i_e}^A F_{i_e}^{\bar{t}_A}$$

$\boxed{10.12}$



$$\underline{c}_e^A = \begin{Bmatrix} c_1^A \\ c_2^A \\ c_3^A \end{Bmatrix}_e \qquad \underline{d}_e^B = \begin{Bmatrix} d_1^B \\ d_2^B \\ d_3^B \end{Bmatrix}_e$$

$(n_{ne} \times n_{sd})$ d.o.f per element.

$$\underline{c}_e = \begin{Bmatrix} \underline{c}_e^1 \\ \underline{c}_e^2 \\ \vdots \\ \underline{c}_e^{n_{ne}} \end{Bmatrix} \qquad \underline{d}_e = \begin{Bmatrix} \underline{d}_e^1 \\ \underline{d}_e^2 \\ \vdots \\ \underline{d}_e^{n_{ne}} \end{Bmatrix} \quad ; \quad \underline{c}_e, \underline{d}_e \in \mathbb{R}^{(n_{ne} \times n_{sd})}$$

24 d.o.f's on trilinear elements

The finite-dimensional weak form — matrix-vector form

$$\sum_e \underline{c}_e^T \underbrace{\underline{K}_e}_{(n_{ne} \times n_{sd}) \times (n_{ne} \times n_{sd})} \underline{d}_e = \sum_e \underline{c}_e^T \underbrace{\underline{F}_e^{\text{int}}}_{(n_{ne} \times n_{sd})} + \sum_{i=1}^{n_{sd}} \sum_{e \in \mathcal{E}_{N_i}} \sum_A c_{i_e}^A F_{i_e}^{\bar{t}_A}$$

general case

each of these is $(n_{sd} \times n_{sd})$

$$\underline{K}_e = \begin{bmatrix} \underline{K}_e^{11} & \underline{K}_e^{12} & \cdots & \underline{K}_e^{1 n_{ne}} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \underline{K}_e^{n_{ne} 1} & \cdots & \cdots & \underline{K}_e^{n_{ne} n_{ne}} \end{bmatrix} ; \quad \underline{F}_e^{\text{int}} = \begin{Bmatrix} \underbrace{\underline{F}_e^{\text{int},1}}_{(3 \times 1)} \\ \vdots \\ \underline{F}_e^{\text{int}\, n_{ne}} \end{Bmatrix}$$

10.13 The Global Matrix-Vector Equation

$$\sum_e \underline{c}_e^T \underline{K}_e \underline{d}_e = \sum_e \underline{c}_e^T \underline{F}_e^{int} + \sum_{i=1}^{n_{sd}} \sum_{e \in \mathcal{E}_{IN}} \sum_A c_{ie}^A F_{ie}^{\mathcal{I}_A}$$

Assembly of the global equations:

No dirichlet BC on first node.

$$\underline{c} = \left\{ \begin{matrix} c_1^1 \\ c_2^1 \\ c_3^1 \\ \vdots \\ c_1^A \\ c_2^A \\ c_3^A \\ \vdots \end{matrix} \right\} \qquad \bar{\underline{d}} = \left\{ \begin{matrix} d_1^1 \\ d_2^1 \\ d_3^1 \\ \vdots \\ d_1^A \\ d_2^A \\ d_3^A \\ \vdots \\ d_1^{n_n} \\ d_2^{n_n} \\ d_3^{n_n} \end{matrix} \right\}$$

Node; spatial dimension

Global $\underline{c}$ & $\bar{\underline{d}}$ vectors.

$\bar{\underline{d}}$ has $(n_n \times n_{sd})$ degrees of freedom (or elements)

$\underline{c}$ has $[(n_n \times n_{sd}) - N_D]$

N_D: # of degrees of freedom w/ Dirichlet B.C.s.

N_D does not need to be a multiple of n_{sd}.

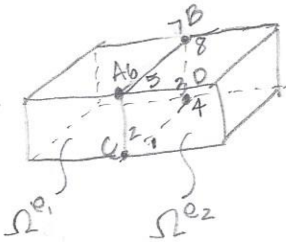
$$\sum_e \underline{c}_e^T \underline{K}_e \underline{d}_e = \underline{c}^T \bar{\underline{K}} \bar{\underline{d}}$$

$$\bar{\underline{K}} = \mathop{\mathsf{A}}_e \underline{K}_e$$ — assembly over global degrees of freedom.

$\bar{\underline{K}}$ has dimension:

$$[(n_n \times n_{sd}) - N_D] \times [(n_n \times n_{sd})]$$

	local node # Ω^{e_1}	Ω^{e_2}
A	6	5
B	7	8
C	2	1
D	3	4



$$\underline{K}_{e_1} = \begin{bmatrix} \underline{K}_{e_1}^{11} & \cdots & \underline{K}_{e_1}^{18} \\ \vdots & \ddots & \vdots \\ \underline{K}_{e_1}^{81} & \cdots & \underline{K}_{e_1}^{88} \end{bmatrix}$$

$$\underline{K}_{e_2} = \begin{bmatrix} \underline{K}_{e_2}^{11} & \cdots & \underline{K}_{e_2}^{18} \\ \vdots & \ddots & \vdots \\ \underline{K}_{e_2}^{81} & \cdots & \underline{K}_{e_2}^{88} \end{bmatrix}$$

$\boxed{10.14}$

$$\underline{\bar{K}} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{bmatrix} \cdots + \underline{K}_{e_1}^{66} + \underline{K}_{e_2}^{55} + \cdots & & & \cdots + \underline{K}_{e_1}^{65} + \underline{K}_{e_2}^{54} \cdots \\ \cdots + \underline{K}_{e_1}^{76} + \underline{K}_{e_2}^{85} + \cdots & & & \\ & & \cdots + \underline{K}_{e_1}^{22} + \underline{K}_{e_2}^{11} + \cdots & \\ & & & \end{bmatrix}$$

$$\begin{array}{cccc} A & B & C & D \end{array}$$

→ contributions from other elements sharing the same node.

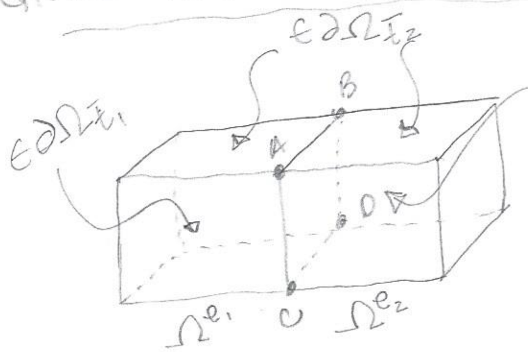
$$\sum_e \underline{C}_e^T \underline{F}_e^{int} = \underline{C}^T \underline{F}^{int} \; ; \quad \underline{F}^{int} = \begin{Bmatrix} F_{e_1}^{int_6} + F_{e_2}^{int_5} \\ \vdots \\ F_{e_1}^{int_2} + F_{e_2}^{int_1} \\ \vdots \end{Bmatrix} \begin{array}{c} A \\ B \\ C \\ D \end{array}$$

$\boxed{10.15}$ Contributions to Global Matrix – Vector Equations from the traction (Neumann) Boundary Condition.

$$\sum_{i=1}^{n_{sd}} \sum_{e \in \mathcal{E}_N} \sum_A c_{ie}^A F_{ie}^{\bar{t}_A} = \underline{C}^T \underline{F}^{\bar{t}}$$

$$\langle c_1^1 \quad \underset{\uparrow A}{c_1^A c_2^A c_3^A} \mid \underset{\uparrow B}{c_1^B c_2^B c_3^B} \mid \underset{\uparrow C}{c_1^C c_2^C c_3^C} \mid \underset{\uparrow D}{c_1^D c_2^D c_3^D} \rangle \begin{Bmatrix} F_{1_e}^{\bar{t}_1} \\ F_{1e_1}^{\bar{t}_6} + F_{1e_2}^{\bar{t}_5} \\ F_{2e_1}^{\bar{t}_6} + F_{2e_2}^{\bar{t}_5} \\ 0 \\ 0 \\ F_{2e_1}^{\bar{t}_7} + F_{2e_2}^{\bar{t}_8} \\ 0 \\ F_{1e_1}^{\bar{t}_2} + F_{1e_2}^{\bar{t}_1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \begin{array}{l} \\ \Big\} A \\ \\ \\ \Big\} B \\ \\ \Big\} C \\ \\ \\ \Big\} D \end{array}$$

Global node A lies in $\partial\Omega_{\bar{t}_1}$ & $\partial\Omega_{\bar{t}_2}$, C lies in $\partial\Omega_{\bar{t}_1}$, B lies in $\partial\Omega_{\bar{t}_2}$, D lies in none of the boundary subsets $\partial\Omega_{\bar{t}_i}$, $i = 1, 2, 3$



NOTE: Node numbering same as before

10.16 Global Matrix – Vector Equations.

$$\underline{c}^T \underline{\bar{K}}\,\underline{\bar{d}} = \underline{c}^T \underline{F}^{int} + \underline{c}^T \underline{F}^{\bar{t}}$$

$\underline{\bar{K}}$: $[(n_n \times n_{sd} - N_D) \times (n_n \times n_{sd})]$

Dirichlet Boundary Conditions:

KNOWN & can be moved to RHS

$$\underline{c}^T \begin{bmatrix} \quad & \underbrace{\begin{Bmatrix} \vdots \end{Bmatrix}}_{\underline{\bar{K}}^{n_{sd} \times A+1}} & \underbrace{\begin{Bmatrix} \vdots \end{Bmatrix}}_{\underline{\bar{K}}^{n_{sd} \times B+2}} & \quad \end{bmatrix} \begin{Bmatrix} \bar{d}^1_1 \\ \bar{d}^1_2 \\ \bar{d}^1_3 \\ \vdots \\ \bar{d}^A_1 \\ \bar{d}^A_2 \\ \bar{d}^A_3 \\ \vdots \\ \bar{d}^B_1 \\ \bar{d}^B_2 \\ \bar{d}^B_3 \\ \vdots \end{Bmatrix}$$

KNOWN

Global dofs $\bar{d}^A_1$, $\bar{d}^B_2$ are known Dirichlet B.C.'s

$$\underline{c}^T \underline{K}\,\underline{d} = \underline{c}^T \underbrace{\left(\underline{F}^{int} + \underline{F}^{\bar{t}} - \bar{d}^A_1 \underline{\bar{K}}^{n_{sd} \times A+1} - \bar{d}^B_2 \underline{\bar{K}}^{n_{sd} \times B+2} \right)}_{\underline{F}}$$

This is now square!

$$\Rightarrow \underline{c}^T (\underline{K}\,\underline{d} - \underline{F}) = \underline{0}, \quad \forall\ \underline{c} \in \mathbb{R}^{n_n \times n_{sd} - N_D}$$

$$\Rightarrow \underline{K}\,\underline{d} = \underline{F} \checkmark$$

$$\underline{d} = \underline{K}^{-1} \underline{F}$$

Solutions exist for $\underline{d}$ iff $\underline{K}$ is invertible.

$\underline{K}$: positive definite $\because$ $\underline{\mathbb{C}}$ is positive definite

& if the dirichlet b.c.'s eliminate rigid body modes.