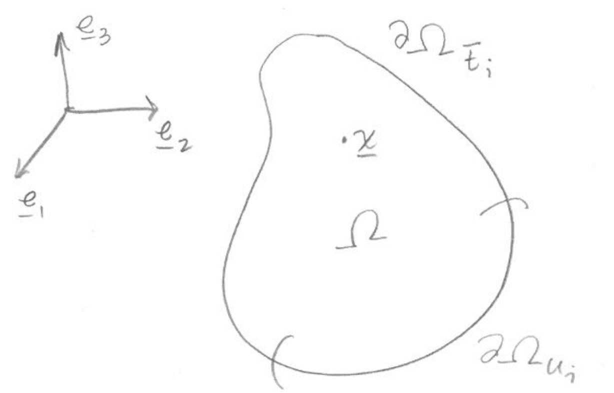
$\boxed{12.01}$ Hyperbolic Linear PDEs in vector unknowns. - Linear Elastodynamics in 3D.



$$\left.\begin{aligned} \partial\Omega &= \partial\Omega_{u_i} \cup \partial\Omega_{\bar{t}_i} \\ \partial\Omega_{u_i} &\cap \partial\Omega_{\bar{t}_i} = \emptyset \end{aligned}\right\} \quad i = 1,2,3$$

mass density

Strong Form: given: $u_i^g, \bar{t}_i, f_i, u_{i_0}, v_{i_0}, \sigma_{ij} = \mathbb{C}_{ijkl}\varepsilon_{kl}, \rho$

$$\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)$$

find: $u_i(\underline{x},t)$ such that:

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \sigma_{ij,j} + f_i \quad \text{in } \Omega \times [0,T]$$

Boundary Conditions: $u_i(\underline{x},t) = u_i^g(\underline{x},t)$ @ $\underline{x} \in \partial\Omega_{u_i}$

$\sigma_{ij}n_j = \bar{t}_i(\underline{x},t)$ @ $\underline{x} \in \partial\Omega_{\bar{t}_i}$

Initial Conditions: $u_i(\underline{x},0) = u_{i_0}(\underline{x}) \;\; \forall \underline{x} \in \Omega$

$\dot{u}_i(\underline{x},0) = v_{i_0}(\underline{x}) \;\; \forall \underline{x} \in \Omega$

Weak Form: Find $u_i \in \mathcal{S} = \{u_i \mid u_i = u_i^g \text{ on } \partial\Omega_{u_i}\}$

such that $\forall\, w_i \in \mathcal{V} = \{w_i \mid w_i = 0 \text{ on } \partial\Omega_{u_i}\}$

$$\int_\Omega w_i \rho \frac{\partial^2 u_i}{\partial t^2}\, dV + \int_\Omega w_{i,j}\sigma_{ij}\, dV = \int_\Omega w_i f_i\, dV + \sum_{i=1}^{3}\int_{\partial\Omega_{\bar{t}_i}} w_i \bar{t}_i\, dS$$

$\boxed{12.02}$ Hyperbolic PDEs in vector unknowns, in 3D – Linear Elastodynamics

Find $u_i^h \in \mathcal{S}^h \subset \mathcal{S}$; $\mathcal{S}^h = \{u_i^h \in H^1(\Omega) \mid u_i^h = u_i^g \text{ on } \partial\Omega_{u_i}\}$ (finite-dimensional problem)

Such that $\forall\ w_i^h \in \mathcal{V}^h \subset \mathcal{V}$; $\mathcal{V}^h = \{w_i^h \in H^1(\Omega) \mid w_i^h = 0 \text{ on } \partial\Omega_{u_i}\}$

$$\underbrace{\int_\Omega w_i^h \rho \frac{\partial^2 u_i^h}{\partial t^2}\, dV}_{} + \underbrace{\int_\Omega w_{i,j}^h \sigma_{ij}^h\, dV}_{\mathbb{C}_{ijkl}\varepsilon_{kl}} = \underbrace{\int_\Omega w_i^h f_i\, dV + \sum_{i=1}^{3} \int_{\partial\Omega_{\bar{t}_i}} w_i^h \bar{t}_i\, dS}_{}$$

FINITE-DIMENSIONAL WEAK FORM.

$$\underline{c}^T \underline{M}\, \ddot{\underline{d}} + \underline{c}^T \underline{K}\, \underline{d} = \underline{c}^T \underline{F}$$

Form of $\underline{M}$: Consider $\displaystyle\int_\Omega w_i^h \rho \frac{\partial^2 u_i^h}{\partial t^2}\, dV = \sum_e \int_{\Omega^e} w_i^h \rho \frac{\partial^2 u_i^h}{\partial t^2}\, dV$

$\boxed{12.03}$ The element Integral:

Recall sum on i

$$\int_{\Omega^e} w_i^h \rho \frac{\partial^2 u_i^h}{\partial t^2}\, dV = \sum_{A,B} c_{e_i}^A \underbrace{\left(\int_{\Omega^e} \rho N^A N^B dV\right)}_{} \ddot{d}_{e_i}^B$$

$$= \sum_{A,B} c_{e_i}^A \left(\int_{\Omega^e} \rho N^A N^B \delta_{ij}\, dV\right) \ddot{d}_{e_j}^B$$

$\langle c_{e_1}^2, c_{e_2}^2, c_{e_3}^2 \rangle$

$$= \langle c_e^{1T}\ c_e^{2T} \dots c_e^{n_{ne}T} \rangle \underbrace{\begin{bmatrix} \int_{\Omega^e} \rho N^1 N^1 dV \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix} & & \\ \dots & \int_{\Omega^e} \rho N^A N^B dV \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix} & \dots \\ & & \end{bmatrix}}_{\underline{M}_e} \begin{Bmatrix} \ddot{d}_e^1 \\ \ddot{d}_e^2 \\ \vdots \\ \ddot{d}_e^{n_{ne}} \end{Bmatrix}$$

$$\ddot{d}_e^2 = \begin{Bmatrix} \ddot{d}_{e_1}^2 \\ \ddot{d}_{e_2}^2 \\ \ddot{d}_{e_3}^2 \end{Bmatrix}$$

size $[n_{ne} \times n_{sd}] \times [n_{ne} \times n_{sd}]$

Assembly proceeds as before over global degrees of freedom.

$$\underline{M}\,\ddot{\underline{d}} + \underline{K}\,\underline{d} = \underline{F}$$

$$\underline{d}(0) = \underline{d}_0$$

specified for every degree of freedom.

$$\dot{\underline{d}}(0) = \underline{v}_0 = \begin{Bmatrix} v_0(\underline{x}^A) \\ \vdots \end{Bmatrix}$$

$\boxed{12.04}$ The matrix-vector problem of linear elastodynamics

$$\underline{M}\,\ddot{\underline{d}} + \underline{K}\,\underline{d} = \underline{F}$$

$$\underline{d}(0) = \underline{d}_0$$

$$\dot{\underline{d}}(0) = \underline{v}_0$$

Including the effect of structural damping:

$$\underline{C} = a \cdot \underline{M} + b \cdot \underline{K} \quad \Big\} \text{ Rayleigh Damping}$$

damping matrix

where a & b are constants

Elastodynamics with structural damping:

$$\underline{M}\,\ddot{\underline{d}} + \underline{C}\,\dot{\underline{d}} + \underline{K}\,\underline{d} = \underline{F} \quad \text{with } \underline{d}(0) = \underline{d}_0 \ \& \ \dot{\underline{d}}(0) = \underline{v}_0$$

Time Discretization: $[0,T] = \cup [t_0, t_1] \cdots [t_{N-1}, t_N]$

$t_0 = 0$, $t_N = T$

d_n: time-discrete approximation to $\underline{d}(t_n)$

Time-discretized matrix-vector equation:

$$\underline{M}\,\underline{a}_{n+1} + \underline{C}\,\underline{v}_{n+1} + \underline{K}\,\underline{d}_{n+1} = \underline{F}_{n+1}$$

acceleration

$\underline{d}_0, \underline{v}_0$ known.

Newmark Family of algorithms for 2nd Order ODEs:

For $\gamma \in [0,1]$; $2\beta \in [0,1]$

$$\underline{d}_{n+1} = \underline{d}_n + \Delta t\, \underline{v}_n + \frac{\Delta t^2}{2}\left((1-2\beta)\underline{a}_n + 2\beta\, \underline{a}_{n+1}\right)$$

$$\underline{v}_{n+1} = \underline{v}_n + \Delta t\left((1-\gamma)\underline{a}_n + \gamma\, \underline{a}_{n+1}\right)$$

The a-method:

Predictors: $\tilde{\underline{d}}_{n+1} = \underline{d}_n + \Delta t\, \underline{v}_n + \frac{\Delta t^2}{2}(1-2\beta)\underline{a}_n$

$\tilde{\underline{v}}_{n+1} = \underline{v}_n + \Delta t(1-\gamma)\underline{a}_n$

Correctors: $\underline{d}_{n+1} = \tilde{\underline{d}}_{n+1} + \Delta t^2 \cdot \beta\, \underline{a}_{n+1}$

$\underline{v}_{n+1} = \tilde{\underline{v}}_{n+1} + \Delta t\, \gamma\, \underline{a}_{n+1}$

Correctors.

Substituting,

$$\underline{M}\, \underline{a}_{n+1} + \underline{C}\left[\tilde{\underline{v}}_{n+1} + \Delta t \gamma\, \underline{a}_{n+1}\right] + \underline{K}\left[\tilde{\underline{d}}_{n+1} + \Delta t^2 \beta\, \underline{a}_{n+1}\right] = \underline{F}_{n+1}$$

$$\Rightarrow \left[\underline{M} + \underline{C}\,\Delta t\,\gamma + \underline{K}\,\Delta t^2 \beta\right]\underline{a}_{n+1} = \underline{F}_{n+1} - \underline{C}\,\tilde{\underline{v}}_{n+1} - \underline{K}\,\tilde{\underline{d}}_{n+1}$$

To get $\underline{a}_0$, use equation:

$$\underline{M}\,\underline{a}_0 = \underline{F}_0 - \underbrace{\underline{C}\,\underline{v}_0}_{\text{known}} - \underbrace{\underline{K}\,\underline{d}_0}_{\text{known.}}$$

$\boxed{12.05}$ Analysis is based on the eigenvalue problem:

$$\omega^2 \underline{M}\,\underline{\Psi} = \underline{K}\,\underline{\Psi}$$

ω: natural frequency of oscillation.

$\underline{\Psi}$: $\underline{M}$-orthogonal eigenvectors, $l = 1, \ldots, n_{df}$

$$\underline{d} = \sum_l d^l \underline{\Psi}^l$$

Reduction to n_{df} single d.o.f modal equations of the time-exact ODE

$$\ddot{d}^l + 2\xi^h_{l}\omega^h_l \dot{d}^l + (\omega^h)^2 d^l = 0 \quad \text{(homogeneous case.)}$$

ω^h_l: finite-dimensional (spatially discretized) natural frequencies

$$\xi^h_{q_\ell} = \frac{a}{\omega^h_\ell} + b\,\omega^h_\ell \quad \Big\} \text{ Modal damping ratio.}$$

Rewrite 2ND ORDER ODE as two 1ST-ORDER ODEs:

$$\underline{y} = \begin{Bmatrix} d \\ \dot{d} \end{Bmatrix}$$

Time-discretized problem in modal form:

$$a^\ell_{n+1} + 2\xi^h_{q_\ell}\,\omega^h_\ell\, v^\ell_{n+1} + (\omega^h_\ell)^2 d^\ell_{n+1} = 0 \qquad \text{(homogeneous problem)}$$

Newmark family Equations relate d^ℓ_{n+1}, v^ℓ_{n+1}, a^ℓ_{n+1}

12.06 Reduction to two 1ST-ORDER ODEs:

$$\underbrace{\underline{y}_{n+1}}_{\begin{Bmatrix} d_{n+1} \\ v_{n+1} \end{Bmatrix}} = \underline{A}\, \underbrace{\underline{y}_n}_{\begin{Bmatrix} d_n \\ v_n \end{Bmatrix}} + \underline{L}_n$$

NOTE: Suppressing Mode Number Index, ℓ.

$\underline{A}$: 2x2 amplification matrix.

Stability: $2\beta \geq \gamma \geq 1/2 \rightarrow$ Unconditional Stability

$$\left.\begin{array}{l} \gamma \geq 1/2 \\ 0 \leq \beta \leq \gamma/2 \end{array}\right\} \text{ Conditional Stability.}$$

$$\omega^h \Delta t \leq \Omega_{crit}$$

$$\Omega_{crit} = \xi^h(\gamma - 1/2) + \Big[\underbrace{\tfrac{\gamma}{2} - \beta + (\xi^h)^2(\gamma - 1/2)}_{\gamma/2-\beta}\Big]^{1/2}$$

damping when $\xi^h > 0$.

Undamped critical frequency: $\Omega^u_{crit} = \left(\frac{\gamma}{2} - \beta\right)^{-1/2}$

$$\Omega^u_{crit} \leq \Omega_{crit}$$

Ω^u_{crit} is a more stringent condition on $\omega^h \Delta t$

Method	Type	β	γ	Ω^u_{crit}	order of Accuracy
Trapezoidal Rule	Implicit	1/4	1/2	uncond.	2
Linear Acceleration	Implicit	1/6	1/2	$2\sqrt{3}$	2
Average Acceleration	Implicit	1/12	1/2	$\sqrt{6}$	2
Central Difference	Explicit	0	1/2	2	2

$\boxed{12.07}$ Stability Analysis is based on an eigenvalue analysis of the amplification matrix. $\underline{A}$

$$\underline{y}_{n+1} = \underline{A}\,\underline{y}_n + \underline{L}_n$$

Define the spectral radius of $\underline{A}$: $r(\underline{A}) = \max_i |\lambda_i(\underline{A})|$ $\} \; i = 1,2$

eigen values of $\underline{A}$

$$= \max_{i=1,2} \sqrt{\lambda_i(\underline{A}) \cdot \underbrace{\overline{\lambda_i(\underline{A})}}_{\text{complex conjugate of } \lambda_i(\underline{A})}}$$

Stability: $r \leq 1$ if λ_1, λ_2 are distinct.
eigenvectors of $\underline{A}$ are linearly independent.

$r < 1$ if $\lambda_1 = \lambda_2$
eigenvectors of $\underline{A}$ are linearly dependent.

(1) Linearly independent eigen vectors:

$$\underline{y}_{n+1} = \underline{A}\,\underline{y}_n \; ; \; \underline{y}_n = \underline{A}\,\underline{y}_{n-1} \; , \; \underline{y}_{n-1} = \ldots$$

$$\vdots$$

$$\underline{y}_{n+1} = \underline{A}^{n+1}\,\underline{y}_0$$

$$\underline{A} = \underline{P}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\underline{P}^{-1} \Rightarrow \underline{A}^n = \underbrace{\underline{P}\begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}\underline{P}^{-1}}_{\text{bounded}}$$

(2) Linearly dependent eigenvectors:

$$\underline{A} = \underline{Q} \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix} \underline{Q}^{-1}$$

$$\Rightarrow \underline{A}^n = \underline{Q} \underbrace{\begin{bmatrix} \lambda^n & n \cdot \lambda^{n-1} \\ & \lambda^n \end{bmatrix}}_{\lambda = 1} \underline{Q}^{-1}$$

then this term becomes n.

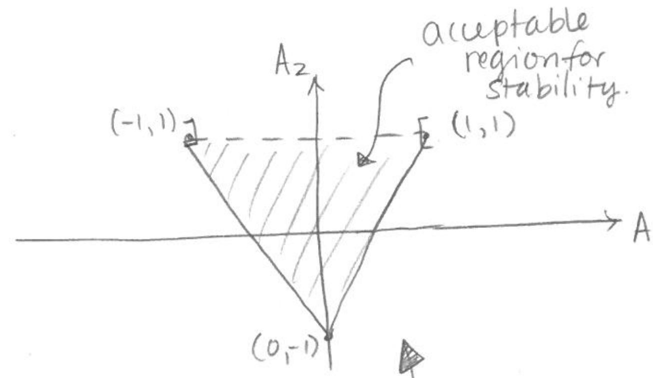
Off-diagonal term diverges as n.

Solving of λ: $\lambda^2 - 2A_1\lambda + A_2 = 0$

$A_1 = \frac{1}{2}\,\mathrm{trace}(\underline{A})$

$A_2 = \det(\underline{A})$

$\hookrightarrow \lambda_1, \lambda_2 = A_1 \pm \sqrt{A_1^2 - A_2}$



Stability in terms of A_1 & A_2:

$-\frac{(A_2+1)}{2} \leq A_1 \leq \frac{(A_2+1)}{2}$ if $A_2 < 1$

$-1 < A_1 < 1$ if $A_2 = 1$

$\boxed{12.08}$ Stability Conditions on $\underline{A}$:

$A_1 = \frac{1}{2}\,\mathrm{trace}(\underline{A})$ & $A_2 = \det(\underline{A})$

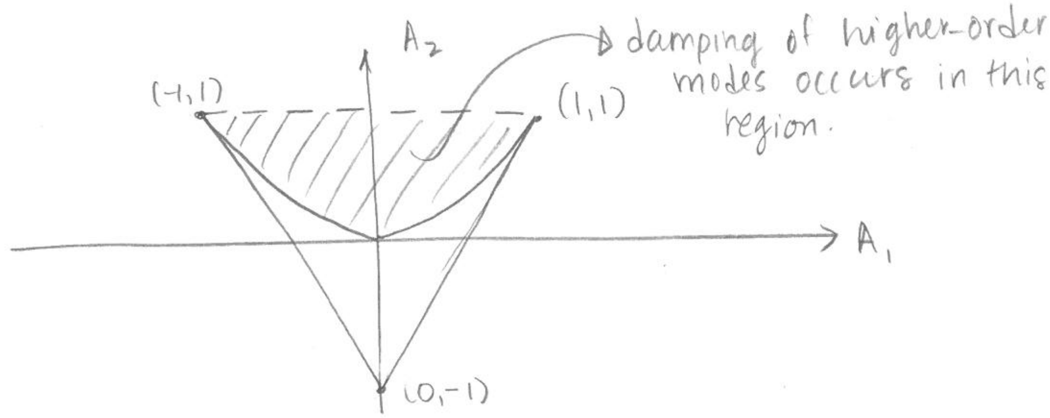
$\lambda_1, \lambda_2 = A_1 \pm \sqrt{A_1^2 - A_2}$

$r = \max_{i=1,2} \sqrt{\lambda_i \cdot \bar{\lambda}_i}$

$r \leq 1$

High-order Modes are nondecaying if λ_1, λ_2 are purely real.

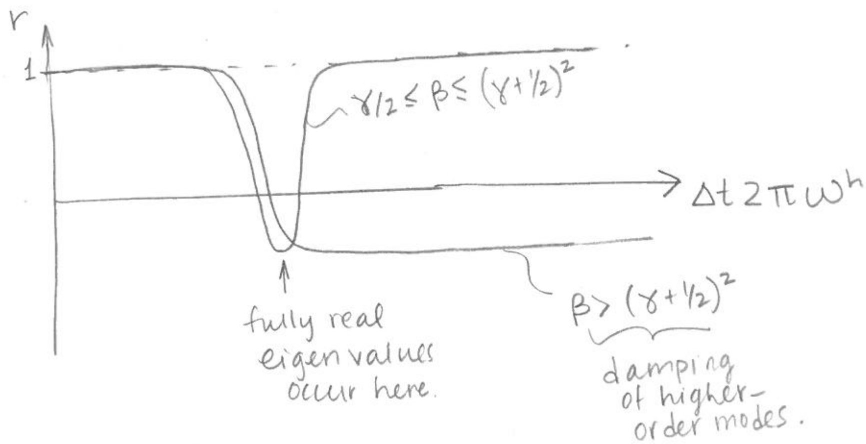
Need $A_1^2 - A_2 < 0$ for damping of higher-order modes.



<u>Stability</u> <u>Requirement</u>: $\beta \geq \frac{\gamma}{2}$ } unconditional

$$\beta \geq \frac{\left(\gamma + \frac{1}{2}\right)^2}{4}$$ } damping of higher-order modes.

↳ <u>Spectral</u> <u>Radius</u>:



<u>Consistency</u>:

$$\underline{y}_{n+1} = \underline{A}\, \underline{y}_n + \underline{L}_n$$ } inhomogeneous problem. & time-discrete modal equation.

corresponding exact sln's

$$\underline{y}(t_{n+1}) = \underline{A}\, \underline{y}(t_n) + \underline{L}_n + \Delta t\, \underline{\tau}(t_n)$$ } time-exact modal eqn..

substituted.

Consistency Requires $\underline{\tau} = \underline{c} \cdot \Delta t^k$, $\begin{Bmatrix} \tau_1(t_n) \\ \tau_2(t_n) \end{Bmatrix} = \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} \Delta t^k$, $k < 0$, C_1, C_2 are constants

k: order of accuracy.

<u>Lax Theorem</u>: Consistency and Stability $\Rightarrow$ Convergence

$$\underline{e}_{n+1} = \underline{A}^{n+1} \underline{e}_0 - \sum_{i=0}^{n} \Delta t\, \underline{A}^i \underline{\tau}(t_n)$$

$$\lim_{\Delta t \to 0} \underline{e}_{n+1} = \underline{0}$$