$n_{int} = 1$, $\xi_1 = 0$, $w_1 = 2$

$n_{int} = 2$, $\xi_1 = -1/\sqrt{3}$, $w_1 = 1$
$\xi_2 = 1/\sqrt{3}$, $w_2 = 1$

$n_{int} = 3$, $\xi_1 = -\sqrt{3/5}$ $w_1 = 5/9$
$\xi_2 = 0$ $w_2 = 8/9$
$\xi_3 = \sqrt{3/5}$ $w_3 = 5/9$

Rules from Gaussian Quadrature.

Gaussian Quadrature Rule with $n_{int}$ points exactly integrates polynomials of order $\leq (2n_{int} - 1)$

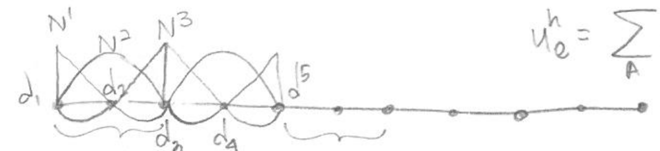
$n_{int} = 1$ ——— linears

$n_{int} = 2$ ——— cubics

$n_{int} = 3$ ——— pentics

⋮

$\boxed{5.01}$ **Norms**: Consider the finite-dimensional trial solution, $u^h$
– Also called the "Finite Element Solution"




$u^h_e = \sum_A N^A d^A_e$ – continuous!
↳ $\underbrace{C^0(\Omega)}$ mathematic way to say funct. is continuous in the domain $\Omega$

$u^h_{e,x}$ – continuous in $\Omega^e$
$C^0(\Omega^e)$ ← continuous over each element (but not over whole domain due to the edges) $u^h_{e,x}$ – not in $C^0(\Omega)$

The Lagrange Polynomial Basis Functions have been constructed to be only $C^0(\Omega)$, not $C^n(\Omega)$ for $\underset{\sim}{n} > 0$.
# of derivatives.

In general, a function is in $C^n(\Omega)$ if its derivatives up to order $n$ are continuous in the domain $\Omega$.

However, does $u^h \in H^1(\Omega)$? ? YES.

$$\int_\Omega \left( (u^h)^2 + \underbrace{m(\Omega)}_{\text{the measure of } \Omega.}^{2/n_{sd}} \cdot (u^h_{,x})^2 \right) dx < \infty$$

discontinuities CAN be integrated.

Recall: $u^h \in \mathcal{S}^h = \{ u^h \in H^1(\Omega) \mid \dots \}$

$\boxed{5.02}$ Define the $H^1$-norm to be

$$\|v\|_1 := \left[ \frac{1}{m(\Omega)^{1/n_{sd}}} \int_\Omega \left( v^2 + m(\Omega)^{2/n_{sd}} (v_{,x})^2 \right) dx \right]^{1/2}$$

designates a norm

spacial dimensions

This the $H^1$ — Hilbert norm of $v$. An example of more general norms called Sobolev norms.

Can extend this to define the $H^n$-norm by including the first $n$ derivatives.

$H^0$-norm: $\|v\|_0 = \left[ \frac{1}{m(\Omega)^{1/n_{sd}}} \int_\Omega v^2 dx \right]^{1/2} \equiv$ $L^2$-norm

may put a zero OR leave blank

Define the energy norm of $v$:

$$\left( \int_\Omega v_{,x} E v_{,x} dx \right)^{1/2} \text{ — "Strain energy of } v\text{"}$$

Equivalence of norms:

$$\underbrace{c_1}_{\text{const.}} \|v\|_1 \leq \left( \int_\Omega v_{,x} E v_{,x} dx \right)^{1/2} \leq \underbrace{c_2}_{\text{second const.}} \|v\|_1$$

we can bound the energy norm from below & above by multiplying $\|v\|_1$ w/ two different constants.

## Inner Product Notation:

$$(w, f) := \int_\Omega (w \cdot f)\, dx \quad \Big\} \text{ leads to the forcing function.}$$

The inner product of w & f — the $L^2$ inner product.

## Bilinear form notation:

linear, linear $\therefore$ bi linear. Since linear in w & u.

$$a(w, u) := \underbrace{\int_\Omega w_{,x} E\, u_{,x}\, dx}_{\text{this leads to the stiffness matrix}}$$

Note

$$\underline{\underline{a(u, u)}} = \int_\Omega u_{,x} E\, u_{,x}\, dx \leftarrow \text{energy norm.}$$

↑ use this notation for Energy Norm of u.

$\boxed{\text{5.02 Question}}$ Does equivalence of $\|v\|_1$ & $a(v, v)$ always hold?

— YES.

Note: $\exists$ $\|v\|_1$ & $a(v, v)$ iff

↑ exist

$\|v\|_1 < \infty$

$a(v, v) < \infty$

$m(\Omega) < \infty$ : our domain $\Omega$ is finite.

$$\left(\frac{1}{m(\Omega)^{1/n_{sd}}} \int_\Omega \left(v^2 + m(\Omega)^{2/n_{sd}} (v_{,x})^2\right) dx\right)^{1/2} \qquad \left(\int_\Omega v_{,x} E\, v_{,x}\, dx\right)^{1/2}$$

↑ This is the difference between the two.

## 5.03 Consistency & the Best approximation Property

Recall the weak form:

$$\underbrace{\int_\Omega w_{,x} E\, u_{,x} A\, dx}_{a(w,u)} = \underbrace{\int_\Omega w f A\, dx}_{(w,f)} + \underbrace{w(L)\, t A}_{:=(w,t)_L}$$

Abstract Notation:

$$a(w,u) = (w,f) + (w,t)_L$$

Finite dimensional weak form:

$$\int_\Omega w^h_{,x} E u^h_{,x} A\, dx = \int_\Omega w^h f A\, dx + w^h(L)\, t A$$

$$\Rightarrow \quad a(w^h, u^h) = (w^h, f) + (w^h, t)_L \quad \text{—— (C)}$$

Consider: $a(w,u) = (w,f) + (w,t)_L \quad \forall\, w \in \mathcal{V}$ —— (A)

But $w^h \in \mathcal{V}^h \subset \mathcal{V}$

So (A) also holds for $w^h \in \mathcal{V}$.

$$\Rightarrow \quad a(w^h, u) = (w^h, f) + (w^h, t)_L \quad \text{—— (B)}$$

Subtract (B) from (C):

$a(w^h, e) = 0$ the projection of the error onto $w^h$ is zero.

$$a(w^h, u^h) - a(w^h, u) = \cancel{(w^h, f)} - (w^h, f) + \cancel{(w^h, t)_L} - (w^h, t)_L$$

$$a(w^h, \underbrace{(u^h - u)}) = 0$$

The difference between the trial solution & the exact solution

$u^h - u : e$ ← error in the FE Solution. because $u$ is the exact solution.

This results from (B) — consistency condition.

The error is orthogonal to $\mathcal{V}^h$.

"outside of"

The condition (B):

$$a(w^h, u) = (w^h, f) + (w^h, t)_L \leftarrow \text{Consistency Condition.}$$

Compare with finite dimensional weak form:

$$a(w^h, \underset{\substack{\uparrow \\ \text{replace with } u \text{ to get back (B).}}}{u^h}) = (w^h, f) + (w^h, t)_L$$

Note: The exact solution $u$ satisfies the finite dimensional weak form.

★ The finite method can recover the exact solution.

Not the case for all numerical methods.

## $\boxed{5.04}$ The "Best Approximation" Property

Let $u^h \in \mathcal{S}^h$ be the FE solution.

$w^h \in \mathcal{V}^h$ be a weighting function.

$$U^h \in \mathcal{S}^h = \{U^h \in H^1(\Omega) \mid U^h(0) = u_0\}$$

NOTE: $U^h = \underset{\substack{\uparrow \\ \text{THE solution} \\ \in H^1}}{u^h} + \underset{\substack{\uparrow \\ \text{ANY weighting} \\ \in H^1}}{w^h}$

$U^h = u^h + w^h \in H^1$ & satisfies the Dirichlet B.C.

THEOREM $$\underbrace{a(\overset{u^h - u}{e}, e)}_{\substack{\text{energy} \\ \text{norm of} \\ \text{the error.}}} \leq \underbrace{a(U^h - u, U^h - u)}_{\forall\, U^h \in \mathcal{S}^h}$$

$\Rightarrow$ Finite Element solution MINIMIZES the energy norm of $U^h - u$ over all members $U^h \in \mathcal{S}^h$

Proof: Consider $a(e+w^h, e+w^h)$ just like expansion of a perfect square.

$$= a(e,e) + \underbrace{a(e,w^h) + a(w^h,e)}_{\text{due to symmetry these are equal.}} + a(w^h,w^h)$$

$$= a(e,e) + \underbrace{2a(w^h,e)}_{=0 \text{ (consistency of the FE method)}} + a(w^h,w^h)$$

$$a(e+w^h, e+w^h) = \underbrace{a(e,e)}_{\geq 0} + \underbrace{a(w^h,w^h)}_{\geq 0}$$

$$\Rightarrow a(e,e) \leq a(e+w^h, e+w^h)$$

NOTE that $a(e+w^h, e+w^h) = a(u^h-u+w^h, u^h-u+w^h)$

$$a(v^h-u, v^h-u)$$

Combining w/ the inequality:

$$a(e,e) \leq a(v^h-u, v^h-u)$$

$$a(u^h-u, u^h-u) \leq a(v^h-u, v^h-u)$$

BEST APPROXIMATION PROPERTY

FEM selects the $u^h$ within $\mathcal{S}^h$ that minimizes the energy norm.

any other function in $\mathcal{S}^h$

## 5.05 The "Pythagorean" Theorem

Corollary 1: $a(u,u) = a(u^h,u^h) + a(e,e)$

for $\mathcal{S}^h = \mathcal{V}^h$ (Dirichlet B.C. are homogeneous)

Proof: $u^h - u = e$ (error)

$\Rightarrow \quad u = u^h - e$

$\Rightarrow \quad a(u,u) = a(u^h - e,\, u^h - e)$

$$a(u,u) = a(u^h,u^h) - \overset{0 \text{ (consistency)}}{\underbrace{2\,a(\cancel{u^h},e)}_{=2a(w^h,e)}} + a(e,e) \checkmark$$

Corollary 2: The Finite Element Solution underestimates the energy norm of the problem.

Proof. From Corollary 1:

$$a(u,u) = \underbrace{a(u^h,u^h)}_{\geq 0} + \underbrace{a(e,e)}_{\geq 0}$$

$\Rightarrow \quad a(u^h,u^h) \leq a(u,u) \checkmark$

NOTE: What are homogeneous Dirichlet boundary conditions?

Homogeneous $\Leftrightarrow$ "zero"

Homogeneous Dirichlet b.c.s $\Rightarrow u^h(0) = u_0 = 0$

$w^h(0) = 0$

Plus $u^h, w^h \in H^1(\Omega)$

$\Rightarrow \mathcal{S}^h = \mathcal{V}^h$

5.06 Sobolev estimates

↑ refers to the type of space.

$$V^h \in \mathcal{S}^h = \{ V^h \in H^n(\Omega) \mid V^h(0) = u_0 \}$$

$V^h$ does NOT necessarily represent $u^h$ (FE sltn)

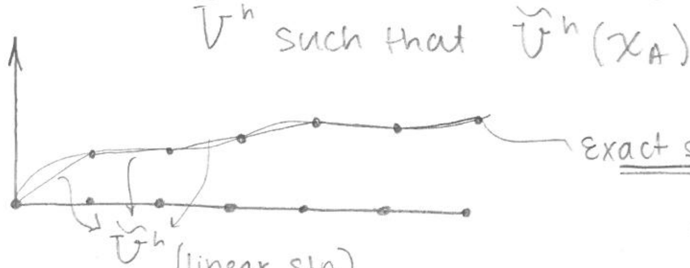
Consider $V^h$ such that $V^h(x_A) = u^h(x_A) = d_A$

A: global degree of freedom

$x_A$: globally numbered node.

$d_A$: globally numbered trial solution degree of freedom.

$\tilde{V}^h$ such that $\tilde{V}^h(x_A) = u(x_A)$ → $\tilde{V}^h(x)$ is "NODALLY EXACT" aka an "interpolate" of the solution.




↓ Linear basis functions.

## Interpolation Error Estimate in Sobolev Spaces.

$$\underbrace{\|\tilde{V} - u\|_m}_{\text{interpolation error}} \leq c(h^e)^{\alpha} \|u\|_r$$

c: a constant; $h^e$: element size; r: regularity of the exact sln.; $\|u\|_r$: measure of smoothness (regularity) of u.

$\alpha$: exponent satisfying $\alpha = \min(k+1-m,\ r-m)$

k: polynomial order of the finite dimensional basis.

If r is large (i.e very smooth exact solution),

$$\alpha = k+1-m$$

$$\|\tilde{V} - u\|_m \leq c(h^e)^{k+1-m} \|u\|_r$$

As $h^e \to 0$ (infinitely refined mesh)

if $k+1-m > 0$ → ensure k is large or m is small.

$\|\tilde{V} - u\|_m \to 0$ at the rate $k+1-m$

PROPERTY OF SOBOLEV SPACE, $\mathcal{S}^h$:

# 5.07 Convergence of the Finite Element Solution

**Recall:** Equivalence of $H^1$- & energy norms:

$$c_1 \|v\|_1 \leq \left(a(v,v)\right)^{1/2} \leq c_2 \|v\|_1$$

This extends to $H^n$-norm

$$c_1 \|v\|_n \leq \left(a(v,v)\right)^{1/2} \leq c_2 \|v\|_n$$

**Theorem:** $\|e\|_n \leq \underset{\uparrow \text{ constant}}{\bar{c}} (h^e)^\alpha \|u\|_r$

**Proof:** $c_1 \|e\|_n \leq \left(a(e,e)\right)^{1/2}$

$$\left(a(e,e)\right)^{1/2} \leq \underbrace{\left(a(\tilde{U}^h - u, \tilde{U}^h - u)\right)^{1/2}}$$ — (BEST APPROXIMATION PROPERTY. & $(\tilde{U}^h \in S^h)$)

$$\leq c_2 \|\tilde{U}^h - u\|_n$$ — (Equivalence of $H^n$- & energy norm)

$$\leq c_2 \cdot c (h^e)^\alpha \|u\|_r$$ — (Sobolev Interpolation Error Estimate)

$$\Rightarrow \|e\|_n \leq \underbrace{\frac{c_2 \cdot c}{c_1}}_{\text{a constant, } \bar{c}} (h^e)^\alpha \|u\|_r \longrightarrow \boxed{\|e\|_n \leq \bar{c} (h^e)^\alpha \|u\|_r}$$ ✓

**Remarks:** For $u$ sufficiently smooth $\alpha = \min(k+1-m, r-n)$

$\alpha = k+1-n$ ← no. of derivatives in norm of error.

↖ polynomial order of basis funct.

- consider $n=1$

$$\|e\|_1 \leq \bar{c}(h^e)^{k+1-1} \|u\|_r$$

for $k=1$,

$$\|e\|_1 \leq \bar{c}(h^e) \|u\|_r$$

for $k=2$,

$$\|e\|_1 \leq \bar{c}(h^e)^2 \|u\|_r$$

↓ converge faster.

- $L^2$-norm $\Leftrightarrow$ $H^0$-norm

Aubin-Nitsche Method

$$\|e\|_{L^2} \leq \bar{c}(h^e)^{k+1} \|u\|_r$$

$k=1$, $\|e\|_{L^2} \leq \bar{c}(h^e)^2 \|u\|_r$ & $k=2$, $\|e\|_{L^2} \leq \bar{c}(h^e)^3 \|u\|_r$