$\boxed{6.01}$ Variational Methods

Consider the following integral:

$$\underset{\substack{\uparrow\\ \text{a function}}}{I}(u) := \int_\Omega \frac{1}{2}\, \underset{\substack{\uparrow\\ \text{Modulus}}}{E}\, \underset{\substack{\uparrow\\ \text{area}}}{A} \underbrace{(u_{,x})}_{\text{strain}}{}^2\, dx \quad \Big\}\ \text{STRAIN ENERGY in linearized elasticity.}$$

Allows us to construct the following integral:

$$\Pi\,[u] := \int_\Omega \underbrace{\frac{1}{2} EA (u_{,x})^2}_{\text{strain energy}} dx - \int_\Omega \underbrace{f \cdot u A}_{\text{work done by } f} dx - \underbrace{tA \cdot u(L)}_{\text{work done by traction.}}$$

where u belongs to $\mathcal{S}$ $\left(u \in \mathcal{S} = \{ u \mid u(0) = u_0 \} \right)$

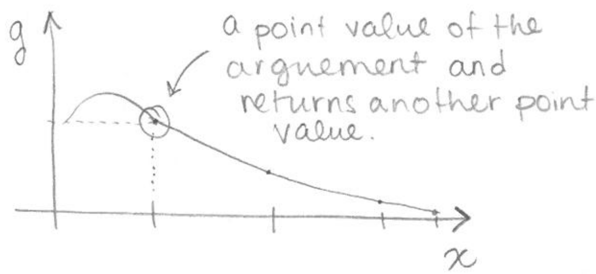
and f, t & constitutive relation $\sigma = E u_{,x}$ are given.

$\Pi[u]$: Gibb's free energy (for purely mechanical problems)

— also called the potential energy in mechanical problems.

★ $\Pi[u]$ is not a function.

A function $g(x)$: $\underbrace{\mathbb{R} \mapsto \mathbb{R}}_{\text{a mapping from } \mathbb{R} \text{ to } \mathbb{R}}$

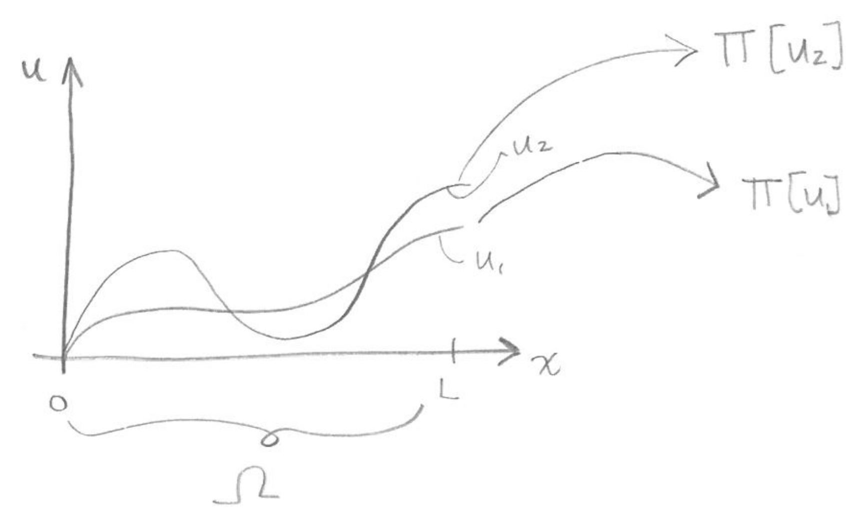


How is $\Pi[u]$ different?

$\Pi[u]$ is a mapping of a field, $u(x)$ to the real numbers.

$\Pi[u] : \mathcal{S} \longmapsto \mathbb{R}$

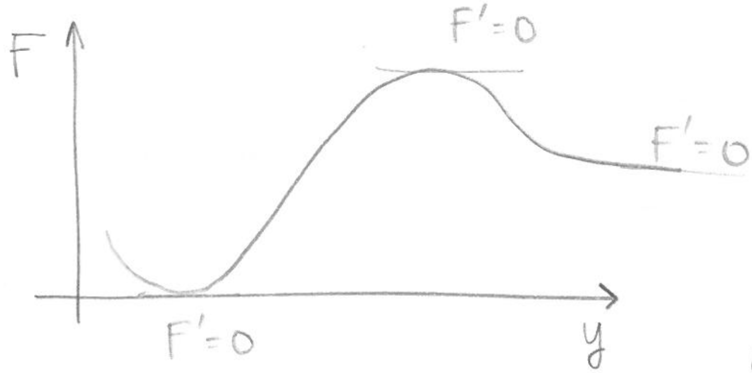
6.02



$\Pi[u]$ is a functional. (integrals & derivatives are functionals.)

{Gibbs free energy functional}

Extrema of free energies characterize states of equilibria of systems.



Extrema occur where $F'=0$.

These are points of equilibrium (either stable, unstable or neutrally stable)

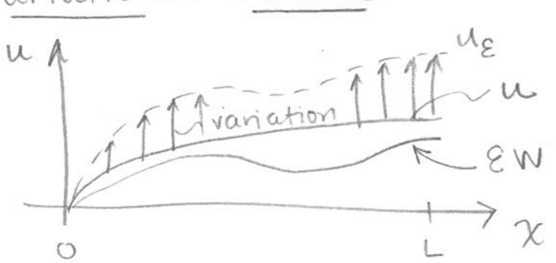
$F''>0$ $\quad$ $F''<0$ $\quad$ $F''=0$

Can we find extrema of $\Pi[u]$?

What is an appropriate notion of a derivative of Π wrt u?

— Differentiation wrt a field, u.

6.03 A variational method



Consider a variation w such that the "varied" or perturbed field is

$$u_\varepsilon := u + \varepsilon w$$

where $\varepsilon \in \mathbb{R}$

w is also a field

NOTE: $w(0)=0$. $\quad w \in \mathcal{V} = \{w \mid w(0)=0\}$.

w vanishes at the Dirichlet boundary so that u_ε satisfies the Dirichlet B.C.

Consider the perturbed functional $\Pi[u_\varepsilon]$

$\frac{d\Pi[u_\varepsilon]}{d\varepsilon}$: the amount of variation in Π for variation in u — having chosen the "form" of the variation w.

$\left.\frac{d}{d\varepsilon}\Pi[u_\varepsilon]\right|_{\varepsilon=0}$: variation in Π wrt u, at u.
— Functional derivative.

$\boxed{6.04}$ Extremization of $\Pi[u]$:

Find $u \in \mathcal{S}$ s.t. $\forall\ w \in \mathcal{V}$

$$\mathcal{S} = \{u \mid u(0) = u_0\}, \quad \mathcal{V} = \{w \mid w(0) = 0\}$$

$$\left.\frac{d}{d\varepsilon}\Pi[u_\varepsilon]\right|_{\varepsilon=0} = 0$$

$$\Rightarrow \frac{d}{d\varepsilon}\left[\int_\Omega \frac{1}{2}EA\left((u+\varepsilon w)_{,x}\right)^2 dx - \int_\Omega f(u+\varepsilon w)A\,dx - tA\cdot(u+\varepsilon w)\right]_{\varepsilon=0} = 0.$$

$$\Rightarrow \left[\int_\Omega \frac{1}{\cancel{2}}EA\,\cancel{2}(u+\overset{0}{\cancel{\varepsilon} w})_{,x}(w_{,x})\,dx - \int_\Omega fwA\,dx - tA\cdot(w)\right]_{\varepsilon=0} = 0$$

$$\Rightarrow \int_\Omega EAu_{,x}w_{,x}\,dx - \int_\Omega fwA\,dx - tAw(L) = 0$$

$$\Rightarrow \int_\Omega w_{,x}\cdot \underbrace{EA\cdot u_{,x}}_{\sigma}\,dx - \int_\Omega wfA\,dx - w(L)\cdot tA = 0$$

$$\Rightarrow \int_\Omega w_{,x}\,\sigma A\,dx - \int_\Omega wfA\,dx - w(L)\cdot tA = 0.$$

This is the weak form!

Remarks: When an extremized principle is available (extremum of the free energy functional) the weak form can be obtained using variational calculus.

A variational principle exists.

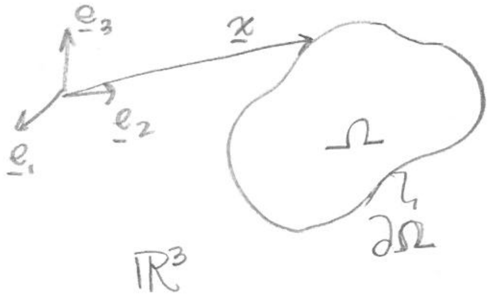
☆ This derivation is not appropriate for the physics of heat conduction of mass diffusion. The mathematics does work... but a physical principle doesn't exist.

Variational Principles Exist for:
- Elasticity @ Steady State
- Schrödinger equation @ Steady State.

$\boxed{7.01}$ Linear, elliptic PDE's in 3-D, with scalar unknown.

- Steady State heat conduction
- Steady state mass diffusion.

Strong Form of the Problem



$\{e_i\}$, $i=1,2,3$ constitutes an orthonormal, Cartesian basis. ($\perp$ unit length)

$e_i \cdot e_j = \delta_{ij}$ the Kronecker Delta.

Ω: open in $\mathbb{R}^3$ & $\partial\Omega$: the boundary of Ω.

Find u, given $f(x)$, u_g, j_n, and the constitutive relation $j_i = -K_{ij}u_{,j}$ $(i,j = 1,2,3)$

such that $-j_{i,i} = f$ in Ω

B.C.s: $u = u_g$ on $\partial\Omega_u$ ← Dirichlet b.c.

$-j \cdot n = j_n$ on $\partial\Omega_j$ ← Neuman b.c.

$\partial\Omega_u \cap \partial\Omega_j = \emptyset$ (intersection; empty set.)

$\partial\Omega = \partial\Omega_u \cup \partial\Omega_j$ (union)