Remarks: When an extremized principle is available (extremum of the free energy functional) the weak form can be obtained using variational calculus.

A variational principle exists.

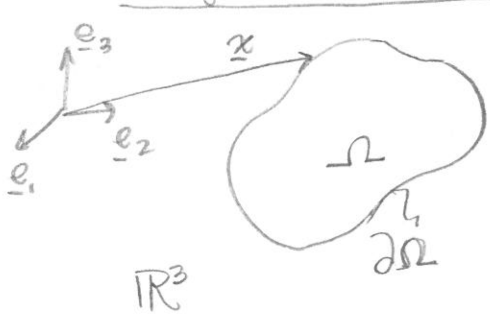
☆ This derivation is not appropriate for the physics of heat conduction of mass diffusion. The mathematics does work... but a physical principle doesn't exist.

Variational Principles Exist for:
- Elasticity @ Steady State
- Schrödinger equation @ Steady State.

$\boxed{7.01}$ Linear, elliptic PDE's in 3-D, with scalar unknown.

- Steady state heat conduction
- Steady state mass diffusion.

Strong Form of the Problem



$\{e_i\}$, $i = 1,2,3$ constitutes an orthonormal, Cartesian basis. ($\perp$ unit length)

$e_i \cdot e_j = \delta_{ij}$ the Kronecker Delta.

Ω: open in $\mathbb{R}^3$ & $\partial\Omega$: the boundary of Ω.

Find u, given $f(x)$, u_g, j_n, and the constitutive relation $j_i = -K_{ij} u_{,j}$ $(i,j = 1,2,3)$

such that $-j_{i,i} = f$ in Ω

B.C.s: $u = u_g$ on $\partial\Omega_u$ ← Dirichlet b.c.

$-j \cdot n = j_n$ on $\partial\Omega_j$ ← Neuman b.c.

$\underset{\text{intersection}}{\partial\Omega_u \cap \partial\Omega_j} = \underset{\text{empty set.}}{\emptyset}$

$\partial\Omega = \partial\Omega_u \underset{\text{union}}{\cup} \partial\Omega_j$

$\boxed{1.02}$

j_i: flux vector in coordinate notation, $i = 1, 2, 3$

$$\underline{j} = \begin{Bmatrix} j_1 \\ j_2 \\ j_3 \end{Bmatrix} \leftarrow \text{Direct Notation} \quad ; \underline{j} \in \mathbb{R}^3 \quad (\underline{j} \text{ is a 3-D vector})$$

likewise $\underline{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$; $\underline{x} \in \mathbb{R}^3$

<u>Consider Heat Conduction @ Steady State in 3D.:</u>

u: temperature

$\underline{j}$: heat flux vector. (amount of heat crossing perpendicular to a unit area per unit)

Constitutive relation: $\underline{j}_i = -\underbrace{K_{ij}}_{\substack{\text{Heat} \\ \text{conductivity} \\ \text{Tensor.}}} \underbrace{u_{,j}}_{\frac{\partial u}{\partial x_j}}$: Fourier Law of Heat conduction.

$\frac{\partial u}{\partial x_j} \rightarrow$ Temperature gradient.

$$\underline{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \rightarrow \text{symmetric} \therefore \underline{K} = \underline{K}^T$$

direct notation.

$K_{12} = K_{21}$ etc.

$\underline{K}_{ij} = \underline{K}_{ji}$

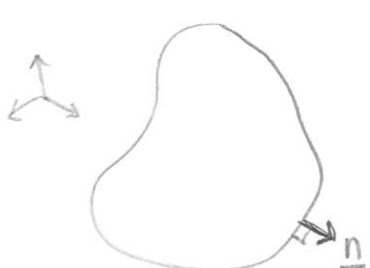
$\underline{K}$: positive semi-definite

$\hookrightarrow$ If $\underline{\xi} \in \mathbb{R}^3$, then $\underline{\xi} \cdot \underline{K}\,\underline{\xi} \geq 0$

Remark: If $\underline{\xi} \cdot \underline{K}\,\underline{\xi} = 0$ for $\underline{\xi} \neq 0$ then there is no heat conduction along $\underline{\xi}$.

$\Rightarrow$ Heat Flows down a temperature grad.

<u>Boundary Conditions.</u> $u = u_g(\underline{x})$ on $\partial\Omega_u$ temperature b.c.



$-\underline{j} \cdot \underline{n} = j_n$ heat influx b.c.

$-j_i n_i = j_n$ } coordinate notation.

$\boxed{7.03}$ Consider Mass Diffusion:

u: concentration. (mass/V or mol/V)
→ or composition (normalized concentration)

$\underline{j}$: mass flux (mass flow ⊥ to $\partial\Omega$ per unit time)
number flux (# of particles flowing ⊥ to $\partial\Omega$ per time)

$$\underline{j} = -\underset{\text{diffusivity tensor}}{\underbrace{\underline{k}}} \nabla \underline{u} = -\underline{k} \frac{\partial u}{\partial \underline{x}} \qquad \text{again, } \underline{k} = \underline{k}^T$$

$u = u_g$ on $\partial\Omega_u$ — concentration b.c.

$-\underline{j} \cdot \underline{n} = j_n$ on $\partial\Omega_j$ — mass influx b.c.

Strong Form in direct notation:

Find u given u_g, j_n, f, the constitutive relation

$$\underline{j} = -\underline{k} \nabla u \quad \text{such that:}$$

$$-\underbrace{\underline{\nabla} \cdot \underline{j}}_{\text{divergence of } \underline{j}} = f \quad \text{in } \Omega$$

B.C.'s: $u = u_g$ on $\partial\Omega_u$ & $-\underline{j} \cdot \underline{n} = j_n$ on $\partial\Omega_j$

Substituting $\underline{j} = -\underline{k} \nabla u$ in pde:

$$-\underline{\nabla} \cdot (-\underline{k} \underline{\nabla} u) = f \quad \text{in } \Omega$$

if $\underline{k}$ is spatially uniform:

$$\Rightarrow \underline{k} : \underline{\nabla}^2 u = f \longrightarrow k_{ij} u_{,ij} = f$$

contracted — Hessian operator

uniform

If $k_{ij} = k \delta_{ij}$

$$\Rightarrow k \delta_{ij} u_{,ij} = f$$

$\Rightarrow K u_{,ii} = f$ in Ω } Poisson Equation

Direct notation: $K \nabla^2 u = f$ in Ω

∇^2 ↑ laplasian

Neumann b.c.: $-\underline{j} \cdot \underline{n} = j_n$

$\Rightarrow \underbrace{+K \underline{\nabla} u \cdot \underline{n}}_{\text{normal gradient of temperature, } u} = j_n$

Remark: $K_{ij} = K \delta_{ij}$ ↑ isotropic heat conduction

$\boxed{7.04}$ The WEAK FORM of the problem.

Find $u \in S = \{u \mid u = u_g \text{ on } \partial\Omega_u\}$ given u_g, j_n, f and the constitutive relation $j_i = -K_{ij} u_{,j}$ such that for all $w \in \mathcal{V} = \{w \mid w = 0 \text{ on } \partial\Omega_u\}$

$$\int_\Omega w_{,i} j_i \, dV = \int_\Omega w f \, dV - \int_{\partial\Omega_j} w j_n \, ds$$

dV ↑ elemental volume; ds ↑ elemental surface area.

Consider the strong form:

Find u given u_g, j_n, f & $j_i = -K_{ij} u_{,j}$ s.t.

$-j_{i,i} = f$ in Ω & B.C.'s: $u = u_g$ on $\partial\Omega_u$

$-j_i n_i = j_n$ on $\partial\Omega_j$

Consider $w \in \mathcal{V} = \{w \mid w = 0 \text{ on } \partial\Omega_u\}$

Multiply strong form of pde by w & integrate by parts.

$$\underbrace{\int_\Omega -w j_{i,i} \, dV}_{\text{integrate by parts. (product rule \& divergence theorem)}} = \int_\Omega w f \, dV$$

$$\int_\Omega -\Big(\underbrace{(w j_i)}_{\text{vector}}{}_{,i} + w_{,i} j_i\Big) dV = \int_\Omega w f \, dV \quad \text{— Product Rule.}$$

$$-\int_{\partial\Omega} w j_i n_i \, ds + \int_\Omega w_{,i} j_i \, dV = \int_\Omega w f \, dV \quad \text{— divergence theorem.}$$

$$\Rightarrow \int_{\Omega} w_{,i} j_i \, dV = \int_{\Omega} w f \, dV + \underbrace{\int_{\partial\Omega_u} \cancel{w} j_i n_i \, dS}_{=0} + \underbrace{\int_{\partial\Omega_j} w \cancel{j_i n_i} \, dS}$$

0 due to b.c. $\qquad -j_n$

$$\Rightarrow \int_{\Omega} w_{,i} j_i \, dV = \int_{\Omega} w f \, dV + \int_{\partial\Omega_j} (-w j_n) \, dS$$

WEAK FORM $\Longleftrightarrow$ STRONG FORM } each implies the other.

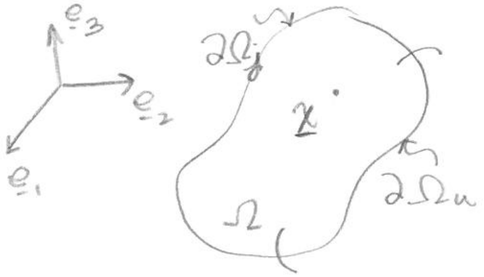
<u>Remark:</u> Recall the pde of the strong form:

$$-j_{i,i} = f \quad \text{-or-} \quad -\underline{\nabla} \cdot \underline{j} = f \quad \text{in } \Omega$$

net influx = source term

$$-\underline{\nabla} \cdot \underline{j} = f$$

$\boxed{7.05}$ <u>The finite-dimensional weak form</u>



Find $u \in \mathcal{S} = \{u \mid u = u_g \text{ on } \partial\Omega_u\}$
given $u_g, j_n, f, j_i = -k_{ij} u_{,j}$
s.t. $\forall\, w \in \mathcal{V} = \{w \mid w(0) = 0 \text{ on } \partial\Omega_u\}$

$$\int_{\Omega} w_{,i} j_i \, dV = \int_{\Omega} w f \, dV - \int_{\partial\Omega_j} w j_n \, dS$$

Find $u^h \in \mathcal{S}^h \subset \mathcal{S}$

finite dimensional function space.

$$\mathcal{S}^h = \{u^h \in H^1(\Omega) \mid u^h = u_g \text{ on } \partial\Omega_u\} \text{ given } u_g, j_n^h, f, j_i = -k_{ij} u_{,j}$$

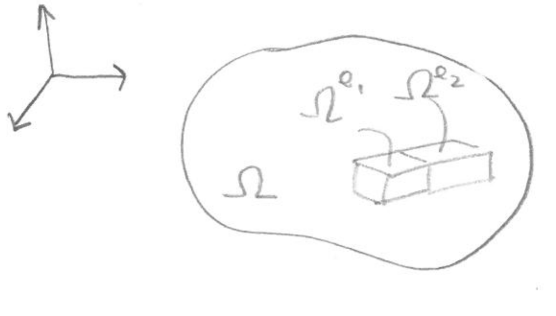
such that $\forall\, w^h \in \mathcal{V}^h \subset \mathcal{V}$, $\mathcal{V}^h = \{w^h \in H^1(\Omega) \mid w^h = 0\}$

$$\int_{\Omega} w^h_{,i} j^h_i \, dV = \int_{\Omega} w^h f \, dV - \int_{\partial\Omega_j} w^h j_n \, dS$$

7.06 The finite-dimensional weak form is the basis of our finite element formulation.

Define $\mathcal{S}^h$ & $\mathcal{V}^h$ by partitioning Ω into subdomains Ω^e.

$$e = 1, \dots, n_{el}, \quad \Omega^e \subset \Omega \subset \mathbb{R}^3$$



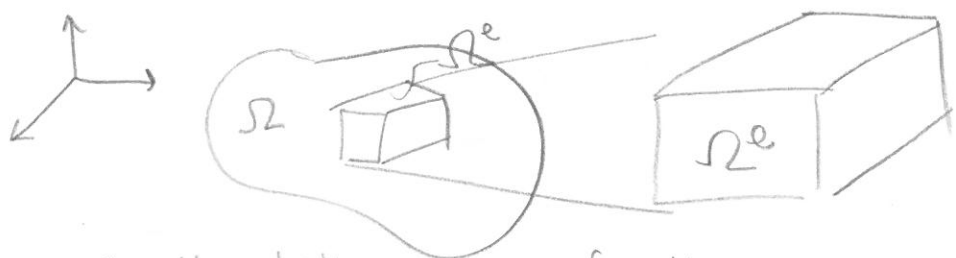
Ω^e is open in $\mathbb{R}^3$.

$$\overline{\Omega} = \overline{\bigcup_{e=1}^{n_{el}} \Omega^e}$$

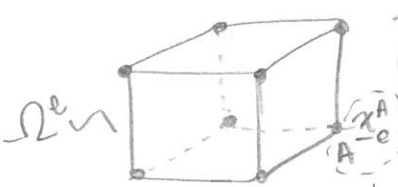
This is the closure of the union

$$\Omega^{e_1} \cap \Omega^{e_2} = \emptyset$$

Consider hexahedral element subdomains, Ω^e, $e = 1, \dots, n_{el}$



Consider trilinear basis functions.



also known as "eight-node bricks."

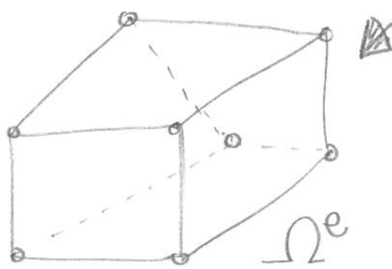
$A = 1, \dots, 8.$

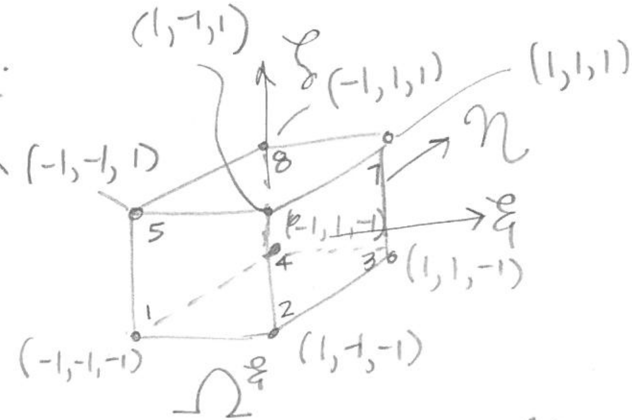
Node A w/ local numbering of nodes.

$$u_e^h = \sum_{A=1}^{n_{nel}=8} N^A(\underline{x})\, d_e^A \quad , \quad w_e^h = \sum_{A=1}^{n_{nel}=8} N^A(\underline{x})\, c_e^A$$

Ω^e is obtained by a mapping from a parent domain Ω^{ξ}

$\boxed{7.07}$ Mapping from Ω^{ξ} to Ω^{e}.





eight-noded irregular hexahedron ↳ "deformed

generally: (ξ^A, η^A, ζ^A) $A = 1, \dots 8$

Ω^{ξ} bi-unit domain $N^A = \underbrace{N^A(\xi, \eta, \zeta)}_{\text{tensor product functions.}} = \frac{1}{8}(1+\xi\xi^A)(1+\eta\eta^A)(1+\zeta\zeta^A)$

$$N^1(\xi,\eta,\zeta) = \tfrac{1}{8}(1-\xi)(1+\eta)(1+\zeta)$$
$$N^2(\quad) = \tfrac{1}{8}(1+\xi)(1-\eta)(1-\zeta)$$
$$N^3(\quad) = \tfrac{1}{8}(1+\xi)(1+\eta)(1-\zeta)$$
$$N^4(\quad) = \tfrac{1}{8}(1-\xi)(1+\eta)(1-\zeta)$$
$$\vdots$$

$N^A(\xi^B, \eta^B, \zeta^B) = \delta^{AB}$

Kronecker delta property

$$\sum_{A=1}^{n_{ne}} N^A(\xi, \eta, \zeta) = 1.$$

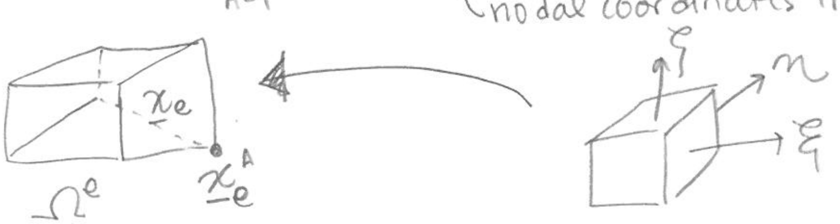
NOTE: • These are Lagrange Polynomial basis functions in $\mathbb{R}^3$
• Note the tri-linearity (linear in each coordinate)

The map from Ω^{ξ} to Ω^{e} is obtained by interpolating

$\underline{x}(\underline{\xi})$ where $\underline{\xi} = \begin{Bmatrix} \xi \\ \eta \\ \zeta \end{Bmatrix}$

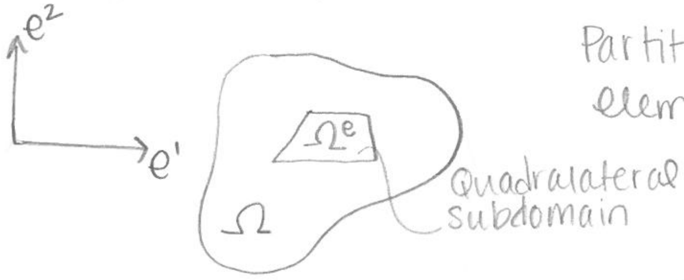
$$x_e(\xi) = \sum_{A=1}^{n_{nel}} N^A(\xi)\, \underline{x}_e^A$$

↖ nodal coordinates in physical domain



Isoparametric Formulation.

$\boxed{7.08}$ Aside: Lagrange Polynomials (bilinears) in 2-D.

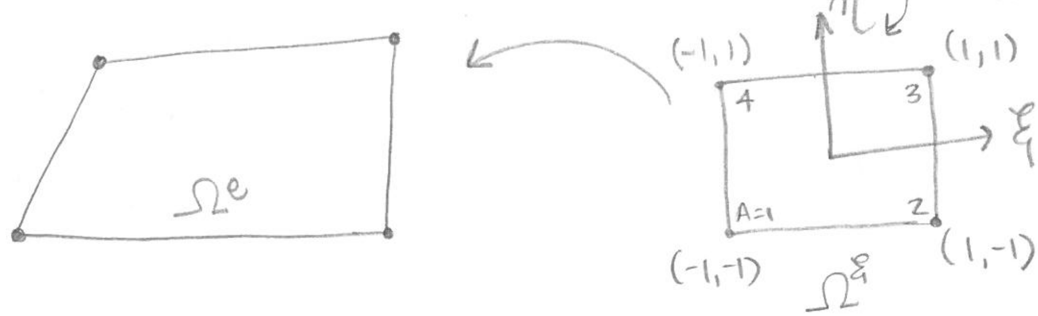


Partition Ω into Quadralateral element subdomains, Ω^e, $e=1,2,n_e$

$$\overline{\Omega} = \overline{\bigcup_e \Omega^e}$$

Consider the use of bilinear basis functions.

parent subdomain is a square



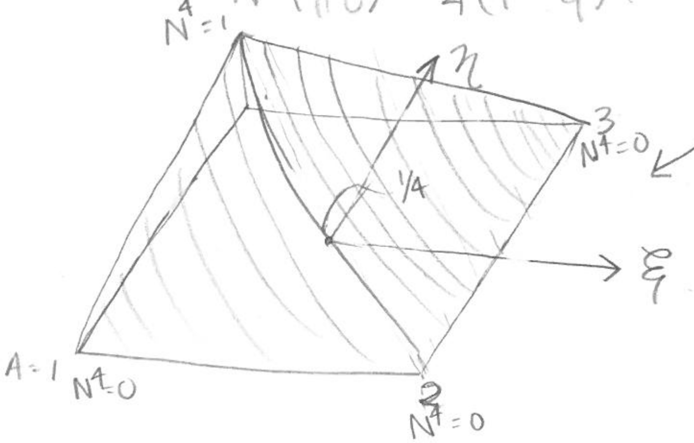
coordinates of local node A:
(ξ^A, η^A)

The bases functions:

$$\left.\begin{aligned} N^1(\xi,\eta) &= \tfrac{1}{4}(1-\xi)(1-\eta) \\ N^2(\xi,\eta) &= \tfrac{1}{4}(1+\xi)(1-\eta) \\ N^3(\xi,\eta) &= \tfrac{1}{4}(1+\xi)(1+\eta) \\ N^4(\xi,\eta) &= \tfrac{1}{4}(1-\xi)(1+\eta) \end{aligned}\right\}$$

$N^A(\xi^B,\eta^B) = \delta^{AB}$ ← Kronecker Delta Property.

$$\sum_{A=1}^{n_{ne}} N^A(\xi,\eta) = 1$$



basis shown: N^4

$\boxed{7.09}$ Return to the Finite-Dimensional Weak Form.

$$\int_{\Omega} w^h_{,i} j^h_i \, dV = \int_{\Omega} w^h f \, dV - \int_{\partial\Omega_j} w^h j_u \, dS$$

$$\Rightarrow \overline{\Omega} = \overline{\bigcup_e \Omega^e} \qquad -K_{ij}u_{,j}$$

$$\sum_e \int_{\Omega^e} w^h_{,i} j^h_i \, dV = \sum_e \int_{\Omega^e} w^h f \, dV - \sum_{e \in \mathcal{E}_N} \int_{\partial\Omega^e_j} w^h j_u \, dS$$

$$e \in \mathcal{E}_N \text{ if } \partial\Omega^e \cap \partial\Omega_j \neq \emptyset \Big\}$$ only considering elements whose surfaces coincide w/ the Neuman boundary

Need function gradients.

Notation: $\begin{Bmatrix} \xi \\ \eta \\ \zeta \end{Bmatrix} = \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix}$ just as $\underline{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$

$$u^h_{e,i} = \sum_{A=1}^{n_{nee}} N^A_{,i} d^A_e \qquad w^h_{e,i} = \sum_{A=1}^{n_{nee}} N^A_{,i} c^A_e$$

use upper case indices for $\underline{\xi} \in \Omega^{\xi}$ $\to \xi_I$, $I = 1,2,3$.

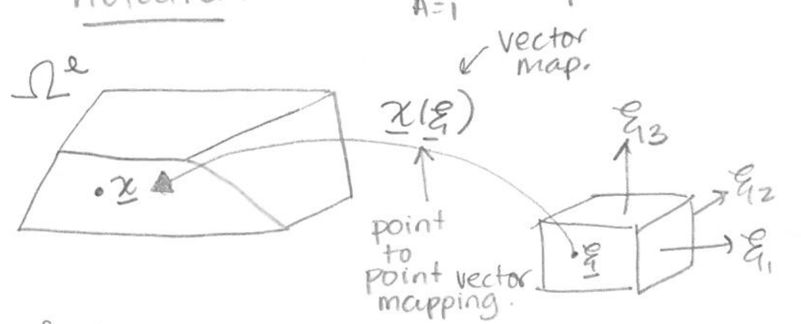
$$N^A_{,i} = \frac{\partial N^A}{\partial x_i} = \underbrace{\frac{\partial N^A}{\partial \xi_I}}_{N^A(\underline{\xi}) \text{ is known.}} \frac{\partial \xi_I}{\partial x_i} \longrightarrow \text{Sum implied on } I = 1,2,3.$$

?

$\boxed{7.10}$ Recall the mapping from $\underline{\xi}$ to $\underline{x}$.

$$\underline{x}(\underline{\xi}) = \sum_A^{n_{ne}} N^A(\underline{\xi}) \underline{x}^A_e$$ coordinate notation: $$x_i = \sum_{A=1}^{n_{ne}} N^A(\underline{\xi}) \underline{x}^A_{e_i}$$

$$\frac{\partial x_i}{\partial \xi_I} = \sum_{A=1}^{n_{nee}} N^A_{,I} x^A_{e_i}$$



gradient of the map: aka "tangent map."

$$\underline{J} := \frac{\partial \underline{x}}{\partial \underline{\xi}} \text{ Jacobian of the map.} \qquad J_{iI} = \frac{\partial x_i}{\partial \xi_I}$$

Represent $\underline{J}$ as a matrix:

$$\underline{J} = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{bmatrix}$$

Map $\underline{x}(\underline{\xi})$ from $\Omega^{\xi} \mapsto \Omega^{e}$ is C^{∞} ← can take ∞ derivatives

$$\underline{J} = \frac{\partial \underline{x}}{\partial \underline{\xi}} \Rightarrow \underset{\text{There exists}}{\exists}\ \underline{J}^{-1} = \frac{\partial \underline{\xi}}{\partial \underline{x}}$$

$$\underline{J}^{-1} = \begin{bmatrix} \frac{\partial \xi_1}{\partial x_1} & \frac{\partial \xi_1}{\partial x_2} & \frac{\partial \xi_1}{\partial x_3} \\ \vdots & \frac{\partial \xi_2}{\partial x_2} & \frac{\partial \xi_2}{\partial x_3} \\ \cdots & \cdots & \frac{\partial \xi_3}{\partial x_3} \end{bmatrix} ; \quad J^{-1}_{Ii} = \frac{\partial \xi_I}{\partial x_i}$$

$\boxed{7.11}$ The Integrals in the finite-dimensional weak form.

$$\int_{\Omega^e} w^h_{,i} j^h_i \, dV = -\int_{\Omega^e} w^h_{,i} K_{ij} u^h_{,j} \, dV = -\int_{\Omega^e} \left(\sum_{A=1}^{n_{ne}} N^A_{,i} c^A_e\right) K_{ij} \left(\sum_{B=1}^{n_{ne}} N^B_{,j} d^B_e\right) dV$$

$$= -\int_{\Omega^e} \Big(\sum_A \underbrace{N^A_{,I}\, \xi_{I,i}}_{\text{chain rule}}\, c^A_e\Big) K_{ij} \Big(\sum_B N^B_{,J}\, \xi_{J,j}\, d^B_e\Big) dV$$

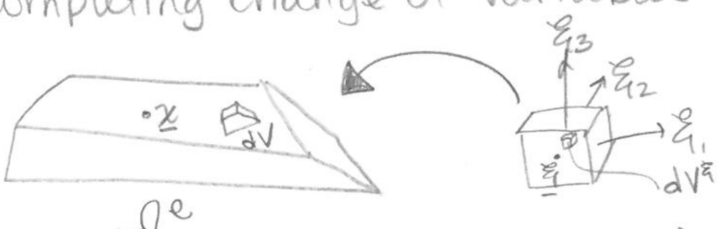
$\boxed{\text{independent of position}}$ (c^A_e, d^B_e)

A or B: physical domain node #

$$= -\sum_{A,B} c^A_e \left(\int_{\Omega^e} N^A_{,I}\, \xi_{I,i}\, K_{ij}\, N^B_{,J}\, \xi_{J,j} \, dV \right) d^B_e$$

Einstein summations

Completing change of variables to $\underline{\xi}$. (need to convert elemental V)



$$\underline{J}(\underline{\xi}) = \frac{\partial \underline{x}}{\partial \underline{\xi}}$$

☆ $dV = \det(\underline{J}(\underline{\xi}))\, dV^{\xi}$ ☆

$\boxed{7.12}$ Integrals in Finite-dimensional weak form.

$$\int_{\Omega^e} w^h_{,i}\, j^h_i \, dV = -\sum_{A,B} c_e^A \left[\int_{\Omega^\xi} N^A_{,I}\, \xi_{I,i}\, K_{ij}\, N^B_{,J}\, \xi_{J,j} \det(\underline{J}(\xi))\, dV^\xi\right] d_e^B$$

$$= -\sum_{A,B} c_e^A \Big[\underbrace{\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} N^A_{,I}\, \xi_{I,i}\, K_{ij}\, N^B_{,J}\, \xi_{J,j} \det[\underline{J}(\xi)]\, d\xi_1\, d\xi_2\, d\xi_3}_{\text{Yields a scalar value.} \; = K_e^{AB}}\Big] d_e^B$$

$$= -\sum_{A,B} c_e^A K_e^{AB} d_e^B$$

Matrix-Vector Notation for local degrees of freedom.

$$\begin{Bmatrix} c_e^1 \\ c_e^2 \\ c_e^3 \\ \vdots \\ c_e^{n_{nee}} \end{Bmatrix} \text{ and } \begin{Bmatrix} d_e^1 \\ d_e^2 \\ d_e^3 \\ \vdots \\ d_e^{n_{nee}} \end{Bmatrix} \Longrightarrow \int_{\Omega^e} w^h_{,j}\, j^h_i\, dV = -\langle c_e^1 \cdots c_e^{n_{nee}} \rangle\, K_e^{AB} \begin{Bmatrix} d_e^1 \\ d_e^2 \\ \vdots \\ d_e^{n_{nee}} \end{Bmatrix}$$

element conductivity or diffusivity matrix.

$$= -\underline{c}_e^T\, \underline{K}_e\, \underline{d}_e$$

Consider RHS:

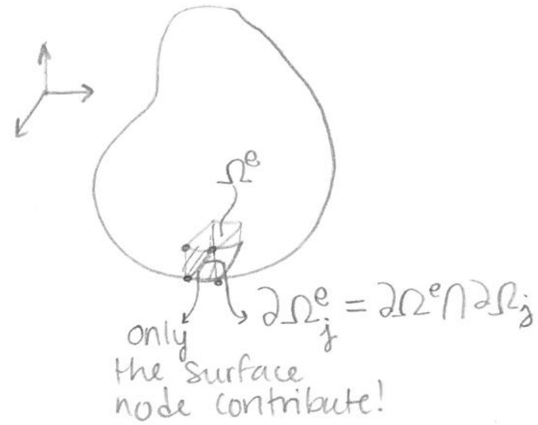
$$\int_{\Omega^e} w^h f\, dV = \int_{\Omega^e} \Big(\sum_A N^A c_e^A\Big) f\, dV = \sum_A c_e^A \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (N^A)\, f(\underline{x}(\xi))\, \det(\underline{J}(\xi))\, d\xi_1\, d\xi_2\, d\xi_3$$

$$= \langle c_e^1\; c_e^2 \cdots c_e^{n_{nee}} \rangle \underbrace{\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \begin{Bmatrix} N^1 \\ N^2 \\ \vdots \\ N^{n_{nee}} \end{Bmatrix} f(\xi)\, \det[\underline{J}(\xi)]\, d\xi_1\, d\xi_2\, d\xi_3}_{F_e^{int}}$$

$$= \langle c_e^1 \cdots c_e^{n_{nee}} \rangle \begin{Bmatrix} F_e^{int_1} \\ F_e^{int_2} \\ \vdots \\ F_e^{int_{n_{nee}}} \end{Bmatrix} = \underline{c}_e^T\, \underline{F}_e^{int}$$

7.13 The Matrix-Vector Weak Form.

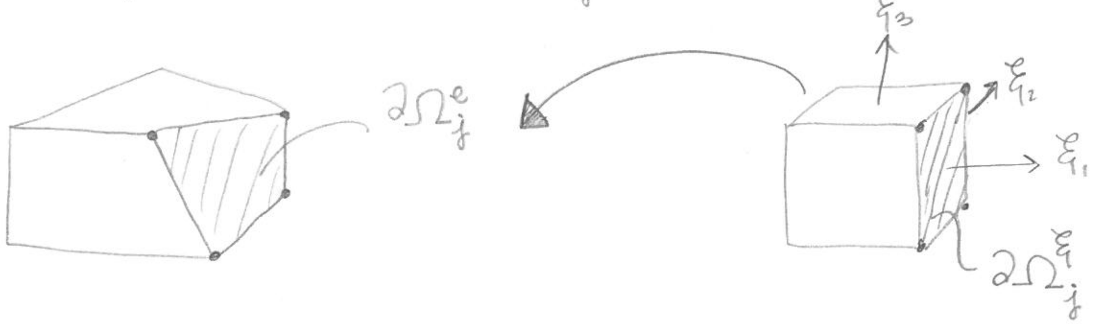
Consider:



$$-\int_{\partial\Omega^e_j} w^h j_n \, dS = -\int_{\partial\Omega^e_j} \left(\sum_{A=1}^{n_{ne}} N^A c^A_e \right) j_n \, dS$$

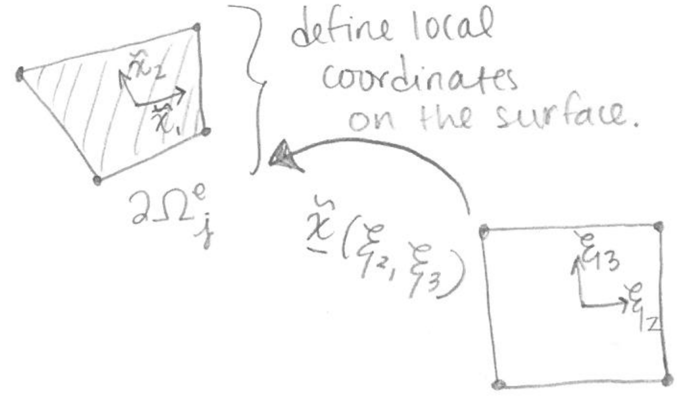
Define $A_N = \{ A \mid \underline{x}^A_e \in \partial\Omega^e_j \}$

$$\Rightarrow -\int_{\partial\Omega^e_j} w^h j_n \, dS = -\sum_{A \in A_N} c^A_e \int_{\partial\Omega^e_j} N^A j_n \, dS$$

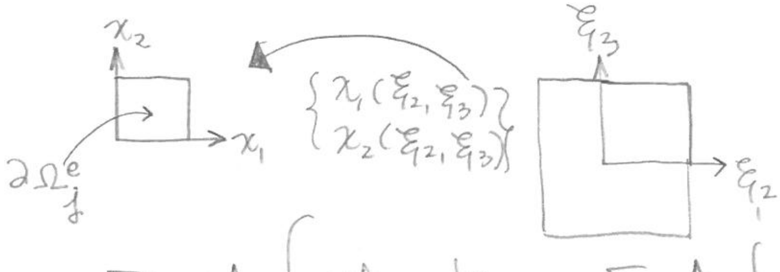


$$-\sum_{A \in A_N} c^A_e \int_{\partial\Omega^e_j} N^A j_n \, dS = -\sum_{A \in A_N} c^A_e \int_{\partial\Omega^\xi_j} N^A j_n \det(\underline{J}_s) \, dS^\xi$$

$$\underline{J}_s = \begin{bmatrix} \tilde{x}_{1,\xi_2} & \tilde{x}_{2,\xi_3} \\ \tilde{x}_{2,\xi_2} & \tilde{x}_{2,\xi_3} \end{bmatrix}$$



7.14 e.g.



$$-\sum_{A \in A_N} c^A_e \int_{\partial\Omega^e_j} N^A j_n \, dS = -\sum_{A \in A_N} c^A_e \int_{\xi_i=-1}^{1} \int_{\xi_j=-1}^{1} N^A_{j_N} \det\left[\underline{J}_s\right] d\xi_i \, d\xi_j$$

$$= \langle c_e^{A_1}\ c_e^{A_2} \cdots c_e^{A_4} \rangle \begin{Bmatrix} F^{j A_1} \\ F^{j A_2} \\ F^{j A_3} \\ F^{j A_4} \end{Bmatrix}, \quad A_1, \ldots, A_4 \in A_N$$

Expanding to include all d.o.f in Ω^e

$$-\int_{\partial\Omega^e_j} w^h j_n \, dS = -\langle c_e^{A_1}, c_e^2, c_e^3, c_e^{A_2}, c_e^{A_3}, c_e^6, c_e^7, c_e^{A_4} \rangle \begin{Bmatrix} F^{j1} \\ 0 \\ 0 \\ F^{j4} \\ F^{j5} \\ 0 \\ 0 \\ F^{j8} \end{Bmatrix}$$

local degrees of freedom $1,4,5,8 = A_1, A_2, A_3, A_4 \in A_N$.

$\boxed{7.15}$ Recall:

$$-\int_{\partial\Omega^e_j} w^h j_n \, dS = -\langle c_e^{A_1}, c_e^2, c_e^3, c_e^{A_2}, c_e^{A_3}, c_e^6, c_e^7, c_e^{A_4} \rangle \begin{Bmatrix} F^{j1} \\ 0 \\ 0 \\ F^{j4} \\ F^{j5} \\ 0 \\ 0 \\ F^{j8} \end{Bmatrix}$$

e.g. local d.o.f $1,4,5,8$ are $\in A_N$.

$$= -\underline{c}_e^T \underline{F}_e^j$$

Use in:

$$+\sum_{e=1}^{n_{el}} \underline{c}_e^T \underline{K}_e \underline{d}_e = \underbrace{-\sum_e^{n_{el}} \underline{c}_e^T \underline{F}_e^{int}} + \sum_{e \in \mathcal{E}^N} \underline{c}_e^T \underline{F}_e^j$$

Aside: The form of equations is obtained from

$$-j_{i,i} = f \quad \text{in } \Omega$$

Steady state equation arrived at from:

$$\underbrace{c}_{\substack{\text{specific}\\ \text{heat}\\ \text{coeffecient}}}\ \underbrace{u_{,t}}_{\substack{\text{rate of}\\ \text{change of}\\ \text{temperature.}}} = \underbrace{-j_{i,i}}_{\substack{\text{influx of}\\ \text{heat (net)}}} - \underbrace{f}_{\substack{\text{local}\\ \text{distributed}\\ \text{heating.}}}$$ —— Time dependent Form.

goes to ϕ for s.s. case.

Redefine $\bar{f} = -f$

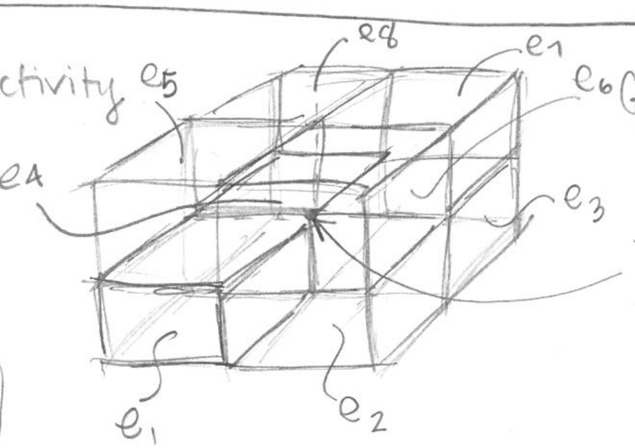
$$-\sum_e \int_{\Omega^e} w^h f \, dV = \sum_e \int_{\Omega^e} w^h (\bar{f}) \, dV$$

$$\Rightarrow \sum_e \underline{c}_e^T \underline{k}_e \underline{d}_e = \underbrace{-\sum_e \underline{c}_e^T \underline{F}_e^{int}}_{\underbrace{+\sum_e \underline{c}_e^T \underline{\bar{F}}_e^{int}}_{= \int_{\Omega_e} w^h \bar{f} \, dV}} + \sum_{e \in \mathcal{E}_N} \underline{c}_e^T \underline{F}_e^{j}$$

$\uparrow$ forcing function.

Assembly of global finite element equations in matrix-vector form.

$\boxed{7.16}$ Mesh Connectivity

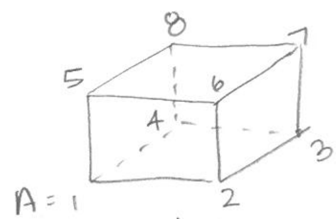


Global d.o.f $\bar{A}$: This node is shared amongst 8 elements $(e_1, \ldots, e_8)$

Typically provided in an input file.

e_1 — — — — — — — $\bar{A}$ —
e_2 — — — — — — — — $\bar{A}$
⋮
e_3 — — — — $\bar{A}$ — — —
e_4 — — — — — $\bar{A}$ — —
⋮
e_5 — $\bar{A}$ — — — — — —
e_6 — — $\bar{A}$ — — — — —
e_7 $\bar{A}$ — — — — — — —
⋮
e_8 — $\bar{A}$ — — — — — —

"Local Destination Array"



local d.o.f. numbering.

7.17 Assembly of Global Matrix-Vector weak form:

$$\sum_e \underline{c}_e^T \underline{K}_e \underline{d}_e = \sum_e \underline{c}_e^T \underline{F}_e^{int} + \sum_{e \in \mathcal{E}_N} \underline{c}_e^T \underline{F}_e^{j}$$

$$\Rightarrow \underline{c}^T \overline{\underline{K}}\, \overline{\underline{d}} = \underline{c}^T \overline{\underline{F}}^{int} + \underline{c}^T \underline{F}^{j}$$

$\uparrow \uparrow$ TO ACCOUNT FOR DIRICHLET B.C. (resizing of these matrices.)

$$\overline{\underline{K}} = \underbrace{\mathop{A}_{e}}_{\text{assembly}} \underline{K}_e \quad ; \quad \overline{\underline{F}}^{int} = \underbrace{\mathop{A}_{e}}_{\text{assembly}} \underline{F}_e^{int}$$

$$\overline{\underline{K}} = \begin{array}{c} \\ \overline{A} \rightarrow \\ \\ \end{array} \begin{bmatrix} & & \\ \cdots & \left(K_{e_1}^{77} + K_{e_2}^{88} + K_{e_3}^{55} + K_{e_4}^{66} + K_{e_5}^{33} + K_{e_6}^{44} + K_{e_7}^{11} \;\; K_{e_8}^{22}\right) & \cdots \\ & & \end{bmatrix}$$

$\uparrow \overline{B}$ $\qquad$ $\overline{A} \uparrow$

$$\left(K_{e_i}^{AB} + K_{e_j}^{CD}\right)$$

Likewise:

$$\underline{F}^{int} = \begin{Bmatrix} \vdots \\ \overline{B} \rightarrow \; \overline{F}_{e_i}^{B\,int} + \overline{F}_{e_j}^{D\,int} \\ \vdots \\ \overline{A} \rightarrow \; \overline{F}_{e_1}^{7\,int} + \overline{F}_{e_2}^{8\,int} + \overline{F}_{e_3}^{5\,int} + \overline{F}_{e_4}^{6\,int} + \overline{F}_{e_5}^{3\,int} + \overline{F}_{e_6}^{4\,int} + \overline{F}_{e_7}^{1\,int} + \overline{F}_{e_8}^{2\,int} \\ \vdots \end{Bmatrix}$$

$$\underline{F}^{j} = \begin{Bmatrix} \vdots \\ \overline{B} \rightarrow \; F_{e_i}^{jB} + F_{e_j}^{jD} \\ \vdots \end{Bmatrix}$$

$\boxed{7.18}$ Return to:

$$\underline{c}^T \underline{K}\,\underline{d} = \underline{c}^T \underline{F}^{int} + \underline{c}^T \underline{F}^{\gamma}$$

Suppose global d.o.f's $\{\bar{A}, \bar{B}, \dots\} \in \bar{A}_D$ the set of global degrees of freedom on which Dirichlet b.c.'s are specified.

If A is the local d.o.f in some element e_i corresponded to global d.o.f. $\bar{A} \in \bar{A}_D$

$$w_{e_i}^h = \sum_{\substack{B=1\\ B\neq A}}^{n_{ne}} N^B c_e^B \Rightarrow \underline{c}^T = \langle \cdots c^{\bar{D}}\ \underbrace{c^{\bar{E}} \cdots c^{\bar{F}} \cdots}_{\bar{A} \text{ dof is missing.}} \rangle$$

Since this.

If measure of $\bar{A}_D$: $m(\bar{A}_D)$: no. of degrees of freedom belonging to $\bar{A}_D = N_D$

$\underline{c}^T$; $\bar{\underline{d}}$

DIM: $n_{sd} \times n_{nodes} - N_D$; $n_{sd} \times n_{nodes}$; $\bar{\underline{d}} = \left\{ d^1, d^2, \dots, \cancel{d^{\bar{A}}}, \cancel{d^{\bar{B}}}, \dots, d^{n_{sd} \times n_{nodes}} \right\}$

KNOWN

$\bar{\underline{K}}$

$$(\underbrace{n_{sd} \times n_{nodes} - N_D}_{\text{rows}}) \times (\underbrace{n_{sd} \times n_{nodes}}_{\text{columns}})$$

$$\underline{c}^T \bar{\underline{K}}\,\bar{\underline{d}} = \underline{c}^T \bar{\underline{F}}^{int} + \underline{c}^T \underline{F}^{\gamma}$$

Scalar

$$\Rightarrow \underbrace{\underline{c}^T \underline{K}\,\underline{d}}_{\text{reduced}} = \underline{c}^T \bar{\underline{F}}^{int} + \underline{c}^T \underline{F}^{\gamma} - \underline{c}^T \underline{K}_{\bar{A}} d^{\bar{A}} - \underline{c}^T \underline{K}_{\bar{B}} d^{\bar{B}}$$

removing $\bar{A}$ & $\bar{B}$ dof from $\bar{\underline{d}}$

$\bar{A}$ & $\bar{B}$ columns of $\bar{\underline{K}}$.

$\forall\, \bar{A}, \bar{B} \in \bar{A}_D$.

$$\underline{c}^T \Big[\underset{\substack{\uparrow \\ (n_{sd} \times n_{nodes} - N_D)^2}}{\underline{K}}\, \underline{d} - \underbrace{\overline{\underline{F}}^{int} - \underline{F}^{\gamma} - \overline{\underline{K}}_{\bar{A}}\, d^{\bar{A}} - \overline{\underline{K}}_{\bar{B}}\, d^{\bar{B}}}_{\text{also } (n_{sd} \times n_{nodes} - N_D)} \Big] = 0 \Big\}$$

dimension of now square matrix $\underline{K}$.

$$\Rightarrow \forall\, \underline{c} \in \mathbb{R}^{n_{sd} \times n_{nodes} - N_D}$$

$$\Rightarrow \quad \underline{K}\,\underline{d} = \underbrace{\overline{\underline{F}}^{int} - \underline{F}^{\gamma} - \overline{\underline{K}}_{\bar{A}}\, d^{\bar{A}} - \overline{\underline{K}}_{\bar{B}}\, d^{\bar{B}}}_{}$$

$\underline{F}$ } forcing vector.

$$\boxed{\underline{K}\,\underline{d} = \underline{F}}$$ } Final Matrix-Vector Equation