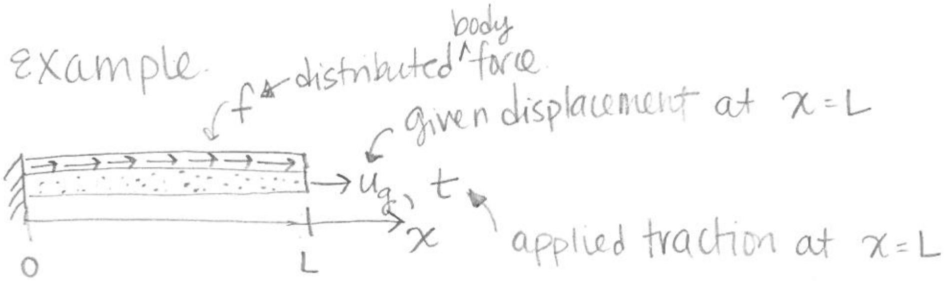
SUMMER 2015

$\boxed{01.01}$ Intro to Finite Element Methods

- Linear Elliptic Differential Equations in One Dimension

for example: (1) 1D Heat conduction @ steady state.
(2) 1D Mass diffusion at steady state.
(3) 1D Elasticity at steady state.
} ALL LINEAR

- Example



$\boxed{01.02}$

Find $\overset{\text{displacement field}}{u(x)}$: $(0, L) \longmapsto \mathbb{R}^1$ } defines a mapping.
specifies 1D space.

$\uparrow$ NOTE: This is an OPEN interval.

given $\underbrace{u \text{ @ } x=0 \text{ is } u_0}_{u(0) = u_0}$, u_g or t, $f(x)$,

& the constitutive relation: $\sigma = E u_{,x}$

such that the following holds:

$$\frac{d\sigma}{dx} + f = 0 \quad \text{in} \quad (0, L)$$ } This is our differential Equation

with the boundary conditions

$$\underset{\substack{\uparrow \\ \text{always holds}}}{u(0) = u_0} \quad \& \quad \underbrace{u(L) = u_g \text{ -OR- } \sigma(L) = t}_{\text{must be one or the other.}}$$

Boundary Conditions:

$u(0) = u_0$, $u(L) = u_g$ } ★ Dirichlet Boundary Conditions. (or displacement B.C.)

→ applied to the primal field of the problem (in this case, $u(x)$)

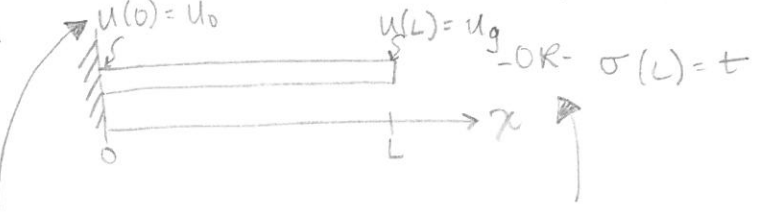
$\sigma(L) = t$ } ★ (or traction B.C.) Neumann Boundary Condition

→ applied to the spacial derivative of the primal field (σ in this case)

follows from the constitutive relation

$$\Rightarrow \left(E u_{,x}\right)\Big|_{x=L} = t$$

1.03



ALWAYS A DIRICHLET B.C @ $u(0)$

CAN BE EITHER DIRICHLET OR NEUMANN B.C.

★ WE DO NOT CONSIDER A CASE WHEN BOTH B.C. ARE NEUMANN! (@ either end)

CONSIDER: Neumann B.C.'s at $x=0$ & $x=L$

$$\sigma(0) = E u_{,x}\Big|_{x=0} = t_0$$

$$\sigma(L) = E u_{,x}\Big|_{x=L} = t_L$$

Consider $u(x)$ satisfying these B.C.'s and the differential equation:

$$\frac{d\sigma}{dx} + f = 0. \quad \text{in } (0, L)$$

$$\frac{d}{dx}\left(E u_{,x}\right) + f = 0$$

But $u(x) + \bar{u}$ is also a solution! ∴ no unique solution.

$\uparrow$ constant wrt x

The solution $u(x)$ is non-unique up to a constant displacemen field (i.e. rigid body motion), $\bar{u}$, which is a constant.

$\Rightarrow$ DIRICHLET B.C.'s guarantee a unique solution!

$\Rightarrow$ Neumann B.C.'s alone can be specified for the <u>time-dependent</u> elasticity problem (this is hyperbolic PDEs)

↖ but right now, we are only considering steady state.

(eliptic PDEs)

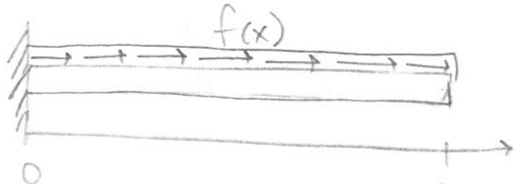
Term in context of elasticity.

Recall the <u>body</u> <u>force</u>, $f(x)$:

$$\frac{d\sigma}{dx} + f(x) = 0 \quad \text{in } (0, L)$$

The "forcing function" in general PDEs

e.g.



$\longrightarrow g$ (acceleration due to gravity)

rotated frame of reference.

density function.

$$f(x) = \rho(x)\, g$$

$\boxed{1.04}$ The differential equation:

$$\frac{d\sigma}{dx} + f(x) = 0 \quad \text{in} \quad (0, L)$$

☆ OPEN interval. (implies that the domain between $x=0$ & $x=L$, excluding the end points 0 & L themselves)

⇓

B.C.'s take care of these points. The problem would be over constrained if the interval were closed.

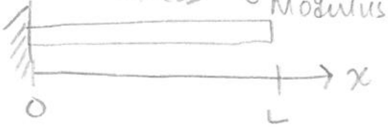
The <u>constitutive</u> <u>relation</u>:

primal field (u)

linearized elasticity — $\left[\ \sigma = E\, u_{,x}\right.$

Stress Young's Modulus.

Strain (often denoted by $\varepsilon = u_{,x}$)



} Domain

The 1D (scalar) linear eliptic problem also models heat or diffusive mass transport.

temperature

↳ Find $u(x): (0, L) \longmapsto \mathbb{R}^1$ given u_0 and u_g -OR- $\bar{j}$

f and the constitutive relation

$$j = -k u_{,x} \text{ such that}$$

heat flux

$$-\frac{dj}{dx} = f(x) \text{ in } (0, L)$$

negative divergence of the heat flux.

BOUNDARY CONDITIONS:

$$u(0) = u_0 \ \& \ \begin{Bmatrix} u(L) = u_g \\ \text{-OR-} \\ j(L) = -\bar{j} \end{Bmatrix}$$

- Dirichlet B.C.'s imply temperature boundary conditions.
- Neumann B.C.'s imply a heat flux boundary condition

$$j(L) = -\bar{j}$$

$\bar{j}$ = heat influx at $x = L$.

$\boxed{1.05}$ The Strong Form of a linear PDE of Eliptic type in one dimension.

Find $u(x)$ given u_0, u_g -OR- t, $f(x)$ and the constitutive relation

$$\sigma = E u_{,x}$$

such that

$$\frac{d\sigma}{dx} + f = 0 \quad \text{in} \quad (0, L)$$

w/ b.c.'s: $u(0) = u_0$ & $\begin{Bmatrix} u(L) = u_g \\ \text{-OR-} \\ \sigma(L) = t \end{Bmatrix}$

This is called the <u>STRONG FORM</u> of the equation.

→ condition must hold at <u>EVERY</u> point.

$$\Rightarrow \boxed{\frac{d}{dx}\left(E\frac{du}{dx}\right)} + f = 0 \qquad \text{in } (0, L)$$

NOTE: u should be a function such that two derivatives are possible (i.e. no discontinuities).

We require <u>STRONG CONDITIONS</u> of "smoothness" on u(x) because the strong form has <u>two</u> spatial derivatives.

We require the PDE to hold pointwise in the interval of interest: (0, L).

<u>AN ANALYTIC SOLUTION:</u>

$$\int_0^y \frac{d\sigma}{dx}dx = \int_0^{y \leftarrow \text{maintains general solution}} (-f\,dx) \qquad \text{where } y \text{ belongs to } [0,L]$$

$$y \in [0,L]$$

$$\Rightarrow \sigma(y) - \sigma(0) = -\int_0^y f\,dx$$

$$\Rightarrow E u_{,x}(y) - E u_{,x}(0) = -\int_0^y f\,dx$$

Write as $E\dfrac{du}{dy} = -\displaystyle\int_0^y f\,dx + E\frac{du}{dx}\Big|_0$

$$\int_0^z E\frac{du}{dy}dy = -\int_0^z\int_0^y f\,dx\,dy + \int_0^z E\frac{du}{dx}\Big|_0$$

$$\Rightarrow Eu(z) - Eu(0) = -\int_0^z\int_0^y f\,dx\,dy + E\frac{du}{dx}\Big|_0 z$$

$$u(z) = \frac{1}{E}\left[-\int_0^z\int_0^y f\,dx\,dy + \underbrace{E\frac{du}{dx}\Big|_0 z}_{\text{determined by B.C. @L}} + E\underbrace{u(0)}_{u_0}\right]$$

$\boxed{1.06}$ The Weak Form of a linear eliptic PDE in one dimension.

Find $u(x) \in \overset{\text{a space of functions}}{\mathcal{S}}$

$$\mathcal{S} = \{ u \mid \underbrace{u(0) = u_0}_{} \}$$

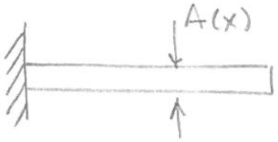
contains all u such that $u(0) = u_0$.
Basically, we bake in the Dirichlet B.C..

given u_0, t, f, and constitutive relation $\sigma = E u_{,x}$

such that for $\underbrace{\text{all } w \text{ belonging to } \mathcal{V}}_{\forall w \in \mathcal{V} = \{ w \mid \underbrace{w(0)=0}_{\text{homogeneous Dirichlet B.C.}} \}}$

THIS IS THE WEAK FORM

$$\left\{ \int_0^L w_{,x} \sigma \, dx = \int_0^L w f \, dx + w(L) t \right.$$



Multiply the above equation through by A(x), creating volume integrals.

$$\int_0^L w_{,x} \sigma \underbrace{A(x)\,dx}_{\text{Volume element}} = \int_0^L w f \underbrace{A(x)\,dx}_{\text{volume element}} + w(L) \underbrace{A(x) \cdot t}_{\text{Boundary force.}}$$

$\boxed{1.07}$ THE FINITE ELEMENT METHOD IS BASED ON THIS WEAK FORM.

It is also the basis of other variationally based numerical methods.

The Strong Form and Weak Form are equivalent!
↳ (each implies the other)

Consider the Strong Form:

$$\frac{d\sigma}{dx} + f = 0 \quad \text{in } (0, L)$$

<u>B.C.'s:</u> $u(0) = u_0$

$\sigma(L) = t$, where $\sigma = E\, u_{,x}$

Introduce $w \in \mathcal{V} = \{ w \mid w(0) = 0 \}$

W: weighting function

$$w \frac{d\sigma}{dx} + wf = 0$$

Multiply Strong form by w and itegrate over $(0, L)$

$$\int_0^L w \underbrace{A(x) \frac{d\sigma}{dx}}\, dx + \int_0^L w A(x) f \, dx = 0$$

INTEGRATE BY PARTS.

$$\boxed{- \int_0^L w_{,x} \sigma A(x) \, dx + w \sigma A(x) \Big|_0^L} + \int_0^L w A(x) f \, dx = 0$$

$$\int_0^L w_{,x} \sigma A(x) \, dx = \int_0^L w A(x) f \, dx + w \sigma A(x) \Big|_0^L$$

$$\int_0^L w_{,x} \sigma A(x) \, dx = \int_0^L w A(x) f \, dx + w(L) \sigma(L) A(L) - \underbrace{\cancelto{0}{w(0) \sigma(0)}}_{\text{Homogeneous Dirichlet}} A(0)$$

HOORAY THE WEAK FORM! →

$$\int_0^L w_{,x} \sigma A(x) \, dx = \int_0^L w A(x) f \, dx + w(L) \overset{\sigma(L) = t}{t} A(L)$$

$\boxed{1.08}$ Now go in reverse & start w/ the <u>WEAK FORM</u>.

Find $u \in \mathcal{S} = \{u \mid u(0) = u_0\}$ such that

$\forall\, w \in \mathcal{V} = \{w \mid w(0) = 0\}$

$$\underbrace{\int_0^L w_{,x}\, \sigma A(x)\, dx}_{\Big\downarrow \text{ Integrate by parts}} = \int_0^L w A(x) f\, dx + w(L)(t) A(L)$$

$$-\int_0^L \frac{d\sigma}{dx} w A\, dx + w\sigma A\Big|_0^L = \int_0^L w A(x) f\, dx + w(L)\sigma(L) A(L)$$

$$-\int_0^L \frac{d\sigma}{dx} w A(x)\, dx + w(L)\sigma\cancel{(L)}A(L) - \cancelto{0}{w(0)}\sigma(0) A(0)$$
$$= \int_0^L w A(x) f dx + w(L)(t) A(L)$$

$$-\int_0^L w A(x) f dy - \int_0^L \frac{d\sigma}{dx} w A(x)\, dx + w(L) A(\sigma(L) - t) = 0$$

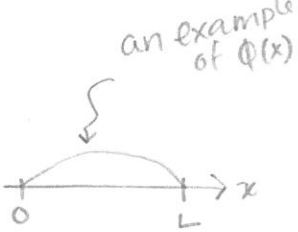
$$\int_0^L w A(x)\left[-f - \frac{d\sigma}{dx}\right] dx + w(L) A(\sigma(L) - t) = 0$$

This holds $\forall\ w \in \mathcal{V} = \{w \mid w(0) = 0\}$

↳ Also holds for $w(x) = \phi(x)(-\sigma_{,x} - f)$

where $\phi(x) > 0$ for $x \in (0, L)$

$\phi(x) = 0$ at $x = \{0, L\}$



ensures that $w(L)=0$ & $w(0)=0$

This gets rid of the term $w(L)\,A\,(\sigma(L)-t)$.

$$\Rightarrow \quad \int_0^L \phi(x)\,(-\sigma_{,x}-f)\,(-\sigma_{,x}-f)\,A\,dx = 0$$

$$\int_0^L \underbrace{\phi(x)}_{\substack{\text{This}\\ \text{is} >0\\ \text{for the}\\ \text{interior domain}}}\,\underbrace{(-\sigma_{,x}-f)^2}_{\substack{\text{must be}\\ >0 \text{ since}\\ \text{it is a}\\ \text{square.}}}\,A(x)\,dx = 0$$

This can only hold if $\underbrace{(-\sigma_{,x}-f)=0}$ for $(0,L)$

This is the STRONG FORM of the PDE. ✓

since strong form must hold → 0

Return to $$\int_0^L w\,(-\sigma_{,x}-f)\,A(x)\,dx + w(L)\,A(L)\,(\sigma(L)-t) = 0$$

$\forall\, w \in \mathcal{V}.$

$$\therefore \quad w(L)\,A(L)\,(\sigma(L)-t) = 0$$

$$\Downarrow$$

$\sigma(L) = t$ ← NEUMANN BC! ✓

(since $w(L)$ & $A(L) \neq 0$)

NOTE: The weak form already has the Dirichlet boundary condition baked in:

$$u \in \mathcal{S} = \{u \mid u(0) = u_0\}. \checkmark$$