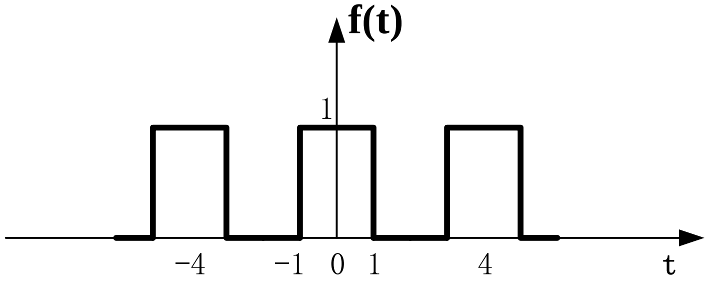
第五次 信号的频域分解

5.1、 用直接计算傅里叶系数的方法，写出下图所示周期函数三种形式的傅立叶级数并画双边频谱图。



$$T = 4, \qquad \Omega = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$F_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jn\Omega t}\,\mathrm{d}t = \frac{1}{4}\int_{-1}^{1} 1\cdot e^{-jn\frac{\pi}{2}t}\,\mathrm{d}t = \frac{1}{4}\frac{1}{\frac{-jn\pi}{2}}\left[e^{j\frac{n\pi}{2}} - e^{-j\frac{n\pi}{2}}\right] = \frac{1}{n\pi}\sin\frac{n\pi}{2} = \frac{1}{2}Sa\left(\frac{n\pi}{2}\right)$$

$$\text{直流}\,a_0 = F_0 = A_0 = \frac{1}{2}$$

$$a_n = 2 \times Re(F_n) = \frac{2}{n\pi}\sin\frac{n\pi}{2} \qquad n>0$$

$$b_n = -2 \times Im(F_n) = 0 \qquad n>0$$

$$A_n = 2|F_n| = \left|\frac{2}{n\pi}\sin\frac{n\pi}{2}\right| \qquad n>0$$

1.复指数形式：

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\Omega t} = \sum_{n=-\infty}^{\infty} \frac{1}{n\pi}\sin\frac{n\pi}{2} e^{j\frac{n\pi}{2}t}$$

2.三角形式：

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\Omega t) + \sum_{n=1}^{\infty} b_n\sin(n\Omega t)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{2}\cos\left(\frac{n\pi}{2}t\right)$$

3 标准三角形式

$$f(t) = A_0 + \sum_{n=0}^{\infty} a_n\cos(n\Omega t + \varphi_n)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\left|\frac{2}{n\pi}\sin\frac{n\pi}{2}\right|\cos\left(\frac{n\pi}{2}t + \varphi_n\right)$$

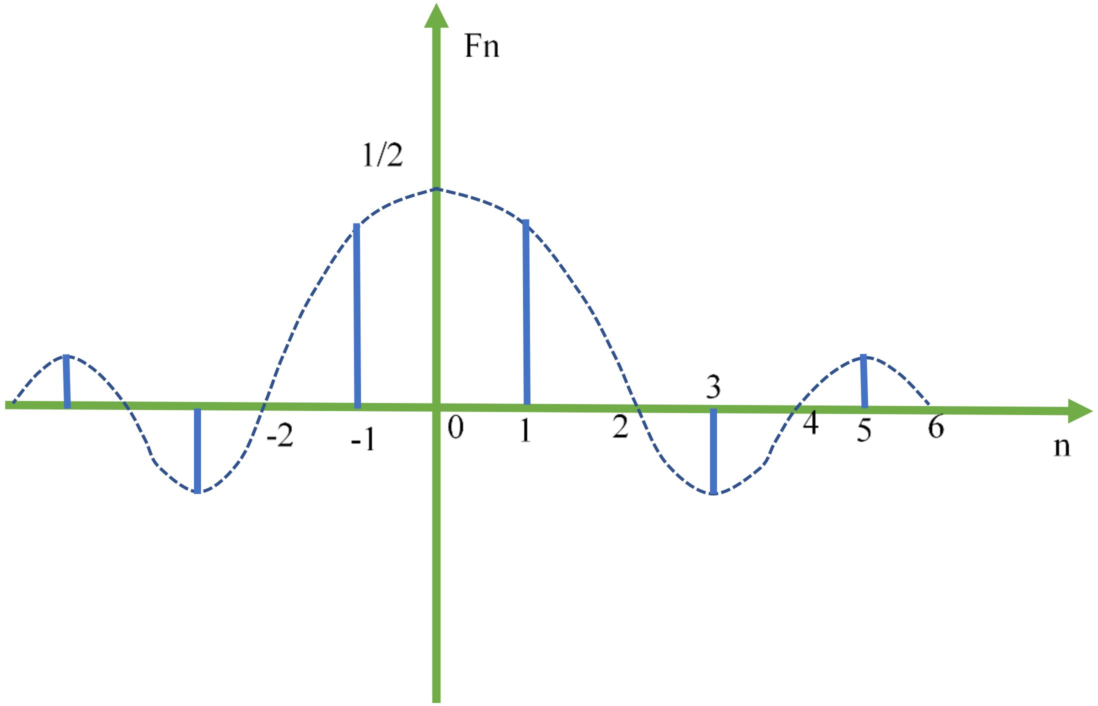
$$= \frac{1}{2} + \frac{2}{\pi}\cos\left(\frac{\pi}{2}t\right) - \frac{1}{\pi}\cos(\pi t) + \frac{2}{3\pi}\cos\left(\frac{3\pi}{2}t\right) - \frac{1}{2\pi}\cos(2\pi t) + \ldots$$

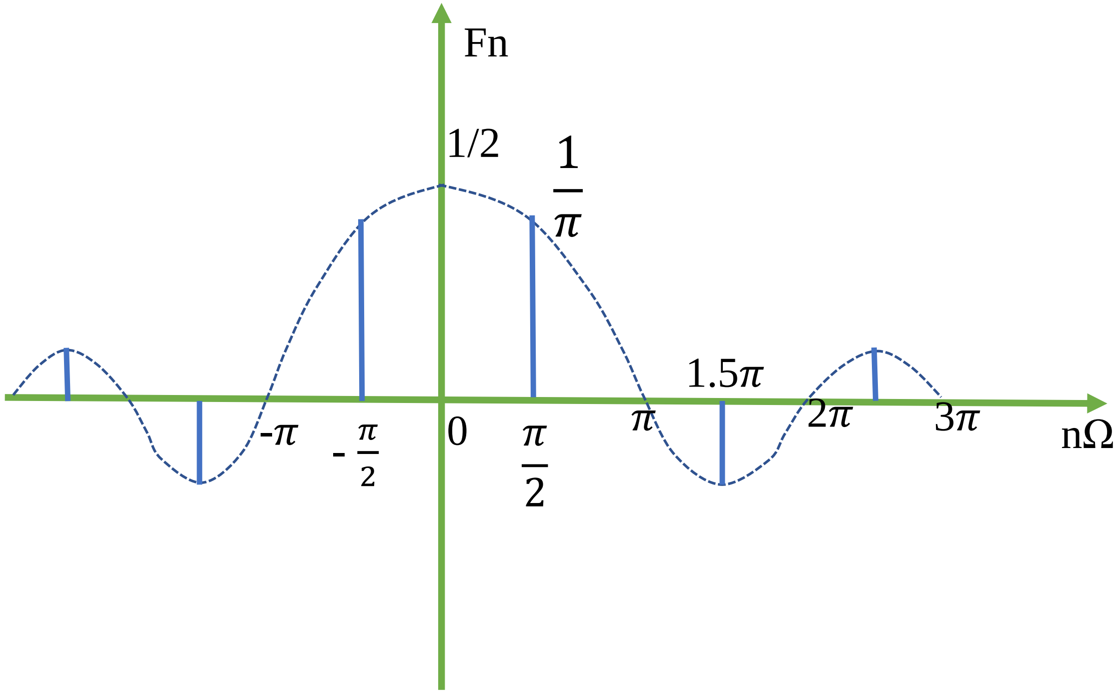
或

$$= \frac{1}{2} + \frac{2}{\pi}\cos\left(\frac{\pi}{2}t\right) + \frac{1}{\pi}\cos(\pi t + \pi)$$

$$+ \frac{2}{3\pi}\cos\left(\frac{3\pi}{2}t\right) + \frac{1}{2\pi}\cos(2\pi t + \pi) + \ldots.$$

$$F_n = \frac{1}{2} Sa\left(\frac{n\pi}{2}\right) \qquad T = 4, \qquad \Omega = \frac{2\pi}{4} = \frac{\pi}{2}$$





5.2、已知： $f(t)=\sum_{n=-\infty}^{\infty}\frac{\sin(n\pi/2)}{n}e^{j2n\pi t}$

1)求 f(t)的周期、直流分量为和频率为 5Hz 的谐波分量；
2)写出 f(t)的时域表达式并画出波形。

$$\Omega=\frac{2\pi}{T}=2\pi$$

$$T=1s$$

$$f(t)=\sum_{n=-\infty}^{\infty}\frac{\sin(n\pi/2)}{n}e^{j2n\pi t}=\frac{\pi}{2}\sum_{n=-\infty}^{\infty}\frac{\sin(n\pi/2)}{n\pi/2}e^{j2n\pi t}$$

$$F_0=\frac{\pi}{2}$$

频率 f=5Hz，即角频率为$2\pi\times5=10\pi$=$2n\pi$，所以 n=5

$$F_5=F_{-5}^{*}=\frac{\sin(5\pi/2)}{5}=0.2$$

$f(t)$ 的 五 次 谐 波 分 量 为 $F_5\times e^{j10\pi t}+F_{-5}\times e^{-j10\pi t}=$

$0.4\cos(10\pi t)$

$$F_n=\frac{1}{T}F_0(n\Omega)=\frac{1}{T}F_0(\omega)\Big|_{\omega=n\Omega}\text{，}\ \Omega=2\pi\text{，}\ T=1s$$

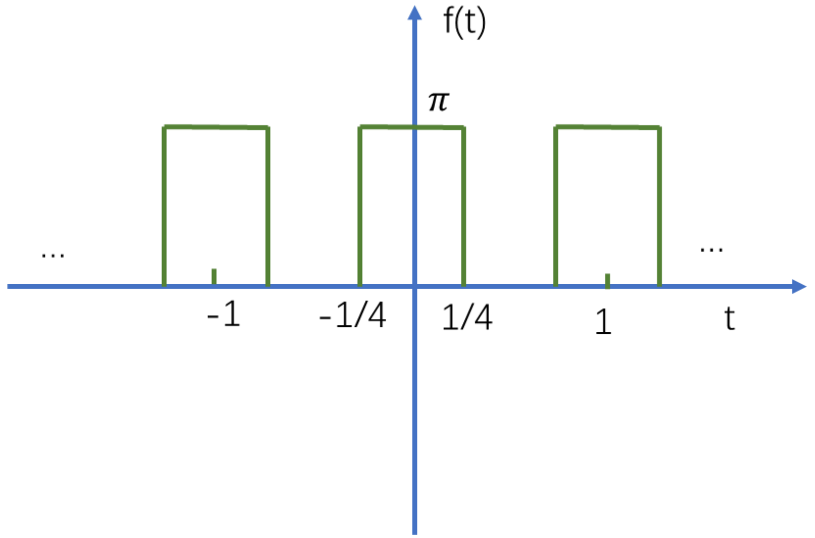
$$Fn=\frac{\pi}{2}\frac{\sin(n\pi/2)}{n\pi/2}=\pi\frac{1}{2}Sa\left(\frac{n\cdot2\pi\cdot0.5}{2}\right)$$

$$F_0(\omega)\Big|_{\omega=n\Omega}=T\cdot Fn=\frac{\pi}{2}Sa\left(\frac{n\cdot\pi}{2}\right)$$

所以$\tau=0.5$

$$G_\tau(t)\leftrightarrow\tau Sa\left(\frac{\omega\tau}{2}\right)$$

$$f_0(t)=\pi G_{0.5}(t)$$



5.3、f(t) 的周期为 0.1s、傅立叶级数系数$F_0 = 5$ $F_3 = F_{-3}^* = 3$ $F_5 = F_{-5}^* = 2j$、其余为 0。试写出此信号的三角表达式 f (t)。

$$T = 0.1s$$

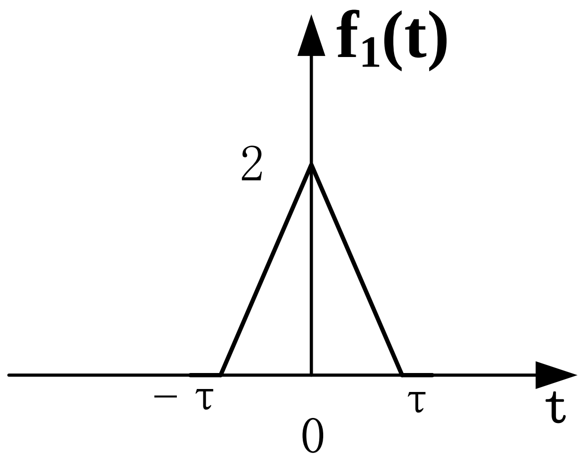
$$\Omega = \frac{2\pi}{T} = 20\pi$$

$$f(t) = 5 + 3e^{j3\times 20\pi} + 3e^{-j3\times 20\pi t} + 2je^{j5\times 20\pi t} - 2je^{-j5\times 20\pi t}$$

$$= 5 + 6\cos(60\pi t) + 2j \times 2\mathrm{j}\sin(100\pi t)$$

$$= 5 + 6\cos(60\pi t) - 4\sin(100\pi t)$$

5.4、 利用$g_\tau(t) \leftrightarrow \tau Sa(\omega\tau/2)$ 求题图所示各信号的傅里叶变换。



(a)

$$g_\tau(t) \leftrightarrow \tau Sa(\omega\tau/2)$$

$$G_\tau(\omega) \leftrightarrow \frac{1}{2\pi}\,\tau Sa(\frac{-t\tau}{2}) = \frac{1}{2\pi}\,\tau Sa(\frac{t\tau}{2})$$

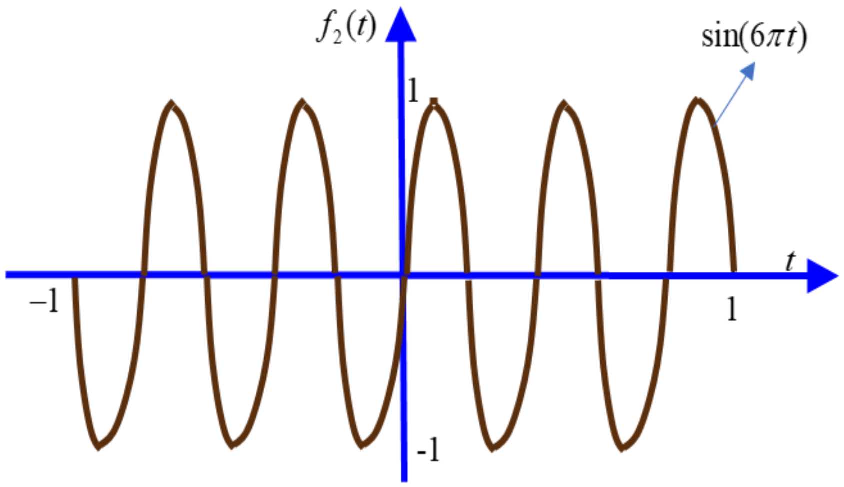
令$\omega_c = \frac{\tau}{2}$

$$G_{2\omega_c}(\omega) \leftrightarrow \frac{\omega_c}{\pi}\,Sa(\omega_c t\Big)$$

或 $$Sa(\omega_c t) \leftrightarrow \frac{\pi}{\omega_c} G_{2\omega_c}(\omega)$$

$$f_1(t) = \frac{2}{\tau} G_\tau(t) * G_\tau(t)$$

$$F_1(\omega) = \frac{2}{\tau}\left[\tau Sa\left(\frac{\omega\tau}{2}\right)\right]^2 = 2\tau\left[Sa\left(\frac{\omega\tau}{2}\right)\right]^2$$

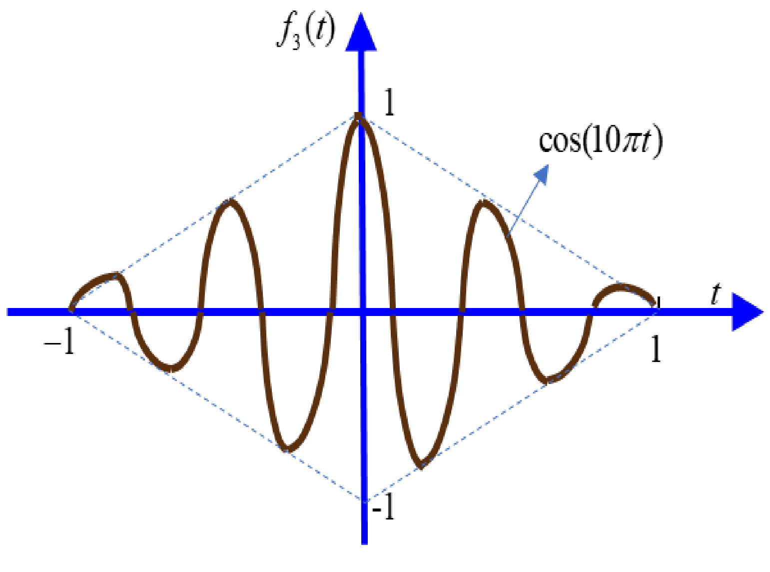


(b)

$$f_2(t) = \sin(6\pi t) \cdot [\varepsilon(t+1) - \varepsilon(t-1)] = \sin(6\pi t) \cdot G_2(t)$$

$$F_2(\omega) = \frac{1}{2j}\left[2Sa\left(\frac{2}{2}(\omega - 6\pi)\right) - 2Sa\left(\frac{2}{2}(\omega + 6\pi)\right)\right]$$

$$= \frac{1}{j}[Sa(\omega - 6\pi) - Sa(\omega + 6\pi)]$$



(c)

$$f_3(t) = G_1(t) * G_1(t) \cdot \cos(10\pi t)$$

$$F_3(\omega) = \frac{1}{2}\left[Sa^2\frac{\omega - 10\pi}{2} + Sa^2\frac{\omega + 10\pi}{2}\right]$$

5.5、若已知 $\mathscr{F}[f(t)] = F(\omega)$ ，试求下列函数的频谱

（1） $t\dfrac{df(t)}{dt}$

为书写方便，部分微分用求导符号代替

方法1：先频域微分： $(-jt)^n f(t) \longleftrightarrow F^{(n)}(\omega)$

$$-jtf(t) \leftrightarrow F'(\omega)$$

再时域微分： $f^{(n)}(t) \longleftrightarrow (j\omega)^n F(\omega)$

微分一次乘一个jw

$$[tf(t)]' \leftrightarrow j\omega jF'(\omega)$$

$$tf'(t) + f(t) \longleftrightarrow -\omega F'(\omega)$$

$$tf'(t) \longleftrightarrow -\omega F'(\omega) - F(\omega)$$

方法2：先时域微分

令 $f_1(t) = \dfrac{df(t)}{dt}$ $\quad f_1(t) \longleftrightarrow F_1(\omega) = j\omega F(\omega)$

先频域微分 $-jtf_1(t) \longleftrightarrow [F_1(\omega)]' = [j\omega F(\omega)]'$

$$= j\omega F'(\omega)] + jF(\omega)$$

$$tf_1(t) \longleftrightarrow -\omega F'(\omega)] - F(\omega)$$

（2） $(1-t)f(1-t)$ $-jtf(t) \longleftrightarrow F'(\omega)$ $f'(t) \longleftrightarrow j\omega F(\omega)$

方法1：先微分、时移，再反转

$$f_1(t) = tf(t) \longleftrightarrow F_1(\omega) = j\frac{dF(\omega)}{d\omega}$$

$$f_2(t) = f_1(t)|_{t=t+1} = (t+1)f(t+1)$$

$$F_2(\omega) = e^{j\omega} F_1(\omega) = je^{j\omega}\frac{dF(\omega)}{d\omega}$$

$$f_3(t) = f_2(-t)=(-t+1)f(-t+1)$$

$$F_3(\omega) = F_2(-\omega) = je^{-j\omega}\frac{dF(-\omega)}{-d\omega} = -je^{-j\omega}\frac{dF(-\omega)}{d\omega}$$

方法2：先频域微分、再反转再时移

$$f_1(t) = tf(t) \longleftrightarrow F_1(\omega) = j\frac{dF(\omega)}{d\omega}$$

$$f_2(t) = f_1(-t) = -tf(-t)$$

$$F_2(\omega) = jF_1(-\omega) = j\frac{dF(-\omega)}{-d\omega}$$

$$f_3(t) = f_2(t)|_{t=t-1} = -(t-1)f(-t+1)$$

$$F_3(\omega) = e^{-j\omega}F_2(\omega) = -je^{-j\omega}\frac{dF(-\omega)}{d\omega}$$

5.6、求函数$F(\omega) = [\varepsilon(\omega) - \varepsilon(\omega - 2)]e^{-j\omega}$ 的傅里叶逆变换.

方法1

$\tau=2$， $F_1(\omega) = G_2(\omega) \leftrightarrow \frac{1}{\pi} Sa(t) = f_1(t)$

$$[\varepsilon(\omega) - \varepsilon(\omega - 2)] = G_2(\omega - 1)$$

根据频移，$f(t)e^{j\omega_0 t} \leftrightarrow F[(\omega - \omega_0)]$

$$\frac{1}{\pi} Sa(t)e^{jt} \leftrightarrow G_2(\omega - 1)$$

根据时移，$f(t-t_0) \leftrightarrow e^{-j\omega t_0}F(\omega)$

$$\frac{1}{\pi} Sa(t)e^{jt}|_{t=t-1} \leftrightarrow e^{-j\omega}G_2(\omega - 1)$$

$$\frac{1}{\pi} Sa(t-1)e^{j(t-1)} \leftrightarrow e^{-j\omega}G_2(\omega - 1)$$

$$F(\omega) = [\varepsilon(\omega) - \varepsilon(\omega - 2)]e^{-j\omega} \leftrightarrow \frac{1}{\pi} Sa(t-1)e^{j(t-1)}$$

方法2：

$$F(\omega)=[\varepsilon(\omega)-\varepsilon(\omega-2)]e^{-j\omega}=e^{-j}\underline{e^{-j(\omega-1)}G_2(\omega-1)}$$

令$F_1(\omega)=\underline{e^{-j\omega}G_2(\omega)}$

$\Rightarrow$ 有$F(\omega)=e^{-j}F_1[(\omega-1)]$

由于 $G_2(\omega)\leftrightarrow\frac{1}{\pi}\mathrm{Sa}(t)$

$$F_1(\omega)=e^{-j\omega}G_2(\omega)\leftrightarrow f_1(t)=\frac{1}{\pi}Sa(t)|_{t=t-1}=\frac{1}{\pi}\mathrm{Sa}(t)$$

$$F(\omega)=e^{-j}F_1[(\omega-1)]\leftrightarrow e^{-j}f_1(t)e^{jt}$$
$$=\frac{1}{\pi}Sa(t-1)e^{j(t-1)}$$

方法3：直接用傅里叶逆变换公式

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{j\omega t}d\omega$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}[\varepsilon(\omega)-\varepsilon(\omega-2)]e^{-j\omega}e^{j\omega t}d\omega$$
$$=\frac{1}{2\pi}\int_{0}^{2}e^{j\omega(t-1)}d\omega=\frac{1}{2\pi}\frac{e^{2j(t-1)}-1}{j(t-1)}$$

根据 $\cos 2\alpha=1-2\sin^2\alpha,\sin 2\alpha=2\sin\alpha\cos\alpha$

可以推出和方法1和方法2一样的写法

$$=\frac{1}{\pi}Sa(t-1)e^{j(t-1)}$$

5.7、 利用能量等式$\int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(j\omega)|^2 d\omega$，计算积分的：

$$\int_{-\infty}^{\infty}[\frac{sin(t)}{t}]^2 dt$$

$$G_\tau(t) \leftrightarrow \tau Sa(\frac{\omega\tau}{2}) \quad \Rightarrow G_\tau(\omega) \leftrightarrow \frac{1}{2\pi}\tau Sa(\frac{t\tau}{2})$$

令 $\tau = 2$ 有$Sa(t) \leftrightarrow \pi G_2(\omega)$　　　　或者$Sa(\omega_c t) \leftrightarrow \frac{\pi}{\omega_c}G_{2\omega_c}(\omega)$

$$\int_{-\infty}^{\infty}[\frac{sin(t)}{t}]^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\pi G_2(\omega)|^2 d\omega = \frac{\pi^2}{2\pi}\int_{-1}^{1}|1|^2 d\omega = \pi$$

5.8、利用傅里叶变换求卷积 $f(t)=Sa(t)*Sa(2t)$。

$$Sa(\omega_c t)\overset{\boxed{}}{\leftrightarrow}\frac{\pi}{\omega_c}G_{2\omega_c}(\omega)$$

$$F(j\omega)=\frac{\pi}{1}G_2(\omega)\cdot\frac{\pi}{2}G_4(\omega)=\frac{\pi}{2}\cdot\frac{\pi}{1}G_2(\omega)$$

方法 1：根据$Sa(\omega_c t)\overset{\boxed{}}{\leftrightarrow}\frac{\pi}{\omega_c}G_{2\omega_c}(\omega)$

$$\frac{\omega_c}{\pi}Sa(\omega_c t)\overset{\boxed{}}{\leftrightarrow}G_{2\omega_c}(\omega)$$

所以有$f(t)=\frac{\pi}{2}\cdot\frac{\pi}{1}\frac{1}{\pi}Sa(t)=\frac{\pi}{2}Sa(t)$

或

因为$F(j\omega)=\frac{\pi}{2}\cdot\frac{\pi}{1}G_2(\omega)$，f(t)一定是 Sa(t)函数。

可以用$f(0)=f(t)|_{t=0}=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\omega)\,d\,\omega$来确定前面的系数

$$f(0)=f(t)|_{t=0}=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\omega)\,d\,\omega=\frac{1}{2\pi}\frac{\pi}{2}\cdot\frac{\pi}{1}\times 2=\frac{\pi}{2}$$

所以有$f(t)=\frac{\pi}{2}Sa(t)$