Compact Relative $\mathrm{SO}_{2,q}^0$-Character Varieties of Punctured Spheres

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张峻铭 Junming Zhang
(joint with 冯宇 Yu Feng)

Chern Institute of Mathematics, Nankai University

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Contents

Introduction

Basic Settings

- $\Sigma_{g,s}$ – the oriented surface of genus g with s punctures
- $\widetilde{\Sigma_{g,s}}$ – the universal cover of $\Sigma_{g,s}$
- $\chi(\Sigma_{g,s})$ – the Euler characteristic of $\Sigma_{g,s}$
- $\Gamma_{g,s} := \pi_1(\Sigma_{g,s})$ – the fundamental group of $\Sigma_{g,s}$

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- $\chi(\Sigma_{g,s})$ – the Euler characteristic of $\Sigma_{g,s}$
- $\Gamma_{g,s} := \pi_1(\Sigma_{g,s})$ – the fundamental group of $\Sigma_{g,s}$
- G – a real reductive Lie group
- $\mathfrak{g} := \mathrm{Lie}(G)$ – the Lie algebra of G
- H – a fixed maximal compact subgroup of G

Character Varieties

There exist $a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_s$ generating $\Gamma_{g,s}$ with the only relation

$$\prod_{i=1}^{g} [a_i, b_i] \cdot \prod_{j=1}^{s} c_j = 1.$$

Therefore, the set of all representations $\mathrm{Hom}(\Gamma_{g,s}, G)$ from $\Gamma_{g,s}$ to G can be viewed as a subvariety of G^{2g+s}. G acts on $\mathrm{Hom}(\Gamma_{g,s}, G)$ by the conjugation. Usually, $\mathrm{Hom}(\Gamma_{g,s}, G)/G$ is non-Hausdorff so we will consider its Hausdorffization $\mathfrak{X}(\Sigma_{g,s}, G)$.

Character Varieties

Definition

A representation $\rho\colon \Gamma_{g,s} \to G$ is called a **completely reducible representation** if $\mathrm{Ad} \circ \rho\colon \Gamma_{g,s} \to \mathrm{GL}(\mathfrak{g})$ decomposes as a direct sum of irreducible representations.

$\mathrm{Hom}^+(\Gamma_{g,s}, G)$ denotes the subspace of $\mathrm{Hom}(\Gamma_{g,s}, G)$ consisting of all completely reducible representations.

Definition

$\mathfrak{X}(\Sigma_{g,s}, G) := \mathrm{Hom}^+(\Gamma_{g,s}, G)/G$ is called the **(absolute) character variety**.

Question

Topological properties of $\mathfrak{X}(\Sigma_{g,s}, G)$*?*

Partial Results

The most classical situation is $g > 1, s = 0, G = \mathrm{PSL}_2\mathbb{R}$ coming from hyperbolic geometry, and there is a component called Teichmüller space.

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- (Hitchin, 1987) the topological type of each component of $\mathfrak{X}(\Sigma_{g,0}, \mathrm{PSL}_2\mathbb{R})$; the Betti numbers of each component of $\mathfrak{X}(\Sigma_{g,0}, \mathrm{PSL}_2\mathbb{C})$.

Euler Number and Toledo Invariant

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$$\mathbb{H}^2 \cong \mathrm{PSO}_2\mathbb{R} \backslash \mathrm{PSL}_2\mathbb{R}.$$

  The Kähler form induces a continuous cohomology class ω of degree 2 on $\mathrm{PSL}_2\mathbb{R}$. Now the pullback $\rho^*\omega$ gives an element in

$$\mathrm{H}^2_{\mathrm{c}}(\Gamma_{g,0}, \mathbb{R}) \cong \mathrm{H}^2(\Sigma_{g,0}, \mathbb{R})$$

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Note that the definition of $\mathrm{eu}(\rho)$ generalizes to representation $\rho\colon \Gamma_{g,0} \to G$ with the target Lie group G whose symmetric space $H\backslash G$ admitting a G-invariant Kähler form, i.e. G is of Hermitian type. This is called the **Toledo invariant** of ρ.

Non-compact Case

When the surface is non-compact, i.e. $s > 0$, Burger, Iozzi, and Wienhard gave a definition of Toledo invariant $\mathrm{Tol}(\rho)$ of a representation ρ from $\Gamma_{g,s}$ to an Hermitian Lie group G by using the bounded cohomology.

Theorem (Burger–Iozzi–Wienhard, 2010)

Let G be an Hermitian Lie group with its maximal compact subgroup H and $\rho\colon \Gamma_{g,s} \to G$ a representation.

(1) *The map $\mathrm{Tol}\colon \mathfrak{X}(\Sigma_{g,s}, G) \to \mathbb{R}$ is continuous.*

(2) *If $\Sigma_{g,s}$ is the connected sum of two connected surfaces Σ_1, Σ_2 along a separating loop, then*

$$\mathrm{Tol}(\rho|_{\pi_1(\Sigma_1)}) + \mathrm{Tol}(\rho|_{\pi_1(\Sigma_2)}) = \mathrm{Tol}(\rho).$$

Rotation Number

Recall c_j denotes the counterclockwise loop around the j-th puncture. When $G = \mathrm{PSL}_2\mathbb{R}$, to get an integer from the Toledo invariant when the surface is non-compact, one need to use the **rotation number**.
We define $\mathrm{rot}\colon \mathrm{PSL}_2\mathbb{R} \to \mathbb{R}$ maps $g \in \mathrm{PSL}_2\mathbb{R}$ to

$$\begin{cases} 0 & \text{if } g \text{ is hyperbolic or positive parabolic.} \\ 1 & \text{if } g \text{ is negative parabolic.} \\ (\text{rotation angle of } g)/2\pi & \text{if } g \text{ is elliptic.} \end{cases}$$

Definition

$\mathrm{Rot}(\rho) := \sum_{j=1}^{s} \mathrm{rot}(\rho(c_j))$ is called the **rotation number** of ρ.

Theorem (Burger–Iozzi–Wienhard, 2010)

For any representation $\rho\colon \Gamma_{g,s} \to \mathrm{PSL}_2\mathbb{R}$,

$$\mathrm{Tol}(\rho) + \mathrm{Rot}(\rho) \in \mathbb{Z}.$$

Deroin–Tholozan Components

Let us fix $\alpha = (\alpha_1, \ldots, \alpha_s) \in (0,1)^s$. We denote by $\mathfrak{X}_\alpha(\Sigma_{0,s}, \mathrm{PSL}_2\mathbb{R})$ the set of conjugacy classes of representations such that $\mathrm{rot}(\rho(c_j)) = \alpha_j$.

Theorem (Benedetto–Goldman for $s = 4$, 1999 & Deroin–Tholozan, 2019)

If $s - 1 < \sum_{j=1}^{s} \alpha_j < s$, then $\mathfrak{X}_\alpha(\Sigma_{0,s}, \mathrm{PSL}_2\mathbb{R})$ is diffeomorphic to $\mathbb{C}P^{s-3}$. Furthermore, any representation ρ in it has the following properties:

(1) $\mathrm{Tol}(\rho) + \mathrm{Rot}(\rho) = s - 1$;

(2) *ρ is* ***totally non-hyperbolic****, i.e. for any element γ in $\Gamma_{0,s}$ freely homotopic to a simple closed curve, $\rho(\gamma)$ is not hyperbolic.*

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here $\mathfrak{X}_\alpha(\Sigma_{0,s}, \mathrm{PSL}_2\mathbb{R}) \cong \mathbb{C}P^{s-3}$ is COMPACT!

Parabolic Higgs Bundles

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- (Mondello, 2018) Reproved the Deroin–Tholozan components are diffeomorphic to $\mathbb{C}P^{s-3}$ by using **parabolic $\mathrm{SL}_2\mathbb{R}$-Higgs bundle**. Moreover, he described the topology of components of the character variety that can contain monodromies of hyperbolic structure.

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Below we denote by $C(g)$ the conjugacy class of an element $g \in G$ in G.

Definition

For an s-tuple $h = (h_1, \ldots, h_s) \in G^s$, the **relative character variety** of h is defined as

$$\mathfrak{X}_h(\Sigma_{0,s}, G) := \{[\rho] \in \mathfrak{X}(\Sigma_{0,s}, G) \mid \rho(c_j) \in C(h_j)\}.$$

In other words, it consists of the representations with prescribed monodromies h around punctures.

Generalization to Hermitian Lie Groups

Theorem (Tholozan–Toulisse, 2021)

Let G be one of $\mathrm{SU}_{p,q}$, $\mathrm{Sp}_{2n}\mathbb{R}$, and SO^*_{2n}. For any $s \geqslant 3$, there exists a tuple $h = (h_1, \ldots, h_s) \in G^s$ such that the relative character variety $\mathfrak{X}_h(\Sigma_{0,s}, G)$ is ***compact*** *and satisfies the following properties:*

(1) *It consists of totally non-hyperbolic representations;*

(2) *It contains a Zariski-dense representation;*

(3) *For any $[\rho] \in \mathfrak{X}_h(\Sigma_{0,s}, G)$, there is a ρ-equivariant holomorphic map from $\widetilde{\Sigma_{0,s}}$ to $H \backslash G$.*

Why not $\mathrm{SO}^0_{2,q}$?

Classical semisimple Hermitian Lie group G	Compact relative G-character variety	Method
$\mathrm{SL}_2\mathbb{R} \cong \mathrm{SU}_{1,1} \cong \mathrm{SO}^0_{2,1}$	DT components	Geometric way & Parabolic G-Higgs bundle
$\mathrm{SU}_{p,q}$	TT components	Parabolic G-Higgs bundle
$\mathrm{Sp}_{2n}\mathbb{R}$		**Tight Embedding**
SO^*_{2n}		
$\mathrm{SO}^0_{2,q}$	Not Known	Not Known

However, the usual embedding $\mathrm{SO}^0_{2,q} \to \mathrm{SU}_{2,q}$ induces a totally real embedding of symmetric spaces, so the pushforward representations must have Toledo invariant 0.

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Method for $\mathrm{SO}^0_{2,q}$: Parabolic $\mathrm{SO}^0_{2,q}$-Higgs bundle

Our Main Results I

We prove the theorem below in the language of parabolic $\mathrm{SO}^0_{2,q}$-Higgs bundle first and then translate it into the language of representations through the non-Abelian Hodge correspondence.

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Theorem (Feng–Z., 2023)

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Main difference: Involving the orthogonal structure on the Higgs bundle.

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For our components, we prove the following theorem:

Theorem (Feng–Z., 2023)

The above $\mathfrak{X}_h(\Sigma_{0,s}, \mathrm{SO}^0_{2,q})$ *is homeomorphic to a projective GIT quotient of an* $\mathrm{SO}_2\mathbb{C} \times \mathrm{SO}_q\mathbb{C}$*-action with suitable linearization.*

Parabolic $\mathrm{SO}^0_{2,q}$-Higgs Bundles

Parabolic G-Higgs Bundles

Let X be a closed Riemann surface with finite marked points $\{x_i\}_{i=1}^s =: D$ on it (we will also use D to denote the divisor $\sum_{i=1}^s x_i$ on X). Any vector bundle or principal bundle we mention below is holomorphic. We denote by $\mathcal{K}$ the canonical line bundle of X.

The concept of parabolic G-Higgs bundle for general G was introduced by O. Biquard, O. García-Prada and I. Mundet i Riera.

We will explain below what a parabolic G-Higgs bundle is for $G = \mathrm{GL}_n\mathbb{C}$ and for our case $G = \mathrm{SO}^0_{2,q}$.

Notations

Definition

Suppose V is a $\mathbb{C}$-linear space. A subspace sequence of V

$$0 = F_k \subsetneq F_{k-1} \subsetneq \cdots \subsetneq F_1 = V, \quad (\text{resp. } 0 = F_1 \subsetneq F_2 \subsetneq \cdots \subset F_k = V)$$

is called a **reverse flag** (resp. **flag**). If V is equipped with a bilinear form Q, then the above reverse flag (resp. flag) is called a **reverse isotropic flag** (resp. **isotropic flag**) if every F_i is isotropic or coisotropic under Q and $F_i = (F_{k+1-i})^{\perp_Q}$.

When $G = \mathrm{GL}_n\mathbb{C}$

A parabolic $\mathrm{GL}_n\mathbb{C}$-Higgs bundle over (X, D) is equivalent to the following data:

(1) a holomorphic vector bundle $\mathcal{E} \to X$ of rank n;

(2) **(parabolic structure and parabolic weights)** a reverse flag $\left(\mathcal{E}_i^j\right)$ of $\mathcal{E}_{x_j}$ equipped with decreasing real numbers $\left(\alpha_i^j\right)$ satisfying that $\alpha_i^j \in [-1/2, 1/2]$ for every marked points $x_j \in D$;

(3) a global section Φ of $\mathrm{End}(\mathcal{E}) \otimes \mathcal{K}(D)$ such that

$$\Phi|_{X \setminus D} \in \mathrm{H}^0(X \setminus D, \mathrm{End}(\mathcal{E}) \otimes \mathcal{K}(D))$$

and with respect to a coordinate chart (U, z), a holomorphic frame $\{e_1, \ldots, e_n\}$ compatible with the reverse flag $\left(\mathcal{E}_i^j\right)$ near x_j,

$$\Phi = \left(O\left(z^{\left\lceil \alpha_k^j - \alpha_l^j \right\rceil}\right)\right)_{1 \leqslant k, l \leqslant n} \otimes \frac{\mathrm{d}z}{z}$$

.

The parabolic structure and parabolic weights tell us the direction and the order of the pole of Φ at D.

Parabolic Degree

Now we give the definition of the parabolic degree of a subbundle contained in a parabolic $\mathrm{GL}(n,\mathbb{C})$-Higgs bundle. It will be used to test the stability condition.

Definition

For any holomorphic subbundle $\mathcal{E}'$ of a parabolic $\mathrm{GL}(n,\mathbb{C})$-Higgs bundle $(\mathcal{E},\mathcal{E}^j_i,\alpha^j_i,\Phi)$, we define the **parabolic degree** of $\mathcal{E}'$ as

$$\mathrm{pardeg}(\mathcal{E}') := \deg(\mathcal{E}') - \sum_{j=1}^{s}\sum_{i=1}^{n}(\alpha^j_i - \alpha^j_{i-1})\dim\left((\mathcal{E}')_{x_j}\cap\mathcal{E}^j_i\right),$$

where we assume $\alpha^j_0 = 0$.

When $G = \mathrm{SO}_{2,q}^0$ I

A parabolic $\mathrm{SO}_{2,q}^0$-Higgs bundle is equivalent to the following data:

(1) the underlying bundle $\mathcal{E} = \mathcal{L}^\vee \oplus \mathcal{L} \oplus \mathcal{V}$, where $\mathcal{L}$ is a holomorphic line bundle, $\operatorname{rank} \mathcal{V} = q$ and $\det(\mathcal{V}) \cong \mathcal{O}$. Furthermore, $\mathcal{V}$ is equipped with a non-degenerate symmetric bilinear form $Q_{\mathcal{V}} \colon \mathcal{V} \otimes \mathcal{V} \to \mathcal{O}$ on $\mathcal{V}$, i.e. it induces an isomorphism $q_{\mathcal{V}} \colon \mathcal{V} \to \mathcal{V}^\vee$;

(2) chosen weights $-\alpha^j$ corresponding to $\mathcal{L}_{x_j}$ and a chosen reverse **isotropic** flag $\left(\mathcal{V}_i^j\right)$ of $\mathcal{V}_{x_j}$ with weights $\{\beta_i^j\}$ at each x_j such that $(\alpha, \beta) = \left(\alpha^j, -\alpha^j, \beta_i^j\right)$ satisfies that $\beta_i^j + \beta_{q+1-i}^j = 0$, $\alpha^j \in [0, 1/2]$ and $\beta_i^j < 1/2$ and β_i^j is non-increasing with respect to i;

When $G = \mathrm{SO}^0_{2,q}$ II

(3) a global section Φ of $\mathrm{End}(\mathcal{E}) \otimes \mathcal{K}(D)$ of the form

$$\begin{pmatrix} 0 & 0 & \eta \\ 0 & 0 & \gamma \\ -\gamma^* & -\eta^* & 0 \end{pmatrix} \in \mathrm{H}^0\left(X \setminus D, \mathrm{End}(\mathcal{E}) \otimes \mathcal{K}(D)\right)$$

under the decomposition $\mathcal{L}^\vee \oplus \mathcal{L} \oplus \mathcal{V}$ for meromorphic (around x_j) sections η, γ of $\mathrm{Hom}(\mathcal{V}, \mathcal{L}^\vee) \otimes \mathcal{K}(D)$ and $\mathrm{Hom}(\mathcal{V}, \mathcal{L}) \otimes \mathcal{K}(D)$ respectively, here

$$\eta = \left(O\left(z^{\lceil \alpha^j - \beta^j_l \rceil}\right)\right)_{1 \leqslant l \leqslant q} \otimes \frac{\mathrm{d}z}{z}, \quad \gamma = \left(O\left(z^{\lceil -\alpha^j - \beta^j_l \rceil}\right)\right)_{1 \leqslant l \leqslant q} \otimes \frac{\mathrm{d}z}{z}$$

over some local holomorphic coordinate (U, z) centered at x_j and with respect to the local holomorphic frame compatible with the chosen reverse isotropic flag.

Note that every parabolic $\mathrm{SO}^0_{2,q}$-Higgs bundle can be viewed as a parabolic $\mathrm{GL}_{2+q}\mathbb{C}$-Higgs bundle naturally.

(Semi-)Stability Condition

For general G, the stability condition of parabolic G-Higgs bundles involves holomorphic reductions and antidominant characters. Here we translate it into the language of vector bundles when $G = \mathrm{SO}^0_{2,q}$.

Proposition

A parabolic $\mathrm{SO}^0_{2,q}$-Higgs bundle $(\mathcal{E} = \mathcal{L}^\vee \oplus \mathcal{L} \oplus \mathcal{V}, \Phi)$ is semistable iff $\mathrm{pardeg}(\mathcal{U}') + \mathrm{pardeg}(\mathcal{V}') \leqslant 0$ for any isotropic subbundles $\mathcal{U}' \subset \mathcal{L}^\vee \oplus \mathcal{L}$, $\mathcal{V}' \subset \mathcal{V}$ satisfying $\mathcal{U}' \oplus \mathcal{V}'$ is Φ-invariant. Moreover, $(\mathcal{E}, \Phi)$ is stable iff the above inequality is strict when $\mathcal{V}'$ is a proper subbundle, i.e. $\mathcal{V}' \neq 0$.

Moduli Space of $\mathrm{SO}_{2,q}^0$-Higgs Bundles

Fix an $\mathrm{SO}_{2,q}^0$-weight $\tau = (\tau^j) = (\alpha^j, \beta^j)$. Let $\mathcal{M}(\alpha, \beta)$ be the moduli space of polystable parabolic $\mathrm{SO}_{2,q}^0$-Higgs bundles over (X, D) with parabolic weights τ.

Remark

This coincides with the S-equivalence classes of semistable parabolic $\mathrm{SO}_{2,q}^0$-Higgs bundles over (X, D) with parabolic weights τ.

Note that there is a continuous map

$$\begin{aligned} f\colon \mathcal{M}(\alpha, \beta) &\longrightarrow \mathbb{Z} \\ [(\mathcal{L}^\vee \oplus \mathcal{L} \oplus \mathcal{V}, \Phi)] &\longmapsto \deg(\mathcal{L}). \end{aligned}$$

Therefore, $\mathcal{M}(\alpha, \beta)$ can be decomposed into $\coprod_{d \in \mathbb{Z}} \mathcal{M}(\alpha, \beta, d)$, where $\mathcal{M}(\alpha, \beta, d) := f^{-1}(d)$.

Non-Abelian Hodge Correspondence

Now for an arbitrary $\mathrm{SO}_{2,q}^0$-weight $\tau = (\alpha, \beta)$, we define

$$h(\alpha, \beta) := \left(\exp(2\pi \mathrm{i} \cdot \tau^j)\right)_{j=1}^{s}.$$

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Theorem (Biquard–García-Prada–Mundet i Riera, 2020)

For any $\mathrm{SO}^0_{2,q}$-weight (α, β) such that $\alpha^j \neq \beta^j_i$ for any i, j, there exists a homeomorphism

$$\mathsf{NAH}\colon \mathcal{M}(\alpha, \beta) \longrightarrow \mathfrak{X}_{h(\alpha,\beta)}\left(\Sigma_{g,s}, \mathrm{SO}^0_{2,q}\right).$$

Through this correspondence, stable, simple Higgs bundles, which are also stable as parabolic $\mathrm{SO}_{2+q}\mathbb{C}$-Higgs bundle, are mapped into irreducible representations.

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For a parabolic $\mathrm{SO}^0_{2,q}$-Higgs bundle $(\mathcal{L}^\vee \oplus \mathcal{L} \oplus \mathcal{V}, \Phi)$,

$$\mathrm{Tol}\left(\mathsf{NAH}\left([(\mathcal{L}^\vee \oplus \mathcal{L} \oplus \mathcal{V}, \Phi)]\right)\right) = 2 \cdot \mathrm{pardeg}(\mathcal{L}).$$

Hitchin Fibration

Definition

Hitchin fibration is defined as

$$\begin{aligned}\Pi_{Hit} \colon \mathcal{M}(\alpha,\beta) &\longrightarrow \bigoplus_{i=1}^{q+2} \mathrm{H}^0(X, \mathcal{K}(D)^i) \\ \big[(\mathcal{E},\Phi)\big] &\longmapsto \big(\mathrm{tr}(\Phi^i)\big)_{i=1}^{q+2}.\end{aligned}$$

It is well-known that:

Theorem

Π_{Hit} *is proper, i.e. the preimage of a compact subset is compact.*

Sketch of Proof

Settings

We assume $X = \mathbb{C}P^1$ be the complex projective line and consider a parabolic $\mathrm{SO}^0_{2,q}$-Higgs bundle $(\mathcal{E} = \mathcal{L}^\vee \oplus \mathcal{L} \oplus \mathcal{V}, \Phi)$ with non-degenerate bilinear form $Q_\mathcal{V}$ on $\mathcal{V}$ and weight $\tau = (\tau^j)$ corresponds to the $\mathrm{SO}^0_{2,q}$-weight (α, β) at x_j, and

$$\Phi = \begin{pmatrix} 0 & 0 & \eta \\ 0 & 0 & \gamma \\ -\gamma^* & -\eta^* & 0 \end{pmatrix}.$$

We define

$$|\alpha| := \sum_{j=1}^{s} \alpha^j, \quad \left|\beta^j\right| := \sum_{\{i|\beta_i^j \geqslant 0\}} \beta_i^j, \quad |\beta| := \sum_{j=1}^{s} \left|\beta^j\right|.$$

Now for any $[(\mathcal{L}^\vee \oplus \mathcal{L} \oplus \mathcal{V}, \Phi)] \in \mathcal{M}(\alpha, \beta, d)$, we know that

$$\mathrm{pardeg}(\mathcal{L}) = d + |\alpha|.$$

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Step 2: Construct stable parabolic $\mathrm{SO}^0_{2,q}$-Higgs bundle with weights (α, β). Recall that suitable rotation numbers α_j are chosen to satisfy that

$$s - 1 < \sum_{j=1}^{s} \alpha_j < s.$$

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$$s - 1 < \sum_{j=1}^{s} \alpha_j < s.$$

Our "nice" weights (α, β) will be chosen to satisfy that

$$\alpha^j > |\beta^j| \text{ for any } j \;\; \& \;\; |\alpha| + |\beta| < 1.$$

If we set $\alpha^j > |\beta^j|$, this implies that $\alpha^j > \beta_1^j$ for all j in particular. Recall that around x_j, we have that

$$\eta = \left(O\left(z^{\lceil \alpha^j - \beta_l^j \rceil - 1}\right)\right)_{1 \leqslant l \leqslant q} \mathrm{d}z, \quad \gamma = \left(O\left(z^{\lceil -\alpha^j - \beta_l^j \rceil - 1}\right)\right)_{1 \leqslant l \leqslant q} \mathrm{d}z,$$

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then

$$\eta \in \mathrm{H}^0\big(X, \mathrm{Hom}(\mathcal{V}, \mathcal{L}^\vee) \otimes \mathcal{K}\cancel{(D)}\big)$$

and γ can be taken as ANY section in $\mathrm{H}^0\big(X, \mathrm{Hom}(\mathcal{V}, \mathcal{L}) \otimes \mathcal{K}(D)\big)$.

Compactness Criterion

Proposition (Feng–Z., 2023)

For any $\mathrm{SO}^0_{2,q}$-weight (α, β) satisfying $\alpha^j > \left|\beta^j\right|$ and $|\alpha| + |\beta| < 1$, if a semistable parabolic $\mathrm{SO}^0_{2,q}$-Higgs bundle $(\mathcal{E}, \Phi) \in \mathcal{M}(\alpha, \beta)$, then η vanishes identically. Moreover, $\mathcal{M}(\alpha, \beta)$ is (maybe empty) compact.

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Idea: Suppose $\eta \neq 0$. Let N and $I \otimes \mathcal{K}$ be the subsheaves of $\mathcal{V}$ and $\mathcal{L}^\vee \otimes \mathcal{K}$ respectively given by the kernel and the image of η. Then use the following short exact sequence of sheaves

$$0 \longrightarrow N \longrightarrow \mathcal{V} \longrightarrow I \otimes \mathcal{K} \longrightarrow 0$$

and the semistability to deduce a contradiction.

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Spoiler: This will be interpreted more concretely for general Lie groups of Hermitian type in the next talk.

Determine the Underlying Bundle

Proposition (Feng–Z., 2023)

For any $\mathrm{SO}_{2,q}^0$*-weight* (α, β) *satisfying* $\alpha^j > \left|\beta^j\right|$ *and* $|\alpha| + |\beta| < 1$*, if* $(\mathcal{L}^\vee \oplus \mathcal{L} \oplus \mathcal{V}, \Phi, \alpha, \beta)$ *is a semistable parabolic* $\mathrm{SO}_{2,q}^0$*-Higgs bundle, then* $\mathcal{L} \cong \mathcal{O}(-1)$ *and* $\mathcal{V} \cong \mathcal{O}^{\oplus q}$.

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Idea: Use the semistability to calculate the degree.

Determine the Reverse Isotropic Flag

Note that when α, β are fixed, the parabolic structure on $\mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus q}$ is uniquely determined by s isotropic flags

$$\left(F_i^j\right)_{j=1}^s = \left(\left(\left(\mathcal{O}^{\oplus q}\right)_i^j\right)^{\perp}\right)_{j=1}^s$$

which correspond to the reverse isotropic flags $\left(\left(\mathcal{O}^{\oplus q}\right)_i^j\right)_{j=1}^s$ at s marked points.

Denote $\left(F_i^j\right)_{j=1}^s$ by F.

Determine the Higgs Field

Since $\mathrm{Hom}(\mathcal{O}^{\oplus q}, \mathcal{O}(-1)) \otimes \mathcal{K}(D) \cong \mathrm{Hom}(\mathcal{O}^{\oplus q}, \mathcal{O}) \otimes \mathcal{O}(s-3)$, by choosing a basis $\{e_1, \ldots, e_{s-2}\}$ of $\mathrm{H}^0(X, \mathcal{O}(s-3))$, we can get a bijection

$$\begin{aligned} \left(\mathbb{C}^{1\times q}\right)^{s-2} &\longrightarrow \mathrm{H}^0(X, \mathrm{Hom}(\mathcal{O}^{\oplus q}, \mathcal{O}(-1)) \otimes \mathcal{K}(D)) \\ \mathrm{A} = (A_i)_{i=1}^{s-2} &\longmapsto \sum_{i=1}^{s-2} A_i \otimes e_i = \gamma_{\mathrm{A}}. \end{aligned}$$

Therefore, every parabolic $\mathrm{SO}_{2,q}^0$-Higgs bundle of weights (α, β) 1-1 corresponds to an (A, F).

Linear-Algebraic Interpretation

By using the above interpretation, we prove that there is a linear-algebraic (semi-)stability for (A, F) which is equivalent to the (semi-)stability of its corresponding Higgs bundle. And it is easy to show that we can construct a stable (A, F) for our linear-algebraic (semi-)stability when $s \geqslant q + 2$. Hence, we get the following corollary.

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Corollary (Feng–Z., 2023)

If $s \geqslant q+2$, then for any $\mathrm{SO}^0_{2,q}$-weight (α, β) satisfying $\alpha^j > \left|\beta^j\right|$ and $|\alpha| + |\beta| < 1$, there exists a $\gamma \in \mathrm{H}^0(X, \mathrm{Hom}(\mathcal{O}^{\oplus q}, \mathcal{O}(-1)) \otimes \mathcal{K}(D))$ such that the $\mathrm{SO}^0_{2,q}$-Higgs bundle of weight (α, β) determined by it is stable. Moreover, $\mathcal{M}(\alpha, \beta)$ is non-empty.

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If $s \geqslant q+2$, then for any $\mathrm{SO}_{2,q}^0$-weight (α, β) satisfying $\alpha^j > \left|\beta^j\right|$ and $|\alpha| + |\beta| < 1$, there exists a $\gamma \in \mathrm{H}^0(X, \mathrm{Hom}(\mathcal{O}^{\oplus q}, \mathcal{O}(-1)) \otimes \mathcal{K}(D))$ such that the $\mathrm{SO}_{2,q}^0$-Higgs bundle of weight (α, β) determined by it is stable. Moreover, $\mathcal{M}(\alpha, \beta)$ is non-empty.

The linear-algebraic stability can be interpreted as some GIT stability. Moreover, this gives the required GIT construction of the moduli space.

Proof of Our Main Results when $s \geqslant q+2$

Define

$$\mathcal{W} := \left\{(\alpha,\beta) \text{ is an } \mathrm{SO}^0_{2,q}\text{-weight} \mid \alpha^j > \left|\beta^j\right|, \forall 1 \leqslant j \leqslant s, |\alpha| + |\beta| < 1\right\}.$$

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Step 1: Existence of compact relative components

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Step 1: Existence of compact relative components

Through non-Abelian Hodge correspondence, we obtain

Theorem (Feng–Z., 2023)

Assume $s \geqslant q+2$. If $(\alpha, \beta) \in \mathcal{W}$, then the relative component

$$\mathfrak{X}_{h(\alpha,\beta)}^{|\alpha|-1}(\Sigma_{0,s}, \mathrm{SO}_{2,q}^0)$$

is compact, non-empty, and contains an irreducible representation.

Proof of Our Main Results when $s \geqslant q+2$

Step 2: Existence of dense representation

Proof of Our Main Results when $s \geqslant q+2$

Step 2: Existence of dense representation

It comes from the following result.

Theorem (Winkelmann, 2002)

Let G be a connected semisimple real Lie group. There exists an open neighbourhood W of the identity element in G and for every $k \geqslant 2$ a subset $Z_k \subset W^k$ of measure zero such that the subgroup generated by $g_1, g_2, \ldots, g_k$ in G is dense in G for all $(g_1, g_2, \ldots, g_k) \in W^k \setminus Z_k$.

Proof of Our Main Results when $s \geqslant q + 2$

Step 3: Holomorphic ρ-equivariant map

It follows from the non-Abelian Hodge correspondence and the complex structure of $(\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_q\mathbb{R})\backslash\mathrm{SO}^0_{2,q}$ directly.

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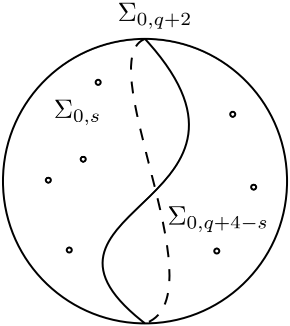
Step 4: Totally non-hyperbolicity
It follows from the holomorphicity of the corresponding harmonic map of the parabolic Higgs bundle, the contraction property of holomorphic maps and the equivalence between the Kobayashi distance and the Killing distance on $(\mathrm{SO}_2\mathbb{R} \times \mathrm{SO}_q\mathbb{R})\backslash\mathrm{SO}^0_{2,q}$.

Proof of Our Main Results when $s \geqslant 3$

Now we try to deduce our main results from the results for $s \geqslant q+2$ by restricting the representations to the subsurface. Assume $3 \leqslant s < q+2$.



There is an restriction map

$$\begin{aligned} \mathrm{Res}\colon \mathfrak{X}(\Sigma_{0,q+2}, \mathrm{SO}^0_{2,q}) &\longrightarrow \mathfrak{X}(\Sigma_{0,s}, \mathrm{SO}^0_{2,q}) \\ [\rho] &\longmapsto \left[\rho|_{\pi_1(\Sigma_{0,s})}\right]. \end{aligned}$$

Proof of Our Main Results when $s \geqslant 3$

Now let Ω' be the open subset in $\mathfrak{X}(\Sigma_{0,q+2}, \mathrm{SO}^0_{2,q})$ we constructed, and then define $\Omega'' \subset \Omega'$ to be the non-empty open subset in Ω' such that

$$\rho(\text{the cut curve})$$

is diagonalizable with distinct eigenvalues.

Theorem (Feng–Z., 2023)

For every class of representation $[\rho]$ in the domain

$$\mathrm{Res}(\Omega'') \subset \mathfrak{X}(\Sigma_{0,s}, \mathrm{SO}^0_{2,q}),$$

the connected component of $[\rho]$ in its relative character variety is compact and contained in $\mathrm{Res}(\Omega'')$.

Then one can check that every class of representation $[\rho]$ in $\mathrm{Res}(\Omega'')$ satisfies the properties we require.

Further Questions

Symplectic Geometry Viewpoint and Mapping Class Group Action

When $G = \mathrm{PSL}_2\mathbb{R}$, the Deroin–Tholozan components are studied more symplectic-geometrically.

- (Deroin–Tholozan, 2019) Using the Atiyah–Bott–Goldman symplectic form to determine the topology of the Deroin–Tholozan components.
- (Maret, 2022) Ergocity of the mapping class group action.
- (Maret, 2024) The action-angle coordinates, which are almost global Darboux coordinates.
- (Bouilly–Faraco–Maret, 2024) Minimality of mapping class group action on infinite orbits.
- (Bronstein–Maret， 2024) Finite orbits of mapping class group action.

Natural (naive) Question: Generalize these to higher rank.

Comparison with Higher Teichmüller Components

$$\Theta\text{-positive} \implies \Theta\text{-Anosov} \implies \text{totally hyperbolic}$$

$$\text{Deroin–Tholozan, and higher rank analogue} \implies \text{totally non-hyperbolic}$$

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Remark

There are already lots of studies on the spectrum distribution of Fuchsian representations or Anosov representations.

Existence of Other Possible Compact Relative Components

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- (Wu, 2025) Higher genus surface, $G = \mathrm{SU}_{p,q}$.
- (García-Prada–González–Z., in progress) Punctured sphere, general Lie groups of Hermitian type.

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Question: Title of this page.

Thank you!