# Anosov property of cyclic $SO_0(2,3)$ -Higgs Bundles (arXiv:2406.08118)

### 张峻铭 Junming Zhang

Chern Institute of Mathematics, Nankai University

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### Classical Teichmüller Theory

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We can view  $\mathcal{T}(S)$  as a connected component of the character variety

$$\operatorname{Hom}^{\operatorname{red}}(\pi_1(S),\operatorname{PSL}(2,\mathbb{R}))/\operatorname{PSL}(2,\mathbb{R}).$$

Moreover,  $\mathcal{T}(S)$  is one of the two connected components which consist entirely of discrete and faithful representations. The other one is  $\mathcal{T}(\overline{S})$ , where  $\overline{S}$  denotes the surface S with the opposite orientation.

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and so on.

In general, the set of discrete and faithful representations is only closed and NOT open.

#### Definition

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We give two famous examples of higher Teichmüller space here.

• (Hitchin; Labourie; Fock–Goncharov) Hitchin components  $\mathcal{T}_{Hit}(S,G)$  for real split G, such as  $\mathrm{SL}(n,\mathbb{R})$ ,  $\mathrm{SO}_0(n,n+1)$ ,  $\mathrm{Sp}(2n,\mathbb{R})$  and so on;

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When  $G = \mathrm{SL}(2,\mathbb{R})$ , the above components coincide with the classical Teichmüller space.

### **Anosov Property**

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The original definition is highly dynamical and we will use an interpreted definition (which is proven equivalent to the original one by Kapovich–Leeb–Porti and Bochi–Potrie–Sambarino).

Also, we will assume G is a semi-simple Lie subgroup of  $\mathrm{SL}(n,\mathbb{C})$  here to avoid involving a Lie-theoretic description.

#### Definition

 $\rho \colon \pi_1(S) \to G$  is  $P_k$ -Anosov if there exist constants D, L > 0 such that

$$\log \frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geqslant D \cdot d_{\mathbb{H}^2}(x_0, \gamma \cdot x_0) - L, \forall \gamma \in \pi_1(S),$$

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### Anosov $\Longrightarrow$ discrete + faithful.

Anosov property is open but not closed. However, all representations in the known higher Teichmüller spaces are Anosov.

Non-Abelian Hodge Correspondence

### Higgs bundles

The Higgs bundle is a useful tool to study the higher Teichmüller space. It is usually used to give a parametrization of the higher Teichmüller space. We fix a complex structure on S such that it becomes a Riemann surface X. Let  $\mathcal{K}_X$  be its canonical line bundle.

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#### Definition

A ( $\mathrm{GL}(n,\mathbb{C})$ -)Higgs bundle over X is a pair  $(\mathcal{E},\Phi)$  consisting of the following data:

- a holomorphic vector bundle  $\mathcal{E}$  over X with  $rank(\mathcal{E}) = n$ ;
- a holomorphic section  $\Phi \in H^0(X, \operatorname{End}(\mathcal{E}) \otimes \mathcal{K}_X)$  called **Higgs field**.

The non-Abelian Hodge correspondence exhibit a homeomorphism between the following moduli spaces:

{reductive representation  $\rho \colon \pi_1(S) \to \mathrm{GL}(n,\mathbb{C})$ }/ $\mathrm{GL}(n,\mathbb{C})$ 

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If we equip additional structure on these objects, we can get the non-Abelian Hodge correspondence for general reductive Lie group G.

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### Theorem (Hitchin-Simpson)

If  $(\mathcal{E}, \Phi)$  is a polystable Higgs bundle with  $\deg(\mathcal{E}) = 0$ , then there exists an Hermitian metric h on  $\mathcal{E}$  such that

$$F(\nabla^h) + [\Phi, \Phi^{*_h}] = 0,$$
 (1)

where  $\nabla^h$  is the Chern connection of the metric h,  $F(\nabla^h)$  denotes its curvature form and  $\Phi^{*_h}$  is the adjoint of  $\Phi$  with respect to h. Moreover, if  $(\mathcal{E}, \Phi)$  is stable, then such h is unique up to a constant scalar.

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If h solves (1), then

$$\nabla^h + \Phi + \Phi^{*_h}$$

gives a flat connection.



From Higgs Bundle to Anosov Representation

Higgs bundle Hitchin's self-dual equation Anosov property?

# Example: Hitchin Component in Higgs Bundle Viewpoint

The Hitchin component for  $SL(3,\mathbb{R})$  consisting of entirely the Higgs bundles of the following form:

$$\mathcal{E} = \mathcal{K}_X \oplus \mathcal{O} \oplus \mathcal{K}_X^{\vee},$$

$$\begin{pmatrix} 0 & q_2 & q_3 \end{pmatrix}$$

$$\Phi = \begin{pmatrix} 0 & q_2 & q_3 \\ 1 & 0 & q_2 \\ 0 & 1 & 0 \end{pmatrix},$$

where  $1: \mathcal{K}_X \to \mathcal{O} \otimes \mathcal{K}_X$  and  $1: \mathcal{O} \to \mathcal{K}_X^{\vee} \otimes \mathcal{K}_X$  are the natural isomorphisms and  $q_i \in \mathrm{H}^0(X, \mathcal{K}_X^i)$ .

It corresponds to the component containing the embedding of Fuchsian representations through the unique irreducible  $SL(2,\mathbb{R}) \to SL(3,\mathbb{R})$ .

# Special $SO_0(2,3)$ -Higgs Bundles

Below we consider the Higgs bundle whose underlying bundle is

$$\mathcal{E} = \mathcal{L}_{-2} \oplus \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2,$$

where  $\mathcal{L}_i$  are line bundles with  $\mathcal{L}_i\cong\mathcal{L}_{-i}^ee$  and  $\mathcal{L}_0\cong\mathcal{O}$  and

$$\Phi = \begin{pmatrix} 0 & 0 & 0 & \gamma & 0 \\ \tau & 0 & 0 & 0 & \gamma \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & \tau & 0 \end{pmatrix}.$$

 $(\mathcal{E},\Phi)$  is a cyclic  $\mathrm{SO}_0(2,3)$ -Higgs bundle and we denote it by

$$\mathcal{L}_{-2} \xrightarrow[\tau]{\gamma} \mathcal{L}_{-1} \xrightarrow[\beta]{\gamma} \mathcal{L}_{0} \xrightarrow[\beta]{\gamma} \mathcal{L}_{1} \xrightarrow[\tau]{\gamma} \mathcal{L}_{2}$$

If the cyclic Higgs bundle

$$\mathcal{L}_{-2} \xrightarrow{\tau} \mathcal{L}_{-1} \xrightarrow{\kappa} \mathcal{L}_{0} \xrightarrow{\beta} \mathcal{L}_{1} \xrightarrow{\tau} \mathcal{L}_{2}$$
 (2)

is (semi-)stable, then the Milnor-Wood inequality implies that

$$|\deg(\mathcal{L}_1)| \leqslant \deg(\mathcal{K}_X).$$

When "=" holds, the Higgs bundle corresponds to a maximal representation and maximal representations are known to be  $P_1$ -Anosov.

If the cyclic Higgs bundle

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When "=" holds, the Higgs bundle corresponds to a maximal representation and maximal representations are known to be  $P_1$ -Anosov.

Such Higgs bundle is called  $\alpha_1$ -cyclic if  $\tau\colon \mathcal{L}_1\to \mathcal{L}_2\otimes \mathcal{K}_X$  is an isomorphism. It is asked by Collier–Tholozan–Toulisse that

### Question

Given a stable  $\alpha_1$ -cyclic  $SO_0(2,3)$ -Higgs bundle (2). When  $|\deg(\mathcal{L}_1)| < \deg(\mathcal{K}_X)$ , is the corresponding representation Anosov?

From a different starting point, S. Filip considered the  $\alpha_1$ -cyclic  $SO_0(2,3)$ -Higgs bundles arising from variation of Hodge structures, i.e.  $\gamma \equiv 0$ .

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### Theorem (Filip)

A stable  $SO_0(2,3)$ -Higgs bundle of the form

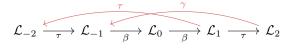
$$\mathcal{L}_{-2} \xrightarrow{\tau} \mathcal{L}_{-1} \xrightarrow{\beta} \mathcal{L}_{0} \xrightarrow{\beta} \mathcal{L}_{1} \xrightarrow{\tau} \mathcal{L}_{2}$$

with  $\tau$  is an isomorphism gives a  $P_2$ -Anosov representation.

Filip proved this theorem by an **analytic method**. Inspired by his method and with some simplification, we extend his results and discover the Anosov property of a general family of  $SO_0(2,3)$ -Higgs bundles.

### Theorem (Z.)

A stable  $\alpha_1$ -cyclic  $SO_0(2,3)$ -Higgs bundle

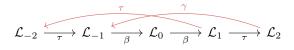


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They form a **non-compact subset (which can go to infinity)** in the character variety.

#### Remark

The trivial line bundle  $\mathcal{L}_0$  above can be replaced by an orthogonal vector bundle of rank n to get an  $\mathrm{SO}(2,n+2)$ -Higgs bundle. With the assumption of stability, the Anosov property still holds.

Lots of results in this section can be generalized when the closed hyperbolic Riemann surface X is replaced by a complete hyperbolic Riemann surface  $X:=\overline{X}\setminus D$  of finite type, i.e. both  $g(\overline{X})$  and #D are finite.

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### Theorem (Z.)

A stable  $\alpha_1$ -cyclic parabolic  $SO_0(2,3)$ -Higgs bundle satisfying specific assumption on parabolic weights and degree gives a  $P_2$ -almost-dominated representation through the non-Abelian Hodge correspondence.

# Thank you!