Anosov property of cyclic $\mathrm{SO}_0(2,3)$-Higgs Bundles
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Classical Teichmüller Theory

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We can view $\mathcal{T}(S)$ as a connected component of the character variety

$$\mathrm{Hom}^{\mathrm{red}}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R}).$$

Moreover, $\mathcal{T}(S)$ is one of the two connected components which consist entirely of discrete and faithful representations. The other one is $\mathcal{T}(\overline{S})$, where $\overline{S}$ denotes the surface S with the opposite orientation.

Move to Higher Rank

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and so on.

In general, the set of discrete and faithful representations is only closed and NOT open.

Higher Teichmüller Spaces

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- (Hitchin; Labourie; Fock–Goncharov) Hitchin components $\mathcal{T}_{Hit}(S, G)$ for **real split** G, such as $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SO}_0(n, n+1)$, $\mathrm{Sp}(2n, \mathbb{R})$ and so on;

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When $G = \mathrm{SL}(2, \mathbb{R})$, the above components coincide with the classical Teichmüller space.

Anosov Property

Anosov property plays an important role in higher Teichmüller theory. Roughly speaking, the Anosov property introduced by F. Labourie means that the representation $\rho\colon \pi_1(S) \to G$ satisfies some special dynamic properties.

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Also, we will assume G is a semi-simple Lie subgroup of $\mathrm{SL}(n, \mathbb{C})$ here to avoid involving a Lie-theoretic description.

Fix a basepoint $x_0 \in \mathbb{H}^2$.

Definition

$\rho\colon \pi_1(S) \to G$ is P_k-**Anosov** if there exist constants $D, L > 0$ such that

$$\log \frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geqslant D \cdot d_{\mathbb{H}^2}(x_0, \gamma \cdot x_0) - L, \forall \gamma \in \pi_1(S),$$

where σ_i denotes the i-th singular value.

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For $g \in \mathrm{SO}_0(2,n)(n > 2)$, it is remarkable that the singular values of g are

$$\sigma_1(g), \sigma_2(g), 1, \ldots, 1, \sigma_2^{-1}(g), \sigma_1^{-1}(g).$$

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Anosov $\Longrightarrow$ discrete + faithful.

Anosov property is open but not closed. However, all representations in the known higher Teichmüller spaces are Anosov.

Non-Abelian Hodge Correspondence

Higgs bundles

The Higgs bundle is a useful tool to study the higher Teichmüller space. It is usually used to give a parametrization of the higher Teichmüller space. We fix a complex structure on S such that it becomes a Riemann surface X. Let $\mathcal{K}_X$ be its canonical line bundle.

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Definition

A **($\mathrm{GL}(n,\mathbb{C})$-)Higgs bundle** over X is a pair $(\mathcal{E}, \Phi)$ consisting of the following data:

- a holomorphic vector bundle $\mathcal{E}$ over X with $\mathrm{rank}(\mathcal{E}) = n$;
- a holomorphic section $\Phi \in \mathrm{H}^0(X, \mathrm{End}(\mathcal{E}) \otimes \mathcal{K}_X)$ called **Higgs field**.

The non-Abelian Hodge correspondence exhibit a homeomorphism between the following moduli spaces:

$$\{\textbf{reductive representation } \rho\colon \pi_1(S) \to \mathrm{GL}(n,\mathbb{C})\}/\mathrm{GL}(n,\mathbb{C})$$

$$\text{associated bundle} \downarrow \quad \uparrow \text{holonomy}$$

$$\{\textbf{reductive} \text{ flat vector bundle } (E \to S, \nabla) \text{ with } \mathrm{rank}(E) = n\}/\mathcal{G}$$

$$\uparrow$$

harmonic metric

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$$\{\textbf{polystable} \text{ Higgs bundle} (\mathcal{E} \to X, \Phi) \text{ with } \mathrm{rank}(\mathcal{E}) = n, \deg(\mathcal{E}) = 0\}/\mathcal{G}$$

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If we equip additional structure on these objects, we can get the non-Abelian Hodge correspondence for general reductive Lie group G.

Kobayashi–Hitchin Correspondence (from Higgs to flat)

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Theorem (Hitchin–Simpson)

If $(\mathcal{E}, \Phi)$ is a polystable Higgs bundle with $\deg(\mathcal{E}) = 0$, then there exists an Hermitian metric h on $\mathcal{E}$ such that

$$F(\nabla^h) + [\Phi, \Phi^{*_h}] = 0, \tag{1}$$

where ∇^h is the Chern connection of the metric h, $F(\nabla^h)$ denotes its curvature form and Φ^{*_h} is the adjoint of Φ with respect to h. Moreover, if $(\mathcal{E}, \Phi)$ is stable, then such h is unique up to a constant scalar.

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If h solves (1), then

$$\nabla^h + \Phi + \Phi^{*_h}$$

gives a flat connection.

From Higgs Bundle to Anosov Representation

$$\text{Higgs bundle} \xrightarrow[\quad]{\textbf{Hitchin's self-dual equation}} \text{Anosov property?}$$

Example: Hitchin Component in Higgs Bundle Viewpoint

The Hitchin component for $\mathrm{SL}(3,\mathbb{R})$ consisting of entirely the Higgs bundles of the following form:

$$\mathcal{E} = \mathcal{K}_X \oplus \mathcal{O} \oplus \mathcal{K}_X^\vee,$$

$$\Phi = \begin{pmatrix} 0 & q_2 & q_3 \\ 1 & 0 & q_2 \\ 0 & 1 & 0 \end{pmatrix},$$

where $1\colon \mathcal{K}_X \to \mathcal{O} \otimes \mathcal{K}_X$ and $1\colon \mathcal{O} \to \mathcal{K}_X^\vee \otimes \mathcal{K}_X$ are the natural isomorphisms and $q_i \in \mathrm{H}^0(X, \mathcal{K}_X^i)$.

It corresponds to the component containing the embedding of Fuchsian representations through the unique irreducible $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R})$.

Special $\mathrm{SO}_0(2,3)$-Higgs Bundles

Below we consider the Higgs bundle whose underlying bundle is

$$\mathcal{E} = \mathcal{L}_{-2} \oplus \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2,$$

where $\mathcal{L}_i$ are line bundles with $\mathcal{L}_i \cong \mathcal{L}_{-i}^\vee$ and $\mathcal{L}_0 \cong \mathcal{O}$ and

$$\Phi = \begin{pmatrix} 0 & 0 & 0 & \gamma & 0 \\ \tau & 0 & 0 & 0 & \gamma \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & \tau & 0 \end{pmatrix}.$$

$(\mathcal{E}, \Phi)$ is a cyclic $\mathrm{SO}_0(2,3)$-Higgs bundle and we denote it by

$$\begin{tikzcd} \mathcal{L}_{-2} \arrow[r, "\tau"'] & \mathcal{L}_{-1} \arrow[r, "\beta"'] & \mathcal{L}_0 \arrow[r, "\beta"'] & \mathcal{L}_1 \arrow[lll, bend right, "\gamma"'] \arrow[r, "\tau"'] & \mathcal{L}_2 \arrow[lll, bend right, "\gamma"'] \end{tikzcd}$$

If the cyclic Higgs bundle

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is (semi-)stable, then the Milnor–Wood inequality implies that

$$|\deg(\mathcal{L}_1)| \leqslant \deg(\mathcal{K}_X).$$

When "$=$" holds, the Higgs bundle corresponds to a maximal representation and maximal representations are known to be P_1-Anosov.

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When "$=$" holds, the Higgs bundle corresponds to a maximal representation and maximal representations are known to be P_1-Anosov.
Such Higgs bundle is called α_1-**cyclic** if $\tau\colon \mathcal{L}_1 \to \mathcal{L}_2 \otimes \mathcal{K}_X$ is an isomorphism. It is asked by Collier–Tholozan–Toulisse that

Question

Given a stable α_1-cyclic $\mathrm{SO}_0(2,3)$-Higgs bundle (2)*. When $|\deg(\mathcal{L}_1)| < \deg(\mathcal{K}_X)$, is the corresponding representation Anosov?*

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Theorem (Filip)

A stable $\mathrm{SO}_0(2,3)$-Higgs bundle of the form

$$\mathcal{L}_{-2} \xrightarrow{\tau} \mathcal{L}_{-1} \xrightarrow{\beta} \mathcal{L}_0 \xrightarrow{\beta} \mathcal{L}_1 \xrightarrow{\tau} \mathcal{L}_2$$

with τ is an isomorphism gives a P_2-Anosov representation.

Filip proved this theorem by an **analytic method**. Inspired by his method and with some simplification, we extend his results and discover the Anosov property of a general family of $\mathrm{SO}_0(2,3)$-Higgs bundles.

Theorem (Z.)

A stable α_1-cyclic $\mathrm{SO}_0(2,3)$-Higgs bundle

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with $\deg(\mathcal{L}_1) < \deg(\mathcal{K}_X)$ gives a P_2-Anosov representation.

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Remark

The trivial line bundle $\mathcal{L}_0$ above can be replaced by an orthogonal vector bundle of rank n to get an $\mathrm{SO}(2,n+2)$-Higgs bundle. With the assumption of stability, the Anosov property still holds.

Non-compact case

Lots of results in this section can be generalized when the closed hyperbolic Riemann surface X is replaced by a complete hyperbolic Riemann surface $X := \overline{X} \setminus D$ of finite type, i.e. both $g(\overline{X})$ and $\#D$ are finite.

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Anosov representation	almost-dominated representation
Higgs bundle	parabolic Higgs bundle

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Theorem (Z.)

A stable α_1-cyclic parabolic $\mathrm{SO}_0(2,3)$-Higgs bundle satisfying specific assumption on parabolic weights and degree gives a P_2-almost-dominated representation through the non-Abelian Hodge correspondence.

Thank you!