Notes on Poncelet's Porism

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Contents

0 Introduction

The following theorem is known as "Poncelet's Closure Theorem" or "Poncelet's Porism", which is named after French engineer and mathematician Jean-Victor Poncelet.

Theorem 0.1. *Let C and D be two plane conics with four different intersections. If it is possible to find, for a given $n > 2$, one n-gon that is simultaneously inscribed in C and circumscribed around D, then it is possible to find infinitely many of them.*

It certainly has a proof by using classical projective geometry techniques (cf.[1]) and also a proof by using some measure theory (cf.[2]). In this note, I will illustrate a proof by using Riemann surfaces. I was first introduced this proof by Longke Tang in a Wechat group, some details in this note is from [3].

1 Poncelet Correspondence

In this section, we will introduce some basic notations and explain how to translate Theorem 0.1 by the language of Riemann surface. Indeed, for our convinience, we have to change our viewpoint into $\mathbb{P}^2_{\mathbb{C}}$ instead of $\mathbb{R}^2$. Let $\mathbb{P}\colon \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2_{\mathbb{C}}$ denote the natural projection.

Definition 1.1. *A* ***conic*** *is a non-singular curve of degree* 2 *in* $\mathbb{P}^2_{\mathbb{C}}$*.*

Definition 1.2. *For any conic* C*, its* ***dual conic*** *is defined as*

$$C^* := \{H \in \mathrm{Gr}(2, \mathbb{C}^3) \mid H \textit{ is tangent to } \mathbb{P}^{-1}(C)\}.$$

By a non-degenerate billinear form on $\mathbb{C}^3$ (e.g. the standard inner product), we can identify $\mathrm{Gr}(2, \mathbb{C}^3)$ with $\mathrm{Gr}(1, \mathbb{C}^3) \cong \mathbb{P}^2_{\mathbb{C}}$ by $H \mapsto H^{\perp}$. By an easy calculation one can see that C^* is also a non-singular conic in $\mathbb{P}^2_{\mathbb{C}}$ under this identification. For instance, if under the standard basis $\mathbb{P}^{-1}(C)$ is defined by an non-singular matrix A, then $\mathbb{P}^{-1}((C^*)^{\perp})$ is given by A^{-1}.

By the definition of C^*, we can also regard it as the collection of all tangent lines of C. For our convinience, we always regard C^* as a conic in $\mathbb{P}^2_{\mathbb{C}}$ below when calculating.

Now we consider the following set.

Definition 1.3. *For two conics* $C, D \subset \mathbb{P}^2_{\mathbb{C}}$*,*

$$\mathcal{P}(C, D) := \{(p, l) \in C \times D^* \mid \mathbb{P}^{-1}(p) \subset (\mathbb{P}^{-1}(l))^{\perp}\}$$

is called the ***Poncelet correspondence*** *for* C *and* D*. We will always use* $\mathcal{P}$ *to denote it below, since there are no other conics.*

There are two natural involutions on $\mathcal{P}$, namely,

$$\begin{aligned} \sigma\colon \mathcal{P} &\to \mathcal{P} \\ (p, l) &\mapsto (p', l) \end{aligned}$$

and

$$\begin{aligned} \tau\colon \mathcal{P} &\to \mathcal{P} \\ (p, l) &\mapsto (p, l'), \end{aligned}$$

where p, p' are two intersections of l and C, l, l' are two tangent lines of D through p. Note that p, p' (resp. l, l') are be the same point (resp. line) iff $p \in C \cap D$. So σ and τ both have exact four fixed points when C and D meet at four different points.

Now we define $\eta := \tau \circ \sigma$, then Theorem 0.1 can be rewritten as

Theorem 1.1. *If C, D meet at four differnt points, then for $n > 3$, $\eta^n : \mathcal{P} \to \mathcal{P}$ has a fixed point iff $\eta^n = \mathrm{id}_{\mathcal{P}}$.*

In the following sections, we will show that $\mathcal{P}$ is a complex torus with dimension 1 and σ, τ, η are all holomorphic maps on it. Once we have proved these two facts, it will be easy to prove Theorem 1.1. Indeed, we have the following lemma.

Lemma 1.1. *Suppose $\mathbb{C}/\Gamma, \mathbb{C}/\Gamma'$ are two complex tori and $f : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$ is a non-constant holomorphic map, then there exists a unique $\alpha \in \mathbb{C}^*$ and some $\beta \in \mathbb{C}$ such that $f(z + \Gamma) = \alpha z + \beta + \Gamma'$.*

Proof. Let $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ and $\pi' : \mathbb{C} \to \mathbb{C}/\Gamma'$ be the natural projections. Let $g := f \circ \pi$, then g is holomorphic. We choose a $\beta \in \mathbb{C}$ such that $g(0) = f(0 + \Gamma) = \beta + \Gamma' = \pi'(\beta)$. Since π' is a covering map, there exists a holomorphic lift $F : \mathbb{C} \to \mathbb{C}$ of g such that $F(0) = \beta$.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "F"] \arrow[d, "\pi"'] \arrow[dr, "g"] & \mathbb{C} \arrow[d, "\pi'"] \\
\mathbb{C}/\Gamma \arrow[r, "f"] & \mathbb{C}/\Gamma'
\end{tikzcd}$$

Hence

$$(\pi' \circ F)(z + \omega) = (f \circ \pi)(z + \omega) = (f \circ \pi)(z) = (\pi' \circ F)(z), \forall \omega \in \Gamma, z \in \mathbb{C}.$$

Thus $F(z + \omega) - F(z) \in \Gamma'$, since $F(z + \omega) - F(z)$ is holomorphic and Γ' is disrete, $F(z + \omega) - F(z)$ is constant for any fixed $\omega \in \Gamma$. So $F'(z)$ is a double periodic funtion with respect to Γ, hence F' is constant. Suppose $F' \equiv \alpha$, then $F(z) = \alpha z + \beta$ due to $F(0) = \beta$. Therefore,

$$f(z + \Gamma) = (f \circ \pi)(z) = (\pi' \circ F)(z) = \alpha z + \beta + \Gamma'.$$

□

Hence we can use it to represent η.

Proof of Theorem 1.1. Suppose $\mathcal{P}$ is isomorphic to $\mathbb{C}/\Gamma$ for a lattice Γ. Then by Lemma 1.1, $\sigma(z + \Gamma) = \alpha_\sigma z + \beta_\sigma + \Gamma$ for a unique $\alpha_\sigma \in \mathbb{C}^*$ and some $\beta_\sigma \in \mathbb{C}$. Since $\sigma^2 = \mathrm{id}_{\mathcal{P}}$, we get

$$\begin{aligned}
&z + \Gamma \\
=&\sigma^2(z + \Gamma) \\
=&\sigma(\alpha_\sigma z + \beta_\sigma + \Gamma) \\
=&\alpha_\sigma^2 z + (\alpha_\sigma + 1)\beta_\sigma + \Gamma.
\end{aligned}$$

So $\alpha_\sigma^2 = 1$ and $\alpha_\sigma = \pm 1$. Note that σ has exact four fixed points, $\alpha_\sigma = -1$. Similarly, τ can be represented as $\tau(z+\Gamma) = -z + \beta_\tau + \Gamma$ for some $\beta_\tau \in \mathbb{C}$. Hence

$$\eta(z+\Gamma) = \tau(\sigma(z+\Gamma)) = z + \beta_\tau - \beta_\sigma + \Gamma.$$

If we denote $\beta_\tau - \beta_\sigma$ by c, then

$$\eta^n(z+\Gamma) = z + nc + \Gamma.$$

Thus η^n has a fixed point iff $nc \in \Gamma$ iff $\eta^n = \mathrm{id}_{\mathcal{P}}$. $\square$

2 Complex Structure on $\mathcal{P}$

In this section, we will show that $\mathcal{P}$ is a complex manifold of dimension 1 with induced complex structure from $C \times D^*$ by showing that it is a non-singular algebraic curve in $C \times D^*$.

2.1 Algebraic Structure

To calculate easily, we consider the Veronese embedding via $\mathcal{O}(2)$ on $\mathbb{P}^1_{\mathbb{C}}$, i.e.

$$\begin{aligned} \varphi \colon \mathbb{P}^1_{\mathbb{C}} &\to \mathbb{P}^2_{\mathbb{C}} \\ (z_0 : z_1) &\mapsto (z_0^2 : z_0 z_1 : z_1^2). \end{aligned}$$

It gives an isomorphism between $\mathbb{P}^1_{\mathbb{C}}$ and the conic V defined by $z_0 z_2 - z_1^2$ in $\mathbb{P}^2_{\mathbb{C}}$. Note that $\mathrm{PGL}(3, \mathbb{C})$ acts transitively on all conics in $\mathbb{P}^2_{\mathbb{C}}$, thus C and D^* can be parameterized by $\mathbb{P}^1_{\mathbb{C}}$ by two isomorphism $p \colon \mathbb{P}^1_{\mathbb{C}} \to C$ and $l \colon \mathbb{P}^1_{\mathbb{C}} \to D^*$ respectively, i.e. suppose $A, B \in \mathrm{GL}(3, \mathbb{C})$ satisfy $A(V) = C$ and $B(V) = D^*$, then $p(z) = A\varphi(z)$ and $l(w) = B\varphi(w)$ for any $z, w \in \mathbb{P}^1_{\mathbb{C}}$. Hence to show $\mathcal{P}$ is an algebraic curve in $C \times D^*$ we only need to prove that

Proposition 2.1. *$X := (p \times l)^{-1}(\mathcal{P})$ is an algebraic curve in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$.*

Proof. Suppose $T = B^{\mathrm{T}} A = (T_{ij})_{0 \leqslant i,j \leqslant 2} \in \mathrm{GL}(3, \mathbb{C})$. We have

$$\begin{aligned} (z, w) \in X &\iff (p(z), l(w)) \in \mathcal{P} \\ &\iff (l(w))^{\mathrm{T}} p(z) = 0 \\ &\iff \begin{pmatrix} w_0^2 & w_0 w_1 & w_1^2 \end{pmatrix} T \begin{pmatrix} z_0^2 \\ z_0 z_1 \\ z_1^2 \end{pmatrix} = 0. \end{aligned}$$

Explicitly, we denote

$$\begin{aligned} H((z_0, z_1), (w_0, w_1)) :=& (T_{00} z_0^2 + T_{01} z_0 z_1 + T_{02} z_1^2) w_0^2 \\ &+ (T_{10} z_0^2 + T_{11} z_0 z_1 + T_{12} z_1^2) w_0 w_1 \\ &+ (T_{20} z_0^2 + T_{21} z_0 z_1 + T_{22} z_1^2) w_1^2 \\ =:& T_0(z_0, z_1) w_0^2 + T_1(z_0, z_1) w_0 w_1 + T_2(z_0, z_1) w_1^2. \end{aligned}$$

Note that T_0, T_1, T_2 cannot have a common zero since $T_0(z) = T_1(z) = T_2(z) = 0$ iff $z_0^2 = z_0 z_1 = z_1^2 = 0$ (due to T is non-singular) iff $z_0 = z_1 = 0$. Thus X is defined by the bi-homogeneous polynomial H of bi-degree $(2, 2)$ in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, which concludes that X is an algebraic curve. $\square$

2.2 Singularity

We now characterize that the set of sigular points X_{sing} of X.

Proposition 2.2. $X_{\text{sing}} = \{(z, w) \in X \mid p(z) \in C \cap D, l(w) \in C^* \cap D^*\}$.

To prove the above proposition, we first prove a lemma.

Lemma 2.1. *Define* $\Delta := T_1^2 - 4T_0T_2$*, then* $\Delta(z) = 0$ *iff* $p(z) \in C \cap D$.

Proof. Note that $p(z) \in C \cap D$ iff $\#\{w \in \mathbb{P}^1_{\mathbb{C}} \mid H(z, w) = 0\} = 1$, we will show the latter one statement is equivalent to $\Delta(z) = 0$.

- If $T_2(z) = 0$, then $0 = \Delta(z) \iff T_1(z) = 0$. Since
$$H(z, w) = T_0(z)w_0^2 + T_1(z)w_0w_1,$$
it has two diffetent roots $(0 : 1)$ and $(T_1(z) : -T_0(z))$ in $\mathbb{P}^1_{\mathbb{C}}$ iff $T_1(z) \neq 0$ and it has a unique root $(0 : 1)$ iff $T_1(z) = 0$. So we get what we require.
- If $T_2(z) \neq 0$, then $H(z, (0 : 1)) \neq 0$. Without loss of generality, we assume that $w_0 = 1$, then the equation is
$$0 = T_0(z) + T_1(z)w_1 + T_2(z)w_1^2,$$
which has a single solution iff $\Delta(z) = 0$.

□

Similarly, if we write $H(z, w) = S_0(w)z_0^2 + S_1(w)z_0z_1 + S_2(w)z_1^2$, where $S_i(w) = T_{0i}w_0^2 + T_{1i}w_0w_1 + T_{2i}w_1^2$, we can obtain that

Lemma 2.2. *Define* $\tilde{\Delta} := S_1^2 - 4S_0S_2$*, then* $\tilde{\Delta}(w) = 0$ *iff* $l(w) \in C^* \cap D^*$.

Now we return to Proposition 2.2.

Proof of Proposition 2.2. Without loss of generality, we only analysis the singularity of X on a standard open set
$$U = \{((z_0 : 1), (w_0 : 1)) \mid z_0, w_0 \in \mathbb{C}\} \cong \mathbb{C}^2.$$
Now $X \cap U$ can be viewed as an affine curve defined by
$$\tilde{H}(z_0, w_0) = H((z_0 : 1), (w_0 : 1)).$$
By an abuse of notation, we will write $T_i(z_0) = T_i((z_0 : 1))$, $S_i(w_0) = S_i((w_0 : 1))$ for $i = 0, 1, 2$. Then
$$\left(\frac{\partial \tilde{H}}{\partial w_0}(z_0, w_0)\right)^2 - 4T_0(z_0)\tilde{H}(z_0, w_0) = \Delta((z_0 : 1))$$

and

$$\left(\frac{\partial \tilde{H}}{\partial z_0}(z_0, w_0)\right)^2 - 4S_0(w_0)\tilde{H}(z_0, w_0) = \tilde{\Delta}((w_0 : 1)).$$

Therefore,

$$\begin{aligned}
&(z, w) = ((z_0 : 1), (w_0 : 1)) \in X_{\mathrm{sing}} \cap U \\
\iff& \tilde{H}(z_0, w_0) = \frac{\partial \tilde{H}}{\partial w_0}(z_0, w_0) = \frac{\partial \tilde{H}}{\partial z_0}(z_0, w_0) = 0 \\
\iff& \tilde{H}(z_0, w_0) = \Delta((z_0 : 1)) = \tilde{\Delta}((w_0 : 1)) = 0 \\
\iff& p(z) \in C \cap D, l(w) \in C^* \cap D^*.
\end{aligned}$$

□

Corollary 2.1. *If C and D meet at four different points, then X is non-singular, i.e. $\mathcal{P}$ is non-singular.*

3 Topology of $\mathcal{P}$

In this section, we will show that $\mathcal{P}$ is a compact Riemann surface with genus 1 when C and D meet at four different points, hence $\mathcal{P}$ is isomorphic to some complex torus $\mathbb{C}/\Gamma$. The compactness can be seen immediately since X is a closed subset of $\mathbb{P}^2_{\mathbb{C}}$. To study other topology property of X, we need the following two maps, i.e.

$$\begin{aligned} \pi_1 \colon \mathcal{P} &\to C \\ (p, l) &\mapsto p \end{aligned}$$

and

$$\begin{aligned} \pi_2 \colon \mathcal{P} &\to D^* \\ (p, l) &\mapsto l. \end{aligned}$$

They are obviously holomorphic since they are both restricted from natural projection. And we can see that both of them are 2-sheeted ramified covering with 4 ramification points with all branch order 1. Explicitly, π_1 has 4 ramification points (p_i, l_i), where $\{p_i\}_{i=1}^4 = C \cap D$ and l_i is the tangent line of D passing through p_i. And π_2 has 4 ramification points (p_i', l_i'), where $\{l_i'\}_{i=1}^4 = C^* \cap D^*$ and l_i' is tangent to C at p_i'.

Proposition 3.1. *$\mathcal{P}$ is connected.*

Proof. It follows from C is connected and π_1 is a ramified covering with non-empty ramified points directly. □

Thus $\mathcal{P}$ is indeed a compact Riemann surface, to compute its genus $g(\mathcal{P})$, we can use the following well-known result.

Theorem 3.1 (Riemann–Hurwitz formula)**.** *Suppose $f \colon X \to Y$ is an n-sheeted holomorphic ramified covering map between compact Riemann surfaces X and Y with total branching order $b := b(f)$. Then*

$$g(X) = \frac{b}{2} + n(g(Y) - 1) + 1.$$

Proposition 3.2. *$\mathcal{P}$ is isomorphic to a complex torus $\mathbb{C}/\Gamma$.*

Proof. Apply Riemann–Hurwitz formula to π_1, we get

$$g(\mathcal{P}) = \frac{4}{2} + 2(0 - 1) + 1 = 1.$$

Hence $\mathcal{P}$ is isomorphic to a complex torus $\mathbb{C}/\Gamma$. □

4 Holomorphicity of σ and τ

In this section, we will show σ and τ are holomorphic. To see that we still focus on X, i.e. we consider $\sigma' = (p \times l)^{-1} \circ \sigma \circ (p \times l)$ and $\tau' = (p \times l)^{-1} \circ \tau \circ (p \times l)$, which maps X into X.

We first note that σ' (resp. τ') interchanges the points of X with the same z-coordinate (resp. w-coordinate).

Proposition 4.1. *σ' and τ' are holomorphic on X.*

Proof. We only show that τ' is holomorphic and the holomorphicity of σ' can be proven by a similar argument.

Let $Y := \{(z, w) \in X \mid p(z) \in C \cap D\}$. We first show that τ' is holomorphic on $X \setminus Y$. We take an arbitrary point $(a, b) \in X \setminus Y$. Note that $\pi_1 \colon X \to \mathbb{P}^1_{\mathbb{C}}$ induced by $\pi_1 \colon \mathcal{P} \to C$ is a 2-sheeted covering map, we can choose a neighborhood V of a such that $\pi_1^{-1}(V) = U_1 \sqcup U_2 \subset X \setminus Y$ and U_1, U_2 are coordinate neighborhoods of (a, b) and $\tau'(a, b)$ respectively. Since

$$\pi_1|_{U_2} \circ \tau' \circ (\pi_1|_{U_1})^{-1} = \mathrm{id}_V$$

is holomorphic, τ' is holomorphic around (a, b).

Now we show that τ' is continuous at Y. Fix a point $p = (a_0, b_0) \in Y$. Without loss of generality, we assume that $a_0, b_0 \in \mathbb{C}$. So $T_0(a_0) = H((a_0, 1), (1, 0)) \neq 0$, then we can take a neighborhood W of p such that $T_0(z) \neq 0$ for any $(z, w) \in W$. Then

$$\tau(a, b) = (a, -T_1(a)/T_0(a) - b)$$

is continuous on W. Therefore, τ' is holomorphic. □

Remark 4.1. *The proof of the continuity of τ' in [3] is wrong, since a subsequence of a sequence which not converges to a chosen point may be converge to it.*

Corollary 4.1. *σ and τ are holomorphic.*

References

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