Solutions to Complex Geometry

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This is a solution to Daniel Huybrechts's book, *Complex Geometry: An Introduction* ([5]).

Contents

1 Local Theory

1.1 Holomorphic Functions of Several Variables

Exercise 1.1.1. Show that every holomorphic map $f : \mathbb{C} \to \mathbb{H} := \{z|\mathrm{Im}(z) > 0\}$ is constant.

Solution. Note that $g(z) := \exp(\sqrt{-1}f(z))$ is holomorphic with $|g| < 1$, so g is a constant by Liouville's Theorem and f is a constant as well. $\square$

Exercise 1.1.2. Show that real and imaginary part u respectively v of a holomorphic function $f = u+\sqrt{-1}v$ are harmonic, i.e. $\sum_i \frac{\partial^2 u}{\partial {x_i}^2} + \sum_i \frac{\partial^2 u}{\partial {y_i}^2} = 0$ and similarly for v.

Solution. By Cauchy-Riemann Formula, we have

$$\begin{aligned}
&\sum_i \frac{\partial^2 u}{\partial {x_i}^2} + \sum_i \frac{\partial^2 u}{\partial {y_i}^2} \\
=&\sum_i \frac{\partial}{\partial x_i}\frac{\partial u}{\partial x_i} + \sum_i \frac{\partial}{\partial y_i}\frac{\partial u}{\partial y_i} \\
=&\sum_i \frac{\partial}{\partial x_i}\frac{\partial v}{\partial y_i} - \sum_i \frac{\partial}{\partial y_i}\frac{\partial v}{\partial x_i} \\
=&0
\end{aligned}$$

for u. So u is harmonic and v is harmonic similarly. $\square$

Exercise 1.1.3. Deduce the maximum principle and the identity theorem for holomorphic functions of several variables from the corresponding one-dimensional results.

Solution. For the identity theorem, suppose f is a holomorphic function in a connected open subset $U \subset \mathbb{C}^n$ such that $f(z) = 0$ for all z in a non-empty open subset $V \subset U$, we only need to prove that $f \equiv 0$ in U. It's obvious to deduce that this holds when U is a polydisc from the corresponding one-dimensional results by finding a piece-wise path connecting two points in it such that every piece of path fix a complex coordinate on it. Now for general U, for any point $z \in U \setminus V$, choose a path $\gamma : [0, 1] \to U$ be a path connecting z and a point $a \in V$, the theorem holds trivially by choosing finite polydisc centered at the point in the image of γ covering the path.

For the maximum principle, if f is a holomorphic function in a connected open subset $U \subset \mathbb{C}^n$ such that $|f|$ attains local maximum at $a \in U$, then there exists a polydisc $V \subset U$ centered at a such that $|f|$ attains maximum at a, then by the similar method above, we can get f is constant in V from the corresponding one-dimensional results. By the identity theorem, we get f is a constant in U. $\square$

Exercise 1.1.4. Prove the chain rule $\frac{\partial(f\circ g)}{\partial z} = \frac{\partial f}{\partial w}\frac{\partial g}{\partial z} + \frac{\partial f}{\partial \bar{w}}\frac{\partial \bar{g}}{\partial z}$; and its analogue for $\frac{\partial}{\partial \bar{z}}$. Use this to show that the composition of two holomorphic functions is holomorphic.

Solution. For $\frac{\partial}{\partial z}$, we have

$$\begin{aligned}
&\frac{\partial(f\circ g)}{\partial z} \\
=&\frac{\partial f(g(z,\bar{z}), \bar{g}(z,\bar{z}))}{\partial z} \\
=&\frac{\partial f}{\partial w}\frac{\partial g}{\partial z} + \frac{\partial f}{\partial \bar{w}}\frac{\partial \bar{g}}{\partial z},
\end{aligned}$$

and for $\frac{\partial}{\partial \bar{z}}$, we have

$$\begin{aligned}&\frac{\partial(f\circ g)}{\partial \bar{z}}\\=&\frac{\partial f(g(z,\bar{z}),\bar{g}(z,\bar{z}))}{\partial \bar{z}}\\=&\frac{\partial f}{\partial w}\frac{\partial g}{\partial \bar{z}}+\frac{\partial f}{\partial \bar{w}}\frac{\partial \bar{g}}{\partial \bar{z}}\\=&\frac{\partial f}{\partial w}\frac{\partial g}{\partial \bar{z}}+\frac{\partial f}{\partial \bar{w}}\overline{\left(\frac{\partial g}{\partial z}\right)}.\end{aligned}$$

If f and g are both holomorphic, then

$$\frac{\partial(f\circ g)}{\partial \bar{z}}=\frac{\partial f}{\partial w}\cdot 0+0\cdot\frac{\partial g}{\partial z}=0$$

shows that $f\circ g$ is holomorphic. □

Exercise 1.1.5. Deduce the implicit function theorem for holomorphic functions $f : U \to \mathbb{C}$ from the Weierstrass preparation theorem.

Solution. Without loss of generality, we can assume $0 \in U$ and $f(0) = 0$, then by Weierstrass preparation theorem, there exists a Weierstrass polynomial $g(z_1, w) = g_w(z_1)$ and a holomorphic function h such that $f = g \cdot h$ in a neighborhood of 0 and $h(0) \neq 0$. By $\frac{\partial f}{\partial z_1}(0) \neq 0$ we deduce that $\deg g = 1$, let $g_w(z_1) = z_1 + \alpha_1(w)$. Then in a small neighborhood of 0, $f = 0$ iff $g = 0$ iff $-\alpha_1(w) = z_1$, that completes the proof. □

Exercise 1.1.6. Consider the function

$$\begin{aligned}f:\mathbb{C}^2&\to\mathbb{C}\\(z_1,z_2)&\mapsto z_1^3z_2+z_1z_2+z_1^2z_2^2+z_2^2+z_1z_2^3\end{aligned}$$

and find an explicit decomposition $f = h \cdot g_w$ as claimed by the WPT.

Solution. Since $f(z_1, 0) \equiv 0$ and $f(0, z_2) = z_2^2$, we should treat z_2 as the variable of the Weierstrass polynomial. When we fix z_1, $f(z_1, z_2)$ has three roots, $a_1(z_1) = 0$, $a_2(z_1) = \frac{-z_1^2-1+\sqrt{-3z_1^4-2z_1^2+1}}{2z_1}$ and $a_3(z_1) = \frac{-z_1^2-1-\sqrt{-3z_1^4-2z_1^2+1}}{2z_1}$. Note that when $z_1 \to 0$, a_1, a_2 also tend to zero, and a_3 tends to ∞. So

$$g(z_1,z_2)=(z_2-a_1(z))(z_2-a_2(z))=z_2\left(z_2+\frac{z_1^2+1-\sqrt{-3z_1^4-2z_1^2+1}}{2z_1}\right)$$

is the Weierstrass polynomial we need, and $h = f/g$. □

Exercise 1.1.7. State and prove the product formula for $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$. Show that the product $f \cdot g$ of two holomorphic functions f and g is holomorphic and that $1/f$ is holomorphic on the complement of the zero set $Z(f)$.

Solution. For $\frac{\partial}{\partial z}$, we have

$$\begin{aligned}\frac{\partial(f\cdot g)}{\partial z}=&\frac{1}{2}\left(\frac{\partial(f\cdot g)}{\partial x}-\sqrt{-1}\frac{\partial(f\cdot g)}{\partial y}\right)\\=&f\frac{\partial g}{\partial z}+g\frac{\partial f}{\partial z},\end{aligned}$$

and for $\frac{\partial}{\partial \bar{z}}$, we have

$$\frac{\partial(f \cdot g)}{\partial \bar{z}} = f\frac{\partial g}{\partial \bar{z}} + g\frac{\partial f}{\partial \bar{z}}.$$

If f, g are both holomorphic, then it's obvious to see that $\frac{\partial(f \cdot g)}{\partial \bar{z}} = 0$, hence $f \cdot g$ is holomorphic. On $\mathbb{C}^n \setminus Z(f)$, apply the product formula for $\frac{\partial}{\partial \bar{z}}$ on f and $1/f$, we can get $\frac{\partial(1/f)}{\partial \bar{z}} = 0$, which implies that $1/f$ is holomorphic. □

Exercise 1.1.8. Let $U \subset \mathbb{C}^n$ be open and connected. Show that for any non-trivial holomorphic function $f : U \to \mathbb{C}$, the complement $U \setminus Z(f)$ of the zero set of f is connected and dense in U.

Solution. Obiviously, $Z(f)$ is closed in U, so by the identity theorem, $Z(f)$ must be nowhere-dense since f is non-trivial. In other words, $U \setminus Z(f)$ is dense.

Now suppose $z_0, w_0 \in U \setminus Z(f)$, consider $g : \mathbb{C} \to \mathbb{C}$, where $g(z) = f(z_0 + z(w_0 - z_0))$. We have $g(0), g(1) \neq 0$. Since g is holomorphic, $Z(g)$ is discrete and at most countable, so there exists a path in $\mathbb{C} \setminus Z(g)$ connected 0 and 1. By this path we find a path in $U \setminus Z(f)$ connected z_0 and w_0, so $U \setminus Z(f)$ is connected. □

Exercise 1.1.9. Let $U \subset \mathbb{C}^n$ be open and connected. Show that the set $K(U)$ of meromorphic functions on U is a field. What is the relation between $K(U)$ and the quotient field of $\mathcal{O}_{\mathbb{C}^n,z}$ for $z \in U$?

Solution. It's easy to see that $K(U)$ is a commutative unital ring, now we only need to show that any $f \in K(U) \setminus \{0\}$ is invertible. Suppose $S \subset U$ is a nowhere dense subset, $\{U_i\}$ is an open cover of U such that there exists relatively prime holomorphic functions $g_i, h_i : U_i \to \mathbb{C}$ such that $h_i|_{U_i \setminus S} \cdot f|_{U_i \setminus S} = g_i|_{U_i \setminus S}$. Since $\mathbb{C}^n$ is paracompact, we can assume $\{U_i\}$ is locally finite, it implies that $S' = S \cup (\cup_i Z(g_i))$ is still nowhere dense. So we can define $h_i|_{U_i \setminus S'}/g_i|_{U_i \setminus S'}$ on each $U_i \setminus S'$, which gives the inverse of f in $K(U)$. So $K(U)$ is a field. Since a meromorphic function locally is the quotient of two holomorphic functions, the quotient field of $\mathcal{O}_{\mathbb{C}^n,z}$ is isomorphism to the stalk at z of presheaf K. □

Exercise 1.1.10. Let $U = B_\varepsilon(0) \subset \mathbb{C}^n$ and consider the ring $\mathcal{O}(U)$ of holomorphic functions on U. Show that $\mathcal{O}(U)$ is naturally contained in $\mathcal{O}_{\mathbb{C}^n,0}$. What is the relation between the localization of $\mathcal{O}(U)$ at the prime ideal of all functions vanishing at the origin and $\mathcal{O}_{\mathbb{C}^n,0}$? Is this prime ideal maximal?

Solution. The map

$$\begin{aligned} i_1 : \mathcal{O}(U) &\to \mathcal{O}_{\mathbb{C}^n,0} \\ f &\mapsto (U, f) \end{aligned}$$

is a natural inclusion. And for any $g \in \mathcal{O}_U$ not vanishing at the origin, $U \setminus Z(g)$ must contain a neighborhood V of the origin since g is continuous. This shows that if $g(g_1 f_2 - g_2 f_1) = 0$, where $g, g_1, g_2 \in \mathcal{O}_U$ not vanishing at the origin and $f_1, f_2 \in \mathcal{O}(U)$, then $f_1/g_1 = f_2/g_2$ in a neighborhood of the origin. Now denote the prime ideal of all functions vanishing at the origin by $\mathfrak{p}$. So the map

$$\begin{aligned} i_2 : \mathcal{O}(U)_\mathfrak{p} &\to \mathcal{O}_{\mathbb{C}^n,0} \\ (f, g) &\mapsto (V, f/g) \end{aligned}$$

is well-defined and injective. However, this inclusion maybe not surjective. For example, $n = 1$, $[z \mapsto \exp(1/(z - \varepsilon/2))] \in \mathcal{O}_{\mathbb{C},0}$ has an essential singular point at $z = \varepsilon/2$, but the element in $\mathcal{O}(U)_\mathfrak{p}$ can only have poles in U. Moreover, $\mathfrak{p}$ is maximal in $\mathcal{O}(U)$ obviously. □

Exercise 1.1.11. The notion of irreducibility for analytic germs generalizes in a straightforward way to the corresponding notion for analytic sets $X \subset \mathbb{C}^n$. Give an example of an irreducible analytic set that does not define irreducible analytic germs at every point and of an analytic set whose induced germs are all irreducible, but the set is not.

Solution. The analytic set $\{(z_1, z_2) : z_1^2 = z_2^2(z_2 + 1)\} \subset \mathbb{C}^2$ induces a reducible analytic germ at the origin since $\sqrt{z_2 + 1}$ can be locally defined at the origin. Since $z_1^2 = z_2^2(z_2 + 1)$ is a Weierstrass polynomial with respect z_1, by Weierstrass preparation theorem, it it is reducible in $\mathcal{O}_{\mathbb{C}^2}$, it must be reducible in $\mathcal{O}_{\mathbb{C}}[z_1]$. However, it is impossible since $\sqrt{z_2 + 1}$ cannot be defined globally on $\mathbb{C}$.

The reducible analytic set $\mathbb{C} \times \{0, 1\} \subset \mathbb{C}^2$ induces irreducible analytic germs at every point. $\square$

Exercise 1.1.12. Let $U \subset \mathbb{C}^n$ be an open subset and let $f : U \to \mathbb{C}$ be holomorphic. Show that for $n \geqslant 2$ the zero set $Z(f)$ cannot consist of a single point. Analogously, show that for a holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$, $n \geqslant 2$ and $w \in \operatorname{im}(f)$ there exists $z \in f^{-1}(w)$ such that $|z| \gg 0$.

Solution. Without loss of generality, suppose f has a single zero at the origin 0. Since it is single, $f = g_w(z_1) \cdot h(z_1, w)$ by Weierstrass preparation theorem, where h does not vanish at a neighborhood V of 0. So $Z(f) \cap V = Z(g) \cap V$. It implies that $Z(f)$ is not single at the origin, contradiction.

For an entire function f, if $Z(f)$ is bounded, then there exists a smallest polydisc centered at origin contains $Z(f)$, i.e. there exists a zero on the boundary of the polydisc, by the argument above we can see a contradiction. $\square$

Exercise 1.1.13. Show that the product of two analytic germs is in a natural way an analytic germ.

Solution. Suppose two analytic germs are given by

$$Z(f_1, \cdots, f_k) \subset \bigoplus_{i=1}^{n} \mathbb{C}z_i$$

and

$$Z(g_1, \cdots, g_l) \subset \bigoplus_{i=n+1}^{n+m} \mathbb{C}z_i,$$

then the analytic germ induced $Z(f_1, \cdots, f_k, g_1, \cdots, g_l) \subset \mathbb{C}^{n+m}$ is the product of these two germs. $\square$

Exercise 1.1.14. Let $X \subset \mathbb{C}^n$ be an irreducible analytic set of dimension d. A point $x \in X$ is called singular if X cannot be defined by $n - d$ holomorphic functions locally around x for which x is regular. Then the set of singular points $X_{\text{sing}} \subset X$ is empty or an analytic subset of dimension $< d$. Although the basic idea behind this result is very simple, its complete proof is rather technical. Try to prove the fact in easy cases, e.g. when X is defined by a single holomorphic function.

If x is a regular point, i.e. $x \in X_{\text{reg}} := X \setminus X_{\text{sing}}$, the $n - d$ holomorphic functions defining X near x can be completed to a local coordinate system.

Solution. Suppose $X = Z(f)$, then $x \in X_{\text{sing}}$ iff $\frac{\partial f}{\partial z_i}(x) = 0$ for any $i = 1, \cdots, n$. So $X_{\text{sing}} = Z\left(f, \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}\right)$.

If $x \in X_{\text{reg}}$, the result follows from Theorem 1.1.30 in [5] directly. $\square$

Exercise 1.1.15. Consider the holomorphic map

$$\begin{aligned} f : \mathbb{C} &\to \mathbb{C}^2 \\ z &\mapsto (z^2 - 1, z^3 - z). \end{aligned}$$

Is the image an analytic set?

Solution. The image of f is the zero set of $z_1^2(z_1+1)=z_2^2$, so it is an analytic set. □

Exercise 1.1.16. Prove that the polydisc $B_{(1,1)}(0)\subset\mathbb{C}^2$ and the unit disc $D:=\{z\in\mathbb{C}^2:\|z\|<1\}$ are not biholomorphic.

Solution. The original exercise in [5] is **wrong**, the group consists of automorphism of a polydisc fixing the center is not abelian.

For the explicit structure of the automorphism group of the ball and the polydisc, see Section I.§4.10 of [8], then the statement holds directly follows that a biholomorphic map will induce an isomorphism between the automorphism groups. □

Exercise 1.1.17. Let $X\subset\mathbb{C}^n$ be an analytic subset. Show that locally around any point $x\in X$ the regular part X_{reg} has finite volume.

Solution. I don't know what the book means by 'finite volume' here. □

Exercise 1.1.18. Let $f:U\to V$ be holomorphic and let $X\subset V$ be an analytic set. Show that $f^{-1}(X)\subset U$ is analytic. What is the relation between the irreducibility of X and $f^{-1}(X)$?

Solution. Suppose $X=Z(g_1,\cdots,g_k)$, where $g_1,\cdots,g_k\in\mathcal{O}(V)$. Then $f^{-1}(X)=Z(g_1\circ f,\cdots,g_k\circ f)$ is an analytic set.

Consider $X=Z(z_1,z_1-1)\subset\mathbb{C}^2$ and $f:\mathbb{C}\cong\{0\}\times\mathbb{C}\subset\mathbb{C}^2$, then $f^{-1}(X)$ is irreducible and X is reducible. Consider $X=Z(z_2^2(z_2+1)-z_1^2)$ and still the above f, then X is irreducible and $f^{-1}(X)=\{0,-1\}$ is reducible. □

Exercise 1.1.19. Let $I\subset\mathcal{O}_{\mathbb{C}^2,0}$ be the ideal generated by $z_1^2-z_2^3+z_1$ and $z_1^4-2z_1z_2^3+z_1^2$. Describe $\sqrt{I}$.

Solution. $2z_1(z_1^2-z_2^3+z_1)-(z_1^4-2z_1z_2^3+z_1^2)=z_1^2(z_1^2-2z_1-1)$ and $1/(z_1^2-2z_1-1)\in\mathcal{O}_{\mathbb{C}^2,0}$ implies that $z_1^2\in I$, so $z_1\in\sqrt{I}$. Hence $z_2^3\in\sqrt{I}$ and $z_2\in\sqrt{I}$. Since $z_1^2-z_2^3+z_1$ and $z_1^4-2z_1z_2^3+z_1^2$ are both vanishing at the origin, $0\in Z(\sqrt{I})$. Therefore $\sqrt{I}$ must be the ideal generated by z_1,z_2 in $\mathcal{O}_{\mathbb{C}^2,0}$. □

Exercise 1.1.20. Let $U\subset\mathbb{C}^n$ be an open subset and $f:U\setminus\mathbb{C}^{n-2}\to\mathbb{C}$ a holomorphic map. Show that there exists a unique holomorphic extension $\tilde{f}:U\to\mathbb{C}$ of f.

Solution. Since the proof of Hartogs's theorem only uses the boundness of two directions and the openness of other directions, this statement holds as well when following the proof of Hartogs's theorem. □

1.2 Complex and Hermitian Structures

Exercise 1.2.1. Let $(V, \langle,\rangle)$ be a four-dimensional euclidian vector space. Show that the set of all compatible almost complex structures consist of two copies of S^2.

Solution. The set of all compatible almost complex structure is equal to $\{A \in \mathfrak{gl}(4;\mathbb{R}) : A^2 = -I_4, AA^{\mathrm{tr}} = I_4\}$. This implies that $A + A^{\mathrm{tr}} = 0$, so one can readily check that A must have the following two forms:

$$\begin{bmatrix} 0 & \cos\varphi & \sin\varphi\cos\theta & \sin\varphi\sin\theta \\ -\cos\varphi & 0 & \sin\varphi\sin\theta & -\sin\varphi\cos\theta \\ -\sin\varphi\cos\theta & -\sin\varphi\sin\theta & 0 & \cos\varphi \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & -\cos\varphi & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & \cos\varphi & \sin\varphi\cos\theta & \sin\varphi\sin\theta \\ -\cos\varphi & 0 & -\sin\varphi\sin\theta & \sin\varphi\cos\theta \\ -\sin\varphi\cos\theta & \sin\varphi\sin\theta & 0 & -\cos\varphi \\ -\sin\varphi\sin\theta & -\sin\varphi\cos\theta & \cos\varphi & 0 \end{bmatrix},$$

where $\varphi \in [0,\pi]$, $\theta \in [0, 2\pi)$. The collection of each form above is diffeomorphism to a sphere S^2. $\square$

Exercise 1.2.2. Show that the two decompositions $\bigwedge^k V^* = \bigoplus_{0\leqslant i} L^i P^{k-2i}$ and $L^i P^{k-2i} = \bigoplus_{p+q=k-2i} L^i P^{p,q}$ are orthogonal with respect to the Hodge-Riemann pairing.

Solution. For the first decomposition, let $\alpha \in P^{k-2i}, \beta \in P^{k-2j}$. Without loss of generality, we assume $i < j$, then

$$\begin{aligned} & Q(L^i\alpha.L^j\beta)\cdot \mathrm{vol} \\ =& \omega^i \wedge \alpha \wedge \omega^j \wedge \beta \wedge \omega^{n-k} \\ =& L^{n-k+2i+1}\alpha \wedge L^{j-1}\beta \\ =& 0 \wedge L^{j-1}\beta \\ =& 0 \end{aligned}$$

by $L^{n-k+2i+1}\alpha = 0$ (cf. Proposition 1.2.30 v) of [5]). The second orthogonality holds directly by Hodge-Riemann bilinear relation since

$$Q(L^i\alpha, L^i\beta) = \omega^i \wedge \alpha \wedge \omega^i \wedge \bar{\beta} \wedge \omega^{n-k} = Q(\alpha,\beta).$$

$\square$

Exercise 1.2.3. Prove the following identities: $*\Pi^{p,q} = \Pi^{n-q,n-p}*$ and $[L, \mathbf{I}] = [\Lambda, \mathbf{I}] = 0$.

Solution. The first identity is trivial by definition. Note that L has bidegree $(1,1)$, Λ has bidegree $(-1,-1)$ and $1-1 = (-1)-(-1) = 0$, so $[L,\mathbf{I}] = [\Lambda,\mathbf{I}] = 0$ directly. $\square$

Exercise 1.2.4. Is the product of two primitive forms again primitive?

Solution. The answer is no. Consider the forms x and y on $\mathbb{C} \cong \mathbb{R}x \oplus \mathbb{R}y$ with $\{x, y\}$ is an orthonormal basis and with volume form $x \wedge y$. x and y must be primitive since they are $1-$forms, but $\Lambda(x\wedge y) = *\omega = *(x\wedge y) \neq 0$. $\square$

Exercise 1.2.5. Let $(V, \langle,\rangle)$ be an euclidian vector space and let I, J, and K be compatible almost complex structures where $K = I\circ J = -J\circ I$. Show that V becomes in a natural way a vector space over the quaternions. The associated fundamental forms are denoted by ω_I, ω_J, and ω_K. Show that $\omega_J + \sqrt{-1}\omega_K$ with respect to I is a form of type $(2,0)$. How many natural almost complex structures do you see in this context?

Solution. Denote the set of quaternions by $\mathbb{H}$. Then the map

$$\begin{aligned} \mathbb{H} \times V &\to V \\ (a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, v) &\mapsto av + bI(v) + cJ(v) + dK(v) \end{aligned}$$

is a well-defined scalar multiplication. Hence V becomes a $\mathbb{H}-$vector space.

Now set $\omega' = \omega_J + \sqrt{-1}\omega_K$, we have

$$\begin{aligned} &\omega'(v, w + \sqrt{-1}I(w)) \\ =&\langle v, J(w) - \sqrt{-1}K(w)\rangle + \langle v, \sqrt{-1}K(w) - J(w)\rangle \\ =&\langle v, 0\rangle \\ =&0, \end{aligned}$$

where $v \in V_{\mathbb{C}}$, $w \in V$, which implies that ω' vanishes on $V^{1,0} \otimes_{\mathbb{C}} V^{0,1}$ and $V^{0,1} \otimes_{\mathbb{C}} V^{0,1}$. Hence ω' is a form of type $(2,0)$.

It's easy to see that any $aI + bJ + cK$ which satisfies $(a,b,c) \in S^2$ is a compatible almost complex structure. $\square$

Exercise 1.2.6. Let $\omega \in \bigwedge^2 V^*$ be non-degenerate, i.e. the induced homomorphism $\tilde{\omega} : V \to V*$ is bijective. Study the relation between the two isomorphisms $L^{n-k} : \bigwedge^k V^* \to \bigwedge^{2n-k} V^*$ and $\bigwedge^k V^* \cong \bigwedge^{2n-k} V \cong \bigwedge^{2n-k} V^*$, where the latter is given by $\wedge^{2n-k}\omega$. Here, $2n = \dim_{\mathbb{R}}(V)$.

Solution. I don't know what the author wants me to study. $\square$

Exercise 1.2.7. Let V be a vector space endowed with a scalar product and a compatible almost complex structure. What is the signature of the pairing $(\alpha, \beta) \mapsto \frac{\alpha\wedge\beta\wedge\omega^{n-2}}{\text{vol}}$ on $\bigwedge^2 V^*$?

Solution. Let $\{x_1, y_1, \cdots, x_i, y_i, \cdots, x_n, y_n\}$ be an orthonormal basis of V and $\bigwedge_{i=1}^n (x^i \wedge y^i)$ be the volume form of V^*. Then $\bigwedge^2 V^*$ can be divided into some orthogonal subspaces, with respect to the pairing, which are spanned by $\{x^1 \wedge y^1, \cdots, x^i \wedge y^i, \cdots, x^n \wedge y^n\}$, $\{x^i \wedge x^j, y^i \wedge y^j\}$ and $\{x^i \wedge y^j, y^i \wedge x^j\}$, where $1 \leqslant i < j \leqslant n$. On the first subspace, the matrix of the pairing is equal to a positive scalar of $[(1-\delta_{ij})]_{1\leqslant i,j\leqslant n}$, where δ_{ij} denotes the Kronecker symbol. So the pairing has 1 positive eigenvalue and $n-1$ negative eigenvalues on the first subspace. It's obvious to see that the pairing has 1 positive eigenvalue and 1 negative eigenvalue on any other subspaces. Hence the pairing has $1 + n(n-1) = n^2 - n + 1$ positive eigenvalues and $n - 1 + n(n-1) = n^2 - 1$ negative eigenvalues on $\bigwedge^2 V^*$. Thus the pairing has signature $(n^2 - n + 1, n^2 - 1)$. $\square$

Exercise 1.2.8. Let $\alpha \in P^k$ and $1 \leqslant s \leqslant r$. Prove the following formula: $\Lambda^s L^r \alpha = r(r-1)\cdots(r-s+1)(n-k-r+1)\cdots(n-k-r+s)L^{r-s}\alpha$.

Solution. The original exercise in [5] has a typo. It states $s \geqslant r$ instead of $s \leqslant r$.

By Corollary 1.2.28 of [5], we have

$$\Lambda L^r \alpha = L^r \Lambda \alpha - [L^r, \Lambda]\alpha = r(n-k-r+1)L^{r-1}\alpha.$$

So the equality required holds directly by induction on the formula above. $\square$

Exercise 1.2.9. (**Wirtinger inequality**) Let $(V, \langle , \rangle)$ be an euclidian vector space endowed with a compatible almost complex structure I and the associated fundamental form ω. Let $W \subset V$ be an oriented subspace of dimension $2m$. The induced scalar product on W together with the chosen orientation define a natural volume form $\mathrm{vol}_W \in \bigwedge^{2m} W^*$. Show that

$$\frac{\omega^m|_W}{\mathrm{vol}_W} \leqslant m!$$

and that equality holds if and only if $W \subset V$ is a complex subspace, i.e. $I(W) = W$, and the orientation is the one induced by the almost complex structure.

Solution. Note that $\omega|_M$ can be viewed as a $2m \times 2m$ skew-symmetric matrix, so there exists an orthonormal basis $\{e_k : 1 \leqslant k \leqslant 2m\}$ such that

$$\omega|_M = \sum_{i=1}^{m} \lambda_i e^{2i-1} \wedge e^{2i},$$

where $\{e^k\}$ is the dual basis of e_k and $\lambda_i \geqslant 0$. Note that

$$\lambda_i = \omega(e_{2i-1}, e_{2i}) = \langle I(e_{2i-1}), e_{2i} \rangle \leqslant \|e_{2i-1}\| \cdot \|e_{2i}\| = 1,$$

hence the original inequality holds. And it turns to be an quality iff $\lambda_i = 1$ for any $i = 1, \cdots, n$, i.e. $I(e_{2i-1}) = e_{2i}$ and the orientation induced by $\{e_k\}$ congruent with th orientation induced by I. $\square$

Exercise 1.2.10. Choose an orthonormal basis x_1, $y_1 = I(x_l)$, $\cdots$, x_n, $y_n = I(x_n)$ of an euclidian vector space V endowed with a compatible almost complex structure I. Show that the dual Lefschetz operator applied to a two-form α is explicitly given by $\Lambda\alpha = \sum \alpha(x_i, y_i)$.

Solution. Since $\Lambda\alpha$ is a $0-$form,

$$\Lambda\alpha = \langle \Lambda\alpha, 1 \rangle = \langle \alpha, \omega \rangle = \alpha \wedge *\omega(x_1, y_1, \cdots, x_n, y_n) = \sum_{i=1}^{n} \alpha(x_i, y_i).$$

$\square$

1.3 Differential Forms

Exercise 1.3.1. Let $f : U \to V$ be a holomorphic map. Show that the natural pull-back $f^* : \mathcal{A}^k(V) \to \mathcal{A}^k(U)$ induces maps $\mathcal{A}^{p,q}(V) \to \mathcal{A}^{p,q}(U)$.

> ***Solution.*** Without loss of generality, we can only consider $\alpha = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_p \wedge \mathrm{d}\bar{z}_{p+1} \wedge \cdots \wedge \mathrm{d}\bar{z}_{p+q} \in \mathcal{A}^{p,q}(V)$, $f^*\alpha = \mathrm{d}(z_1 \circ f) \wedge \cdots \wedge \mathrm{d}(z_p \circ f) \wedge \mathrm{d}(\bar{z}_{p+1} \circ f) \wedge \cdots \wedge \mathrm{d}(\bar{z}_{p+q} \circ f)$. Note that $z_i \circ f$ is holomorphic and $\bar{z}_i \circ f$ is anti-holomorphic we can deduce that $f^*\alpha \in \mathcal{A}^{p,q}(U)$. □

Exercise 1.3.2. Show that $\overline{\partial\alpha} = \bar{\partial}\bar{\alpha}$. In particular, this implies that a real $(p,p)-$form $\alpha \in \mathcal{A}^{p,p}(U) \cap \mathcal{A}^{2p}(U)$ is $\partial-$closed (exact) if and only if α is $\bar{\partial}-$closed (exact). Formulate the $\partial-$versions of the three Poincaré lemmas.

> ***Solution.*** For a $(p,q)-$form α, $\overline{\partial\alpha} = \overline{\Pi^{p+1,q}(\mathrm{d}\alpha)} = \Pi^{q,p+1}(\mathrm{d}\bar{\alpha}) = \bar{\partial}\bar{\alpha}$. So the general case holds by the linearity. Now let α be a real $(p,p)-$form. $\partial\alpha = 0$ iff $\bar{\partial}\alpha = \bar{\partial}\bar{\alpha} = 0$. If α is $\partial-$exact, i.e. there exists β such that $\partial\beta = \alpha$, then $\bar{\partial}\bar{\beta} = \overline{\partial\beta} = \bar{\alpha} = \alpha$, so α is also $\bar{\partial}-$exact. The other direction is similar.
>
> $\partial-$**Poincaré lemma in one variable**: Consider an open neighbourhood of the closure of a bounded one-dimensional disc $B_\varepsilon \subset \overline{B_\varepsilon} \subset U \subset \mathbb{C}$. For $\alpha = f\mathrm{d}z \in \mathcal{A}^{0,1}(U)$ the function
>
> $$g(z) := \frac{1}{2\pi\sqrt{-1}} \int_{B_\varepsilon} \frac{\bar{f}(w)}{w-z} \mathrm{d}w \wedge \mathrm{d}\bar{w}$$
>
> on B_ε satisfies $\alpha = \partial\bar{g}$.
>
> $\partial-$**Poincaré lemma in several variables**: Let U be an open neighbourhood of the closure of a bounded polydisc $B_\varepsilon \subset \overline{B_\varepsilon} \subset U \subset \mathbb{C}^n$. If $\alpha \in \mathcal{A}^{p,q}(U)$ is $\partial-$closed and $p > 0$, then there exists a form $\beta \in \mathcal{A}^{p-1,q}(B_\varepsilon)$ with $\alpha = \partial\beta$ on B_ε.
>
> $\partial-$**Poincaré lemma on the open disc**: Let B denote a polydisc B_ε which can be unbounded, i.e. $\varepsilon_i = \infty$ is allowed. If $\alpha \in \mathcal{A}^{p,q}(B)$ is $\partial-$closed and $p > 0$, then there exists a form $\beta \in \mathcal{A}^{p-1,q}(B)$ with $\alpha = \partial\beta$ on B_ε. □

Exercise 1.3.3. Let $B \subset \mathbb{C}^n$ be a polydisc and let $\alpha \in \mathcal{A}^{p,q}(B)$ be a $\mathrm{d}-$closed form with $p, q \geqslant 1$. Show that there exists a form $\gamma \in \mathcal{A}^{p-1,q-1}(B)$ such that $\partial\bar{\partial}\gamma = \alpha$.

> ***Solution.*** By Poincaré lemma, $\mathrm{d}\alpha = 0$ implies that there exists β satisfying $\mathrm{d}\beta = \alpha$. Since only $(p, q-1)-$part and $(p-1, q)-$part turn into $(p,q)-$part after d, we can suppose that β only has $(p, q-1)-$part $\beta^{(p,q-1)}$ and $(p-1,q)-$part $\beta^{(p-1,q)}$. Now by comparing the bidegree of the equality
>
> $$\alpha = \mathrm{d}\beta = (\partial + \bar{\partial})(\beta^{(p,q-1)} + \beta^{(p-1,q)})$$
>
> we can get $\partial\beta^{(p,q-1)} = \bar{\partial}\beta^{(p-1,q)} = 0$ and $\bar{\partial}\beta^{(p,q-1)} + \partial\beta^{(p-1,q)} = \alpha$. So there exists $\gamma_1, \gamma_2 \in \mathcal{A}^{p-1,q-1}$ such that $\partial\gamma_1 = \beta^{(p,q-1)}(B)$ and $\bar{\partial}\gamma_2 = \beta^{(p-1,q)}$. Hence $\alpha = \bar{\partial}\beta^{(p,q-1)} + \partial\beta^{(p-1,q)} = \partial\bar{\partial}(\gamma_2 - \gamma_1)$, i.e. $\gamma = \gamma_2 - \gamma_1$ is what we need. □

Exercise 1.3.4. Show that for a polydisc $B \subset \mathbb{C}^n$ the sequence

$$\mathcal{A}^{p-1,q-1}(B) \xrightarrow{\partial\bar{\partial}} \mathcal{A}^{p,q}(B) \xrightarrow{\mathrm{d}} \mathcal{A}_{\mathbb{C}}^{p+q+1}(B)$$

is exact.

> ***Solution.*** From Exercise 1.3.3 above, we get that $\mathrm{Ker}\ d \subset \mathrm{Im}\partial\bar{\partial}$. In the other direction, $\mathrm{d}\partial\bar{\partial} = 0$ shows that the sequence is exact. □

Exercise 1.3.5. Show that $\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(|z|^2 + 1) \in \mathcal{A}^{1,1}(\mathbb{C})$ is the fundamental form of a compatible metric g that osculates to order two in any point.

Solution. A direct calculate shows that $\omega = \frac{\sqrt{-1}}{2\pi} \cdot \frac{1}{(|z|^2+1)^2} \mathrm{d}z \wedge \mathrm{d}\bar{z}$, so the corresponding compatible metric is $g = \frac{1}{\pi(|z|^2+1)^2} \mathrm{d}z \otimes \mathrm{d}\bar{z}$. It's easy to see that $\mathrm{d}\omega = 0$ since $\mathrm{d}\partial\bar{\partial} = 0$, hence g osculates to order two in any point. $\square$

Exercise 1.3.6. Analogously to Exercise 1.3.5, study the form $\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(1 - |z|^2)$ on $B_1 \subset \mathbb{C}$.

Solution. Similarly, $\omega = \frac{\sqrt{-1}}{2\pi} \cdot \frac{-1}{(1-|z|^2)^2} \mathrm{d}z \wedge \mathrm{d}\bar{z}$, its corresponding compatible metric is $g = -\frac{1}{\pi(1-|z|^2)^2} \mathrm{d}z \otimes \mathrm{d}\bar{z}$. However, it's not positive definite.
I think there is a typo in [5], a '-' should be added into ω here. $\square$

Exercise 1.3.7. Let $\omega = \frac{\sqrt{-1}}{2\pi} \sum \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$ be the standard fundamental form on $\mathbb{C}^n$. Show that one can write $\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi$ for some positive function φ and determine φ. The function φ is called the Kähler potential.

Solution. It is equivalent to solve $\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} = \delta_{ij}$, and $\varphi = \sum |z_i|^2$ is a trivial solution. $\square$

Exercise 1.3.8. Let $\omega \in \mathcal{A}^{1,1}(B)$ be the fundamental form associated to a compatible metric on a polydisc $B \subset \mathbb{C}^n$ which osculates in every point $z \in B$ to order two. Show that $\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi$ for some real function $\varphi \in \mathcal{A}^0(B)$.

Solution. We know that $\mathrm{d}\omega = 0$ since the corresponding osculates in every point to order two. So there exists a function $\varphi_0 \in \mathcal{A}^{0,0}(B)$ such that $\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi_0$ by Exercise 1.3.3. Since ω is real, we have $\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi_0 = \omega = \bar{\omega} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\bar{\varphi}_0$. Hence $\varphi = (\varphi_0 + \bar{\varphi}_0)/2$ is a real function satisfying $\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi$. $\square$

Exercise 1.3.9. Let g be a compatible metric on $U \subset \mathbb{C}^n$ that osculates to order two in any point $z \in U$. For which real function f has the conformally equivalent metric $\mathrm{e}^f \cdot g$ the same property?

Solution. Suppose the fundamental forms of g and $\mathrm{e}^f \cdot g$ are ω and ω' respectively. Then $\omega' = \mathrm{e}^f \cdot g(I(), ()) = \mathrm{e}^f \omega$ and $\mathrm{d}\omega = 0$. If the conformal metric has the same property, we can get $L(\mathrm{d}\mathrm{e}^f) = \omega \wedge (\mathrm{d}\mathrm{e}^f) = \mathrm{d}(\mathrm{e}^f \omega) = 0$. If $n = 1$, then this equlity holds automatically. If $n \geqslant 2$, since L^{n-1} is an isomorphism between $\bigwedge^1 T^*U$ and $\bigwedge^{n-1} T^*U$, we get that $\mathrm{d}\mathrm{e}^f = 0$, hence e^f is a constant, i.e. f is a constant. $\square$

2 Complex Manifolds

2.1 Complex Manifolds: Definition and Examples

Exercise 2.1.1. Show that $\mathbb{P}^n$ is a compact complex manifold. Describe a diffeomorphism of $\mathbb{P}^1$ with the two-dimensional sphere S^2. Conclude that $\mathbb{P}^1$ is simply connected.

Solution. We regard S^{2n+1} as the subset of $\mathbb{C}^{n+1}$ by the natural inclusion $S^{2n-1} \subset \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$. Note that $\mathbb{P}^n$ is diffeomorphism to the orbit space of the S^1-action on S^{2n+1}, i.e. $z \cdot (z_0, \cdots, z_n) = (zz_0 . \cdots, zz_n)$, $\mathbb{P}^n$ is the continuous image of a compact space. Hence $\mathbb{P}^n$ is compact. The diffeomorphism between $\mathbb{P}^1$ and S^2 can be described by the well-known stereographic projection, hence $\mathbb{P}^1$ is simply-connected. $\square$

Exercise 2.1.2. Show that $\mathbb{C}^n$ does not have any compact submanifolds of positive dimension.

Solution. Suppose X is a compact submanifold of $\mathbb{C}^n$, without loss of generality, we assume X is connected. Then the embedding $X \subset \mathbb{C}^n$ is holomorphic. By Liouville's Theorem it must be constant, so X must be 0-dimensional. $\square$

Exercise 2.1.3. Determine the algebraic dimension of the following manifolds: $\mathbb{P}^1$, $\mathbb{P}^n$, and $\mathbb{C}/(\mathbb{Z}+\sqrt{-1}\mathbb{Z})$. How big is the function field of $\mathbb{C}$?

Solution. Consider the identity map of $\mathbb{C}$, it can be extended into a meromorphic function on $\mathbb{P}^1$, so $a(\mathbb{P}^1) \geqslant 1$. By Siegel's Theorem, it must be 1. SImilarly, consider the map z_i/z_0, $i = 1, \cdots, n$, defined on $\mathbb{P}^n$. It's trivial that they are well-defined and meromorphic. Since z_i, $i = 0, \cdots, n$ are algebraically independent on $\mathbb{C}^{n+1}$, the above n functions are algebraically independent on $\mathbb{P}^n$. So $a(\mathbb{P}^n) \geqslant n$ and by Siegle's Theorem $a(\mathbb{P}^n) = n$ again. For any complex torus $\mathbb{C}/\Gamma$ with dimension 1, its function field can be generated by the corresponding Weierstrass $\wp-$function and its derivative (see Theorem 1.8 of Chapter 9 in [9]), and they are algebraically dependent, hence $a(\mathbb{C}/\Gamma) = 1$. The algebraic dimension of $\mathbb{C}$ is at least $\aleph_0$ since $\exp(z^k)$ $(k \in \mathbb{N})$, are algebraic independent element in the function field of $\mathbb{C}$. $\square$

Exercise 2.1.4. Show that any holomorphic map from $\mathbb{P}^1$ into a complex torus is constant. What about maps from $\mathbb{P}^n$ into a complex torus?

Solution. Since $\mathbb{P}^n$ is simply connected from that it has a cell structure with only even cells, the holomorphic map from $\mathbb{P}^n$ to a complex torus can be lifted into a holomorphic map from $\mathbb{P}^n$ to a affine space, then it must be constant by Exercise 2.1.2. $\square$

Exercise 2.1.5. Consider the Hopf curve $X = \mathbb{C}^*/\mathbb{Z}$, where $k \in \mathbb{Z}$ acts by $z \mapsto \lambda^k z$, for $\lambda \in \mathbb{R}_{>0}$. Show that X is isomorphic to an elliptic curve $\mathbb{C}/\Gamma$ and determine Γ explicitly.

Solution. Let Γ be generated by $2\pi\sqrt{-1}$ and $\log \lambda$, and consider the following diagram.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\exp"] \arrow[d, "\pi"'] & \mathbb{C}^* \arrow[d, "\pi'"] \\ \mathbb{C}/\Gamma \arrow[r, dashed, "f"] & X \end{tikzcd}$$

Since $\exp(z + m \cdot 2\pi\sqrt{-1} + n \log \lambda) = \lambda^n \exp(z)$, where $m, n \in \mathbb{Z}$, there exists unique holomorphic map $f : \mathbb{C}/\Gamma \to X$ such that the diagram above is commutative. And it's trivial to see that f is bijective hence is biholomorphic. $\square$

Exercise 2.1.6. Generalize the construction of the Hopf manifolds by considering the action of $\mathbb{Z}$ given by $(z_1, \cdots, z_n) \mapsto (\lambda_1^k z_1, \cdots, \lambda_n^k z_n)$, where $0 < \lambda_i < 1$. Show that the quotient $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$ is again diffeomorphic to $S^1 \times S^{2n-1}$.

Solution. Now we regard S^1 as $\mathbb{R}/\mathbb{Z}$ and S^{2n-1} be the unit sphere in $\mathbb{C}^n$. Then the diffeomorphism from $S^1 \times S^{2n-1}$ to the Hopf manifold can be expressed by $(t, z_1, \cdots, z_n) \mapsto (\lambda_1^t z_1, \cdots, \lambda_n^t z_n)$. □

Exercise 2.1.7. Show that any Hopf surface contains elliptic curves.

Solution. The projection of $\mathbb{C}^* \times \{0\} \subset \mathbb{C}^2 \setminus \{0\}$ is an elliptic curve contained in Hopf surface by Exercise 2.1.5. □

Exercise 2.1.8. Describe the quotient of the torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ by the involution $z \mapsto -z$ locally and globally. Using the Weierstrass function $\wp$ again, one can show that the quotient is isomorphic to $\mathbb{P}^1$. What happens in higher dimensions?

Solution. It's easy to see that the quotient of the torus is still a complex manifold with dimension 1. Since the Weierstrass function $\wp$ is a proper holomorphic even function from $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ to $\mathbb{P}^1$ with multiplicity 2, we can claim that $\wp$ induces a biholomorphic map between the quotient and $\mathbb{P}^1$.

I don't know what happens in higher dimensions, but the quotient should still be a complex manifold. □

Exercise 2.1.9. Let E_1, E_2 be two elliptic curves and let $G = \mathbb{Z}/2\mathbb{Z}$ act by translation on E_1 and by the involution $z \mapsto -z$ on E_2. Study the quotient $(E_1 + E_2)/G$.

Solution. I don't know what the author wants me to study. □

Exercise 2.1.10. Describe connected complex manifolds X and Y together with a holomorphic map $X \to Y$ such that every complex torus of dimension one is isomorphic to one of the fibres.

Solution. Consider a $\mathbb{Z}^2-$action an $\mathbb{C} \times \mathbb{H}$ such that

$$(m, n) \cdot (z, \tau) = (z + m\tau + n, \tau).$$

Since it is a properly discontinuous action, its orbit space X is still a complex manifold, and the projection from $\mathbb{C} \times \mathbb{H}$ to $\mathbb{H}$ induces a holomorphic map f from X to $\mathbb{H}$, whose fibre at $\tau \in \mathbb{H}$ is isomorphic to $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. Since every complex torus with dimension 1 is isomorphic to a $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, f is the map we need. □

Exercise 2.1.11. Show that $(W \subset V) \mapsto (\bigwedge^k W \subset \bigwedge^k V)$ defines a (holomorphic) embedding $\mathrm{Gr}_k(V) \hookrightarrow \mathbb{P}(\bigwedge^k V)$. (This is called the *Plücker embedding* $\mathrm{Gr}_k(V)$. In particular, the Grassmannians are all projective.)

Solution. Choose a basis of V, then $W \in \mathrm{Gr}_k(V)$ can be expressed by a $k \times (n-k)$ matrix A, and the Plücker embedding of A can be expressed by $A \mapsto (\det A_i)_{i=1}^{\binom{n}{k}}$, where A_i denotes the $i-$th $k \times k$ minor of A. Under this expression, it's easy to verify that this map is holomorphic, injective and nowhere singular, hence is an embedding. □

Exercise 2.1.12. Let ρ be a fifth root of unity. The group $G = \langle \rho \rangle \cong \mathbb{Z}/5\mathbb{Z}$ acts on $\mathbb{P}^3$ by

$$(z_0 : z_1 : z_2 : z_3) \mapsto (z_0 : \rho z_1 : \rho^2 z_2 : \rho^3 z_3).$$

Describe all fix points of this action. Show that the surface Y defined by $\sum_{i=0}^{3} z_i^5 = 0$ is $G-$invariant and that the induced action is fix point free.

Solution. If $(z_0 : z_1 : z_2 : z_3)$ is a fix point of this action, then it's obvious to see that there must be three zeroes in z_0, z_1, z_2, z_3, since $z_i z_j = 0$ for any $0 \leqslant i < j \leqslant 3$. So this action has only four fix points on $\mathbb{P}^3$, $(0 : 0 : 0 : 1)$, $(0 : 0 : 1 : 0)$, $(0 : 1 : 0 : 0)$ and $(1 : 0 : 0 : 0)$. Since $z_0^5 + (\rho z_1)^5 + (\rho^2 z_2)^5 + (\rho^3 z_3)^5 = z_0^5 + z_1^5 + z_2^5 + z_3^5$, Y is $G-$invariant. And the four fix points above do not belong to Y. □

Exercise 2.1.13. Let $G = \langle \rho \rangle$ as before and let $\tilde{G}$ be the following subgroup of G^5:

$$\tilde{G} = \left\{ (\xi_0, \cdots, \xi_4) | \xi_i \in G, \prod_{i=1}^{4} \xi_i = 1 \right\}.$$

We let $\tilde{G}$ act on $\mathbb{P}^4$ by $(z_0 : z_1 : z_2 : z_3 : z_4) \mapsto (\xi_0 z_0 : \xi_1 z_1 : \xi_2 z_2 : \xi_3 z_3 : \xi_4 z_4)$. Describe the subgroup H that acts trivially. Show that the hypersurface

$$X = V\left(\sum_{i=0}^{4} z_i^5 - 5t \prod_{i=0}^{4} z_i \right)$$

with $t \in \mathbb{C}$, is invariant under $\tilde{G}$. Study the action of $\tilde{G}/H$ on X, in particular the points with non-trivial stabilizer.

Solution. If (ξ_i) acts trivially on $\mathbb{P}^4$, then for any $0 \leqslant i < j \leqslant 4$ we have $\xi_i z_i z_j = \xi_j z_i z_j$, hence $\xi_0 = \xi_1 = \xi_2 = \xi_3 = \xi_4$. So

$$H = \{(\rho^i, \rho^i, \rho^i, \rho^i, \rho^i) | 0 \leqslant i \leqslant 4\}.$$

For any $(z_0 : z_1 : z_2 : z_3 : z_4) \in \mathbb{P}^4$, $(\xi_0, \xi_1, \xi_2, \xi_3, \xi_4) \in \tilde{G}$, we have

$$\sum_{i=0}^{4} (\xi_i z_i)^5 - 5t \prod_{i=0}^{4} (\xi_i z_i) = \sum_{i=0}^{4} z_i^5 - 5t \prod_{i=0}^{4} z_i = 0,$$

so X is invariant under $\tilde{G}$.

If $(z_0 : z_1 : z_2 : z_3 : z_4)$ has all coordinates nonzero, then its stabilizer must be trivial in $\tilde{G}/H$.

If $(z_0 : z_1 : z_2 : z_3 : z_4)$ only has one coordinate zero, without loss of generality, we assume $z_4 = 0$, then the element (ξ_i) of its stabilizer must have $\xi_0 = \xi_1 = \xi_2 = \xi_3$, then $\prod_{i=0}^{4} \xi_i = 1$ implies that $(\xi_i) \in H$ and hence the stabilizer is trivial in $\tilde{G}/H$.

If $(z_0 : z_1 : z_2 : z_3 : z_4)$ only has two coordinates zero, without loss of generality, we assume $z_3 = z_4 = 0$, then the element (ξ_i) of its stabilizer must have $\xi_0 = \xi_1 = \xi_2$. We can assume that they all equal to 1, otherwise we can choose the element in the same coset with it. Then its stablizer is generated by $(1, 1, 1, \rho, \rho^4)$ in $\tilde{G}/H$.

If $(z_0 : z_1 : z_2 : z_3 : z_4)$ only has three coordinates zero, without loss of generality, we assume $z_2 = z_3 = z_4 = 0$, then the element (ξ_i) of its stabilizer must have $\xi_0 = \xi_1$. We can assume that they all equal to 1, otherwise we can choose the element in the same coset with it. Then its stablizer is generated by $(1, 1, 1, \rho, \rho^4)$ and $(1, 1, \rho, 1, \rho^4)$in $\tilde{G}/H$.

For $(z_0 : z_1 : z_2 : z_3 : z_4)$ has only one coordinate nonzero, it can't belong to X. □

2.2 Holomorphic Vector Bundles

Exercise 2.2.1. Let E and F be vector bundles determined by the cocycles $\{(U_i, \psi_{ij} : U_i \cap U_j \to \mathrm{GL}(r, \mathbb{C})\}$ respectively $\{(U_i, \psi'_{ij} : U_i \cap U_j \to \mathrm{GL}(r', \mathbb{C})\}$. Verify the cocycle description for the constructions below:

i) The direct sum $E \oplus F$ corresponds to $\psi_{ij} \oplus \psi'_{ij} : U_i \cap U_j \to \mathrm{GL}(r + r', \mathbb{C})\}$.

ii) The tensor product $E \otimes F$ corresponds t o $\psi_{ij} \otimes \psi'_{ij} : U_i \cap U_j \to \mathrm{GL}(r \cdot r', \mathbb{C})\}$.

iii) The dual bundle E^* corresponds to $(\psi_{ij}^{\mathrm{t}})^{-1}$.

iv) The determinant bundle $\det(E)$ corresponds to $\det(\psi_{ij})$.

Solution. Suppose the trivialization of E and F are $\{(U_i, \varphi_i)\}$ and $\{(U_i, \varphi'_i)\}$ respectively.

i) $(\varphi_i \oplus \varphi'_i) \circ (\varphi_j \oplus \varphi'_j)^{-1}(x, \cdot) = (\varphi_i \circ \varphi_j^{-1}) \oplus (\varphi'_i \circ {\varphi'_j}^{-1})(x, \cdot) = \psi_{ij} \oplus \psi'_{ij}$.

ii) $(\varphi_i \otimes \varphi'_i) \circ (\varphi_j \otimes \varphi'_j)^{-1}(x, \cdot) = (\varphi_i \circ \varphi_j^{-1}) \otimes (\varphi'_i \circ {\varphi'_j}^{-1})(x, \cdot) = \psi_{ij} \otimes \psi'_{ij}$.

iii) $(\varphi_i^{\mathrm{t}})^{-1} \circ \varphi_j^{\mathrm{t}}(x, \cdot) = ((\varphi_i \circ \varphi_j^{-1})^{\mathrm{t}})^{-1}(x, \cdot) = (\psi_{ij}^{\mathrm{t}})^{-1}$.

iv) $\det(\varphi_i) \det(\varphi_j^{-1})(x, \cdot) = \det(\varphi_j \circ \varphi_j^{-1})(x, \cdot) = \det(\psi_{ij})$.

□

Exercise 2.2.2. Show that any short exact sequence of holomorphic vector bundles $0 \to L \to E \to F \to 0$, where L is a line bundle, induces short exact sequences of the form

$$0 \longrightarrow L \otimes \bigwedge^{i-1} F \longrightarrow \bigwedge^{i} E \longrightarrow \bigwedge^{i} F \longrightarrow 0.$$

Solution. Let the map $L \to E$ and $E \to F$ denote by f, g respectively in the short exact sequence. It's trivial to see that the surjective map g can induce a surjective map $\tilde{g}_i : \bigwedge^i E \longrightarrow \bigwedge^i F$. Now consider (U, φ_L), (U, φ_E), (U, φ_F) as trivializations of L, E, F respectively. For a point $x \in U$, we define a map $L \otimes \bigwedge^{i-1} F(x) \to \bigwedge^i E(x)$ as follows: it maps $l \otimes v \in L \otimes \bigwedge^{i-1} F(x)$ to $f(l) \wedge \tilde{g}_{i-1}^{-1}(v)$ and extends by linearity. Since L is a line bundle, this map is independent on the choice of the preimage of v and this map is independent on the choice of the local trivialization, hence can be glued to a bundle map $h : L \otimes \bigwedge^{i-1} F \to \bigwedge^i E$. Now we claim that h is injective at every fibre to deduce that h is injective. If $h(l \otimes v) = 0$, then there exists a component v_j of v such that $f(l) = g^{-1}(v_j)$ or $f(l) = 0$ or $g^{-1}(v_j) = 0$, which implies $v_j = 0$ or $l = 0$ by the exactness of the original sequence, and implies that $l \otimes v = 0$. Actually, the decomposable case above implies the injectivity of h since we can choose a basis of $E(x)$. The exactness of the new sequence at $\bigwedge^i E$ follows from the exactness of the original sequence at E directly. □

Exercise 2.2.3. Show that for any holomorpic vector bundle E of rank r there exists a non-degenerate pairing

$$\bigwedge^k E \otimes \bigwedge^{r-k} E \longrightarrow \det(E).$$

Deduce from this the existence of a natural isomorphism of holomorphic vector bundles $\bigwedge^k E \cong \bigwedge^{r-k} E^* \otimes \det(E)$.

Solution. Trivial. The pairing is just the exterior product. □

Exercise 2.2.4. Show that any homomorphism $f : E \to F$ of holomorphic vector bundles E and F induces natural homomorphisms $f \otimes \mathrm{id}_G : E \otimes G \to F \otimes G$ for any holomorphic vector bundle G. If f is injective, then so is $f \otimes \mathrm{id}_G$.

Solution. Under a local trivialization, f can be expressed by a $\dim E \times \dim F$ matrix function A holomorphic with respect to the base manifold. Then $f \otimes \mathrm{id}_G$ can be expressed by the $(\dim E \cdot \dim G) \times (\dim F \cdot \dim G)$ matrix function $B := \mathrm{diag}(A, \cdots, A)$. It is also holomorphic with respect to the base manifold. So we can deduce that $f \otimes \mathrm{id}_G$ is indeed a bundle map. If f is injective, then A has full rank and so does B, hence $f \otimes \mathrm{id}_G$ is injective. $\square$

Exercise 2.2.5. Let L be a holomorphic line bundle on a compact complex manifold X. Show that L is trivial if and only if L and its dual L^* admit non-trivial global sections.

Solution. Suppose L and L^* admit non-trivial global sections s and σ repectively, then $s \otimes \sigma$ is a global section of $L \otimes L^*$, which is isomorphic to the trivial bundle on X. Since X is compact, $s \otimes \sigma$ must be a non zero constant section. Hence s and σ are both nonvanish on X. So L is trivial.

If L is trivial, then L^* is also trivial, hence they both admit non-trivial global sections. $\square$

Exercise 2.2.6. Let $L \in \mathrm{Pic}(X)$ and $Y \subset X$ a submanifold of codimension at least two. Show that the restriction $H^0(X, L) \to H^0(X \setminus Y, L)$ is bijective.

Solution. For any $s \in H^0(X \setminus Y, L)$, we first suppose U is a local trivialization of L and (U, φ) be a chart such that $\varphi(Y \cap U) \subset \varphi(U)$ is the restriction of the natural embedding $\mathbb{C}^{\dim X - \dim Y} \to \mathbb{C}^{\dim X}$. Then $s|_{U \setminus Y}$ can be viewed as a holomorphic function and can be extended to an element of $H^0(U, L)$ by Exercise 1.1.20. Since such U can be collected into an open cover of X, by the Identity Theorem, the extensions with respect to every U can be glued into an element of $H^0(X, L)$. $\square$

Exercise 2.2.7. Let L_1 and L_2 be two holomorphic line bundles on a complex manifold X . Suppose that $Y \subset X$ is a submanifold of codimension at least two such that L_1 and L_2 are isomorphic on $X \setminus Y$. Prove that $L_1 \cong L_2$.

Solution. The isomorphism $f : L_1 \to L_2$ on $X \setminus Y$ can be viewed as an element in $H^0(X \setminus Y, L_1^* \otimes L_2)$. By Exercise 2.2.6, f can be extended into $\tilde{f} \in H^0(X, L_1^* \otimes L_2)$. Now since f is an isomorphism, $\tilde{f}$ is nonvanishing in $X \setminus Y$, i.e. $Z(\tilde{f}) \subset Y$. However $Z(\tilde{f})$ has codimension 1 if it is not empty, so $Z(\tilde{f})$ must be empty. Hence $L_1 \cong L_2$.

The isomorphism may also be deduced by that the corresponding locally free $\mathcal{O}_X-$modules are isomorphism by Exercise 2.2.6. $\square$

Exercise 2.2.8. Show that any non-trivial homogeneous polynomial

$$0 \neq s \in \mathbb{C}[z_0, \cdots, z_n]_k$$

of degree k can be considered as a non-trivial section of $\mathcal{O}(k)$ on $\mathbb{P}^n$.

Solution. Since global linear coordinates$z_0, \cdots, z_n$ on $\mathbb{C}^{n+1}$ define natural sections of $\mathcal{O}(1)$, any non-trivial homogeneous polynomial $0 \neq s \in \mathbb{C}[z_0, \cdots, z_n]_k$ of degree k can be viewed as a non-trivial section of $\mathcal{O}(k)$. $\square$

Exercise 2.2.9. Show that $\mathcal{O}(-1) \setminus s(\mathbb{P}^n)$ is naturally identified with $\mathbb{C}^{n+1} \setminus \{0\}$, where $s : \mathbb{P}^n \to \mathcal{O}(-1)$ is the zero-section. Use this to construct a submersion $S^{2n+1} \to \mathbb{P}^n$ with fibre S^1.

Solution. It's easy to see that the projection $\mathcal{O}(-1) \setminus s(\mathbb{P}^n) \subset \mathbb{P}^n \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ with respect to the second coordinate gives the identity between $\mathcal{O}(-1) \setminus s(\mathbb{P}^n)$ and $\mathbb{C}^{n+1} \setminus \{0\}$. Hence the sphere bundle of $\mathcal{O}(-1)$ gives the submersion $S^{2n+1} \to \mathbb{P}^n$ with fibre S^1. $\square$

Exercise 2.2.10. Let $\{(U_i, \varphi_i)\}$ be an atlas of the complex manifold X. Use the cocycle description of the holomorphic tangent bundle $\mathcal{T}_X$ to show that for $x \in U_i \subset X$ the fibre $\mathcal{T}_X(x)$ can be identified with $T_{\varphi_i(x)}\varphi_i(U_i) \cong T^{1,0}_{\varphi_i(x)}\varphi_i(U_i)$. In particular, the vectors $\partial/\partial z_i$ can be viewed as a basis of $\mathcal{T}_X(x)$.

Solution. Note that the transition function of $\mathcal{T}_X$ on $U_i \cap U_j$, which is $J(\varphi_{ij})(\varphi_j(x))$, maps $T_{\varphi_j(x)}\varphi_j(U_j)$ into $T_{\varphi_i(x)}\varphi_i(U_i)$, hence we can identify $\mathcal{T}_X(x)$ with $T_{\varphi_i(x)}\varphi_i(U_i)$. □

Exercise 2.2.11. Let $Y \subset X$ be a submanifold of dimension k of a complex manifold X of dimension n. Assume that $\{(U_i, \varphi_i)\}$ is a compatible atlas in the sense of Definition 2.1.16 in [5]. Show that for $x \in Y \cap U_i$ the tangent vectors $\partial/\partial z_{k+1}, \cdots \partial/\partial z_n \in T^{1,0}_{\varphi_i(x)}\varphi_i(U_i)$ form in a natural way a basis of $\mathcal{N}_{Y/X}(x)$.

Solution. It directly follows from the cocycle description of the normal bundle, i.e. transition function can be expressed by a quasi-upper triangular matrices. (c.f. Lemma 2.2.15 of [5]) □

Exercise 2.2.12. Let $Y \subset X$ be a submanifold locally in $U \subset X$ defined by holomorphic functions $f_1, \cdots, f_{n-k}$ (i.e. 0 is a regular value of $(f_1, \cdots, f_{n-k}) : U \to \mathbb{C}^{n-k}$ and Y is the pre-image of it.). Show that $f_1, \cdots, f_{n-k}$ naturally induce a basis of $\mathcal{N}_{Y/X}(x)^*$ for any $x \in U \cap Y$. Go on and prove the existence of a natural isomorphism $\mathcal{N}^*_{Y/X} \cong \mathcal{I}_Y/\mathcal{I}^2_Y$.

Solution. Consider the differential map induced by f_j, i.e. $[\partial/\partial z_i \mapsto \partial f_j \partial z_i]$, which is a linear function on $\mathcal{N}_{Y/X}(x)$. Since 0 is a regular value of $(f_1, \cdots, f_{n-k})$, the differential map induced by them form a basis of $\mathcal{N}_{Y/X}(x)^*$.

To give the isomorphism between $\mathcal{N}^*_{Y/X}$ and $\mathcal{I}_Y/\mathcal{I}^2_Y$ (I think $\mathcal{N}^*_{Y/X}$ here means the corresponding sheaf of sections.), we only need to take differential of a function in $\mathcal{I}_Y(U)$ in a compatible atlas U. □

Exercise 2.2.13. Show that the holomorphic tangent bundle of a complex torus $X = \mathbb{C}^n/\Gamma$ is trivial, i.e. isomorphic to the trivial vector bundle $\mathcal{O}^{\oplus n}$. Compute $\mathrm{kod}(X)$.

Solution. By choosing the suitable atlas on X with transition function be translation, one can observe that the holomorphic tangent bundle is trivial directly. Since $\mathcal{T}_X$ is trivial, then K_X is also trivial. So $R(X) = \mathbb{C}[s]$ due to X is compact, where s denote the constant section of K_X. Hence $\mathrm{kod}(X) = 0$. □

Exercise 2.2.14. Show that any submanifold of a complex torus has non-negative Kodaira dimension.

Solution. A complex submanifold embedding $Y \hookrightarrow X$ induces the field embedding $Q(X) \hookrightarrow Q(Y)$, hence $\mathrm{kod}(Y) \geqslant \mathrm{kod}(X) = 0$. □

Exercise 2.2.15. A parallelizable complex manifold is a manifold whose holomorphic tangent bundle is trivial. Thus, complex tori are parallelizable. Show that the Iwasawa manifold is parallelizable. Compute the Kodaira dimension of a compact parallelizable manifold.

Solution. It's well-known that every complex Lie group is parallelizable. Just like the complex torus, one can readily prove that the Kodaira dimension of a compact parallelizable manifold is 0. □

Exercise 2.2.16. What can you say at this point about the algebraic dimension of a projective manifold?

Solution. I don't know what to say. □

2.3 Divisors and Line Bundles

Exercise 2.3.1. Show that the natural map $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ is not injective if and only if $a(X) > 0$.

Solution. We use the notation of Corollary 2.3.10 in [5] in the following argument. If the natural map is not injective, then there is a nontrivial $D \in \mathrm{Div}(X)$ such that $\mathcal{O}(D)$ is trivial. This happens iff there are $g_i \in \mathcal{O}_{U_i}^*$ such that $f_i \cdot f_j^{-1} = g_i \cdot g_j^{-1}$ on $U_i \cap U_j$ for any i, j, which implies that there exists $h \in K(X)$ such that $h = f_i \cdot g_i^{-1}$ on every U_i. Since f_i is not holomorphic, h cannot be a constant. So $a(X) > 0$.

If $a(X) > 0$, then it's clear that the divisor of a non-constant meromorphic function is mapped to the trivial line bundle by the natural map. □

Exercise 2.3.2. Let $Y \subset X$ be a smooth hypersurface defined by a section $s \in H^0(X, L)$ for some holomorphic line bundle $L \in \mathrm{Pic}(X)$. Show that the normal bundle $\mathcal{N}_{Y/X}$ is isomorphic to $L|_Y$.

Solution. It follows from the cocycle description of the normal bundle directly. For a complete proof, see Proposition 2.4.7 in [5]. □

Exercise 2.3.3. Determine the normal bundle of a complete intersection $X \subset \mathbb{P}^N$ defined by irreducible homogeneous polynomials $f_1, \cdots, f_k$ of degree $d_1, \cdots, d_k$.

Solution. Consider the standard cover $\{U_i, \varphi_i\}$ of $\mathbb{P}^N$. For a point $x = (z_0 : \cdots : z_{N+1}) \in X \cap U_i \cap U_j$, without loss of generality, suppose $\det \left(\frac{\partial f_i}{\partial z_j}\right)_{1 \leqslant i,j \leqslant k} \neq 0$. Now let $\psi_i : U_i \to \mathbb{C}^N$ maps x to $(f_1/z_i^{d_1}, \cdots, f_k/z_i^{d_k}, z_{k+1}/z_i, \cdots, z_{N+1}/z_i)$ and omits the $i-$th components. So (U_i, ψ_i) forms a new coordinate system around x. Now it is easy to obverse that $\mathcal{N}_{X/\mathbb{P}^N}$ is $\bigoplus_{i=1}^k \mathcal{O}(d_i)|_X$ by verifying the transition function. □

Exercise 2.3.4. Show that the image of $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ consists of those line bundles admitting non-trivial meromorphic sections.

Solution. For $D \in \mathrm{Div}(X)$, the corresponding element in $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ gives the non-trivial meromorphic section of $\mathcal{O}(D)$.

If there is a non-trivial meromorphic section (f_i) on $L \in \mathrm{Pic}(X)$, then $L = \mathcal{O}((f_i))$. □

Exercise 2.3.5. Prove that the Veronese map $\mathbb{P}^n \to \mathbb{P}^N$ is an embedding.

Solution. For $(z_0^{i_0} \cdots z_n^{i_n})_{i_1+\cdots+i_n=d}$, without loss of generality, we assume that $z_0 \neq 0$, then $z_0^d : z_0^{d-1} z_i$ determines $z_0 : z_i$. Hence the Veronese map is injective. Consider the standard open cover $\{z_0 \neq 0\}$ and $\{z_0^d \neq 0\}$ on $\mathbb{P}^n$ and $\mathbb{P}^N$ respectively. The Jacobian under these two coordinate system has a block I_n, hence the Veronese map is non-singular. Since $\mathbb{P}^n$ is compact, the Veronese map must be an embedding. □

Exercise 2.3.6. Let L_1 and L_2 be holomorphic line bundles on complex manifolds X_1 and X_2, respectively. To two sections $s_i \in H^0(X_i, L_i)$, $i = 1, 2$, one associates a section $s_1 \cdot s_2 \in H^0(X_1 \times X_2, p_1^*(L_1) \otimes p_2^*(L_2))$. Show that if $s_1^1, \cdots, s_1^k$ and $s_2^1, \cdots, s_2^l$ are linearly independent sections of L_1 and L_2, respectively, then $s_1^i s_2^j$ form linearly independent sections of $p_1^*(L_1) \otimes p_2^*(L_2)$. Here, p_1 and p_2 are the two projections.

Solution. Suppose there are $a_{ij} \in \mathbb{C}$ such that

$$\sum_{i,j} a_{ij} s_1^i(x_1) \otimes s_2^j(x_2) = 0$$

for any $(x_1, x_2) \in X_1 \times X_2$, then since s_2^j are linearly independent, we get that for each j,

$$\sum_i a_{ij} s_1^i(x_1) = 0$$

for any $x_1 \in X_1$. So $a_{ij} = 0$ by s_1^i are linearly independent. □

Exercise 2.3.7. Let $x \in \mathbb{P}^n$ and consider the linear system

$$\{s \in H^0(\mathbb{P}^n, \mathcal{O}(1)) | s(x) = 0\}.$$

Show that it defines a holomorphic map $\varphi : \mathbb{P}^n \setminus \{x\} \to \mathbb{P}^{n-1}$. Describe this map geometrically.

Solution. Since $s \in H^0(\mathbb{P}^n, \mathcal{O}(1))$ can be expressed by the linear combination of the coordinate function on $\mathbb{P}^n$, the linear system defined above is a linear space with dimension n. So we can choose a basis s_i of this linear space. Then $y \mapsto (s_1(y) : \cdots : s_n(y))$ is the map we need, by noting that $s_i(y)$ cannot be all 0. In fact, it can be viewed as the map induced by the projection $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}/\mathbb{C}x$. □

Exercise 2.3.8. (Bézout's theorem) Let $C \subset \mathbb{P}^2$ be a smooth curve defined by a homogeneous polynomial f of degree d. Show that the line bundle $\mathcal{O}(1)$ restricted to C is of degree d. Let $D \subset \mathbb{P}^2$ be a second smooth curve different from C defined by a homogeneous polynomial g of degree e. Show that

$$d \cdot e = \sum_{p \in C \cap D} \dim \mathcal{O}_{\mathbb{P}^2,p}/(f, g).$$

Solution. Consider the global section $z_0 + \lambda z_1$ of $\mathcal{O}(1)$ such that $z_0 + \lambda z_1 \nmid f$, then the degree of $\mathcal{O}(1)|_C$ is equal to the number of intersections (with multiplicities) of C and $z_0 + \lambda z_1 = 0$. Hence it can be viewed as the number of zeroes of a one-variable polynomial with degree d on $\mathbb{P}^1$, which is equal to d.

See Example I.9 of [1] for a proof of Bézout's theorem. □

Exercise 2.3.9. Show that the image of $\varphi_{\mathcal{O}(3)} : \mathbb{P}^1 \to \mathbb{P}^3$ is not a complete intersection.

Solution. $\varphi_{\mathcal{O}(3)}$ maps $(z_0 : z_1)$ to $(z_0^3 : z_0^2 z_1 : z_0 z_1^2 : z_1^3)$, we denote the image by X. Hence $f_1(w_0 : w_1 : w_2 : w_3) = w_0 w_2 - w_1^2$ and $f_2(w_0 : w_1 : w_2 : w_3) = w_0 w_3 - w_1 w_2$ vanish on X. Since $Z(f_1, f_2) \cap U_0 = X \cap U_0$, where $U_0 := \{(w_0 : w_1 : w_2 : w_3) | w_0 \neq 0\}$, if X is a complete intersection, it must be equal to $Z(f_1, f_2)$. However, $(0 : 0 : 1 : 1) \in Z(f_1, f_2) \setminus X$, contradiction. □

Exercise 2.3.10. Let $C = \varphi_{\mathcal{O}(2)}(\mathbb{P}^1) \subset \mathbb{P}^2$ and consider the restriction of the linear system in Exercise 2.3.7 to C. Study the induced map $C \setminus \{x\} \to \mathbb{P}^1$.

Solution. If $x = (x_0^2 : x_0 x_1 : x_1^2) \in C$, then $x_1 w_0 - x_0 w_1$, $x_1 w_1 - x_0 w_2$ can be regarded as a basis of the linear system defined in Exercise 2.3.7, where w_0, w_1, w_2 are the standard coordinate in $\mathbb{C}^3$. So for any $z = (z_0^2 : z_0 z_1 : z_1^2)$, it is mapped to $((x_1 z_0^2 - x_0 z_0 z_1) : (x_1 z_0 z_1 - x_0 z_1^2)) = (z_0 : z_1)$. Hence this map is the left inverse of $\varphi_{\mathcal{O}(2)}$.

If $x = (x_0 : x_1 : x_2) \notin C$, then $x_1 w_0 - x_0 w_1$, $x_2 w_1 - x_1 w_2$ can be regarded as a basis of the linear system defined in Exercise 2.3.7. So for any $z = (z_0^2 : z_0 z_1 : z_1^2)$, it is mapped to $((x_1 z_0^2 - x_0 z_0 z_1) : (x_2 z_0 z_1 - x_1 z_1^2))$. Hence the map

$$\mathbb{P}^1 \xrightarrow{\varphi_{\mathcal{O}(2)}} C \longrightarrow \mathbb{P}^1$$

can be regarded as the meromorphic function $w \mapsto \frac{x_2 w - x_1 w^2}{x_1 - x_0 w}$. □

Exercise 2.3.11. Show that on a compact curve $X \ncong \mathbb{P}^1$ there always exist divisors D with $\deg(D) = 0$, but which are not principal.

Solution. Without loss of generality, we assume X is connected. If $X \ncong \mathbb{P}^1$, then $\mathcal{O}(x - x_0) \ncong \mathcal{O}$ for any $x \in X \setminus \{x_0\}$ since the Abel-Jacobi map is injective by Proposition 2.3.34 of [5]. So $x - x_0$ is not a principle divior but $\deg(x - x_0) = 0$. □

2.4 The Projective Space

Exercise 2.4.1. Show that the canonical bundle K_X of a complete intersection $X = Z(f_1) \cap \cdots \cap Z(f_k) \subset \mathbb{P}^m$ is isomorphic to $\mathcal{O}(\sum \deg(f_i) - n - 1)|_X$. What can you deduce from this for the Kodaira dimension of X?

Solution. By the adjunction formula, $K_X \cong K_{\mathbb{P}}^n|_X \otimes \det(\mathcal{N}_{X/\mathbb{P}^n})$, so $K_X \cong \mathcal{O}(-n-1)|_X \otimes \det(\bigoplus \mathcal{O}(\deg(f_i)))|_X \cong \mathcal{O}(\sum \deg(f_i) - n - 1)|_X$ by Exercise 2.3.3.

If X is a point, then its Kodaira dimension is 0. If $\sum \deg(f_i) < n+1$ and X has positive dimension, $H^0(X, K_X) = H^0(X, \mathcal{O}(\sum \deg(f_i) - n - 1))$. The global section of X must has zeroes but X is compact, so $H^0(X, K_X) = 0$. Hence $\mathrm{kod}(X) = -\infty$. If $\sum \deg(f_i) = n+1$, $K_X \cong \mathcal{O}|_X$, hence its Kodaira dimension is 0. If $\sum \deg(f_i) > n+1$, I don't know how to compute Kodaira dimension explicitly. But I hear that by Serre Vanishing and the basic properties of the Hilbert polynomial one can find $\mathrm{kod}(X) = \dim X$. □

Exercise 2.4.2. Are there holomorphic vector fields on $\mathbb{P}^n$, i.e. global sections of $\mathcal{T}_{\mathbb{P}^n}$, which vanish only in a finite number of points? If yes, in how many?

Solution. Note that

$$H^0(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}) = \left\{ \sum_{i=0}^n l_i \frac{\partial}{\partial z_i} | l_i \in \mathbb{C}[z_0, \cdots, z_n]_1 \right\} / \mathbb{C} \sum_{i=0}^n z_i \frac{\partial}{\partial z_i}.$$

So for a nontrivial holomorphic vector field on $\mathbb{P}^n$, w is a zero of it means that $(l_1(w) : \cdots : l_n(w)) = (w_1 : \cdots : w_n)$, which is equivalent to w is an eigenvector of the linear transformation $(l_1, \cdots, l_n)$. Hence it has $n+1$ zeroes (counting multiplicity).

Remark: One can also find the number of zeroes by using Poincaré-Hopf Index Theorem, since the Euler characteristic of $\mathbb{P}^n$ is $n+1$. □

Exercise 2.4.3. Compute the Kodaira dimension of the following smooth hypersurfaces:

i) $Z(\sum_{i=0}^2 x_i^2) \subset \mathbb{P}^2$,

ii) $Z(\sum_{i=0}^2 x_i^3) \subset \mathbb{P}^2$,

iii) $Z(\sum_{i=0}^3 x_i^3) \subset \mathbb{P}^3$, and

iv) $Z(\sum_{i=0}^4 x_i^5) \subset \mathbb{P}^4$.

Solution. The solution has been contained in Exercise 2.4.1.

i) $-\infty$;

ii) 0;

iii) $-\infty$;

iv) 0.

□

Exercise 2.4.4. Show that $H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}) = 0$ for $q > 0$.

Solution. Since $H^0(\mathbb{P}^n, \mathcal{T}^q_{\mathbb{P}^n})$ is nontrivial and $\mathcal{T}^q_{\mathbb{P}^n}$ is nontrivial, $H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n})$ must be trivial. □

Exercise 2.4.5. The surface $\Sigma_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ is called the $n-$th *Hirzebruch surface*. Show that Σ_n is isomorphic to the hypersurface $Z(x_0^n y_1 - x_1^n y_2) \subset \mathbb{P}^1 \times \mathbb{P}^2$, where $(x_0 : x_1)$ and $(y_0 : y_1 : y_2)$ are the homogeneous coordinates of $\mathbb{P}^1$ respectively $\mathbb{P}^2$.

Solution. Let $\{U_0, U_1\}$ be the standard cover of $\mathbb{P}^1$. Define

$$\begin{aligned}\varphi_0 : \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))|_{U_0} \cong U_0 \times \mathbb{P}^1 &\to \mathbb{P}^1 \times \mathbb{P}^2\\ ((x_0 : x_1), (v : w)) &\mapsto \left((x_0 : x_1), \left(v : w \cdot \frac{x_1^n}{x_0^n} : w\right)\right)\end{aligned}$$

and

$$\begin{aligned}\varphi_1 : \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))|_{U_1} \cong U_1 \times \mathbb{P}^1 &\to \mathbb{P}^1 \times \mathbb{P}^2\\ ((x_0 : x_1), (v : w)) &\mapsto \left((x_0 : x_1), \left(v : w : w \cdot \frac{x_0^n}{x_1^n}\right)\right).\end{aligned}$$

By the transition function of $\mathcal{O}_{\mathbb{P}^1}(n)$, φ_0 and φ_1 can be glued into a holomorphic injective map φ from $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ to $\mathbb{P}^1 \times \mathbb{P}^2$. And φ is onto $Z(x_0^n y_1 - x_1^n y_2)$. Hence Σ_n is isomorphic to $Z(x_0^n y_1 - x_1^n y_2)$. $\square$

Exercise 2.4.6. Describe the tangent, cotangent, and canonical bundle of $\mathbb{P}^n \times \mathbb{P}^m$.

Solution. Consider the projection $\mathbb{P}^n \xleftarrow{p_1} \mathbb{P}^n \times \mathbb{P}^m \xrightarrow{p_2} \mathbb{P}^m$. Then $\mathcal{T}_{\mathbb{P}^n \times \mathbb{P}^m}$ can be described by the direct sum of Euler sequence, for instance,

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{O} \longrightarrow p_1^*\left(\bigoplus_{i=1}^{n} \mathcal{O}(1)\right) \oplus p_2^*\left(\bigoplus_{j=1}^{m} \mathcal{O}(1)\right) \longrightarrow \mathcal{T}_{\mathbb{P}^n \times \mathbb{P}^m} \longrightarrow 0.$$

Then the cotangent bundle can be described by its dual. Then its canonical bundle is $p_1^*(\mathcal{O}(-n-1)) \otimes p_2^*(\mathcal{O}(-m-1))$. $\square$

Exercise 2.4.7. Determine all complete intersections $X = H_1 \cap \cdots \cap H_{n+m-2} \subset \mathbb{P}^n \times \mathbb{P}^m$ with $H_i \in |p_1^*\mathcal{O}(d_i) \otimes p_2^*\mathcal{O}(e_i)|$, $d_i, e_i > 0$ with trivial canonical bundle.

Solution. Similar as one projective space, K_X is $(p_1^*\mathcal{O}(\sum d_i - m - 1) \otimes p_2^*\mathcal{O}(\sum e_i - n - 1))|_X$, so it is trivial iff (I think it is true similar as the situation of $\mathbb{P}^n$) $\sum d_i = m + 1$ and $\sum e_i = n + 1$. Since $\sum d_i \geqslant n + m - 2$ and $\sum e_i \geqslant n + m - 2$, we can get $m, n \leqslant 3$ when K_X is trivial.

Without loss of generality, we assume that $m \geqslant n$. If $m = n = 3$, then all d_i and e_i must be 1. If $m = 3, n = 2$, then (d_i) must be $(2, 1, 1)$ or its permutation and all e_i must be 1. If $m = 3, n = 1$, then (d_i) can be $(2, 2)$ or $(3, 1)$ or their permutation, all e_i must be 1. If $m = 2, n = 2$, then (d_i) and (e_i) can be chosen in $(2, 1)$ and $(1, 2)$. If $m = 2, n = 1$, then $(d_1, e_1) = (3, 2)$. $\square$

Exercise 2.4.8. Let $Y \subset X$ be a smooth hypersurface in a complex manifold X of dimension n and let α be a meromorphic section of K_X with at most simple poles along Y. Locally one can write

$$\alpha = h \cdot \frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_2 \wedge \cdots \wedge \mathrm{d}z_n$$

with z_1 defining Y. One sets $\mathrm{Res}_Y(\alpha) = (h \cdot \mathrm{d}z_2 \wedge \cdots \wedge \mathrm{d}z_n)|_Y$.

a) Show that $\mathrm{Res}_Y(\alpha)$ is well defined and that it yields an element in $H^0(Y, K_Y)$.

b) Consider α as an element in $H^0(X, K_X \otimes \mathcal{O}(Y))$ and compare the definition of the residue with the adjunction formula: $K_Y \cong (K_X \otimes \mathcal{O}(Y))|_Y$.

c) Consider a smooth hypersurface $Y \subset \mathbb{P}^n$ defined by a homogeneous polynomial $f \in H^0(\mathbb{P}^n, \mathcal{O}(n+1))$. Prove that $\alpha := \sum(-1)^i z_i f^{-1} \mathrm{d}z_0 \wedge \cdots \wedge \widehat{\mathrm{d}z_i} \wedge \cdots \wedge \mathrm{d}z_n$ is a meromorphic section of $K_{\mathbb{P}^n}$ with simple poles along Y. Furthermore, show that $\mathrm{Res}_Y(\alpha) \in H^0(Y, K_Y)$ defines a holomorpic volume form on Y, i.e. a trivializing section of K_Y.

Solution.

a) Suppose $\alpha = h \cdot \dfrac{\mathrm{d}z_1}{z_1} \wedge \beta = h' \cdot \dfrac{\mathrm{d}w_1}{w_1} \wedge \beta'$ locally at $x \in Y$ in two different coordinate system U and V. Then there is a non vanish holomorphic function in $U \cap V$ such that $z_1 = hw_1$. Hence $\dfrac{\mathrm{d}z_1}{z_1} = \dfrac{\mathrm{d}w_1}{w_1} + \dfrac{\mathrm{d}h}{h} = \dfrac{\mathrm{d}w_1}{w_1}$. So $\mathrm{Res}_Y(\alpha)$ is well-defined.

b) $1/z_1$ can be regarded as the local section 1 of $\mathcal{O}_Y$, and it corresponds to $\frac{\partial}{\partial z_1} \in \mathcal{N}_{X/Y}$ by the isomorphism

$$\begin{aligned} H^0(U, \mathcal{N}_{X/Y}) &\to H^0(U, \mathcal{O}(Y)|_Y) \\ \sigma &\mapsto \sigma(f) \end{aligned},$$

where f is the defining function of Y on U. Hence $h \cdot \dfrac{\mathrm{d}z_1}{z_1} \wedge \beta$, which corresponds to $(h \cdot \mathrm{d}z_1 \wedge \beta) \otimes \frac{\partial}{\partial z_1}$, is mapped to $h \cdot \beta = \mathrm{Res}(\alpha)$ by the adjunction formula.

c) Since $\deg(f) = n+1$, α is invariant under the left action by $\mathbb{C} \setminus \{0\}$, it defines a meromorphic section of $K_{\mathbb{P}^n}$. α has simple poles along Y since in any standard coordinate U_i, $f \cdot \alpha$ is a holomorphic form. Now we identify U_0 with the affine hypersurface $(1, w_1, \cdots, w_n)$, then $\alpha|_{U_0}$ can be written as $f^{-1}\mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_n$. Since $\displaystyle\sum_{i=1}^{n} \frac{\partial f}{\partial w_i}\mathrm{d}w_i = \mathrm{d}f = 0$ on Y, $\mathrm{Res}_Y(\alpha)$ can be written as $(-1)^{i-1}\dfrac{\mathrm{d}w_1 \wedge \cdots \wedge \widehat{\mathrm{d}w_i} \wedge \cdots \wedge \mathrm{d}w_n}{\partial f / \partial w_i}$ on the open set $\left\{ \dfrac{\partial f}{\partial w_i} \neq 0 \right\}$ of Y. Since 0 is a regular value of f, $\mathrm{Res}_Y(\alpha)$ is non-vanishing on each U_i, hence is a holomorphic volume form. □

Exercise 2.4.9. Let E be a holomorphic vector bundle on a complex manifold X. Construct the relative tautological holomorphic line bundle $\mathcal{O}_\pi(-1)$ on $\mathbb{P}(E)$. Here, $\pi : \mathbb{P}(E) \to X$ denotes the projection and the fibre $\mathcal{O}_\pi(-1)(l)$ for a line $l \subset E(X)$ representing a point $l \in \pi^{-1}(x)$ should be, by construction, identified with l.

Solution. $\mathcal{O}_\pi(-1) = \{(l, v) | v \in l\} \subset \mathbb{P}(E) \times E$ is the required bundle. □

Exercise 2.4.10. This exercise generalizes the Euler sequence to Grassmannians. Show that on $\mathrm{Gr}_k(V)$ there exists a natural short exact sequence (the tautological sequence) of holomorphic vector bundles

$$0 \longrightarrow S \longrightarrow \mathcal{O} \otimes V \longrightarrow Q \longrightarrow 0$$

such that over a point $w \in \mathrm{Gr}_k(V)$ corresponding to $W \subset V$ the inclusion $S(w) \subset (\mathcal{O} \otimes V)(w) = V$ is just $W \subset V$. Moreover, prove that $\mathrm{Hom}(S, Q)$ is isomorphic to the holomorphic tangent bundle of $\mathrm{Gr}_k(V)$. Observe that this generalize what has been said about the Euler sequence on $\mathbb{P}^n$ which corresponds to the case $k = 1$.

Solution. Let S be the tautological bundle

$$S := \{(W, v) | v \in W\} \subset \mathrm{Gr}_k(V) \times V.$$

Then Q can also be defined such that the fibre of $W \in \mathrm{Gr}_k(V)$ is V/W.

Now consider the Plücker embedding $\mathrm{Gr}_k(V) \to \mathbb{P}\left(\bigwedge^k V\right)$. The image of $\mathrm{Gr}_k(V)$ under the Plücker embedding are precisely all lines in $\mathbb{P}\left(\bigwedge^k V\right)$ which have a generator of the form $v_1 \wedge \cdots \wedge v_k$ for $v_i \in V$. Now pick a point $W = \langle v_1 \wedge \cdots \wedge v_k \rangle$. Complete $\{v_1, \cdots, v_k\}$ to a basis $\{v_1, \cdots, v_n\}$ of V. Given a linear map $\varphi : W \to V$, we can define a curve $\varphi_c : (-\varepsilon, \varepsilon) \to \mathrm{Gr}_k(V)$ by $\varphi_c(t) =$

$\langle (v_1 + t\varphi(v_1)) \wedge \cdots \wedge (v_k + t\varphi(v_k)) \rangle$. Since $\varphi_c(0) = W$, this determines a tangent vector $\varphi_c'(0) = \langle \sum_{i=1}^k (v_1 \wedge \cdots \wedge \varphi(v_i) \wedge \cdots \wedge v_k) \rangle$. Two curves given by φ_c, ψ_c determine the same tangent vector if and only if the image of their difference $\varphi - \psi$ lies in $W' = \langle v_1, \cdots, v_k \rangle$, so the tangent space naturally contains $\mathrm{Hom}(W', V/W')$ as a subspace. But this subspace has dimension $k(n-k)$, which is the dimension of $\mathrm{Gr}_k(V)$, so in fact they are equal. Hence we conclude that the tangent bundle of $\mathrm{Gr}_k(V)$ is isomorphic to $\mathrm{Hom}(S, Q)$. □

Exercise 2.4.11. Let X be a complex manifold and let $\mathcal{O}_X \otimes V \twoheadrightarrow E$ be a surjection of vector bundles where V is a vector space and E is a vector bundle of rank k. Show that there exists a natural morphism $\varphi : X \to \mathrm{Gr}_k(V)$ with $\varphi^*(\mathcal{O} \otimes V \to Q) = \mathcal{O}_X \otimes V \to E$.

Solution. I think here is a typo, $\mathrm{Gr}_k(V)$ should be $\mathrm{Gr}_{n-k}(V)$. $\varphi(x)$ can be defined to be $\mathrm{Ker}\, E(x)$. □

2.5 Blow-ups

Exercise 2.5.1. Let $\hat{X} \to X$ be the blow-up of a surface X in a point $x \in X$. Show that the pull-back of sections defines an isomorphism $H^0(X, K_X) \cong H^0(\hat{X}, K_{\hat{X}})$. More generally, one has $H^0(X, K_X^{\otimes m}) = H^0(\hat{X}, K_{\hat{X}}^{\otimes m})$.

Solution. By pulling back along $\sigma : \hat{X} \to X$, we can get a homomorphism from $H^0(X, K_X^{\otimes m}) \to H^0(\hat{X}, K_{\hat{X}}^{\otimes m})$. Then by restriction we can get the homomorphism from $H^0(\hat{X}, K_{\hat{X}}^{\otimes m})$ to $H^0(X \setminus \{x\}, K_{X\setminus\{x\}}^{\otimes m})$. By Hartog's Theorem, $H^0(X \setminus \{x\}, K_{X\setminus\{x\}}^{\otimes m}) \cong H^0(X, K_X^{\otimes m})$. So To show that the decomposition of these two map is identity, it suffices to note that for the exceptional divisor E, $H^0(E, K_E) = H^0(E, \sigma^* K_X \otimes \mathcal{O}(E)|_E) = 0$, which is implied by $\sigma^* K_X$ is trivial along E and $\mathcal{O}(E)|_E = \mathcal{O}(-1)$. □

Exercise 2.5.2. Show that $\mathcal{O}(E)$ of the exceptional divisor $E = \mathbb{P}(\mathcal{N}_{Y/X})$ of a blow-up $\mathrm{Bl}_Y(X) \to X$ of a compact manifold X admits (up to scaling) only one section. (You might reduce to the case of the blow-up of a point.)

Solution. We first construct the global section f of $\mathcal{O}(E)$ which defines $\mathcal{O}(E)$. For any other section g of $\mathcal{O}(E)$, since $\mathcal{O}(E)|_E$ has negative degree, g must vanish along E, hence $(g) \geqslant E = (f)$. This implies that g/f is a holomorphic function on $\mathrm{Bl}_Y(X)$. Since X is compact, the map $\sigma : \mathrm{Bl}_Y(X) \to X$ is proper, we get $\mathrm{Bl}_Y(X)$ is compact. Then global holomorphic function on $\mathrm{Bl}_Y(X)$ must be constant. □

Exercise 2.5.3. Consider the $\mathbb{Z}/2\mathbb{Z}-$action $z \mapsto -z$ on $\mathbb{C}^2$ which has one fixed point and whose quotient $\mathbb{C}^2/\pm$ is not smooth. Show that the action lifts to a $\mathbb{Z}/2\mathbb{Z}-$action on the blow-up $\mathrm{Bl}_0(\mathbb{C}^2)$ and prove that the quotient $\mathrm{Bl}_0(\mathbb{C}^2)/\pm$ is a manifold.

Solution. I don't know the exact meaning of "not smooth" here. □

Exercise 2.5.4. Let $C \subset \mathbb{C}^2$ be the reducible curve defined by $z_1 \cdot z_2 = 0$. Show that the closure of $C \setminus \{0\} \subset \mathbb{C}^2 \setminus \{0\} = \mathrm{Bl}_0(\mathbb{C}^2) \setminus E$ in $\mathrm{Bl}_0(\mathbb{C}^2)$ is a smooth curve.

Solution. By definition, the pre-image of $C \setminus \{0\}$ is

$$\{(z_1, 0), (1:0) | z_1 \in \mathbb{C} \setminus \{0\}\} \cup \{(0, z_2), (0:1) | z_2 \in \mathbb{C} \setminus \{0\}\} \subset \mathrm{Bl}_0(\mathbb{C}^2).$$

Hence its closure is $(\mathbb{C} \times \{0\}) \times \{(1:0)\} \cup (\{0\} \times \mathbb{C}) \times \{(0:1)\}$, is obviously a smooth curve. □

Exercise 2.5.5. Let X be a K3 surface, i.e. X is a compact complex surface with $K_X \cong \mathcal{O}_X$ and $h^1(X, \mathcal{O}_X) = 0$. Show that X is not the blow-up of any other smooth surface.

Solution. Suppose X is the blow-up of a surface Y along a point x (Note that if X is the blow-up of Y along a smooth divisor, then $Y \cong X$), and denote the blow-up map and the exceptional divisor by σ and E respectively. Then $K_{Y\setminus\{x\}} \cong K_{X\setminus E} \cong \mathcal{O}_{X\setminus E} \cong \mathcal{O}_{Y\setminus\{x\}}$, hence $K_Y \cong \mathcal{O}_Y$. Since Y must be compact, K_Y only admits trivial section. Since $\mathcal{O}(E)$ has a nontrivial section, $K_X \cong \sigma^* K_Y \otimes \mathcal{O}(E)$ admits nontrivial section, which contradicts to $K_X \cong \mathcal{O}_X$. □

2.6 Differential Calculus on Complex Manifolds

Exercise 2.6.1. Show that any almost complex structure is induced by at most one complex structure.

Solution. Note that a smooth map is holomorphic if and only if its differential is $\mathbb{C}$-linear, we can get the identity map between the manifold endowed with two complex structures which induce the same almost complex structure is biholomorphic. $\square$

Exercise 2.6.2. Show that any oriented Riemannian surface admits a natural almost complex structure. Use the result of Newlander and Nierenberg to show that any almost complex structure on a Riemann surface is induced by a complex structure.

Solution. For an oriented Riemannian surface X, we can choose a volume form on it. Then we can define Hodge-$*$ operator on it. Note that $*^2 = -\operatorname{id}$ on T^*X. Since the Riemannian metric g gives an isomorphism $\Phi_g : TX \to T^*X$, then $J = \Phi_g^{-1} \circ * \circ \Phi_g$ is a natural almost complex structure on X.

Since $T^{0,1}X$ is a line bundle, $\mathcal{A}^{0,2}(X)$ is zero. So $\bar{\partial}^2 = 0$ automatically and J is integrable. $\square$

Exercise 2.6.3. Compare the almost complex structure on S^2 as given in the previous exercise with the natural complex structure on $\mathbb{P}^1$.

Solution. Consider the standard metric induced by $S^2 \hookrightarrow \mathbb{R}^3$. Since S^2 is invariant under $SO(3)$, we only consider the almost complex structure defined at $(0,0,-1)$. Now consider the coordinate system defined by the stereographic projection

$$\begin{aligned} f : \mathbb{C} &\to \mathring{S} \\ x + \sqrt{-1}y &\mapsto \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1} \right). \end{aligned}$$

It is easy to compute that the standard metric at is just $g|_{(0,0,-1)} = \mathrm{d}x^2 + \mathrm{d}y^2$, so $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ is a orthonormal basis at $(0,0,-1)$ and the volume form at $(0,0,-1)$ is $\mathrm{d}x \wedge \mathrm{d}y$. So $J = \Phi_g^{-1} \circ * \circ \Phi_g$ sends $\frac{\partial}{\partial x}$ to $\frac{\partial}{\partial y}$ and sends $\frac{\partial}{\partial y}$ to $-\frac{\partial}{\partial x}$. So J defined by the standard metric on S^2 equals to the almost complex structure induced by $\mathbb{P}^1$. $\square$

Exercise 2.6.4. Let $f : X \to Y$ be a surjective holomorphic map between connected complex manifolds. Let $(X,f)_{\text{reg}}$ be the open set of f-smooth points, i.e. the set of points where f looks locally like the projection of a product . Show that $(X,f)_{\text{reg}}$ is dense in X and that its complement can be described as the zero set of a global section of a holomorphic vector bundle on X.

Solution. Note that the complement of $(X,f)_{\text{reg}}$ is the zero set of all $\dim Y \times \dim Y$ minors of the differential map of f, the statement follows directly. $\square$

Exercise 2.6.5. Use the Poincaré lemma to show that any hypersurface $D \subset \mathbb{C}^n$ is defined by a global holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$.

Solution. By Poincaré lemma we have $H^{0,1}(\mathbb{C}^n) = 0$, then $H^1(\mathbb{C}^n, \mathcal{O}) \cong H^{0,1}(\mathbb{C}^n) = 0$. Hence by the long exact sequence one get that $H^1(\mathbb{C}^n, \mathcal{O}^*_{\mathbb{C}^n}) = 0$ and the projection map $H^0(\mathbb{C}^n, \mathcal{K}^*_{\mathbb{C}^n}) \to H^0(\mathbb{C}^n, \mathcal{K}^*_{\mathbb{C}^n}/\mathcal{O}^*_{\mathbb{C}^n})$ is surjective. Hence every hypersurface is defined by a global holomorphic function. $\square$

Exercise 2.6.6. Show that the exterior product induces a multiplication on the full Dolbeault cohomology $\bigoplus_{p,q} H^{p,q}(X)$ and that this yields a $\mathbb{Z}^2$-graded $\mathbb{Z}/2\mathbb{Z}$-commutative algebra $\bigoplus_{p,q} H^{p,q}(X)$ for any complex manifold X.

Solution. We only need to prove that the $\bar{\partial}$-closed form is close under the exterior product and the multiplication is not depend on the choice in the cohomology class. Suppose $\bar{\partial}\alpha = \bar{\partial}\beta = 0$ (without loss of generality, suppose α, β are decomposable), then $\bar{\partial}(\alpha\wedge\beta) = \bar{\partial}(\alpha)\wedge\beta + (-1)^{\deg\alpha+\deg\beta}\alpha\wedge\bar{\partial}(\beta) = 0$. On the other hand, $(\alpha + \bar{\partial}\alpha_1)\wedge\beta = \alpha + \bar{\partial}(\alpha_1\wedge\beta)$ since $\bar{\partial}\beta = 0$. $\square$

Exercise 2.6.7. Let X be a complex manifold. Verify that the following definition of the *Bott-Chern cohomology*

$$H^{p,q}_{\mathrm{BC}}(X) := \frac{\{\alpha \in \mathcal{A}^{p,q}(X) | \mathrm{d}\alpha = 0\}}{\partial\bar{\partial}\mathcal{A}^{p-1,q-1}(X)}$$

makes sense. Deduce from Exercise 1.3.4 that $H^{p,q}_{\mathrm{BC}}(B) = 0$ for a polydisc $B \subset \mathbb{C}^n$ and $p, q \geqslant 1$. Show that there are natural maps

$$H^{p,q}_{\mathrm{BC}}(X) \longrightarrow H^{p,q}(X) \quad \text{and} \quad H^{p,q}_{\mathrm{BC}}(X) \longrightarrow H^{p+q}(X, \mathbb{C}).$$

Solution. Since $\mathrm{d}\partial\bar{\partial} = (\partial + \bar{\partial})\partial\bar{\partial} = \partial\partial\bar{\partial} - \bar{\partial}\bar{\partial}\partial = 0$, $H^{p,q}_{\mathrm{BC}}(X)$ makes sense.

For a polydisc $B \subset \mathbb{C}^n$, $H^{p,q}_{\mathrm{BC}}(B) = 0$ follows from Exercise 1.3.4 directly.

Now if $\mathrm{d}\alpha = 0$, then $\bar{\partial}\alpha = 0$. If $\alpha = \partial\bar{\partial}\beta$, then $\alpha = \bar{\partial}(-\partial\beta)$. So there is natural map $H^{p,q}_{\mathrm{BC}}(X) \to H^{p,q}(X)$. If $\alpha = \partial\bar{\partial}\beta$ then $\alpha = \mathrm{d}(\bar{\partial}\beta)$. Hence there is natural map $H^{p,q}_{\mathrm{BC}}(X) \to H^{p+q}(X, \mathbb{C})$ as well. $\square$

Exercise 2.6.8. Let M be the real manifold described as a hypersurface $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ in $\mathbb{P}^3$. We denote the naturally induced complex structure by I. Show that (M, I) and $(M, -I)$ define isomorphic complex manifolds.

Solution. The isomorphic is given by $(x_0 : x_1 : x_2 : x_3) = (\overline{x_0} : \overline{x_1} : \overline{x_2} : \overline{x_3})$. One can readily check it is an isomorphic by considering the tangent vector as a local curve. $\square$

Exercise 2.6.9. Let G be a real Lie group. Consider the map $\mathrm{Ad} : G \to \mathrm{Gl}(T_eG)$ with $\mathrm{Ad}(g) : T_eG \to T_eG$ defined as the differential of $G \to G, h \mapsto ghg^{-1}$.

i) Show that a connected Lie group is commutative if and only if $\mathrm{Ad} \equiv \mathrm{id}$.

ii) Show that for a complex Lie group the map Ad is holomorphic. Deduce that $\mathrm{Ad} \equiv \mathrm{id}$ for any connected compact complex Lie group.

iii) Conclude that any connected compact complex Lie group is abelian.

Solution.

i) If a connected real Lie group G is abelian, then $h \mapsto ghg^{-1}$ is identity map, hence $\mathrm{Ad} \equiv \mathrm{id}$. If $\mathrm{Ad} \equiv \mathrm{id}$, then $h \mapsto ghg^{-1}$ has the same differential with the identity map of G, hence they are the same homomorphism. So G is abelian.

ii) Note that the differential map of Ad is just the adjoint map of Lie algebra, i.e. $\mathrm{ad} : T_eG \to \mathfrak{gl}(T_eG), x \mapsto [x, \cdot]$. It is $\mathbb{C}$-linear, hence Ad is holomorphic. If G is connected compact, then every component of Ad is a holomorphic function on G, it must be a constant. So $\mathrm{Ad}(g) = \mathrm{Ad}(e) = \mathrm{id}$.

iii) From i) and ii), it is trivial.

$\square$

Exercise 2.6.10. Let M be a real four-dimensional manifold and let $\sigma \in \mathcal{A}^2_{\mathbb{C}}(M)$ be a closed form such that $\sigma \wedge \sigma = 0$ and $\sigma \wedge \bar{\sigma}$ everywhere non-zero. Show that there exists a unique complex structure I on M such that σ is a holomorphic two-form on (M, I).

Solution. We first consider σ as a map from $T_{\mathbb{C}}M \to \bigwedge_{\mathbb{C}} M$. Since σ is a form, which is skew-symmetry, the rank of σ is even. $\sigma \wedge \sigma = 0$ implies that σ is not symplectic so the $\mathrm{Ker}(\sigma)$ is non-trivial. $\sigma \wedge \bar{\sigma}$ is a volume form implies that $\dim \mathrm{Ker}(\sigma) \neq 4$. Hence we can regard $\mathrm{Ker}(\sigma)$ as $T^{0,1}M$.

Now we let $T^{1,0}M$ be the complex conjugate of $T^{0,1}M$ in $T_{\mathbb{C}}M$. It is trivial that $T^{1,0}M = \mathrm{Ker}(\bar{\sigma})$. Since $\sigma \wedge \bar{\sigma}$ is nowhere vanish, $T^{1,0}M + T^{0,1}M$ is a direct sum. This direct sum decomposition induces an almost complex structure I on M

Finally, to check the integrability, it suffices to see that $[T_M^{0,1}, T_M^{0,1}] \subset T_M^{0,1}$. Let $X, Y \in T_M^{0,1}$ and Z be an arbitrary section of $T_{\mathbb{C}}M$. Now the formula $0 = \mathrm{d}\sigma(X, Y, Z) = X\sigma(Y, Z) - Y\sigma(X, Z) + Z\sigma(X, Y) - \sigma([X, Y], Z) + \sigma([X, Z], Y) - \sigma([Y, Z], X) = -\sigma([X, Y], Z)$ proves the integrability.

If σ is to be holomorphic, then it must vanish on $T^{0,1}M$. This gives the uniqueness. $\square$

Exercise 2.6.11. For this exercise you need to be acquainted with the basics of hypercohomology. Consider the holomorphic de Rham complex

$$\mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_X \xrightarrow{\mathrm{d}} \Omega_X^2 \to \cdots .$$

(Note that on Ω^i one has $\mathrm{d} = \partial$.) Show that this complex is quasi-isomorphic to $\mathbb{C}$. Conclude that $H^k(X, \mathbb{C}) = \mathbb{H}^k(X, (\Omega_X^*, \mathrm{d}))$. Similarly, show that $H^k(X, \mathbb{C}) = \mathbb{H}^k(X, (\Omega_X^*, t \cdot \mathrm{d}))$ for any $t \in \mathbb{C}^*$. Prove that in the limit $t = 0$ one has $\mathbb{H}^k(X, (\Omega_X^*, 0) = \bigoplus_{p+q=k} H^{p,q}(X)$.

Solution. See Chapter 3 of [4].

I am not familiar with hypercohomology. And I think I won't learn it in the foreseeable future. $\square$

3 Kähler Manifolds

3.1 Kähler Identities

Exercise 3.1.1. Show that any complex manifold admits an hermitian structure.

Solution. One can easily define compatible hermitian structure locally, then glue them up by partition of unity. $\square$

Exercise 3.1.2. Let X be a connected complex manifold of dimension $n > 1$ and let g be a Kähler metric. Show that g is the only K̈ahler metric in its conformal class, i.e. if $g' = e^f \cdot g$ is Kähler then f is constant (cf. Exercise 1.3.9).

Solution. It holds directly by Exercise 1.3.9. $\square$

Exercise 3.1.3. Let (X, g) be a compact hermitian manifold of dimension n. Show that d-harmonicity equals ∂-harmonicity and $\bar{\partial}$-harmonicity for forms of degree 0 and $2n$.

Solution. We only prove this statement for 0-forms, and it is similar for $2n$-forms. Choose an orthonormal basis $\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right\}$ of T_xX at a point $x \in X$. Then $\left\{\sqrt{2}\frac{\partial}{\partial z_i}, \sqrt{2}\frac{\partial}{\partial \bar{z}_i}\right\}$ is an orthonormal basis of $T_xX_{\mathbb{C}}$. Hence $\left\{\frac{1}{\sqrt{2}}\mathrm{d}z_i, \frac{1}{\sqrt{2}}\mathrm{d}\bar{z}_i\right\}$ is an orthonormal basis of $T_xX_{\mathbb{C}}^*$. And the volume form $\bigwedge_{i=1}^n(\mathrm{d}x_i \wedge \mathrm{d}y_i) = (-2\sqrt{-1})^n \bigwedge_{i=1}^n(\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i)$. Denote $(-2\sqrt{-1})^n$ by C.

For a form f of degree 0,

$$
\begin{aligned}
\varDelta f &= (\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)(f) = \mathrm{d}^*\mathrm{d}f \\
&= - * \circ \mathrm{d} \circ * \left(\sum_{i=1}^n \frac{\partial f}{\partial z_i}\mathrm{d}z_i + \sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i}\mathrm{d}\bar{z}_i\right) \\
&= - * \circ \mathrm{d} \circ * \left(\sum_{i=1}^n \frac{\partial f}{\partial z_i}\mathrm{d}z_i + \sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i}\mathrm{d}\bar{z}_i\right) \\
&= - 2C \cdot * \circ \mathrm{d}(-\sum_{i=1}^n \frac{\partial \bar{f}}{\partial z_i}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1) \wedge \cdots \wedge (\mathrm{d}z_i) \wedge \cdots \wedge (\mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n) \\
&\quad + \sum_{i=1}^n \frac{\partial \bar{f}}{\partial \bar{z}_i}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1) \wedge \cdots \wedge (\mathrm{d}\bar{z}_i) \wedge \cdots \wedge (\mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n)) \\
&= - 2 \cdot * \left(\sum_{i=1}^n \frac{\partial^2 f}{\partial z_i \partial \bar{z}_i} \cdot \mathrm{vol}\right) \\
&= - 2\sum_{i=1}^n \frac{\partial^2 f}{\partial z_i \partial \bar{z}_i}
\end{aligned}
$$

Similarly, one can get $\varDelta_\partial = \varDelta_{\bar{\partial}} = -\sum_{i=1}^n \frac{\partial^2 f}{\partial z_i \partial \bar{z}_i}$. Hence d-harmonicity equals ∂-harmonicity and $\bar{\partial}$-harmonicity for forms of degree 0. $\square$

Exercise 3.1.4. Prove $\int_{\mathbb{P}^n} \omega_{\mathrm{FS}}^n = 1$.

Solution. Since $\det(A + uv^T) = (1 + v^T A^{-1} u)\det(A)$,

$$\det\left(\frac{(1+|z|^2)\delta_{ij} - \bar{z}_i z_j}{(1+|z|^2)^2}\right) = \frac{1}{(1+|z|^2)^{n+1}},$$

hence

$$\begin{aligned}
\int_{U_0} \omega_{\mathrm{FS}}^n &= \int_{\mathbb{C}^n} \left(\frac{\sqrt{-1}}{2\pi}\right)^n n! \det\left(\frac{(1+|z|^2)\delta_{ij} - \bar{z}_i z_j}{(1+|z|^2)^2}\right) \bigwedge_{i=1}^{n} (\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i) \\
&= \left(\frac{\sqrt{-1}}{2\pi}\right)^n n!(-2\sqrt{-1})^n \mathrm{vol}(S_{2n-1}) \int_0^\infty \frac{r^{2n-1}}{(1+r^2)^{n+1}} \mathrm{d}r \\
&= 2n \int_0^{\pi/2} \sin^{2n-1}\theta \cos\theta d\theta \\
&= 1.
\end{aligned}$$

Note that $\mathbb{P}^n \setminus U_0$ has zero measure, so $\int_{\mathbb{P}^n} \omega_{\mathrm{FS}}^n = \int_{U_0} \omega_{\mathrm{FS}}^n = 1$. □

Exercise 3.1.5. Let $\mathbb{C}^n \subset \mathbb{C}^{n+1}$ be the standard inclusion

$$(z_0, \cdots, z_{n-1}) \mapsto (z_0, \cdots, z_{n-1}, 0)$$

and consider the induced inclusion $\mathbb{P}^{n-1} \subset \mathbb{P}^n$. Show that restricting the Fubini–Study Kähler form $\omega_{\mathrm{FS}}(\mathbb{P}^n)$ on $\mathbb{P}^n$ yields the Fubini–Study Kähler form on $\mathbb{P}^{n-1}$.

Solution. For any $i \neq n$,

$$\omega_i(\mathbb{P}^n)|_{\{z_n=0\}} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{l=0}^{n-1}\left|\frac{z_l}{z_i}\right|^2\right) = \omega_i(\mathbb{P}^{n-1}),$$

hence $\omega_{\mathrm{FS}}(\mathbb{P}^n)|_{\mathbb{P}^{n-1}} = \omega_{\mathrm{FS}}(\mathbb{P}^{n-1})$. □

Exercise 3.1.6. Let $A \in \mathrm{Gl}(n+1, \mathbb{C})$ and denote the induced isomorphism by $F_A : \mathbb{P}^n \cong \mathbb{P}^n$. Show that $F_A^*(\omega_{\mathrm{FS}}) = \omega_{\mathrm{FS}}$ if and only if $A \in \mathrm{U}(n+1)$.

Solution. If $A \in \mathrm{U}(n+1)$, denote the natural projection by $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. Then we get $F_A^*(\omega_{\mathrm{FS}}) = \pi_* \pi^* F_A^*(\omega_{\mathrm{FS}}) = \pi_* A^* \pi^*(\omega_{\mathrm{FS}}) = \pi_*\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\|Az\|^2)\right) = \pi_*\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\|z\|^2)\right) = \omega_{\mathrm{FS}}$.

The other direction is wrong, since a constant scalar of a unitary matrix also does the same thing. If $F_A^*(\omega_{\mathrm{FS}}) = \omega_{\mathrm{FS}}$, note that the Fubini–Study distance between two point in $\mathbb{P}^n$ is induced by the angle of the two subspaces spaned by them in $\mathbb{C}^{n+1}$, hence A must preserve angle, i.e. it must lie in $\mathbb{C}^*\,\mathrm{U}(n+1)$ □

Exercise 3.1.7. Show that L, d, and d^* acting on $\mathcal{A}^*(X)$ of a Kahler manifold X determine the complex structure of X.

Solution. By the Kähler identities, $[\mathrm{d}^*, L] = [\bar{\partial}^*, L] + [\partial^*, L] = \sqrt{-1}(\partial - \bar{\partial})$. So we can compute ∂ and $\bar{\partial}$ by the equality above and $\mathrm{d} = \partial + \bar{\partial}$. □

Exercise 3.1.8. Show that on a compact Kähler manifold X of dimension n the integral $\int_X \omega^n$ is $n! \cdot \mathrm{vol}(X)$ (cf. Exercise 1.2.9). Conclude from this that there exists an injective ring homomorphism $\mathbb{R}[x]/x^{n+1} \to$

$H^*(X,\mathbb{R})$. In particular, $b_2(X) \geqslant 1$. Deduce from this that S^2 is the only sphere that admits a Kähler structure.

Solution. Let the generating element x be mapped to $\omega \in H^2(X,\mathbb{R})$, then extend this map linearly and by multiplication. Then it must be a ring homomorphism. If $\sum_{i=0}^{n} c_i\omega^i = 0$, then $\omega^{n-i}\sum_{i=0}^{n} c_i\omega^i = 0$ for every i, hence by $\int_X \omega^n = n! \cdot \mathrm{vol}(X)$ we know that ω^n is not exact and every c_i equals 0. So this gives an injective ring homomorphism. So $b_2(X) \geqslant 1$ and S^2 is the only sphere that admits a Kähler structure. □

Exercise 3.1.9. Conclude $[\bar\partial^*, L] = \sqrt{-1}\partial$, $[\partial^*, L] = -\sqrt{-1}\bar\partial$ from $[\Lambda, \bar\partial] = -\sqrt{-1}\partial^*$, $[\Lambda, \partial] = \sqrt{-1}\bar\partial^*$.

Solution. We only prove $[\partial^*, L] = -\sqrt{-1}\bar\partial$ from $[\Lambda, \bar\partial] = -\sqrt{-1}\partial^*$, the other equation is similar. On $\bigwedge^k X$,

$$\begin{aligned}
&[\partial^*, L]\\
=&\partial^* \circ L - L \circ \partial^*\\
=&- * \circ \bar\partial \circ * \circ L + L \circ * \circ \bar\partial \circ *\\
=&- * \circ \bar\partial \circ * \circ * \circ \Lambda \circ *^{-1} + * \circ \Lambda \circ *^{-1} \circ * \circ \bar\partial \circ *\\
=&(-1)^{k(2n-k)+(k-2)(2n+2-k)+1} * \circ \bar\partial \circ \Lambda \circ * + * \circ \Lambda \circ \bar\partial \circ *\\
=&* \circ [\Lambda, \bar\partial] \circ *\\
=&\sqrt{-1} *^2 \circ \bar\partial \circ *^2\\
=&(-1)^{k(2n-k)+(k+1)(2n-k-1)}\sqrt{-1}\bar\partial\\
=&-\sqrt{-1}\bar\partial.
\end{aligned}$$

Hence the equation holds generally by linearity. □

Exercise 3.1.10. Let X and Y be two Kähler manifolds. Show that the product $X \times Y$ admits a natural Kähler structure, too.

Solution. Consider the natural projection $X \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} Y$ and the fundamental forms ω_X and ω_Y of X and Y respectively. Then $\pi_X^*\omega_X + \pi_Y^*\omega_Y$ is a fundamental form of $X \times Y$. □

Exercise 3.1.11. Fill in the details of Remark 3.1.14 of [5].

Solution. See Section 7 in Chapter 0 of [4]. □

Exercise 3.1.12. Let X be a complex manifold endowed with a Kähler form and let α be a closed $(1,1)$-form which is primitive. Show that α is harmonic, i.e. $\Delta(\alpha) = 0$.

Solution. Since X is Kähler, we only need to prove that $\Delta_\partial(\alpha) = 0$ by $\Delta_\partial = \frac{1}{2}\Delta$. Then

$$\begin{aligned}
\Delta_\partial(\alpha) =&\partial\partial^*\alpha + \partial^*\partial\alpha\\
=&\partial\partial^*\alpha \quad \text{(since } \alpha \text{ is closed)}\\
=&\sqrt{-1}\partial[\Lambda, \bar\partial](\alpha) \quad \text{(by Kähler identities)}\\
=&\sqrt{-1}\partial\Lambda\bar\partial\alpha - \sqrt{-1}\partial\bar\partial\Lambda\alpha = 0 \quad \text{(since } \alpha \text{ is closed and primitive).}
\end{aligned}$$

□

Exercise 3.1.13. Let M be a differentiable manifold of dimension $2n$. A closed two-form ω on M is a symplectic structure (or form) if ω is everywhere non-degenerate, i.e. ω^n is a volume form. Show that any Kähler manifold (X, g) possesses a natural symplectic structure.

Solution. Trivially, the fundamental form of the Kähler manifold X is the required symplectic structure. □

3.2 Hodge Theory on Kähler Manifolds

Exercise 3.2.1. Let (X, g) be a Kähler manifold. Show that the Kähler form ω is harmonic.

Solution.

$$
\begin{aligned}
&\Delta\omega \\
=&\mathrm{d}\mathrm{d}^*\omega + \mathrm{d}^*\mathrm{d}\omega \\
=&-\mathrm{d} * \mathrm{d} * \omega \\
=&-\frac{1}{(n-1)!}\mathrm{d} * \mathrm{d}\omega^{n-1} \\
=&0.
\end{aligned}
$$

□

Exercise 3.2.2. Show that for a compact hermitian manifold (X, g) of dimension n there exists a natural isomorphism $H^{p,q}(X) \cong H^{n-p,n-q}(X)^*$. And if (X, g) is also Kähler, with respect to complex conjugation on $H^*(X, \mathbb{C}) = H^*(X, \mathbb{R}) \otimes \mathbb{C}$ one has $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

Solution. $H^{p,q}(X) \cong \mathcal{H}^{p,q}_{\bar\partial}(X) \cong \mathcal{H}^{n-p,n-q}_{\bar\partial}(X)^* \cong H^{n-p,n-q}(X)^*$. $\overline{H^{p,q}(X)} \cong \overline{\mathcal{H}^{p,q}_{\bar\partial}(X)} = \mathcal{H}^{q,p}_{\partial}(X) = \mathcal{H}^{q,p}_{\bar\partial}(X) \cong H^{q,p}(X)$. □

Exercise 3.2.3. Let X be a compact Kähler manifold X of dimension n. Let $H^{p,q}(X) \cong H^{n-p,n-q}(X)^*$ be given by Serre duality. Observe that the direct sum decomposition of these isomorphisms yields Poincaré duality

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X) \cong \bigoplus_{p+q=k} H^{n-p,n-q}(X)^* = H^{2n-k}(X, \mathbb{C})^*.$$

Solution. I don't know what should I prove. □

Exercise 3.2.4. Recall Exercise 2.6.11 and show that on a compact Kähler manifold "the limit $\lim_{t\to 0}$ commutes with hypercohomology" , i.e.

$$\lim_{t\to 0} \mathbb{H}^k(X, (\Omega^*_X, t \cdot \mathrm{d})) = \mathbb{H}^k(X, \lim_{t\to 0}(\Omega^*_X, t \cdot \mathrm{d})).$$

Solution. I am not familiar with hypercohomology. And I think I won't learn it in the foreseeable future. □

Exercise 3.2.5. Show that for a complex torus of dimension one the decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

does depend on the complex structure.

Solution. Consider two different almost complex structure (automatically integrable) on $\mathbb{R}^2/\mathbb{Z}^2$ given by the dual of $I_1(\mathrm{d}x) = \mathrm{d}y$, $I_1(\mathrm{d}y) = -\mathrm{d}x$ and $I_2(\mathrm{d}x) = \mathrm{d}x + \mathrm{d}y$, $I_2(\mathrm{d}y) = -2\mathrm{d}x - \mathrm{d}y$. Then it is easy to observe that the decomposition of $H^1(X, \mathbb{C})$ with respect to these two almost complex structures are

$$\mathbb{C}\langle \mathrm{d}x - \sqrt{-1}\mathrm{d}y\rangle \oplus \mathbb{C}\langle \mathrm{d}x + \sqrt{-1}\mathrm{d}y\rangle$$

and

$$\mathbb{C}\langle (1 + \sqrt{-1})\mathrm{d}x + \mathrm{d}y\rangle \oplus \mathbb{C}\langle (1 - \sqrt{-1})\mathrm{d}x + \mathrm{d}y\rangle$$

respectively. □

Exercise 3.2.6. Show that the odd Betti numbers b_{2i+1} of a compact Kähler manifold are even.

Solution. It follows from $H^{2i+1}(X, \mathbb{C}) = \bigoplus_{p+q=2i+1} H^{p,q}(X)$ and $\dim H^{p,q}(X) = \dim H^{q,p}(X)$ since $\overline{H^{p,q}(X)} = H^{q,p}(X)$ directly. $\square$

Exercise 3.2.7. Are Hopf surfaces Kähler manifolds?

Solution. Since Hopf surface diffeomorphic to $S^1 \times S^3$, its first Betti number $b_1 = 1$, hence it cannot be a Kähler manifold by Exercise 3.2.6. $\square$

Exercise 3.2.8. Show that holomorphic forms, i.e. elements of $H^0(X, \Omega^p)$, on a compact Kähler manifold X are harmonic with respect to any Kähler metric.

Solution. By Hodge decomposition, for any $\alpha \in H^0(X, \Omega^p)$, there exists $\alpha_1 \in \mathcal{H}^{1,0}_{\bar{\partial}}(X, g) = \mathcal{H}^{1,0}(X, g)$ and $\beta \in \mathcal{A}^{1,1}(X)$ such that $\alpha = \alpha_1 + \bar{\partial}^*\beta$. Hence $0 = \bar{\partial}\alpha = \bar{\partial}\alpha_1 + \bar{\partial}\bar{\partial}^*\beta = \bar{\partial}\bar{\partial}^*\beta$. So $(\bar{\partial}^*\beta, \bar{\partial}^*\beta) = (\beta, \bar{\partial}\bar{\partial}^*\beta) = 0$ and $\bar{\partial}^*\beta = 0$. Hence $\alpha = \alpha_1$ is harmonic. $\square$

Exercise 3.2.9. Can you deduce the Hodge decomposition for compact Kähler manifolds from the Hodge decomposition for compact oriented Riemannian manifolds and the $\partial\bar{\partial}$-lemma?

Solution. Recall that the Hodge decomposition for compact oriented Riemannian manifolds (M, g) is

$$\mathcal{A}^k(M) = \mathrm{d}(\mathcal{A}^{k-1}(M)) \oplus \mathcal{H}^k(M, g) \oplus \mathrm{d}^*(\mathcal{A}^{k+1}(M)).$$

Now suppose (X, g) is a compact Kähler manifold. By the Hodge decomposition for compact oriented Riemannian manifolds, any $\alpha \in \mathcal{A}^{p,q}(X) \subset \mathcal{A}^{p+q}_{\mathbb{C}}(X)$ can be decomposed into $\mathrm{d}\alpha_1 + \alpha_2 + \mathrm{d}^*\alpha_3$, where $\alpha_1 \in \mathcal{A}^{p+q-1}_{\mathbb{C}}(X)$, $\alpha_2 \in \mathcal{H}^{p,q}(X, g) = \mathcal{H}^{p,q}_{\bar{\partial}}(X, g)$, $\alpha_3 \in \mathcal{A}^{p+q+1}_{\mathbb{C}}(X)$. Since $\mathrm{d} = \partial + \bar{\partial}$, $\mathrm{d}^* = \partial^* + \bar{\partial}^*$, $\alpha_1 = \partial\beta_1 + \bar{\partial}\beta_2$, $\alpha_3 = \partial^*\gamma_1 + \bar{\partial}^*\gamma_2$, where $\beta_1 \in \mathcal{A}^{p-1,q}(X)$, $\beta_2 \in \mathcal{A}^{p,q-1}(X)$, $\gamma_1 \in \mathcal{A}^{p+1,q}(X)$, $\gamma_2 \in \mathcal{A}^{p,q+1}(X)$. Hence by $\partial\bar{\partial}$-lemma, α_1 is ∂-exact, α_3 is ∂^*-exact. Similarly, the $\bar{\partial}$-version Hodge decomposition holds. $\square$

Exercise 3.2.10. Let (X, g) be a compact hermitian manifold. Show that any d-harmonic (p, q)-form is also $\bar{\partial}$-harmonic.

Solution. Since (X, g) is compact hermitian, $\partial^*, \bar{\partial}^*$ are the dual operator of $\partial, \bar{\partial}$ respectively with respect to the hermitian product $(,)$. Hence $\mathrm{d}^* = \partial^* + \bar{\partial}^*$ is also the dual operator of d. So $(\Delta(\alpha), \alpha) = \|\mathrm{d}\alpha\|^2 + \|\mathrm{d}^*\alpha\|^2$, which implies that α is d-harmonic iff $\mathrm{d}\alpha = \mathrm{d}^*\alpha = 0$. Hence $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$ since α is a (p, q)-form. Thus $\Delta_{\bar{\partial}}(\alpha) = 0$. $\square$

Exercise 3.2.11. Show that $H^{p,q}(\mathbb{P}^n) = 0$ except for $p = q \leqslant n$. In the latter case, the space is one-dimensional. Use this and the exponential sequence to show that $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$.

Solution. Note that $H^{2k}(\mathbb{P}^n) \cong \mathbb{C}$ and $H^{2k-1}(\mathbb{P}^n) = 0$. Since $\mathbb{P}^n$ is compact Kähler, $\dim H^{p,q}(\mathbb{P}^n) = \dim H^{q,p}(\mathbb{P}^n)$, it is easy to observe that $H^{p,q}(\mathbb{P}^n) = 0$ except for $p = q \leqslant n$ and $\dim H^{p,p}(\mathbb{P}^n) = 1$. By the exponential sequence, one can get the long exact sequence

$$\cdots \to H^1(\mathbb{P}^n, \mathcal{O}) \to \mathrm{Pic}(\mathbb{P}^n) \to H^2(\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^n, \mathcal{O}) \to \cdots$$

Since $H^q(\mathbb{P}^n, \mathcal{O}) \cong H^{0,q}(\mathbb{P}^n) = 0$ for $q > 0$, $\mathrm{Pic}(\mathbb{P}^n) \cong H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$. $\square$

Exercise 3.2.12. Let X be a compact Kähler manifold and consider $H^{1,1}(X)$ as a subspace of $H^2(X,\mathbb{C})$. Show that the Kähler $\mathcal{K}_X$ cone is an open convex cone in $H^{1,1}(X,\mathbb{R}) := H^{1,1}(X) \cap H^2(X,\mathbb{R})$ and that $\mathcal{K}_X$ not contain any line $\{\alpha + t\beta | t \in \mathbb{R}\}$ for any $\alpha, \beta \in H^{1,1}(X,\mathbb{R})$ with $\beta \neq 0$.

Furthermore, show that $t\alpha + \beta$ is a Kähler class for $t \gg 0$ for any Kähler class α and any β.

Solution. It follows from Corollary 3.1.8 of [5] directly. Also note that a neighborhood of positive (resp. negative) definite matrix is also positive (resp. negative) definite. $\square$

Exercise 3.2.13. Prove the dd^c-lemma: If $\alpha \in \mathcal{A}^k(X)$ is a d^c-exact and d-closed form on a compact Kähler manifold X then there exists a form $\beta \in \mathcal{A}^{k-2}(X)$ such that $\alpha = \mathrm{dd}^c\beta$.

Solution. See Lemma 3.A.22 of [5]. $\square$

Exercise 3.2.14. Let X be compact and Kähler. Show that the two natural homomorphisms $H^{p,q}_{\mathrm{BC}}(X) \to H^{p,q}(X)$ and $\bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X) \to H^k(X,\mathbb{C})$ introduced in Exercise 2.6.7 are bijective.

Use this to show again that the bidegree decomposition in Corollary 3.2.12 is independent of the Kähler structure.

Solution. It follows from $\partial\bar{\partial}$-lemma directly. $\square$

Exercise 3.2.15. Let (X, g) be a compact hermitian manifold and let $[\alpha] \in H^{p,q}(X)$. Show that the harmonic representative of $[\alpha]$ is the unique $\bar{\partial}$-closed form with minimal norm $\|\alpha\|$.

Solution. For any $\beta \in [\alpha]$, where $\alpha \in \mathcal{H}^{p,q}_{\bar{\partial}}(X)$, there exists $\gamma \in \mathcal{A}^{p,q-1}(X)$ such that $\beta = \alpha + \bar{\partial}\gamma$. Note that it is just the Hodge decomposition, hence orthogonal. Then $(\beta,\beta) = (\alpha,\alpha) + (\bar{\partial}\gamma, \bar{\partial}\gamma) \geqslant (\alpha,\alpha)$. $\square$

Exercise 3.2.16. Let X be a compact Kähler manifold. Show that for two cohomologous Kähler forms ω and ω', i.e. $[\omega] = [\omega'] \in H^2(X,\mathbb{R})$, there exists a real function f such that $\omega = \omega' + \sqrt{-1}\partial\bar{\partial}f$.

Solution. Since $\omega - \omega' \in d\mathcal{A}^1(X)$, by $\partial\bar{\partial}$-lemma, there exists $g \in \mathcal{A}^{0,0}(X)$ such that $\omega - \omega' = \partial\bar{\partial}g$. Since $\omega - \omega'$ is real, $\omega - \omega' = -\partial\bar{\partial}\bar{g}$. Hence $\omega - \omega' = \sqrt{-1}\partial\bar{\partial}\mathrm{Im}g$, where the imaginary part $\mathrm{Im}g$ of g is a real function. $\square$

3.3 Lefschetz Theorems

Exercise 3.3.1. Let X be a compact hermitian manifold. The two natural maps $H^k(X,\mathbb{C}) \to H^k(X,\mathcal{O}_X)$, induced by $\mathbb{C} \subset \mathcal{O}_X$, and $H^k(X,\mathbb{C}) \to H^{0,k}(X)$, given by the projection in bidegree decomposition, coincide.

Solution. Note that without Kähler condition, we cannot get the bidegree decomposition of X, hence the map $H^k(X,\mathbb{C}) \to H^{0,k}(X)$ here may be given on the form level. It follows directly from the acyclic resolution $\mathcal{A}^\bullet_{\mathbb{C}}(X)$ and $\mathcal{A}^{0,\bullet}(X)$ of $\mathbb{C}$ and $\mathcal{O}_X$ which have chain map $\Pi^{0,\bullet}$ between them. $\square$

Exercise 3.3.2. Let X be a Hopf surface. Show that the Jacobian, i.e.

$$H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$$

is not a compact torus in a natural way. In fact, $H^1(X,\mathbb{Z}) = \mathbb{Z}$ and $H^1(X,\mathcal{O}_X) = \mathbb{C}$.

Solution. Since X is diffeomorphic to $S^1 \times S^3$, one can get $H^1(X,\mathbb{Z}) \cong \mathbb{Z}$ and $b_1(X) = 1$. Now consider $h_{0,1}(X) = \dim_{\mathbb{C}} H^{0,1}(X) = \dim_{\mathbb{C}} H^1(X,\mathcal{O}_X)$. Let π be the natural projection from $\mathbb{C}^2 \setminus \{0\}$ to X, then $\pi_*\left(\frac{\bar\partial\|z\|^2}{\|z\|^2}\right) = \pi_*\left(\frac{z_1 \mathrm{d}\bar z_1 + z_2 \mathrm{d}\bar z_2}{|z_1|^2+|z_2|^2}\right)$ gives a nontrivial cohomology class in $H^{0,1}(X)$, hence $h^{0,1} = 1$. Moreover, $H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z}) \cong \mathbb{C}/\mathbb{Z}$ is not a complex torus since $\mathbb{Z}$ cannot be a lattice in $\mathbb{C}$. $\square$

Exercise 3.3.3. Show that the oriented differentiable manifold given by the connected sum $\mathbb{P}^2\sharp\mathbb{P}^2$ does not underly any Kähler surface. In fact, it cannot even be a complex surface, but in order to see this one would have to use the fact that any complex surface with even b_1 is in fact Kähler.

Solution. Note that $\mathrm{sgn}(\mathbb{P}^2) = 1$ and $\mathrm{sgn}(\mathbb{P}^2\sharp\mathbb{P}^2) = 2$, so if $\mathbb{P}^2\sharp\mathbb{P}^2$ is a Kähler surface, by Hodge index theorem we get $h^{1,1}(\mathbb{P}^2\sharp\mathbb{P}^2)$ must be $2h^{2,0}(\mathbb{P}^2\sharp\mathbb{P}^2)$, which implies that $4|b_2(\mathbb{P}^2\sharp\mathbb{P}^2) = 2$, contradiction. $\square$

Exercise 3.3.4. Let X be a Kähler surface with $\mathrm{kod}(X) = -\infty$. Show that its signature is $(1, h^{1,1}(X)-1)$.

Solution. Note that $H^{2,0}(X) = H^0(X,\Omega_X^2) = H^0(X,K_X) = 0$ since $\mathrm{kod}(X) = -\infty$, hence $h^{2,0}(X) = 0$. So by Hodge index theorem its signature is $(1, h^{1,1}(X)-1)$. $\square$

Exercise 3.3.5. Let X be a compact Kähler manifold of dimension n and let $Y \subset X$ be a smooth hypersurface such that $[Y] \in H^2(X,\mathbb{R})$ is a Kähler class. Show that the canonical restriction map $H^k(X,\mathbb{R}) \to H^k(Y,\mathbb{R})$ is injective for $k \leqslant n-1$.

Solution. See Proposition 5.2.6 of [5]. $\square$

Exercise 3.3.6. Construct a complex torus $X = \mathbb{C}^2/\Gamma$ such that $\mathrm{NS}(X) = 0$. Conclude that such a torus cannot be projective and, moreover, that $K(X) = \mathbb{C}$.

Solution. Let Γ be generated by $(1,0)$, $(0,1)$, $(\sqrt{-2},\sqrt{-3})$ and $(\sqrt{-5},\sqrt{-7})$. Then $X = \mathbb{C}^2/\Gamma$ has trivial Néron-Severi group by Appell-Humbert Theorem. For explicit calculation, see Appendix of [2], moreover one can see $a(X) = 0$ in it. Then by Corollary 5.3.3 of [5] we know that X is not projective. $\square$

Exercise 3.3.7. Show that any complex line bundle on $\mathbb{P}^n$ can be endowed with a unique holomorphic structure. Find an example of a compact (Kähler, projective) manifold and a complex line bundle that does not admit a holomorphic complex structure.

Solution. Note that for any complex manifold X, the first Chern class map factors through the group of complex line bundles, which is isomorphic to $H^2(X, \mathbb{Z})$ since the sheaf of smooth complex-valued function is a fine sheaf, so every complex line bundle admits a unique holomorphic structure if and only if $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is an isomorphism and there exists a complex line bundle does not admit a holomorphic structure if and only if c_1 is not surjective.

Now by Exercise 3.2.11 we know that c_1 of $\mathbb{P}^n$ is an isomorphism since $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \cong H^{0,1}(\mathbb{P}^n) = 0$ and $H^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \cong H^{0,2}(\mathbb{P}^n) = 0$.

Since the image of c_1 is contained in $H^{1,1}(X, \mathbb{Z})$, then if $H^{1,1}(X, \mathbb{Z}) \subsetneq H^2(X, \mathbb{Z})$, we can get a complex line bundle that does not admit a holomorphic structure. The torus defined in Exercise 3.3.6 is a Kähler example. $\square$

Exercise 3.3.8. Show that on a complex torus $\mathbb{C}^n/\Gamma$ the trivial complex line bundle admits many (how many?) holomorphic non-trivial structures.

Solution. Note that the holomorphic structure on the trivial complex line bundle is classified by its Jacobian $\mathrm{Pic}^0(\mathbb{C}^n/\Gamma)$, which is a complex torus whose dimension is $b_1(\mathbb{C}^n/\Gamma)/2 = n$. $\square$

Exercise 3.3.9. Let X be a compact Kähler manifold. Show that the fundamental class $[Y] \in H^{p,p}(X)$ of any compact complex submanifold $Y \subset X$ of codimension p is non-trivial. Is this true for Hopf manifolds?

Solution. Let ω be the Kähler form of X, by Wirtinger's theorem,

$$0 < \mathrm{vol}(Y) = \int_Y \omega^p = \int_X \omega^p \wedge [Y],$$

so $[Y]$ cannot be trivial.

Consider the Hopf surface X which diffeomorphic to $S^1 \times S^3$, then $H^2(X, \mathbb{Z})$ is trivial. However, the image of $\mathbb{C}^* \times \{0\}$ under the natural projection $\mathbb{C}^2 \setminus \{0\} \to X$ is a complex tori, its fundamental class must be trivial. $\square$

Exercise 3.3.10. Let X be a complex torus $\mathbb{C}^n/\Gamma$. Show that $\mathrm{Pic}^0(\mathrm{Pic}^0(X))$ is naturally isomorphic to X.

Solution. See Section 6 in Chapter 2 of [4]. Note that $\mathrm{Pic}^0(X)$ can be viewed as quotient of the dual space of $\mathbb{C}^n$ by the dual lattice of Γ. $\square$

Exercise 3.3.11. Show that $\mathrm{Alb}(X)$ and $\mathrm{Pic}^0(X)$ of a compact Kähler manifold X are dual to each other, i.e. $\mathrm{Pic}^0(\mathrm{Alb}(X)) \cong \mathrm{Pic}^0(X)$.

Solution. See Section 6 in Chapter 2 of [4]. $\square$

4 Vector Bundles

4.1 Hermitian Vector Bundles and Serre Duality

Exercise 4.1.1. Let L be a holomorphic line bundle which is globally generated by sections $s_1, \cdots, s_k \in H^0(X, L)$. Then L admits a canonical hermitian structure h defined in Example 4.1.2 of [5]. The dual bundle L^* obtains a natural hermitian structure h' via the inclusion $L^* \subset \mathcal{O}^{\oplus k}$. Describe h' and and show that $h' = h^*$.

Solution. Note that for any $v \in L^*(x)$, $(v(s_i(x)))_i$ gives the embedding $L^* \subset \mathcal{O}^{\oplus k}$. Hence for any $s, t \in L(x)$, suppose ψ is a local trivialization around x, then $h_x^*(h_x(\cdot, s), h_x(\cdot, t)) = h_x(t, s) = \dfrac{\psi(t)\overline{\psi(s)}}{\sum\limits_i |\psi(s_i)|^2}$ and

$$\begin{aligned}
&h'_x(h_x(\cdot, s), h_x(\cdot, t)) \\
=&\langle (h_x(s_i, s))_i, (h_x(s_i, t))_i \rangle_x \\
=&\sum_i h_x(s_i, s)\overline{h_x(s_i, t)} \\
=&\sum_i \frac{\psi(s_i)\overline{\psi(s)}}{\sum\limits_i |\psi(s_i)|^2} \cdot \frac{\psi(t)\overline{\psi(s_i)}}{\sum\limits_i |\psi(s_i)|^2} \\
=&\frac{\psi(t)\overline{\psi(s)}}{\sum\limits_i |\psi(s_i)|^2} \\
=&h_x^*(h_x(\cdot, s), h_x(\cdot, t)),
\end{aligned}$$

where $\langle \cdot, \cdot \rangle_x$ denote the standard hermitian product at $x \in X$. Thus $h' = h^*$. $\square$

Exercise 4.1.2. Let L be a holomorphic line bundle of degree $d > 2g(C) - 2$ on a compact curve C. Show that $H^1(C, L) = 0$. Here, for our purpose we define the genus $g(C)$ of C by the formula $\deg(K_C) = 2g(C) - 2$.

In other words, $H^1(C, K_C \otimes L) = 0$ for any holomorphic line bundle L with $\deg(L) > 0$. In this form, it will later be generalized to the Kodaira vanishing theorem for arbitrary compact Kähler manifolds.

Solution. By Serre duality, $(H^1(C, L))^* \cong H^0(C, K_C \otimes L^*)$. Since $\deg(K_C \otimes L^*) = \deg(K_C) - \deg(L) < 0$, if $f \in H^0(K_C \otimes L^*)$ is nontrivial, then $0 = \deg((f)) \geqslant -\deg(K_C \otimes L^*) > 0$, contradiction. $\square$

Exercise 4.1.3. Show, e.g. by writing down an explicit basis, that

$$\dim H^n(\mathbb{P}^n, \mathcal{O}(k)) = \begin{cases} 0 & k > -n-1 \\ \binom{-k-1}{-n-1-k} & k \leqslant -n-1 \end{cases}$$

Solution. By Serre duality, $H^n(\mathbb{P}^n, \mathcal{O}(k)) \cong (H^0(\mathbb{P}^n, \mathcal{O}(-k-n-1)))^*$, then the dimension and the basis can be written directly. $\square$

Exercise 4.1.4. Let E be an hermitian holomorphic vector bundle on a compact Kähler manifold X. Show that any section $s \in H^0(X, \Omega^p \otimes E)$ is harmonic.

Solution. Note that the isomorphism $H^0(X, \Omega^p \otimes E) \cong H^{p,0}(X, E)$ is given by the identity, and the natural projection (identity in fact) $\mathcal{H}^{p,q}(X, E) \to H^{p,q}(X, E)$ is bijective, hence the statement follows. $\square$

Exercise 4.1.5. Compare this section with the discussion in Sections 3.2 and 3.3 of [5]. In particular, check whether the Lefschetz operator L is defined on $H^{p,q}(X,E)$ and whether it defines isomorphisms $H^{p,k-p}(X,E) \to H^{n+p-k,n-p}(X,E)$ (cf. Remark 3.2.7,iii) of [5]).

Solution. For $\alpha \otimes s \in \mathcal{A}^{p,q}(X,E)$, we can define $L_E(\alpha \otimes s)$ as $L(\alpha) \otimes s$.

I think L_E^{n-k} defines an isomorphism as well since all the operators $(L_E, \Lambda_E, \bar{\partial}_E)$ can be calculated locally, and it maps harmonic forms to harmonic forms. $\square$

4.2 Connections

Exercise 4.2.1.

i) Show that any (hermitian) vector bundle admits an (hermitian) connection.

ii) Show that a connection ∇ is given by its action on the space of global sections $\mathcal{A}^0(X, E)$.

Solution.

i) One can define the (hermitian) connection locally and then glue them by the partition of unity.

ii) For any open subset $U \subset X$ and $s \in \mathcal{A}^0(U, E)$, one can find $\{U_i\}$ covering U and a collection of functions $\phi_i : X \to \mathbb{R}$ only supported in U_i, then one can glue $\nabla(\phi_i \cdot s|_{U_i})$ to define $\nabla(s)$.

$\square$

Exercise 4.2.2. Let ∇_i be connections on vector bundles E_i, $i = 1, 2$. Change both connections by one-forms $\alpha_i \in \mathcal{A}^1(X, \mathrm{End}(E_i))$ and compute the new connections on the associated bundles $E_1 \oplus E_2$, $E_1 \otimes E_2$, and $\mathrm{Hom}(E_1, E_2)$.

Solution. $(\nabla_1 + \alpha_1)(s_1) \oplus (\nabla_2 + \alpha_2)(s_2) - \nabla_1(s_1) \oplus \nabla_2(s_2) = \alpha_1(s_1) \oplus \alpha_2(s_2)$, so $\nabla_1 \oplus \nabla_2$ is changed by $\alpha_1 \oplus \alpha_2$. Similarly, one can see that $\nabla_1 \otimes \nabla_2$ is changed by $\alpha_1 \otimes \mathrm{id}_{E_2} + \mathrm{id}_{E_1} \otimes \alpha_2$. And for any section s_1 of $\mathcal{A}^0(E_1)$, we can get $\mathrm{Hom}(\nabla_1 + \alpha_1, \nabla_2 + \alpha_2)(f)(s_1) - \mathrm{Hom}(\nabla_1, \nabla_2)(f)(s_1) = \alpha_2(f(s_1)) - f(\alpha_1(s_1))$. $\square$

Exercise 4.2.3. Prove that connections on bundles E_i, $i = 1, 2$ which are compatible with given hermitian or holomorphic structures induce compatible connections on the associated bundles $E_1 \oplus E_2$, $E_1 \otimes E_2$, and $\mathrm{Hom}(E_1, E_2)$.

Solution. Suppose ∇_1, ∇_2 are connections compatible with (E_1, h_1) and (E_2, h_2) respectively. Let s_1, s_2 are sections of E_1, t_1, t_2 are sections of E_2.

- On $E_1 \oplus E_2$,

$$\begin{aligned}
&\mathrm{d}((h_1 \oplus h_2)(s_1 \oplus t_1, s_2 \oplus t_2)) = \mathrm{d}(h_1(s_1, s_2) + h_2(t_1, t_2))\\
=&h_1(\nabla_1(s_1), s_2) + h_1(s_1, \nabla_1(s_2)) + h_2(\nabla_2(t_1), t_2) + h_2(t_1, \nabla_2(t_2))\\
=&(h_1 \oplus h_2)((\nabla_1 \oplus \nabla_2)(s_1 \oplus t_1), s_2 \oplus t_2)\\
&+ (h_1 \oplus h_2)(s_1 \oplus t_1, (\nabla_1 \oplus \nabla_2)(s_2 \oplus t_2)).
\end{aligned}$$

- On $E_1 \otimes E_2$,

$$\begin{aligned}
&\mathrm{d}((h_1 \otimes h_2)(s_1 \otimes t_1, s_2 \otimes t_2)) = \mathrm{d}(h_1(s_1, s_2) \cdot h_2(t_1, t_2))\\
=&(h_1(\nabla_1(s_1), s_2) + h_1(s_1, \nabla_1(s_2))) \cdot h_2(t_1, t_2)\\
&+ h_1(s_1, s_2) \cdot (h_2(\nabla_2(t_1), t_2) + h_2(t_1, \nabla_2(t_2)))\\
=&(h_1 \otimes h_2)(\nabla_1(s_1) \otimes t_1 + s_1 \otimes \nabla_2(s_2), s_2 \otimes t_2)\\
&+ (h_1 \otimes h_2)(s_1 \otimes t_1, \nabla_1(s_2) \otimes t_2 + s_2 \otimes \nabla_2(t_2))\\
=&(h_1 \otimes h_2)((\nabla_1 \otimes \nabla_2)(s_1 \otimes s_2), t_1 \otimes t_2)\\
&+ (h_1 \otimes h_2)(s_1 \otimes s_2, (\nabla_1 \otimes \nabla_2)(t_1 \otimes t_2)).
\end{aligned}$$

- On $\mathrm{Hom}(E_1, E_2)$, we denote $\mathrm{Hom}(h_1, h_2)$ and $\mathrm{Hom}(\nabla_1, \nabla_2)$ by h and ∇ respectively. First note that

$$\begin{aligned}
&\nabla(h_1(\cdot, s_1) \cdot t_1)(s)\\
=&\nabla_2(h_1(s, s_1) \cdot t_1) - h_1(\nabla_1(s), s_1) \cdot t_1\\
=&\mathrm{d}(h_1(s, s_1)) \cdot t_1 + h(s, s_1) \cdot \nabla_2(t_1) - h_1(\nabla_1(s), s_1) \cdot t_1\\
=&h_1(s, \nabla_1(s_1)) \cdot t_1 + h(s, s_1) \cdot \nabla_2(t_1).
\end{aligned}$$

So
$$\begin{aligned}&h(\nabla(h_1(\cdot,s_1)\cdot t_1),h_1(\cdot,s_2)\cdot t_2)\\=&h(h_1(\cdot,\nabla_1(s_1))\cdot t_1+h_1(\cdot,s_1)\cdot\nabla_2(t_1),h_1(\cdot,s_2)\cdot t_2)\\=&h_1(\nabla_1(s_1),s_2)\cdot h_2(t_1,t_2)+h_1(s_1,s_2)\cdot h_2(\nabla(t_1),t_2)\end{aligned}$$
and then
$$\begin{aligned}&\mathrm{d}(h(h_1(\cdot,s_1)\cdot t_1,h_1(\cdot,s_2)\cdot t_2))\\=&\mathrm{d}(h_1(s_1,s_2)\cdot h_2(t_1,t_2))\\=&(h_1(\nabla_1(s_1),s_2)+h_1(s_1,\nabla_1(s_2)))\cdot h_2(t_1,t_2)\\&+h_1(s_1,s_2)\cdot(h_2(\nabla_2(t_1),t_2)+h_2(t_1,\nabla_2(t_2)))\\=&h(\nabla(h_1(\cdot,s_1)\cdot t_1),h_1(\cdot,s_2)\cdot t_2)+h(h_1(\cdot,s_1)\cdot t_1,\nabla(h_1(\cdot,s_2)\cdot t_2)).\end{aligned}$$
Similarly, one can check the holomorphic situation. □

Exercise 4.2.4. Study connections on an hermitian holomorphic vector bundle (E,h) that admits local holomorphic trivialization which are at the same time orthogonal with respect to the hermitian structure.

Solution. This means the second fundamental form with respect to this splitting vanish, and the connection splits to the connection on each subbundle. □

Exercise 4.2.5. Let (E,h) be an hermitian vector bundle. If $E=E_1\oplus E_2$, then E_1 and E_2 inherit natural hermitian structures h_1 and h_2. Are the induced connections ∇_i on E_i again hermitian with respect to these hermitian structures? What can you say about the second fundamental form?

Solution. Suppose s_1,s_2 are two sections of E_1, then $\mathrm{d}h_1(s_1,s_2)=\mathrm{d}h(s_1,s_2)=h(\nabla(s_1),s_2)+h(s_1,\nabla(s_2))$. So if the decomposition is orthogonal with respect to h, then $h(\nabla(s_1),s_2)=h(\nabla_1(s_1)+b_1(s_1),s_2)=h_1(\nabla(s_1),s_2)$ and similarly $h(s_1,\nabla(s_2))=h_1(s_1,\nabla_1(s_2))$, where b_1 denotes the second fundamental form in $\mathcal{A}^1(M,\mathrm{Hom}(E_1,E_2))$. Hence ∇_1 is hermitian with respect to h_1. And similarly ∇_2 is hermitian with respect to h_2.

We still suppose that E_1 and E_2 are orthogonal with respect to h. And let b_1,b_2 denote the second fundamental forms in $\mathcal{A}^1(M,\mathrm{Hom}(E_1,E_2))$ and $\mathcal{A}^1(M,\mathrm{Hom}(E_2,E_1))$ respectively. Suppose s,t are sections of E_1 and E_2 respectively. So
$$0=\mathrm{d}h(s,t)=h(\nabla(s),t)+h(s,\nabla(t))=h_2(b_1(s),t)+h_1(s,b_2(t)).$$
And similarly, $h_1(b_2(s),t)+h_2(s,b_1(t))=0$. □

Exercise 4.2.6. Let ∇ be a connection on E. Describe the induced connections on $\bigwedge^2 E$ and $\det(E)$.

Solution. We view $\bigwedge^2 E$ as a quotient bundle of $E\otimes E$ with projection π, let $s,s_1,\dots,s_n$ be sections of E, since $\nabla(s\otimes s)=\nabla(s)\otimes s+s\otimes\nabla(s)$ is still symmetric, we can induce a natural connection on $\bigwedge^2 E$. Then
$$\begin{aligned}&\nabla(s_1\wedge s_2)\\=&\nabla(\pi(s_1\otimes s_2))\\=&\pi(\nabla(s_1)\otimes s_2+s_1\nabla(s_2))\\=&\nabla(s_1)\wedge s_2+s_1\wedge\nabla(s_2).\end{aligned}$$
Similarly, suppose $n:=\mathrm{rk}(E)$, then on $\det(E)=\bigwedge^n E$, we have $\nabla(s_1\wedge\cdots\wedge s_n)=\nabla(s_1)\wedge s_2\wedge\cdots\wedge s_n+\cdots+s_1\wedge\cdots\wedge s_{n-1}\wedge\nabla(s_n)$. □

Exercise 4.2.7. Show that the pull-back of an hermitian connection is hermitian with respect to the pull-back hermitian structure. Analogously, the pull-back of a connection compatible with the holomorphic structure on a holomorphic vector bundle under a holomorphic map is again compatible with the holomorphic structure on the pull-back bundle.

Solution. The conclusions follow directly from d commutes with pull-back and the chain rule. $\square$

Exercise 4.2.8. Show that a connection ∇ on an hermitian bundle (E, h) is hermitian if and only if $\nabla(h) = 0$, where by ∇ we also denote the naturally induced connection on the bundle $(E \otimes \bar{E})^*$.

Solution. Suppose s_1, s_2 are arbitrary sections of E. Then

$$\begin{aligned}
&\nabla(h)(s_1 \otimes \bar{s}_2) \\
=&\mathrm{d}(h(s_1, s_2)) - h(\nabla(s_1 \otimes \bar{s}_2)) \\
=&\mathrm{d}(h(s_1, s_2)) - h(\nabla(s_1), s_2) - h(s_1, \nabla(s_2))
\end{aligned}$$

implies the required conclusion directly. $\square$

Exercise 4.2.9. Show that the definition of the Atiyah class does not depend on the chosen trivialization.

Solution. Note that the Atiyah class equal to the coholomogy class defined by the curvature of Chern connection on E, so it automatically independent of the choice of trivialization.

Remark: One can also prove the conclusion by checking definition. $\square$

4.3 Curvature

Exercise 4.3.1. Show that $\nabla^2 : \mathcal{A}^k(E) \to \mathcal{A}^{k+2}(E)$ is given by taking the exterior product with the form part of the curvature $F_\nabla \in \mathcal{A}^2(M, \mathrm{End}(E))$ and applying its endomorphism part to E.

Solution. Suppose t is an arbitrary section of $\mathcal{A}^0(E)$ and β is a k-form, then

$$\begin{aligned}&\nabla^2(\beta \wedge t)\\ =&\nabla(\mathrm{d}(\beta) \wedge t + (-1)^k \beta \wedge \nabla(t))\\ =&(-1)^{k+1}\mathrm{d}(\beta) \wedge \nabla(t) + (-1)^k \mathrm{d}(\beta) \wedge \nabla(t) + \beta \wedge \nabla^2(t)\\ =&F_\nabla \wedge (\beta \wedge t).\end{aligned}$$

So the conclusion holds on $\mathcal{A}^k(E)$. □

Exercise 4.3.2. Let E_1, E_2 be vector bundles endowed with connections ∇_1 and ∇_2, respectively. Prove that

i) The curvature of the induced connection on the direct sum $E_1 \oplus E_2$ is given by $F_{\nabla_1 \oplus \nabla_2} = F_{\nabla_1} \oplus F_{\nabla_2}$.

ii) For the induced connection ∇^* on the dual bundle E^* one has $F_{\nabla^*} = -F_\nabla^{\mathrm{tr}}$.

iii) Compute the connection and the curvature of the determinant bundle.

Solution.

i) Note that $(\nabla_1 \oplus \nabla_2)^2 = \nabla_1^2 \oplus \nabla_2^2$.

ii) Note that

$$\begin{aligned}&F_{\nabla^*}(f)(s)\\ =&\nabla^*(\nabla^*(f))(s)\\ =&\mathrm{d}(\nabla^*(f)(s)) - \nabla^*(f)(\nabla(s))\\ =&\mathrm{d}(\nabla^*(f)(s)) + \mathrm{d}(f(\nabla(s))) - f(\nabla^2(s))\\ =&\mathrm{d}^2(f(s)) - f(F_\nabla(s))\\ =&f(F_\nabla(s)).\end{aligned}$$

So under the dual basis of E and E^*, we have $F_{\nabla^*} = -F_\nabla^{\mathrm{tr}}$.

iii) Suppose locally $\nabla = \mathrm{d} + A$, $\{e_1, \ldots, e_n\}$ is a local trivialization of E where $n := \mathrm{rk}(E)$, then $A(e_1 \wedge \cdots \wedge e_n) = (Ae_1) \wedge \cdots \wedge e_n + \cdots + e_1 \wedge \cdots \wedge A(e_n) = \mathrm{tr}(A) e_1 \wedge \cdots \wedge e_n$. So the induced connection equals to $d + \mathrm{tr}(A)$. Then the curvature form equals to $\mathrm{d}\,\mathrm{tr}(A)$.

□

Exercise 4.3.3. Let (E_1, h_1) and (E_2, h_2) be two hermitian holomorphic vector bundles endowed with hermitian connections ∇_1, ∇_2 such that the curvature of both is (semi-)positive. Prove the following assertions.

i) The curvature of the induced connection ∇^* on the dual vector bundle E^* is (semi-)negative.

ii) The curvature on $E_1 \otimes E_2$ is (semi-)positive and it is positive if at least one of the two connections has positive curvature.

iii) The curvature on $E_1 \oplus E_2$ is (semi-)positive.

Solution.

i) For any sections s, t of E_1, we have

$$\nabla^*(h_1(\cdot, s))(t) = \mathrm{d}(h_1(t, s)) - h_1(\nabla(t), s) = h_1(t, \nabla(s)),$$

without loss of generality, suppose $\nabla(s) = s_1 \otimes \alpha$, then

$$\begin{aligned}
&\nabla^*(h_1(\cdot, \nabla(s)))(t) \\
=&\mathrm{d}(h_1(t, \nabla(s))) - h_1(\nabla(t), \nabla(s)) \\
=&\mathrm{d}(h_1(t, s_1)\alpha) - h_1(\nabla(t), \nabla(s)) \\
=&\mathrm{d}(h_1(t, s_1)) \wedge \alpha - h_1(t, s_1)\mathrm{d}\alpha - h_1(\nabla(t), \nabla(s)) \\
=&h_1(t, \nabla(s_1) \wedge \alpha) - h_1(t, s_1 \otimes \mathrm{d}\alpha) \\
=&h_1(t, -F_\nabla(s)).
\end{aligned}$$

Thus for any $v \in T^{1,0}X$ and section $h_1(\cdot, s)$ of E_1,

$$\begin{aligned}
&h_1^*(F_{\nabla^*}(h_1(\cdot, s)), h_1(\cdot, s))(v, \bar{v}) \\
=&h_1^*(h_1(\cdot, -F_\nabla(s)), h_1(\cdot, s))(v, \bar{v}) \\
=&-h_1(F\nabla(s), s)(v, \bar{v}).
\end{aligned}$$

Hence F_{∇^*} is (semi-)negative.

ii) Suppose s_i, t_i are local orthogonal frame of E_1 and E_2 with respect to h_1 and h_2, then any local section of $E_1 \otimes E_2$ can be written as $\sum_i s_i \otimes t_i$, then for any $v \in T^{1,0}X$, we have

$$\begin{aligned}
&(h_1 \otimes h_2)\left(F_{\nabla_1 \otimes \nabla_2}\left(\sum_i s_i \otimes t_i\right), \sum_i s_i \otimes t_i\right)(v, \bar{v}) \\
=&\left(\sum_{i,j}(h_1(F_{\nabla_1}(s_i), s_j)h_2(t_i, t_j) + h_1(s_i, s_j)h_2(F_{\nabla_2}(t_i), t_j))\right)(v, \bar{v}) \\
=&\left(\sum_i(h_1(F_{\nabla_1}(s_i), s_i)h_2(t_i, t_i) + h_1(s_i, s_i)h_2(F_{\nabla_2}(t_i), t_i))\right)(v, \bar{v}) \geqslant 0.
\end{aligned}$$

And when at least one of F_{∇_1} and F_{∇_2} is positive, we have the tensor curvature is also positive.

iii) Suppose s_i, t_i are local frame of E_1 and E_2, then any local section of $E_1 \oplus E_2$ can be written as $\sum_i (s_i, t_i)$, then for any $v \in T^{1,0}X$, we have

$$\begin{aligned}
&(h_1 \oplus h_2)\left(F_{\nabla_1 \oplus \nabla_2}\left(\sum_i (s_i, t_i)\right), \sum_i (s_i, t_i)\right)(v, \bar{v}) \\
=&\left(\sum_{i,j}(h_1(F_{\nabla_1}(s_i), s_j) + h_2(F_{\nabla_2}(t_i), t_j))\right)(v, \bar{v}) \\
=&\left(h_1\left(F_{\nabla_1}\left(\sum_i s_i\right), \sum_i s_i\right) + h_2\left(F_{\nabla_2}\left(\sum_i t_i\right), \sum_i t_i\right)\right)(v, \bar{v}) \geqslant 0.
\end{aligned}$$

□

Exercise 4.3.4. Find an example of two connections ∇_1 and ∇_2 on a vector bundle E, such that F_{∇_1} is positive and F_{∇_2} is negative.

Solution. Consider the tangent bundle E of the open unit disk D in $\mathbb{C}$, the Fubini–Study metric on $\mathbb{P}^1$ resticts to a metric with positive curvature on it and in Exercise 4.3.8 we will see that it also admits a metric with negative curvature. $\square$

Exercise 4.3.5. Let L be a holomorphic line bundle on a complex manifold. Suppose L admits an hermitian structure whose Chern connection has positive curvature. Show that X is Kähler. If X is in addition compact prove $\int_X A(L)^n > 0$.

Solution. Note that the curvature F_∇ of the Chern connection ∇ belongs to $\mathcal{A}^{1,1}_{\mathbb{R}}(X, \mathrm{End}(L,h))$ and $\mathrm{End}(L,h)$ is a trivial real line bundle. And note that by Bianchi identity we get $\bar\partial(F_\nabla) = 0$ and then $\partial(F_\nabla) = -\overline{\bar\partial(F_\nabla)} = 0$, hence F_∇ is d-closed form. And also the positivity of F_∇ implies that is nondegenerate. So F_∇ is a Kähler form on X. If X is compact, since F_∇ is positive, and its Dolbeault cohomology class is just $A(L)$, we have $\int_X A(L)^n = \int_X F_\nabla^n > 0$ by the positivity of F_∇ (which can be calculated locally by partition of unity). $\square$

Exercise 4.3.6. Show that the canonical bundle of $\mathbb{P}^n$ comes along with a natural hermitian structure such that the curvature of the Chern connection is negative.

Solution. Since the canonical bundle of $\mathbb{P}^n$ is isomorphic to $\mathcal{O}(-n-1)$ and the curvature of the hermitian structure on $\mathcal{O}(1)$ is positive, thus by Exercise 4.3.3 we get the curvature of the hermitian structure on $\mathcal{O}(-n-1)$ is negative. $\square$

Exercise 4.3.7. Show that the curvature of a complex torus $\mathbb{C}^n/\Gamma$ endowed with a constant hermitian structure is trivial. Is this true for any hermitian structure on $\mathbb{C}^n/\Gamma$?

Solution. The connection on $\mathbb{C}^n/\Gamma$ is given by the standard exterior differential d, then $\mathrm{d}^2 = 0$.

It is wrong for any hermitian structure on $\mathbb{C}^n/\Gamma$. Suppose $n = 1$ and we consider a positive smooth function f on it. Then we regard f as an hermitian metric on it, the curvature of f is $-\sqrt{-1}\partial\bar\partial f$, which cannot be always zero. $\square$

Exercise 4.3.8. Show that the curvature of a the natural metric on the unit disk $D^n \subset \mathbb{C}^n$ induced by the Kähler form

$$\omega = -\frac{\sqrt{-1}}{2}\partial\bar\partial \log(1 - \|z\|^2)$$

is negative.

Solution. Note that D^n with this Kähler form has a transitive isometry transformation group $\mathrm{SU}(1,n)$, hence we only need to check the curvature is negative at 0.

$$\begin{aligned}\frac{2}{\sqrt{-1}}\omega &= -\partial\bar\partial \log\left(1 - \sum_{i=1}^n |z_i|^2\right)\\ &= \sum_{i,j} h_{ij}\mathrm{d}z_i \wedge \mathrm{d}\bar z_j,\end{aligned}$$

where $H = (h_{ij}) = \left(\frac{\delta_{ij}}{(1-\|z\|^2)} + \frac{\bar z_i z_j}{(1-\|z\|^2)^2}\right)$. Then one can get $H(0) = I_n$, $\partial H(0) = \bar\partial H(0) = \partial \bar H(0) = 0$ and

$$\bar\partial\partial(\bar H)_{ij}(0) = \left(\sum_i \mathrm{d}\bar z_i \wedge \mathrm{d}z_i\right)\delta_{ij} + \mathrm{d}\bar z_i \wedge \mathrm{d}z_j.$$

Since

$$
\begin{aligned}
F_\nabla =&\bar{\partial}(\bar{H}^{-1}\partial(\bar{H}))\\
=&\bar{\partial}(\bar{H}^{-1})\partial(\bar{H})+\bar{H}^{-1}\bar{\partial}\partial(\bar{H})\\
=&-\bar{H}^{-1}\bar{\partial}(\bar{H})\bar{H}^{-1}\partial(\bar{H})+\bar{H}^{-1}\bar{\partial}\partial(\bar{H}),
\end{aligned}
$$

F_∇ is just $\bar{\partial}\partial(\bar{H})_{ij}(0)$ under the basis $\{\partial/\partial z_i\}$ at (0). So suppose $s=\sum_i s_i\frac{\partial}{\partial z_i}\neq 0$ and $v=\sum_i v_i\frac{\partial}{\partial z_i}\neq 0$. Then at 0,

$$
\begin{aligned}
h(F_\nabla(s),s)=&\sum_{i,j}h_{ij}(0)\left(\sum_k s_k\bar{\partial}\partial(\bar{H})_{ik}(0)\right)\bar{s}_j\\
=&\sum_{i,j}\delta_{ij}\left(\sum_k s_k\bar{\partial}\partial(\bar{H})_{ik}(0)\right)\bar{s}_j\\
=&\sum_{i,j}\bar{s}_i s_j\bar{\partial}\partial(\bar{H})_{ij}(0)
\end{aligned}
$$

and

$$
\begin{aligned}
&h(F_\nabla(s),s)(v,\bar{v})\\
=&\sum_{i,j,k,l}v_k\bar{v}_l\bar{s}_i s_j\left[\left(\sum_i \mathrm{d}\bar{z}_i\wedge\mathrm{d}z_i\right)\delta_{ij}+\mathrm{d}\bar{z}_i\wedge\mathrm{d}z_j\right]\left(\frac{\partial}{\partial z_k},\frac{\partial}{\partial\bar{z}_l}\right)\\
=&-\sum_{i\neq j}\bar{v}_i\bar{s}_i v_j s_j-2\sum_i|s_i|^2|v_i|^2\\
=&-\sum_i|s_i|^2|v_i|^2-\left|\sum_i v_i s_i\right|^2<0.
\end{aligned}
$$

Thus, F_∇ is negative. □

Exercise 4.3.9. Let X be a compact Kähler manifold with $b_1(X)=0$. Show that there exists a unique fiat connection ∇ on the trivial holomorphic line bundle $\mathcal{O}$ with $\nabla^{0,1}=\bar{\partial}$. Moreover, up to isomorphism $\mathcal{O}$ is the only line bundle with trivial Chern class $c_1\in H^2(X,\mathbb{Z})$.

Solution. Note that d is a connection on $\mathcal{O}$ satisfying the condition. If there exists another connection ∇ such that $\nabla^{0,1}=\bar{\partial}$, suppose $\nabla=\mathrm{d}+A$, then $F_\nabla=\mathrm{d}A+A\wedge A=0$ and A is holomorphic. Thus $(F_\nabla)^{1,1}=\bar{\partial}A=0$. So A represents a cohomology class in $H^{1,0}(X)$. But since X is compact Kähler manifold and $b_1(X)=0$, $0=H^1(X)=H^{1,0}(X)\oplus H^{0,1}(X)$. So $A=0$.

The next assertion follows from the long exact sequence induced from exponential sequence. □

Exercise 4.3.10. Let ∇ be a connection on a (complex) line bundle L on a manifold M. Show that L locally admits trivializing parallel sections if and only if $F_\nabla=0$.

Solution. If M admits parallel frame s_i on U, then any section on U can be written as $\sum_i f_i s_i$ and

$$
F_\nabla(s)=\sum_i f_i F_\nabla(s_i)=0.
$$

If $F_\nabla=0$, then on a contractible domain U of X, one can get a parallel frame from a basis of the fibre at one point $x\in U$ by parallel transporting well-definedly. □

4.4 Chern Classes

Exercise 4.4.1. Let C be a connected compact curve. Then there is a natural isomorphism $H^2(C,\mathbb{Z}) \cong \mathbb{Z}$. Show that with respect to this isomorphism (or, rather, its $\mathbb{R}$-linear extension) one has $c_1(L) = \deg(L)$ for any line bundle L on C.

Solution. The isomorphism between $H^2(C,\mathbb{Z})$ is given by $\alpha \mapsto \int_C \alpha$. Since the natural homomorphism $\mathrm{Div}(C) \to \mathrm{Pic}(C)$ is surjective, there exists a divisor D such that $L = \mathcal{O}(D)$, then

$$\deg(L) = \deg \mathcal{O}(D) = \deg D = \int_D 1 = \int_C [D] = \int_C c_1(\mathcal{O}(D)) = \int_C c_1(L).$$

□

Exercise 4.4.2. Show that for a base-point free line bundle L on a compact complex manifold X the integral $\int_X c_1(L)^n$ is non-negative.

Solution. Since L is base point free, there exists a holomorphic map $\varphi_L : X \to \mathbb{P}^N$, where $N := \dim H^0(X,L)$ such that $\varphi_L^*\mathcal{O}_{\mathbb{P}^N}(1) \cong L$. Thus

$$\int_X c_1(L)^n = \int_X \varphi_L^*(c_1(\mathcal{O}_{\mathbb{P}^N}(1)))^n = \int_X \varphi_L^*(\omega_{\mathrm{FS}})^n.$$

Since $\omega_{\mathrm{FS}}(\cdot, I(\cdot))$ is positive definite, its pullback along φ_L must be semi-positive definite. Thus by computing the integration locally and summing them via partition of unity,

$$\int_X \varphi_L^*(\omega_{\mathrm{FS}})^n \geqslant 0.$$

□

Exercise 4.4.3. Show that $\mathrm{td}(E_1 \oplus E_2) = \mathrm{td}(E_1) \cdot \mathrm{td}(E_2)$.

Solution. Note that $F_{\nabla_E} = F_{\nabla_{E_1}} \oplus F_{\nabla_{E_2}}$ and $\exp(A \oplus B) = \exp(A) \oplus \exp(B)$, hence

$$\begin{aligned}
\mathrm{td}(E) =& \frac{\det(\sqrt{-1}/2\pi \cdot F_{\nabla_E})}{\det(\mathrm{id} - \exp(\sqrt{-1}/2\pi \cdot F_{\nabla_E}))} \\
=& \frac{\det(\sqrt{-1}/2\pi \cdot F_{\nabla_{E_1}})}{\det(\mathrm{id} - \exp(\sqrt{-1}/2\pi \cdot F_{\nabla_{E_1}}))} \cdot \frac{\det(\sqrt{-1}/2\pi \cdot F_{\nabla_{E_2}})}{\det(\mathrm{id} - \exp(\sqrt{-1}/2\pi \cdot F_{\nabla_{E_2}}))} \\
=& \mathrm{td}(E_1) \cdot \mathrm{td}(E_2).
\end{aligned}$$

□

Exercise 4.4.4. Compute the Chern classes of (the tangent bundle of) $\mathbb{P}^n$ and $\mathbb{P}^n \times \mathbb{P}^m$. Try to interpret $\int_{\mathbb{P}^n} c_n(\mathbb{P}^n)$ and $\int_{\mathbb{P}^n \times \mathbb{P}^m} c_{n+m}(\mathbb{P}^n \times \mathbb{P}^m)$.

Solution. Consider the Euler sequence, one can get

$$c(\mathbb{P}^n) \cdot c(\mathcal{O}) = c(\mathcal{O}(1))^{n+1} = (1 + \omega_{\mathrm{FS}})^{n+1},$$

thus $c_k(\mathbb{P}^n) = \binom{n+1}{k}\omega_{\mathrm{FS}}^k$. So

$$\int_{\mathbb{P}^n} c_n(\mathbb{P}^n) = (n+1)\int_{\mathbb{P}^n} \omega_{\mathrm{FS}}^n = n+1.$$

Since $c(\mathbb{P}^n \times \mathbb{P}^m) = c(\mathbb{P}^n) \cdot c(\mathbb{P}^m) = (1+\alpha)^{n+1}(1+\beta)^{m+1}$, where α, β denotes the Fubini–Study metric on $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively, $c_{n+m}(\mathbb{P}^n \times \mathbb{P}^m) = c_n(\mathbb{P}^n) \cdot c_m(\mathbb{P}^m) = (n+1)(m+1)\alpha^n\beta^m$. So

$$\int_{\mathbb{P}^n \times \mathbb{P}^m} c_{n+m}(\mathbb{P}^n \times \mathbb{P}^m) = (n+1)(m+1).$$

$\square$

Exercise 4.4.5. Prove the following explicit formulae for the first three terms of $\mathrm{ch}(E)$ and $\mathrm{td}(E)$ in terms of $c_i(E)$:

$$\mathrm{ch}(E) = \mathrm{rk}(E) + c_1(E) + \frac{c_1(E)^2 - 2c_2(E)}{2} + \frac{c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)}{6} + \cdots$$
$$\mathrm{td}(E) = 1 + \frac{c_1(E)}{2} + \frac{c_1^2(E) + c_2(E)}{12} + \frac{c_1(E)c_2(E)}{24} + \cdots$$

Solution. By splitting principle, one can assume that E is the direct sum of line bundles L_i with first Chern classes γ_i. So

$$\begin{aligned}\mathrm{ch}_2(E) =& \frac{1}{2}\sum_i \gamma_i^2 \\ =& \frac{1}{2}\left(\sum_i \gamma_i\right)^2 - \frac{1}{2}\sum_{i\neq j}\gamma_i\gamma_j \\ =& \frac{c_1(E)^2}{2} - c_2(E)\end{aligned}$$

and

$$\begin{aligned}\mathrm{ch}_3(E) =& \frac{1}{6}\sum_i \gamma_i^3 \\ =& \frac{1}{6}\left(\sum_i \gamma_i\right)^3 - \frac{1}{2}\sum_{i\neq j}\gamma_i^2\gamma_j - \frac{1}{6}\sum_{i\neq j, j\neq k, k\neq i}\gamma_i\gamma_j\gamma_k \\ =& \frac{c_1(E)^3}{6} - \frac{1}{2}\left(\sum_{i\neq j}\gamma_i^2\gamma_j + \frac{1}{2}\sum_{i\neq j, j\neq k, k\neq i}\gamma_i\gamma_j\gamma_k\right) + \frac{1}{12}\sum_{i\neq j, j\neq k, k\neq i}\gamma_i\gamma_j\gamma_k \\ =& \frac{c_1(E)^3}{6} - \frac{1}{2}\left(\sum_i \gamma_i\right)\left(\frac{1}{2}\sum_{i\neq j}\gamma_i\gamma_j\right) + \frac{c_3(E)}{2} \\ =& \frac{c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)}{6}.\end{aligned}$$

Similarly,

$$\mathrm{td}_1(E) = \sum_i \frac{\gamma_i}{2} = \frac{c_1(E)}{2},$$

$$\begin{aligned}\mathrm{td}_2(E) =& \frac{B_1}{2}\sum_i \gamma_i^2 + \frac{1}{4}\sum_{i>j}\gamma_i\gamma_j \\ =& \frac{c_1(E)^2 - 2c_2(E)}{12} + \frac{1}{4}c_2(E) \\ =& \frac{c_1(E)^2 + c_2(E)}{12}\end{aligned}$$

and

$$\begin{aligned}\operatorname{td}_3(E) =&\frac{B_1}{4}\sum_{i\neq j}\gamma_i^2\gamma_j+\frac{1}{8}\sum_{i>j>k}\gamma_i\gamma_j\gamma_k\\ =&\frac{1}{24}\left(\sum_{i\neq j}\gamma_i^2\gamma_j+\frac{1}{2}\sum_{i\neq j,j\neq k,k\neq i}\gamma_i\gamma_j\gamma_k\right)\\ =&\frac{c_1(E)c_2(E)}{24}.\end{aligned}$$

□

Exercise 4.4.6. Let E be a vector bundle and L a line bundle. Show

$$c_i(E\otimes L)=\sum_{j=0}^{i}\binom{\operatorname{rk}(E)-j}{i-j}c_j(E)c_1(L)^{i-j}.$$

Solution. By splitting principle, one can assume that E is the direct sum of line bundles L_i with first Chern classes γ_i and $c_1(L)=\gamma$. Then

$$c(E\otimes L)=\prod_i c(L_i\otimes L)=\prod_i(1+\gamma+\gamma_i)$$

and the above formula follows directly. □

Exercise 4.4.7. Show that on $\mathbb{P}^n$ one has $c_1(\mathcal{O}(1))=[\omega_{\mathrm{FS}}]\in H^2(\mathbb{P}^n,\mathbb{R})$.

Solution. Endow $\mathcal{O}(1)$ with natural Hermitian metric generated by the coordinate function and Chern connection ∇, then the associated curvature form $F_\nabla=\frac{2\pi}{\sqrt{-1}}\omega_{\mathrm{FS}}$. Thus $c_1(\mathcal{O}(1))=\left[\frac{\sqrt{-1}}{2\pi}F_\nabla\right]=[\omega_{\mathrm{FS}}]$. □

Exercise 4.4.8. Prove that a polynomial P of degree k on the space of $r\times r$ matrices is invariant if and only if $\sum_i P(A_1,\cdots,A_{i-1},[A,A_i],A_{i+1},\cdots,A_k)=0$ for all matrices $A_1,\cdots,A_k,A$.

Solution. This follows directly by the exponential map of $\mathrm{GL}(r,\mathbb{C})$ is surjective and take derivative on the one-parameter group action. □

Exercise 4.4.9. Show that $c_1(\operatorname{End}(E))=0$ on the form level and compute

$$c_2(\operatorname{End}(E))$$

in terms of $c_i(E)$, $i=1,2$. Compute $(4c_2-c_1^2)(L\oplus L)$ for a line bundle L. Show that $c_{2k+1}(E)=0$, if $E\cong E^*$.

Solution. Since $\operatorname{End}(E)\cong E^*\otimes E$, one can choose an hermitian metric and its Chern connection ∇ on E and the corresponding dual connection ∇^* on E^*. Then $F_{\nabla^*}=-F_\nabla^{\mathrm{tr}}$, thus $\operatorname{tr}F_\nabla=-\operatorname{tr}F_{\nabla^*}$. So $F_{\nabla^{\operatorname{End}(E)}}$ has zero trace and $c_1(\operatorname{End}(E),\nabla^{\mathrm{End(E)}})=0$.

Since $\operatorname{ch}_2(\operatorname{End}(E))=\operatorname{ch}_2(E)\operatorname{rk}(E)+\operatorname{ch}_2(E^*)\operatorname{rk}(E^*)+\operatorname{ch}_1(E)\cdot\operatorname{ch}_1(E^*)$ and $\operatorname{ch}_2(E^*)=\frac{c_1(E^*)^2-2c_2(E^*)}{2}=\frac{c_1(E)^2-2c_2(E)}{2}$, $\operatorname{ch}_1(E)=c_1(E)=-c_1(E^*)=-\operatorname{ch}_1(E)$, one have

$$\frac{c_1(\operatorname{End}(E))^2-2c_2(\operatorname{End}(E))}{2}=\operatorname{rk}(E)(c_1(E)^2-2c_2(E))-c_1(E)^2,$$

thus $c_2(\mathrm{End}(E)) = \mathrm{rk}(E)(2c_2(E) - c_1(E)^2) + c_1(E)^2$.

Since $c(L \oplus L) = c(L)^2 = 1 + 2c_1(L) + c_1(L)^2$, $c_2(L \oplus L) = c_1(L)^2$ and $c_1(L \oplus L) = 2c_1(L)$. So $(4c_2 - c_1^2)(L \oplus L) = 0$.

Note that $c_k(E^*) = (-1)^k c_k(E)$, the last statement follows directly. $\square$

Exercise 4.4.10. Let L be a holomorphic line bundle on a compact Kähler manifold X. Show that for any closed real $(1,1)$-form α with $\alpha = c_1(L)$ there exists an hermitian structure on L such that the curvature of the Chern connection ∇ on L satisfies $(\sqrt{-1}/2\pi)F_\nabla = \alpha$.

Solution. Fix an hermitian metric h_0 on L, then another hermitian metric h can be expressed by the form $e^f \cdot h_0$. Then then Chern connection ∇^h can be locally expressed by $d + \partial \log h = d + \partial(f + \log h_0)$, and its curvature $F_{\nabla^h} = d\partial(f + \log h_0) + \partial(f + \log h_0) \wedge \partial(f + \log h_0) = \bar{\partial}\partial f + F_{\nabla^{h_0}}$. Therefore, the $\partial\bar{\partial}$-lemma implies the conclusion directly. $\square$

Exercise 4.4.11. Let X be a compact Kähler manifold. Show that via the natural inclusion $H^k(X, \Omega_X^k) \subset H^{2k}(X, \mathbb{C})$ one has

$$\mathrm{ch}_k(E) = \frac{1}{k!}\left(\frac{\sqrt{-1}}{2\pi}\right)^k \mathrm{tr}\left(A(E)^{\otimes k}\right).$$

Here, $A(E)^{\otimes k}$ is obtained as the image of $A(E) \otimes \cdots \otimes A(E)$ under the natural map $H^1(X, \Omega_X \otimes \mathrm{End}(E)) \times, \ldots, H^1(X, \Omega_X \otimes \mathrm{End}(E)) \to H^k(X, \Omega_X^k \otimes \mathrm{End}(E))$ which is induced by composition in $\mathrm{End}(E)$ and exterior product in Ω_X^*.

Solution. Note that $A(E)$ is just defined for a holomorphic vector bundle. Since for the curvature F_∇ of the of the Chern connection on an hermitian holomorphic vector bundle (E, h), one have $[F_\nabla] = A(E)$, thus this statement follows from the relation between Chern classes and Chern characters. $\square$

Exercise 4.4.12. Let X be a compact Kähler manifold and let E be a holomorphic vector bundle admitting a holomorphic connection $D : E \to \Omega_X \otimes E$. Show that $c_k(E) = 0$ for all $k > 0$.

Solution. Note that E admits a holomorphic connection if and only if $A(E) = 0$, then $c(E) = 0$. $\square$

5 Applications of Cohomology

5.1 Hirzebruch–Riemann–Roch Theorem

Exercise 5.1.1. Let X be a K3 surface. Show that $b_2(X) = 22$. Prove that the Picard number $\rho(X)$ is bounded by 20.

Solution. First we admit the fact that K3 surface must be Kähler which is proved by Siu. Then by Poincaré duality and the Hodge decomposition we get $b_1(X) = b_3(X) = h^{1,2}(X) = h^{2,1}(X) = h^{1,0}(X) = h^{0,1}(X) = h^1(X, \mathcal{O}_X) = 0$ and $b_0(X) = b_4(X) = h^{0,0}(X) = h^{2,2}(X) = 1$. By the Serre duality of $\mathcal{O}_X$, we get $H^0(X, \mathcal{O}_X) \cong (H^2(X, K_X))^* \cong (H^2(X, \mathcal{O}_X))^*$, thus $h^{0,2}(X) = h^{0,0}(X) = 1$. So $h^{2,0}(X) = 1$. Note that $c_1(X) = c_1(\mathcal{T}_X) = c_1(\det \mathcal{T}_X) = -c_1(K_X) = -c_1(\mathcal{O}_X) = 0$, we apply the Hirzebruch–Riemann–Roch Theorem on $\mathcal{O}_X$, then we can get

$$
\begin{aligned}
&\int_X \frac{c_1^2(X) + c_2(X)}{12} \\
=&h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) \\
=&h^{0,0}(X) - h^{0,1}(X) + h^{0,2}(X) = 2.
\end{aligned}
$$

So $\displaystyle\int_X c_2(X) = 24$. Therefore $c_1(\Omega_X) = 0$ and $\displaystyle\int_X c_2(\Omega_X) = 24$ since $\Omega_X = \mathcal{T}_X^*$ and $c_2(\mathcal{T}_X) = c_2(\Omega_X)$. Now we apply the Hirzebruch–Riemann–Roch Theorem on Ω_X, then

$$
\begin{aligned}
&h^{1,0}(X) - h^{1,1}(X) + h^{1,2}(X) \\
=&h^0(X, \Omega_X) - h^1(X, \Omega_X) + h^2(X, \Omega_X) \\
=&\chi(X, \Omega_X) \\
=&\int_X \mathrm{ch}_2(\Omega_X)\,\mathrm{td}_0(X) + \mathrm{ch}_1(\Omega_X)\,\mathrm{td}_1(X) + \mathrm{ch}_0(\Omega_X)\,\mathrm{td}_2(X) \\
=&\int_X -\frac{5}{6} c_2(X) \\
=&-20.
\end{aligned}
$$

Hence $h^{1,1}(X) = 20$. Therefore, $\rho(X) \leqslant h^{1,1}(X) = 20$ and $b_2(X) = h^{2,0}(X) + h^{1,1}(X) + h^{0,2}(X) = 22$. $\square$

Exercise 5.1.2. Let $X = \mathbb{C}^n/\Gamma$ be a complex torus and $L \in \mathrm{Pic}(X)$. Consider $c_1(L) \in H^2(X, \mathbb{R})$ as an alternating form on $H^1(X, \mathbb{R})^*$ and choose a basis such that it corresponds to the matrix

$$
\begin{pmatrix}
0 & \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \\
\begin{pmatrix} -\lambda_1 & & \\ & \ddots & \\ & & -\lambda_n \end{pmatrix} & 0
\end{pmatrix}.
$$

Show that $\chi(X, L) = \lambda_1 \times \cdots \times \lambda_n$.

Solution. I think here we need the dual basis of the basis of $H^1(X, \mathbb{R})^*$ we choose need to form a volume form which has total integration 1 on X.

Suppose the basis we choose is $\{e_1, \ldots, e_{2n}\}$, its dual basis is $\{e_1^*, \ldots, e_{2n}^*\}$, then

$$
c_1(L) = \sum_{i=1}^n \lambda_i e_{2i-1}^* \wedge e_{2i}^*,
$$

hence by the Hirzebruch–Riemann–Roch Theorem we get

$$\chi(X,L)=\int_X \frac{c_1(L)^n}{n!}=\lambda_1\times\cdots\times\lambda_n\int_X e_1^*\wedge\cdots\wedge e_{2n}^*=\lambda_1\times\cdots\times\lambda_n.$$

□

Exercise 5.1.3. The Hilbert polynomial of a polarized manifold (X, L), i.e. L is an ample line bundle on X, is defined as the function

$$\begin{aligned}\mathbb{Z}&\to\mathbb{Z}\\ m&\mapsto P_{(X,L)}(m):=\chi(X,L^{\otimes m}).\end{aligned}$$

Show that $P_{(X,L)}$ is indeed a polynomial in m. Determine its degree and its leading coefficient.

Solution. By Hirzebruch–Riemann–Roch Theorem we get

$$\begin{aligned}&\chi(X,L^{\otimes m})\\ =&\int_X[\mathrm{ch}(L^{\otimes m})\,\mathrm{td}(X)]_{2n}\\ =&\int_X\left[\sum_{i=0}^n\frac{c_1(L^{\otimes m})^i}{i!}\cdot\mathrm{td}(X)\right]_{2n}\\ =&\int_X\left[\sum_{i=0}^n\frac{c_1(L)^i m^i}{i!}\cdot\mathrm{td}(X)\right]_{2n}\\ =&\sum_{i=0}^n\left[\int_X\frac{c_1(L)^i}{i!}\cdot\mathrm{td}_{n-i}(X)\right]m^i,\end{aligned}$$

hence it is indeed a polynomial with degree n and its leading coefficient is $\displaystyle\int_X\frac{c_1(L)^n}{n!}$. □

Exercise 5.1.4. Compute the Hilbert polynomial of a hypersurface $Y\subset\mathbb{P}^n$ of degree k.

Solution. First note that the Hilbert polynomial of Y is defined as $P_Y(m):=\chi(Y,\mathcal{O}_Y(m))$. By Example 5.2.5 in [5] one can get $\chi(\mathbb{P}^n,\mathcal{O}(d))=\binom{n+d}{n}$. So due to the short exact sequence

$$0\to\mathcal{O}_{\mathbb{P}^n}(-k)\to\mathcal{O}_{\mathbb{P}^n}\to i_*\mathcal{O}_Y\to 0,$$

where $i: Y\to\mathbb{P}^n$ is the inclusion, we have another short exact sequence by twisting $\mathcal{O}_{\mathbb{P}^n}(m)$

$$0\to\mathcal{O}_{\mathbb{P}^n}(m-k)\to\mathcal{O}_{\mathbb{P}^n}(m)\to i_*\mathcal{O}_Y(m)\to 0,$$

which implies that

$$\begin{aligned}&\chi(\mathbb{P}^n,i_*\mathcal{O}_Y(m))\\ =&\chi(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(m))-\chi(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(m-k))\\ =&\binom{n+m}{n}-\binom{n+m-k}{n}\end{aligned}$$

Also, by Leray spectral sequence, since i is a closed embedding, the pullback homomorphisms $H^i(\mathbb{P}^n,i_*\mathcal{O}_Y(m))\to H^i(Y,\mathcal{O}_Y(m))$ are isomorphisms. Hence

$$P_Y(m)=\binom{n+m}{n}-\binom{n+m-k}{n}.$$

□

Exercise 5.1.5. Let L be a line bundle on a compact connected curve C with $\deg(L) > g(C) - 1$. Show that L admits non-trivial global holomorphic sections.

> ***Solution.*** Riemann–Roch Theorem tells us that $h^0(X, L) - h^1(X, L) = \chi(X, L) = \deg(L) + 1 - g(C) > 0$, hence $h^0(X, L) > 0$. Thus L admits nontrivial global sections. $\square$

Exercise 5.1.6. Let L be a line bundle on a compact connected curve C with $\deg(L) > 2g(C) - 1$. Show that L is globally generated.

> ***Solution.*** Suppose $D \in \operatorname{Div}(C)$ satisfies that $L = \mathcal{O}(D)$, then L is globally generated iff for every $x \in C$, there exists $f \in H^0(C, \mathcal{O}(D))$ such that $\operatorname{ord}_x(f) = -D(x)$. Suppose $D' = D - x$, then $\deg D' = \deg D - 1 > 2g - 2$, thus by Exercise 4.1.2, we have $H^1(C, \mathcal{O}(D')) = H^1(C, \mathcal{O}(D)) = 0$. Hence by Riemann–Roch Theorem, $h^0(C, \mathcal{O}(D)) = h^0(C, \mathcal{O}(D')) + 1$, which means that there exists $f \in H^0(C, \mathcal{O}(D))$ such that $-D(x) + 1 > \operatorname{ord}_x(f) \geqslant -D(x)$, so $\operatorname{ord}_x(f) = -D(x)$. And this implies that L is globally generated. $\square$

Exercise 5.1.7. Let L be a line bundle on a compact surface X with $\displaystyle\int_X c_1(L)^2 > 0$. Show that for $m \gg 0$ either $L^{\otimes m}$ or $L^{\otimes(-m)}$ admits non-trivial global holomorphic sections.

> ***Solution.*** By Hirzebruch–Riemann–Roch Theorem we get $h^0(X, L^{\otimes m}) + h^2(X, L^{\otimes m}) \geqslant \chi(X, L) > am^2$ for some $a > 0$. Hence by Serre duality we have $h^0(X, L^{\otimes m}) + h^0(X, K_X \otimes L^{\otimes -m}) > am^2$. Siimilarly, $h^0(X, L^{\otimes -m}) + h^0(X, K_X \otimes L^{\otimes m}) > am^2$. We take an M such that $h^0(X, K_X^2) < aM^2$. Then for any $m > M$, if $h^0(X, K_X \otimes L^{\otimes -m}) > 0$, we have $h^0(X, K_X \otimes L^{\otimes m}) \leqslant h^0(X, K_X^2) < am^2$, this implies that $h^0(X, L^{\otimes -m}) > 0$, else if $h^0(X, K_X \otimes L^{\otimes -m}) = 0$, then $h^0(X, L^{\otimes m}) > am^2 > 0$ directly. $\square$

Exercise 5.1.8. Let X be a compact surface such that $c_1(X) \in 2H^2(X, \mathbb{Z})$. Show that $\displaystyle\int_X c_1(L)^2$ is even for any line bundle L on X.

> ***Solution.*** Since $\chi(X, L)$ and $\chi(X, \mathcal{O}_X)$ are both integers,
> $$\int_X \frac{c_1(L)(c_1(L) + c_1(X))}{2}$$
> is also integer by Hirzebruch–Riemann–Roch Theorem. Since $c_1(X) \in 2H^2(X, \mathbb{Z})$,
> $$\int_X \frac{c_1(L)c_1(X)}{2}$$
> is an integer, hence
> $$\int_X c_1(L)^2$$
> is even. $\square$

Exercise 5.1.9. Let X and Y be compact complex manifolds and let $f\colon X \to Y$ be a smooth finite morphism of degree d. In other words, $f\colon X \to Y$ is smooth surjective with $\dim(X) = \dim(Y)$ and every fibre $f^{-1}(y)$ contains d points. Show that $\operatorname{td}(X) = f^* \operatorname{td}(Y)$ and deduce $\chi(X, \mathcal{O}_X) = d \cdot \chi(Y, \mathcal{O}_Y)$. In particular, if X and Y are K3 surfaces, then $d = 1$.

Solution. Since f is smooth (here smooth means a holomorphic map with $\mathcal{T}_X \to f^*\mathcal{T}_Y$ is surjective everywhere) and $\dim(X) = \dim(Y)$, $\mathcal{T}_X \to f^*\mathcal{T}_Y$ is an isomorphism. So $\operatorname{td}(X) = \operatorname{td}(\mathcal{T}_X) = \operatorname{td}(f^*\mathcal{T}_Y) = f^* \operatorname{td}(\mathcal{T}_Y) = f^* \operatorname{td}(Y)$.

Hence

$$\begin{aligned}
&\chi(X, \mathcal{O}_X) \\
=&\int_X [\operatorname{ch}(\mathcal{O}_X) \operatorname{td}(X)]_{2n} \\
=&\int_X [\operatorname{ch}(f^*\mathcal{O}_Y) \cdot f^* \operatorname{td}(Y)]_{2n} \\
=&\int_X f^*[\operatorname{ch}(\mathcal{O}_Y) \operatorname{td}(Y)]_{2n} \\
=&d\int_Y [\operatorname{ch}(\mathcal{O}_Y) \operatorname{td}(Y)]_{2n} \\
=&d\chi(Y, \mathcal{O}_Y).
\end{aligned}$$

For K3 surface X, we have proven $\chi(X, \mathcal{O}_X) = 2$, so the degree of a smooth map between two K3 surfaces must be 1. □

5.2 Kodaira Vanishing Theorem and Applications

Exercise 5.2.1. Let (E, h) be an hermitian holomorphic vector bundle on a compact Kähler manifold X. Suppose that the curvature F_∇ of the Chern connection is trivial, i.e. the Chern connection is flat. Prove that the Lefschetz operator Λ preserves the harmonicity of forms and thus defines a map $\Lambda\colon \mathcal{H}^{p,q}(X, E) \to \mathcal{H}^{p-1,q-1}(X, E)$. Deduce from this the existence of a Lefschetz decomposition on $H^{*,*}(X, E)$.

Solution. Note that a form $\beta \in \mathcal{A}^{p,q}(X, E)$ is harmonic iff $\bar\partial_E\beta = \bar\partial_E^*\beta = 0$. Thus for any $\alpha \in \mathcal{H}^{p,q}(X, E)$, $-\|\bar\partial\Lambda(\alpha)\|^2 = (\sqrt{-1}F_\nabla\Lambda(\alpha), \alpha) = 0$. And note that $[\Lambda, \bar\partial_E^*] = 0$ by Kähler identity (since it holds locally), $\bar\partial_E^*\Lambda(\alpha) = \Lambda\bar\partial_E^*(\alpha) = 0$. Hence $\Lambda(\alpha)$ is also harmonic.

Similarly, one can prove L preserves harmonicity as well by $[\partial_E, L] = 0$, $[\bar\partial_E^*, L] = \sqrt{-1}\nabla^{1,0}$ (can be proven by taking adjoint on the Nakano identity with respect to the standard inner product on $\mathcal{A}^{p,q}(X, E)$) and $\sqrt{-1}(L(\alpha), F_\nabla(\alpha)) = -\|\bar\partial_E^*L(\alpha)\|^2$ which follows directly from previous identity. Thus one can deduce the existence of Lefschetz decomposition similarly as the case of trivial E. $\square$

Exercise 5.2.2. Let C be an elliptic curve. Show that $H^1(C, \mathcal{O}_C) = \mathbb{C}$ and use this to construct a non-splitting extension $0 \to \mathcal{O} \to E \to \mathcal{O} \to 0$. Prove that E cannot be written as a direct sum of two holomorphic line bundles.

Solution. $H^1(C, \mathcal{O}_C) \cong H^0(K_C)^*$. Since $\bigwedge^{0,1} C = K_C$ is trivial with global section $\mathrm{d}\bar z$, $H^0(K_C) = \mathbb{C}\mathrm{d}\bar z \cong \mathbb{C}$.

Note that the isomorphism classes of $\mathcal{E} = (E, \bar\partial_E)$ of the form

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{Q} \to 0$$

is in bijection with $\mathbb{P}(H^{0,1}(C, \mathrm{Hom}(Q, F)))$ by using the second fundamental form. Since $H^{0,1}(C, \mathrm{Hom}(\mathcal{O}_C, \mathcal{O}_C)) = H^{0,1}(C) = H^1(C, \mathcal{O}_C) \cong \mathbb{C}$, one can choose a nonzero element $[\beta]$ in it to construct $(E, \bar\partial_E)$ by defining $\bar\partial_E = \begin{pmatrix} \bar\partial & \beta \\ 0 & \bar\partial \end{pmatrix}$. For our convenience, we suppose $\beta = \mathrm{d}\bar z \otimes (e_2^* \otimes e_1)$, where e_1, e_2 be the respective nowhere vanishing global section of two $\mathcal{O}_C$.

Suppose $\phi \in \mathrm{End}(\mathcal{E})$. Consider $ae_1 + be_2$ as a holomorphic section of $\mathcal{E}$, where $a, b \in \mathcal{A}^0(C)$. Then $0 = \bar\partial_E(ae_1 + be_2) = (\bar\partial a + b\mathrm{d}\bar z)e_1 + \bar\partial b e_2$. So b is a constant and it must be 0 since $\mathrm{d}\bar z$ is not $\bar\partial$-exact and a must be a constant. Hence $H^0(X, \mathcal{E}) \cong \mathbb{C}$. Suppose ϕ induced the multiplication by a on $H^0(X, \mathcal{E})$, then we consider $\phi' := \phi - a\,\mathrm{id}_E$. Since ϕ' induce the zero map on $H^0(X, \mathcal{E})$, the first $\mathcal{O}_C$ must be mapped into 0, hence ϕ' factor through the second $\mathcal{O}_C$, denoted by $\tilde\phi'\colon \mathcal{O}_C \to \mathcal{E}$. Since the exact sequence defining $\mathcal{E}$ is not split, $\mathcal{O}_C \xrightarrow{\tilde\phi'} \mathcal{E} \to \mathcal{O}_C$ must be 0. This means that

$$\mathcal{E} \to \mathcal{O}_C \xrightarrow{\tilde\phi'} \mathcal{E} \to \mathcal{O}_C$$

is zero, hence the image of ϕ' must lie in the first $\mathcal{O}_C$. Thus ϕ is of the form $\begin{pmatrix} t & \alpha \\ 0 & t \end{pmatrix}$, where $\alpha \in \mathrm{Hom}(\mathcal{O}_C, \mathcal{O}_C)$.

Now suppose $\mathcal{E}$ splits into $\mathcal{F} \oplus \mathcal{Q}$. Then both $f_1 := (\mathrm{id}_F, 0)$ and $f_2 := (0, \mathrm{id}_Q)$ are not isomorphism but their sum, id_E, is an isomorphism. However, suppose $f_1 = \begin{pmatrix} t & \alpha \\ 0 & t \end{pmatrix}$, then $f_2 = \begin{pmatrix} 1-t & -\alpha \\ 0 & 1-t \end{pmatrix}$ under the defining exact sequence. Without loss of generality, we assume $t \neq 0$ (otherwise, consider f_2), then f_1 has inverse $\begin{pmatrix} t^{-1} & t^{-2}\alpha \\ 0 & t^{-1} \end{pmatrix}$, contradiction. $\square$

Exercise 5.2.3. Show that on $\mathbb{P}^2$ there exists a rank two vector bundle which is not isomorphic to the direct sum of holomorphic line bundles.

Solution. By the Euler sequence, one can deduce that $h^0(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}) = 3 \times 3 - 1 = 8$. Since $h^0(\mathbb{P}^2, \mathcal{O}(m))$ equals to either 0 (when $m < 0$) or $\binom{2+m}{2}$ (when $m \geqslant 0$), the sum of two of these cannot be 8. $\square$

Exercise 5.2.4. Let $C \subset \mathbb{P}^2$ be a smooth curve defined by a homogeneous polynomial of degree d. Show that the genus $g(C) = \dim H^0(C, K_C)$ is given by the formula

$$g(C) = \frac{1}{2}(d-1)(d-2).$$

Use this to show that there are curves which are not plane, i.e. not isomorphic to a smooth curve in $\mathbb{P}^2$. Prove that for a smooth curve $C \subset X$ in a K3 surface X one has $g(C) = ([C]^2 + 2)/2$.

Solution. We first compute K_C,

$$\begin{aligned} K_C =& K_{\mathbb{P}^2}|_C \otimes \det(\mathcal{N}_{C/\mathbb{P}^2}) \quad \text{(adjuction formula)} \\ =& \mathcal{O}(-3)|_C \otimes \mathcal{O}(d)|_C \quad (K_{\mathbb{P}^n} = \mathcal{O}(-n-1) \text{ and Exercise 2.4.1}) \\ =& \mathcal{O}(d-3)|_C. \end{aligned}$$

Thus $\deg K_C = d(d-3)$ by Exercise 2.3.8. Due to Riemann–Roch Theorem,

$$g(C) = \frac{\deg(K_C)}{2} + 1 = \frac{(d-1)(d-2)}{2}.$$

Hence any curve, i.e. Riemann surface, with genus 2, cannot be embedded into $\mathbb{P}^2$.

For any curve C in a K3 surface X,

$$\begin{aligned} K_C =& K_X|_C \otimes \det(\mathcal{N}_{C/X}) \quad \text{(adjunction formula)} \\ =& \mathcal{O}_X|_C \otimes \mathcal{O}([C])|_C \quad \text{(definition of K3 surface and Exercise 2.3.2)} \\ =& \mathcal{O}([C])|_C. \end{aligned}$$

So

$$\deg K_C = \int_C c_1(\mathcal{O}([C])|_C) = \int_C i^* c_1(\mathcal{O}([C])) = \int_X c_1(\mathcal{O}([C]))^2 =: [C]^2,$$

where $i\colon C \to X$ is the inclusion, by Poincaré duality. Hence $g(C) = ([C]^2 + 2)/2$ by Riemann–Roch Theorem again. $\square$

Exercise 5.2.5. Show that hypersurfaces in $\mathbb{P}^n$ with $n \geqslant 3$ do not admit non-trivial holomorphic one-forms. In particular, the Albanese of any such hypersurface is trivial.

Solution. Weak Lefschetz Theorem tells us the restriction map $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) \to H^0(Y, \Omega^1_Y)$ is bijective for any hypersurface Y in $\mathbb{P}^n$ since $1 \leqslant n-2$. Then the statements hold directly from $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) \cong H^{1,0}(\mathbb{P}^n) = 0$. $\square$

Exercise 5.2.6. Which complex tori could possibly be realized as complete intersections in $\mathbb{P}^n$?

Solution. See Corollary 5.3.5 of [5]. $\square$

Exercise 5.2.7. Let L be an ample line bundle on a K3 surface X. Show that $h^0(X, L) = 2 + \frac{1}{2}\int_X c_1(L)^2$. Study ample line bundles on complex tori.

Solution. By Hirzebruch–Riemann–Roch Theorem and $K_X \cong \mathcal{O}_X$ we have $\chi(X, L) = 2 + \frac{1}{2}\int_X c_1(L)^2$, hence we only need to prove that $h^1(X, L) = h^2(X, L) = 0$. By Kodaira embedding Theorem, L is positive. So by Kodaira vanishing Theorem, $h^1(X, L) = h^1(X, K_X \otimes L) = 0$ and $h^2(X, L) = h^2(X, K_X \otimes L) = 0$.

Similarly, for a complex torus X we have

$$\chi(X, L) = \int_X \frac{c_1(L)^n}{n!}$$

due to Hirzebruch–Riemann–Roch Theorem, and similarly by Kodaira vanishing Theorem we can get $h^i(X, L) = 0$ for any $i > 0$ since we still have $K_X \cong \mathcal{O}_X$. Hence we have

$$h^0(X, L) = \int_X \frac{c_1(L)^n}{n!}.$$

□

Exercise 5.2.8. Use Serre duality to give a direct algebraic proof of the Kodaira vanishing theorem for curves.

Solution. Suppose C is a compact curve and L is a positive line bundle on it, then $H^1(C, K_C \otimes L) \cong (H^0(C, L^*))^*$ by Serre duality. Since

$$\deg(L^*) = -\deg(L) = -\int_C c_1(L) < 0,$$

$H^0(C, L^*) = 0$. □

Exercise 5.2.9. Prove that $H^q(\mathbb{P}^n, \Omega^p(m)) = 0$ for $p + q > n$, $m > 0$ and for $p + q < n$, $m < 0$.

Solution. For $p + q > n$, $m > 0$, $H^q(\mathbb{P}^n, \Omega^p(m)) = 0$ directly by Kodaira vanishing Theorem. For $p + q < n$, $m < 0$, Serre duality tells us $H^q(\mathbb{P}^n, \Omega^p(m)) \cong H^{n-q}(\mathbb{P}^m, \Omega^{n-p}(-m))^*$. Then it equals to 0 by Kodaira vanishing Theorem again. □

Exercise 5.2.10. Let Y be a hypersurface of a compact complex manifold X with $\mathcal{O}(Y)$ positive. Suppose that $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$ are torsion free. Prove that the restriction induces an isomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ if $\dim(X) \geqslant 4$.

Solution. Consider the exponential sequences on X and Y and the induced long exact sequence, note that the map in these two exact sequences commutes with the restriction map induced by the inclusion $Y \to X$, then we can get the following commutative diagram

$$\begin{tikzcd}
H^1(X,\mathbb{Z}) \arrow[r] \arrow[d] & H^1(X,\mathbb{C}) \arrow[r] \arrow[d] & \mathrm{Pic}(X) \arrow[r] \arrow[d] & H^2(X,\mathbb{Z}) \arrow[r] \arrow[d] & H^2(X,\mathbb{C}) \arrow[d] \\
H^1(Y,\mathbb{Z}) \arrow[r] & H^1(Y,\mathbb{C}) \arrow[r] & \mathrm{Pic}(Y) \arrow[r] & H^2(Y,\mathbb{Z}) \arrow[r] & H^2(Y,\mathbb{C})
\end{tikzcd}.$$

By weak Lefschetz Theorem and "five" lemma, we get $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is an isomorphism when $\dim X \geqslant 4$. □

Exercise 5.2.11. Let X be a projective manifold of dimension n and let $L \in \mathrm{Pic}(X)$ be an ample line bundle.

i) Show that $m \mapsto h^0(X, L^{\otimes m})$ for $m \gg 0$ is a polynomial of degree n with positive leading coefficient.

ii) Deduce from this that $a(X) = \dim(X) = n$, i.e. X is Moishezon.

Solution.

i) It follows directly from Hilbert polynomial and Serre vanishing Theorem.

ii) In fact by another definition of Kodaira dimension, i.e. the growth rate of $h^0(X, L^{\otimes m})$, we know that $n = \mathrm{kod}(X, L) \leqslant a(X) \leqslant \dim X = n$. However, I don't know how to prove the equivalence between different definitions of Kodaira dimension. See [7].

$\square$

5.3 Kodaira Embedding Theorem

Exercise 5.3.1. Show that a complex torus $\mathbb{C}^2/\Gamma$ is abelian, i.e. projective, if and only if there exists a line bundle L with $c_1^2(L) > 0$.

Solution. If $X := \mathbb{C}^2/\Gamma$ is projective, then X admits an ample line bundle L, i.e. for some $m > 0$, $L^{\otimes m}$ is basepoint free and Kodaira embedding Theorem tells us L is positive. Then by Hirzebruch–Riemann–Roch Theorem and Kodaira vanishing Theorem we know that

$$m^2 \int_X \frac{c_1^2(L)}{2} = \int_X \frac{c_1^2(L^{\otimes m})}{2} = \chi(X, L^{\otimes m}) = h^0(X, L^{\otimes m}) > 0.$$

Thus $c_1^2(L) > 0$.

If $c_1^2(L) > 0$, then by Hirzebruch–Riemann–Roch Theorem, we know $\chi(X, L) > 0$, hence Exercise 5.1.2 tells us that either $c_1(L)$ or $-c_1(L)$ is positive. Thus either L or L^* is positive. Therefore Kodaira embedding Theorem shows that X is projective. $\square$

Exercise 5.3.2. Show that any vector bundle E on a projective manifold X can be written as a quotient $(L^k)^{\oplus \ell} \twoheadrightarrow E$ with L an ample line bundle, $k \ll 0$ and $\ell \gg 0$.

Solution. Let L be an ample bundle on X. Since if $L^k \otimes E$ can be globally generated by ℓ sections in $H^0(X, L^k \otimes E)$, then E^* can be embedded into $(L^k)^{\oplus \ell}$ by them and then E can be viewed as a quotient of $(L^{-k})^{\oplus \ell}$. Thus we only need to prove that $L^k \otimes E$ is globally generated for sufficiently large $k \gg 0$.

Suppose $\sigma : \hat{X} \to X$ is the blow-up of X in $x \in X$ with the exceptional divisor D and $\dim_{\mathbb{C}} X =: n$. If $n = 1$, then σ is an isomorphism, thus $H^0(X, L^k \otimes E) \cong H^0(\hat{X}, \sigma^*(L^k \otimes E))$, otherwise $n > 1$, Hartogs' theorem shows that $H^0(X, L^k \otimes E) \cong H^0(\hat{X}, \sigma^*(L^k \otimes E))$. Since $L^k \otimes E$ can be generated by global sections at x iff the natural restriction $H^0(X, L^k \otimes E) \to (L^k \otimes E)(x)$ is surjective, and we have the commutative diagram

$$\begin{tikzcd}
H^0(X, L^k \otimes E) \arrow[r] \arrow[d, "\sigma^*"'] & (L^k \otimes E)(x) \arrow[d, "\sim"] \\
H^0(\hat{X}, \sigma^*(L^k \otimes E)) \arrow[r] & H^0(D, \mathcal{O}_D) \otimes (L^k \otimes E)(x)
\end{tikzcd}$$

it suffices to prove that the bottom map is surjective for a $k \gg 0$. However, its cokernel is contained in $H^1(\hat{X}, \sigma^*(L^k \otimes E) \otimes \mathcal{O}(-D))$ by the short exact sequence

$$0 \to \sigma^*(L^k \otimes E) \otimes \mathcal{O}(-D) \to \sigma^*(L^k \otimes E) \to (L^k \otimes E)(x) \otimes \mathcal{O}_D \to 0.$$

Therefore, we will show that $H^1(\hat{X}, \sigma^*(L^k \otimes E) \otimes \mathcal{O}(-D)) = 0$ for a $k \gg 0$.

Let α be the positive form with $[\alpha] = c_1(L)$, then α is a Kähler form on X. Lemma 5.3.2 of [5] shows that there eixsts an hermitian metric on $\mathcal{O}(-D)$ with Chern curvature form F which is positive on D and semi-positive around D. Thus we can take $\varepsilon > 0$ small enough such that $\omega := \sigma^*\alpha + \varepsilon\frac{\sqrt{-1}}{2\pi}F$ is positive on $\hat{X}$, i.e. it is a Kähler form on $\hat{X}$. Thus for the induced connection ∇ on $V_k := \sigma^*(L^k \otimes E) \otimes K_{\hat{X}}^* \otimes \mathcal{O}(-D)$ and a harmonic form $\gamma \in \mathcal{H}^{p,q}(\hat{X}, V_k)$, we have (compare the proof of Proposition 5.2.7 of [5])

$$\begin{aligned}
0 \leqslant& \frac{\sqrt{-1}}{2\pi}([\Lambda, F_\nabla](\gamma), \gamma) \\
=& \frac{\sqrt{-1}}{2\pi}\left(\left[\Lambda, F_{\nabla_{\sigma^*E \otimes K_{\hat{X}}^*}}\right](\gamma), \gamma\right) - (k-1)\varepsilon\frac{\sqrt{-1}}{2\pi}([\Lambda, F](\gamma), \gamma) + k([\Lambda, L_\omega](\gamma), \gamma) \\
\leqslant& \frac{\left\|\left[\Lambda, F_{\nabla_{\sigma^*E \otimes K_{\hat{X}}^*}}\right]\right\|}{2\pi} + k(n - (p+q)) \cdot \|\gamma\|^2.
\end{aligned}$$

Hence for $p = n$, there exists $k \gg 0$ such that for any $q > 0$, the last formula is smaller than 0 unless

$\gamma = 0$. Hence $H^q(\hat{X}, \sigma^*(L^k \otimes E) \otimes \mathcal{O}(-D)) \cong H^{n,q}(\hat{X}, V_k) = \mathcal{H}^{n,q}(\hat{X}, V_k) = 0$ for this k. That completes the proof.

Remark: One can also prove this without using blow-up but considering the natural restriction $L^k \otimes E \to L^k \otimes E \otimes \mathbb{C}_x$, where $\mathbb{C}_x$ is the skyscraper sheaf supported at x. Then its cokernel is contained in $H^1(\mathcal{I}_x \otimes L^k \otimes E)$, where $\mathcal{I}_x$ is the ideal sheaf. Then by the coherent sheaf version of Serre vanishing we can show $L^k \otimes E \to L^k \otimes E \otimes \mathbb{C}_x$ is surjective for sufficiently large k. $\square$

Exercise 5.3.3. Let $\sigma\colon \hat{X} \to X$ be the blow-up of X in $x \in X$ with the exceptional divisor E and let L be an ample line bundle on X. Show that $\sigma^* L^k \otimes \mathcal{O}(-E)$ is ample for $k \gg 0$.

Solution. This comes from Kodaira embedding Theorem, Lemma 5.3.2 of [5], and $\hat{X}$ is still Kähler (cf. Exercise 5.3.2) directly. $\square$

Exercise 5.3.4. Continue Exercise 5.1.6 and show, by using the techniques of this section, that any line bundle L of degree $\deg(L) > 2g$ on a compact curve C of genus g is very ample, i.e. the linear system associated with L embeds C.

Conclude that any elliptic curve is isomorphic to a plane curve, i.e. to a hypersurface in $\mathbb{P}^2$.

Solution. We only need to show φ_L is injective. Suppose $x_1 \neq x_2$ are two points of X. Let $D - x_2$. Since $\deg D' = \deg D - 1 > 2g - 1$, $\mathcal{O}(D')$ is globally generated by Exercise 5.1.6, hence there exists an $f \in H^0(X, \mathcal{O}(D'))$ such that $\mathrm{ord}_{x_1}(f) = -D(x_1)$. By the definition of D' we have $\mathrm{ord}_{x_2}(f) \geqslant -D(x_2) + 1$. f also belongs to $H^0(X, \mathcal{O}_D)$, thus by choosing a basis $\{s_i\}_{i=0}^N$ of $H^0(X, L)$, we can write $f = \sum\limits_i \lambda_i s_i$ where $\lambda_i \in \mathbb{C}$. Let (V_1, z_1) and (V_2, z_2) be coordinate neighborhoods of x_1 and x_2 respectively such that $z_1(x_1) = z_2(x_2) = 0$. Since $\mathcal{O}_D$ is globally generated, we have

$$k_j := \min_i \mathrm{ord}_{x_j}(s_i) = -D(x_j)$$

for $j = 1, 2$. Write $s_i = z_j^{k_j} g_{ji}$ and $f = z_j^{k_j} g_j$ in a neighborhood of x_j. Then $\varphi_L(x_j) = (g_{j0}(x_j) : \cdots : g_{jN}(x_j))$ and

$$\sum_{i=0}^N \lambda_i g_{ji}(x_j) = g_j(x_j).$$

Note that $g_1(x_1) \neq 0$ and $g_2(x_2) = 0$, hence $\varphi_L(x_1) \neq \varphi_L(x_2)$.

For any elliptic curve $X := \mathbb{C}/\Gamma$, we can use the derivative $\wp'_\gamma$ of its corresponding Weierstrass p function $\wp_\gamma$ to define a divisor $D = 3P$ for a point P on it. Then the line bundle $L := \mathcal{O}(3P)$ gives an embedding to $\mathbb{P}^{h^0(X,L)-1}$. Since $\deg L > 2g(X) - 2 = 0$, $h^1(X, L) = h^0(X, K_X \otimes L^*) = 0$. Hence by Riemann–Roch Theorem we know that $h^0(X, L) = 1 - g(X) + \deg L = 3$, i.e. φ_L is an embedding into $\mathbb{P}^2$. For instant, $(1 : \wp_\Gamma : \wp'_\Gamma)$ gives an embedding. $\square$

Exercise 5.3.5. Show that there exists a complex torus X of dimension n such that X is projective and, therefore, $a(X) = n$.

Solution. Suppose $X = \mathbb{C}^n/\Gamma$ with Γ is the free abelian group generated by $e_t, \sqrt{-1}e_t$, where $\{e_t\}_{t=1}^n$ is the standard basis of $\mathbb{C}^n$. Then it is easy to verify that for $u = u_1^t e_t + u_2^t \sqrt{-1} e_t$ and $v = v_1^t e_t + v_2^t \sqrt{-1} e_t$,

$$\omega\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$$

$$(u, v) \mapsto \sum_{t=1}^n (u_1^t v_2^t - u_2^t v_1^t)$$

is a Riemann form. Therefore, X is projective. $\square$

6 Deformations of Complex Structures

6.1 The Maurer-Cartan Equation

Exercise 6.1.1. Compute the dimensions of $H^1(X, \mathcal{T}_X)$ in the following cases: i) $X = \mathbb{P}^n$, ii) X a compact complex torus, iii) X a curve.

Solution.

i) For $X = \mathbb{P}^n$, by the Euler sequence, we get

$$H^1(X, \mathcal{O}(1)^{\oplus(n+1)}) \to H^1(X, \mathcal{T}_X) \to H^2(X, \mathcal{O})$$

is exact. Since $H^1(X, \mathcal{O}(1)) = 0$ ny Kodaira vanishing Theorem and $H^2(X, \mathcal{O}) \cong H^{0,2}(X) = 0$ due to $h^{0,2} = h^{2,0}$ and $b_2 = 1$, $H^1(X, \mathcal{T}_X) = 0$.

ii) For X a compact complex torus $\mathbb{C}^n/\Gamma$, $\mathcal{T}_X$ is trivial and thus $H^1(X, \mathcal{T}_X) = H^1(X, \mathcal{O}) = H^{0,1}(X) = \mathcal{H}^{0,1}(X)$. Since for any $\alpha = f^i \mathrm{d}\bar{z}_i \in \mathcal{A}^{0,1}(X)$, $\Delta(\alpha) = \Delta(f_i)\mathrm{d}\bar{z}_i$, so α is harmonic iff f_i is harmonic functions iff f_i are constants due to X is compact. Thus $\dim H^1(X, \mathcal{T}_X) = h^{0,1}(X) = n$.

Remark: One can also deduce that from $h^{0,1}(X) + h^{1,0}(X) = b_1(X) = 2 \cdot \dim X$ and $h^{0,1}(X) = h^{1,0}(X)$.

iii) We first suppose that X is a compact curve. If $g(X) = 0$, then X is isomorphic to $\mathbb{P}^1$ and this case is contained in i). If $g(X) = 1$, then X is isomorphic to a compact complex torus with dimension 1 and this case is contained in ii). If $g(X) > 1$, then $\deg \mathcal{T}_X = 2 - 2g(X) < 0$ so $h^0(X, \mathcal{T}_X) = 0$. Therefore, by Riemann–Roch Theorem,

$$h^1(X, \mathcal{T}_X) = h^0(X, \mathcal{T}_X) - (1 - g(X) + \deg \mathcal{T}_X) = 3g(X) - 3.$$

Next we consider the noncompact case. It is well-known that any holomorphic line bundle on X is holomorphically trivial and $H^1(X, \mathcal{O}) = 0$. See Theorem 30.3 and Theorem 26.1 of [3].

$\square$

Exercise 6.1.2. Let X be a compact complex manifold (not necessarily Kähler) with a everywhere non-degenerate holomorphic two-form $\sigma \in H^0(X, \Omega_X^2)$, i.e. the induced homomorphism $\mathcal{T}_X \to \Omega_X$ is an isomorphism. Show that also in this case there exists a formal solution $\sum \varphi_i t^i$ satisfying the Maurer-Cartan equation, such that $[\varphi_1] = v \in H^1(X, \mathcal{T}_X)$ is a given class.

Solution. The existence of σ tells us that $n := \dim_{\mathbb{C}} X$ is even and then $\sigma^{n/2}$ gives a holomorphic volume form on X, hence X is Calabi–Yau. So the statement holds directly when X is Kähler. If X is not Kähler, I do not find any good results since we do not have $\partial\bar{\partial}$-lemma and Hodge decomposition. But I see that the statement holds when $H^{0,2}(X)$ is generated by ∂-closed $(0,2)$-forms, all ∂-exact holomorphic 3-forms on X vanish and all classes in $H^{1,1}(X)$ are represented by ∂-closed $(1,1)$-forms, see [6]. $\square$

Exercise 6.1.3. Let X be a compact complex manifold with $H^2(X, \mathcal{T}_X) = 0$. Show that any $v \in H^1(X, \mathcal{T}_X)$ can formally be integrated. This in particular applies to complex curves.

Solution. We can choose an arbitrary $\bar{\partial}$-closed form $\phi_1 \in \mathcal{A}^{0,1}(\mathcal{T}_X)$ presenting v to start. Note that if we have already chosen $\phi_1, \ldots, \phi_k$ satisfying Maurer–Cartan equation, then since $-\sum_{0<i<k} [\phi_i, \phi_{k-i}] \in \mathcal{A}^{0,2}(\mathcal{T}_X)$ is always $\bar{\partial}$-closed, we can choose $\phi_{k+1} \in \mathcal{A}^{0,1}(\mathcal{T}_X)$ satisfying Maurer–Cartan equation due to $H^2(X, \mathcal{T}_X) = 0$. $\square$

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