Anosov property of cyclic $SO_0(2,3)$ -Higgs Bundles (arXiv:2406.08118)

张峻铭 Junming Zhang

Chern Institute of Mathematics, Nankai University

Geometry & Topology Seminar 2025.02.28 at UC Riverside, Riverside



Contents

Anosov Property and Higher (rank) Teichmüller Theory

Non-Abelian Hodge Correspondence

From Higgs Bundle to Anosov Representation

Anosov Property and Higher (rank) Teichmüller Theory

Anosov Property and Higher (rank) Teichmüller Theory

Classical Teichmüller Theory

Let S be a closed connected oriented surface whose genus is larger than 2. Its Teichmüller space $\mathcal{T}(S)$ is the moduli space of marked hyperbolic structure over S.

Classical Teichmüller Theory

Let S be a closed connected oriented surface whose genus is larger than 2. Its Teichmüller space $\mathcal{T}(S)$ is the moduli space of marked hyperbolic structure over S.

The holonomy representations corresponding to the point in $\mathcal{T}(S)$ are called **Fuchsian** representations. Furthermore, we can view $\mathcal{T}(S)$ as a connected component of the character variety

 $\operatorname{Hom}(\pi_1(S), \operatorname{PSL}(2, \mathbb{R})) /\!\!/ \operatorname{PSL}(2, \mathbb{R}).$

Classical Teichmüller Theory

Let S be a closed connected oriented surface whose genus is larger than 2. Its Teichmüller space $\mathcal{T}(S)$ is the moduli space of marked hyperbolic structure over S.

The holonomy representations corresponding to the point in $\mathcal{T}(S)$ are called **Fuchsian** representations. Furthermore, we can view $\mathcal{T}(S)$ as a connected component of the character variety

$$\operatorname{Hom}(\pi_1(S),\operatorname{PSL}(2,\mathbb{R}))/\!\!/\operatorname{PSL}(2,\mathbb{R}).$$

Moreover, $\mathcal{T}(S)$ is one of the two connected components which consist entirely of discrete and faithful representations. The other one is $\mathcal{T}(\overline{S})$, where \overline{S} denotes the surface S with the opposite orientation.

Anosov Property of Fuchsian Representation

For a Fuchsian representation $\rho \colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$, we fix a base point $x_0 = (0,1) \in \mathbb{H}^2 \cong \widetilde{S}$ on the universal cover of S. We have

$$\log \frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))} = d_{\mathbb{H}^2}(x_0, \rho(\gamma)(x_0)),$$

 σ_i denotes the *i*-th singular value.

Instead of focusing on representations from $\pi_1(S)$ into $\mathrm{PSL}(2,\mathbb{R})$, we replace the target by a semi-simple Lie group G of higher rank, such as

6/25

Instead of focusing on representations from $\pi_1(S)$ into $\mathrm{PSL}(2,\mathbb{R})$, we replace the target by a semi-simple Lie group G of higher rank, such as

$$SL(n, \mathbb{R}) \quad (n \geqslant 3),$$

Instead of focusing on representations from $\pi_1(S)$ into $\mathrm{PSL}(2,\mathbb{R})$, we replace the target by a semi-simple Lie group G of higher rank, such as

$$SL(n, \mathbb{R}) \quad (n \geqslant 3),$$

$$\operatorname{Sp}(2n,\mathbb{R}) \quad (n \geqslant 2),$$

6/25

Instead of focusing on representations from $\pi_1(S)$ into $\mathrm{PSL}(2,\mathbb{R})$, we replace the target by a semi-simple Lie group G of higher rank, such as

$$SL(n, \mathbb{R}) \quad (n \geqslant 3),$$

$$\operatorname{Sp}(2n,\mathbb{R}) \quad (n \geqslant 2),$$

$$SO_0(p,q) \quad (\min\{p,q\} \geqslant 2)$$

and so on.

Instead of focusing on representations from $\pi_1(S)$ into $\mathrm{PSL}(2,\mathbb{R})$, we replace the target by a semi-simple Lie group G of higher rank, such as

$$SL(n, \mathbb{R}) \quad (n \geqslant 3),$$

$$\operatorname{Sp}(2n,\mathbb{R}) \quad (n \geqslant 2),$$

$$SO_0(p,q) \quad (\min\{p,q\} \geqslant 2)$$

and so on.

In general, the set of discrete and faithful representations is only closed and NOT open.

Definition

A **higher Teichmüller space** is a subset of $\operatorname{Hom}(\pi_1(S),G)/\!\!/ G$, which is a union of connected components that consist entirely of discrete and faithful representations.

Definition

A **higher Teichmüller space** is a subset of $\operatorname{Hom}(\pi_1(S),G)/\!\!/ G$, which is a union of connected components that consist entirely of discrete and faithful representations.

We give two famous examples of higher Teichmüller space here.

Definition

A higher Teichmüller space is a subset of $\text{Hom}(\pi_1(S), G) /\!\!/ G$, which is a union of connected components that consist entirely of discrete and faithful representations.

We give two famous examples of higher Teichmüller space here.

• (Hitchin, 1992; Labourie, 2006; Fock-Goncharov, 2006) Hitchin components $\mathcal{T}_{Hit}(S,G)$ for real split G, such as $SL(n,\mathbb{R})$, $SO_0(n,n+1)$, $Sp(2n,\mathbb{R})$ and so on:

Definition

A **higher Teichmüller space** is a subset of $\operatorname{Hom}(\pi_1(S), G) /\!\!/ G$, which is a union of connected components that consist entirely of discrete and faithful representations.

We give two famous examples of higher Teichmüller space here.

- (Hitchin, 1992; Labourie, 2006; Fock–Goncharov,2006) Hitchin components $\mathcal{T}_{Hit}(S,G)$ for real split G, such as $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SO}_0(n,n+1)$, $\mathrm{Sp}(2n,\mathbb{R})$ and so on;
- (Burger–lozzi–Wienhard, 2003) Maximal components $\mathcal{T}_{max}(S,G)$ for Hermitian G, such as $\mathrm{SU}(p,q)$, $\mathrm{SO}_0(2,n)$ and so on.

7 / 25

Definition

A **higher Teichmüller space** is a subset of $\operatorname{Hom}(\pi_1(S), G) /\!\!/ G$, which is a union of connected components that consist entirely of discrete and faithful representations.

We give two famous examples of higher Teichmüller space here.

- (Hitchin, 1992; Labourie, 2006; Fock–Goncharov,2006) Hitchin components $\mathcal{T}_{Hit}(S,G)$ for real split G, such as $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SO}_0(n,n+1)$, $\mathrm{Sp}(2n,\mathbb{R})$ and so on:
- (Burger–lozzi–Wienhard, 2003) Maximal components $\mathcal{T}_{max}(S,G)$ for Hermitian G, such as $\mathrm{SU}(p,q)$, $\mathrm{SO}_0(2,n)$ and so on.

When $G = \mathrm{SL}(2,\mathbb{R})$, the above components coincide with the classical Teichmüller space.

Anosov Property

Anosov property plays an important role in higher Teichmüller theory. Roughly speaking, the Anosov property introduced by F. Labourie means that the representation $\rho\colon\pi_1(S)\to G$ satisfies some special dynamic properties.

Anosov Property

Anosov property plays an important role in higher Teichmüller theory. Roughly speaking, the Anosov property introduced by F. Labourie means that the representation $\rho\colon \pi_1(S)\to G$ satisfies some special dynamic properties.

The original definition is highly dynamical and we will use an interpreted definition (which is proven equivalent to the original one by Kapovich–Leeb–Porti and Bochi–Potrie–Sambarino).

Anosov Property

Anosov property plays an important role in higher Teichmüller theory. Roughly speaking, the Anosov property introduced by F. Labourie means that the representation $\rho\colon\pi_1(S)\to G$ satisfies some special dynamic properties.

The original definition is highly dynamical and we will use an interpreted definition (which is proven equivalent to the original one by Kapovich–Leeb–Porti and Bochi–Potrie–Sambarino).

Also, we will assume G is a semi-simple Lie subgroup of $\mathrm{SL}(n,\mathbb{C})$ here to avoid involving a Lie-theoretic description.

 $\rho \colon \pi_1(S) \to G$ is $P_k ext{-Anosov}$ if there exist constants D, L>0 such that

$$\log \frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geqslant D \cdot d_{\mathbb{H}^2}(x_0, \gamma \cdot x_0) - L, \forall \gamma \in \pi_1(S),$$

where σ_i denotes the *i*-th singular value.

 $\rho \colon \pi_1(S) \to G$ is P_k -Anosov if there exist constants D, L > 0 such that

$$\log \frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geqslant D \cdot d_{\mathbb{H}^2}(x_0, \gamma \cdot x_0) - L, \forall \gamma \in \pi_1(S),$$

where σ_i denotes the *i*-th singular value.

For $g \in SO_0(2, n)(n > 2)$, it is remarkable that $\sigma_i(g) = (\sigma_{n+3-i}(g))^{-1}$ and $\sigma_3(g) = \cdots = \sigma_n(g) = 1$.

 $\rho \colon \pi_1(S) \to G$ is P_k -Anosov if there exist constants D, L > 0 such that

$$\log \frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geqslant D \cdot d_{\mathbb{H}^2}(x_0, \gamma \cdot x_0) - L, \forall \gamma \in \pi_1(S),$$

where σ_i denotes the *i*-th singular value.

For $g \in SO_0(2, n)(n > 2)$, it is remarkable that $\sigma_i(g) = (\sigma_{n+3-i}(g))^{-1}$ and $\sigma_3(g) = \cdots = \sigma_n(g) = 1$.

Anosov \Longrightarrow discrete + faithful.

 $\rho \colon \pi_1(S) \to G$ is $P_k ext{-Anosov}$ if there exist constants D, L>0 such that

$$\log \frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))} \geqslant D \cdot d_{\mathbb{H}^2}(x_0, \gamma \cdot x_0) - L, \forall \gamma \in \pi_1(S),$$

where σ_i denotes the *i*-th singular value.

For $g \in SO_0(2, n)(n > 2)$, it is remarkable that $\sigma_i(g) = (\sigma_{n+3-i}(g))^{-1}$ and $\sigma_3(g) = \cdots = \sigma_n(g) = 1$.

Anosov \Longrightarrow discrete + faithful.

Anosov property is open but not closed. However, all representations in the known higher Teichmüller spaces are Anosov.

2025.02.28 UCR (Riverside)

Non-Abelian Hodge Correspondence

Higgs bundles

The Higgs bundle is a useful tool to study the higher Teichmüller space. It is usually used to give a parametrization of the higher Teichmüller space. We fix a complex structure on S such that it becomes a Riemann surface X. Let \mathcal{K}_X be its canonical line bundle.

Higgs bundles

The Higgs bundle is a useful tool to study the higher Teichmüller space. It is usually used to give a parametrization of the higher Teichmüller space. We fix a complex structure on S such that it becomes a Riemann surface X. Let \mathcal{K}_X be its canonical line bundle.

Definition

A (GL (n, \mathbb{C}) -)Higgs bundle over X is a pair (\mathcal{E}, Φ) consisting of the following data:

- a holomorphic vector bundle \mathcal{E} over X with $rank(\mathcal{E}) = n$;
- a holomorphic section $\Phi \in H^0(X, \operatorname{End}(\mathcal{E}) \otimes \mathcal{K}_X)$ called **Higgs field**.

The non-Abelian Hodge correspondence exhibit a homeomorphism between the following moduli spaces:

{reductive representation $\rho \colon \pi_1(S) \to \operatorname{GL}(n,\mathbb{C})$ }/ $\operatorname{GL}(n,\mathbb{C})$

12 / 25

The non-Abelian Hodge correspondence exhibit a homeomorphism between the following moduli spaces:

If we equip additional structure on these objects, we can get the non-Abelian Hodge correspondence for general reductive Lie group G.

Kobayashi–Hitchin Correspondence (from Higgs to flat)

We only explain how to get a flat bundle from a Higgs bundle here. The key point is the Hitchin's self-dual equation.

13 / 25

Kobayashi–Hitchin Correspondence (from Higgs to flat)

We only explain how to get a flat bundle from a Higgs bundle here. The key point is the Hitchin's self-dual equation.

Theorem (Hitchin–Simpson)

If (\mathcal{E}, Φ) is a polystable Higgs bundle with $\deg(\mathcal{E}) = 0$, then there exists an Hermitian metric h on \mathcal{E} such that

$$F(\nabla^h) + [\Phi, \Phi^{*_h}] = 0,$$
 (1)

where ∇^h is the Chern connection of the metric h, $F(\nabla^h)$ denotes its curvature form and Φ^{*_h} is the adjoint of Φ with respect to h. Moreover, if (\mathcal{E}, Φ) is stable, then such h is unique up to a constant scalar.

Kobayashi–Hitchin Correspondence (from Higgs to flat)

We only explain how to get a flat bundle from a Higgs bundle here. The key point is the Hitchin's self-dual equation.

Theorem (Hitchin-Simpson)

If (\mathcal{E}, Φ) is a polystable Higgs bundle with $\deg(\mathcal{E}) = 0$, then there exists an Hermitian metric h on \mathcal{E} such that

$$F(\nabla^h) + [\Phi, \Phi^{*_h}] = 0, \tag{1}$$

where ∇^h is the Chern connection of the metric h, $F(\nabla^h)$ denotes its curvature form and Φ^{*_h} is the adjoint of Φ with respect to h. Moreover, if (\mathcal{E}, Φ) is stable, then such h is unique up to a constant scalar.

If h solves (1), then

$$\nabla^h + \Phi + \Phi^{*_h}$$

gives a flat connection.



From Higgs Bundle to Anosov Representation

Higgs bundle Higgs bundle Anosov property?

Example: Hitchin Component in Higgs Bundle Viewpoint

Let us fix a square root $\mathcal{K}_X^{1/2}$ of \mathcal{K}_X , then the Hitchin component for $\mathrm{SL}(n,\mathbb{R})$ consisting of entirely the Higgs bundles of the following form:

$$\mathcal{E} = \mathcal{K}_{X}^{(n-1)/2} \oplus \mathcal{K}_{X}^{(n-3)/2} \oplus \cdots \oplus \mathcal{K}_{X}^{(1-n)/2},$$

$$\Phi = \begin{pmatrix} 0 & q_{2} & q_{3} & q_{4} & \cdots & q_{n} \\ 1 & 0 & q_{2} & q_{3} & \ddots & q_{n-1} \\ & 1 & 0 & q_{2} & \ddots & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & 1 & 0 & q_{2} \\ & & & & 1 & 0 \end{pmatrix},$$

where $1: \mathcal{K}_X^{(n-1)/2-i} \to \mathcal{K}_X^{(n-1)/2-(i+1)} \otimes \mathcal{K}_X$ is the natural isomorphism and $q_i \in \mathrm{H}^0(X, \mathcal{K}_X^i)$.

It corresponds to the component containing the embedding of Fuchsian representations through the unique irreducible $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$.

16 / 25

Some Recent Progress

• (Filip, 2021) some weight 3 variations of Hodge structure;

- (Filip, 2021) some weight 3 variations of Hodge structure;
- \bullet (Collier–Virgilio, 2023+) conformal $\mathrm{SU}(1,n)$ -Higgs bundle in the Slodowy slice:

- (Filip, 2021) some weight 3 variations of Hodge structure;
- (Collier–Virgilio, 2023+) conformal $\mathrm{SU}(1,n)$ -Higgs bundle in the Slodowy slice;
- (Z. 2024) α_1 -cyclic $SO_0(2,3)$ -Higgs bundle;

- (Filip, 2021) some weight 3 variations of Hodge structure;
- (Collier–Virgilio, 2023+) conformal $\mathrm{SU}(1,n)$ -Higgs bundle in the Slodowy slice;
- (Z. 2024) α_1 -cyclic $SO_0(2,3)$ -Higgs bundle;
- (Bronstein–Davalo, 2025) a slice of deformations of Barbot representations.

Special $SO_0(2,3)$ -Higgs Bundles

Below we consider the Higgs bundle whose underlying bundle is

$$\mathcal{E} = \mathcal{L}_{-2} \oplus \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2,$$

where \mathcal{L}_i are line bundles with $\mathcal{L}_i \cong \mathcal{L}_{-i}^{\vee}$ and $\mathcal{L}_0 \cong \mathcal{O}$. Note that there is a natural pairing on \mathcal{E} defined by

$$Q = \begin{pmatrix} & & & & -1 \\ & & & 1 & \\ & & -1 & & \\ & 1 & & & \\ -1 & & & & \end{pmatrix}$$

Suppose that the Higgs field Φ projects to 0 in $\mathrm{H}^0(X,\mathrm{Hom}(\mathcal{L}_i,\mathcal{L}_j)\otimes\mathcal{K}_X)$ for any i,j have the same parity and Φ is compatible with Q, then polystable (\mathcal{E},Φ) gives an $\mathrm{SO}_0(2,3)$ -representation.

In addition, if (\mathcal{E},Φ) comes from a variation of Hodge structure, then we have

$$(\mathcal{E}, \Phi) = \mathcal{L}_{-2} \xrightarrow{\alpha} \mathcal{L}_{-1} \xrightarrow{\beta} \mathcal{L}_{0} \xrightarrow{\beta} \mathcal{L}_{1} \xrightarrow{\alpha} \mathcal{L}_{2} .$$

Such Higgs bundle is maximal if and only if β is an isomorphism, and maximal representations are known to be Anosov. From a different starting point, S. Filip considered the Higgs bundle of the same form, but instead α is an isomorphism.

In addition, if (\mathcal{E},Φ) comes from a variation of Hodge structure, then we have

$$(\mathcal{E}, \Phi) = \mathcal{L}_{-2} \xrightarrow{\alpha} \mathcal{L}_{-1} \xrightarrow{\beta} \mathcal{L}_{0} \xrightarrow{\beta} \mathcal{L}_{1} \xrightarrow{\alpha} \mathcal{L}_{2} .$$

Such Higgs bundle is maximal if and only if β is an isomorphism, and maximal representations are known to be Anosov. From a different starting point, S. Filip considered the Higgs bundle of the same form, but instead α is an isomorphism.

Theorem (Filip, 2021)

A stable $SO_0(2,3)$ -Higgs bundle of the form

$$\mathcal{L}_{-2} \xrightarrow{\alpha} \mathcal{L}_{-1} \xrightarrow{\beta} \mathcal{L}_0 \xrightarrow{\beta} \mathcal{L}_1 \xrightarrow{\alpha} \mathcal{L}_2$$

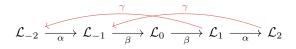
with α is an isomorphism gives a P_2 -Anosov representation.

Filip proved this theorem by an analytic method. Inspired by his method and with some simplification, we extend his results and discover the Anosov property of a general family of $SO_0(2,3)$ -Higgs bundles.

Filip proved this theorem by an analytic method. Inspired by his method and with some simplification, we extend his results and discover the Anosov property of a general family of $\mathrm{SO}_0(2,3)$ -Higgs bundles.

Theorem (Z.)

A stable $SO_0(2,3)$ -Higgs bundle of the form



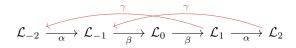
with α is an isomorphism gives a P_2 -Anosov representation.

They form a **non-compact subset (which can go to infinity)** in the character variety.

Filip proved this theorem by an analytic method. Inspired by his method and with some simplification, we extend his results and discover the Anosov property of a general family of $\mathrm{SO}_0(2,3)$ -Higgs bundles.

Theorem (Z.)

A stable $SO_0(2,3)$ -Higgs bundle of the form



with α is an isomorphism gives a P_2 -Anosov representation.

They form a **non-compact subset (which can go to infinity)** in the character variety.

Remark

The trivial line bundle \mathcal{L}_0 above can be replaced by an orthogonal vector bundle of rank n to get an $\mathrm{SO}(2,n+2)$ -Higgs bundle. With the assumption of stability, the Anosov property still holds.

4 D > 4 A > 4 B > 4 B >

Sketch of Proof

The core idea is to estimate the norm of the flat section with respect to the associated flat bundle.

Sketch of Proof

The core idea is to estimate the norm of the flat section with respect to the associated flat bundle.

Sketch of Proof

The core idea is to estimate the norm of the flat section with respect to the associated flat bundle.

It suffices to show that there exist constants $C_1, C_2, \varepsilon > 0$ such that

$$||v(x)||^2 \geqslant C_1 \cdot \exp(\varepsilon \cdot d_{\mathbb{H}^2}(x, x_0)) - C_2,$$

then the corresponding ρ is P_2 -Anosov.

Filip's estimate (simplified)

Lemma (Filip, 2021)

Given a complete Riemannian manifold (M,g) with the distance function $d\colon M\times M\to \mathbb{R}$ and a smooth function $f\in C^\infty(M;\mathbb{R})$. Suppose that f satisfies

- (S1) f is a non-negative function with all zeroes isolated;
- (S2) $\|\mathrm{d}f\|_g \gtrsim f$;
- (S3) $\|df\|_g \gtrsim f^{1/2}$.

Then f achieves its unique minimum at x_{\min} and there exist constants $C_1,C_2,\varepsilon>0$, such that

$$f(x) \geqslant C_1 \cdot \exp(\varepsilon \cdot d(x_{\min}, x)) - C_2, \forall x \in M.$$

For Filip's case, we want to check f_v satisfies (S1)-(S3). Note that $\|v_1\| \gtrsim \max\{\|v_2\|, 1\}$.

$$\mathcal{L}_{-2} \xrightarrow{\alpha} \mathcal{L}_{-1} \xrightarrow{\beta} \mathcal{L}_0 \xrightarrow{\beta} \mathcal{L}_1 \xrightarrow{\alpha} \mathcal{L}_2$$

For Filip's case, we want to check f_v satisfies (S1)-(S3). Note that $\|v_1\|\gtrsim \max\{\|v_2\|,1\}$.

$$\mathcal{L}_{-2} \xrightarrow{\alpha} \mathcal{L}_{-1} \xrightarrow{\beta} \mathcal{L}_0 \xrightarrow{\beta} \mathcal{L}_1 \xrightarrow{\alpha} \mathcal{L}_2$$

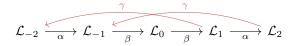
• (S2)(S3) $\Longleftarrow \|\alpha\| > C$ for some constant C > 0. (α is nowhere vanishing)

For Filip's case, we want to check f_v satisfies (S1)-(S3). Note that $\|v_1\|\gtrsim \max\{\|v_2\|,1\}$.

$$\mathcal{L}_{-2} \xrightarrow{\alpha} \mathcal{L}_{-1} \xrightarrow{\beta} \mathcal{L}_0 \xrightarrow{\beta} \mathcal{L}_1 \xrightarrow{\alpha} \mathcal{L}_2$$

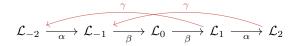
- (S2)(S3) $\Longleftarrow \|\alpha\| > C$ for some constant C > 0. (α is nowhere vanishing)
- (S1) $\longleftarrow v_2$ is a holomorphic section. (by flatness)

For our case, we want to check f_v satisfies (S1)-(S3). Note that $\|v_1\| \gtrsim \max\{\|v_2\|, 1\}$.



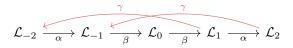
- (S1) $> \sim v_2$ is a holomorphic section.

For our case, we want to check f_v satisfies (S1)-(S3). Note that $\|v_1\| \gtrsim \max\{\|v_2\|, 1\}$.



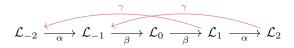
- (S2)(S3) $\iff \|\alpha\| \|\gamma\| > C$ for some constant C > 0.
- (S1) $> \sim v_2$ is a holomorphic section.

A global flat section v decomposes as $\sum_{i=-2}^2 v_i$ with $v_i \colon \mathbb{H}^2 \to \mathcal{L}_i$. Choose a real positive initial vector $v \in \mathcal{E}_{x_0}$, i.e. $2\|v_1\|^2 - \left(\|v_0\|^2 + 2\|v_2\|^2\right) > 0$ and we take $f_v := \|v_2\|^2$ for the lemma above. For our case, we want to check f_v satisfies (S1)-(S3). Note that $\|v_1\| \gtrsim \max\{\|v_2\|, 1\}$.



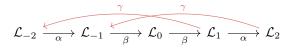
- (S2)(S3) $\iff \|\alpha\| \|\gamma\| > C$ for some constant C > 0. (by maximum principle and Hitchin's self-dual equation)
- (S1) v_2 is a holomorphic section.

A global flat section v decomposes as $\sum_{i=-2}^2 v_i$ with $v_i : \mathbb{H}^2 \to \mathcal{L}_i$. Choose a real positive initial vector $v \in \mathcal{E}_{x_0}$, i.e. $2\|v_1\|^2 - (\|v_0\|^2 + 2\|v_2\|^2) > 0$ and we take $f_v := ||v_2||^2$ for the lemma above. For our case, we want to check f_v satisfies (S1)-(S3). Note that $||v_1|| \ge \max\{||v_2||, 1\}.$



- (S2)(S3) $\Leftarrow \|\alpha\| \|\gamma\| > C$ for some constant C > 0. (by maximum principle and Hitchin's self-dual equation)
- (S1) $\Leftarrow= f_v = ||v_2||^2$ is a Morse function.

A global flat section v decomposes as $\sum_{i=-2}^2 v_i$ with $v_i \colon \mathbb{H}^2 \to \mathcal{L}_i$. Choose a real positive initial vector $v \in \mathcal{E}_{x_0}$, i.e. $2\|v_1\|^2 - \left(\|v_0\|^2 + 2\|v_2\|^2\right) > 0$ and we take $f_v := \|v_2\|^2$ for the lemma above. For our case, we want to check f_v satisfies (S1)-(S3). Note that $\|v_1\| \gtrsim \max\{\|v_2\|, 1\}$.



- (S2)(S3) $\iff \|\alpha\| \|\gamma\| > C$ for some constant C > 0. (by maximum principle and Hitchin's self-dual equation)
- (S1) $\longleftarrow f_v = ||v_2||^2$ is a Morse function. (by $||\alpha|| ||\gamma|| > C$)

$$D = \emptyset \text{ (closed surface)} \quad D \neq \emptyset \text{ (non-compact surface of finite type)}$$

$D=\emptyset$ (closed surface)	$D \neq \emptyset$ (non-compact surface of finite type)
Anosov representation	almost-dominated representation

$D=\emptyset$ (closed surface)	$D eq \emptyset$ (non-compact surface of finite type)
Anosov representation	almost-dominated representation
Higgs bundle	

Lots of results in this section can be generalized when the closed hyperbolic Riemann surface X is replaced by a hyperbolic Riemann surface $X:=\overline{X}\setminus D$ of finite type, i.e. both $g(\overline{X})$ and #D are finite.

$D = \emptyset$ (closed surface)	$D \neq \emptyset$ (non-compact surface of finite type)
Anosov representation	almost-dominated representation
Higgs bundle	parabolic Higgs bundle

Lots of results in this section can be generalized when the closed hyperbolic Riemann surface X is replaced by a hyperbolic Riemann surface $X := \overline{X} \setminus D$ of finite type, i.e. both $q(\overline{X})$ and #D are finite.

$D = \emptyset$ (closed surface)	$D \neq \emptyset$ (non-compact surface of finite type)
Anosov representation	almost-dominated representation
Higgs bundle	parabolic Higgs bundle

Theorem (Z.)

A stable α_1 -cyclic parabolic $SO_0(2,3)$ -Higgs bundle gives a P_2 -almost-dominated representation through the non-Abelian Hodge correspondence.

Lots of results in this section can be generalized when the closed hyperbolic Riemann surface X is replaced by a hyperbolic Riemann surface $X:=\overline{X}\setminus D$ of finite type, i.e. both $g(\overline{X})$ and #D are finite.

$D = \emptyset$ (closed surface)	$D \neq \emptyset$ (non-compact surface of finite type)
Anosov representation	almost-dominated representation
Higgs bundle	parabolic Higgs bundle

Theorem (Z.)

A stable α_1 -cyclic parabolic $SO_0(2,3)$ -Higgs bundle gives a P_2 -almost-dominated representation through the non-Abelian Hodge correspondence.

Key point: Set suitable weight such that

$$\|\alpha\| - \|\gamma\| > C.$$

Thank you!