GEOMETRIC GROUP THEORY: SOLUTIONS

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This is an unofficial solution for the exercises of the short course, Geometric Group Theory, which is organized by Qiongling Li in the summer of 2021. More information can be found on `http://www.cim.nankai.edu.cn/2021/0611/c11453a372030/page.htm`.

Exercise 1. Let (X,d) be a proper length metric space. Given $o \in X$, let $p_n : [0,\infty)$ be a sequence of length parameterized geodesic rays with the same origin $(p_n)_- = o$. Prove that there exists a subsequence of p_n which converges locally uniformly to a geodesic ray $p_\infty : [0,\infty)$ with $p_\infty(0) = o$.

Solution. Consider the compact ball $B_m := \overline{B(o,m)}$. We will construct the convergent subsequence by induction. By Arzelà-Ascoli Theorem, there exists a subsequence $p_{1,n}$ of p_n such that $p_{1,n}$ uniformly converges to a geodesic segment $p_{1,\infty}$ in B_1. Now if the sequence $p_{m,n}$ are chosen, then by Arzelà-Ascoli Theorem again, there exists a subsequence $p_{m+1,n}$ of $p_{m,n}$ such that $p_{m+1,n}$ uniformly converges to a geodesic segment $p_{m+1,\infty}$ in B_{m+1}. Note that $p_{m+1,n}$ is a subsequence of $p_{m,n}$, thus $p_{m,\infty} = p_{m+1,\infty}|_{[0,m]}$. Hence $p_{m,m}$ converges locally uniformly to a geodesic ray p_∞, which is equal to $p_{m,\infty}$ when restricts on $[0,m]$, with $p_\infty = o$. □

Exercise 2. Denote by $QI(X)$ the set of equivalent classes of quasi-isometries of X. Prove that the set $QI(X)$ with the composition operation is a group. Moreover, there exists a homomorphism from the isometry group $Isom(X)$ of X into the group $QI(X)$.

Solution. a) We first prove that $QI(X)$ is closed under composition. Suppose $\psi_1, \psi_2 \in QI(X)$ are (λ_1, c_1), (λ_2, c_2)-quasi-isometry respectively. Then $f = \psi_1 \circ \psi_2$ satisfies that

$$d_X(f(x), f(x')) \leqslant \lambda_2 d_X(\psi_1(x), \psi_1(x')) + c_2 \leqslant \lambda_1\lambda_2 d_X(x,x') + \lambda_2 c_1 + c_2,$$
$$d_X(f(x), f(x')) \geqslant \lambda_2^{-1} d_X(\psi_1(x), \psi_1(x')) - c_2 \geqslant (\lambda_1\lambda_2)^{-1} d_X(x,x') - \lambda_2^{-1} c_1 - c_2$$

for any $x, x' \in X$. Note that $\lambda_2^{-1} c_1 \leqslant \lambda_2 c_1$, we get f is a $(\lambda_1\lambda_2, \lambda_2 c_1 + c_2)$ quasi-isometric embedding. Suppose $X \subset N_R(\psi_2(X))$. For any $x \in X$, there exists $x' \in X$ such that $d_X(\psi_2(x), x') \leqslant R$. Hence

$$d_X(f(x), \psi_1(x')) \leqslant \lambda_1 R + c,$$

i.e. $X \subset N_{\lambda_1 R + c}(f(x))$. Thus $f \in QI(X)$.

b) We also would like to prove that composition is well-defined up to equivalent class, i.e. if f_1, f_2 are equivalent and g_1, g_2 are equivalent, then $f_1 \circ g_1$ is equivalent to $f_2 \circ g_2$. Suppose f_1 is (λ, c)-quasi-isometry and $d_X(f_1, f_2) \leqslant R_1$, $d_X(g_1, g_2) \leqslant R_2$. Then

$$\begin{aligned}
&d_X(f_1 \circ g_1(x), f_2 \circ g_2(x)) \\
\leqslant &d_X(f_1 \circ g_1(x), f_1 \circ g_2(x)) + d_X(f_2 \circ g_2(x), f_1 \circ g_2(x)) \\
\leqslant &\lambda d_X(g_1(x), g_2(x)) + c + R_1 \\
\leqslant &\lambda R_2 + c + R_1,
\end{aligned}$$

which means that $f_1 \circ g_1$ and $f_2 \circ g_2$ are equivalent.

c) Now we want to prove that quasi-inverse is a suitable inverse. Fix a (λ, c)-quasi-isometry $f \in QI(X)$ with its two different quasi-inverse g and g' with $d_X(f \circ g(x), x) \leqslant R$ and $d_X(f \circ g'(x), x) \leqslant R$. From

$$\lambda^{-1} d_X(g(x), g'(x)) - c \leqslant d_X(f \circ g(x), f \circ g'(x)) \leqslant 2R$$

we get g, g' are equivalent. Hence quasi-inverse is well-defined up to equivalent class. By definition, $f \circ g$ is equivalent to the identity map. Besides, if $X \subset N_{R'}(f(X))$, then

$$\lambda^{-1} d_X(g \circ f(x), x) - c \leqslant d_X(f \circ g \circ f(x), f(x)) \leqslant R'$$

tells us $g \circ f$ is equivalent to the identity map.

Thus $QI(X)$ is a group equipped with the composition and quasi-inverse as product and inverse respectively. Because every isometry is indeed a quasi-isometry, $Isom(X)$ can embedded into $QI(X)$. □

Exercise 3. Suppose two metric spaces X, Y are quasi-isometric. Prove that $QI(X)$ is isomorphic to $QI(Y)$.

Solution. As the argument in Exercise 2, we can show that if $f_1, f_2 : X \to Y$ are equivalent quasi-isometry and $g_1, g_2 : Y \to Z$ are equivalent quasi-isometry, then $g_1 \circ f_1$ and $g_2 \circ f_2$ are equivalent. Now suppose $f : X \to Y$ is a quasi-isometry and f^{-1} is its quasi-inverse. Consider the map

$$\begin{aligned} \tilde{f} : QI(X) &\to QI(Y) \\ \phi &\mapsto f^{-1} \circ \phi \circ f \end{aligned}.$$

$\tilde{f}$ is indeed a homomorphism since $f \circ f^{-1}$ is equivalent to id_X. Conversely,

$$\begin{aligned} \tilde{f}^{-1} : QI(Y) &\to QI(X) \\ \phi &\mapsto f \circ \phi \circ f^{-1} \end{aligned}$$

is a homomorphism as well. Moreover, $\tilde{f}\tilde{f}^{-1}$ and $\tilde{f}^{-1}\tilde{f}$ are both identity map. Thus $QI(X)$ is isomorphic to $QI(Y)$. $\square$

Exercise 4. Let $n \geqslant 3$ be an integer. Prove that any two trees with vertices of degree between 3 and n are quasi-isometric.

Solution. Denote T_3 by the 3-regular tree. We will prove that any tree T with vertices degree between 3 and n is quasi-isometric to T_3 by constructing a quasi-isometry $q : T_3 \to T$. Fix a vertex $v_{0,1} \in T_3$ and $w_{0,1} \in T$. Suppose $S(v_{0,1}, k) = \{v_{k,i} : 1 \leqslant i \leqslant m'_k\}$ and $S(w_{0,1}, k) = \{w_{k,i} : 1 \leqslant i \leqslant m_k\}$, where $m'_k = \#S(v_{0,1}, k)$ and $m_k = \#S(w_{0,1}, k)$. And let $q(v_{0,1}) = w_{0,1}$. By collapsing the path $v_{0,1}v_{1,1} \cdots v_{m_1-3,1}$ to the vertex $w_{0,1}$ and define $q(v_{m_1-i-2,i+1}) = w_{1,i+1}$, where $0 \leqslant i \leqslant m_1 - 3$,and $q(v_{1,2}) = w_{1,m_1-1}$, $q(v_{1,m_1})$. By the induction we can construct a surjective map $q : T_3 \to T$. It's obvious that

$$\frac{1}{n-2} d(x, y) - 1 \leqslant d(q(x), q(y)) \leqslant d(x, y).$$

Thus q is a quasi-isometry. $\square$

Exercise 5. Prove the following two statements.

(1) The growth function of a finitely generated group always dominates that of any finitely generated subgroup.

(2) The growth function of a finitely generated group always dominates that of any quotient group.

Solution. Suppose G is a finitely generated group with a finite symmetric generating set $1 \notin S$ and growth function $\phi(n)$.

(1) Let H be a finitely generated subgroup of G with a finite symmetric generating set $1 \notin S'$ and growth function $\psi(n)$. Let C be the length of the longest word in S' with respect to S. Thus $\psi(n) \leqslant \phi(Cn)$.

(2) Let G/N be a quotient group of G with growth function $\psi(n)$. S is still the generating set of G/N. Thus $\psi(n) \leqslant \phi(n)$. $\square$

Exercise 6. A finitely generated group is of exponential growth if and only if the growth rate with respect to some (or any) generating set is positive.

Solution. Suppose G is a finitely generated group with a finite symmetric generating set $1 \notin S$ and growth function $\phi(n)$.

If G is of exponential growth, i.e. there exists $C > 1$ such that $e^n \leqslant C\phi(Cn)$. Then

$$\delta_{G,S} = \lim_{n\to\infty} \frac{\ln \phi(n)}{n} = \lim_{n\to\infty} \frac{\ln \phi(Cn)}{Cn} \geqslant \lim_{n\to\infty} \frac{n - \ln C}{Cn} = \frac{1}{C} > 0.$$

If $\delta_{G,S} > 0$, then there exists N such that for any $n > N$, $\dfrac{\ln \phi(n)}{n} > \dfrac{\delta_{G,S}}{2}$. Hence there exists $C' > 0$ such that $\dfrac{\ln \phi(n)}{n} > C'$ for any $n \in \mathbb{N}$. So $\phi(n) \geqslant e^{C'n}$. Hence G is of exponential growth. $\square$

Exercise 7. Let F be a free group of rank n.

(1) Let S be the standard generating set of F. Prove that $\delta_{F,S} = \ln(2n - 1)$.

(2) Let T be a finite generating set of F. Prove that $\delta_{F,T} \geqslant \ln(2n - 1)$. Therefore, we see that δ_F is realized by some generating set $\delta_F = \delta_{F,S} = \ln(2n - 1)$.

(3) By the second statement, explain that the value of the growth rate is not invariant under quasi-isometries.

Solution.
(1) Let ϕ be the growth function of F with respect to S. Then $\delta(0) = 1$ and when $m \geqslant 1$,

$$\phi(m) = 1 + 2n \sum_{i=0}^{m-1} (2n-1)^i.$$

If $n = 1$, then $\phi(m) = 2m + 1$, $\delta_{F,S} = 0 = \ln(2 \times 1 - 1)$. If $n \geqslant 2$, then

$$\phi(m) = 1 + \frac{n}{n-1}[(2n-1)^m - 1]$$

and $\delta_{F,S} = \ln(2n-1)$.

(2) Consider the natural homomorphism $\pi : F \to F/[F,F]$. Then $\pi(T)$ is a generating set of the free abelian group $F/[F,F]$. Thus T contains an n-element subset T_1 such that $\pi(T_1)$ generates a free abelian subgroup of rank n in $F/[F,F]$ by extending $F/[F,F]$ to a $\mathbb{R}$-linear space and choosing basis. Now T_1 generates a free subgroup F_1 of F. Now $\pi(F_1)$ is at least rank n as a free abelian group, hence F_1 is rank n as a free group. So $\delta_{F,T} \geqslant \delta_{F_1,T_1} = \ln(2n-1)$.

(3) Note that the Cayley graph of any finite rank free group with respect to the standard generators are quasi-isometry by Exercise 4. However, any two different finite rank free group has different growth rate by (2). Hence the growth rate is not invariant under quasi-isometries. □

Exercise 8. Suppose G acts co-boundedly on a proper length space (X, d). Fix a basepoint $o \in X$. Then there exists a (possibly infinite) generating set S of G such that the map

$$\begin{aligned}(G, d_S) &\to (Go, d)\\ g &\mapsto go\end{aligned}$$

is a G-equivariant quasi-isometric map.

Solution. Let K be the bounded subset of X such that $G \cdot K = X$ with $\mathrm{Diam}K = R$. Suppose $S := \{s \in G : d(o, so) \leqslant 2R + 1\}$. Note that every proper length space is a geodesic space, so for any $g \in G$, there is a geodesic $p : [0, \mathrm{Len}(p)] \to X$ from o to go. Set $n := [\mathrm{Len}(p)]$. For any $p(i)$, there exists a $g_i \in G$ such that $d(p(i), g_i o) \leqslant R$, where $1 \leqslant i \leqslant n$. Hence

$$\begin{aligned}&d(o, g_i^{-1} g_{i+1} o)\\ =&d(g_i o, g_{i+1} o)\\ \leqslant&d(p(i), g_i o) + d(p(i), p(i+1)) + d(p(i+1), g_{i+1} o)\\ \leqslant&2R + 1\end{aligned}$$

for any $1 \leqslant i \leqslant n-1$ shows that $g_i^{-1} g_{i+1} \in S$. Similarly, $g_1, g_n^{-1} g \in S$. Thus G is generated by S. Now for any $g, h \in G$, let $g^{-1}h = h_1 \cdots h_m$, where $h_i \in S$ and $m = d_S(g, h)$. Then

$$d(go, ho) = d(o, g^{-1}ho) \leqslant \sum_{i=1}^{m} d(o, h_i o) \leqslant (2R+1) d_S(g, h).$$

On the other hand, by the above argument about generators and the geodesic we have

$$d_S(g, h) = |g^{-1}h| \leqslant d(o, g^{-1}ho) + 1 = d(go, ho) + 1.$$

Therefore, $[g \mapsto go]$ is a quasi-isometric embedding. Note that $X \subset N_R(Go)$, $[g \mapsto go]$ is a quasi-isometry. □

Exercise 9. We would like to prove the uniform boundedness of k-centers in δ-hyperbolic space.

(1) Let $\triangle = \triangle(abc)$ be a geodesic triangle with vertices $a, b, c \in X$ and o be a k-center for $k > 0$. Prove that

$$d(c, o) - 2k \leqslant (a, b)_c \leqslant d(c, o) + k.$$

(2) Let p, q be the two k-taut paths in X with same endpoints x and y. Let $z \in p$, $w \in q$ be two points such that $d(z, x) = d(w, x)$. Prove that

$$d(z, w) \leqslant 2k + 16\delta.$$

(3) Prove that the set of k-centers is of uniformly diameter depending only on k and δ.

Solution.

(1) Let $w \in [a,b]$ satisfy $d(o,w) \leqslant k$. Then

$$\begin{aligned}(a,b)_c =&\frac{1}{2}(d(a,c)-d(a,w)+d(b,c)-d(b,w))\\ \leqslant&\frac{1}{2}\cdot 2d(c,w)\\ =&d(c,w)\\ \leqslant&d(c,o)+d(o,w)\\ \leqslant&d(c,o)+k.\end{aligned}$$

Let $x \in [a,c]$ satisfy that $d(o,x) \leqslant k$. Note that

$$\begin{aligned}d(a,c) =&d(a,x)+d(c,x)\\ \geqslant&d(a,o)-d(x,o)+d(c,o)-d(x,o)\\ \geqslant&d(a,o)+d(c,o)-2k,\end{aligned}$$

and similarly, $d(b,c) \geqslant d(b,o)+d(c,o)-2k$, we can get

$$d(a,c)+d(b,c) \geqslant d(a,o)+d(b,o)+2d(c,o)-4k \geqslant d(a,b)+2d(c,o)-4k.$$

Hence $(a,b)_c \geqslant d(c,o)-2k$.

(2) If $d(x,z) < d(x,y)$, there exists a point $v \in [x,y]$ such that $d(x,v) = d(x,z)$. Since X is δ-hyperbolic and p is a k-taut path, there exists a point $u \in [x,y]$ such that $d(z,u) \leqslant \dfrac{k}{2}+4\delta$. So

$$d(u,v) = |d(x,u)-d(x,v)| = |d(x,u)-d(x,z)| \leqslant d(z,u) \leqslant \frac{k}{2}+4\delta.$$

Hence $d(z,v) \leqslant d(u,v)+d(z,u) \leqslant k+8\delta$. Similarly we have $d(w,v) \leqslant k+8\delta$. Thus

$$d(z,w) \leqslant d(z,v)+d(w,v) \leqslant 2k+16\delta.$$

If $d(x,z) \geqslant d(x,y)$, then

$$d(x,z)+d(z,y) \leqslant Len(p) \leqslant d(x,y)+k$$

implies that $d(z,y) \leqslant k$. Similarly, $d(w,y) \leqslant k$, hence

$$d(z,w) \leqslant d(z,y)+d(w,y) \leqslant 2k \leqslant 2k+16\delta.$$

(3) Suppose there are two k-centers o, o' with respect to $\triangle(abc)$. Then The path $[c,o]\cup[o,a]$ and $[c,o']\cup[o',a]$ are two $2k$-taut paths. Because $d(c,o) \geqslant (a,b)_c - k$ and $d(c,o') \geqslant (a,b)_c - k$, there exists $p \in [c,o]$ and $p' \in [c,o']$ such that $d(c,p) = d(c,p') = (a,b)_c - k$. So $d(p,p') \leqslant 4k+16\delta$. Since $d(o,p) = d(c,o)-d(c,p) \leqslant 3k$ and $d(o',p') \leqslant 3k$ similarly, we have

$$d(o,o') \leqslant d(o,p)+d(p,p')+d(o',p') \leqslant 10k+16\delta.$$

Thus the diameter of the set of k-centers is smaller than $10k+16\delta$. $\square$

Exercise 10. Let (X,d) be a geodesic metric space with δ-thin triangle property. Prove that there exists a constant $\delta' > 0$ such that every geodesic triangle is δ'-thinner than a comparison geodesic triangle in a tree.

Solution. Let $\triangle = \triangle(abc)$ be a geodesic triangle in X and x,y,z be the congruent points on the respect sidelines. Since X has δ-thin triangle property, there is a point $p \in [a,b]\cup[a,c]$ such that $d(p,x) \leqslant \delta$. Without loss of generality, suppose $p \in [a,b]$. Hence

$$d(p,z) = |d(p,b)-d(b,z)| = |d(b,x)-d(b,p)| \leqslant d(p,x) \leqslant \delta.$$

So $d(x,z) \leqslant d(p,z)+d(p,x) \leqslant 2\delta$. Similarly, we can prove $\min\{d(x,y),d(y,z)\} \leqslant 2\delta$. Hence $\max\{d(x,y),d(y,z),d(z,x)\} \leqslant 4\delta$. Now let $\pi : \triangle \to T_\triangle$ be the comparison map from $\triangle$ to the tree $T_\triangle$. Suppose $q \in T_\triangle$ is a point satisfying $d(\pi(a),q) \in (0,(b,c)_a)$. Then $\pi^{-1}(q) = \{q_1,q_2\}$ and $q_1 \in [a,y]$, $q_2 \in [a,z]$. Without loss of generality, we only need to prove that $d_{q_1,q_2} \leqslant 4\delta$. To prove that, we would like to construct a geodesic triangle in X such that q_1,q_2 are two congruent points of it. Let $c : [0,d(a,c)] \to X$ be an arc parameterization of $[a,c]$ and consider the geodesic triangle $\triangle(abc(t))$. Since $(b,c(0))_a = 0$, $(b,c(d(a,c)))_a = (b,c)_a$ and $(b,c(t))_a$ is continuous, there is a $t_0 \in (0,d(a,c))$ such that q_1,q_2 are congruent points of $\triangle(abc(t_0))$. Hence $d(q_1,q_2) \leqslant 4\delta$. $\square$

Exercise 11. Let p be a path in a δ-hyperbolic space. Given a non-decreasing function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, let p be a path such that $Len(q) \leqslant f(d(q_-, q_+))$ for any subpath q of p. Assume that f is sub-exponential (i.e. $\lim\limits_{n\to\infty} \ln f(n)/n = 0$). Prove that p is a quasi-geodesic path.

Solution. We first prove a claim: Any path q satisfying $Len(r) \leqslant f(d(r_-, r_+))$ for any subpath r of q is contained in a uniform neighborhood in $[q_-, q_+]$. Let $x \in [q_-, q_+]$ be the point maximize $d(x, q)$ and suppose $d(x, q) = t$. Since $q_- \in q$, there exists a point $a_0 \in [q_-, x]$ such that $d(x, a_0) = t$. Noe let $a_1 \in [q_-, a_0]$ be the point satisfying $d(x, a_1) = \min\{2t, d(q_-, x)\}$. Moreover, there exists a point $a_2 \in q$ which realizes $d(a_1, q)$ and $d(a_1, a_2) \leqslant t$. Similarly, we can define b_0, b_1 on $[q_+, x]$ and $b_2 \in q$. Denote q' by the path which is the restricted part of q from a_2 to b_2. Hence $Len(q') \leqslant f(d(a_2, b_2)) \leqslant f(6t)$. Since $q' \cup [a_0, a_1] \cup [a_1, a_2] \cup [b_0, b_1] \cup [b_1, b_2]$ is a path outside the t-neighborhood of x,

$$Len(q') + 4t \geqslant C_1 e^{C_2 d(a_0, b_0)} = C_1 e^{2C_2 t}$$

by the exponential divergence of path, where C_1, C_2 are two constants only depending on δ. Thus

$$f(6t) + 4t \geqslant C_1 e^{2C_2 t}.$$

That means t has an upper bound D only depending on δ because f is a sub-exponential function.

Now suppose q is a finite subpath of p. Set $n := [d(q_-, q_+)]$ and let $r : [0, d(q_-, q_+)] \to X$ be the geodesic from q_- to q_+. For any $1 \leqslant i \leqslant n$, there exists a point $q_i \in q$ such that $d(q_i, r(i)) \leqslant D$. Hence the length of the path which is the subpath of q from q_i to q_{i+1} is smaller than $f(d(q_i, q_{i+1})) \leqslant f(2D + 1)$. And similarly, the length of the path which is the subpath of q from q_- to q_1 (or from q_n to q_+) is smaller than $f(D + 1) \leqslant f(2D + 1)$. Note that the union of the subpaths of q from q_- to q_1, from q_i to q_{i+1}, from q_n to q_+ must be longer than q itself, thus

$$d(q_-, q_+) \leqslant Len(q) \leqslant (n + 1)f(2D + 1) \leqslant f(2D + 1)(d(q_-, q_+) + 1).$$

So p is a quasi-geodesic. □

Exercise 12. We would like to prove that there are only finitely many conjugacy classes of finite subgroups in a hyperbolic group in the following steps. Assume that a group G acts geometrically on a proper hyperbolic space (X, d).

(1) Define a notion of the center for any bounded set B in a metric space X. Define first the radius of B:

$$r_B := \inf\{r : B \subset B(x, r), r \geqslant 0, x \in X\},$$

where $B(x, r)$ is the closed ball of radius r at x. The center of B is then defined to be set of points $o \in X$ such that $B \subset B(o, r_B + 1)$.

(2) Prove that if X is δ-hyperbolic space, the center of any bounded set is bounded by a constant depending only on δ.

(3) Apply the assertion (2) to the orbit $B = F \cdot x$ of a finite subgroup F of G, and conclude the proof that there are finitely many conjugacy classes of finite subgroups F.

Solution. Suppose $Y \subset X$ is a bounded set with radius r_y and two centers o, o'. Note that in the proper metric space, $\overline{Y}$ is a compact subspace so $d(p, \cdot)$ has maximum value on it for any $p \in Y$. Hence for any $p \in Y$, there is a point $p' \in Y$ such that $d(p, p') \geqslant r_Y$. Now let m be the midpoint of $[o, o']$. There exists a point $y \in Y$ such that $d(m, y) \geqslant r_Y$. Since X is a δ-hyperbolic space, $\triangle(oo'y)$ has 6δ-thin property. Thus there is a point $q \in [o, y] \cup [o', y]$ such that $d(m, q) \leqslant 6\delta$. Without loss of generality, suppose $q \in [o, y]$. Then we have

$$d(y, q) = d(y, o) - d(o, q) \leqslant r_Y + 1 - d(o, m) + d(m, q) \leqslant r_Y + 1 + 6\delta - \frac{d(o, o')}{2}.$$

Consequently, we get

$$r_Y \leqslant d(y, m) \leqslant d(y, q) + d(q, m) \leqslant 12\delta + r_Y + 1 - \frac{d(o, o')}{2}.$$

Therefore, $d(o, o') \leqslant 24\delta + 2$.

Without loss of generality, we can suppose there is a compact subset K with diameter smaller than 1 such that $G \cdot K$ (If diameter is $R > 1$, we can define a new metric $d' = d/R$ on X, then X is still hyperbolic but the hyperbolic constants become smaller). Suppose the set of the centers of $B = F \cdot x$ is C. Now by (2) we know that the diameter of C is bounded by $24\delta + 2$. If $o \in X$ satisfies that $B \subset B(o, r_B)$, then for any point $o' \in Y$ which satisfies $d(o', o) \leqslant 1$, o' is a center of B. In particular, there exists a $g \in G$ such that gx is a center of B. Note that C is invariant under

F, hence $g^{-1}C$ is invariant under $g^{-1}Fg$. Since id $\in g^{-1}Fg$, $g^{-1}Fgx \subset g^{-1}C$. So every finite subgroup F is conjugate to a finite subgroup which is contained in

$$S := \{g \in G : d(x, gx) \leqslant 24\delta + 2\}.$$

Note that $\#S < \infty$ since G-action is proper, we get that there are only finitely many conjugacy classes of finite subgroups. □

Exercise 13. Let (X, d) be a geodesic metric space. If there exists $\delta > 0$ such that the following inequality holds

$$(x, y)_w \geqslant \min\{(x, z)_w, (y, z)_w\} - \delta$$

for any $x, y, z, w \in X$, then (X, d) is Gromov-hyperbolic. Now suppose (X, d) is a Gromov-hyperbolic space. We would like to prove that X is a hyperbolic space.

(1) Prove first that there exists a point $w \in [x, y]$ such that $(x, z)_w, (y, z)_w \leqslant \delta$.

(2) Then prove that if $(x, z)_w \leqslant \delta$, then $d(w, [x, z])$ is bounded by a constant depending on δ.

Solution. It's obvious that by (1) and (2) we can prove the previous statement. Precisely, w is a δ'-center of $\triangle(xyz)$, where δ' is the constant bounding $d(w, [x, z])$.

(1) Let $w(t)$ be an arc parameterization of $[x, y]$ such that $w(0) = x$, $w(d(x, y)) = y$. Note that $f(t) := (x, z)_{w(t)} - (y, z)_{w(t)}$ is a continuous function and $f(0) = -(y, z)_x \leqslant 0$, $f(d(x, y)) = (x, z)_y \geqslant 0$. There exists a point $0 \leqslant t_0 \leqslant d(x, y)$ such that $w := w_{t_0}$ satisfies $(x, z)_w = (y, z)_w$. Since $(x, y)_w = 0$ and X is a Gromov-hyperbolic space, $(x, z)_w = (y, z)_w \leqslant \delta$.

(2) By the similar argument in (1), there exists a point $u \in [x, z]$ such that $(x, u)_w = (z, u)_w$ since $(x, x)_w - (z, x)_w = (z, w)_x \geqslant 0$ and $(x, z)_w - (z, z)_w = -(x, w)_z \leqslant 0$. Thus

$$(x, z)_w + d(w, u) = (x, u)_w + (z, u)_w = 2(x, u)_w \leqslant 2(x, z)_w + 2\delta.$$

So

$$d(w, [x, z]) \leqslant d(w, u) \leqslant (x, z)_w + 2\delta \leqslant 3\delta.$$

□

Exercise 14. If any infinite set $\{g_n : n \in \mathbb{N}\}$ in G has the convergence property, prove that G-action (on a compact metrizable space M) is a convergence group action.

Solution. Suppose G-action is not a convergence group action, then there exists a compact subset $K_0 \in \Theta^3(M)$ and an infinite sequence $\{g_n : n \in \mathbb{N}\}$ such that $g_n K_0 \cap K_0 \neq \emptyset$. Hence there exists a sequence $\{(x_n, y_n, z_n)\} \subset K_0$ such that $g_n(x_n, y_n, z_n) \in K_0$ for all $n \in \mathbb{N}$. By the compactness of K_0, we can find a subsequence $\{(x_{n_i}, y_{n_i}, z_{n_i})\}$ of $\{(x_n, y_n, z_n)\}$ which converges to $(x, y, z) \in K_0$ and $\{g_{n_i}(x_{n_i}, y_{n_i}, z_{n_i})\}$ converges to $(x', y', z') \in K_0$. Now by the convergence property of $\{g_{n_i}\}$, there exists a subsequence $\{g_{n_{i_j}}\}$ of $\{g_{n_i}\}$ and $a, b \in M$ such that $g_{n_{i_j}}$ converges to b locally uniformly on $M \setminus \{a\}$. Since $(x, y, z) \in \Theta^3(M)$, there are at least two elements in $\{x, y, z\}$ not equal to a. Without loss of generality, suppose $a \notin \{x, y\}$. Similarly, there is at least one element in $\{x', y'\}$ not equal to b. Without loss of generality, suppose $x' \neq b$. Then $\{x_{n_{i_j}}\} \cup \{x\}$ is a compact subset in $M \setminus \{a\}$. However, $\{g_{n_{i_j}} x_{n_{i_j}}\}$ converges to x' instead of b, a contradiction. □

Exercise 15. Let G be acting on a compact metrizable space M as a convergence group. Then any infinite set $\{g_n : n \in \mathbb{N}\}$ in G contains a subsequence $\{g_{n_i}\}$ and points $a, b \in M$ so that

(1) g_{n_i} converges to b locally uniformly in $M \setminus \{a\}$, and

(2) $g_{n_i}^{-1}$ converges to a locally uniformly in $M \setminus \{b\}$.

Solution. There exists a subsequence $\{g_{n_i}\}$ and points $a, b, x, y \in M$ such that g_{n_i} converges to b locally uniformly in $M \setminus \{a\}$, and $g_{n_i}^{-1}$ converges to x locally uniformly in $M \setminus \{y\}$ since G-action is a convergence group action. If $x \neq a$, then consider a point $x' \in M \setminus \{y, b\}$. $g_{n_i}^{-1}x'$ converges to x, hence $g_{n_i}x$ converges to x', which contradicts to $x \in M \setminus \{a\}$ and $x' \neq b$. Similarly we can prove that $y = b$, hence g_{n_i} is the subsequence we need. □

Exercise 16. Prove that any finite group acts on a tree with a global fixed point.

Solution. Let F be a finite group acts on a tree T. For any $x \in T$, its orbit $B := F \cdot x$ is a bounded set. Note that there exists an o which satisfies $B \subset B(o, r_B)$, where r_B is the radius defined in Exercise 12. Such a point o is called a center. If there exists two centers o, o' in T, consider their midpoint m. For any point $p \in T$, let $q \in [o, o']$ be the point realizing $d(p, [o, o'])$. If $q \in [o', m]$, then $d(p, m) = d(p, o) - \dfrac{d(o, o')}{2}$ and similar thing happens when $q \in [o, m]$ since geodesic connecting two points is unique in a tree. Thus the center is unique. Since B is invariant under F, o is fixed under F. $\square$

Exercise 17. Let a finitely generated group G act without inversion on a tree T. Then there exists a minimal G-invariant subtree in T.

Solution. If every element in G has a fixed point, then they have a common fixed point by the Corollary 2 of Chap.I.6.5 in [1]. Hence the common fixed point is a minimal G-invariant subtree in T. If there exists an element g which fixes no point in T, by the Proposition 25 of Chap.I.6.4 in [1] g has a unique axis A_g in T where g acts on as a translation. Such a g is called a hyperbolic element in G. Now let T_G be the union of all axes of the hyperbolic elements in G. Note that every G-invariant subtree contains T_G since every g-invariant subtree contains A_g for ant hyperbolic element $g \in G$. Besides, T_G is G-invariant since for any $h \in G$ and a hyperbolic element $g \in G$, $hA_g = A_{hgh^{-1}}$. Now it suffices to prove that T_G is indeed a tree, i.e. T_G is connected. If two axes A_g and A_h are disjoint, then by the Proposition 1.2 in [2] we can get gh is also a hyperbolic element and A_{gh} intersects with both A_g and A_h. Hence T_G is a minimal G-invariant subtree. $\square$

Exercise 18. Prove that $\mathrm{SL}(n, \mathbb{Z})$ for $n \geqslant 3$ is not a hyperbolic group.

Solution. Since $\mathrm{SL}(3, \mathbb{Z})$ is a subgroup for any $\mathrm{SL}(n, \mathbb{Z})$ when $n \geqslant 3$ and a hyperbolic group cannot contain a subgroup isomorphic to $\mathbb{Z}^2$, it suffices to prove that $\mathrm{SL}(3, \mathbb{Z})$ contains a subgroup isomorphic to $\mathbb{Z}^2$. Now it's easy to verify that

$$\left\{ \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{bmatrix} : a, b \in \mathbb{Z} \right\}$$

is a subgroup we need since

$$\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a' & 0 \\ 0 & 1 & 0 \\ 0 & b' & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+a' & 0 \\ 0 & 1 & 0 \\ 0 & b+b' & 1 \end{bmatrix}.$$

$\square$

References

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