Bài tập

1. Chứng minh các đẳng thức sau

a) $C_n^0 = C_n^n = 1$, với mọi $n \in \mathbb{N}$;

b) $C_n^k + C_n^{k-1} = C_{n+1}^k$, với mọi $n \in \mathbb{N}$, $k = 1,\ldots,n$;

c) $C_n^k = C_{n-2}^{k-2} + 2C_{n-2}^{k-1} + C_{n-2}^k$, với mọi $n \in \mathbb{N}$, $k = 2,\ldots,n-2$;

d) $C_n^0 + C_n^1 + \ldots + C_n^{n-1} + C_n^n = 2^n$;

e) $C_n^0 - C_n^1 + \ldots + (-1)^{n-1}C_n^{n-1} + (-1)^n C_n^n = 0$.

ĐS: a) $C_n^0 = C_n^n = \dfrac{n!}{0!n!} = 1$

b) $$C_n^k + C_n^{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-(k-1))!}$$

$$= \frac{n![n+1-k+k]}{k!((n+1)-k)!} = \frac{(n+1)!}{k!((n+1)-k)!} = C_{n+1}^k$$

c) $C_{n-2}^{k-2} + 2C_{n-2}^{k-1} + C_{n-2}^k = \left(C_{n-2}^{k-2} + C_{n-2}^{k-1}\right) + \left(C_{n-2}^{k-1} + C_{n-2}^k\right) = C_{n-1}^{k-1} + C_{n-1}^k = C_n^k$

d) $2^n = (1+1)^n = \sum_{k=0}^{n} C_n^k 1^k 1^{n-k} = C_n^0 + C_n^1 + \ldots + C_n^{n-1} + C_n^n$

e) $0 = (1-1)^n = \sum_{k=0}^{n} (-1)^k C_n^k 1^k 1^{n-k} = C_n^0 - C_n^1 + \ldots + (-1)^{n-1} C_n^{n-1} + (-1)^n C_n^n$

2. Cho a, b là hai số thực bất kỳ. Chứng tỏ

a) $2 \cdot |ab| \le a^2 + b^2$

b) $\sqrt{a^2+b^2} \le |a| + |b| \le \sqrt{2}\sqrt{a^2+b^2}$

c) $\max\{a,b\} = \dfrac{a+b+|a-b|}{2}$

d) $\min\{a,b\} = \dfrac{a+b-|a-b|}{2}$

ĐS: a) $2 \cdot |ab| = 2|a||b| \le |a|^2 + |b|^2 = a^2 + b^2$

b) $a^2 + b^2 \le |a|^2 + |b|^2 + 2|a||b| = (|a|+|b|)^2$ và $(|a|+|b|)^2 = a^2 + b^2 + 2|a||b| \le 2(a^2+b^2)$

c) $\dfrac{a+b+|a-b|}{2} = \begin{cases} \frac{a+b+(a-b)}{2} & \text{khi} \quad a \geq b \\ \frac{a+b-(a-b)}{2} & \text{khi} \quad a < b \end{cases} = \begin{cases} a & \text{khi} \quad a \geq b \\ b & \text{khi} \quad a < b \end{cases} = \max\{a, b\}$

d) $\dfrac{a+b-|a-b|}{2} = \begin{cases} \frac{a+b-(a-b)}{2} & \text{khi} \quad a \geq b \\ \frac{a+b+(a-b)}{2} & \text{khi} \quad a < b \end{cases} = \begin{cases} b & \text{khi} \quad a \geq b \\ a & \text{khi} \quad a < b \end{cases} = \min\{a, b\}$

3. Chứng minh mệnh đề 1.4.

i) Nếu λ là số thực độc lập với các chỉ số của tổng hữu hạn, ta có

$$\sum_{k=1}^{n} \lambda = n\lambda; \quad \sum_{k=1}^{n} \lambda a_k = \lambda \sum_{k=1}^{n} a_k$$

ii) $\displaystyle\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$; $\displaystyle\prod_{k=1}^{n} (a_k \cdot b_k) = \left(\prod_{k=1}^{n} a_k\right)\left(\prod_{k=1}^{n} b_k\right)$

iv) Với $(a_{ij})_{\substack{i=1,2,\ldots,n \\ j=1,2,\ldots,m}}$ là họ gồm $n \times m$ số thực, ta có

$$\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} = \sum_{j=1}^{m}\sum_{i=1}^{n} a_{ij}$$

ĐS: i) $\displaystyle\sum_{k=1}^{1} \lambda = \lambda$ và nếu $\displaystyle\sum_{k=1}^{n} \lambda = n\lambda$ thì $\displaystyle\sum_{k=1}^{n+1} \lambda = \lambda + \sum_{k=1}^{n} \lambda = \lambda + n\lambda = (n+1)\lambda$.

$\displaystyle\sum_{k=1}^{1} \lambda a_k = \lambda a_1 = \lambda \sum_{k=1}^{1} a_k$ và nếu $\displaystyle\sum_{k=1}^{n} \lambda a_k = \lambda \sum_{k=1}^{n} a_k$ thì

$$\sum_{k=1}^{n+1} \lambda a_k = \sum_{k=1}^{n} \lambda a_k + \lambda a_{n+1} = \lambda \sum_{k=1}^{n} a_k + \lambda a_{n+1} = \lambda\left(\sum_{k=1}^{n} a_k + a_{n+1}\right) = \lambda \sum_{k=1}^{n+1} a_k$$

ii) $\displaystyle\sum_{k=1}^{1} (a_k + b_k) = a_1 + b_1 = \sum_{k=1}^{1} a_k + \sum_{k=1}^{1} b_k$ và nếu $\displaystyle\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$ thì

$$\begin{aligned} \sum_{k=1}^{n+1} (a_k + b_k) &= \sum_{k=1}^{n} (a_k + b_k) + (a_{n+1} + b_{n+1}) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k + (a_{n+1} + b_{n+1}) \\ &= \left(\sum_{k=1}^{n} a_k + a_{n+1}\right) + \left(\sum_{k=1}^{n} a_k + b_{n+1}\right) = \sum_{k=1}^{n+1} a_k + \sum_{k=1}^{n+1} b_k \end{aligned}$$

$\displaystyle\prod_{k=1}^{1} (a_k \cdot b_k) = a_1 \cdot b_1 = \left(\prod_{k=1}^{1} a_k\right)\left(\prod_{k=1}^{1} b_k\right)$ và nếu $\displaystyle\prod_{k=1}^{n} (a_k \cdot b_k) = \left(\prod_{k=1}^{n} a_k\right)\left(\prod_{k=1}^{n} b_k\right)$ thì

$$\prod_{k=1}^{n+1}\left(a_k\cdot b_k\right)=\prod_{k=1}^{n}\left(a_k\cdot b_k\right)\times\left(a_{n+1}\cdot b_{n+1}\right)=\left(\prod_{k=1}^{n}a_k\right)\left(\prod_{k=1}^{n}b_k\right)\times\left(a_{n+1}\cdot b_{n+1}\right)$$
$$=\left(\prod_{k=1}^{n}a_k\times a_{n+1}\right)\left(\prod_{k=1}^{n}b_k\times b_{n+1}\right)=\left(\prod_{k=1}^{n+1}a_k\right)\left(\prod_{k=1}^{n+1}b_k\right)$$

iv) Quy nạp theo n : $\sum_{i=1}^{1}\sum_{j=1}^{m}a_{ij}=\sum_{j=1}^{m}a_{1j}=\sum_{j=1}^{m}\sum_{i=1}^{1}a_{ij}$ và nếu $\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}=\sum_{j=1}^{m}\sum_{i=1}^{n}a_{ij}$ thì

$$\sum_{i=1}^{n+1}\sum_{j=1}^{m}a_{ij}=\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}+\sum_{j=1}^{m}a_{(n+1)j}=\sum_{j=1}^{m}\sum_{i=1}^{n}a_{ij}+\sum_{j=1}^{m}a_{(n+1)j}$$
$$=\sum_{j=1}^{m}\left(\sum_{i=1}^{n}a_{ij}+a_{(n+1)j}\right)=\sum_{j=1}^{m}\sum_{i=1}^{n+1}a_{ij}$$

4. Cho $a>1$. Chứng minh rằng

a) với mọi $n\in\mathbb{N}$, $a^n-1\ge n\left(a-1\right)$.

b) với mọi $n\in\mathbb{N}$, $a-1\ge n\left(a^{1/n}-1\right)$.

ĐS: a) $a^n=\left[1+\left(a-1\right)\right]^n\ge 1+n\left(a-1\right)$

b) Thay a bằng $a^{1/n}$ trong a), ta có $a-1=\left(a^{1/n}\right)^n-1\ge n\left(a^{1/n}-1\right)$

5. Cho $a_n=\left(1+\frac{1}{n}\right)^n$ và $b_n=\left(1+\frac{1}{n}\right)^{n+1}$, $n\in\mathbb{N}$.

a) Chứng tỏ rằng $a_n\le a_{n+1}$ và $b_{n+1}\le b_n$, với mọi $n\in\mathbb{N}$.

b) Chứng minh $a_n\le b_m$, với mọi $m,n\in\mathbb{N}$.

ĐS: a)

$$\frac{a_{n+1}}{a_n}=\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n}=\frac{\left(1+\frac{1}{n+1}\right)^n}{\left(1+\frac{1}{n}\right)^n}\left(1+\frac{1}{n+1}\right)=\left(\frac{\frac{n+2}{n+1}}{\frac{n+1}{n}}\right)^n\left(1+\frac{1}{n+1}\right)$$
$$=\left(\frac{n\left(n+2\right)}{\left(n+1\right)^2}\right)^n\left(1+\frac{1}{n+1}\right)=\left(1-\frac{1}{\left(n+1\right)^2}\right)^n\left(1+\frac{1}{n+1}\right)$$
$$\ge\left(1-\frac{n}{\left(n+1\right)^2}\right)\left(1+\frac{1}{n+1}\right)=\frac{n^2+n+1}{\left(n+1\right)^2}\frac{n+2}{n+1}=\frac{n^3+3n^2+3n+2}{\left(n+1\right)^3}\ge 1$$

$$\frac{b_n}{b_{n+1}}=\frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)^{n+2}}=\frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)^{n+1}}\frac{1}{\left(1+\frac{1}{n+1}\right)}=\left(\frac{\frac{n+1}{n}}{\frac{n+2}{n+1}}\right)^{n+1}\frac{n+1}{n+2}$$

$$=\left(\frac{\left(n+1\right)^2}{n\left(n+2\right)}\right)^{n+1}\frac{n+1}{n+2}=\left(1+\frac{1}{n\left(n+2\right)}\right)^{n+1}\frac{n+1}{n+2}$$

$$\geq\left(1+\frac{n+1}{n\left(n+2\right)}\right)\frac{n+1}{n+2}=\frac{n^2+3n+1}{n\left(n+2\right)}\frac{n+1}{n+2}=\frac{n^3+4n^2+4n+1}{n\left(n+2\right)^2}\geq 1$$

b) Ta có, $a_n \leq b_n$, với mọi $n \in \mathbb{N}$ và với $m \leq n$, $a_m \leq a_n$ và $b_n \leq b_m$.

Do đó, với $m, n \in \mathbb{N}$, nếu $m \leq n$ thì $a_n \leq b_n \leq b_m$ và nếu $n \leq m$ thì $a_n \leq a_m \leq b_m$.