Modeling Aggregate Demand for Products and Services based on Consumer Preferences for Product Features

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Abstract

This article presents a model of aggregate demand for products and services based on consumer preferences for product features. Organizations offering services and products can use this model to prioritize feature development and service improvement initiatives by estimating the potential impact of product changes on aggregate demand.

Introduction

In this article, we present a mathematical model of aggregate demand for products and services based on consumer preferences for product features. We show how to use this model to estimate the potential impact that changes to product features will have on consumer demand. Using these techniques, organizations can prioritize product development and service improvement efforts.

The Model

We start by assuming that there is a set of products A_i and potential consumers in a marketplace. We characterize the products using a set of common feature dimensions, such as weight, cost, and performance. Thus, every product is associated with a vector listing its feature values $[x_{j,i}]$. Every potential consumer assigns a different weight to each feature dimension. These weights are represented by the consumer's preference vector $[k_{l,j}]$. Consumers assign a quality score to each product $s_{l,i} = \sum_{j=0}^{n} k_{l,j} x_{j,i}$ that is a linear combination of the product's feature values and the consumer's feature weights.

We assume that, for every feature dimension j, consumer preference weights $k_{l,j}$ are normally distributed and that preferences across dimensions are independent.[1]

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[1]In general, preferences will not be independent across feature dimensions. For example, the importance that a consumer assigns to the weight of a product might correlate with the weight that they assign to its size. Thus, in practice, we will typically need to perform factor analysis such as Principal Component Analysis (PCA) to identify a set of independent feature

Because the quality scores are linear combinations of independent normally distributed variables, the quality scores for each product will also be normally distributed.

Let μ_j and σ_j represent the mean and standard deviation of weights that consumers give to feature dimension j. The distribution of consumer quality scores s_i for a product A_i will equal [2][3]

$$s_i \sim \mathcal{N}(\sum_{j=0}^{n} x_{j,i} \; \mu_j, \sum_{j=0}^{n} x_{j,i}^2 \; \sigma_j^2). \tag{1}$$

When presented with two products A_i and A_j, a potential consumer will prefer A_i over A_j when $s_i > s_j$. If $s_i = s_j$, the consumer will be indifferent between A_i and A_j and will choose one of them randomly with probability $\frac{1}{2}$.

Let the quality scores s_i for product A_i be normally distributed with mean μ_i and standard deviation σ_i. Additionally, let the quality scores s_j for another product A_j be normally distributed with mean μ_j and standard deviation σ_j. We can calculate the probability $p_{i,j}$ that a consumer will choose A_i over A_j as

$$p_{i,j} = \int \int_{0}^{\infty} \varphi(\frac{s - \mu_j}{\sigma_j}) \; \varphi(\frac{s + \delta - \mu_i}{\sigma_i}) \; d\delta \; ds. \tag{2}$$

(1) is equivalent to

$$p_{i,j} = \Phi(\frac{\mu_i - \mu_j}{\sqrt{\sigma_i^2 + \sigma_j^2}}). \tag{3}$$

The derivation is presented in Appendix 1.

Let p represent the probability that a randomly selected consumer chooses either product A_i or A_j. Then, if there are n consumers, in np times, a consumer will choose either A_i or A_j. Let n_i and n_j represent the number of times that a consumer chooses A_i and A_j respectively.

$$np \; p_{i,j} = n_i$$
$$np \; p_{j,i} = n_j.$$

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dimensions.

[2]Note that the second argument to $\mathcal{N}$ gives the distribution's variance.

[3]Soch, J. (2021, June 2). Proof: Linear combination of independent normal random variables. The Book of Statistical Proofs. https://statproofbook.github.io/P/norm-lincomb.html

Equating both equations gives:

$$np = \frac{n_i}{p_{i,j}} = \frac{n_j}{p_{j,i}}$$

Rearranging gives the relative frequency $k_{i,j}$ with which consumers choose A_i over A_j.

$$k_{i,j} = \frac{p_{i,j}}{p_{j,i}} = \frac{n_i}{n_j}$$

Let p_i represent the absolute probability that a consumer will choose product A_i. Then, assuming that every consumer chooses a product, we can form the equation:

$$p_0 + p_1 + \cdots + p_n = 1.$$

We can substitute the likelihood ratios into this equation to form:

$$
\begin{aligned}
p_0 + k_{1,0}p_0 + \cdots + k_{n,0}p_0 &= 1 \\
\cdots & \\
k_{0,n}p_n + k_{1,n}p_n + \cdots + p_n &= 1
\end{aligned}
$$

We can easily solve these equations to calculate the absolute probability that a consumer will choose a each product. For example:

$$p_0 = \frac{1}{1 + k_{1,0} + \cdots + k_{n,0}}.$$

Thus, we can calculate the proportion of consumers who will choose each product offered within a marketplace based on their preferences for product features. Using these equations, we can calculate the effect that feature changes will have on aggregate demand for various products.

Example Use

Imagine that we want to model the aggregate demand for three computer processors. Each processor is characterized by myriad features however, we determine that four factors, price, power, area, and performance, explain most consumer preference. We conduct a series of surveys and determine that the weights assigned to these factors are independent and normally distributed.

j	Factor	Mean μ	Standard Deviation σ
0	Price	-3	1
1	Power	-2	3
2	Performance	4	3
3	Area	-1	2

Table 1: Example Factor Weights for Computer Processors. This table presents example factor weights to illustrate how product features influence consumer product preferences.

We then measure three computer processors along each of these four dimensions and produce the following measurements.

Product	Price $x_{0,i}$	Power $x_{1,i}$	Performance $x_{2,i}$	Area $x_{3,i}$
A_0	1	1	1	1
A_1	10	3	5	3
A_2	2	2	3	2

Table 2: Example Product Factor Values. This table presents example factor measurements for three hypothetical computer processors to illustrate how the aggregate demand model can be used.

Based on these factor values, we can use (1) to calculate the score distribution for each product.

Product	Mean Score μ	Score Standard Deviation σ
A_0	-2	4.7958
A_1	-19	21.0238
A_2	0	11.7047

Table 3: Example Product Quality Scores. This table presents probability distribution parameters for example product quality scores (utilities).

Once we have derived the distributions for the product quality scores, we can calculate the probability $p_{i,j}$ that a randomly selected consumer will choose one product over another for each pair of products A_i and A_j using (3). We can

record these probabilities in a matrix like the following

$$P := \begin{bmatrix} 0.5000 & 0.7848 & 0.4372 \\ 0.2152 & 0.5000 & 0.2149 \\ 0.5628 & 0.7851 & 0.5000 \end{bmatrix}$$

.

Dividing along the diagonals, we can calculate the likelihood ratios $k_{i,j}$ for each pair of products

$$K := \begin{bmatrix} 1.0000 & 3.6459 & 0.7768 \\ 0.2743 & 1.0000 & 0.2737 \\ 1.2874 & 3.6538 & 1.0000 \end{bmatrix}$$

.

Adding elements within each column, we can calculate the probability that a randomly selected consumer will choose each product

$$A_0 = \frac{1}{1.0000 + 0.2743 + 1.2874} = 0.3904$$

$$A_1 = \frac{1}{3.6459 + 1.0000 + 3.6538} = 0.1205$$

$$A_2 = \frac{1}{0.7768 + 0.2737 + 1.0000} = 0.4877$$

However, we can do more than calculate relative market share for each product. We can go further and calculate the impact that changes to product features will have on market share. For example, we can use this model to calculate the price that maximizes the gross earnings for the company that produces the second product A_1.

Let's assume that the market size consists of 10 million units and that the following function is used to convert price scores into actual prices

$$price = 5(e^{priceScore/10} - 1)$$

Then each product has the following gross earnings

Product	Gross Earnings
A_0	$2,052,795.88
A_1	$10,351,453.05
A_2	$5,398,845.97

Table 4: Example Product Gross Earnings. This table presents example gross earnings for a set of hypothetical products.

However, our model indicates that the company that manufactures A_1 will actually maximize their gross earnings if they can reduce their price from $8.59 to $6.56. Doing so will increase their market share to 16.06% and increase their gross earnings to $10,529,874.61.

Conclusion

In this article, we have presented a mathematical model of aggregate product demand based on consumer preferences for product features. We have shown how to use this model to calculate the effect that product changes will have on consumer demand. This knowledge can be used to prioritize product development initiatives and to support planning.

While we have focused on products, the same methods can be used to model demand for services. Additionally, while our examples have assumed that the total number of units sold within a market is fixed, we can easily extend the model to represent variable demand by introducing a "null" product.

In practice, the primary difficulty using this model, as is often the case, lies in parameterization. Unfortunately, it is not possible to "run the equations backwards" and infer the input variables such as product feature scores from output variables such as market share. In general, there are many different product "configurations" that will produce the same market share divisions. As a result, the methods introduced in this article are best used for qualitative and approximate modeling. Input variables such as product scores, and consumer weights, can be approximated using consumer surveys and expert judgement.

Lastly, we hope that our mathematical solutions to the equations are valuable. While are confident that these equations have been presented and solved in other contexts, we were unable to find solutions to them in our readings. Hence, we hope that this article will make their derivation and proofs easier to find.

Appendix 1: Solving the Choice Equation

Assume that a population of consumers choose between two products A_i and A_j. Every consumer assigns a quality score s_i and s_j to products A_i and A_j respectively. These quality scores are normally distributed with means μ_i and μ_j and standard deviations σ_i and σ_j respectively. A consumer will choose A_i over A_j if $s_i > s_j$. If $s_i = s_j$ the consumer will be indifferent and will choose one

of them randomly with probability $\frac{1}{2}$. The probability that a randomly selected consumer will choose A_i over A_j is given by

$$p_{i,j} = \int \int_0^\infty \varphi(\frac{s - \mu_j}{\sigma_j}) \; \varphi(\frac{s + \delta - \mu_i}{\sigma_i}) \; d\delta \; ds \tag{4}$$

In this section, we will show that (4) equals

$$\Phi(\frac{\mu_i - \mu_j}{\sqrt{\sigma_i^2 + \sigma_j^2}}). \tag{5}$$

To prove this equality, we will calculate the Fourier transform for (4) and (5) and show that they are the same.[4]

Recall that the Fourier transform for the Normal Cumulative Density function (CDF) is

$$k(\xi) := \int \Phi(\frac{x_0 - \mu}{\sigma}) \; e^{-2\pi i \xi x_0} dx_0 = \frac{e^{-2\pi i \xi \mu} e^{-2(\pi \xi \sigma)^2}}{2\pi i \xi} + \frac{1}{2}\delta(\xi) \tag{6}$$

where δ is the Dirac Delta function. In Appendix 2, we show how we can derive (6). From this equation, we can use the Inverse Fourier Transform to express the Normal CDF as

$$\Phi(\frac{x_0 - \mu}{\sigma}) = \int \frac{e^{2\pi i \xi (x_0 - \mu)} e^{-2(\pi \xi \sigma)^2}}{2\pi i \xi} d\xi + \frac{1}{2}. \tag{7}$$

From (6) we see that the Inverse Fourier Transform for (4) equals

$$\Phi(\frac{\mu_i - \mu_j}{\sqrt{\sigma_i^2 + \sigma_j^2}}) = \int \frac{e^{2\pi i \xi (\mu_i - \mu_j)} e^{-2(\pi \xi \sqrt{\sigma_i^2 + \sigma_j^2})^2}}{2\pi i \xi} d\xi + \frac{1}{2}. \tag{8}$$

We will show that the Fourier transform for (4) has the same form as (8). Let $k(\xi)$ represent the Fourier transform for $\varphi(\frac{s - \mu_j}{\sigma_j})$ and replace this term in (3) with its Fourier transform:

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[4]Whenever two functions have identical Fourier transforms, we know that they are equal.

$$p_{i,j} = \int\int_0^\infty \varphi(\frac{s - \mu_j}{\sigma_j}) \; \varphi(\frac{s + \delta - \mu_i}{\sigma_i}) \; d\delta \; ds$$

$$= \int\int_0^\infty (\int k(\xi)e^{2\pi i \xi s} \; d\xi) \; \varphi(\frac{s + \delta - \mu_i}{\sigma_i}) \; d\delta \; ds$$

(*substitute the Inverse Fourier transform*)

$$= \int k(\xi)\int_0^\infty \int e^{2\pi i \xi s} \; \varphi(\frac{s + \delta - \mu_i}{\sigma_i}) \; ds \; d\delta \; d\xi$$

$$= \int k(\xi)\int_0^\infty e^{2\pi i \xi(\mu_i - \delta)}e^{-2(\pi \xi \sigma_i)^2} \; d\delta \; d\xi$$

(*similar to the derivation for the Fourier transform of the Normal PDF*)

$$= \int k(\xi)e^{2\pi i \xi \mu_i}e^{-2(\pi \xi \sigma_i)^2} \int_0^\infty e^{-2\pi i \xi \delta} \; d\delta \; d\xi$$

$$= \int \frac{k(\xi)e^{2\pi i \xi \mu_i}e^{-2(\pi \xi \sigma_i)^2}}{2\pi i \xi}d\xi$$

$$= \int \frac{e^{-2\pi i \xi \mu_j} \; e^{-2(\pi \xi \sigma_j)^2}e^{2\pi i \xi \mu_i}e^{-2(\pi \xi \sigma_i)^2}}{2\pi i \xi}d\xi$$

(*expand $k(\xi)$*)

$$= \int \frac{e^{2\pi i \xi(\mu_i - \mu_j)} \; e^{-2(\pi \xi \sqrt{\sigma_i^2 + \sigma_j^2})^2}}{2\pi i \xi}d\xi.$$

All that remains is to set the constant of integration, which in this instance is $\frac{1}{2}$. Thus, we see that the Fourier inverse transform for (4) is equivalent to the inverse transform for (5).

Appendix 2: The Fourier Transforms of the Normal PDF and CDF

In this section, we will show how we can calculate the Fourier Transform for the Normal Probability Density Function (PDF) and Cumulative Density Function (CDF).

We can calculate the Fourier Transform for the PDF by expanding terms and completing the square as follows

$$k(\xi) := \int \varphi(\frac{x-\mu}{\sigma})e^{-2\pi i\xi x}dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int e^{-\frac{(x-\mu)^2}{s\sigma^2}}e^{-2\pi i\xi x}dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int e^{-\frac{1}{2\sigma^2}(x^2-2\mu x+\mu^2)-2\pi i\xi x}dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int e^{-\frac{1}{2\sigma^2}(x^2-2\mu x+\mu^2+4\pi i\xi\sigma^2 x)}dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int e^{-\frac{1}{2\sigma^2}(x^2-2x(\mu-2\pi i\xi\sigma^2)+(\mu-2\pi i\xi\sigma^2)^2-(\mu-2\pi i\xi\sigma^2)^2+\mu^2)}dx$$

(complete the square)

$$= \frac{1}{\sigma\sqrt{2\pi}}\int e^{-\frac{1}{2\sigma^2}((x-(\mu-2\pi i\xi\sigma^2))^2-(\mu-2\pi i\xi\sigma^2)^2+\mu^2)}dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int e^{-\frac{1}{2\sigma^2}((x-(\mu-2\pi i\xi\sigma^2))^2+4\mu\pi i\xi\sigma^2+4(\pi\xi\sigma^2)^2)}dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int e^{-\frac{1}{2\sigma^2}((x-(\mu-2\pi i\xi\sigma^2))^2}e^{-2\mu\pi i\xi-2(\pi\xi\sigma)^2}dx$$

$$= e^{-2\mu\pi i\xi-2(\pi\xi\sigma)^2}\int \varphi(\frac{x-(\mu-2\pi i\xi\sigma^2)}{\sigma})\ dx$$

$$= e^{-2\mu\pi i\xi-2(\pi\xi\sigma)^2}.$$

We can then use this equation to calculate the Inverse Fourier transform for the Normal PDF.

$$\varphi(\frac{x-\mu}{\sigma}) = \int k(\xi)e^{2\pi i\xi x}\ d\xi$$

$$= \int e^{-2\mu\pi i\xi-2(\pi\xi\sigma)^2}e^{2\pi i\xi x}\ d\xi$$

$$= \int e^{2\pi i\xi(x-\mu)-2(\pi\xi\sigma)^2}\ d\xi$$

We can actually simplify this expression further. To do so, note that the integral is quasi symmetric about $\xi = 0$. Rewrite the integral so that it spans from 0 to ∞ and move the expression for both $+\xi$ and $-\xi$ into it. Next, expand the complex exponential into sines and cosines. The sine terms will cancel out. The result will be

$$\varphi(\frac{x-\mu}{\sigma}) = 2\int_0^\infty cos(2\pi\xi(x-\mu))e^{-2(\pi\xi\sigma)^2}d\xi$$

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From the above calculations, we see that the Inverse Fourier transform for (3) must equal

$$\varphi(\frac{\mu_i - \mu_j}{\sqrt{\sigma_i^2 + \sigma_j^2}}) = \int e^{2\pi i \xi (\mu_i - \mu_j)} \ e^{-2(\pi \xi \sqrt{\sigma_i^2 + \sigma_j^2})^2} d\xi. \tag{9}$$

We can use the Integration Property of the Fourier Transform to calculate the Fourier Transform for the Normal CDF. Given a continuous function $f(x)$, the Integration Property says

$$\mathcal{F}(\int_{-\infty}^{x_0} f(x)dx) = \frac{\mathcal{F}(f(x))}{2\pi i \xi} + c \ \delta(\xi) \tag{10}$$

Where $\mathcal{F}$ denotes the Fourier Transform operator, δ represents the Dirac delta function, and c is a constant that depends on f. In the standard proof the integral in (10) is represented by a convolution of f and the step-function. We then use the Convolution Property of the Fourier Transform to express the Fourier Transform as the product of the Fourier Transform of f and the step function. The Fourier Transform of the step function reduces to a sum involving the Dirac Delta function. A detailed derivation can be found in [5].

Applying the Integration Property of the Fourier Transform to the Normal CDF gives

$$\Phi(\frac{x_0 - \mu}{\sigma}) = \int \frac{e^{2\pi i \xi (x_0 - \mu)} e^{-2(\pi \xi \sigma)^2}}{2\pi i \xi} d\xi + \frac{1}{2}. \tag{11}$$

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[5]Bevelacqua, P. (n.d.). *The Integration Property of the Fourier Transform.* The Fourier Transform.com. https://www.thefouriertransform.com/transform/integration.php