Geophysical Fluid Dynamics: Final Exam

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Due May 16, 2008

1.

(a)

The geostrophic velocities can be expressed using pressure fields as

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y}$$

$$v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x}$$

So the divergence of the energy flux due to the O(1) geostrophic velocity and pressure fields is

$$\begin{aligned}
\nabla \cdot (p\mathbf{V_g}) &= \nabla p \cdot \mathbf{V_g} + p\nabla \cdot \mathbf{V_g} \\
&= \frac{\partial p}{\partial x}(-\frac{1}{f\rho}\frac{\partial p}{\partial y}) + \frac{\partial p}{\partial y}(\frac{1}{f\rho}\frac{\partial p}{\partial x}) + (-\frac{1}{f\rho}\frac{\partial^2 p}{\partial y \partial x}) + \frac{1}{f\rho}\frac{\partial^2 p}{\partial y \partial x} \\
&= 0
\end{aligned}$$

which exactly vanishes.

(b)

The linear, shallow water quasigeostrophic potential vorticity equation is

$$\frac{\partial}{\partial t}(\nabla^2 \psi - \lambda_d^{-2}\psi) + \beta\frac{\partial \psi}{\partial x} = 0$$

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Multiplying $-\psi$ in both sides of this equation yields

$$-\psi\frac{\partial}{\partial t}(\nabla^2\psi - \lambda_d^{-2}\psi) - \beta\psi\frac{\partial\psi}{\partial x} = 0$$

As

$$\begin{aligned}\psi\frac{\partial}{\partial t}\nabla^2\psi &= \nabla\cdot(\psi\frac{\partial}{\partial t}\nabla\psi) - \nabla\psi\frac{\partial}{\partial t}\nabla\psi \\ &= \nabla\cdot(\psi\nabla\psi_t) - \frac{\partial}{\partial t}[\frac{(\nabla\psi)^2}{2}]\end{aligned}$$

$$\psi\frac{\partial}{\partial t}(\lambda_d^{-2}\psi) = \frac{\partial}{\partial t}[\frac{\psi^2}{2\lambda_d^2}]$$

$$\beta\psi\frac{\partial\psi}{\partial x} = \beta\frac{\partial}{\partial x}(\frac{\psi^2}{2}) = \nabla\cdot(\frac{\beta\psi^2\mathbf{e_x}}{2})$$

So

$$\frac{\partial}{\partial t}[\frac{(\nabla\psi)^2}{2} + \frac{\psi^2}{2\lambda_d^2}] + \nabla\cdot(-\psi\nabla\psi_t - \frac{\beta\psi^2\mathbf{e_x}}{2}) = 0$$

or

$$\frac{\partial E}{\partial t} + \nabla\cdot\mathbf{S} = 0$$

where $E = (\nabla\psi)^2/2 + \psi^2/(2\lambda_d^2)$ is the energy density and $\mathbf{S}$ is the energy flux. E includes kinetic energy $(\nabla\psi)^2/2$ plus potential energy $\psi^2/(2\lambda_d^2)$.

(c)

Suppose the plane Rossby wave has the expression

$$\psi = A\cos(kx + ly - \omega t) = A\cos\theta$$

where A is the amplitude and θ is the phase. The kinetic energy is

$$KE = \frac{(\nabla\psi)^2}{2} = \frac{A^2(k^2+l^2)}{2}\sin^2\theta$$

and averaged over a period,

$$< KE >= \frac{A^2}{4}(k^2+l^2)$$

Similarly,

$$< PE >=< \psi^2/(2\lambda_d^2) >= \frac{A^2}{4\lambda_d^2}$$

Thus the total energy in the Rossby wave, averaged over a period is

$$< E >=< KE > + < PE >= \frac{A^2}{4}(k^2 + l^2 + \lambda_d^{-2})$$

The energy flux is

$$\begin{aligned} \mathbf{S} &= -\psi\nabla\psi_t - \frac{\beta\psi^2\mathbf{e_x}}{2} \\ &= -\omega A^2(\mathbf{k}+\mathbf{l})\cos^2\theta - \frac{\beta A^2\mathbf{e_x}}{2}\cos^2\theta \end{aligned}$$

and averaged over a period

$$\begin{aligned} < \mathbf{S} > &= \frac{\omega A^2}{2}(\mathbf{k}+\mathbf{l}) + \frac{\beta A^2}{4}\mathbf{e_x} \\ &= \frac{\beta A^2}{2}\frac{k(\mathbf{k}+\mathbf{l})}{k^2+l^2+\lambda_d^{-2}} - \frac{\beta A^2}{4}\mathbf{e_x} \\ &= \frac{\beta A^2}{4}\frac{2k^2\mathbf{e_x} + 2kl\mathbf{e_y} - (k^2+l^2+\lambda_d^{-2})\mathbf{e_x}}{k^2+l^2+\lambda_d^{-2}} \\ &= \frac{\beta A^2(k^2+l^2+\lambda_d^{-2})}{4}[\frac{k^2-l^2-\lambda_d^{-2}}{(k^2+l^2+\lambda_d^{-2})^2}\mathbf{e_x} + \frac{2kl}{(k^2+l^2+\lambda_d^{-2})^2}\mathbf{e_y}] \\ &= < E > [\frac{\beta(k^2-l^2-\lambda_d^{-2})}{(k^2+l^2+\lambda_d^{-2})^2}\mathbf{e_x} + \frac{2\beta kl}{(k^2+l^2+\lambda_d^{-2})^2}\mathbf{e_y}] \end{aligned}$$

Here, we have used the dispersion relation

$$\omega = -\frac{\beta k}{k^2+l^2+\lambda_d^{-2}}$$

from which we can obtain the group velocity

$$c_{gx} = \frac{\beta(k^2-l^2-\lambda_d^{-2})}{(k^2+l^2+\lambda_d^{-2})^2}$$

and

$$c_{gy} = \frac{2\beta kl}{(k^2+l^2+\lambda_d^{-2})^2}$$

Substituting these two expressions into the energy flux formula yields

$$< S >=< E > (c_{gx}\mathbf{e_x} + c_{gy}\mathbf{e_y}) =< E > \mathbf{c_g}$$

Now the energy conservation equation became

$$\frac{\partial < E >}{\partial t} + \nabla\cdot(< E > \mathbf{c_g}) = 0$$

2.

(a)

The linear quasigeostrophic potential vorticity equation is

$$\frac{\partial}{\partial t}(\nabla^2\psi - \lambda_d^{-2}\psi) + \beta\frac{\partial\psi}{\partial x} = 0$$

According to the boundary condition at $x = 0$,

$$u = U_0\cos(\pi y/L)e^{-i\omega t}, \;\; 1 \le y \le L$$

we suppose the solution for $x > 0$ is

$$\psi = \psi_0\sin(\pi y/L)e^{i(kx-\omega t)}$$

where ψ_0 can be determined by the relation $u = -\partial\psi/\partial y$ and the boundary condition at $x = 0$:

$$\psi_0 = -U_0L/\pi$$

As the wave source is at the origin, it requires that the wave propagates energy to the east for $x > 0$, which is equivalent to $c_{gx} > 0$ at $x > 0$. Substituting the expression of ψ into the potential vorticity equation yields the dispersion relation:

$$\omega = -\frac{\beta k}{k^2 + \pi^2/L^2 + \lambda_d^{-2}}$$

or

$$k^2 + \beta k/\omega + \pi^2/L^2 + \lambda_d^{-2} = 0$$

The two solutions for k are

$$k_1 = -\frac{\beta}{2\omega} + \sqrt{\frac{\beta^2}{4\omega^2} - \pi^2/L^2 - \lambda_d^{-2}}$$

$$k_2 = -\frac{\beta}{2\omega} - \sqrt{\frac{\beta^2}{4\omega^2} - \pi^2/L^2 - \lambda_d^{-2}}$$

Because $k_1k_2 = \pi^2/L^2 + \lambda_d^{-2}$ and $|k_1| < |k_2|$, we have $k_1^2 < \pi^2/L^2 + \lambda_d^{-2}$ and $k_2^2 > \pi^2/L^2 + \lambda_d^{-2}$. As the group velocity in x direction is

$$c_{gx} = \frac{\beta(k^2 - \pi^2/L^2 - \lambda_d^{-2})}{(k^2 + \pi^2/L^2 + \lambda_d^{-2})^2}$$

It can be seen that

$$c_{gx}(k_1) < 0$$

and

$$c_{gx}(k_2) > 0$$

So the solution for $x > 0$ is

$$\psi = \psi_0 \sin(\pi y/L) e^{i(k_2 x - \omega t)}$$

where

$$k_2 = -\frac{\beta}{2\omega} - \sqrt{\frac{\beta^2}{4\omega^2} - \pi^2/L^2 - \lambda_d^{-2}}$$

(b)

we suppose the solution for $x < 0$ is

$$\psi = \psi_0 \sin(\pi y/L) e^{i(kx - \omega t)}$$

where ψ_0 can be determined by the relation $u = -\partial\psi/\partial y$ and the boundary condition at $x = 0$:

$$\psi_0 = -U_0 L/\pi$$

As the wave source is at the origin, it requires that the wave propagates energy to the west for $x < 0$, which is equivalent to $c_{gx} < 0$ for $x < 0$. Substituting the expression of ψ into the potential vorticity equation yields the dispersion relation:

$$\omega = -\frac{\beta k}{k^2 + \pi^2/L^2 + \lambda_d^{-2}}$$

or

$$k^2 + \beta k/\omega + \pi^2/L^2 + \lambda_d^{-2} = 0$$

The two solutions for k are

$$k_1 = -\frac{\beta}{2\omega} + \sqrt{\frac{\beta^2}{4\omega^2} - \pi^2/L^2 - \lambda_d^{-2}}$$

$$k_2 = -\frac{\beta}{2\omega} - \sqrt{\frac{\beta^2}{4\omega^2} - \pi^2/L^2 - \lambda_d^{-2}}$$

Because $k_1 k_2 = \pi^2/L^2 + \lambda_d^{-2}$ and $|k_1| < |k_2|$, we have $k_1^2 < \pi^2/L^2 + \lambda_d^{-2}$ and $k_2^2 > \pi^2/L^2 + \lambda_d^{-2}$. As the group velocity in x direction is

$$c_{gx} = \frac{\beta(k^2 - \pi^2/L^2 - \lambda_d^{-2})}{(k^2 + \pi^2/L^2 + \lambda_d^{-2})^2}$$

It can be seen that

$$c_{gx}(k_1) < 0$$

and

$$c_{gx}(k_2) > 0$$

So the solution for $x < 0$ is

$$\psi = \psi_0 \sin(\pi y/L) e^{i(k_1 x - \omega t)}$$

where

$$k_1 = -\frac{\beta}{2\omega} + \sqrt{\frac{\beta^2}{4\omega^2} - \pi^2/L^2 - \lambda_d^{-2}}$$

3.

(a)

The equivalent nondimensional shallow water equations on the equatorial beta-plane for a stratified, incompressible fluid with a heat source given by $Q(x, y, z) \sim \hat{Q}(x, y) \sin mz$ are:

$$u_t - yv/2 = -\eta_x - \epsilon u$$

$$v_t + yu/2 = -\eta_y - \epsilon v$$

$$\eta_t + u_x + v_y = -\epsilon\eta - \hat{Q}$$

where subscripts denotes derivatives. The length scale is $(c/2\beta)^{1/2}$ and the time scale is $(2\beta c)^{-1/2}$. ϵ is the dissipation coefficient.

(b)

Fourier transform the equations in the x-direction, and we have

$$u_t - yv/2 = -ik\eta - \epsilon u$$

$$v_t + yu/2 = -\eta_y - \epsilon v$$

$$\eta_t + iku + v_y = -\epsilon\eta - Q$$

Notice that u, v, η and Q here represent the Fourier-transformed variables. Discretizing these equations in a implicit scheme yields:

$$\frac{u_j^{n+1} - u_j^n}{\delta t} - \frac{y_i}{2} \times \frac{1}{4}(v_{j-1/2}^n + v_{j+1/2}^n + v_{j-1/2}^{n+1} + v_{j+1/2}^{n+1}) = -ik\frac{\eta_j^{n+1} + \eta_j^n}{2} - \epsilon\frac{u_j^{n+1} + u_j^n}{2}$$

$$\frac{v_{j+1/2}^{n+1} - v_{j+1/2}^n}{\delta t} + \frac{1}{2} \times \frac{1}{4}(y_j u_j^{n+1} + y_{j+1}u_{j+1}^{n+1} + y_j u_j^n + y_{j+1}u_{j+1}^n)$$

$$= -\frac{1}{2\delta y}(\eta_{j+1}^{n+1} - \eta_j n + 1 + \eta_{j+1}^n - \eta_j^n) - \epsilon\frac{v_{j+1/2}^n + v_{j+1/2}^n}{2}$$

$$\frac{\eta_j^{n+1} - \eta_j^n}{\delta t} + ik\frac{u_j^{n+1} + u_j^n}{2} + \frac{1}{2\delta y}(v_{j+1/2}^{n+1} - v_{j-1/2}^{n+1} + v_{j+1/2}^n - v_{j-1/2}^n) = -Q_j - \epsilon\frac{\eta_j^{n+1} + \eta_j^n}{2}$$

where v is discretized as $v_{j+1/2},\ j = 1, 2, \cdots, N_y$, with $v_{1/2} = 0$ and $v_{N+1+1/2} = 0$ locating at south and north boundaries. u, η, Q and y are discretized as u_j, η_j, Q_j and y_j, with $j = 1, 2, \cdots, N_y + 1$.

(c)

Reorganizing the discrete version equations above yields

$$(\frac{1}{\delta t}+\frac{\epsilon}{2})\eta_j^{n+1}+\frac{ik}{2}u_j^{n+1}+\frac{1}{2\delta y}(-v_{j-1/2}^{n+1}+v_{j+1/2}^{n+1}) = (\frac{1}{\delta t}-\frac{\epsilon}{2})\eta_j^n-\frac{ik}{2}u_j^n-\frac{1}{2\delta y}(-v_{j-1/2}^n+v_{j+1/2}^n)-Q_j$$

$$\frac{ik}{2}\eta_j^{n+1} + (\frac{1}{\delta t} + \frac{\epsilon}{2})u_j^{n+1} - \frac{y_j}{8}(v_{j-1/2}^{n+1} + v_{j+1/2}^{n+1}) = -\frac{ik}{2}\eta_j^n + (\frac{1}{\delta t} - \frac{\epsilon}{2})u_j^n + \frac{y_j}{8}(v_{j-1/2}^n + v_{j+1/2}^n)$$

$$\frac{1}{2\delta y}(-\eta_j^{n+1} + \eta_{j+1}^{n+1}) + \frac{1}{8}(y_i u_j^{n+1} + y_{j+1}u_{j+1}^{n+1}) + (\frac{1}{\delta t} + \frac{\epsilon}{2})v_{j+1/2}^{n+1}$$

$$= -\frac{1}{2\delta y}(-\eta_j^n + \eta_{j+1}^n) - \frac{1}{8}(y_i u_j^n + y_{j+1}u_{j+1}^n) + (\frac{1}{\delta t} - \frac{\epsilon}{2})v_{j+1/2}^n$$

which originate from the continuity, x-momentum and y-momentum equation, respectively. We can rewrite these equations in matrix forms:

$$\begin{aligned}
E^+\eta^{n+1} + Ku^{n+1} + A_1 v^{n+1} &= E^-\eta^n - Ku^n - A_1 v^n - Q && (1)\\
K\eta^{n+1} + E^+u^{n+1} + A_2 v^{n+1} &= -K\eta^n + E^-\eta^n - A_2 v^n && (2)\\
-A_1^T\eta^{n+1} - A_2^T u^{n+1} + E^{v+}v^{n+1} &= A_1^T\eta^n + A_2^T u^n + E^{v-}v^n && (3)
\end{aligned}$$

where superscript T denotes matrix transpose, but not conjugate transpose and

$$\eta^n = (\eta_1^n, \eta_2^n, \cdots, \eta_{N_y+1}^n)^T$$

$$u^n = (u_1^n, u_2^n, \cdots, u_{N_y+1}^n)^T$$

$$v^n = (v_{1+1/2}^n, v_{2+1/2}^n, \cdots, v_{N_y+1/2}^n)^T$$

$$E^+ = (\frac{1}{\delta t} + \frac{\epsilon}{2})I_{(N_y+1)\times(N_y+1)}$$

$$E^{-} = (\frac{1}{\delta t} - \frac{\epsilon}{2})I_{(N_y+1)\times(N_y+1)}$$

$$E^{v+} = (\frac{1}{\delta t} + \frac{\epsilon}{2})I_{N_y\times N_y}$$

$$E^{v-} = (\frac{1}{\delta t} - \frac{\epsilon}{2})I_{N_y\times N_y}$$

$$K = \frac{ik}{2}I_{(N_y+1)\times(N_y+1)}$$

$$A_1 = \frac{1}{2\delta y}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ & & & \vdots & \\ 0 & \cdots & 0 & -1 & 1 \\ 0 & \cdots & 0 & 0 & -1 \end{pmatrix}_{(N_y+1)\times N_y}$$

$$A_2 = -\frac{\delta y}{8}\begin{pmatrix} -\frac{N}{2} & 0 & 0 & \cdots & 0 \\ -\frac{N}{2}+1 & -\frac{N}{2}+1 & 0 & \cdots & 0 \\ 0 & -\frac{N}{2}+2 & -\frac{N}{2}+2 & \cdots & 0 \\ & & & \vdots & \\ 0 & \cdots & 0 & \frac{N}{2}-1 & \frac{N}{2}-1 \\ 0 & \cdots & 0 & 0 & \frac{N}{2} \end{pmatrix}_{(N_y+1)\times N_y}$$

Taking the summation and difference of equation (1) and equation (2), we have

$$(E^+ + K)(\eta^{n+1} + u^{n+1}) + (A_1 + A_2)v^{n+1} = (E^- - K)(\eta^n + u^n) - (A_1 + A_2)v^n - Q$$

$$(E^+ - K)(\eta^{n+1} - u^{n+1}) + (A_1 - A_2)v^{n+1} = (E^- + K)(\eta^n - u^n) - (A_1 - A_2)v^n - Q$$

Using the inverse matrix operations, we have

$$\eta^{n+1}+u^{n+1} = -(E^+ + K)^{-1}(A_1+A_2)(v^{n+1}+v^n)+(E^+ + K)^{-1}(E^- - K)(\eta^n+u^n)-(E^+ + K)^{-1}Q$$

$$\eta^{n+1}-u^{n+1} = -(E^+ - K)^{-1}(A_1-A_2)(v^{n+1}+v^n)+(E^+ - K)^{-1}(E^- + K)(\eta^n-u^n)-(E^+ - K)^{-1}Q$$

So we have

$$\begin{aligned}\eta^{n+1} \quad = \quad & -[(E^+ + K)^{-1}(A_1 + A_2) + (E^+ - K)^{-1}(A_1 - A_2)](v^{n+1} + v^n)/2 \\ & +[(E^+ + K)^{-1}(E^- - K) + (E^+ - K)^{-1}(E^- + K)]\eta^n/2 \\ & +[(E^+ + K)^{-1}(E^- - K) - (E^+ - K)^{-1}(E^- + K)]u^n/2 \\ & -[(E^+ + K)^{-1} + (E^+ - K)^{-1}]Q/2\end{aligned}$$

$$
\begin{aligned}
u^{n+1} \;=\; & -[(E^{+}+K)^{-1}(A_1+A_2)-(E^{+}-K)^{-1}(A_1-A_2)](v^{n+1}+v^{n})/2 \\
& +[(E^{+}+K)^{-1}(E^{-}-K)-(E^{+}-K)^{-1}(E^{-}+K)]\eta^{n}/2 \\
& +[(E^{+}+K)^{-1}(E^{-}-K)+(E^{+}-K)^{-1}(E^{-}+K)]u^{n}/2 \\
& -[(E^{+}+K)^{-1}-(E^{+}-K)^{-1}]Q/2
\end{aligned}
$$

Substituting η^{n+1} and u^{n+1} into equation (3), we can obtain the formula for v^{n+1}:

$$
\begin{aligned}
& [E^{v+}+(A_1^T+A_2^T)(E^{+}+K)^{-1}(A_1+A_2)/2+(A_1^T-A_2^T)(E^{+}-K)^{-1}(A_1-A_2)/2]v^{n+1} \\
= \; & [A_1^T+(A_1^T+A_2^T)(E^{+}+K)^{-1}(E^{-}-K)/2+(A_1^T-A_2^T)(E^{+}-K)^{-1}(E^{-}+K)/2]\eta^{n} \\
+ \; & [A_1^T+(A_1^T+A_2^T)(E^{+}+K)^{-1}(E^{-}-K)/2-(A_1^T-A_2^T)(E^{+}-K)^{-1}(E^{-}+K)/2]u^{n} \\
+ \; & [E^{v-}-(A_1^T+A_2^T)(E^{+}+K)^{-1}(A_1+A_2)/2-(A_1^T-A_2^T)(E^{+}-K)^{-1}(A_1-A_2))/2]v^{n} \\
- \; & [(A_1^+A_2^T)(E^{+}+K)^{-1}+(A_1^T-A_2^T)(E^{+}-K)^{-2}]Q/2
\end{aligned}
$$

which can be used to solve v^{n+1} from η^n, u^n and v^n. After obtaining v^{n+1}, we can compute η^{n+1} and u^{n+a} from their formulae. This process can be looped for different values of wavenumber k. As soon as we know the Fourier transformed variables, we can obtain the variables in physical space using inverse Fourier transform.

(d)

Formulae for solving u^{n+1} and η^{n+1} are given in the last subsection.

(e)

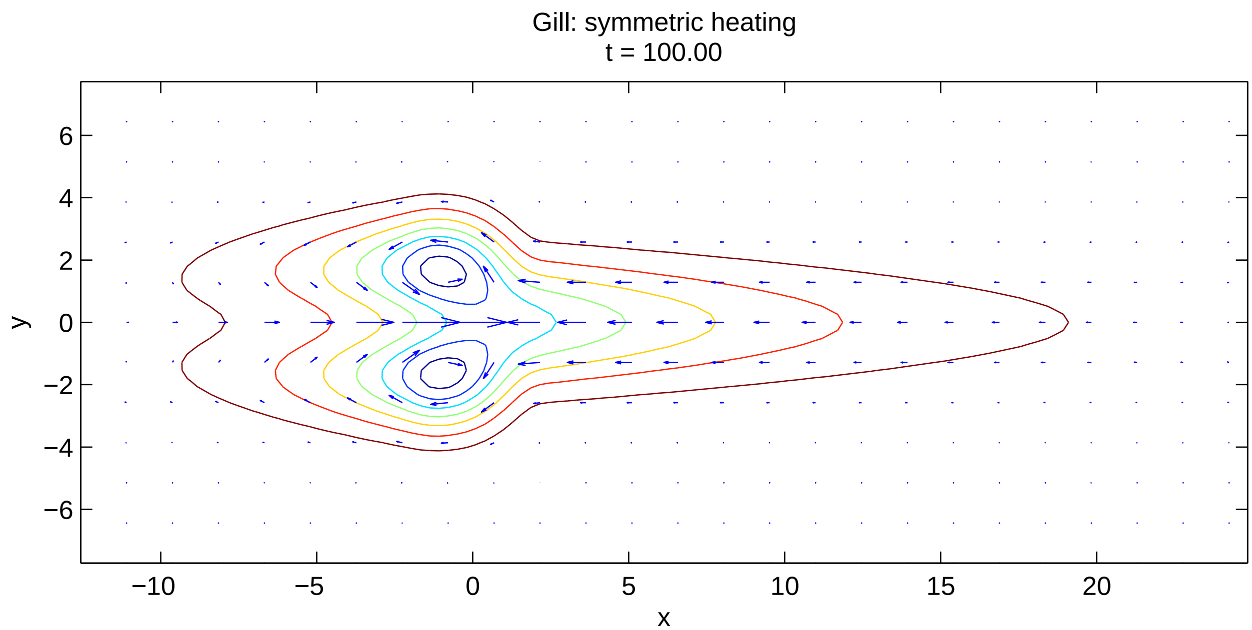
I use MATLAB to numerically solve these equations. In MATLAB, v^{n+1} for each k is expressed as a N_y-length row vector while η^{n+1} and u^{n+1} for each k are expressed as N_y+1-length row vectors.

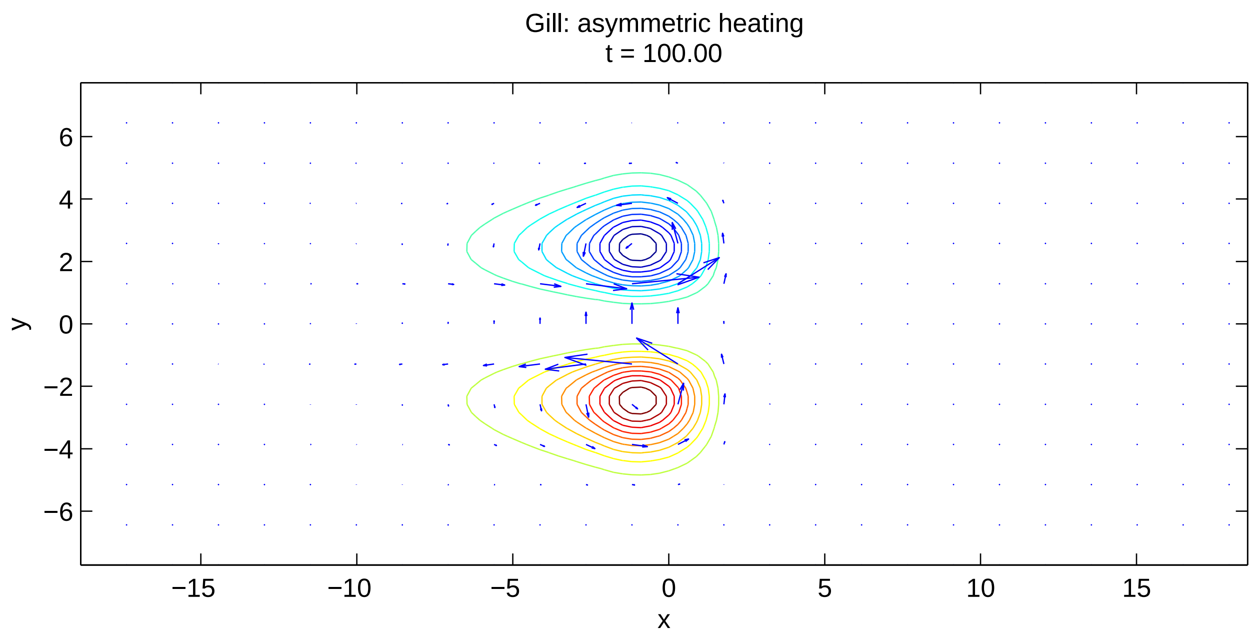
(f)

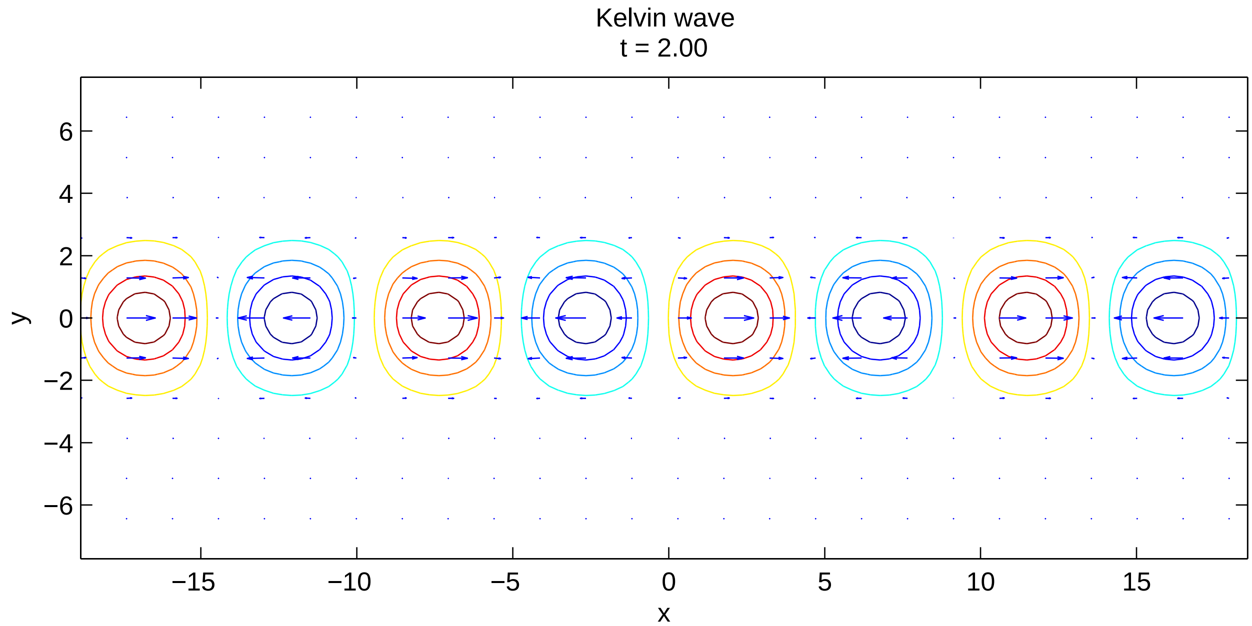
My program is included in two files, `equ_mot.m` and `matgen.m`. The former is the main operating file and the later is a function that outputs some matrices used for solving the linear equations. For long time, my solutions converge to those shown in Fig. 1 and Fig. 2 of Gill, as shown in figures below. The physical interpretation for case (b) is as follows. Comparing to case (a), heating influence does not extend much to the east because of lack of a long wave (Kelvin wave) which propagates east. The heating influence extends west a little because there is a long wave (Rossby wave) which propagates west. For this pattern Rossby wave, the decay rate is bigger than that of in case(a) so the west extend is not as long as that in case(a).

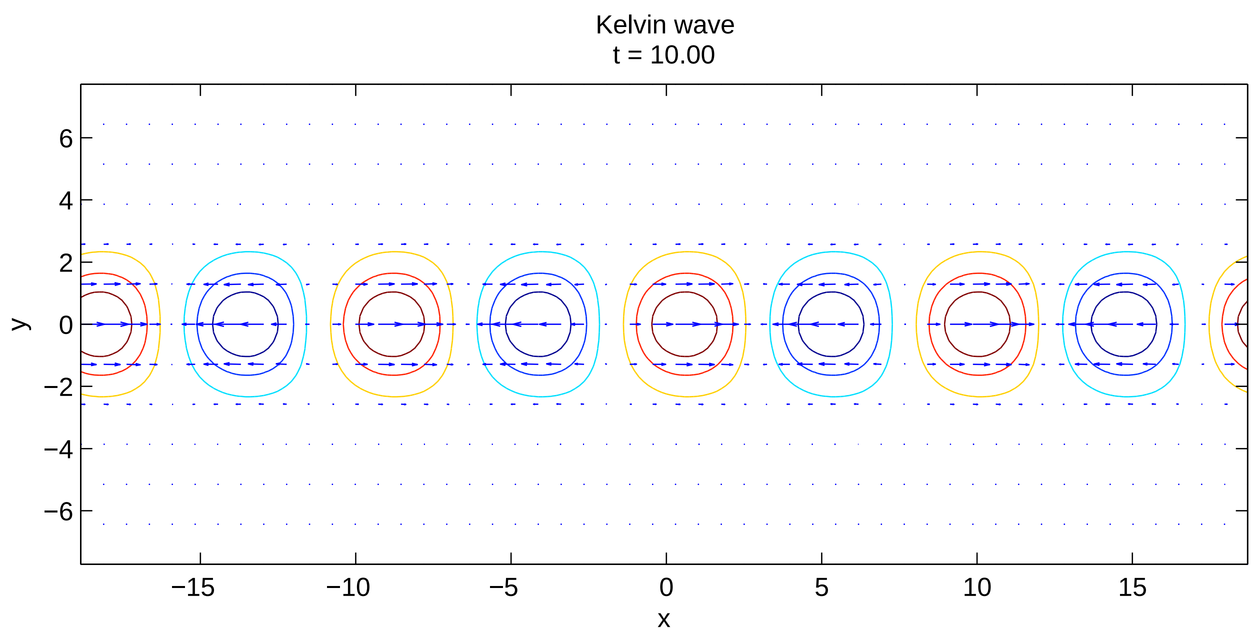
(g)

The numerical phase speed is approximately 1 for the nondimensional unit, which is consistent with the theoretical value. The snapshots are shown below.






Kelvin wave
t = 2.00
6
4
2
0
−2
−4
−6
y
−15
−10
−5
0
5
10
15
x


Kelvin wave
t = 10.00
6
4
2
0
−2
−4
−6
y
−15
−10
−5
0
5
10
15
x