Department of Applied Physics and Applied Mathematics
Columbia University
APPH E4210. Geophysical Fluid Dynamics
Spring 2008
Final Exam
Due date: May 16, 2008

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1. (25 points) Consider small amplitude, quasigeostrophic motions of a shallow layer of homogeneous fluid on the mid-latitude beta-plane.

   (a) Show that the divergence of the energy flux due to the O(1) geostrophic velocity and pressure fields exactly vanishes. Recall that the energy flux is given by the rate of working by the pressure.

   (b) Using the linear, shallow water quasigeostrophic potential vorticity equation, derive an equation for the conservation of energy. The equation should be of the form: time rate of change of energy density = - (divergence of a flux). Label the various terms in the energy equation. (You could start with the momentum equations, but it is much more convenient to work directly with the QGPV equation.)

   (c) Using the result of part (b), show that for a plane Rossby wave, the energy flux vector is given by the (energy density)$\times$(group velocity).

2. (25 points) Consider a homogeneous fluid in an infinitely long zonal channel of width L on the mid-latitude beta-plane. At $x = 0$, a wave maker oscillating at a known frequency ω, produces a zonal velocity of the form:

$$u = U_o \cos(\pi y/L) e^{-i\omega t}, \;\; 0 \leq y \leq L.$$

   Assuming that the wave maker has been on for a very long time (so the steady state response is at the same frequency), and that the dynamics are linear and quasigeostrophic,

   (a) find the resulting Rossby wave for $x > 0$.

   (b) Repeat (a) for the region $x < 0$.

3. (50 points) In this problem, you will develop and implement a numerical model for the forced shallow water equations on the unbounded equatorial beta-plane.

   (a) Write down the "equivalent" nondimensional shallow water equations (SWEs) on the equatorial beta-plane for a stratified, incompressible fluid with a heat source given by $Q(x, y, z) \sim \hat{Q}(x, y) \sin mz$.

(b) Fourier transform the equations in the x-direction, and discretize in time and the y-direction as discussed in class. Be sure to use a staggered grid in which u, η, and Q are co-located in space. (In the following, the symbols u, v, etc., refer to the Fourier-transformed variables.)

(c) Eliminate u and η to derive a single equation for v^{n+1} in terms of u^n, η^n, v^n, and Q^n. You should get a linear system of equations with a tridiagonal coefficient matrix. (Systems with tridiagonal coefficients are very easy to solve, e.g., see http://www.nrbook.com/a/bookfpdf/f2-4.pdf.) This is the discrete version of the second order ODE for v that we derived in class. (Recall that the solutions of the ODE turned out to be parabolic cylinder functions.) To solve this equation, we require two boundary conditions. In the continuum ODE case, we set $v \to 0$ as $|y| \to \infty$. In the discrete case, we set $v(\pm y_{\max}) = 0$, where $\pm y_{\max}$ are the end points of the domain. I use a $y_{\max}$ corresponding to 70° latitude.

(d) To compute u^{n+1} and η^{n+1}, it is convenient to introduce variables $r \equiv u + \eta$ and $q \equiv u - \eta$ as we did for the continuum equations. Derive the discrete equations that relate r and q to v^{n+1}. The time-stepping procedure is to first solve the tridiagonal system for v^{n+1}, then compute r and q, and from these recover u^{n+1} and η^{n+1}.

(e) Implement the above equations in a programming language of your choice. I suggest FORTRAN, but MATLAB is also convenient. If you use the former, I recommend the FFT and tridiagonal solver routines from Numerical Recipes (email me if you don't have access to them).

(f) Verify that your code works by computing time-dependent solutions to the SWEs, with an equivalent phase speed $c = 60$ m/s, a damping time scale of 2 days, and heating patterns given by (a) eq. 3.12 of Gill, and (b) eq. 3.13 of Gill. Show that for long times, your solutions converge to those shown in Fig. 1 and Fig. 2 of Gill. For case (b), provide a physical interpretation of your solution in terms of equatorially trapped waves.

(g) Initial value problem: Set $Q = 0$, and initialize u, v, and η as a sinusoidal wave in x, with wavelength $2\pi R/4$ (R=radius of the earth), and the meridional structure of an equatorial Kelvin wave. Compute the numerical phase speed of your time-dependent solution, and compare it with the theoretically expected phase speed for this wave.