Error between NURBS curve and Discrete Points

SLAPStack

June 7, 2025

The problem: Given an vehicle's hypothetical path and the real path taken, what is the error and how do we define that?
The approach: We define the error between a vehicle's hypothetical path and the real path taken to be the maximum distance between the hypothetical and real path. That is, we consider the worst-case deviation between the two paths over the course of the vehicle's motion.
Mathematically: Given a NURBS curve (the hypothetical path)

$$\mathbf{C}(t) : [0, 1] \to \mathbb{R}^2$$

(e.g., a circular arc), and a finite set of fixed points (the real path)

$$\{\mathbf{P}_1, \ldots, \mathbf{P}_N\} \subset \mathbb{R}^2,$$

we seek to find the maximum distance from any point $\mathbf{P}_j$ to the curve. That is,

$$\max_{j=1,\ldots,N} \min_{t\in[0,1]} \|\mathbf{C}(t) - \mathbf{P}_j\| \,.$$

It is important to note that $\mathbf{C}(t)$ is a rational polynomial (as per the definition of a NURBS curve), so this hints at the idea of using numerical techniques later on.

We provide an outline for a general idea and practical approach to solve our problem:

The General Idea

Treating $\mathbf{C}(t)$ and $\mathbf{P}$ as vectors, we minimize the squared Euclidean distance between the point $\mathbf{P}$ and the curve:

$$f(t) = \|\mathbf{C}(t) - \mathbf{P}\|^2 = (\mathbf{C}(t) - \mathbf{P}) \cdot (\mathbf{C}(t) - \mathbf{P})$$

We differentiate $f(t)$ with respect to t:

$$f'(t) = 2\,(\mathbf{C}(t) - \mathbf{P}) \cdot \mathbf{C}'(t)$$

Set $f'(t) = 0$ to find candidate minima:

$$(\mathbf{C}(t) - \mathbf{P}) \cdot \mathbf{C}'(t) = 0$$

This condition also means that the vector from the point $\mathbf{P}$ to the curve is orthogonal to the curve's tangent vector at t (which checks with our intuition).
To find the shortest distance, solve for all $t \in [0, 1]$ such that

$$(\mathbf{C}(t) - \mathbf{P}) \cdot \mathbf{C}'(t) = 0$$

Then evaluate the distance $\|\mathbf{C}(t) - \mathbf{P}\|$ at each solution, as well as at the endpoints $t = 0$ and $t = 1$, and choose the minimum.

The Algorithmic Approach and Brent's Method

In actual implementation, finding $(\mathbf{C}(t) - \mathbf{P}) \cdot \mathbf{C}'(t) = 0$ is costly with direct methods. $\mathbf{C}(t)$ is already a rational polynomial; so then multiplying it by its derivative and then trying to solve for t is mathematically and computationally difficult. So, we utilize Brent's minimization method[1]. Python's `scipy.optimize` package implements this method via `minimize_scalar`. Brent's method is a robust and efficient algorithm for univariate minimization on a bounded interval, combining parabolic interpolation with golden section search.

Our objective function, $f(t) = \|\mathbf{C}(t) - \mathbf{P}\|^2$, satisfies the key requirements:

i. **Univariate**: It depends only on the curve parameter $t \in [0, 1]$.

ii. **Sufficiently unimodal**: While Brent's method assumes unimodality, it is tolerant of mild deviations and generally converges to the global minimum in practice, especially for smooth rational curves like ours.

Additionally, since our end goal is to compute

$$\max_{j=1,\ldots,N} \min_{t\in[0,1]} \|\mathbf{C}(t) - \mathbf{P}_j\|,$$

any inaccuracies in a few local optimizations are unlikely to affect the global maximum. Thus, Brent's method offers a worthwhile tradeoff between accuracy and simplicity for our needs.

Using the `minimize_scalar` method we can iterate through all points $\mathbf{P}_j$ with respect to the function $f(t)$ and select the maximum value of f.

Simplicial Homology Global Optimization

If accuracy is of utmost importance (such as in post-mortem evaluations), Brent's method can be replaced with Simplicial Homology Global Optimization (SHGO). This method is implemented in the `shgo` function within the `scipy.optimize` package. Unlike Brent's method, which assumes

[1]Source: Wikipedia Brent's Method https://tinyurl.com/a3tyjech

unimodality, SHGO performs a global search by evaluating the function over multiple regions of the domain. This increases the likelihood of overcoming local minima and locating the true global minimum. The tradeoff is that this method is far slower than Brent's method.

Further Optimization: Coarse-to-Fine Sampling

While computing

$$\max_{j=1,\ldots,N} \min_{t\in[0,1]} \|\mathbf{C}(t) - \mathbf{P}_j\|$$

over all points $\mathbf{P}_j$ guarantees accuracy, it may be computationally expensive, especially when the number of test points N is large. In our case, however, the set of test points is not arbitrary: we observe $\mathbf{P}_j$ lie on or near a smooth curve (by the vehicle's natural path). This structure can be exploited to reduce computation through a coarse-to-fine sampling strategy.

Coarse-to-Fine Procedure:

i. **Coarse pass:** Sample every k-th point (e.g., $k = 10$) and compute its minimal distance to the curve using Brent's method.

ii. **Candidate selection:** Identify a small number of coarse points with the largest distances.

iii. **Fine pass:** For each selected coarse point, evaluate its neighboring points (e.g., the 2–5 indices on either side), and recompute the minimum distance.

iv. **Final selection:** From the refined set, select the point with the overall maximum minimum distance to the curve.

This strategy can significantly reduce the number of optimization calls, while preserving accuracy.