Econ 600: taught by Prof. Shaowei Ke

Linfeng Li
llinfeng@umich.edu

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Disclaimer

This is a personal note of mine. I will try to follow professor Ke's lecture as close as possible. However, neither is this an official lecture note, nor will Linfeng be responsible for any errors + typos. Nevertheless, corrections and suggestions are always welcomed.

As this lecture note will be maintained on Github, PLEASE:

- Use the "Issues" feature on Github to post suggestions;
- Feel free to fork this repo and send me pull requests.

Paragraphs starting with "Note that ..." are most likely my personal reflections. Please be aware of this.

1 Lecture 1: Logic, Sets and some Real Analysis[1]

1.1 Logic

Definition 1.1. Proposition is a sentence that is either *true* or *false*. It cannot be both true and false.

Note: "true" and "false" may not necessarily be based on any (objective/subjective) factual basis. However, to give a concrete example, contextually correct propositions are usually employed.

Definition 1.2. Logic Connectives: $\wedge$ and $\vee$. Let P and Q be propositions

- Conjunction of P and Q is denoted as $P \wedge Q$;
- Disjunction of P and Q is denoted as $P \vee Q$.
- Negation of P is denoted as: $\neg P$.

P	Q	$P \wedge Q$	$P \vee Q$	$\neg P$
1	1	1	1	0
1	0	0	1	0
0	1	0	1	0
0	0	0	0	1

Table 1: Truth Table for logic connectives

Truth Table is vaguely defined, with each row being a possible "state of the world". On top of this,

Definition 1.3 (Conditionals and Biconditionals)**.** Let P, Q, R be propositions,

1. Conditional of P and Q is $P \implies Q$;
2. Biconditional of P and Q is $P \iff Q$.

P	Q	$P \implies Q$	$P \iff Q$
1	1	1	1
1	0	0	0
0	1	1	0
0	0	1	1

Table 2: Truth Table for Conditionals and Biconditionals

Note that, the two 1's are obtained for free. Conditional of P and Q are trivially true if P is false (thus the conditional is not entered, thereby cannot be disproved?). ⇐Check This.

Additionally, from an external source (← click me!):

> Conditionals are FALSE only when the first condition (if) is true and the second condition (then) is false. All other cases are TRUE.

Definition 1.4. Two propositions are **equivalent** if they have the same truth table, denoted using "≡".

Example 1. Claim: that $P \implies Q$ and $\neg Q \implies \neg P$ are equivalent.

Proof. Refer to table 3: that by definition, the truth table of the two conditionals are the same. □

Note, (it seems that)[a] truth tables are the same if the two "column vectors" denoting the true/false status are the same.

[a]Since "truth table" was not explicitly defined.

Definition 1.5 (Tautology)**.** A proposition whose truth table consists only 1's is called **tautology**.

[1]Relation, Function, Correspondence and Sequences in $\mathbb{R}$

Table 3: Truth Table: equivalence of $P \implies Q$ and $\neg Q \implies \neg P$

P	Q	$P \implies Q$	$\neg Q \implies \neg P$
1	1	1	1
1	0	0	0
0	1	1	1
0	0	1	1

Example 2. Claim: $Q \implies (P \implies Q)$ is a tautology.

Proof. Refer to Table 4 □

Table 4: Truth Table: Tautology

P	Q	$P \implies Q$	$Q \implies (P \implies Q)$
1	1	1	1
1	0	0	1
0	1	1	1
0	0	1	1

Remark 1.6. We introduce the following 4 types of proof:

1. Direct proof: to follow the direction of the statement.
   - **Proposition**: For odd integers x, y, $x + y$ is an even integer.
2. Proof by contrapositive: (restate the proposition and prove the easier direction).
   - **Proposition**: If n^2 is an odd integer (P), then n is an odd integer.

     Proof. Prove instead that: "if n is an even integer, then n^2 is an even integer". □
3. Proof by contradiction: (construct a structure that leads to contradiction between derived conditions and given conditions.).
   - That $\sqrt{2}$ is rational number[2].
4. Proving a "if and only if" statement/proposition to be true: either one of the following 4 are valid strategies:

   (a) $P \implies Q$ and $Q \implies P$;

   (b) $P \implies Q$ and $\neg P \implies \neg Q$;

   (c) $\neg Q \implies \neg P$ and $Q \implies P$;

   (d) $\neg Q \implies \neg P$ and $\neg P \implies \neg Q$.

[2]The set of rational numbers is denoted as Q.

1.2 Sets

Remark 1.7 (Russell's paradox)**.** The barber is a man who shaves all those and only those who do not shave themselves.

In terms of set theory, let $R = \{x : x \notin x\}$, then:

$$R \in R \iff R \notin R$$

which is very problematic.

Definition 1.8 (Sets)**.** There are two definition of sets:

1. (Enumerating all elements)

   A set is a collection of objects, e.g. $\{1, 2, \ldots\}$ [3] or $\{1, 2\}$ [4].

2. (Describing properties to be satisfied by elements in the set)

   If A is a set of all objects that satisfies property P, then we can write

   $$A = \{x : P(x)\}$$

   where the colon means "such that", and $P(x)$ means that x satisfies property P.

Now, we can define the following **sets** using the two definitions of sets:

- (Natural Number) $N = \{1, 2, \ldots\}$;
- (Integer) $Z = \{x : x = n \text{ or } x = -n \text{ or } x = 0, \text{ for some } n \in N\}$;
- (Rational number) $Q = \{x : x = \frac{m}{n}, m, n \in Z\}$.

Definition 1.9 (Set Equality)**.** Two sets A and B are equal if they have the same elements. That is:

$$A = B \text{ if and only if } x \in A \iff x \in B, \forall x$$

Note, that the notion $\forall x$ was used sloppily here. Without loss of generality, it shall better be $\forall x \in A \bigcup B$.

Definition 1.10 (Set Containment)**.** A set A is contained in a set B, denoted by $A \subseteq B$, if $\forall x \in A \implies x \in B$.

As a consequence, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Definition 1.11 (Cardinality (finite case))**.** If a set A has $n \in N$[5] distinct elements, then n is the cardinality of A and we call A a finite set. The **cardinality of** A is denoted by $|A|$.

Definition 1.12 (Empty set $\emptyset$)**.** The empty set is the set with no element.

[3] a countably infinite set.
[4] a finite set.
[5] Natural number.

Definition 1.13 (Power set 2^A)**.** Let A be a set. The **power set of** A is the collection of all subsets of A.

Note that, A is an arbitrary set. It could be finite, in which case 2^A easy to envision; At the other extreme, it could be a uncountable set. Nevertheless, the following equality shall hold:

$$|2^A| = 2^{|A|}$$

Example 3. Let $A = \{1, 3\}$, then $2^A = \big\{\emptyset, \{1\}, \{3\}, \{1, 3\}\big\}$. In terms of notation, note that 1 is an element in A, thus $1 \in A$; yet, $\{1\}$ is a subset of A, thus $\{1\} \subset A$.

Definition 1.14 (Operations on sets: $\bigcup$, $\bigcap$, $\setminus$ and $\cdot^c$.)**.** Let A and B be two sets:

- Union: $A \bigcup B \coloneqq \{x : x \in A \vee x \in B\}$;
- Intersection: $A \bigcap B \coloneqq \{x : x \in A \wedge x \in B\}$;
- A and B is disjoint if $A \bigcup B = \emptyset$;
- Difference of A and B is defined as: $A \setminus B \coloneqq \{x \in A \wedge x \notin B\}$;
- Complements of A: $A^c \coloneqq \{x : x \notin A\}$.

Side note: **Index set** I is a countable set.

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}$$

Definition 1.15 (de Morgan's law)**.**

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} (A_i^c) \text{ and } \left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} (A_i^c)$$

Exercise 1.16. Prove that $(A \bigcup B)^c = A^c \bigcap B^c$.

Proof. Prove mutual containment using element argument. $\square$

Counters reset

1.3 Relation, Function and Correspondence

Definition 1.1 (Ordered pair)**.** For two sets A and B, an ordered pair is (a, b) such that $a \in A$ and $b \in B$.

Definition 1.2 (n-taple)**.** Let there be n sets: $A_1, \dots, A_n$, an n-taple is $(a_1, \dots, a_n)$ such that $a_i \in A_i$, $\forall i = 1, 2, \dots n$.

Definition 1.3 (Cartesian Product)**.** Let $A_1, \dots, A_n$ be non-empty sets. Cartesion product of $A_1, \dots, A_n$ is $A_1 \times \cdots \times A_n$, defined as:

$$\Pi_{i=1}^n A_i = \{(a_1, \dots, a_n) : a_i \in A_i, \forall i = 1, \dots, n\}$$

Definition 1.4 (Relation)**.** A relation from set A to set B is a subset of $A \times B$, denoted by R.

$$aRb \iff (a, b) \in R$$

A relation on A is a subset of $A \times A$.

Definition 1.5. A relation $R \subseteq A \times A$ is said to be:

- *reflective* if aRa $\forall a \in A$. (That is, $(a, a) \in R$, $\forall a \in A$.);
- *complete* if either aRb or bRa, $\forall a, b \in A$;
- *symmetric* if $\forall a, b \in A$, $aRb \implies bRa$;
- *antisymmetric* if $\forall a, b \in A$, aRb and $bRa \implies a = b$.
- *transitive* if $\forall a, b, c \in A$ s.t. aRb and bRc, aRc (is implied).

Table 5: Property of common relations

	$<$	$\leq$	$\in$	$\subseteq$	$\succeq$
reflective	✗	✓	✗	✓	✓
complete	✗	✓	✗	✗	✓
symmetric	✗	✗	✗	✗	✗
antisymmetric	✓	✓	✓	✓	✗
transitive	✓	✓	✗	✓	✓

Note that, $<$ and $\leq$ are defined on $\mathbb{R}$; $\in$ and $\subseteq$ are defined on sets; $\succeq$ is preference relation that represents "weakly prefer".

Also note that, completeness implies reflectiveness.

Definition 1.6 (Equivilance relation)**.** An **equivalence** on set A is a relation E that is *reflective, symmetric and transitive*. It is denoted as $\sim$.

For any $a \in A$, the **equivalence class** of a with respect to $\sim$ is defined to be the set

$$E_\sim(a) = \{b \in A, b \sim a\}$$

Remark: by construction in Definition 1.4, equivalence ($\sim$) is defined as "a relation on A", which is thereby defined in the Cartesian space.

Definition 1.7 (Fucntion: defined using Relation from A to B)**.** A function from set A to set B is a relation f from A to B such that:

(i) $\forall a \in A$, $\exists b \in B$ such that $(a, b) \in f$, i.e. afb

(ii) $\forall a \in A$, if $(a, b) \in f$ and $(a, c) \in f$ for some $b, c \in B$, then $b = c$.

Note that, alternatively, the two conditions could be written in short as:

(iii) $\forall a \in A$, $\exists! b \in B$ such that $(a, b) \in f$, i.e. afb

Convention for f: If $(a, b) \in f$, we write $f(a) = b$. And, f could be interpreted as a "mapping": "$f : A \to B$".

Definition 1.8 (Domain and Rnage)**.** If f is a function from A to B, then A is called the **domain** of f and B is the **codomain** of f. The **range** of f is the set:

$$Ran(f) = \{b \in B : \exists a \in A \text{ s.t. } f(a) = b\}.$$

Definition 1.9 (Propoteries of functions)**.** Let f be a function, then:

(i) f is **surjective** if $Ran(f) = B$; onto

(ii) f is **injective** if $a_1 \neq a_2 \in A \implies f(a_1) \neq f(a_2)$; 1-to-1

(iii) f is bijective if f is subjective and injective.

Side note: a *indicator function* is defined as following: for A being a set and $S \subseteq A$,

$$\mathcal{X}_S(a) = \begin{cases} 1 & \text{if } a \in S \\ 0 & \text{otherwise} \end{cases}$$

Definition 1.10 (Image and Preimage)**.** For $f : A \to B$ and $C \subseteq A$, the **image** of C under f is

$$f(C) = \{b \in B : \exists a \in C \text{ s.t. } f(a) = b\}$$

The **preimage** of $D \subseteq B$ is

$$f^{-1}(D) = \{a \in A : f(a) \in D\}$$

Exercise. Prove that

1. $f^{-1}(f(A)) = A$, and
2. $f(f^{-1}(B)) = B$ if and only if f is subjective.

Proposition 1.11. Given $f : A \to B$, then $f^{-1} : B \to A$ is a function if and only if f is bijective.

Definition 1.12 (Sequence). A sequence is a function $f : N \to A$, denoted by $\{a_1, a_2, \ldots\} = \{a_i\}_{i=1}^{\infty}$[6] i.e. the set of all sequence is the following set:

$$A^{\infty} = A \times A \times \cdots$$

Definition 1.13 (Cardinality, for (infinite) sequences). Two sets A, B have the same cardinality if $\exists$ a bijective function $f : A \to B$.

Then, $|A| \geq |B|$ if there exists an injective function $f : B \to A$. (Example: $|Z| \geq |N|$ by using identify mapping from N to Z; $|N| \geq |Z|$ by enumerating elements in Z using N. Thus, $|Z| = |N|$.) Eventually, we have:

$$|\mathbb{R}^2| = |\mathbb{R}| > |Q| = |Z| = |N|$$

Definition 1.14 (Correspondence). $T : A \rightrightarrows B$ is a correspondence such that $T : A \to 2^A \backslash \emptyset$.

1.4 Sequences

Definition 1.1 (Sequence in $\mathbb{R}$). A sequence of real number is a function $a : N \to \mathbb{R}$ s.t. $a(i) = a_i$ is the i-th component of the sequence $\{a_j\}_{j=1}^{\infty}$.

Definition 1.2 (Increasing sequence). A real sequence is increasing if $a_{n+1} \geq a_n$ $\forall n \in N$.

Definition 1.3 (Bounded and Bounded (from) above/below). A real sequence is

- **bounded above** if $\exists \bar{m} \in \mathbb{R}$ s.t. $a_n \leq \bar{m}$ $\forall n \in N$.
- **bounded below** if $\exists \underline{m} \in \mathbb{R}$ s.t. $a_n \geq \underline{m}$ $\forall n \in N$.
- **bounded** if it is bounded above and bounded below.

Definition 1.4 (Least upper bound). $a \in \mathbb{R}$ is the least upper bound of a sequence $\{a_n\}$ if

(i) a is an upper bound;

(ii) a is the smallest upper bound, i.e. $\nexists b \in \mathbb{R}$ s.t. $b < a$ and b is a upper bound of $\{a_n\}$.

Axiom 1.5 (Axiom of Real Number: completeness axiom). If S is a nonempty set of real numbers that is bounded above, then there exists a least upper bound that is also a real number.

Definition 1.6 (Convergence sequences). A real sequence $\{a_n\}$ converges to the limit $a \in \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n \geq N$

$$|a_n - a| < \varepsilon$$

We write $\lim\limits_{n\to\infty} a_n = a$ or $a_n \to a$.

- If a sequence does not converge, then it diverges. (To $+\infty$ or $-\infty$.)

Theorem 1. A monotone bounded sequence converges.

Proof. Discuss two cases where 1) $\{a_n\}$ is an increasing sequence, and 2) $\{a_n\}$ is a decreasing sequence. Then, proof is completed through using either least upper bound (for increasing sequence) or largest lower bound (for decreasing sequence). $\square$

[6] This is an ordered set.