Econ 600: taught by Prof. Shaowei Ke

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Disclaimer

This is a personal note of mine. I will try to follow professor Ke's lecture as close as possible. However, neither is this an official lecture note, nor will Linfeng be responsible for any errors + typos. Nevertheless, corrections and suggestions are always welcomed.

As this lecture note will be maintained on Github, PLEASE:

- Use the "Issues" feature on Github to post suggestions;
- Feel free to fork this repo and send me pull requests.

Paragraphs starting with "Note that . . . " are most likely my personal reflections. Please be aware of this.

1 Lecture 1: Logic, Sets and some Real Analysis[1]

1.1 Logic

Definition 1.1. Proposition is a sentence that is either *true* or *false*. It cannot be both true and false.

Note: "true" and "false" may not necessarily be based on any (subjective) factual basis. However, to give a concrete example, contextually correct propositions are employed.

Definition 1.2. Logic Connectives: $\wedge$ and $\vee$. Let P and Q be propositions

- Conjunction of P and Q is denoted as $P \wedge Q$;
- Disjunction of P and Q is denoted as $P \vee Q$.
- Negation of P is denoted as: $\neg P$.

P	Q	$P \wedge Q$	$P \vee Q$	$\neg P$
1	1	1	1	0
1	0	0	1	0
0	1	0	1	0
0	0	0	0	1

Table 1: Truth Table for logic connectives

Truth Table is vaguely defined, with each row being a possible "state of the world". On top of this,

Definition 1.3 (Conditionals and Biconditionals)**.** Let P, Q, R be propositions,

1. Conditional of P and Q is $P \implies Q$;
2. Biconditional of P and Q is $P \iff Q$.

P	Q	$P \implies Q$	$P \iff Q$
1	1	1	1
1	0	0	0
0	1	1	0
0	0	1	1

Table 2: Truth Table for Conditionals and Biconditionals

Note that, the two 1's are obtained for free. Conditional of P and Q are trivially true if P is false (thus the conditional is not entered, thereby cannot be disproved?). ⇐Check This.

Additionally, from an external source (← click me!):

> Conditionals are FALSE only when the first condition (if) is true and the second condition (then) is false. All other cases are TRUE.

Definition 1.4. Two propositions are **equivalent** if they have the same truth table, denoted using "≡".

Example 1. Claim: that $P \implies Q$ and $\neg Q \implies \neg P$ are equivalent.

Proof. Refer to table 3: that by definition, the truth table of the two conditionals are the same. □

Note, (it seems that)[a] truth tables are the same if the two "column vectors" denoting the true/false status are the same.

[a]Since "truth table" was not explicitly defined.

[1]Relation, Function, Correspondence and Sequences in $\mathbb{R}$

Table 3: Truth Table: equivalence of $P \implies Q$ and $\neg Q \implies \neg P$

P	Q	$P \implies Q$	$\neg Q \implies \neg P$
1	1	1	1
1	0	0	0
0	1	1	1
0	0	1	1

Definition 1.5 (Tautology)**.** A proposition whose truth table consists only 1's is called **tautology**.

Example 2. Claim: $Q \implies (P \implies Q)$ is a tautology.

Proof. Refer to Table 4 □

Table 4: Truth Table: Tautology

P	Q	$P \implies Q$	$Q \implies (P \implies Q)$
1	1	1	1
1	0	0	1
0	1	1	1
0	0	1	1

Remark 1.6. We introduce the following 4 types of proof:

1. Direct proof: to follow the direction of the statement.
   - **Proposition**: For odd integers x, y, $x + y$ is an even integer.
2. Proof by contrapositive: (restate the proposition and prove the easier direction).
   - **Proposition**: If n^2 is an odd integer (P), then n is an odd integer.

     Proof. Prove instead that: "if n is an even integer, then n^2 is an even integer". □
3. Proof by contradiction: (construct a structure that leads to contradiction between derived conditions and given conditions.).
   - That $\sqrt{2}$ is rational number[2].
4. Proving a "if and only if" statement/proposition to be true: either one of the following 4 are valid strategies:

   (a) $P \implies Q$ and $Q \implies P$;

   (b) $P \implies Q$ and $\neg P \implies \neg Q$;

   (c) $\neg Q \implies \neg P$ and $Q \implies P$;

   (d) $\neg Q \implies \neg P$ and $\neg P \implies \neg Q$.

1.2 Sets

Remark 1.7 (Russell's paradox)**.** The barber is a man who shaves all those and only those who do not shave themselves.

In terms of set theory, let $R = \{x : x \notin x\}$, then:

$$R \in R \iff R \notin R$$

which is very problematic.

Definition 1.8 (Sets)**.** There are two definition of sets:

1. (Enumerating all elements)

   A set is a collection of objects, e.g. $\{1, 2, \ldots\}$ [3] or $\{1, 2\}$ [4].

[2]The set of rational numbers is denoted as Q.
[3]a countably infinite set.
[4]a finite set.

2. (Describing properties to be satisfied by elements in the set)

   If A is a set of all objects that satisfies property P, then we can write

$$A = \{x : P(x)\}$$

   where the colon means "such that", and $P(x)$ means that x satisfies property P.

Now, we can define the following **sets** using the two definitions of sets:

- (Natural Number) $N = \{1, 2, \ldots\}$;
- (Integer) $Z = \{x : x = n \text{ or } x = -n \text{ or } x = 0, \text{ for some } n \in N\}$;
- (Rational number) $Q = \{x : x = \frac{m}{n}, m, n \in Z\}$.

Definition 1.9 (Set Equality)**.** Two sets A and B are equal if they have the same elements. That is:

$$A = B \text{ if and only if } x \in A \iff x \in B, \forall x$$

Note, that the notion $\forall x$ was used sloppily here. Without loss of generality, it shall better be $\forall x \in A \bigcup B$.

Definition 1.10 (Set Containment)**.** A set A is contained in a set B, denoted by $A \subseteq B$, if $\forall x \in A \implies x \in B$.

As a consequence, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Definition 1.11 (Cardinality (finite case))**.** If a set A has $n \in N$[5] distinct elements, then n is the cardinality of A and we call A a finite set. The **cardinality of** A is denoted by $|A|$.

Definition 1.12 (Empty set $\emptyset$)**.** The empty set is the set with no element.

Definition 1.13 (Power set 2^A)**.** Let A be a set. The **power set of** A is the collection of all subsets of A.

Note that, A is an arbitrary set. It could be finite, in which case 2^A easy to envision; At the other extreme, it could be a uncountable set. Nevertheless, the following equality shall hold:

$$|2^A| = 2^{|A|}$$

Example 3. Let $A = \{1, 3\}$, then $2^A = \big\{\emptyset, \{1\}, \{3\}, \{1, 3\}\big\}$. In terms of notation, note that 1 is an element in A, thus $1 \in A$; yet, $\{1\}$ is a subset of A, thus $\{1\} \subset A$.

[5] Natural number.

Definition 1.14 (Operations on sets: $\bigcup$, $\bigcap$, $\setminus$ and $\cdot^c$.)**.** Let A and B be two sets:

- Union: $A \bigcup B \coloneqq \{x : x \in A \vee x \in B\}$;
- Intersection: $A \bigcap B \coloneqq \{x : x \in A \wedge x \in B\}$;
- A and B is disjoint if $A \bigcup B = \emptyset$;
- Difference of A and B is defined as: $A \setminus B \coloneqq \{x \in A \wedge x \notin B\}$;
- Complements of A: $A^c \coloneqq \{x : x \notin A\}$.

Side note: **Index set** I is a countable set.

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}$$

Definition 1.15 (de Morgan's law)**.**

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} (A_i^c) \text{ and } \left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} (A_i^c)$$

Exercise 1.16. Prove that $(A \bigcup B)^c = A^c \bigcap B^c$.

Proof. Prove mutual containment using element argument. □

Counters reset

1.3 Relation, Function and Correspondence

Definition 1.1 (Ordered pair)**.** For two sets A and B, an ordered pair is (a, b) such that $a \in A$ and $b \in B$.

Definition 1.2 (n-taple)**.** Let there be n sets: $A_1, \ldots, A_n$, an n-taple is $(a_1, \ldots, a_n)$ such that $a_i \in A_i$, $\forall i = 1, 2, \ldots n$.

Definition 1.3 (Cartesian Product)**.** Let $A_1, \ldots, A_n$ be non-empty sets. Cartesion product of $A_1, \ldots, A_n$ is $A_1 \times \cdots \times A_n$, defined as:

$$\Pi_{i=1}^n A_i = \{(a_1, \ldots, a_n) : a_i \in A_i, \forall i = 1, \ldots, n\}$$

Definition 1.4 (Relation)**.** A relation from set A to set B is a subset of $A \times B$, denoted by R.

$$aRb \iff (a, b) \in R$$

A relation on A is a subset of $A \times A$.

Definition 1.5. A relation $R \subseteq A \times A$ is said to be:

- *reflective* if $aRa\ \forall a \in A$. (That is, $(a, a) \in R$, $\forall a \in A$.);
- *complete* if either aRb or bRa, $\forall a, b \in A$;
- *symmetric* if $\forall a, b \in A$, $aRb \implies bRa$;
- *antisymmetric* if $\forall a, b \in A$, aRb and $bRa \implies a = b$.
- *transitive* if $\forall a, b, c \in A$ s.t. aRb and bRc, aRc (is implied).

Table 5: Property of common relations

	$<$	$\leq$	$\in$	$\subseteq$	$\succeq$
reflective	✗	✓	✗	✓	✓
complete	✗	✓	✗	✗	✓
symmetric	✗	✗	✗	✗	✗
antisymmetric	✓	✓	✓	✓	✗
transitive	✓	✓	✗	✓	✓

Note that, $<$ and $\leq$ are defined on $\mathbb{R}$; $\in$ and $\subseteq$ are defined on sets; $\succeq$ is preference relation that represents "weakly prefer".
Also note that, completeness implies reflectiveness.