Econ 600: taught by Prof. Shaowei Ke

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Disclaimer

This is a personal note of mine. I will try to follow professor Ke's lecture as close as possible. However, neither is this an official lecture note, nor will Linfeng be responsible for any errors + typos. Nevertheless, corrections and suggestions are always welcomed.

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- Use the "Issues" feature on Github to post suggestions;
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Paragraphs starting with "Note that ..." are most likely my personal reflections. Please be aware of this.

1 Lecture 1: Logic, Sets and some Real Analysis[1]

1.1 Logic

Definition 1.1. Proposition is a sentence that is either *true* or *false*. It cannot be both true and false.

Note: "true" and "false" may not necessarily be based on any (objective/subjective) factual basis. However, to give a concrete example, contextually correct propositions are usually employed.

Definition 1.2. Logic Connectives: $\wedge$ and $\vee$. Let P and Q be propositions

- Conjunction of P and Q is denoted as $P \wedge Q$;
- Disjunction of P and Q is denoted as $P \vee Q$.
- Negation of P is denoted as: $\neg P$.

P	Q	$P \wedge Q$	$P \vee Q$	$\neg P$
1	1	1	1	0
1	0	0	1	0
0	1	0	1	0
0	0	0	0	1

Table 1: Truth Table for logic connectives

Truth Table is vaguely defined, with each row being a possible "state of the world". On top of this,

Definition 1.3 (Conditionals and Biconditionals)**.** Let P, Q, R be propositions,

1. Conditional of P and Q is $P \implies Q$;
2. Biconditional of P and Q is $P \iff Q$.

P	Q	$P \implies Q$	$P \iff Q$
1	1	1	1
1	0	0	0
0	1	1	0
0	0	1	1

Table 2: Truth Table for Conditionals and Biconditionals

Note that, the two 1's are obtained for free. Conditional of P and Q are trivially true if P is false (thus the conditional is not entered, thereby cannot be disproved?). ⇐Check This.

Additionally, from an external source (← click me!):

> Conditionals are FALSE only when the first condition (if) is true and the second condition (then) is false. All other cases are TRUE.

Definition 1.4. Two propositions are **equivalent** if they have the same truth table, denoted using "≡".

Example 1. Claim: that $P \implies Q$ and $\neg Q \implies \neg P$ are equivalent.

Proof. Refer to table 3: that by definition, the truth table of the two conditionals are the same. □

Note, (it seems that)[a] truth tables are the same if the two "column vectors" denoting the true/false status are the same.

[a]Since "truth table" was not explicitly defined.

Definition 1.5 (Tautology)**.** A proposition whose truth table consists only 1's is called **tautology**.

[1]Relation, Function, Correspondence and Sequences in $\mathbb{R}$

Table 3: Truth Table: equivalence of $P \implies Q$ and $\neg Q \implies \neg P$

P	Q	$P \implies Q$	$\neg Q \implies \neg P$
1	1	1	1
1	0	0	0
0	1	1	1
0	0	1	1

Example 2. Claim: $Q \implies (P \implies Q)$ is a tautology.

Proof. Refer to Table 4 □

Table 4: Truth Table: Tautology

P	Q	$P \implies Q$	$Q \implies (P \implies Q)$
1	1	1	1
1	0	0	1
0	1	1	1
0	0	1	1

Remark 1.6. We introduce the following 4 types of proof:

1. Direct proof: to follow the direction of the statement.
   - **Proposition**: For odd integers x, y, $x + y$ is an even integer.
2. Proof by contrapositive: (restate the proposition and prove the easier direction).
   - **Proposition**: If n^2 is an odd integer (P), then n is an odd integer.

     Proof. Prove instead that: "if n is an even integer, then n^2 is an even integer". □
3. Proof by contradiction: (construct a structure that leads to contradiction between derived conditions and given conditions.).
   - That $\sqrt{2}$ is rational number[2].
4. Proving a "if and only if" statement/proposition to be true: either one of the following 4 are valid strategies:
   (a) $P \implies Q$ and $Q \implies P$;
   (b) $P \implies Q$ and $\neg P \implies \neg Q$;
   (c) $\neg Q \implies \neg P$ and $Q \implies P$;
   (d) $\neg Q \implies \neg P$ and $\neg P \implies \neg Q$.

[2]The set of rational numbers is denoted as Q.

1.2 Sets

Remark 1.7 (Russell's paradox)**.** The barber is a man who shaves all those and only those who do not shave themselves.

In terms of set theory, let $R = \{x : x \notin x\}$, then:

$$R \in R \iff R \notin R$$

which is very problematic.

Definition 1.8 (Sets)**.** There are two definition of sets:

1. (Enumerating all elements)
   A set is a collection of objects, e.g. $\{1, 2, \dots\}$ [3] or $\{1, 2\}$ [4].

2. (Describing properties to be satisfied by elements in the set)
   If A is a set of all objects that satisfies property P, then we can write

   $$A = \{x : P(x)\}$$

   where the colon means "such that", and $P(x)$ means that x satisfies property P.

Now, we can define the following **sets** using the two definitions of sets:

- (Natural Number) $N = \{1, 2, \dots\}$;
- (Integer) $Z = \{x : x = n \text{ or } x = -n \text{ or } x = 0, \text{ for some } n \in N\}$;
- (Rational number) $Q = \{x : x = \frac{m}{n}, m, n \in Z\}$.

Definition 1.9 (Set Equality)**.** Two sets A and B are equal if they have the same elements. That is:

$$A = B \text{ if and only if } x \in A \iff x \in B, \forall x$$

Note, that the notion $\forall x$ was used sloppily here. Without loss of generality, it shall better be $\forall x \in A \bigcup B$.

Definition 1.10 (Set Containment)**.** A set A is contained in a set B, denoted by $A \subseteq B$, if $\forall x \in A \implies x \in B$.

As a consequence, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Definition 1.11 (Cardinality (finite case))**.** If a set A has $n \in N$[5] distinct elements, then n is the cardinality of A and we call A a finite set. The **cardinality of** A is denoted by $|A|$.

Definition 1.12 (Empty set $\emptyset$)**.** The empty set is the set with no element.

[3] a countably infinite set.
[4] a finite set.
[5] Natural number.

Definition 1.13 (Power set 2^A)**.** Let A be a set. The **power set of** A is the collection of all subsets of A.

Note that, A is an arbitrary set. It could be finite, in which case 2^A easy to envision; At the other extreme, it could be a uncountable set. Nevertheless, the following equality shall hold:

$$|2^A| = 2^{|A|}$$

Example 3. Let $A = \{1, 3\}$, then $2^A = \big\{\emptyset, \{1\}, \{3\}, \{1, 3\}\big\}$. In terms of notation, note that 1 is an element in A, thus $1 \in A$; yet, $\{1\}$ is a subset of A, thus $\{1\} \subset A$.

Definition 1.14 (Operations on sets: $\bigcup$, $\bigcap$, $\setminus$ and $\cdot^c$.)**.** Let A and B be two sets:

- Union: $A \bigcup B \coloneqq \{x : x \in A \vee x \in B\}$;
- Intersection: $A \bigcap B \coloneqq \{x : x \in A \wedge x \in B\}$;
- A and B is disjoint if $A \bigcup B = \emptyset$;
- Difference of A and B is defined as: $A \setminus B \coloneqq \{x \in A \wedge x \notin B\}$;
- Complements of A: $A^c \coloneqq \{x : x \notin A\}$.

Side note: **Index set** I is a countable set.

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}$$

Definition 1.15 (de Morgan's law)**.**

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} (A_i^c) \text{ and } \left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} (A_i^c)$$

Exercise 1.16. Prove that $(A \bigcup B)^c = A^c \bigcap B^c$.

Proof. Prove mutual containment using element argument. $\square$

Counters reset

1.3 Relation, Function and Correspondence

Definition 1.1 (Ordered pair)**.** For two sets A and B, an ordered pair is (a, b) such that $a \in A$ and $b \in B$.

Definition 1.2 (n-taple)**.** Let there be n sets: $A_1, \dots, A_n$, an n-taple is $(a_1, \dots, a_n)$ such that $a_i \in A_i$, $\forall i = 1, 2, \dots n$.

Definition 1.3 (Cartesian Product)**.** Let $A_1, \dots, A_n$ be non-empty sets. Cartesion product of $A_1, \dots, A_n$ is $A_1 \times \dots \times A_n$, defined as:

$$\Pi_{i=1}^n A_i = \{(a_1, \dots, a_n) : a_i \in A_i, \forall i = 1, \dots, n\}$$

Definition 1.4 (Relation)**.** A relation from set A to set B is a subset of $A \times B$, denoted by R.

$$aRb \iff (a, b) \in R$$

A relation on A is a subset of $A \times A$.

Definition 1.5. A relation $R \subseteq A \times A$ is said to be:

- *reflective* if aRa $\forall a \in A$. (That is, $(a, a) \in R$, $\forall a \in A$.);
- *complete* if either aRb or bRa, $\forall a, b \in A$;
- *symmetric* if $\forall a, b \in A$, $aRb \implies bRa$;
- *antisymmetric* if $\forall a, b \in A$, aRb and $bRa \implies a = b$.
- *transitive* if $\forall a, b, c \in A$ s.t. aRb and bRc, aRc (is implied).

Table 5: Property of common relations

	$<$	$\leq$	$\in$	$\subseteq$	$\succeq$
reflective	✗	✓	✗	✓	✓
complete	✗	✓	✗	✗	✓
symmetric	✗	✗	✗	✗	✗
antisymmetric	✓	✓	✓	✓	✗
transitive	✓	✓	✗	✓	✓

Note that, $<$ and $\leq$ are defined on $\mathbb{R}$; $\in$ and $\subseteq$ are defined on sets; $\succeq$ is preference relation that represents "weakly prefer".

Also note that, completeness implies reflectiveness.

Definition 1.6 (Equivilance relation)**.** An **equivalence** on set A is a relation E that is *reflective, symmetric and transitive.* It is denoted as $\sim$.

For any $a \in A$, the **equivalence class** of a with respect to $\sim$ is defined to be the set

$$E_{\sim}(a) = \{b \in A, b \sim a\}$$

Remark: by construction in Definition 1.4, equivalence ($\sim$) is defined as "a relation on A", which is thereby defined in the Cartesian space.

Definition 1.7 (Fucntion: defined using Relation from A to B)**.** A function from set A to set B is a relation f from A to B such that:

(i) $\forall a \in A$, $\exists b \in B$ such that $(a, b) \in f$, i.e. afb

(ii) $\forall a \in A$, if $(a, b) \in f$ and $(a, c) \in f$ for some $b, c \in B$, then $b = c$.

Note that, alternatively, the two conditions could be written in short as:

(iii) $\forall a \in A$, $\exists! b \in B$ such that $(a, b) \in f$, i.e. afb

Convention for f: If $(a, b) \in f$, we write $f(a) = b$. And, f could be interpreted as a "mapping": "$f : A \to B$".

Definition 1.8 (Domain and Rnage)**.** If f is a function from A to B, then A is called the **domain** of f and B is the **codomain** of f. The **range** of f is the set:

$$Ran(f) = \{b \in B : \exists a \in A \text{ s.t. } f(a) = b\}.$$

Definition 1.9 (Propoteries of functions)**.** Let f be a function, then:

(i) f is **surjective** if $Ran(f) = B$; onto

(ii) f is **injective** if $a_1 \neq a_2 \in A \implies f(a_1) \neq f(a_2)$; 1-to-1

(iii) f is bijective if f is subjective and injective.

Side note: a *indicator function* is defined as following: for A being a set and $S \subseteq A$,

$$\mathcal{X}_S(a) = \begin{cases} 1 & \text{if } a \in S \\ 0 & \text{otherwise} \end{cases}$$

Definition 1.10 (Image and Preimage)**.** For $f : A \to B$ and $C \subseteq A$, the **image** of C under f is

$$f(C) = \{b \in B : \exists a \in C \text{ s.t. } f(a) = b\}$$

The **preimage** of $D \subseteq B$ is

$$f^{-1}(D) = \{a \in A : f(a) \in D\}$$

Exercise. Prove that

1. $f^{-1}(f(A)) = A$, and
2. $f(f^{-1}(B)) = B$ if and only if f is subjective.

Proposition 1.11. Given $f : A \to B$, then $f^{-1} : B \to A$ is a function if and only if f is bijective.

Definition 1.12 (Sequence)**.** A sequence is a function $f: N \to A$, denoted by $\{a_1, a_2, \dots\} = \{a_i\}_{i=1}^{\infty}$[6] i.e. the set of all sequence is the following set:

$$A^{\infty} = A \times A \times \cdots$$

Definition 1.13 (Cardinality, for (infinite) sequences)**.** Two sets A, B have the same cardinality if $\exists$ a bijective function $f: A \to B$.

Then, $|A| \geq |B|$ if there exists an injective function $f: B \to A$. (Example: $|Z| \geq |N|$ by using identify mapping from N to Z; $|N| \geq |Z|$ by enumerating elements in Z using N. Thus, $|Z| = |N|$.) Eventually, we have:

$$|\mathbb{R}^2| = |\mathbb{R}| > |Q| = |Z| = |N|$$

Definition 1.14 (Correspondence)**.** $T: A \rightrightarrows B$ is a correspondence such that $T: A \to 2^A \backslash \emptyset$.

1.4 Sequences

Definition 1.1 (Sequence in $\mathbb{R}$)**.** A sequence of real number is a function $a: N \to \mathbb{R}$ s.t. $a(i) = a_i$ is the i-th component of the sequence $\{a_j\}_{j=1}^{\infty}$.

Definition 1.2 (Increasing sequence)**.** A real sequence is increasing if $a_{n+1} \geq a_n$ $\forall n \in N$.

Definition 1.3 (Bounded and Bounded (from) above/below)**.** A real sequence is

- **bounded above** if $\exists \bar{m} \in \mathbb{R}$ s.t. $a_n \leq \bar{m}$ $\forall n \in N$.
- **bounded below** if $\exists \underline{m} \in \mathbb{R}$ s.t. $a_n \geq \underline{m}$ $\forall n \in N$.
- **bounded** if it is bounded above and bounded below.

Definition 1.4 (Least upper bound)**.** $a \in \mathbb{R}$ is the least upper bound of a sequence $\{a_n\}$ if

(i) a is an upper bound;

(ii) a is the smallest upper bound, i.e. $\nexists b \in \mathbb{R}$ s.t. $b < a$ and b is a upper bound of $\{a_n\}$.

Axiom 1.5 (Axiom of Real Number: completeness axiom)**.** If S is a nonempty set of real numbers that is bounded above, then there exists a least upper bound ~~that is also a real number~~.

Note, that, claiming that the upper bound is in $\mathbb{R}$ is redundant.

Definition 1.6 (Convergence sequences)**.** A real sequence $\{a_n\}$ converges to the limit $a \in \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n \geq N$

$$|a_n - a| < \varepsilon$$

We write $\lim_{n \to \infty} a_n = a$ or $a_n \to a$.

- If a sequence does not converge, then it diverges. (To $+\infty$ or $-\infty$.)

Theorem 1. A monotone bounded sequence converges.

Proof. Discuss two cases where 1) $\{a_n\}$ is an increasing sequence, and 2) $\{a_n\}$ is a decreasing sequence. Then, proof is completed through using either least upper bound (for increasing sequence) or largest lower bound (for decreasing sequence). $\square$

[6] This is an ordered set.

2 Lecture 2: convergence and more

Definition 2.1. A set $S \subset X$ is a linearly ordered set if there is a relation "$\leq$" on X s.t.

$$\leq \text{ is complete, transitive and antisymmetric.}$$

Note that, given the linear ordering, we can define $<$ accordingly. (For arbitrary $a, b \in X$ and $a \leq b$, then we say $a < b$ if $a \leq b$ and $a \neq b$.)

Definition 2.2 (Boundedness for an arbitrary set.)**.** Let X be a linearly ordered set and $S \subset X$, then $a \in X$ is the **supremum** (or *least upper bound)* of X if:

1. a itself is an upper bound of S, i.e.
2. for $b \in X$, $b < a$, then b is not an upper bound of S.

Corolloary: For $a = \sup X$, $\forall \varepsilon > 0$, there exists $x \in S$ s.t. $x > a - \varepsilon$.

Axiom 2.3 (Completeness Axiom)**.** If S is a nonempty set of real numbers that is bounded above, then there exists a least upper bound.

Definition 2.4 (Sequence in $\mathbb{R}$)**.** A sequence of real number is a function $a : N \to \mathbb{R}$ s.t. $a(i) = a_i$ is the i-th component of the sequence $\{a_j\}_{j=1}^{\infty}$.

Remark 2.5. $\{a_n\}$ is bounded if $a(N)$ is bounded.

Note, here N is the set of all natural numbers $\{1, 2, \ldots, \}$. Thus, we hereby define the boundedness of a sequence using the our previous definition of set-boundedness.

Lemma 2.6. A monotone bounded sequence converges.

Definition 2.7 (Subsequence)**.** A subsequence $\{a_{n_i}\}$ of $\{a_n\}$ is a sequence s.t. $1 \leq n_1 \leq n_2 \leq \ldots$. That is:

$\exists$ conversion function $\Phi : N \to N$ s.t. $n_i = \Phi(i)$ and $\Phi(i) < \Phi(j)$ whenever $i < j$. We can also write: $a_{n_i} = a_{\Phi(i)}$.

Lemma 2.8. Every sequence of $\mathbb{R}$ has a monotone subsequence.

Proof. Proof by doodling: try to construct a decreasing sequence first, if failed (cannot identify infinitely many of elements as candidate of the sequence), construct an increasing one.

Formally: let $S = \{i : \text{ if } j > i, \text{ then } a_j < a_i\}$.

- if $|S| = |N|$ (countably infinite) [1], we have found a monotone (decreasing) sequence.
- If $|S| < \infty$, let $\max S = N$, then by construction, $\exists n_1$ s.t. $a_{n_1} \geq a_{N+1}$. Since $n_1 \notin X$, there exists $n_2 > n_1$ s.t. $a_{n_2} \geq a_{n_1} \geq a_N$.

  We can construct an increasing sequence in this fashion.

[1] Writing $|S| = \infty$ is not rigorous enough, since uncountably infinite could also be denoted similarly.

□

Theorem 2.9 (Bolzano-Weierstrass Theorem)**.** A bounded sequence of $\mathbb{R}$ has a convergent subsequence.

Proof. By Lemma 2.8, such bounded sequence of $\mathbb{R}$ has a monotone subsequence, which inebriates the boundedness property.

Thus, by Lemma 2.6, such bounded monotone sequence converges. □

Remark 2.10 (Properties of Limites)**.** For $a_n \to a$ and $b_n \to b$ (two convergent sequences):

(i) $c \cdot a_n \to c \cdot a$, for $c \in \mathbb{R}$;

(ii) $a_n + b_n \to a + b$

(iii) $a_n \cdot b_n \to a \cdot b$

(iv) $\frac{a_n}{b_n} \to \frac{a}{b}$ s.t. $b \neq 0$ and $b_n \neq 0$ $\forall n$.

(v) $\forall n \in N$, if $c \leq a_n$, then $c \leq a$. (Note that we have defined only one linear ordering $\leq$.)

However, $a_n > c$ does not imply $a > c$. (e.g.: $\frac{1}{n} > 0, \forall n$, yet $\frac{1}{n} \to 0 = 0$.)

(vi) $\forall n$, if $b_n \leq a_n$, then $b \leq a$.

Definition 2.11 (Cauchy sequence)**.** $\{a_n\}$ is a Cauchy sequence if $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall m, n \geq N$, $|a_m - a_n| < \varepsilon$.

Note that, since the definition of convergent sequence relies on knowing the limit a, when such limit is not attainable, Cauchy becomes handy.

Theorem 2.12. Every convergent sequence is Cauchy.

Proof. Given $\{a_n\} \to a$, thus $\forall \frac{\varepsilon}{2} > 0$ $\exists N$ s.t. $|a_n - a| < \frac{\varepsilon}{2}$, $\forall n > N$.

Now, for any $m, n \geq N$, we have:

$$\begin{aligned} |a_m - a_n| &= |a_m - a + a - a_n| \\ &\leq |a_m - a| + |a_n - a| < \varepsilon \end{aligned}$$

□

Example : Prove that $a_{n+1} = \frac{a_n + 2a_{n-1}}{3}$ converges for $a_1 = 0$, $a_2 = 1$.

Proof. Step 1 First observe that: a_n is an average of two real numbers that are in $[0, 1]$. Thus, $a_n \in [0, 1]$.

Step 2 Also observe that by rearranging the terms in the equality, we have:

$$\frac{a_{n+1} - a_n}{a_n - a_{n-1}} = -\frac{2}{3}$$

At this point, we check definition of Cauchy sequence by showing that: for arbitrary ε, we can find a N such that $|a_m - a_n| < \varepsilon$. Deriving the functional form of $|a_m - a_n|$ suffices. (We can then use this functional form to find a proper N.)

Without loss of generality, let $m > n$, then:

$$\begin{aligned}
|a_m - a_n| &= |a_n - a_{n+1} + a_{n+1} - \cdots - a_m| \\
&\leq |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + \cdots + |a_{m-1} - a_m| \\
&\leq \left(\frac{2}{3}\right)^{n-1} + \left(\frac{2}{3}\right)^{n} + \cdots + \left(\frac{2}{3}\right)^{m-2} \\
&= \frac{\left(\frac{2}{3}\right)^{n-1}\left(1 - \left(\frac{2}{3}\right)^{m-n+2}\right)}{1 - \frac{2}{3}} \\
&= O(\left(\frac{2}{3}\right)^{n})
\end{aligned}$$

By now, we can easily demonstrates that the definition of Cauchy sequence could be satisfied by choosing a proper N for any given ε. □

Theorem 2.13. Every Cauchy sequence is bounded.

Proof. Let $\{a_n\}$ be an arbitrary Cauchy sequence. Then, for for arbitrary $\varepsilon > 0$, we know that $\exists N_\varepsilon > 0$ such that $\forall m, n > N$, $|a_m - a_n| < \varepsilon$.

Now, to construct an upper bound for $\{a_n\}$, without loss of generality, let $\varepsilon = 1$. Then, we know that there exists $N_1 > 0$ such that $\forall n, m > N_1$, $|a_n - a_m| < 1$. Then, let M_1 denote the bound (either upper or lower). Then, in absolute value, we can define it to be:

$$|M_1| = \max\{|a_1|, \ldots, |a_{N_1}|, |a_{N_1+1}| + 1\}$$

Through more careful, yet unnecessary, discussions, we can derive the exact bound using the absolute value $|M_1|$.

Note that, the bound we found above is only *one of the upper bound*. It is not necessarily the sup nor inf. □

Theorem 2.14. Every Cauchy sequence `in` $\mathbb{R}$ [2] converges.

[2] Note that, for $\{\frac{1}{n}\}$ defined on $(0, 1]$, it does not converge in this space since $0 \notin (0, 1]$.