

# Econ 600: taught by Prof. Shaowei Ke

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August 25, 2015

## Disclaimer

This is a personal note of mine. I will try to follow professor Ke's lecture as close as possible. However, neither is this an official lecture note, nor will Linfeng be responsible for any errors + typos. Nevertheless, corrections and suggestions are always welcomed.

As this lecture note will be maintained on Github, PLEASE:

- Use the “Issues” feature on Github to post suggestions;
- Feel free to fork this repo and send me pull requests.

Paragraphs starting with “Note that ...” are most likely my personal reflections. Please be aware of this.

## 1 Lecture 1: Logic, Sets and some Real Analysis<sup>1</sup>

### 1.1 Logic

**Definition 1.1. Proposition** is a sentence that is either *true* or *false*. It cannot be both true and false.

Note: “true” and “false” may not necessarily be based on any (subjective) factual basis. However, to give a concrete example, contextually correct propositions are employed.

**Definition 1.2.** Logic Connectives:  $\wedge$  and  $\vee$ . Let  $P$  and  $Q$  be propositions

- Conjunction of  $P$  and  $Q$  is denoted as  $P \wedge Q$ ;
- Disjunction of  $P$  and  $Q$  is denoted as  $P \vee Q$ .
- Negation of  $P$  is denoted as:  $\neg P$ .

$P$	$Q$	$P \wedge Q$	$P \vee Q$	$\neg P$
1	1	1	1	0
1	0	0	1	0
0	1	0	1	0
0	0	0	0	1

Table 1: Truth Table for logic connectives

**Truth Table** is vaguely defined, with each row being a possible “state of the world”. On top of this,

**Definition 1.3** (Conditionals and Biconditionals). Let  $P, Q, R$  be propositions,

1. Conditional of  $P$  and  $Q$  is  $P \implies Q$ ;
2. Biconditional of  $P$  and  $Q$  is  $P \iff Q$ .

$P$	$Q$	$P \implies Q$	$P \iff Q$
1	1	1	1
1	0	0	0
0	1	1	0
0	0	1	1

Table 2: Truth Table for Conditionals and Biconditionals

Note that, the two 1's are obtained for free. Conditional of  $P$  and  $Q$  are trivially true if  $P$  is false (thus the conditional is not entered, thereby cannot be disproved?).

Additionally, from [an external source](#) ( $\leftarrow$  click me!):

Conditionals are FALSE only when the first condition (if) is true and the second condition (then) is false. All other cases are TRUE.

$\Leftarrow$ Check This.

**Definition 1.4.** Two propositions are **equivalent** if they have the same truth table, denoted using “ $\equiv$ ”.

**Example 1.** Claim: that  $P \implies Q$  and  $\neg Q \implies \neg P$  are equivalent.

*Proof.* Refer to table 3: that by definition, the truth table of the two conditionals are the same.  $\square$

Note, (it seems that)<sup>a</sup> truth tables are the same if the two “column vectors” denoting the true/false status are the same.

<sup>a</sup>Since “truth table” was not explicitly defined.

**Definition 1.5** (Tautology). A proposition whose truth table consists only 1's is called **tautology**.

<sup>1</sup>Relation, Function, Correspondence and Sequences in  $\mathbb{R}$

Table 3: Truth Table: equivalence of  $P \implies Q$  and  $\neg Q \implies \neg P$

$P$	$Q$	$P \implies Q$	$\neg Q \implies \neg P$
1	1	1	1
1	0	0	0
0	1	1	1
0	0	1	1

**Example 2.** Claim:  $Q \implies (P \implies Q)$  is a tautology.

*Proof.* Refer to Table 4

□

Table 4: Truth Table: Tautology

$P$	$Q$	$P \implies Q$	$Q \implies (P \implies Q)$
1	1	1	1
1	0	0	1
0	1	1	1
0	0	1	1

**Remark 1.6.** We introduce the following 4 types of proof:

1. Direct proof: to follow the direction of the statement.

• **Proposition:** For odd integers  $x, y$ ,  $x + y$  is an even integer.

2. Proof by contrapositive: (restate the proposition and prove the easier direction).

• **Proposition:** If  $n^2$  is an odd integer ( $P$ ), then  $n$  is an odd integer.

*Proof.* Prove instead that: “if  $n$  is an even integer, then  $n^2$  is an even integer”. □

3. Proof by contradiction: (construct a structure that leads to contradiction between derived conditions and given conditions.).

• That  $\sqrt{2}$  is rational number<sup>2</sup>.

4. Proving a “if and only if” statement/proposition to be true: either one of the following 4 are valid strategies:

- (a)  $P \implies Q$  and  $Q \implies P$ ;
- (b)  $P \implies Q$  and  $\neg P \implies \neg Q$ ;
- (c)  $\neg Q \implies \neg P$  and  $Q \implies P$ ;
- (d)  $\neg Q \implies \neg P$  and  $\neg P \implies \neg Q$ .

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<sup>2</sup>The set of rational numbers is denoted as  $\mathbb{Q}$ .

## 1.2 Sets

**Remark 1.7** (Russell's paradox). The barber is a man who shaves all those and only those who do not shave themselves.

In terms of set theory, let  $R = \{x : x \notin x\}$ , then:

$$R \in R \iff R \notin R$$

which is very problematic.

**Definition 1.8** (Sets). There are two definition of sets:

1. (Enumerating all elements)

A set is a collection of objects, e.g.  $\{1, 2, \dots\}$ <sup>3</sup> or  $\{1, 2\}$ <sup>4</sup>.

2. (Describing properties to be satisfied by elements in the set)

If  $A$  is a set of all objects that satisfies property  $P$ , then we can write

$$A = \{x : P(x)\}$$

where the colon means “such that”, and  $P(x)$  means that  $x$  satisfies property  $P$ .

Now, we can define the following **sets** using the two definitions of sets:

- (Natural Number)  $N = \{1, 2, \dots\}$ ;
- (Integer)  $Z = \{x : x = n \text{ or } x = -n \text{ or } x = 0, \text{ for some } n \in N\}$ ;
- (Rational number)  $Q = \{x : x = \frac{m}{n}, m, n \in Z\}$ .

**Definition 1.9** (Set Equality). Two sets  $A$  and  $B$  are equal if they have the same elements. That is:

$$A = B \text{ if and only if } x \in A \iff x \in B, \forall x$$

Note, that the notion  $\forall x$  was used sloppily here. Without loss of generality, it shall better be  $\forall x \in A \cup B$ .

**Definition 1.10** (Set Containment). A set  $A$  is contained in a set  $B$ , denoted by  $A \subseteq B$ , if  $\forall x \in A \implies x \in B$ .

As a consequence,  $A = B$  if and only if  $A \subseteq B$  and  $B \subseteq A$ .

**Definition 1.11** (Cardinality (finite case)). If a set  $A$  has  $n \in \mathbb{N}$ <sup>5</sup> distinct elements, then  $n$  is the cardinality of  $A$  and we call  $A$  a finite set. The **cardinality of**  $A$  is denoted by  $|A|$ .

**Definition 1.12** (Empty set  $\emptyset$ ). The empty set is the set with no element.

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<sup>3</sup>a countably infinite set.

<sup>4</sup>a finite set.

<sup>5</sup>Natural number.

**Definition 1.13** (Power set  $2^A$ ). Let  $A$  be a set. The **power set of  $A$**  is the collection of all subsets of  $A$ .

Note that,  $A$  is an arbitrary set. It could be finite, in which case  $2^A$  easy to envision; At the other extreme, it could be a uncountable set. Nevertheless, the following equality shall hold:

$$|2^A| = 2^{|A|}$$

**Example 3.** Let  $A = \{1, 3\}$ , then  $2^A = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\}$ . In terms of notation, note that 1 is an element in  $A$ , thus  $1 \in A$ ; yet,  $\{1\}$  is a subset of  $A$ , thus  $\{1\} \subset A$ .

**Definition 1.14** (Operations on sets:  $\cup$ ,  $\cap$ ,  $\setminus$  and  $\cdot^c$ ). Let  $A$  and  $B$  be two sets:

- Union:  $A \cup B := \{x : x \in A \vee x \in B\}$ ;
- Intersection:  $A \cap B := \{x : x \in A \wedge x \in B\}$ ;
- $A$  and  $B$  is disjoint if  $A \cap B = \emptyset$ ;
- Difference of  $A$  and  $B$  is defined as:  $A \setminus B := \{x \in A \wedge x \notin B\}$ ;
- Complements of  $A$ :  $A^c := \{x : x \notin A\}$ .

Side note: **Index set**  $I$  is a countable set.

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}$$

**Definition 1.15** (de Morgan's law).

$$\left( \bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} (A_i^c) \quad \text{and} \quad \left( \bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} (A_i^c)$$

**Exercise 1.16.** Prove that  $(A \cup B)^c = A^c \cap B^c$ .

*Proof.* Prove mutual containment using element argument. □

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Counters reset

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### 1.3 Relation, Function and Correspondence

**Definition 1.1** (Ordered pair). For two sets  $A$  and  $B$ , an ordered pair is  $(a, b)$  such that  $a \in A$  and  $b \in B$ .

**Definition 1.2** ( $n$ -tuple). Let there be  $n$  sets:  $A_1, \dots, A_n$ , an  $n$ -tuple is  $(a_1, \dots, a_n)$  such that  $a_i \in A_i, \forall i = 1, 2, \dots, n$ .

**Definition 1.3** (Cartesian Product). Let  $A_1, \dots, A_n$  be non-empty sets. Cartesian product of  $A_1, \dots, A_n$  is  $A_1 \times \dots \times A_n$ , defined as:

$$\prod_{i=1}^n A_i = \{(a_1, \dots, a_n) : a_i \in A_i, \forall i = 1, \dots, n\}$$

**Definition 1.4** (Relation). A relation from set  $A$  to set  $B$  is a subset of  $A \times B$ , denoted by  $R$ .

$$aRb \iff (a, b) \in R$$

A relation on  $A$  is a subset of  $A \times A$ .

**Definition 1.5.** A relation  $R \subseteq A \times A$  is said to be:

- *reflective* if  $aRa \forall a \in A$ . (That is,  $(a, a) \in R, \forall a \in A$ );
- *complete* if either  $aRb$  or  $bRa, \forall a, b \in A$ ;
- *symmetric* if  $\forall a, b \in A, aRb \implies bRa$ ;
- *antisymmetric* if  $\forall a, b \in A, aRb$  and  $bRa \implies a = b$ .
- *transitive* if  $\forall a, b, c \in A$  s.t.  $aRb$  and  $bRc, aRc$  (is implied).

Table 5: Property of common relations

	$<$	$\leq$	$\in$	$\subseteq$	$\succeq$
reflective	$\times$	$\checkmark$	$\times$	$\checkmark$	$\checkmark$
complete	$\times$	$\checkmark$	$\times$	$\times$	$\checkmark$
symmetric	$\times$	$\times$	$\times$	$\times$	$\times$
antisymmetric	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$
transitive	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\checkmark$

Note that,  $<$  and  $\leq$  are defined on  $\mathbb{R}$ ;  $\in$  and  $\subseteq$  are defined on sets;  $\succeq$  is preference relation that represents “weakly prefer”.

Also note that, completeness implies reflectiveness.

**Definition 1.6** (Equivalence relation). An **equivalence** on set  $A$  is a relation  $E$  that is *reflective, symmetric and transitive*. It is denoted as  $\sim$ .

For any  $a \in A$ , the **equivalence class** of  $a$  with respect to  $\sim$  is defined to be the set

$$E_{\sim}(a) = \{b \in A, b \sim a\}$$

Remark: by construction in Definition 1.4, equivalence ( $\sim$ ) is defined as “a relation on  $A$ ”, which is thereby defined in the Cartesian space.

**Definition 1.7** (Function: defined using Relation from  $A$  to  $B$ ). A function from set  $A$  to set  $B$  is a relation  $f$  from  $A$  to  $B$  such that:

- (i)  $\forall a \in A, \exists b \in B$  such that  $(a, b) \in f$ , i.e.  $afb$
- (ii)  $\forall a \in A$ , if  $(a, b) \in f$  and  $(a, c) \in f$  for some  $b, c \in B$ , then  $b = c$ .

Note that, alternatively, the two conditions could be written in short as:

- (iii)  $\forall a \in A, \exists! b \in B$  such that  $(a, b) \in f$ , i.e.  $afb$

**Convention for  $f$ :** If  $(a, b) \in f$ , we write  $f(a) = b$ . And,  $f$  could be interpreted as a “mapping”: “ $f : A \rightarrow B$ ”.

**Definition 1.8** (Domain and Range). If  $f$  is a function from  $A$  to  $B$ , then  $A$  is called the **domain** of  $f$  and  $B$  is the **codomain** of  $f$ . The **range** of  $f$  is the set:

$$\text{Ran}(f) = \{b \in B : \exists a \in A \text{ s.t. } f(a) = b\}.$$

**Definition 1.9** (Properties of functions). Let  $f$  be a function, then:

- (i)  $f$  is **surjective** if  $\text{Ran}(f) = B$ ; onto
- (ii)  $f$  is **injective** if  $a_1 \neq a_2 \in A \implies f(a_1) \neq f(a_2)$ ; 1-to-1
- (iii)  $f$  is bijective if  $f$  is surjective and injective.

Side note: a *indicator function* is defined as following: for  $A$  being a set and  $S \subseteq A$ ,

$$\chi_S(a) = \begin{cases} 1 & \text{if } a \in S \\ 0 & \text{otherwise} \end{cases}$$

**Definition 1.10** (Image and Preimage). For  $f : A \rightarrow B$  and  $C \subseteq A$ , the **image** of  $C$  under  $f$  is

$$f(C) = \{b \in B : \exists a \in C \text{ s.t. } f(a) = b\}$$

The **preimage** of  $D \subseteq B$  is

$$f^{-1}(D) = \{a \in A : f(a) \in D\}$$

**Exercise.** Prove that

1.  $f^{-1}(f(A)) = A$ , and
2.  $f(f^{-1}(B)) = B$  if and only if  $f$  is surjective.

**Proposition 1.11.** Given  $f : A \rightarrow B$ , then  $f^{-1} : B \rightarrow A$  is a function if and only if  $f$  is bijective.

**Definition 1.12** (Sequence). A sequence is a function  $f : N \rightarrow A$ , denoted by  $\{a_1, a_2, \dots\} = \{a_i\}_{i=1}^\infty$ <sup>6</sup> i.e. the set of all sequence is the following set:

$$A^\infty = A \times A \times \dots$$

**Definition 1.13** (Cardinality, for (infinite) sequences). Two sets  $A, B$  have the same cardinality if  $\exists$  a bijective function  $f : A \rightarrow B$ .

Then,  $|A| \geq |B|$  if there exists an injective function  $f : B \rightarrow A$ . (Example:  $|Z| \geq |N|$  by using identify mapping from  $N$  to  $Z$ ;  $|N| \geq |Z|$  by enumerating elements in  $Z$  using  $N$ . Thus,  $|Z| = |N|$ .) Eventually, we have:

$$|\mathbb{R}^2| = |\mathbb{R}| > |Q| = |Z| = |N|$$

**Definition 1.14** (Correspondence).  $T : A \rightrightarrows B$  is a correspondence such that  $T : A \rightarrow 2^A \setminus \emptyset$ .

## 1.4 Sequences

**Definition 1.1** (Sequence in  $\mathbb{R}$ ). A sequence of real number is a function  $a : N \rightarrow \mathbb{R}$  s.t.  $a(i) = a_i$  is the  $i$ -th component of the sequence  $\{a_j\}_{j=1}^\infty$ .

**Definition 1.2** (Increasing sequence). A real sequence is increasing if  $a_{n+1} \geq a_n \forall n \in N$ .

**Definition 1.3** (Bounded and Bounded (from) above/below). A real sequence is

- **bounded above** if  $\exists \bar{m} \in \mathbb{R}$  s.t.  $a_n \leq \bar{m} \forall n \in N$ .
- **bounded below** if  $\exists \underline{m} \in \mathbb{R}$  s.t.  $a_n \geq \underline{m} \forall n \in N$ .
- **bounded** if it is bounded above and bounded below.

**Definition 1.4** (Least upper bound).  $a \in \mathbb{R}$  is the least upper bound of a sequence  $\{a_n\}$  if

- (i)  $a$  is an upper bound;
- (ii)  $a$  is the smallest upper bound, i.e.  $\nexists b \in \mathbb{R}$  s.t.  $b < a$  and  $b$  is a upper bound of  $\{a_n\}$ .

**Axiom 1.5** (Axiom of Real Number: completeness axiom). If  $S$  is a nonempty set of real numbers that is bounded above, then there exists a least upper bound that is also a real number.

**Definition 1.6** (Convergence sequences). A real sequence  $\{a_n\}$  converges to the limit  $a \in \mathbb{R}$  if  $\forall \varepsilon > 0, \exists N$  s.t.  $\forall n \geq N$

$$|a_n - a| < \varepsilon$$

We write  $\lim_{n \rightarrow \infty} a_n = a$  or  $a_n \rightarrow a$ .

- If a sequence does not converge, then it diverges. (To  $+\infty$  or  $-\infty$ .)

**Theorem 1.** A monotone bounded sequence converges.

*Proof.* Discuss two cases where 1)  $\{a_n\}$  is an increasing sequence, and 2)  $\{a_n\}$  is a decreasing sequence. Then, proof is completed through using either least upper bound (for increasing sequence) or largest lower bound (for decreasing sequence).  $\square$

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<sup>6</sup>This is an ordered set.