

Econ 600: taught by Prof. Shaowei Ke

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Disclaimer

This is a personal note of mine. I will try to follow professor Ke's lecture as close as possible. However, neither is this an official lecture note, nor will Linfeng be responsible for any errors + typos. Nevertheless, corrections and suggestions are always welcomed.

As this lecture note will be maintained on Github, PLEASE:

- Use the “Issues” feature on Github to post suggestions;
- Feel free to fork this repo and send me pull requests.

Paragraphs starting with “Note that ...” are most likely my personal reflections. Please be aware of this.

1 Lecture 1: Logic, Sets and some Real Analysis¹

1.1 Logic

Definition 1.1. Proposition is a sentence that is either *true* or *false*. It cannot be both true and false.

Note: “true” and “false” may not necessarily be based on any (subjective) factual basis. However, to give a concrete example, contextually correct propositions are employed.

Definition 1.2. Logic Connectives: \wedge and \vee . Let P and Q be propositions

- Conjunction of P and Q is denoted as $P \wedge Q$;
- Disjunction of P and Q is denoted as $P \vee Q$.
- Negation of P is denoted as: $\neg P$.

P	Q	$P \wedge Q$	$P \vee Q$	$\neg P$
1	1	1	1	0
1	0	0	1	0
0	1	0	1	0
0	0	0	0	1

Table 1: Truth Table for logic connectives

Truth Table is vaguely defined, with each row being a possible “state of the world”. On top of this,

Definition 1.3 (Conditionals and Biconditionals). Let P, Q, R be propositions,

1. Conditional of P and Q is $P \implies Q$;
2. Biconditional of P and Q is $P \iff Q$.

P	Q	$P \implies Q$	$P \iff Q$
1	1	1	1
1	0	0	0
0	1	1	0
0	0	1	1

Table 2: Truth Table for Conditionals and Biconditionals

Note that, the two 1’s are obtained for free. Conditional of P and Q are trivially true if P is false (thus the conditional is not entered, thereby cannot be disproved?).

Additionally, from [an external source](#) (← click me!):

Conditionals are FALSE only when the first condition (if) is true and the second condition (then) is false. All other cases are TRUE.

⇐Check
This.

Definition 1.4. Two propositions are **equivalent** if they have the same truth table, denoted using “ \equiv ”.

Example 1. Claim: that $P \implies Q$ and $\neg Q \implies \neg P$ are equivalent.

Proof. Refer to table 3: that by definition, the truth table of the two conditionals are the same. □

Note, (it seems that)^a truth tables are the same if the two “column vectors” denoting the true/false status are the same.

^aSince “truth table” was not explicitly defined.

¹Relation, Function, Correspondence and Sequences in \mathbb{R}

Table 3: Truth Table: equivalence of $P \implies Q$ and $\neg Q \implies \neg P$

P	Q	$P \implies Q$	$\neg Q \implies \neg P$
1	1	1	1
1	0	0	0
0	1	1	1
0	0	1	1

Definition 1.5 (Tautology). A proposition whose truth table consists only 1's is called **tautology**.

Example 2. Claim: $Q \implies (P \implies Q)$ is a tautology.

Proof. Refer to Table 4

□

Table 4: Truth Table: Tautology

P	Q	$P \implies Q$	$Q \implies (P \implies Q)$
1	1	1	1
1	0	0	1
0	1	1	1
0	0	1	1

Remark 1.6. We introduce the following 4 types of proof:

1. Direct proof: to follow the direction of the statement.

• **Proposition:** For odd integers x, y , $x + y$ is an even integer.

2. Proof by contrapositive: (restate the proposition and prove the easier direction).

• **Proposition:** If n^2 is an odd integer (P), then n is an odd integer.

Proof. Prove instead that: “if n is an even integer, then n^2 is an even integer”.

□

3. Proof by contradiction: (construct a structure that leads to contradiction between derived conditions and given conditions.).

• That $\sqrt{2}$ is rational number².

4. Proving a “if and only if” statement/proposition to be true: either one of the following 4 are valid strategies:

(a) $P \implies Q$ and $Q \implies P$;

(b) $P \implies Q$ and $\neg P \implies \neg Q$;

(c) $\neg Q \implies \neg P$ and $Q \implies P$;

(d) $\neg Q \implies \neg P$ and $\neg P \implies \neg Q$.

1.2 Sets

Remark 1.7 (Russell’s paradox). The barber is a man who shaves all those and only those who do not shave themselves.

In terms of set theory, let $R = \{x : x \notin x\}$, then:

$$R \in R \iff R \notin R$$

which is very problematic.

Definition 1.8 (Sets). There are two definition of sets:

1. (Enumerating all elements)

A set is a collection of objects, e.g. $\{1, 2, \dots\}$ ³ or $\{1, 2\}$ ⁴.

²The set of rational numbers is denoted as Q .

³a countably infinite set.

⁴a finite set.

2. (Describing properties to be satisfied by elements in the set)

If A is a set of all objects that satisfies property P , then we can write

$$A = \{x : P(x)\}$$

where the colon means “such that”, and $P(x)$ means that x satisfies property P .

Now, we can define the following **sets** using the two definitions of sets:

- (Natural Number) $N = \{1, 2, \dots\}$;
- (Integer) $Z = \{x : x = n \text{ or } x = -n \text{ or } x = 0, \text{ for some } n \in N\}$;
- (Rational number) $Q = \{x : x = \frac{m}{n}, m, n \in Z\}$.

Definition 1.9 (Set Equality). Two sets A and B are equal if they have the same elements. That is:

$$A = B \text{ if and only if } x \in A \iff x \in B, \forall x$$

Note, that the notion $\forall x$ was used sloppily here. Without loss of generality, it shall better be $\forall x \in A \cup B$.

Definition 1.10 (Set Containment). A set A is contained in a set B , denoted by $A \subseteq B$, if $\forall x \in A \implies x \in B$.

As a consequence, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Definition 1.11 (Cardinality (finite case)). If a set A has $n \in \mathbb{N}$ ⁵ distinct elements, then n is the cardinality of A and we call A a finite set. The **cardinality of** A is denoted by $|A|$.

Definition 1.12 (Empty set \emptyset). The empty set is the set with no element.

Definition 1.13 (Power set 2^A). Let A be a set. The **power set of** A is the collection of all subsets of A .

Note that, A is an arbitrary set. It could be finite, in which case 2^A easy to envision; At the other extreme, it could be a uncountable set. Nevertheless, the following equality shall hold:

$$|2^A| = 2^{|A|}$$

Example 3. Let $A = \{1, 3\}$, then $2^A = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\}$. In terms of notation, note that 1 is an element in A , thus $1 \in A$; yet, $\{1\}$ is a subset of A , thus $\{1\} \subset A$.

⁵Natural number.

Definition 1.14 (Operations on sets: \cup , \cap , \setminus and \cdot^c). Let A and B be two sets:

- Union: $A \cup B := \{x : x \in A \vee x \in B\}$;
- Intersection: $A \cap B := \{x : x \in A \wedge x \in B\}$;
- A and B is disjoint if $A \cup B = \emptyset$;
- Difference of A and B is defined as: $A \setminus B := \{x \in A \wedge x \notin B\}$;
- Complements of A : $A^c := \{x : x \notin A\}$.

Side note: **Index set** I is a countable set.

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}$$

Definition 1.15 (de Morgan's law).

$$\left(\bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} (A_i^c) \quad \text{and} \quad \left(\bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} (A_i^c)$$

Exercise 1.16. Prove that $(A \cup B)^c = A^c \cap B^c$.

Proof. Prove mutual containment using element argument. □

Counters reset

This is a side note

1.3 Relation, Function and Correspondence

Definition 1.1 (Ordered pair). For two sets A and B , an ordered pair is (a, b) such that $a \in A$ and $b \in B$.

Definition 1.2 (n -tuple). Let there be n sets: A_1, \dots, A_n , an n -tuple is (a_1, \dots, a_n) such that $a_i \in A_i, \forall i = 1, 2, \dots, n$.

Definition 1.3 (Cartesian Product). Let A_1, \dots, A_n be non-empty sets. Cartesian product of A_1, \dots, A_n is $A_1 \times \dots \times A_n$, defined as:

$$\prod_{i=1}^n A_i = \{(a_1, \dots, a_n) : a_i \in A_i, \forall i = 1, \dots, n\}$$

Definition 1.4 (Relation). A relation from set A to set B is a subset of $A \times B$, denoted by R .

$$aRb \iff (a, b) \in R$$

A relation on A is a subset of $A \times A$.

Definition 1.5. A relation $R \subseteq A \times A$ is said to be:

- *reflective* if $aRa \ \forall a \in A$. (That is, $(a, a) \in R, \forall a \in A$.);
- *complete* if either aRb or $bRa, \forall a, b \in A$;
- *symmetric* if $\forall a, b \in A, aRb \implies bRa$;
- *antisymmetric* if $\forall a, b \in A, aRb$ and $bRa \implies a = b$.
- *transitive* if $\forall a, b, c \in A$ s.t. aRb and bRc, aRc (is implied).

Table 5: Property of common relations

	$<$	\leq	\in	\subseteq	\succsim
reflective	\times	\checkmark	\times	\checkmark	\checkmark
complete	\times	\checkmark	\times	\times	\checkmark
symmetric	\times	\times	\times	\times	\times
antisymmetric	\checkmark	\checkmark	\checkmark	\checkmark	\times
transitive	\checkmark	\checkmark	\times	\checkmark	\checkmark

Note that, $<$ and \leq are defined on \mathbb{R} ; \in and \subseteq are defined on sets; \succsim is preference relation that represents “weakly prefer”.

Also note that, completeness implies reflectiveness.

Definition 1.6 (Equivalence relation). An **equivalence** on set A is a relation E that is *reflective, symmetric and transitive*. It is denoted as \sim .

For any $a \in A$, the **equivalence class** of a with respect to \sim is defined to be the set

$$E_{\sim}(a) = \{b \in A, b \sim a\}$$

Remark: by construction in Definition 1.4, equivalence (\sim) is defined as “a relation on A ”, which is thereby defined in the Cartesian space.

Definition 1.7 (Fucntion: defined using Relation from A to B). A function from set A to set B is a relation f from A to B such that:

- $\forall a \in A, \exists b \in B$ such that $(a, b) \in f$, i.e. afb
- $\forall a \in A$, if $(a, b) \in f$ and $(a, c) \in f$ for some $b, c \in B$, then $b = c$.

Note that, alternatively, the two conditions could be written in short as:

- $\forall a \in A, \exists! b \in B$ such that $(a, b) \in f$, i.e. afb

Convention for f : If $(a, b) \in f$, we write $f(a) = b$. And, f could be interpreted as a “mapping”: “ $f : A \rightarrow B$ ”.

Definition 1.8 (Domain and Range). If f is a function from A to B , then A is called the **domain** of f and B is the **codomain** of f . The **range** of f is the set:

$$\text{Ran}(f) = \{b \in B : \exists a \in A \text{ s.t. } f(a) = b\}.$$

Definition 1.9 (Properties of functions). Let f be a function, then:

- (i) f is **surjective** if $\text{Ran}(f) = B$; onto
- (ii) f is **injective** if $a_1 \neq a_2 \in A \implies f(a_1) \neq f(a_2)$; 1-to-1
- (iii) f is bijective if f is surjective and injective.

Side note: a *indicator function* is defined as following: for A being a set and $S \subseteq A$,

$$\chi_S(a) = \begin{cases} 1 & \text{if } a \in S \\ 0 & \text{otherwise} \end{cases}$$

Definition 1.10 (Image and Preimage). For $f : A \rightarrow B$ and $C \subseteq A$, the **image** of C under f is

$$f(C) = \{b \in B : \exists a \in C \text{ s.t. } f(a) = b\}$$

The **preimage** of $D \subseteq B$ is

$$f^{-1}(D) = \{a \in A : f(a) \in D\}$$

Exercise. Prove that

1. $f^{-1}(f(A)) = A$, and
2. $f(f^{-1}(B)) = B$ if and only if f is surjective.

Proposition 1.11. Given $f : A \rightarrow B$, then $f^{-1} : B \rightarrow A$ is a function if and only if f is bijective.

Definition 1.12 (Sequence). A sequence is a function $f : \mathbb{N} \rightarrow A$, denoted by $\{a_1, a_2, \dots\} = \{a_i\}_{i=1}^{\infty}$ ⁶ i.e. the set of all sequence is the following set:

$$A^{\infty} = A \times A \times \dots$$

Definition 1.13 (Cardinality, for (infinite) sequences). Two sets A, B have the same cardinality if \exists a bijective function $f : A \rightarrow B$.

Then, $|A| \geq |B|$ if there exists an injective function $f : B \rightarrow A$. (Example: $|Z| \geq |N|$ by using identify mapping from N to Z ; $|N| \geq |Z|$ by enumerating elements in Z using N . Thus, $|Z| = |N|$.) Eventually, we have:

$$|\mathbb{R}^2| = |\mathbb{R}| > |Q| = |Z| = |N|$$

Definition 1.14 (Correspondence). $T : A \rightrightarrows B$ is a correspondence such that $T : A \rightarrow 2^B \setminus \emptyset$.

⁶This is an ordered set.

1.4 Sequences

Definition 1.1 (Sequence in \mathbb{R}). A sequence of real number is a function $a : N \rightarrow \mathbb{R}$ s.t. $a(i) = a_i$ is the i -th component of the sequence $\{a_j\}_{j=1}^{\infty}$.

Definition 1.2 (Increasing sequence). A real sequence is increasing if $a_{n+1} \geq a_n \forall n \in N$.

Definition 1.3 (Bounded and Bounded (from) above/below). A real sequence is

- **bounded above** if $\exists \bar{m} \in \mathbb{R}$ s.t. $a_n \leq \bar{m} \forall n \in N$.
- **bounded below** if $\exists \underline{m} \in \mathbb{R}$ s.t. $a_n \geq \underline{m} \forall n \in N$.
- **bounded** if it is bounded above and bounded below.