# Econ 600: taught by Prof. Shaowei Ke

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August 28, 2015

## Disclaimer

This is a personal note of mine. I will try to follow professor Ke's lecture as close as possible. However, neither is this an official lecture note, nor will Linfeng be responsible for any errors + typos. Nevertheless, corrections and suggestions are always welcomed.

As this lecture note will be maintained on Github, PLEASE:

- Use the "Issues" feature on Github to post suggestions;
- Feel free to fork this repo and send me pull requests.

Paragraphs starting with "Note that ..." are most likely my personal reflections. Please be aware of this.

## 1 Lecture 1: Logic, Sets and some Real Analysis[1]

### 1.1 Logic

**Definition 1.1. Proposition** is a sentence that is either *true* or *false*. It cannot be both true and false.

Note: "true" and "false" may not necessarily be based on any (objective/subjective) factual basis. However, to give a concrete example, contextually correct propositions are usually employed.

**Definition 1.2.** Logic Connectives: $\wedge$ and $\vee$. Let $P$ and $Q$ be propositions

- Conjunction of $P$ and $Q$ is denoted as $P \wedge Q$;
- Disjunction of $P$ and $Q$ is denoted as $P \vee Q$.
- Negation of $P$ is denoted as: $\neg P$.

| $P$ | $Q$ | $P \wedge Q$ | $P \vee Q$ | $\neg P$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 | 1 |

Table 1: Truth Table for logic connectives

**Truth Table** is vaguely defined, with each row being a possible "state of the world". On top of this,

**Definition 1.3** ( Conditionals and Biconditionals)**.** Let $P, Q, R$ be propositions,

1. Conditional of $P$ and $Q$ is $P \implies Q$;
2. Biconditional of $P$ and $Q$ is $P \iff Q$.

| $P$ | $Q$ | $P \implies Q$ | $P \iff Q$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 |
| 0 | 0 | 1 | 1 |

Table 2: Truth Table for Conditionals and Biconditionals

Note that, the two 1's are obtained for free. Conditional of $P$ and $Q$ are trivially true if $P$ is false (thus the conditional is not entered, thereby cannot be disproved?). ⇐Check This.

Additionally, from an external source (← click me!):

> Conditionals are FALSE only when the first condition (if) is true and the second condition (then) is false. All other cases are TRUE.

**Definition 1.4.** Two propositions are **equivalent** if they have the same truth table, denoted using "≡".

**Example 1.** Claim: that $P \implies Q$ and $\neg Q \implies \neg P$ are equivalent.

*Proof.* Refer to table 3: that by definition, the truth table of the two conditionals are the same. □

Note, (it seems that)[a] truth tables are the same if the two "column vectors" denoting the true/false status are the same.

[a]Since "truth table" was not explicitly defined.

**Definition 1.5** (Tautology)**.** A proposition whose truth table consists only 1's is called **tautology**.

[1]Relation, Function, Correspondence and Sequences in $\mathbb{R}$

Table 3: Truth Table: equivalence of $P \implies Q$ and $\neg Q \implies \neg P$

| $P$ | $Q$ | $P \implies Q$ | $\neg Q \implies \neg P$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 |

**Example 2.** Claim: $Q \implies (P \implies Q)$ is a tautology.

*Proof.* Refer to Table 4 □

Table 4: Truth Table: Tautology

| $P$ | $Q$ | $P \implies Q$ | $Q \implies (P \implies Q)$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 |

**Remark 1.6.** We introduce the following 4 types of proof:

1. Direct proof: to follow the direction of the statement.
   - **Proposition**: For odd integers $x, y$, $x + y$ is an even integer.
2. Proof by contrapositive: (restate the proposition and prove the easier direction).
   - **Proposition**: If $n^2$ is an odd integer ($P$), then $n$ is an odd integer.

     *Proof.* Prove instead that: "if $n$ is an even integer, then $n^2$ is an even integer". □
3. Proof by contradiction: (construct a structure that leads to contradiction between derived conditions and given conditions.).
   - That $\sqrt{2}$ is rational number[2].
4. Proving a "if and only if" statement/proposition to be true: either one of the following 4 are valid strategies:
   (a) $P \implies Q$ and $Q \implies P$;
   (b) $P \implies Q$ and $\neg P \implies \neg Q$;
   (c) $\neg Q \implies \neg P$ and $Q \implies P$;
   (d) $\neg Q \implies \neg P$ and $\neg P \implies \neg Q$.

[2]The set of rational numbers is denoted as $Q$.

## 1.2 Sets

**Remark 1.7** (Russell's paradox)**.** The barber is a man who shaves all those and only those who do not shave themselves.

In terms of set theory, let $R = \{x : x \notin x\}$, then:

$$R \in R \iff R \notin R$$

which is very problematic.

**Definition 1.8** (Sets)**.** There are two definition of sets:

1. (Enumerating all elements)

   A set is a collection of objects, e.g. $\{1, 2, \dots\}$ [3] or $\{1, 2\}$ [4].

2. (Describing properties to be satisfied by elements in the set)

   If $A$ is a set of all objects that satisfies property $P$, then we can write

   $$A = \{x : P(x)\}$$

   where the colon means "such that", and $P(x)$ means that $x$ satisfies property $P$.

Now, we can define the following **sets** using the two definitions of sets:

- (Natural Number) $N = \{1, 2, \dots\}$;
- (Integer) $Z = \{x : x = n \text{ or } x = -n \text{ or } x = 0, \text{ for some } n \in N\}$;
- (Rational number) $Q = \{x : x = \frac{m}{n}, m, n \in Z\}$.

**Definition 1.9** (Set Equality)**.** Two sets $A$ and $B$ are equal if they have the same elements. That is:

$$A = B \text{ if and only if } x \in A \iff x \in B, \forall x$$

Note, that the notion $\forall x$ was used sloppily here. Without loss of generality, it shall better be $\forall x \in A \bigcup B$.

**Definition 1.10** (Set Containment)**.** A set $A$ is contained in a set $B$, denoted by $A \subseteq B$, if $\forall x \in A \implies x \in B$.

As a consequence, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

**Definition 1.11** (Cardinality (finite case))**.** If a set $A$ has $n \in N$[5] distinct elements, then $n$ is the cardinality of $A$ and we call $A$ a finite set. The **cardinality of** $A$ is denoted by $|A|$.

**Definition 1.12** (Empty set $\emptyset$)**.** The empty set is the set with no element.

[3] a countably infinite set.

[4] a finite set.

[5] Natural number.

**Definition 1.13** (Power set $2^A$). Let $A$ be a set. The **power set of** $A$ is the collection of all subsets of $A$.

Note that, $A$ is an arbitrary set. It could be finite, in which case $2^A$ easy to envision; At the other extreme, it could be a uncountable set. Nevertheless, the following equality shall hold:

$$|2^A| = 2^{|A|}$$

**Example 3.** Let $A = \{1, 3\}$, then $2^A = \big\{\emptyset, \{1\}, \{3\}, \{1, 3\}\big\}$. In terms of notation, note that 1 is an element in $A$, thus $1 \in A$; yet, $\{1\}$ is a subset of $A$, thus $\{1\} \subset A$.

**Definition 1.14** (Operations on sets: $\bigcup$, $\bigcap$, $\setminus$ and $\cdot^c$.). Let $A$ and $B$ be two sets:

- Union: $A \bigcup B \coloneqq \{x : x \in A \vee x \in B\}$;
- Intersection: $A \bigcap B \coloneqq \{x : x \in A \wedge x \in B\}$;
- $A$ and $B$ is disjoint if $A \bigcup B = \emptyset$;
- Difference of $A$ and $B$ is defined as: $A \setminus B \coloneqq \{x \in A \wedge x \notin B\}$;
- Complements of $A$: $A^c \coloneqq \{x : x \notin A\}$.

Side note: **Index set** $I$ is a countable set.

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}$$

**Definition 1.15** (de Morgan's law).

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} (A_i^c) \text{ and } \left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} (A_i^c)$$

**Exercise 1.16.** Prove that $(A \bigcup B)^c = A^c \bigcap B^c$.

*Proof.* Prove mutual containment using element argument. $\square$

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Counters reset

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## 1.3 Relation, Function and Correspondence

**Definition 1.1** (Ordered pair)**.** For two sets $A$ and $B$, an ordered pair is $(a, b)$ such that $a \in A$ and $b \in B$.

**Definition 1.2** ($n$-taple)**.** Let there be $n$ sets: $A_1, \dots, A_n$, an $n$-taple is $(a_1, \dots, a_n)$ such that $a_i \in A_i$, $\forall i = 1, 2, \dots n$.

**Definition 1.3** (Cartesian Product)**.** Let $A_1, \dots, A_n$ be non-empty sets. Cartesion product of $A_1, \dots, A_n$ is $A_1 \times \cdots \times A_n$, defined as:

$$\Pi_{i=1}^{n} A_i = \{(a_1, \dots, a_n) : a_i \in A_i, \forall i = 1, \dots, n\}$$

**Definition 1.4** (Relation)**.** A relation from set $A$ to set $B$ is a subset of $A \times B$, denoted by $R$.

$$aRb \iff (a, b) \in R$$

A relation on $A$ is a subset of $A \times A$.

**Definition 1.5.** A relation $R \subseteq A \times A$ is said to be:

- *reflective* if $aRa$ $\forall a \in A$. (That is, $(a, a) \in R$, $\forall a \in A$.);
- *complete* if either $aRb$ or $bRa$, $\forall a, b \in A$;
- *symmetric* if $\forall a, b \in A$, $aRb \implies bRa$;
- *antisymmetric* if $\forall a, b \in A$, $aRb$ and $bRa \implies a = b$.
- *transitive* if $\forall a, b, c \in A$ s.t. $aRb$ and $bRc$, $aRc$ (is implied).

Table 5: Property of common relations

| | $<$ | $\leq$ | $\in$ | $\subseteq$ | $\succeq$ |
|---|---|---|---|---|---|
| reflective | ✗ | ✓ | ✗ | ✓ | ✓ |
| complete | ✗ | ✓ | ✗ | ✗ | ✓ |
| symmetric | ✗ | ✗ | ✗ | ✗ | ✗ |
| antisymmetric | ✓ | ✓ | ✓ | ✓ | ✗ |
| transitive | ✓ | ✓ | ✗ | ✓ | ✓ |

Note that, $<$ and $\leq$ are defined on $\mathbb{R}$; $\in$ and $\subseteq$ are defined on sets; $\succeq$ is preference relation that represents "weakly prefer".

Also note that, completeness implies reflectiveness.

**Definition 1.6** (Equivilance relation)**.** An **equivalence** on set $A$ is a relation $E$ that is *reflective, symmetric and transitive.* It is denoted as $\sim$.

For any $a \in A$, the **equivalence class** of $a$ with respect to $\sim$ is defined to be the set

$$E_{\sim}(a) = \{b \in A, b \sim a\}$$

Remark: by construction in Definition 1.4, equivalence ($\sim$) is defined as "a relation on $A$", which is thereby defined in the Cartesian space.

**Definition 1.7** (Fucntion: defined using Relation from $A$ to $B$)**.** A function from set $A$ to set $B$ is a relation $f$ from $A$ to $B$ such that:

(i) $\forall a \in A$, $\exists b \in B$ such that $(a, b) \in f$, i.e. $afb$

(ii) $\forall a \in A$, if $(a, b) \in f$ and $(a, c) \in f$ for some $b, c \in B$, then $b = c$.

Note that, alternatively, the two conditions could be written in short as:

(iii) $\forall a \in A$, $\exists! b \in B$ such that $(a, b) \in f$, i.e. $afb$

**Convention for $f$:** If $(a, b) \in f$, we write $f(a) = b$. And, $f$ could be interpreted as a "mapping": "$f : A \to B$".

**Definition 1.8** (Domain and Rnage)**.** If $f$ is a function from $A$ to $B$, then $A$ is called the **domain** of $f$ and $B$ is the **codomain** of $f$. The **range** of $f$ is the set:

$$Ran(f) = \{b \in B : \exists a \in A \text{ s.t. } f(a) = b\}.$$

**Definition 1.9** (Propoteries of functions)**.** Let $f$ be a function, then:

(i) $f$ is **surjective** if $Ran(f) = B$; onto

(ii) $f$ is **injective** if $a_1 \neq a_2 \in A \implies f(a_1) \neq f(a_2)$; 1-to-1

(iii) $f$ is bijective if $f$ is subjective and injective.

Side note: a *indicator function* is defined as following: for $A$ being a set and $S \subseteq A$,

$$\mathcal{X}_S(a) = \begin{cases} 1 & \text{if } a \in S \\ 0 & \text{otherwise} \end{cases}$$

**Definition 1.10** (Image and Preimage)**.** For $f : A \to B$ and $C \subseteq A$, the **image** of $C$ under $f$ is

$$f(C) = \{b \in B : \exists a \in C \text{ s.t. } f(a) = b\}$$

The **preimage** of $D \subseteq B$ is

$$f^{-1}(D) = \{a \in A : f(a) \in D\}$$

**Exercise.** Prove that

1. $f^{-1}(f(A)) = A$, and
2. $f(f^{-1}(B)) = B$ if and only if $f$ is subjective.

**Proposition 1.11.** Given $f : A \to B$, then $f^{-1} : B \to A$ is a function if and only if $f$ is bijective.

**Definition 1.12** (Sequence)**.** A sequence is a function $f : N \to A$, denoted by $\{a_1, a_2, \dots\} = \{a_i\}_{i=1}^{\infty}$[6] i.e. the set of all sequence is the following set:

$$A^\infty = A \times A \times \cdots$$

**Definition 1.13** (Cardinality, for (infinite) sequences)**.** Two sets $A, B$ have the same cardinality if $\exists$ a bijective function $f : A \to B$.

Then, $|A| \geq |B|$ if there exists an injective function $f : B \to A$. (Example: $|Z| \geq |N|$ by using identify mapping from $N$ to $Z$; $|N| \geq |Z|$ by enumerating elements in $Z$ using $N$. Thus, $|Z| = |N|$.) Eventually, we have:

$$|\mathbb{R}^2| = |\mathbb{R}| > |Q| = |Z| = |N|$$

**Definition 1.14** (Correspondence)**.** $T : A \rightrightarrows B$ is a correspondence such that $T : A \to 2^A \backslash \emptyset$.

## 1.4 Sequences

**Definition 1.1** (Sequence in $\mathbb{R}$)**.** A sequence of real number is a function $a : N \to \mathbb{R}$ s.t. $a(i) = a_i$ is the $i$-th component of the sequence $\{a_j\}_{j=1}^{\infty}$.

**Definition 1.2** (Increasing sequence)**.** A real sequence is increasing if $a_{n+1} \geq a_n$ $\forall n \in N$.

**Definition 1.3** (Bounded and Bounded (from) above/below)**.** A real sequence is

- **bounded above** if $\exists \bar{m} \in \mathbb{R}$ s.t. $a_n \leq \bar{m}$ $\forall n \in N$.
- **bounded below** if $\exists \underline{m} \in \mathbb{R}$ s.t. $a_n \geq \underline{m}$ $\forall n \in N$.
- **bounded** if it is bounded above and bounded below.

**Definition 1.4** (Least upper bound)**.** $a \in \mathbb{R}$ is the least upper bound of a sequence $\{a_n\}$ if

(i) $a$ is an upper bound;

(ii) $a$ is the smallest upper bound, i.e. $\nexists b \in \mathbb{R}$ s.t. $b < a$ and $b$ is a upper bound of $\{a_n\}$.

**Axiom 1.5** (Axiom of Real Number: completeness axiom)**.** If $S$ is a nonempty set of real numbers that is bounded above, then there exists a least upper bound ~~that is also a real number~~.

Note, that, claiming that the upper bound is in $\mathbb{R}$ is redundant.

**Definition 1.6** (Convergence sequences)**.** A real sequence $\{a_n\}$ converges to the limit $a \in \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n \geq N$

$$|a_n - a| < \varepsilon$$

We write $\lim\limits_{n\to\infty} a_n = a$ or $a_n \to a$.

- If a sequence does not converge, then it diverges. (To $+\infty$ or $-\infty$.)

**Theorem 1.** A monotone bounded sequence converges.

*Proof.* Discuss two cases where 1) $\{a_n\}$ is an increasing sequence, and 2) $\{a_n\}$ is a decreasing sequence. Then, proof is completed through using either least upper bound (for increasing sequence) or largest lower bound (for decreasing sequence). □

[6] This is an ordered set.

# 2 Lecture 2: convergence and more

**Definition 2.1.** A set $S \subset X$ is a linearly ordered set if there is a relation "$\leq$" on $X$ s.t.

$$\leq \text{ is complete, transitive and antisymmetric.}$$

Note that, given the linear ordering, we can define $<$ accordingly. (For arbitrary $a, b \in X$ and $a \leq b$, then we say $a < b$ if $a \leq b$ and $a \neq b$.)

**Definition 2.2** (Boundedness for an arbitrary set.)**.** Let $X$ be a linearly ordered set and $S \subset X$, then $a \in X$ is the **supremum** (or *least upper bound)* of $X$ if:

1. $a$ itself is an upper bound of $S$, i.e.
2. for $b \in X$, $b < a$, then $b$ is not an upper bound of $S$.

**Corolloary:** For $a = \sup X$, $\forall \varepsilon > 0$, there exists $x \in S$ s.t. $x > a - \varepsilon$.

**Axiom 2.3** (Completeness Axiom)**.** If $S$ is a nonempty set of real numbers that is bounded above, then there exists a least upper bound.

**Definition 2.4** (Sequence in $\mathbb{R}$)**.** A sequence of real number is a function $a : N \to \mathbb{R}$ s.t. $a(i) = a_i$ is the $i$-th component of the sequence $\{a_j\}_{j=1}^{\infty}$.

**Remark 2.5.** $\{a_n\}$ is bounded if $a(N)$ is bounded.

Note, here $N$ is the set of all natural numbers $\{1, 2, \dots,\}$. Thus, we hereby define the boundedness of a sequence using the our previous definition of set-boundedness.

**Lemma 2.6.** A monotone bounded sequence converges.

**Definition 2.7** (Subsequence)**.** A subsequence $\{a_{n_i}\}$ of $\{a_n\}$ is a sequence s.t. $1 \leq n_1 \leq n_2 \leq \dots$. That is:

$\exists$ conversion function $\Phi : N \to N$ s.t. $n_i = \Phi(i)$ and $\Phi(i) < \Phi(j)$ whenever $i < j$. We can also write: $a_{n_i} = a_{\Phi(i)}$.

**Lemma 2.8.** Every sequence of $\mathbb{R}$ has a monotone subsequence.

*Proof.* Proof by doodling: try to construct a decreasing sequence first, if failed (cannot identify infinitely many of elements as candidate of the sequence), construct an increasing one.

Formally: let $S = \{i : \text{ if } j > i, \text{ then } a_j < a_i\}$.

- if $|S| = |N|$ (countably infinite) [1], we have found a monotone (decreasing) sequence.
- If $|S| < \infty$, let $\max S = N$, then by construction, $\exists n_1$ s.t. $a_{n_1} \geq a_{N+1}$. Since $n_1 \notin X$, there exists $n_2 > n_1$ s.t. $a_{n_2} \geq a_{n_1} \geq a_N$.

  We can construct an increasing sequence in this fashion.

[1] Writing $|S| = \infty$ is not rigorous enough, since uncountably infinite could also be denoted similarly.

□

**Theorem 2.9** (Bolzano-Weierstrass Theorem)**.** A bounded sequence of $\mathbb{R}$ has a convergent subsequence.

*Proof.* By Lemma 2.8, such bounded sequence of $\mathbb{R}$ has a monotone subsequence, which inebriates the boundedness property.

Thus, by Lemma 2.6, such bounded monotone sequence converges. □

**Remark 2.10** (Properties of Limites)**.** For $a_n \to a$ and $b_n \to b$ (two convergent sequences):

(i) $c \cdot a_n \to c \cdot a$, for $c \in \mathbb{R}$;

(ii) $a_n + b_n \to a + b$

(iii) $a_n \cdot b_n \to a \cdot b$

(iv) $\frac{a_n}{b_n} \to \frac{a}{b}$ s.t. $b \neq 0$ and $b_n \neq 0$ $\forall n$.

(v) $\forall n \in N$, if $c \leq a_n$, then $c \leq a$. (Note that we have defined only one linear ordering $\leq$.)
However, $a_n > c$ does not imply $a > c$. (e.g.: $\frac{1}{n} > 0, \forall n$, yet $\frac{1}{n} \to 0 = 0$.)

(vi) $\forall n$, if $b_n \leq a_n$, then $b \leq a$.

**Definition 2.11** (Cauchy sequence)**.** $\{a_n\}$ is a Cauchy sequence if $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall m, n \geq N$, $|a_m - a_n| < \varepsilon$.

Note that, since the definition of convergent sequence relies on knowing the limit $a$, when such limit is not attainable, Cauchy becomes handy.

**Theorem 2.12.** Every convergent sequence is Cauchy.

*Proof.* Given $\{a_n\} \to a$, thus $\forall \frac{\varepsilon}{2} > 0$ $\exists N$ s.t. $|a_n - a| < \frac{\varepsilon}{2}$, $\forall n > N$.

Now, for any $m, n \geq N$, we have:

$$\begin{aligned} |a_m - a_n| &= |a_m - a + a - a_n| \\ &\leq |a_m - a| + |a_n - a| < \varepsilon \end{aligned}$$

□

**Example** : Prove that $a_{n+1} = \frac{a_n + 2a_{n-1}}{3}$ converges for $a_1 = 0$, $a_2 = 1$.

*Proof.* Step 1 First observe that: $a_n$ is an average of two real numbers that are in $[0, 1]$. Thus, $a_n \in [0, 1]$.

Step 2 Also observe that by rearranging the terms in the equality, we have:

$$\frac{a_{n+1} - a_n}{a_n - a_{n-1}} = -\frac{2}{3}$$

At this point, we check definition of Cauchy sequence by showing that: for arbitrary $\varepsilon$, we can find a $N$ such that $|a_m - a_n| < \varepsilon$. Deriving the functional form of $|a_m - a_n|$ suffices. (We can then use this functional form to find a proper $N$.)

Without loss of generality, let $m > n$, then:

$$\begin{aligned}
|a_m - a_n| &= |a_n - a_{n+1} + a_{n+1} - \cdots - a_m| \\
&\leq |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + \cdots + |a_{m-1} - a_m| \\
&\leq \left(\frac{2}{3}\right)^{n-1} + \left(\frac{2}{3}\right)^{n} + \cdots + \left(\frac{2}{3}\right)^{m-2} \\
&= \frac{\left(\frac{2}{3}\right)^{n-1}\left(1 - \left(\frac{2}{3}\right)^{m-n+2}\right)}{1 - \frac{2}{3}} \\
&= O(\left(\frac{2}{3}\right)^{n})
\end{aligned}$$

By now, we can easily demonstrates that the definition of Cauchy sequence could be satisfied by choosing a proper $N$ for any given $\varepsilon$. □

**Theorem 2.13.** Every Cauchy sequence is bounded.

*Proof.* Let $\{a_n\}$ be an arbitrary Cauchy sequence. Then, for for arbitrary $\varepsilon > 0$, we know that $\exists N_\varepsilon > 0$ such that $\forall m, n > N$, $|a_m - a_n| < \varepsilon$.

Now, to construct an upper bound for $\{a_n\}$, without loss of generality, let $\varepsilon = 1$. Then, we know that there exists $N_1 > 0$ such that $\forall n, m > N_1$, $|a_n - a_m| < 1$. Then, let $M_1$ denote the bound (either upper or lower). Then, in absolute value, we can define it to be:

$$|M_1| = \max\{|a_1|, \ldots, |a_{N_1}|, |a_{N_1+1}| + 1\}$$

Through more careful, yet unnecessary, discussions, we can derive the exact bound using the absolute value $|M_1|$.

Note that, the bound we found above is only *one of the upper bound*. It is not necessarily the sup nor inf. □

**Theorem 2.14.** Every Cauchy sequence `in` $\mathbb{R}$ [2] converges.

**Remark 2.15** (Useful limits)**.** Limits of sequences as $n \to \infty$:

Refer to page 57 of [Rudin(1976)] Theorem 3.20.

**Definition 2.16** (limsup, liminf)**.** Let $\{a_n\}$ be a sequence in $\mathbb{R}$, we say:

$$\limsup\{a_n\} = a$$

if $\sup S = a$, where $S = \{b \in \mathbb{R} : \exists \text{ subsequence } \{a_{n_i}\} s.t. a_{n_i} \to b\}$.

[2] Note that, for $\{\frac{1}{n}\}$ defined on $(0, 1]$, it does not converge in this space since $0 \notin (0, 1]$.

**Exercise: equivalent definition of limsup** Prove that $\limsup a_n = a$ if and only if:

(i) $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $a_n < a + \varepsilon$, $\forall n > N$;

(ii) $\forall \varepsilon > 0$, $\forall n \in N$, $\exists k > n$ s.t. $a_k > a - \varepsilon$.

Note that, (i) specified a property for subsequence; and (ii) is merely about the existence of one element in the sequence, to be found for all $(\varepsilon, n) \in \mathbb{R}_{++} \times N$.

*Proof.* The iff statement will be established in the following three steps:

- Prove that $\limsup a_n = a$ implies (i).

WTS: $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $a_n < a + \varepsilon$, $\forall n > N$;

First, suppose that $a = +\infty$, that is $\{a_n\}$ is not bounded from above. Then we are done.

Then, suppose that $\{a_n\}$ is bounded from above. We now prove by contradiction. Suppose that $\exists \varepsilon > 0$ s.t. no such $N \in \mathbb{N}$ exists. Then, we know that 1 cannot serve the role of $N$. So, for some $n_1 > 1$,

$$a_{n_1} \geq a + \varepsilon$$

Still, $n_1 + 1$ cannot serve the role of $N$, then for some $n_2 > n_1 + 1$,

$$a_{n_2} \geq a + \varepsilon$$

By induction, we can construct a subsequence that is bounded from below by $a + \varepsilon$. Note that, the original sequence is bounded from above, by Bolzano-Weierstrass Theorem, we know that a bounded sequence converges. However, the limit of such subsequence shall be larger than $a$, contradicting $\limsup a_n = a$.

Thus what we assumed is wrong. We thereby proved the original claim in (ii).

- Prove that $\limsup a_n = a$ implies (ii).

WTS: Given that $\limsup a_n = a$, $\forall \varepsilon > 0$, $\forall n \in N$, $\exists k > n$ s.t. $a_k > a - \varepsilon$.

Now, for arbitrary $\varepsilon > 0$, by definition of limsup, we know that $\exists a' \in (a - \frac{\varepsilon}{2}, a)$ s.t. $\exists \{a_{n_j}\}$ (a subsequence of $\{a_n\}$) s.t. $a_{n_j} \to a'$.

For this convergent subsequence per se, given the arbitrary $\varepsilon$ we have specified in the very beginning, we know that $\exists J > 0$ s.t.

$$|a_{n_j} - a'| < \frac{\varepsilon}{2}, \qquad \text{for all } j > J$$

Now, for arbitrary $n \in N$, we can always find a $k = n_i$ with $i > J$, such that $a_k = a_{n_i}$ is within $\frac{\varepsilon}{2}$ distance away from $a'$. Combining this fact with the construction that $a' \in (a - \frac{\varepsilon}{2}, a)$, it is clear the $a_k$ we found specifically for $\varepsilon$ and $n \in N$ satisfies: $a_k > a - \varepsilon$.

- Prove that (i) and (ii) implies that $\limsup a_n = a$.

To prove that $\limsup a_n = a$, we first show that $a$ is the limit of a subsequence of $\{a_n\}$; then we show that $\nexists a' > a$ s.t. $a'$ is the limit of a subsequence of $\{a_n\}$.

Firstly, by (i) and (ii), for arbitrary $\varepsilon > 0$, we can find a subsequence $\{a_{n_j}\}$ with certain $N \in \mathbb{N}$ such that $a - \varepsilon < a_{n_j} < a + \varepsilon$, $\forall n_j > N$. (Step 1: by (i), we can find a $N^\varepsilon$ for arbitrary $\varepsilon > 0$, so that: $a_n < a + \varepsilon \ \forall n > N^\varepsilon$; Step 2, for the $\varepsilon$ and all $\tilde{n} \geq N^\varepsilon$, we can find a $a_{k_{\tilde{n}}}$ s.t. $a - \varepsilon < a_{k_{\tilde{n}}}$. Thus, we have composed a subsequence $\{a_{k_{\tilde{n}}}\}$.)

Then, suppose $\exists a' > a$ as the limsup, then $\forall \varepsilon > 0 \ \exists N'$ s.t. $\forall n' > N'$, $|a_{n'} - a'| < \varepsilon$. However, (i) is violated when $\varepsilon < \frac{a'-a}{2}$: suppose that $a_{n_k} \to a'$. Then, $\exists N' > 0$ s.t. $\forall k > N'$, $|a_{n_k} - a'| < \varepsilon$. Given that $\varepsilon < \frac{a'-a}{2}$, there does not exist a $N$ that may satisfy (i). (The "$\forall n > N$" statement is violated due to the subsequence that converges to $a'$.)

# References

[Rudin(1976)] Walter Rudin. Principles of mathematical analysis. *New York: McGraw-Hill,[1976]*, 1976.

$\square$