Math 625
Professor Erhan Bayraktar

Linfeng Li
llinfeng@umich.edu

September 2, 2014

1 Week 1

1.1 Tue: 2014-09-02

1.1.1 Basics

Exam (in class, 90 minutes): Tue, Oct 7; Tue, Nov 11; Tue, Dec 9. —— 25%, 30 % and 40 %. 5% attendance.

Textbook: Probability and Statistics (took the class from the writer, available through library). Probability with **Martingale**, the latter is the emphasis.

Exercises: Book has exercises, but not graded homework.

1.1.2 Beginning: measure theory

Sigma-algebra Given a set E (a universal set), $\mathcal{E}$, a nonempty collection of subsets of E, is called a σ-algebra if closed under complements & countable unions.

- The most trivial sigma-algebra $\{\emptyset, E\}$ is called the trivial σ-algebra.

Definition 1.1.1 (σ algebra generated by $\mathcal{C}$)**.** *Given a collection $\mathcal{C}$ of subsets (of E), $\sigma(\mathcal{C})$ will denote the smaller σ-algebra containing $\mathcal{C}$.*

Definition 1.1.2. *σ-algebra generated by open sets is called **Borel σ-algebra**.*

p-system A collection of $\mathcal{C}$ which is closed under (finite) intersection.

"p" for product, could also use π-system. The latter in Greek.

d-system A collection $\mathcal{D}$ is called a d-system if

(i) $E \in \mathcal{D}$,

(ii) $A, B \in \mathcal{D}$, $A \supset B \implies A \setminus B \in \mathcal{D}$

(iii) $(A_n) \subset \mathcal{D}$ and $A_n \uparrow A \implies A \subset \mathcal{D}$.

("d" for Dynkin)

Note: curly characters are for collection of sets.

Proposition 1.1.3. *$\mathcal{E}$ is a σ-algebra if and only if it is a p-system and a d-system.*

Proof. $\Rightarrow$ is trivial.
$\Leftarrow$: Let $\mathcal{E}$ be a collection which is a p-system and a d-system.

1. Closed under complements (to be a sigma algebra). Let $A \in \mathcal{E} \implies E \setminus A \in \mathcal{E}$ by (ii) for property of d-system.
2. Closed under finite unions: $A, B \in \mathcal{E} \implies A \bigcup B = (A^c \cap B^c)^c$ by 1 above and property of p-system.
3. Closed under countable unions: for $(A_n) \subset \mathcal{E}$, $\bigcup_n A_n \in \mathcal{E}$? We construct an increasing sequence of (B_n):

   Let $B_1 = A_1$, $B_2 = A_1 \bigcup A_2 \in \mathcal{E} \ldots \bigcup_n A_n = \bigcup_n B_n$. Then by (iii) for property of d-system, the conclusion follows.

□

Lemma 1.1.4. *For $\mathcal{D}$, a d-system, fix $D \in \mathcal{D}$. Define $\hat{\mathcal{D}} := \{A \in \mathcal{D} : A \cap D \in \mathcal{D}\}$. Then, $\hat{\mathcal{D}}$ is also a d-system.*

Monotone Class Theorem [Very useful tool in showing an arbitrary collection of sets is a σ-algebra]

Theorem 1.1.5. *If a d-system contains a p-system, then it also contains the σ-algebra generated by the p-system.*

Proof. Symbolic expression: $\mathcal{C} \subset \mathcal{D} \implies \sigma(\mathcal{C}) \subset \mathcal{D}$.
Step 1:
Let $\mathcal{C}$ be a p-system. $\mathcal{D}$ is the smallest d-system that contains $\mathcal{C}$. [1]
Enough to show $\mathcal{D} \supset \sigma(\mathcal{C})$.
If fact, we will show $\mathcal{D}$ is a σ-algebra. By proposition, it is enough to show it is a p-system.
Step 2:
Fix $B \in \mathcal{C}$ and let $\mathcal{D}_1 := \{A \in \mathcal{D} : A \bigcap B \in \mathcal{D}\}$. 1) By the lemma 1.1.4, $\mathcal{D}_1$ is a d-system. 2) $\mathcal{C} \subset \mathcal{D}_1$.
1) and 2) $\implies \mathcal{D}_1 = \mathcal{D}$.
Setp 3:
Fix $A \in \mathcal{D}$, let $\mathcal{D}_2 := \{B \in \mathcal{D} : B \bigcap A \in \mathcal{D}\}$.
1) by the lemma, $\mathcal{D}_2$ is a d-system. 2) by **step 2**, $\mathcal{C} \subset D_2$.
1) and 2) $\implies \mathcal{D}_2 = \mathcal{D}$.
Step 1-3 gives that $\mathcal{D}$ is a p-system.
In here, $\mathcal{D} = \sigma(\mathcal{C})$. [But in the theorem, this is not a necessary conclusion.] □

Measurable space $(E, \mathcal{E})$ is a measurable space. [$\mathcal{E}$ is a σ-algebra on E.]

Products of measure spaces $(E, \mathcal{E})$, $(F, \mathcal{F})$. Then $(E \times F, \mathcal{E} light - product \mathcal{F})$ where $\times$ is regular set product; and the light-product is σ (generated by measurable rectangles)

Measurable functions (random variables)

Lemma 1.1.6. *A mapping $f : E \to F$ and (inverse mapping) $f^{-1}(A) := \{x \in E : f(x) \in A\}$. Then, $f^{-1}\emptyset = \emptyset$. $f^{-1}(F) = E$. $f^{-1}(B \setminus C) = f^{-1}(B) \setminus f^{-1}(C)$. $f^{-1}\left(\bigcup_i B_i\right) = \bigcup_i f^{-1}(B_i)$ and $f^{-1}\left(\bigcap B_i\right) = \bigcap_i f^{-1}(B_i)$.*
(set operation passes through the inverse function operation.)

[1] (***: little result – the intersections of d-systems is a d-system [to obtain the "smallest"]. Also, the "smallest" matters.)

Definition 1.1.7. $(E, \mathcal{E})$, $(F, \mathcal{F})$. $f : E \to F$ *is "measurable" relative to* $\mathcal{E}\&\mathcal{F}$ *if* $f^{-1}(B) \in \mathcal{E}$, $\forall B \in \mathcal{F}$.