Math 625
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September 4, 2014

No responding to emails!

1 Week 1

1.1 Tue: 2014-09-02

1.1.1 Basics

Exam (in class, 90 minutes): Tue, Oct 7; Tue, Nov 11; Tue, Dec 9. —— 25%, 30 % and 40 %. 5% attendance.

Textbook: Probability and Statistics (took the class from the writer, available through library). Probability with **Martingale**, the latter is the emphasis.

Exercises: Book has exercises, but not graded homework.

1.1.2 Beginning: measure theory

Sigma-algebra Given a set E (a universal set), $\mathcal{E}$, a nonempty collection of subsets of E, is called a σ-algebra if closed under complements & countable unions.

- The most trivial sigma-algebra $\{\emptyset, E\}$ is called the trivial σ-algebra.

Definition 1.1.1 (σ algebra generated by $\mathcal{C}$)**.** *Given a collection $\mathcal{C}$ of subsets (of E), $\sigma(\mathcal{C})$ will denote the smaller σ-algebra containing $\mathcal{C}$.*

Definition 1.1.2. *σ-algebra generated by open sets is called **Borel σ-algebra**.*

***p*-system** A collection of $\mathcal{C}$ which is closed under (finite) intersection.

"p" for product, could also use π-system. The latter in Greek.

***d*-system** A collection $\mathcal{D}$ is called a d-system if

(i) $E \in \mathcal{D}$,

(ii) $A, B \in \mathcal{D}$, $A \supset B \implies A \setminus B \in \mathcal{D}$

(iii) $(A_n) \subset \mathcal{D}$ and $A_n \uparrow A \implies A \subset \mathcal{D}$.

("d" for Dynkin)

Note: curly characters are for collection of sets.

Proposition 1.1.3. *$\mathcal{E}$ is a σ-algebra if and only if it is a p-system and a d-system.*

Proof. $\Rightarrow$ is trivial.
$\Leftarrow$: Let $\mathcal{E}$ be a collection which is a p-system and a d-system.

1. Closed under complements (to be a sigma algebra). Let $A \in \mathcal{E} \implies E \setminus A \in \mathcal{E}$ by (ii) for property of d-system.
2. Closed under finite unions: $A, B \in \mathcal{E} \implies A \bigcup B = (A^c \cap B^c)^c$ by 1 above and property of p-system.
3. Closed under countable unions: for $(A_n) \subset \mathcal{E}$, $\bigcup_n A_n \in \mathcal{E}$? We construct an increasing sequence of (B_n):
   Let $B_1 = A_1$, $B_2 = A_1 \bigcup A_2 \in \mathcal{E} \ldots \bigcup_n A_n = \bigcup_n B_n$. Then by (iii) for property of d-system, the conclusion follows.

□

Lemma 1.1.4. *For $\mathcal{D}$, a d-system, fix $D \in \mathcal{D}$. Define $\hat{\mathcal{D}} := \{A \in \mathcal{D} : A \cap D \in \mathcal{D}\}$. Then, $\hat{\mathcal{D}}$ is also a d-system.*

Monotone Class Theorem [Very useful tool in showing an arbitrary collection of sets is a σ-algebra]

Theorem 1.1.5. *If a d-system contains a p-system, then it also contains the σ-algebra generated by the p-system.*

Proof. Symbolic expression: $\mathcal{C} \subset \mathcal{D} \implies \sigma(\mathcal{C}) \subset \mathcal{D}$.
Step 1:
Let $\mathcal{C}$ be a p-system. $\mathcal{D}$ is the smallest d-system that contains $\mathcal{C}$. [1]
Enough to show $\mathcal{D} \supset \sigma(\mathcal{C})$.
If fact, we will show $\mathcal{D}$ is a σ-algebra. By proposition, it is enough to show it is a p-system.
Step 2:
Fix $B \in \mathcal{C}$ and let $\mathcal{D}_1 := \{A \in \mathcal{D} : A \bigcap B \in \mathcal{D}\}$. 1) By the lemma 1.1.4, $\mathcal{D}_1$ is a d-system. 2) $\mathcal{C} \subset \mathcal{D}_1$.
1) and 2) $\implies \mathcal{D}_1 = \mathcal{D}$.
Setp 3:
Fix $A \in \mathcal{D}$, let $\mathcal{D}_2 := \{B \in \mathcal{D} : B \bigcap A \in \mathcal{D}\}$.
1) by the lemma, $\mathcal{D}_2$ is a d-system. 2) by **step 2**, $\mathcal{C} \subset D_2$.
1) and 2) $\implies \mathcal{D}_2 = \mathcal{D}$.
Step 1-3 gives that $\mathcal{D}$ is a p-system.
In here, $\mathcal{D} = \sigma(\mathcal{C})$. [But in the theorem, this is not a necessary conclusion.] □

Measurable space $(E, \mathcal{E})$ is a measurable space. [$\mathcal{E}$ is a σ-algebra on E.]

Products of measure spaces $(E, \mathcal{E})$, $(F, \mathcal{F})$. Then $(E \times F, \mathcal{E} light - product \mathcal{F})$ where $\times$ is regular set product; and the light-product is σ (generated by measurable rectangles)

Measurable functions (random variables)

Lemma 1.1.6. *A mapping $f : E \to F$ and (inverse mapping) $f^{-1}(A) := \{x \in E : f(x) \in A\}$. Then, $f^{-1}\emptyset = \emptyset$. $f^{-1}(F) = E$. $f^{-1}(B \setminus C) = f^{-1}(B) \setminus f^{-1}(C)$. $f^{-1}\left(\bigcup_i B_i\right) = \bigcup_i f^{-1}(B_i)$ and $f^{-1}\left(\bigcap B_i\right) = \bigcap_i f^{-1}(B_i)$.*
(set operation passes through the inverse function operation.)

[1] (***: little result – the intersections of d-systems is a d-system [to obtain the "smallest"]. Also, the "smallest" matters.)

Definition 1.1.7. $(E, \mathcal{E})$, $(F, \mathcal{F})$. *$f : E \to F$ is "measurable" relative to $\mathcal{E}$&$\mathcal{F}$ if $f^{-1}(B) \in \mathcal{E}$, $\forall B \in \mathcal{F}$.*

1.1.3 Thu: 2014-09-04

measurable functions (To "measure" a measurable function: just to integrate it).

Proposition 1.1.8. *A function $f : E \to F$ is meausrbale if and only if for some collection $\mathcal{F}_0$ with $\mathcal{F} = \sigma(\mathcal{F}_0)$, $f^{-1}(B) \in \mathcal{E}$.*

Proof. Necessicty is trivial; (by definitoin)

First collect all the sets s.t. $\mathcal{F}_1 = \left\{B \in \mathcal{F} : f^{-1}(B) \in \mathcal{E}\right\} \supset \mathcal{F}$. We show this by showing that this is a sigma algebra. [through checking the properties of inverse fucntions.] □

Lemma 1.1.9 (Composition of measurable functions are measurbal)**.** *[2.5 Proposition]*

Let $m(\mathcal{E})$ note the collection of measurable functions. Abuse of notation: let $\mathcal{E}$ also note $m(\mathcal{E})$ since the context would be clear.

Theorem 1.1.10. *Proof.* **Step 1** the sup would exist:

...

□

Approximation of measurbale functions

Lemma 1.1.11. *For $r \in \mathbb{R}_+$, $d_n(r) = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} 1_{[\frac{k-1}{2^n}, \frac{k}{2^n}]}(r) + n1_{[n,\infty)}(r)$. THen $d_n(r) \to r$ as $n \to \infty$.*

Theorem 1.1.12. *A positive function is measurable if and only if it is a limit of positive simple functions ($\sum_{i=1}^{n} a_i 1_{A_i}$ for $a_i \in \mathbb{R}$ and $A_i \in \mathcal{E}$).*

Proof. Sufficiency is given by the previous theorem.

Necessity: Let $f_n = d_n \circ f$ where $f_n \uparrow f$. (By construction of $d_n(r)$, f_n is simple measurable function.) □

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Lemma 1.1.13. *If $h_1, h_2, h_3 \in \mathcal{E}$, $h_1 + h_2, h_1\dot{h}_2, \lambda h \in \mathcal{E}$ for $\lambda \in \mathbb{R}$.*

Decomposition of positive part and negative part of function f Let $f = f^+ - f^-$ where $f^+ = f \ldots$

Monotone Class Theorem for Functions

Definition 1.1.14. *For $\mathcal{M}$, a collection of functions is a monotone class if*

1. *$1 \in \mathcal{M}$ (1 is the function assigning all elements in E to 1)*
2. *$f, g \in \mathcal{M}_b \implies af + bg \in \mathcal{M}_b$. where $\mathcal{M}_b$ denote bounded functions in the set of functions denoted by $\mathcal{M}$.*
3. *$(f_n) \subset \mathcal{M}_+$, and $f_n \uparrow f \implies f \in \mathcal{M}_+$. where $\mathcal{M}_+$ denote non-negative functions in $\mathcal{M}$.*

Theorem 1.1.15 (Monotone Convergence Theorem)**.** *Let $\mathcal{M}$ be a monotone class. Suppose that for some p-system $\mathcal{C}$, $\mathcal{E} = \sigma(\mathcal{C})$.*

$1_A \in \mathcal{M}, \qquad \forall A \in \mathcal{C} \implies$ *$\mathcal{M}$ includes all positive measurable functions ($\mathcal{E}_+$) and all bounded measurable functions ($\mathcal{E}$*

1_A here is an indicator function

Proof. **Step 1:** we want to show that, for all $1_A \in \mathcal{M}$, $\forall A \in \mathcal{E}$.

Define

use the defection of a monotone class to show that $\mathcal{D}$ is a d-system.

$\mathcal{D}$ being a d-system implies that ...

Step 2 Simple functions are also in $\mathcal{M}$. [find a reason to this.]

Step 3 By the previous theorem on the simple function, we see that for arbitrary $f \in \mathcal{E}_+$, $\exists (f_n) \uparrow f$ where f_n is simple measurable functions.

Then, by (3) in definition of monotone class (of functions) , $f \in \mathcal{M}_+$.

Step 4 For $f \in \mathcal{E}_b$, as $f = f^+ - f^-$ by (2) in definition of monotone class, we have $f \in \mathcal{M}_b$. □

Definition 1.1.16. $X : \Omega \to (E, \mathcal{E})$, $\sigma(X) = X^{-1}\mathcal{E} := \left\{X^{-1}(A) : A \in \mathcal{E}\right\}$ *is called the σ-algebra generated by X. Note that X here is a*

Hereby we define a new σ-algebra on Ω.

Proposition 1.1.17. *Let $X : \Omega \to (E, \mathcal{E})$ and another mapping $V : \Omega \to \bar{\mathbb{R}}$ belongs to $\sigma(X)$ if and only if $V = f \circ X$ for some function $f \in \mathcal{E}$.*

Proof. Sufficiency part is trivial;

Necessity: let $\mathcal{M}$ be the collection of all $V f \circ X$ for some $f \in \mathcal{E}$. (This is enough to show if Y is bounded measurable w.r.t. $\sigma(X)$, $\exists$ a bounded measurable function f s.t. $Y = f(X)$)

Step 1 Show that $\mathcal{M}$ is a monotone class:

Step 2 $\mathcal{M}$ includes every indicator function in $\sigma(X)$. The set $H \in \sigma(X)$, $H = X^{-1}(A)$ for $A \in \mathcal{E}$. Then

$$1_H = 1_A \circ X$$

Step 3 Use MCT

□

Measures

Definition 1.1.18. *In a measurable space $(E, \mathcal{E})$, $\mathcal{E}$, for $\mu : \mathcal{E} \to \bar{\mathbb{R}}_+$, if*

(a) $\mu(\emptyset) = 0$

(b) $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$ *for A_n's being disjoint.*

μ is called a measure. (Note that this measure is infinite as $\mu : \mathcal{E} \to \bar{\mathbb{R}}_+$, with a bar overhead of $\mathbb{R}$.)

Proposition 1.1.19. *For A and B being measurable sets,*

(i) (Monotonicity) $A \subset B \implies \mu(A) \leq \mu(B)$; *[Implied by finite additivity.]*

(ii) (Continuity under increasing limits) $A_n \uparrow A \to \mu(A_n) \uparrow \mu(A)$.

Proof. For $B_1 = A_1$, $B_2 = A_2 \setminus A_1$, ..., $\bigcup_{n=1}^n B_n = \bigcup_{n=1}^n A_n$ [This is finite additivity.] [Could finish the proof by taking limits at both sides.] □

(iii) (Sub-additivity) $\mu(\bigcup_n A_n) \leq \sum(A_n)$.

Notation $\mu(E) \leq \infty \implies \mathcal{M}$ is a finite measure. $\mu(E) = 1$ implies that μ is a probability measure.
σ-finite if $\exists$ a partition[2] (E_n) of E s.t. $\sum(E_n) < \infty$.
Σ-finite if $\exists \mu_n$ s.t. $\mu = \sum_n \mu_n$ for $\mu_n(E) < \infty$.
σ-finite $\implies$ Σ-finite.

Theorem 1.1.20. *Let $(E, \mathcal{E})$ be a measurable space and measures μ and ν are $\mu(E) = \nu(E) < \infty$. Moreover, μ and ν agree on $\mathcal{C}$, which is a p-system satisfying $\mathcal{E} = \sigma(\mathcal{C})$, then*

$$\mu = \nu$$

This is why we can specify the Lebesgue measure by only assigning measure to the intervals. The above theorem would generalize the measure.

¡++¿ Again is a consequence of monotone class theorem

[2] Only countable, not necessarily finite. Each element in the partition is disjoint.