Logic in Computer Science - Assignment 2

李鹏达 10225101460

For soundness proof, complete the proof of the following proof rule cases:

(1) $\wedge e_1$: It must be the case that $\chi_k = \chi_1$ with $\chi_1 \wedge \chi_2$ appearing at line $l < k$. The formula $\chi_1 \wedge \chi_2$ has the shorter proof, and therefore, using the induction hypothesis, it has the truth value T. Using the truth table for $\wedge$, we can conclude that the truth value of χ_1 is T.

(2) $\wedge e_2$: Same as (1), but with $\chi_k = \chi_2$.

(3) $\bot e$: It must be the case that $\chi_k = \phi$ with $\bot$ appearing at line $l < k$, which means that $\chi_1, \neg\chi_1$ appears at line $m, n < l$. Using the induction hypothesis, both χ_1 and $\neg\chi_1$ have the truth value T. However, according to the truth table of $\neg$, if χ_1 has the truth value T, then $\neg\chi_1$ must have the truth value F, which contradicts the fact that $\neg\chi_1$ has the truth value T. Therefore, this case cannot happen, and χ_k is trivially true.

(4) $\neg i$: It must be the case that $\chi_k = \neg\phi$ with ϕ as an assumption at line $l < k$ and $\bot$ appearing at line $m < k(m > l, l \in d_m)$. If ϕ is T, then by the induction hypothesis, $\bot$ must be T since the assumption is T. However, that is impossible, so we can get χ_k is T trivially. If ϕ is F, then by the truth table of $\neg$, $\neg\phi$ is T. Therefore, in both cases, χ_k is T.

(5) $\neg\neg e$: It must be the case that $\chi_k = \phi$ with $\neg\neg\phi$ appearing at line $l < k$. Since it has a shorter proof, using the induction hypothesis, $\neg\neg\phi$ is T, then by the truth table of $\neg$, $\neg\phi$ is F. Then, by the truth table of $\neg$ again, we can get that ϕ is T. Therefore, χ_k is T.

(6) $\vee i_1$: It must be the case that $\chi_k = \chi_1 \vee \chi_2$ with χ_1 appearing at line $l < k$. Since χ_1 has a shorter proof, by the induction hypothesis, χ_1 is T. Then, by the truth table of $\vee$, we can get that $\chi_1 \vee \chi_2$ is T. Therefore, χ_k is T.

(7) $\vee i_2$: Same as (6), but with χ_2 appearing at line $l < k$.

(8) $\to i$: It must be the case that $\chi_k = \phi \to \psi$ with ψ appearing at line $l < k$ and ϕ as an assumption at line $m < l$ $(m \in d_l)$. If ϕ is F, then by the truth table of $\to$, $\phi \to \psi$ is T. If ϕ is T, then by the induction hypothesis, ψ is T since the assumption is T. Then, by the truth table of $\to$, we can get that $\phi \to \psi$ is T. Therefore, χ_k is T.

(9) $\to e$: It must be the case that $\chi_k = \psi$ with $\phi \to \psi$ appearing at line $l < k$ and ϕ appearing

at line $m < k$. Since both $\phi \to \psi$ and ϕ have shorter proofs, by the induction hypothesis, they are both T. Then, by the truth table of $\to$, we can get that ψ is T. Therefore, χ_k is T.

(10) $\neg$e : It must be the case that $\chi_k = \perp$ with ϕ appearing at line $l < k$ and $\neg\phi$ appearing at line $m < k$. Since both ϕ and $\neg\phi$ have shorter proofs, by the induction hypothesis, they are both T. However, this case cannot happen, so the conclusion is trivially proved.