Efficient Calibration for Libor Market Models: Alternative strategies and implementation issues

Thomas Weber
SciComp Inc.
Weber & Partner
Austin, TX./Heidelberg, Germany
weber@scicomp.com

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SciComp Inc.

- 1994 spin off from Schlumberger Oil Exploration
- Knowledge based system for solving PDEs and SDEs
- Since 1999 adaption of the core system on derivative pricing
- Main customers are Merill Lynch, Morgan Stanley, ING

Current Suite of Calibrators

SV/SVJ-Calibrator (SciCal) Estimation of up to five parameters for SV and SVJ-models for typical processes used in equity and fx using simulated annealing

STCDO-Calibrator Extracts hazard rates and correlation from Single Tranche CDOs. Based on (Andersen and Sidenius 2004) approach

Short Rate Calibrator Allows the parameter estimation for one an two factor Gaussian (Hull/White), CIR, Black/Karasinski, Black/Derman/Toy models

LMM Calibrator Estimation of volatility and correlation parameters in the sense of (Brigo and Mercurio 2001), (Brigo and Morini 2004) and (Brigo, Mercurio, and Morini 2005)

Plan of attack

- Short Introduction into Libor Market Model to motivate the calibration approach
- Give an update on the progress since the MathFinance 2002 presentation of Lutz Molgedey
- Report on the decision made during the implementation

The Libor Market Model

Assume that the change in forward rates under the T_k-forward adjusted probability measure Q^k can be described as

$$\frac{dF_k(t)}{F_k(t)} = \sigma_k(t) dZ_k(t) \quad (1)$$

where Z_k^k is the k-th component of a M-dimensional Brownian motion $Z^k(t)$. Other forward rates i under same measure Q^k follow the dynamics

$$\frac{dF_i(t)}{F_i(t)} = -\sigma_i(t) \sum_{j=k+1}^{i} \frac{\rho_{i,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)} dt + \sigma_i(t) dZ_i^k(t), \qquad i < k \quad (2)$$

$$\frac{dF_i(t)}{F_i(t)} = \sigma_i(t) \sum_{j=k+1}^{i} \frac{\rho_{i,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)} dt + \sigma_i(t) dZ_i^k(t), \qquad i > k \quad (3)$$

where

- $0 = T_0 < T_1 < \ldots < T_M$ is the tenor structure and $\tau_i = T_{i+1} - T_i$.
- $F_i(t) = \frac{1}{\tau_i}\left(\frac{P_i(t)}{P_{i+1}(t)} - 1\right)$ are the forward rates at time t for the maturity period $T_{i+1} - T_i$.
- $P_i(t)$ are the zero coupon bond prices at time t with maturity T_i.

Pricing Caps and Floors with the Libor Market Modell

The discounted payoff at time 0 of a cap with first reset date T_α and payment dates $T_{\alpha+1}, \ldots, T_\beta$ is given by

$$\sum_{i=\alpha+1}^{\beta} \tau_i P_0(T_i) \left(F_{T-1}(T-1+\tau) - K\right)^+$$

This can be done with Blacks (1976) formula setting

$$\hat{\sigma}_T^2 = \frac{1}{T} \int_0^T \sigma_u^2 du \tag{4}$$

for the volatility parameter in the formula.

The Swap market in term of the Libor market model

From the LMM point of view swap rate can be considered as

$$S_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_i(t) F_i(t) \tag{5}$$

where

$$\begin{aligned} w_i(t) &= \frac{\tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j+1 F_j(t)}}{\sum_{k=\alpha+1}^{\beta} \tau_k \prod_{j=\alpha+1}^{k} \frac{1}{1+\tau_j F_j(t)}} \\ &= \frac{P_{i+1}(t)}{\sum_{k=\alpha+1}^{\beta} P_{k+1}(t)} \end{aligned} \tag{6}$$

Swaption volatilities in term of Libor rates

From the dynamics of (5) volatilities for the swap rates in term of Libor rates can be derived through

- Simulation
- Analytical
  - Rebonato (1998) approximation assuming constant (freezing) $w_i(t)$
  - Hull/White (1999) exact expression for the volatility
  - Jäckel/Rebonato (2000) approximation assuming that the volatility of the swap rate viewing it as a weighted sum of forward rate covariances

Swaption volatilities in term of Libor rates (II)

Testing different approches against each other lead to the conclusion that Rebonato's approximation is sufficient accurat.

Compare (Brigo and Mercurio 2001) and (Brigo, Mercurio, and Morini 2005)

Rebonato's approximating function for the Black swaption volatility

Assuming that

$$S_{\alpha,\beta}(t) \approx \sum_{i=\alpha+1}^{\beta} w_i(0) F_i(t)$$

it follows that

$$\left(v_{\alpha,\beta}^{\mathsf{LMM}}\right)^2 = \frac{1}{T_\alpha} \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0) w_j(0) F_i(0) F_j(0) \rho_{i,j}}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t) \sigma_j(t) dt \qquad (7)$$

Calibration task

For practical use of the Libor Market Modell it is necessary to estimate

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_M \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,M} \\ \rho_{2,1} & 1 & \cdots & \rho_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{M,1} & \rho_{M,2} & \cdots & 1 \end{pmatrix} \tag{8}$$

$\frac{M^2-M}{2} + M$ parameters have to be estimated, if no method for reduction is used!

M	correlation	all together
3	3	6
5	10	15
10	45	55

Estimation strategy I

- Estimate volatilities from Caps/Floors
- Rescale volatilities to the needed Libor maturity
- Extract correlation parameters from swaption volatilities

Estimation of volatilities

- In the LM-model implied average volatility $\hat{\sigma}_T^2$ can be extracted from caplet prices using Black (1976) formula ...
- ... but instantaneous volatilities σ_t^2 are needed. Therefore some further assumptions on the relation of average volatility and instantaneous volatility are necessary.

Recall that

$$\hat{\sigma}_T^2 = \frac{1}{T} \int_0^T \sigma_u^2 du$$

Assuming functional forms for the volatility function I

Example 1: Piecewise Constant Form[1]

$$\sigma(F, t, T, \cdot) \equiv \bar{\sigma}(t, T)$$

	$(0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	...	$(T_{M-2, T_{M-1}}]$
$F_1(t)$	$\sigma_{1,1}$	-	-	...	-
$F_2(t)$	$\sigma_{2,1}$	$\sigma_{2,2}$	-	...	-
$\vdots$	...	...	...	...	-
$F_M(t)$	$\sigma_{M,1}$	$\sigma_{M,1}$	$\sigma_{M,3}$	...	$\sigma_{M,M}$

Table 1: Assuming a piecewise-constant form

[1] compare (Brigo and Mercurio 2001, p. 195)

Assuming functional forms for the volatility function II

Example 2: Parametric Form[2]

$$\sigma_i(t) = \Phi_i \psi\left(T_{i-1} - t; a, b, c, d\right) \equiv \Phi_i \left(\left[a\left(T_{i-1} - t\right) + d\right] e^{-b\left(T_{i-1}-t\right)} + c \right)$$

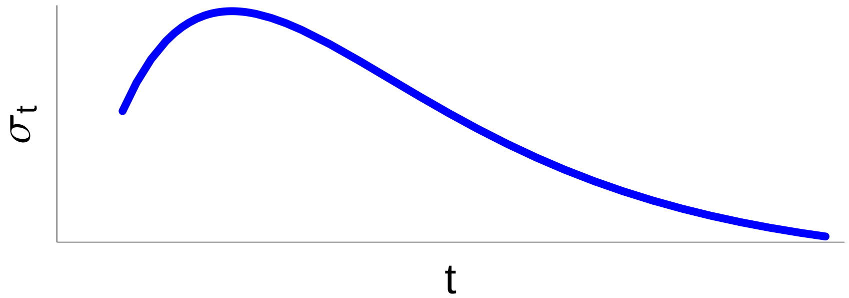

Figure 1: Typical shape of the suggested functional form

[2] (Brigo and Mercurio 2001), Formulation 7

Target function for the volatility extraction

The aim is to minimizes the (weighted) sum of the squared volatility differences.

The objective function for the extraction of volatility parameter using the parametric form is:

$$\min_{a,b,c,d} \sum_i \eta_i \left(\sigma_i^{\mathsf{Market}} \sqrt{t_i} - \Phi_i \sqrt{\int_0^{T_i} \left(\left[a \left(T_{i-1} - t \right) + d \right] e^{-b \left(T_{i-1} - t \right)} + c \right)^2 dt} \right)^2 \tag{9}$$

The weights η_i are useful to account for the different quality of option prices (and volatilities)

Bootstrapping caplet volatilities (I)

Problem: The market quotes 'flat' implied volatility $\hat{\sigma}(T)$ for a cap of strike K and maturity T, that is

$$\text{Cap}(\hat{\sigma}_T) = \sum_{i=\alpha+1}^{\beta} \text{Caplet}_i(\sigma_{t_i}) = 0$$

A first order Taylor approximation for each $\text{Caplet}_i(\sigma_{t_i})$ about σ_{t_i} yields[3]

$$\sum_{i=\alpha+1}^{\beta} \text{Caplet}_i(\sigma_{t_i}) \approx \sum \left[\text{Caplet}_i(\hat{\sigma}) + (\sigma_{t_i} - \hat{\sigma}_T) \frac{\partial \text{Caplet}_i(\sigma_{t_i})}{\partial \sigma_{t_i}} |_{\sigma_{t_i} = \hat{\sigma}_T} \right]$$

...

[3] Compare (Alexander 2002)

Bootstrapping caplet volatilities (II)

...

From the flatness it follows that

$$\sum_{i=\alpha+1}^{\beta} \left[(\sigma_{t_i} - \hat{\sigma}_T) \frac{\partial \mathsf{Caplet}_i(\sigma_{t_i})}{\partial \sigma_{t_i}} |_{\sigma_{t_i}=\hat{\sigma}_T} \right] \approx 0 \qquad \text{and} \qquad \sum_{i=\alpha+1}^{\beta} \sigma_{t_i} \nu_i \approx \hat{\sigma}_T \sum_{i=\alpha+1}^{\beta} \nu_i$$

by writing ν_i instead of $\frac{\partial \mathsf{Caplet}_i(\sigma_{t_i})}{\partial \sigma_{t_i}}$.

$\Rightarrow$ the flat volatility can be regarded as a vega weighted sum of each caplet volatility:

$$\hat{\sigma}_T \approx \frac{\sum_{i=\alpha+1}^{\beta} \nu_i \sigma_{t_i}}{\sum_{i=\alpha+1}^{\beta} \nu_i}$$

Bootstrapping caplet volatilities (III)

Algorithm 1. *Bootstrapping caplet volatilities:*

1. *Set* $\sigma_1 = \hat{\sigma}_{T=1}$

2. *Solve for* σ_2 *from* $\frac{\nu_1\sigma_1+\nu_2\sigma_2}{\nu_1+\nu_2} = \hat{\sigma}_{T=2}$

3. *and so forth ...*

Rescale extracted volatilities to the needed Libor maturity

From

$$F_i(t) = \left(1 + \frac{1}{2} f_{2i-1}(t)\right)\left(1 + \frac{1}{2} f_{2i}(t)\right) - 1$$

where f_i denotes semi-annual forward rates and
$dF = x_{2i-1} df_{2i-1}/f_{2i-1} + x_{2i} df_{2i}/f_{2i}$ where $x_{2i-1} = f_{2i-1}/2 + f_{2i-1}f_{2i}/4$ and
$x_{2i} = f_{2i}/2 + f_{2i-1}f_{2i}/4$. (3) it follows from (3) that

$$\sigma_i^2 \approx \frac{1}{F_i^2}\left(x_{2i-1}^2\sigma_{2i-1}^2 + x_{2i}^2\sigma_{2i}^2 + 2\rho_{2i-1,2}x_{2i-1}, x_{2i}\sigma_{2i-1}\sigma_{2i}\right) \qquad (10)$$

ρ might be assumed, or the annual volatilities have to be estimated together with the correlation matrix.

Estimation Methods for the Correlation Matrices

- Historical (time series analysis)
  - Typically Principal Component Analysis (PCA) is used.
  - Well established (Theil 1971). Comparativly simple to implement
  - PCA was first used by HJM (compare (Heath 1991))
  - Empirical studies by (Weber 1996) and (Alexander 2003) show that the results are very sensitive to choice of period, interpolation rules etc.
  - New approaches like Functional Data Analysis might give more structure to the estimation

- Implied from market data
  - Predefined functional form is needed
  - correlation dependent products are needed

Functional Form for the Correlation Matrix[4]

- (Brigo and Mercurio 2001), (Rebonato and Joshi 2001), (Jäckel 2002) assume that $\rho_{i,j} = e^{-\rho|i-j|}$

- (Schoenmakers and Coffey 2000) propose a semi-parametric form for a correlation matrix of M forward rates is based on an increasing sequence of M real numbers, to allow for the fact that correlation of forward rates tends to increase with maturity.

- The increasing correlation of forward rates with longer maturity can also be captured with $\rho_{i,j} = \left(\psi^{j-1}\rho\right)^{|i-j|}$

- Rebonato's angles formulation $\rho_{i,j} = \cos\left(\theta_i - \theta_j\right)$

[4] Compare (Brigo 2002)

Target function for the correlation extraction

The objective for the correlation is then to choose parameters ρ and ψ such that:

$$\min_{\rho,\psi} \sum_{n,m} \omega_{n,m} \left(\sigma_{n,m}^{\mathsf{market}} - \sigma_{n,m}^{\mathsf{model}}\right)^2 \tag{11}$$

The search for the minimum can efficiently be done using the Levenberg-Marquardt algorithm.

Summary

- The overall picture is still close to the one drawn from Lutz Molgedey in 2002.
- There are several result-influencing watch-outs in the details
- There is still a lot to learn how the parts are related to each other

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