An Introduction to Modern Portfolio Theory

Certificate in Quantitative Finance

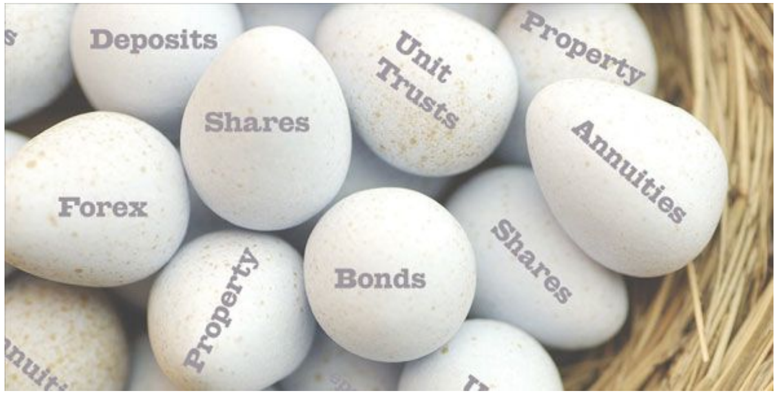
In this lecture, we will see…

- The core concepts of portfolio management and Modern Portfolio Theory:
  - Risky and risk-free assets;
  - Mean-variance analysis;
  - Optimal portfolio;
  - Diversification;
  - Opportunity set and efficient frontier;
  - Tangency and market portfolio;
  - Sharpe ratio and market price of risk;
  - The linear model and the CAPM.
  - The APT
  - Measuring risk-adjusted performance

- The drawbacks of MPT: dimensionality and parameter estimation.

Historical note

- Modern Portfolio Theory (MPT) was pioneered by Markowitz in 1952.
- Although "***Don't put all your eggs in the same basket***" was a popular say in the investment world even before Markowitz, portfolios tended to be constructed as collections of individual securities selected for their return potential and with little regards for their risks or interactions.



http://maverickinvestors.files.wordpress.com/2012/01/portfolio_diversification_investment.jpg

Historical note

- Markowitz showed that risk and return are equally important.
  - To produce more returns it is necessary to take more risk (the idea of "**risk-return trade-off**")
  - The only sure way to reduce risk without sacrificing too much return is through **diversification** across enough securities.
  - In Markowitz' framework, diversification benefits depend on the correlation of securities returns via the variance of the portfolio returns .

Historical note

- His student William Sharpe then proposed a linear factor model as well as an economic “equilibrium” model called the Capital Asset Pricing Model (CAPM), establishing a clear connection between securities pricing and portfolio selection;

- Other early contributors to the development of MPT include Jack Treynor, who developed the CAPM before Sharpe but never published it, as well as John Lintner and Jan Mossin.

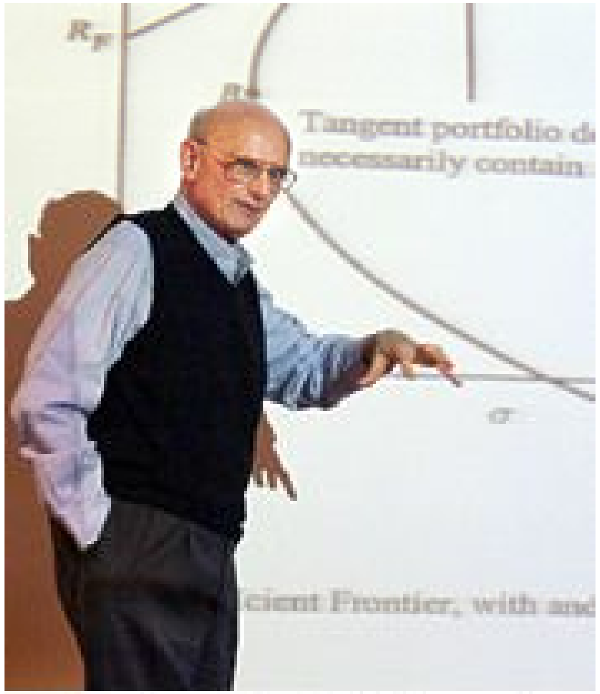
- The development of MPT marked not only the dawn of financial economics[1] but also of quantitative finance as fields of study.

- Sharpe and Markowitz were awarded the Nobel Prize of Economics in 1990 for their contribution to the theory of financial economics.

[1] *The other breakthrough of the 1950s were Arrow and Debreu’s state securities and the Modigliani-Miller Theorem published in 1958*

Markowitz and Sharpe

- Harry Markowitz (1927-)



Keynote speaker Nobel laureate
Harry M. Markowitz

- William Sharpe (1934-)

The Setting

The setting

- We are in an economy where $N \geq 2$ assets are traded.
- We start with a wealth of £W.
- Our objective is to make the "best" investment of our wealth for a period of T years.
- To achieve this objective, we will constitute a portfolio by buying (or shorting) some or all of the N possible assets.
- We will not revisit our decision up until the end of the period (a one-period or "buy and hold" investment model).

Assets? Which assets?

- The definition of assets used here is very wide, encompassing all tradable assets on Earth, including:

  - **Financial assets**: equity shares, bonds, currencies…
  - **Real assets**: commodities, real estate, collectibles (artwork, fine wine…), manufacturing plants, consumer goods…
  - **Intangible assets**: labour income.

- Of course, in practice, portfolio managers do not tend to use this definition since they are often limited to a single asset class and country (U.K. equity, U.S. bonds…).

Portfolio weights

- To establish a portfolio, it is generally more convenient to deal with proportions of total wealth than amounts:
  - Proportion of wealth: "10% of my assets are in stock XYZ"
  - Amount: "I have $10,000 invested in bond ABC"

- Denote by w_i the **portfolio weight** (another name for the proportion of wealth) invested in asset i, i =1,…,N.

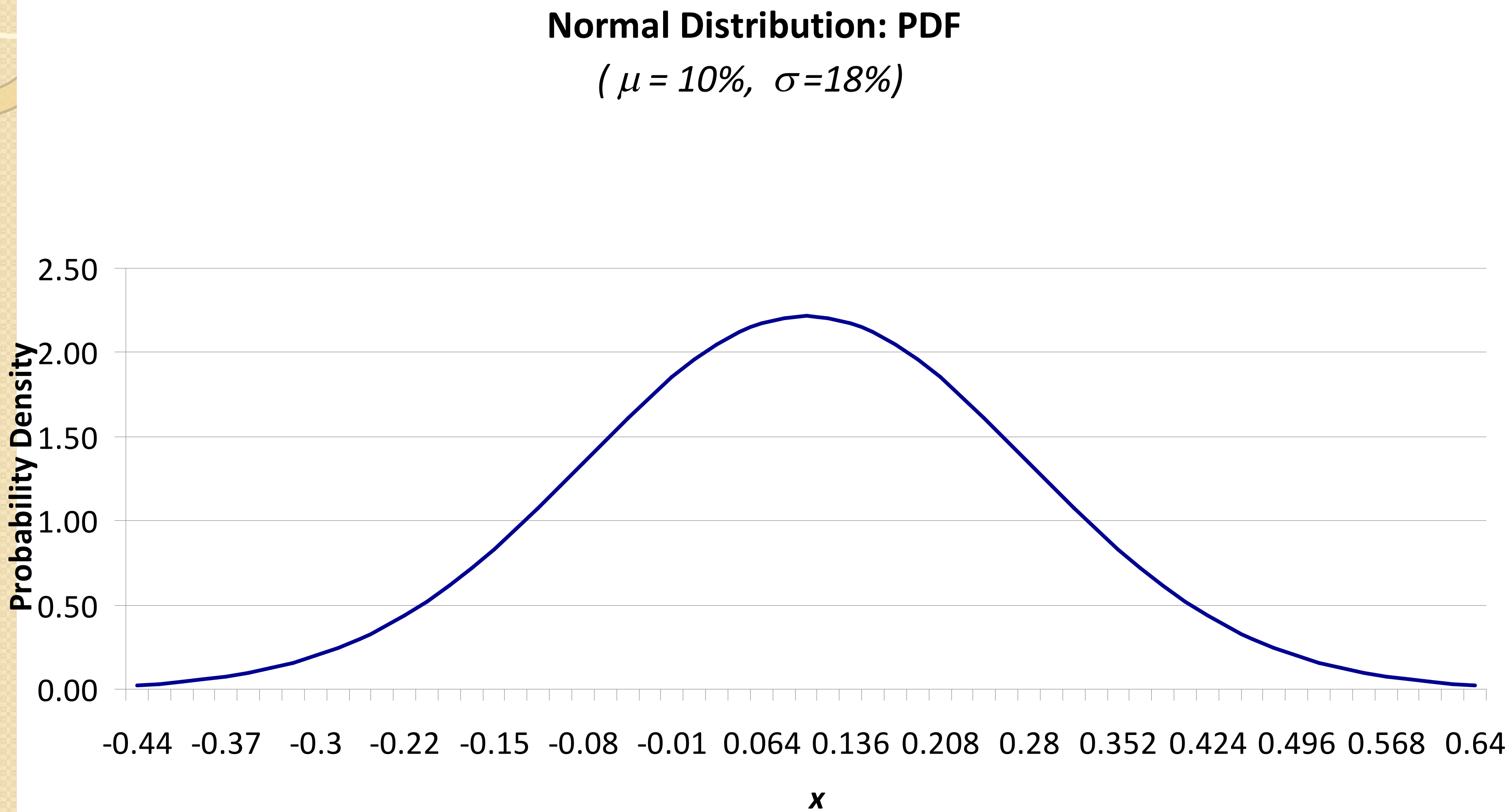
$$w_i = \frac{\text{Market Value of Investment in Asset } i}{\text{Total Market Value of the Portfolio}}$$

- Because all of the wealth must be invested in the assets, the proportion of wealth invested or "weights" invested in the various assets must equal 100% of wealth.

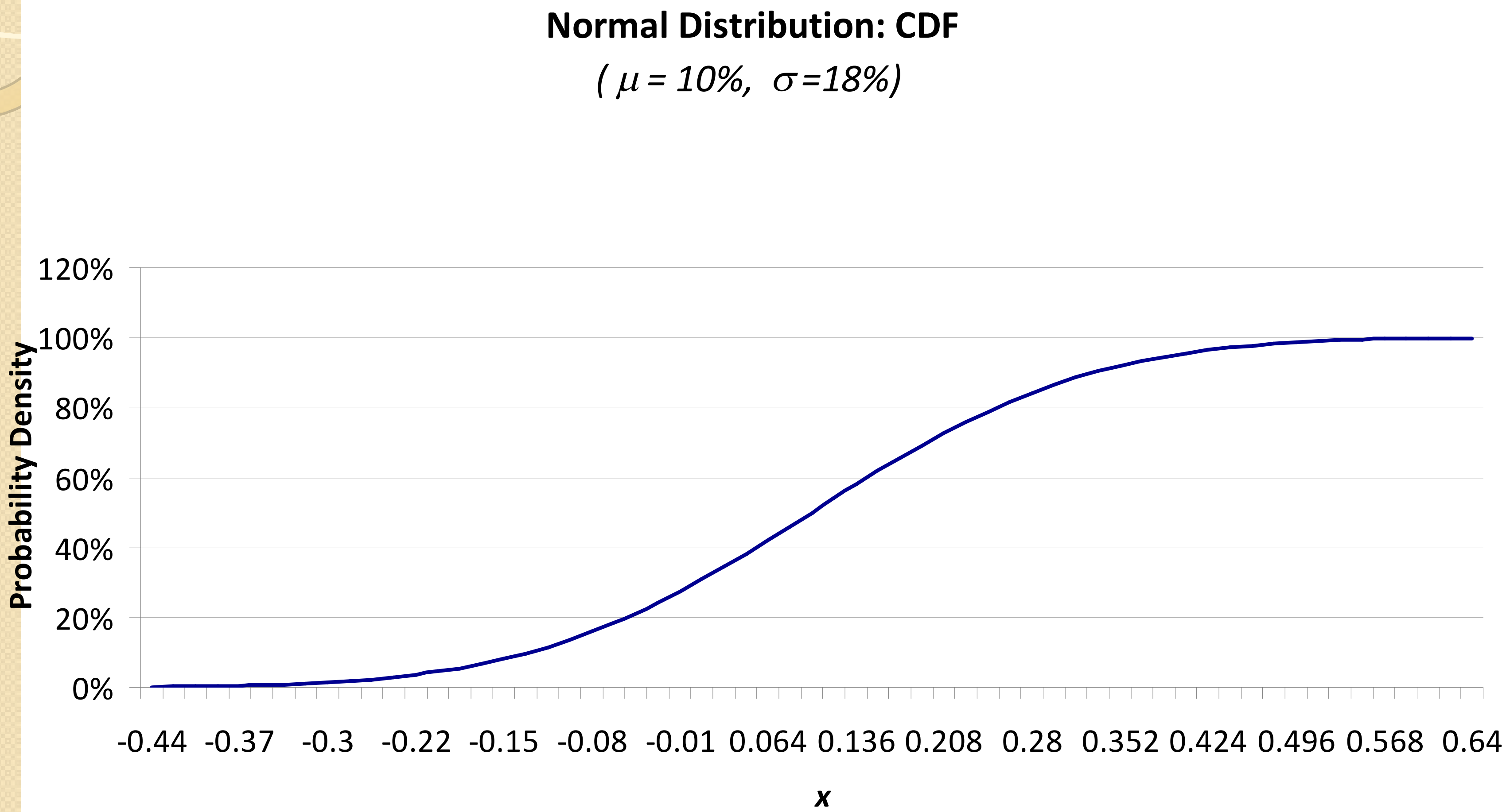
The core assumption of the MPT

- Our first, and main, assumption is that all the risky assets are fully characterized by:
  - Their expected return (denoted by μ_i, for Asset i, i = 1,…,N);
  - The standard deviation of their returns (denoted by σ_i, for Asset i, $i = 1,\ldots,N$);
  - The correlation of their return with the return of any other asset (the return correlation of Assets I and j is denoted by ρ_{ij}, for $i,j = 1,\ldots,N$).
- Note that this assumption is satisfied as long as the distribution of asset returns is Elliptical
  - Elliptical distribution are an important family of probability distributions;
  - Pre-eminent members include the Normal and *t* distributions.

Normal Distribution: Probability Density Function



Normal Distribution: Cumulative Density Function



MPT as a Mean-Variance optimization problem

- For Markowitz, the objective of any investor is to achieve either:
  - The highest return for a given risk budget;
  - The lowest level of risk for a given return objective.

- Under the assumptions made in the previous slide, we could say

Return := Expected return of the asset / portfolio

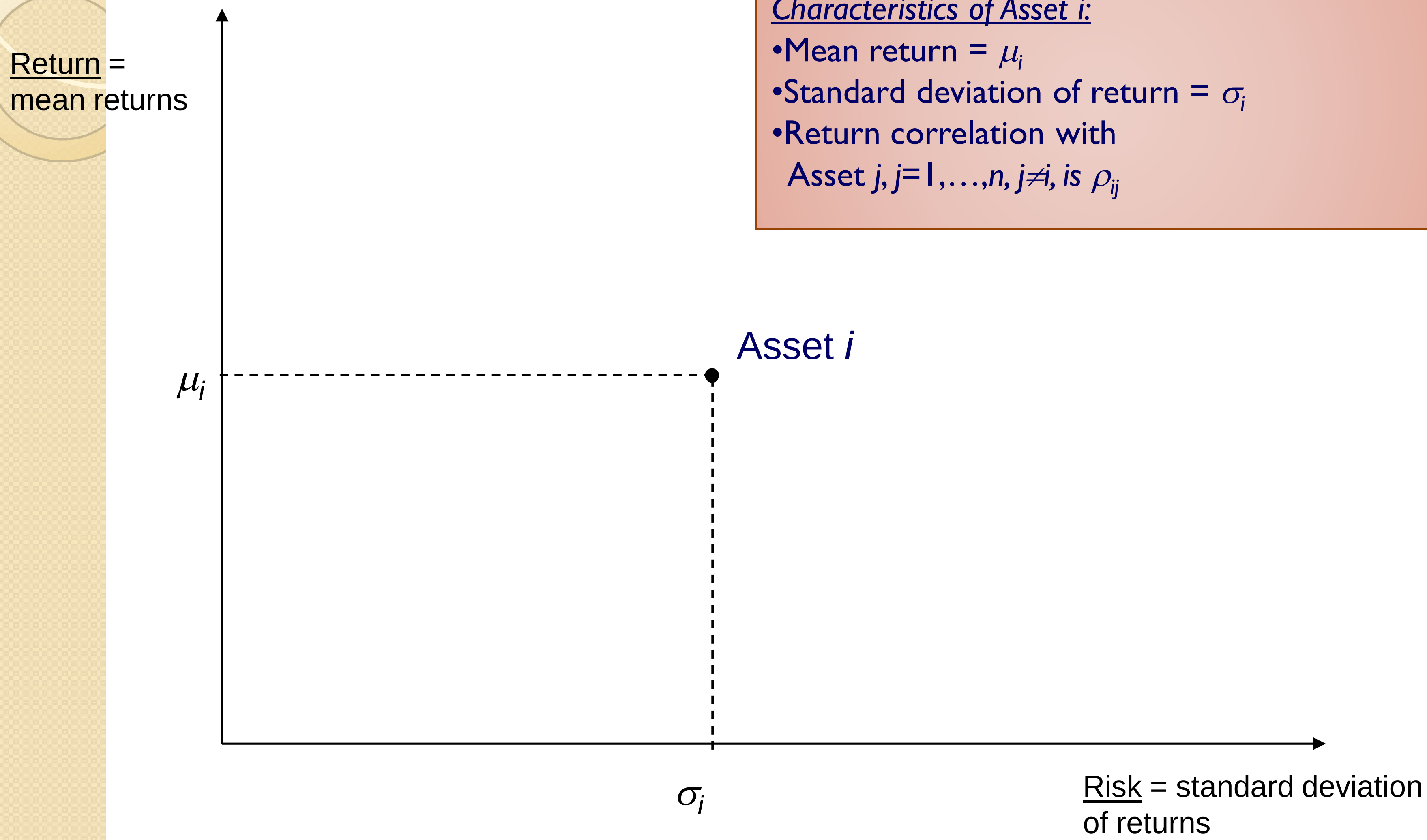
Risk := Variance of the returns of the asset / portfolio

- With these definitions of "risk" and "return," we can express the investor's objective as a Mean-Variance optimization problem.

Representing a Risky Asset

Characteristics of Asset i:

- Mean return = μ_i
- Standard deviation of return = σ_i
- Return correlation with Asset *j*, $j=1,\ldots,n$, $j \neq i$, is ρ_{ij}



A very special security: the risk-free asset

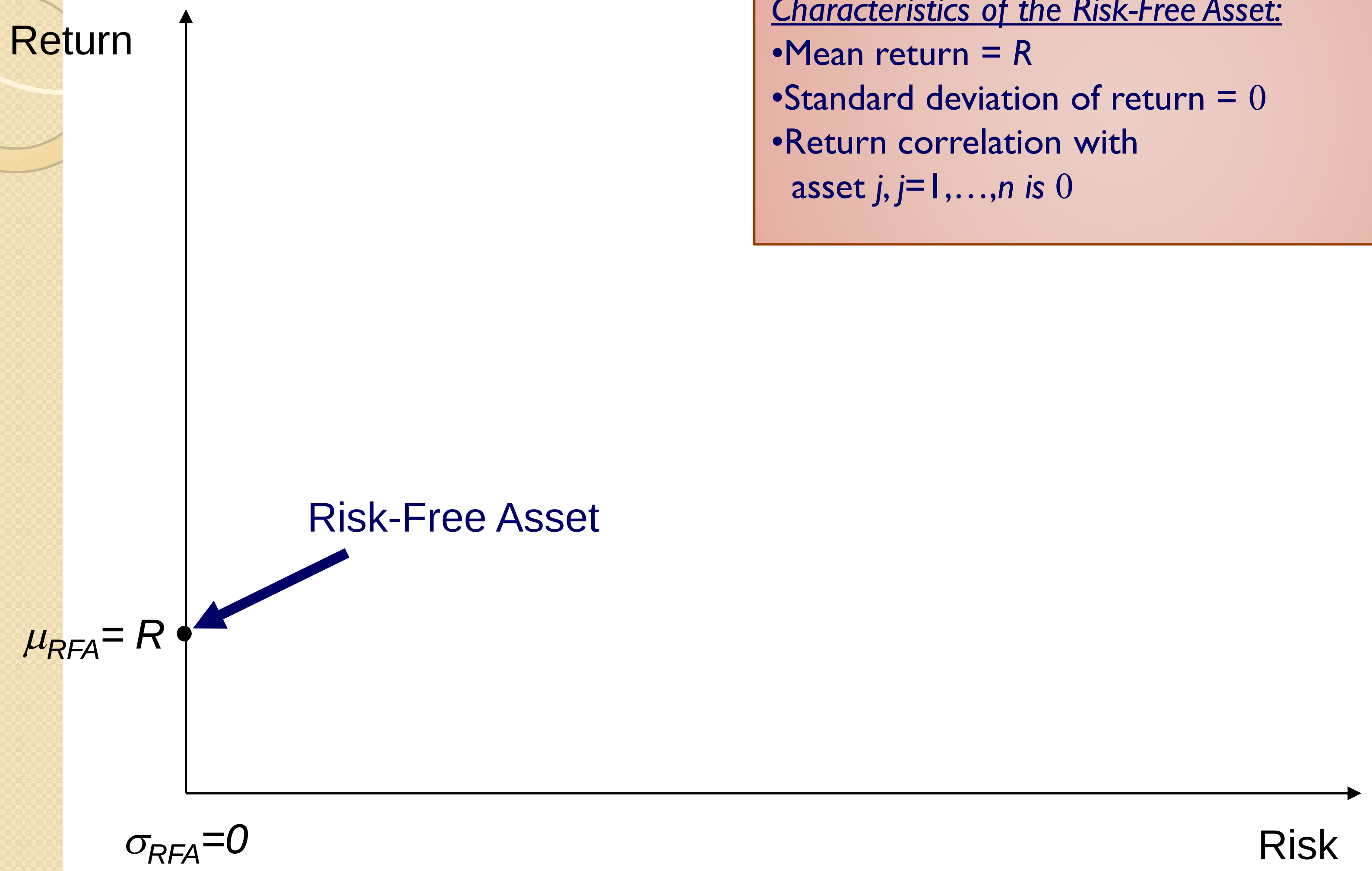
- What if you do not want to invest in any risky asset? In fact, what if you only want to deposit your money in some bank account at a fixed rate *R*?

- If this "bank" does not have any risk of defaulting, then your deposit does not carry any risk: it is a **risk-free asset** (RFA).

- As a result,
  - The **expected return** of the RFA, called the risk-free rate, is equal to **R**;
  - The **volatility** of the RFR is equal to **0** (since it does not carry any risk!);
  - The **correlation** of the RFA with any other asset is also **0**.

- The concept of risk-free asset is often used in financial economics as a proxy for a secure term deposit.

Representing the risk-free asset



Characteristics of the Risk-Free Asset:

•Mean return = R

•Standard deviation of return = 0

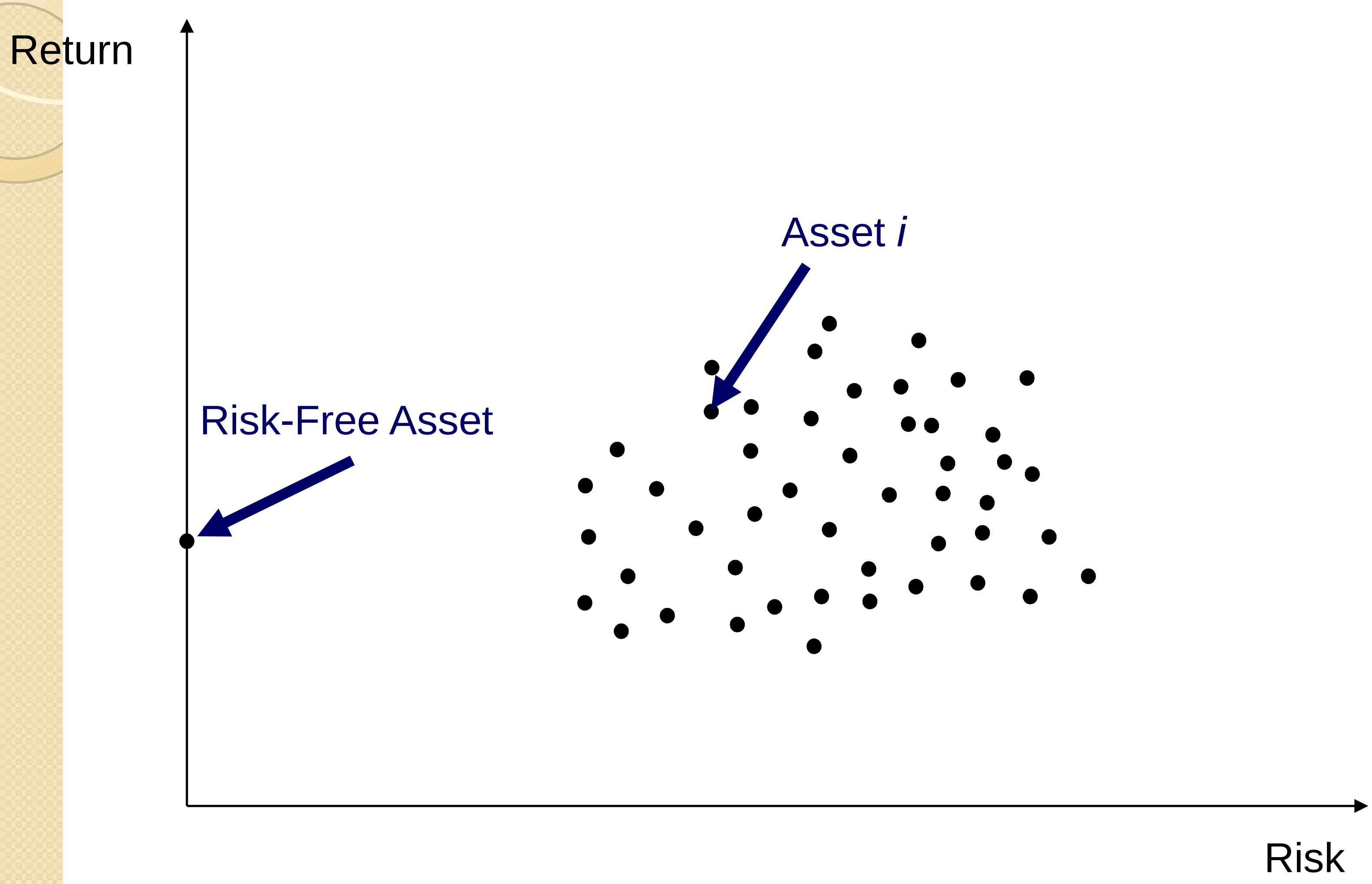
•Return correlation with asset j, $j=1,\ldots,n$ is 0

Additional assumptions

Furthermore, we assume that:

- All statistics are based on total returns, i.e. all dividends and interest paid out are reinvested in the securities.
- Fractional investing is possible;
- Investors can deposit and borrow freely at the risk-free rate;
- There is no penalty or restriction on short-selling of risky securities;
- The market is “frictionless” in the sense that there is not tax, no transaction fees, and no need for collateral or margins.

The investment universe



A Simpler Problem:
2 assets and the risk-free asset

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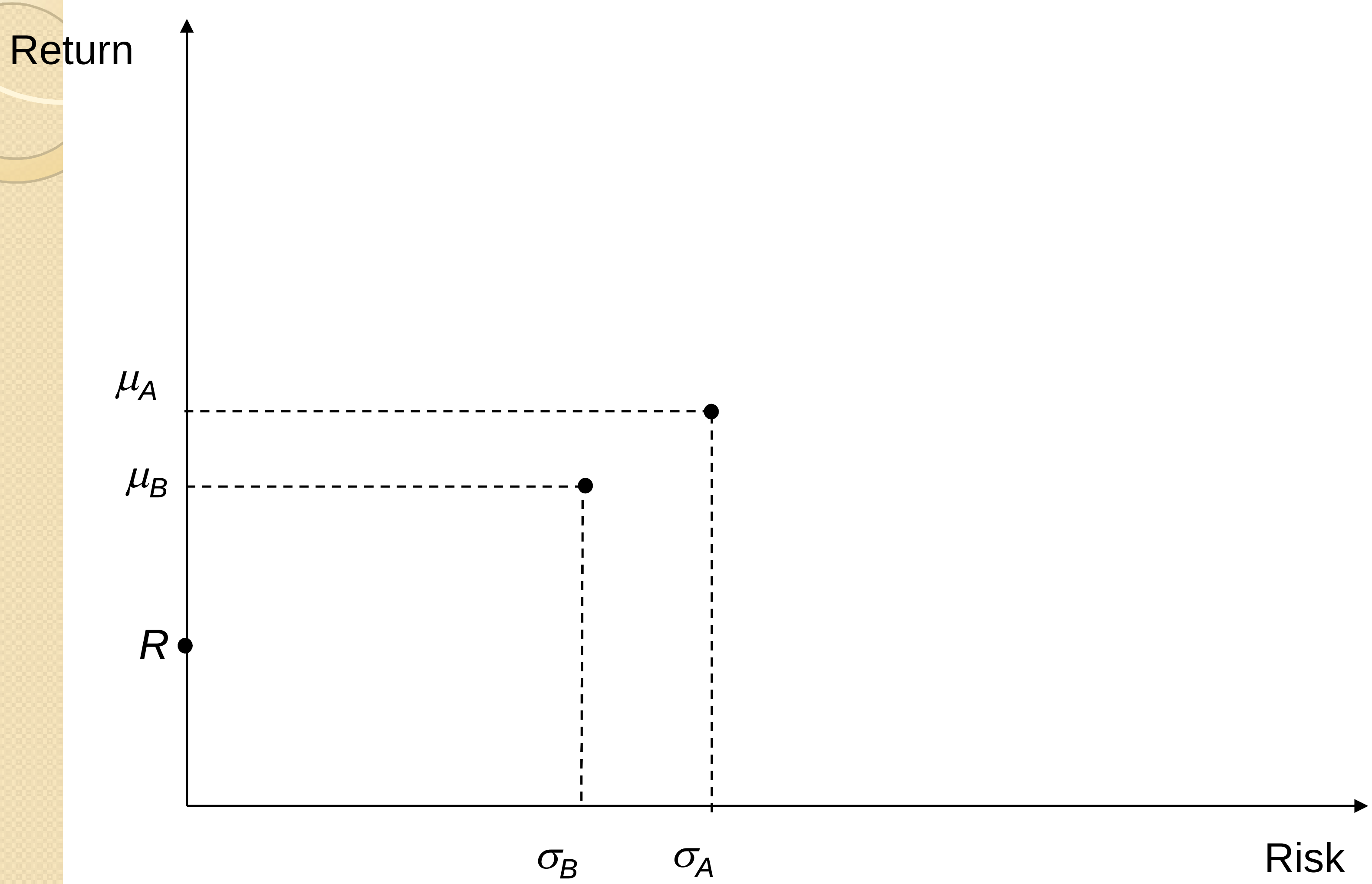


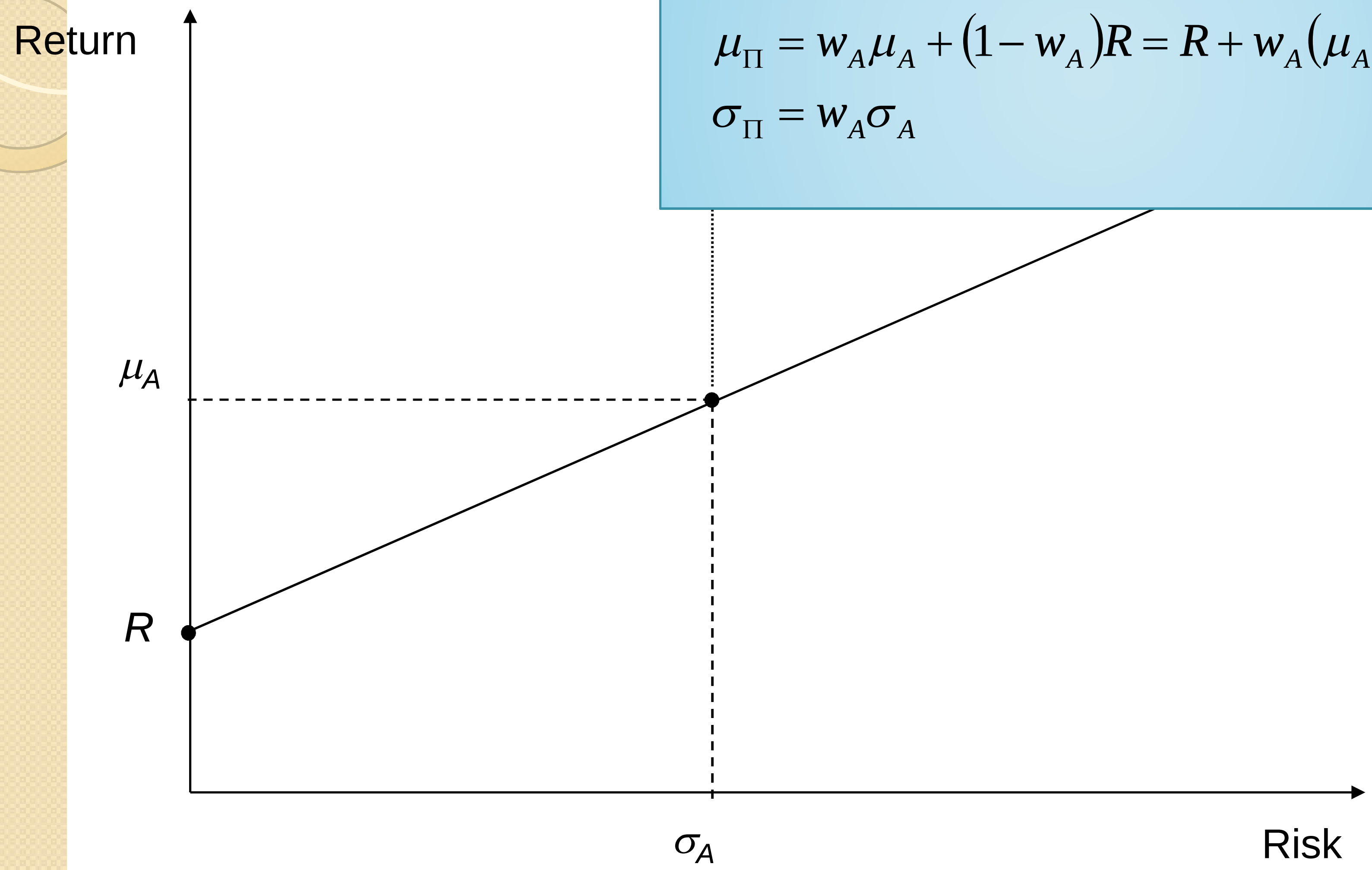
Denote by w_A the proportion of the portfolio invested in asset A and w_B the proportion invested in portfolio B.

The Risk-Free Asset and Risky Asset *A*

Line parametrized by w_A:

$$\mu_\Pi = w_A \mu_A + (1 - w_A) R = R + w_A (\mu_A - R)$$

$$\sigma_\Pi = w_A \sigma_A$$



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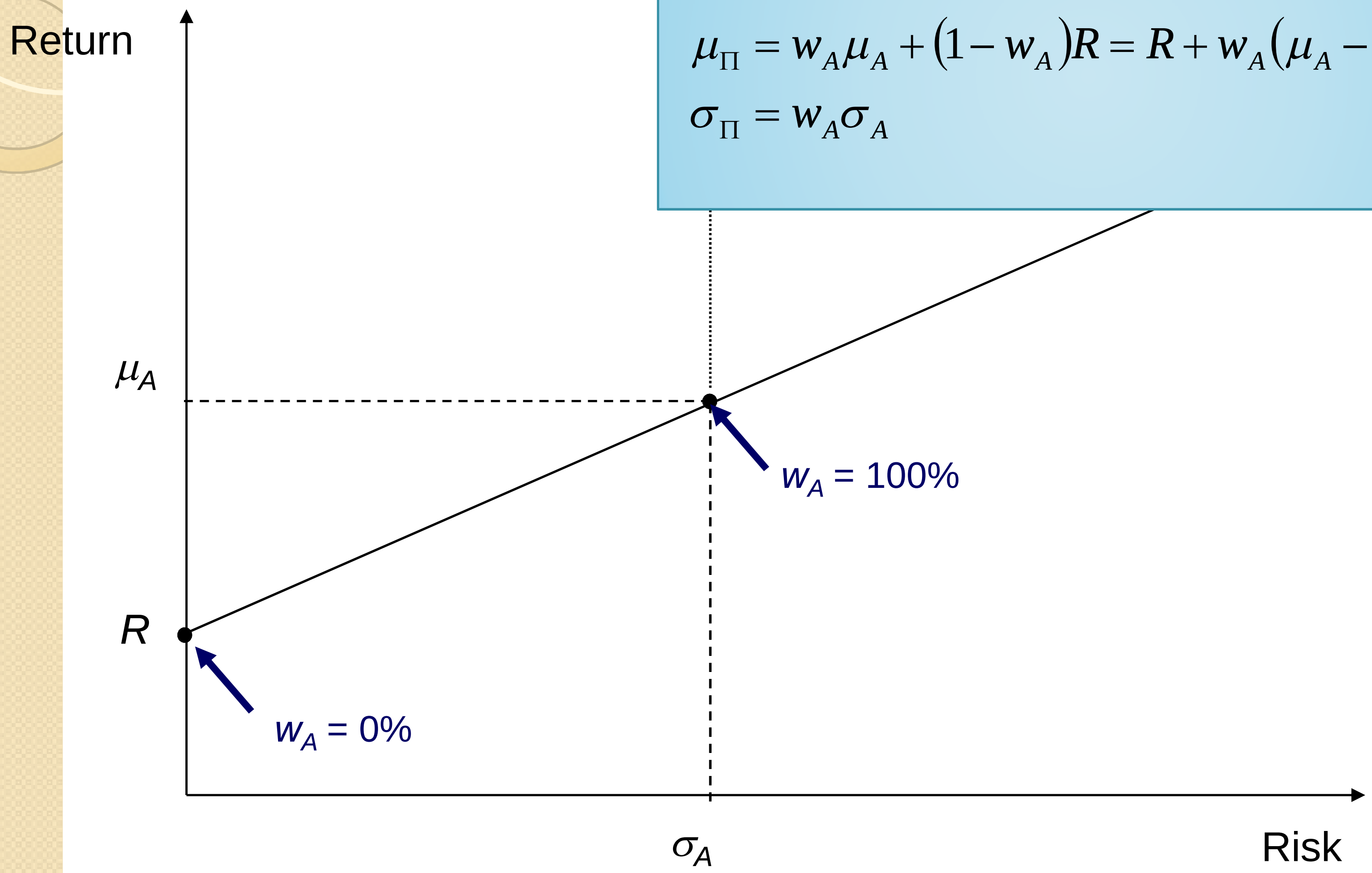

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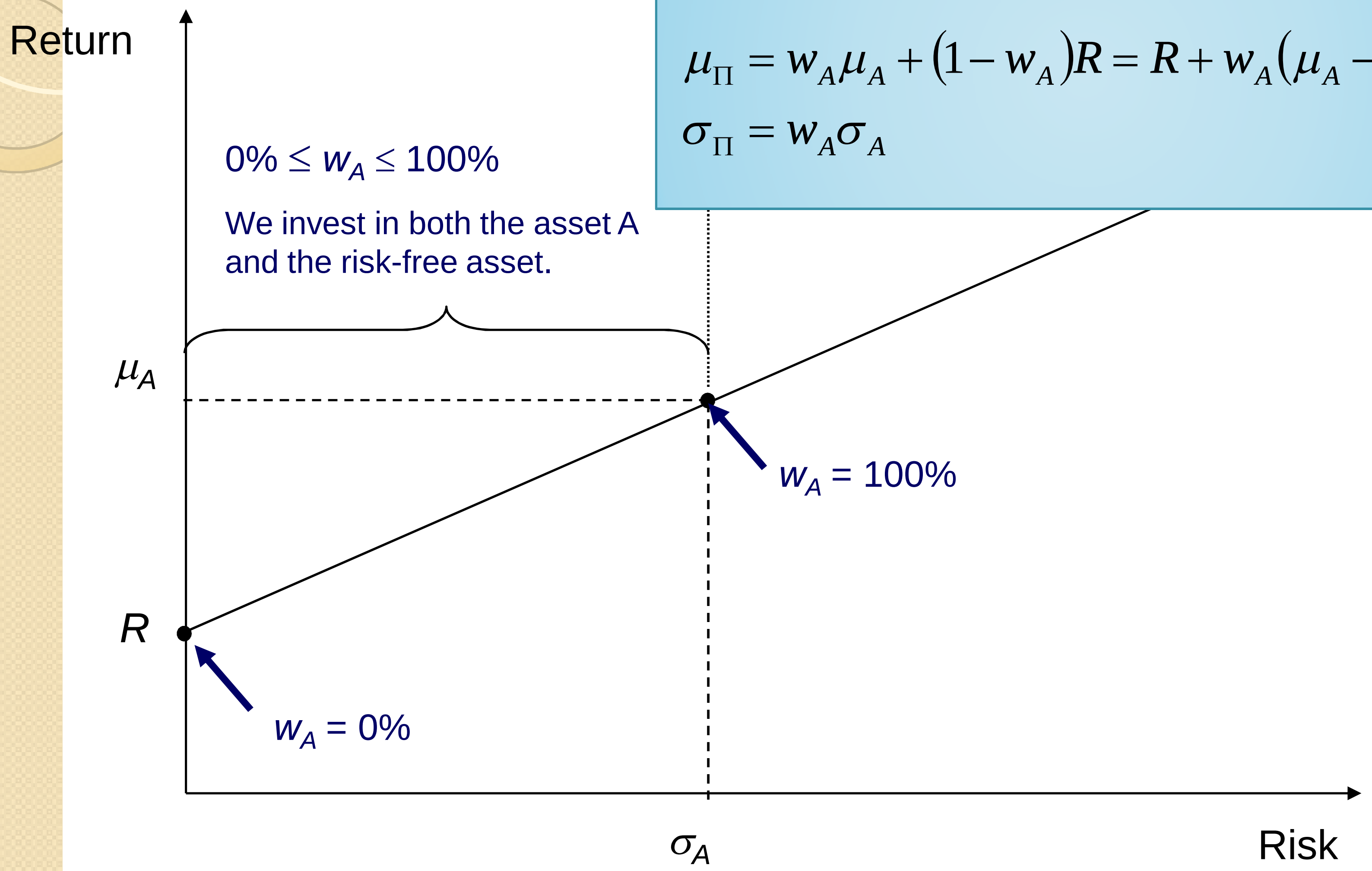

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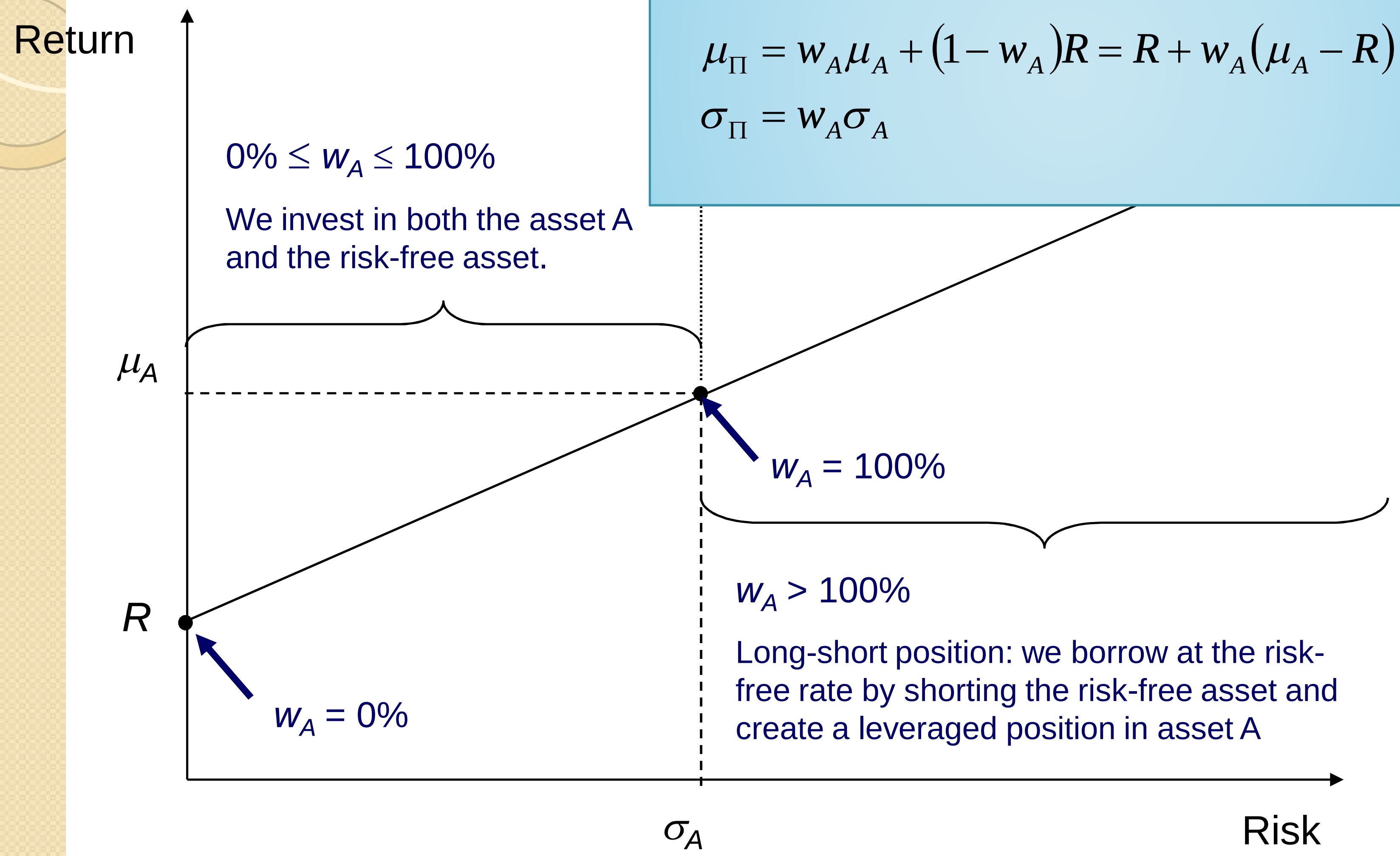
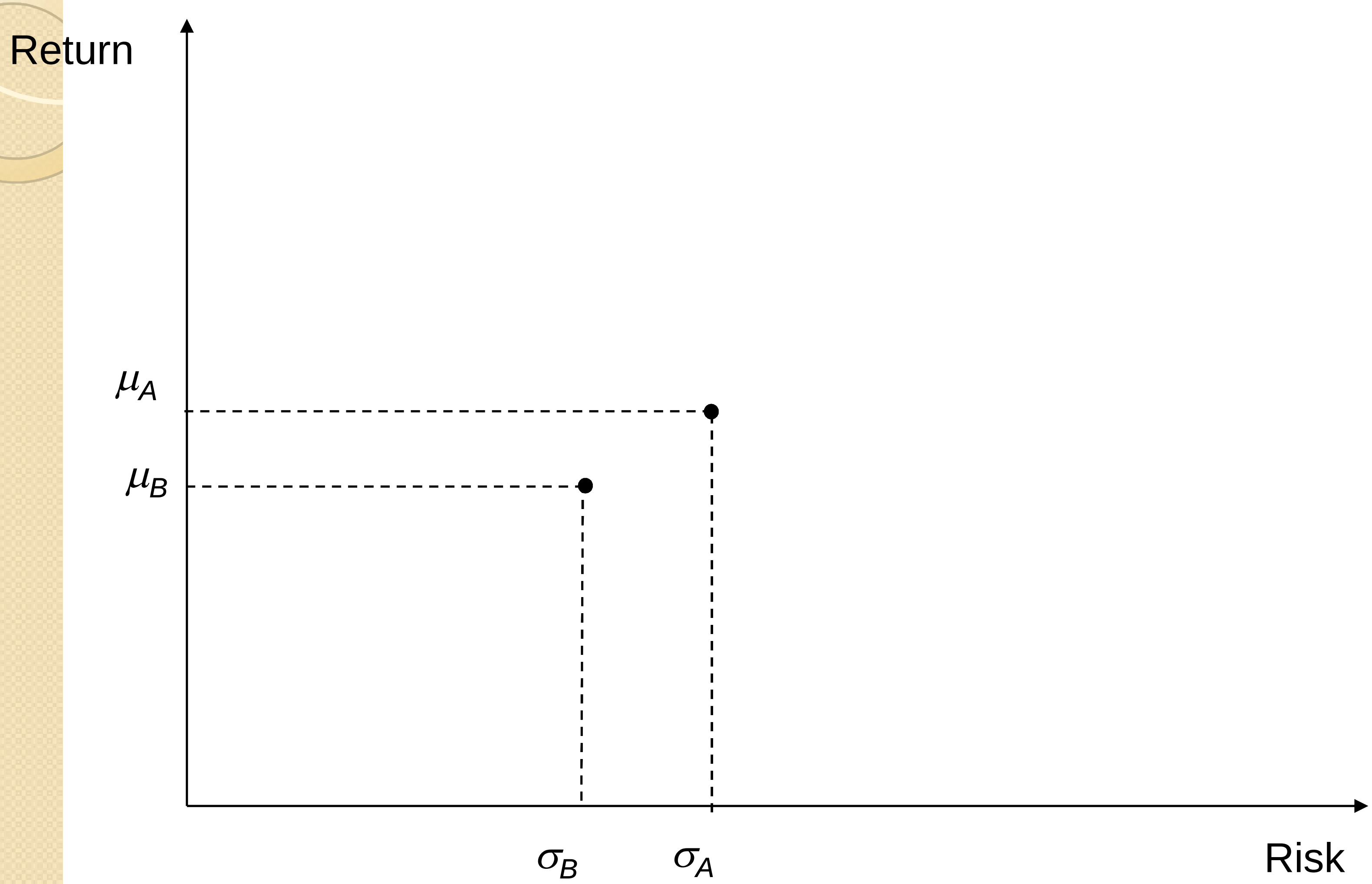


The case with two risky assets

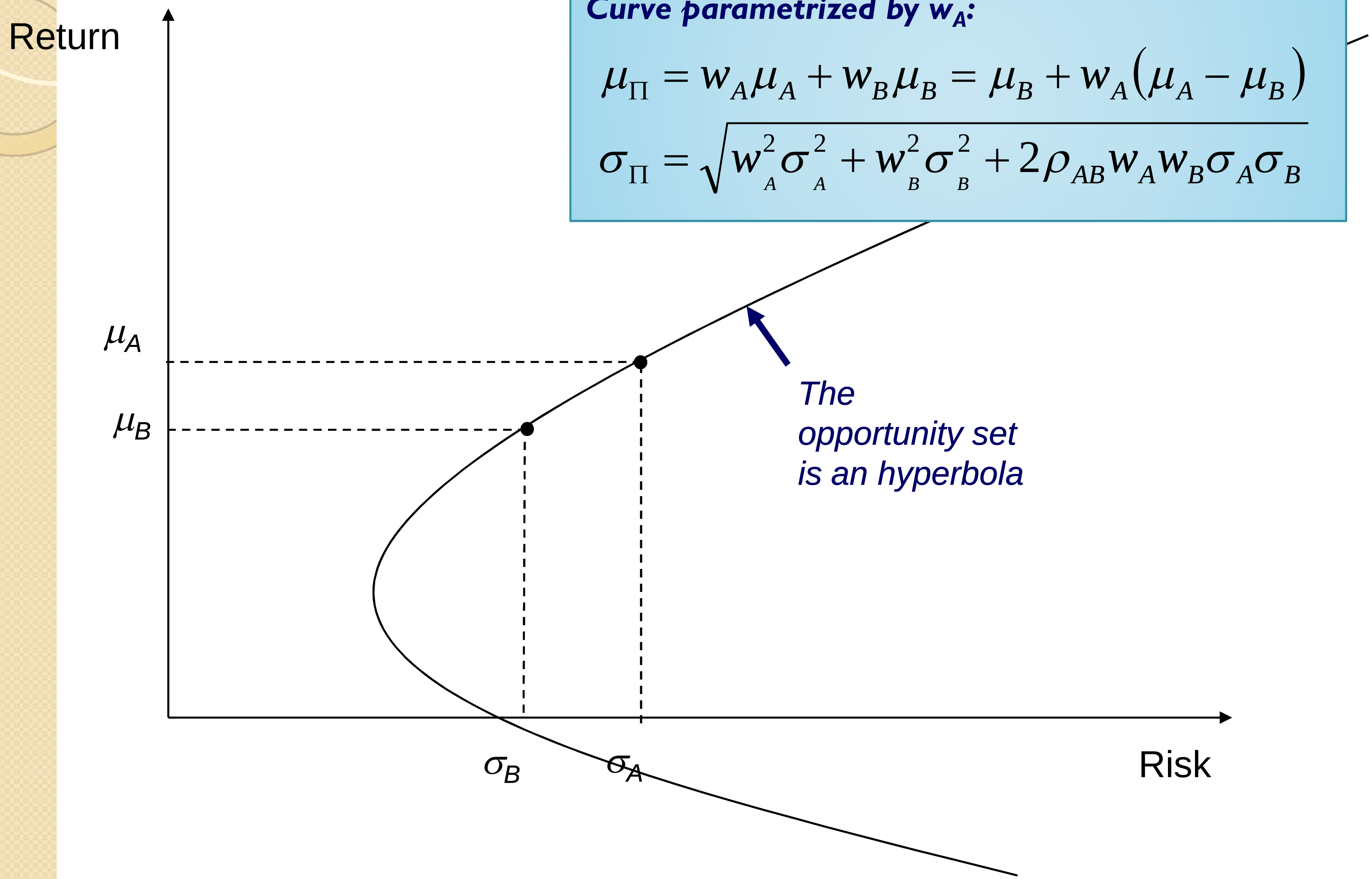


The opportunity set

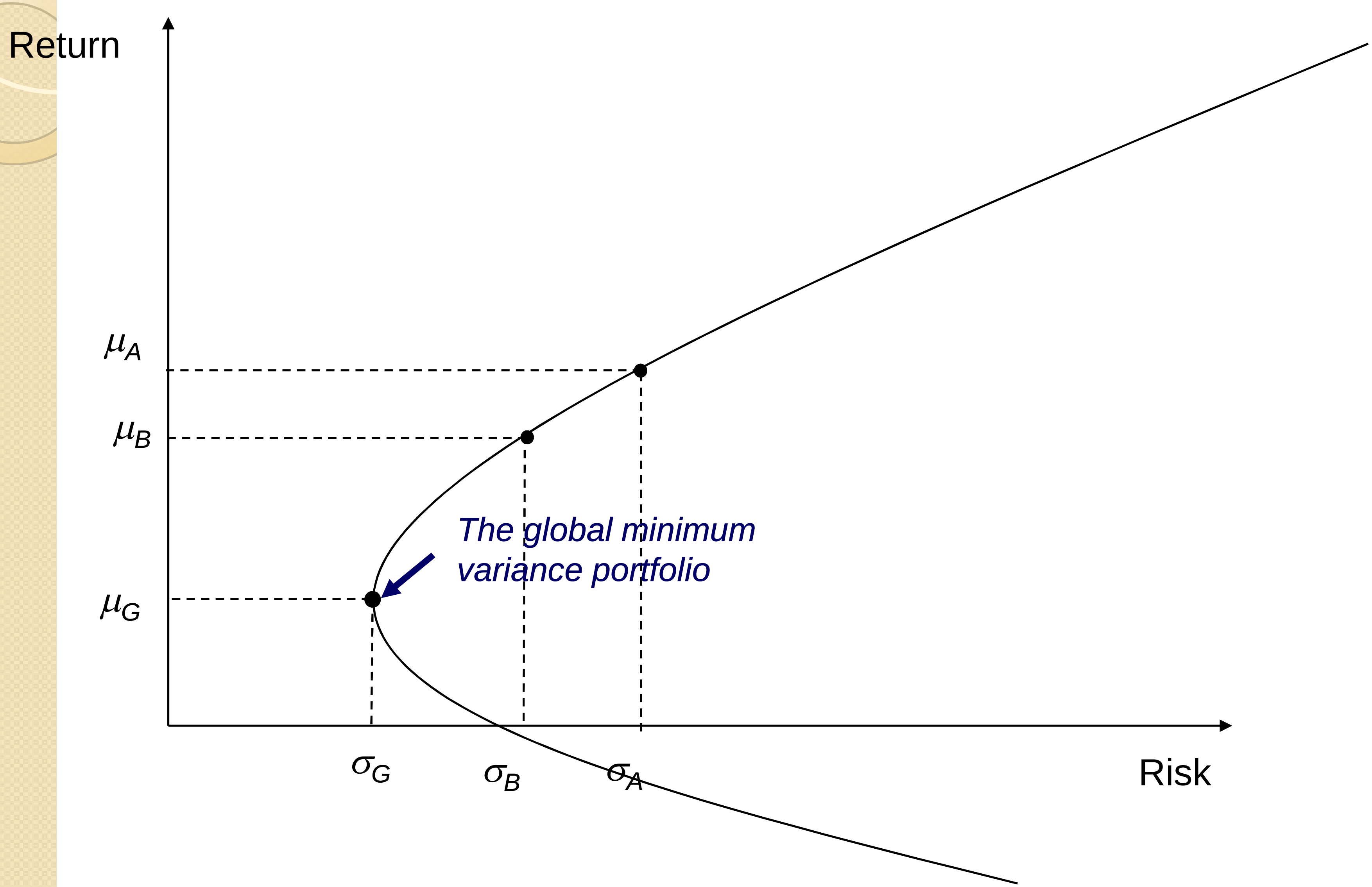
Curve parametrized by w_A:

$$\mu_\Pi = w_A\mu_A + w_B\mu_B = \mu_B + w_A(\mu_A - \mu_B)$$

$$\sigma_\Pi = \sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2\rho_{AB}w_Aw_B\sigma_A\sigma_B}$$

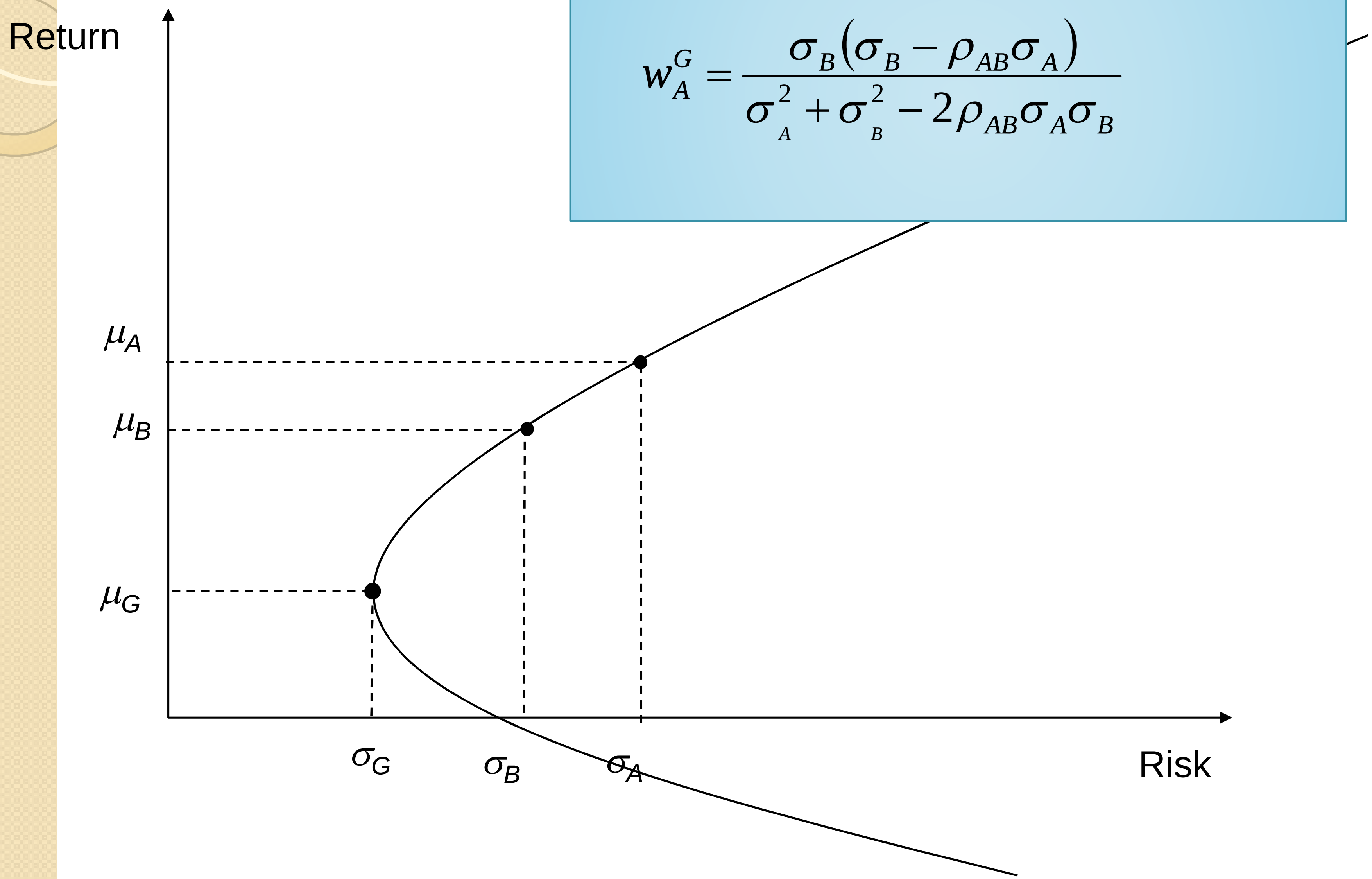


The global minimum variance portfolio

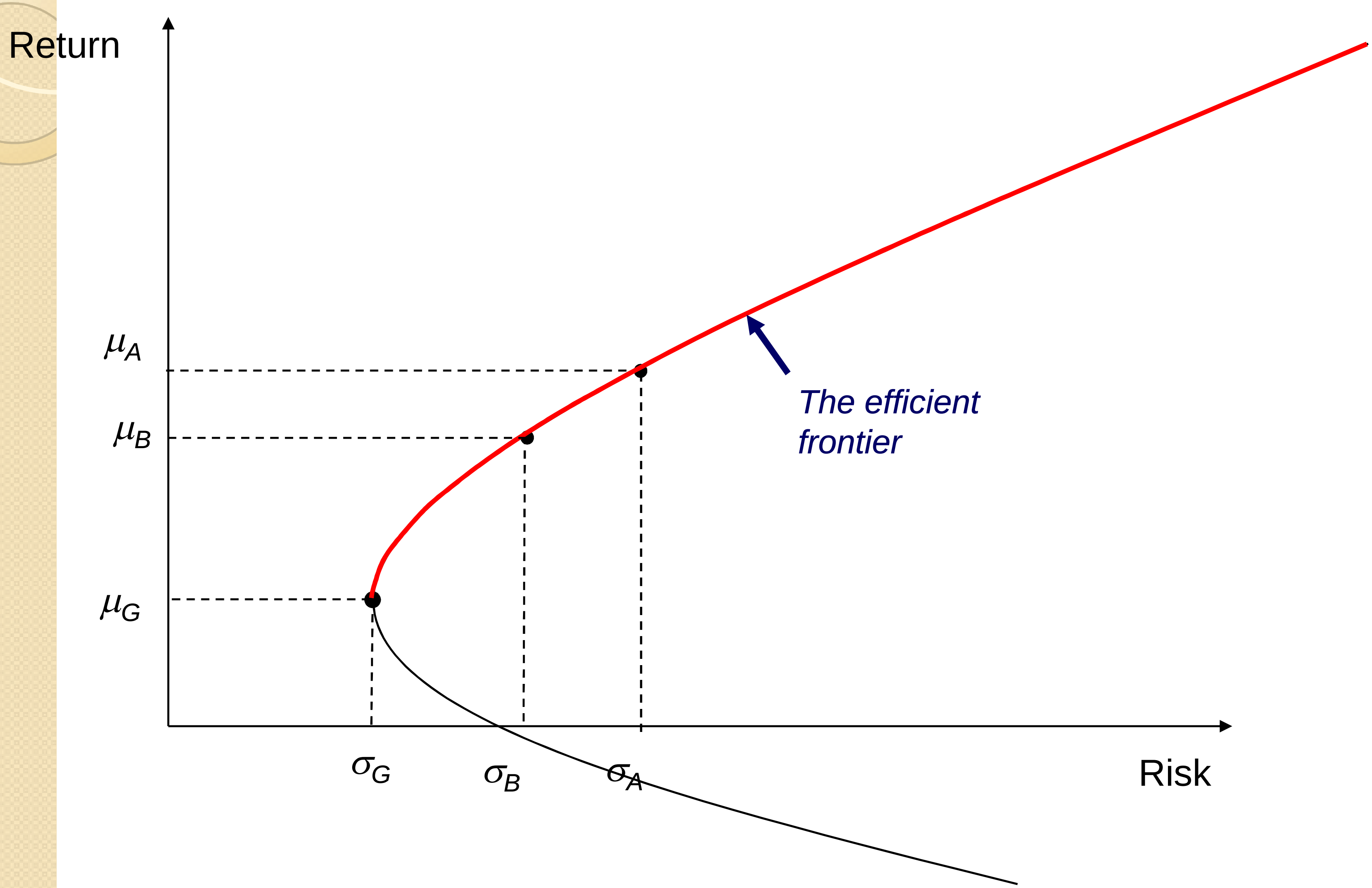


The global minimum variance portfolio's allocation

$$w_A^G = \frac{\sigma_B\left(\sigma_B - \rho_{AB}\sigma_A\right)}{\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B}$$



The efficient frontier



Case 1: ρ_{AB} = 1

Portfolio characteristics:

$$\mu_{\Pi} = w_A \mu_A + w_B \mu_B = \mu_B + w_A(\mu_A - \mu_B)$$

$$\sigma_{\Pi} = |w_A \sigma_A + w_B \sigma_B| = |\sigma_B + w_A(\sigma_A - \sigma_B)|$$

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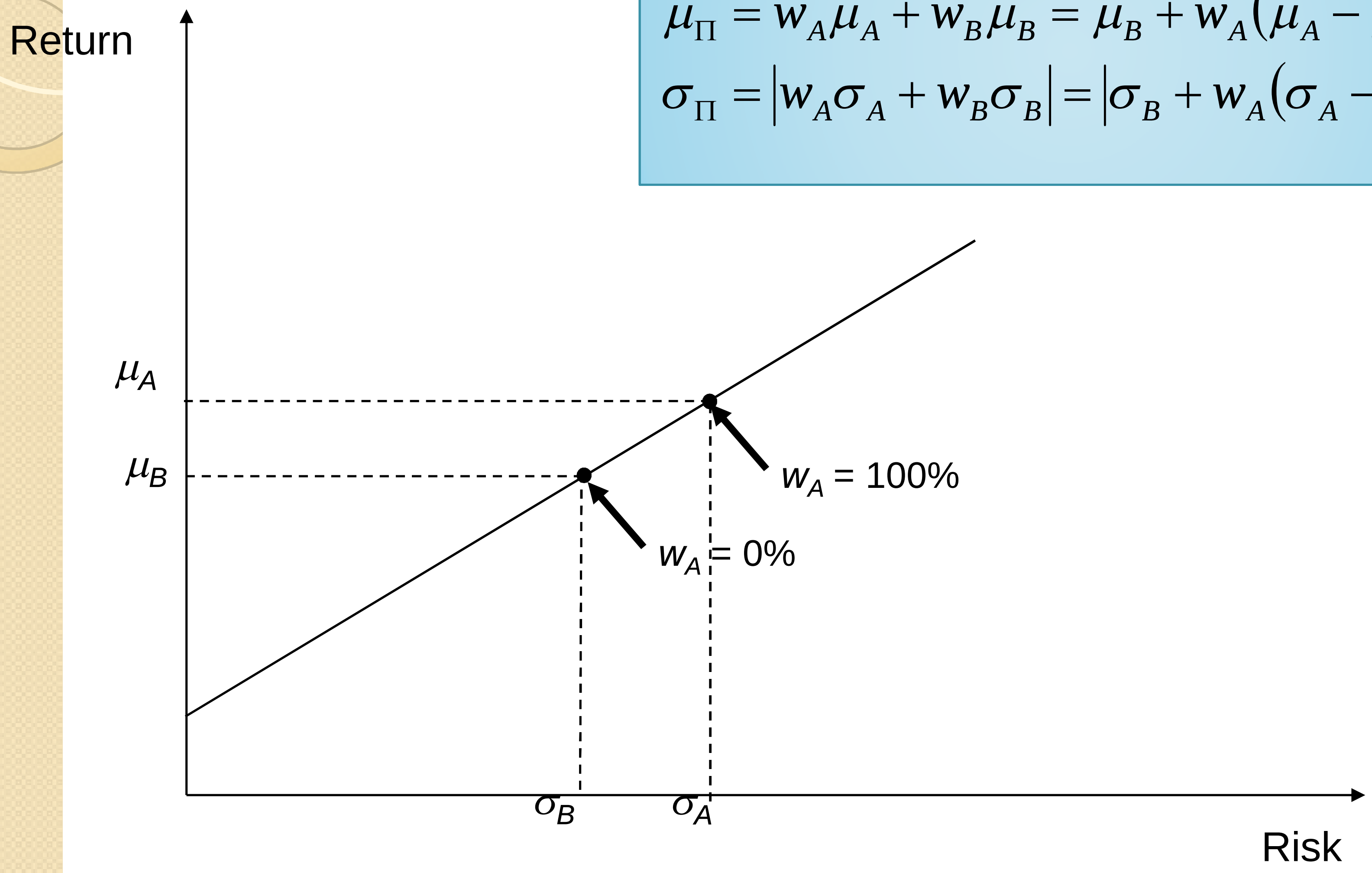

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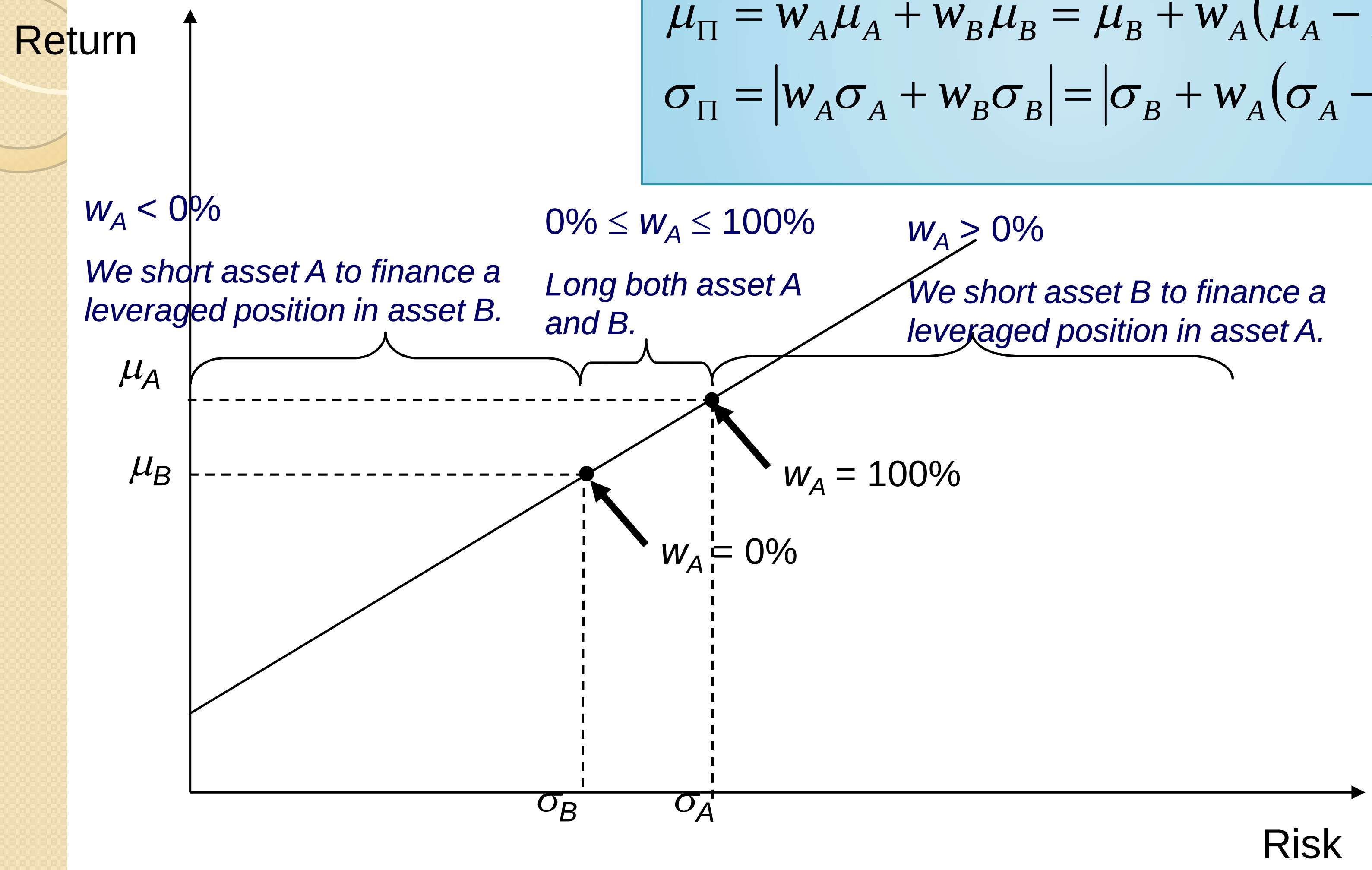


Case 2: ρ_{AB} = -1

Portfolio characteristics:

$$\mu_{\Pi} = w_A \mu_A + w_B \mu_B = \mu_B + w_A(\mu_A - \mu_B)$$

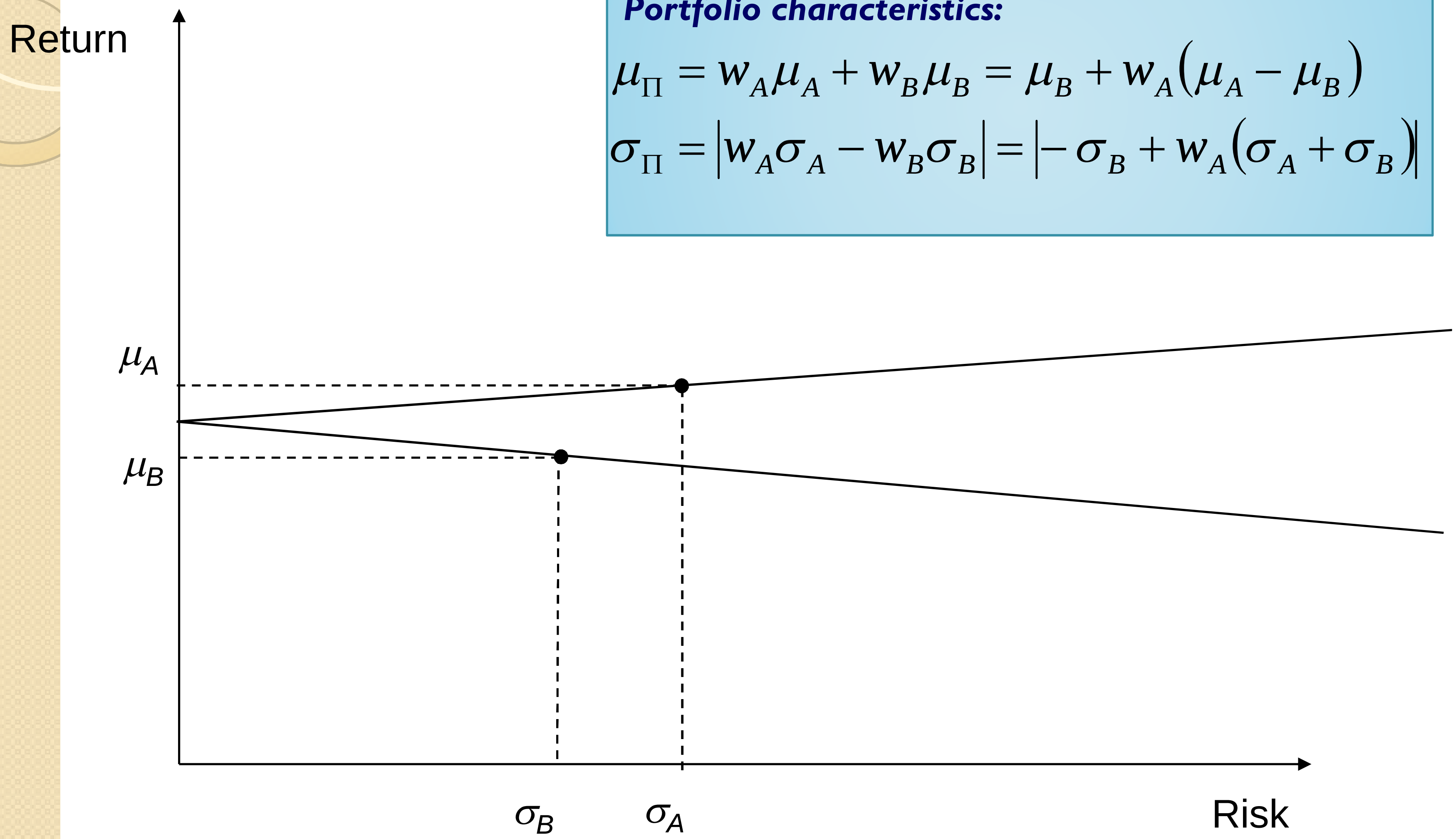
$$\sigma_{\Pi} = \left| w_A \sigma_A - w_B \sigma_B \right| = \left| -\sigma_B + w_A(\sigma_A + \sigma_B) \right|$$

Case 2: ρ_{AB} = -1

Portfolio characteristics:

$$\mu_\Pi = w_A\mu_A + w_B\mu_B = \mu_B + w_A(\mu_A - \mu_B)$$

$$\sigma_\Pi = \left|w_A\sigma_A - w_B\sigma_B\right| = \left|-\sigma_B + w_A(\sigma_A + \sigma_B)\right|$$

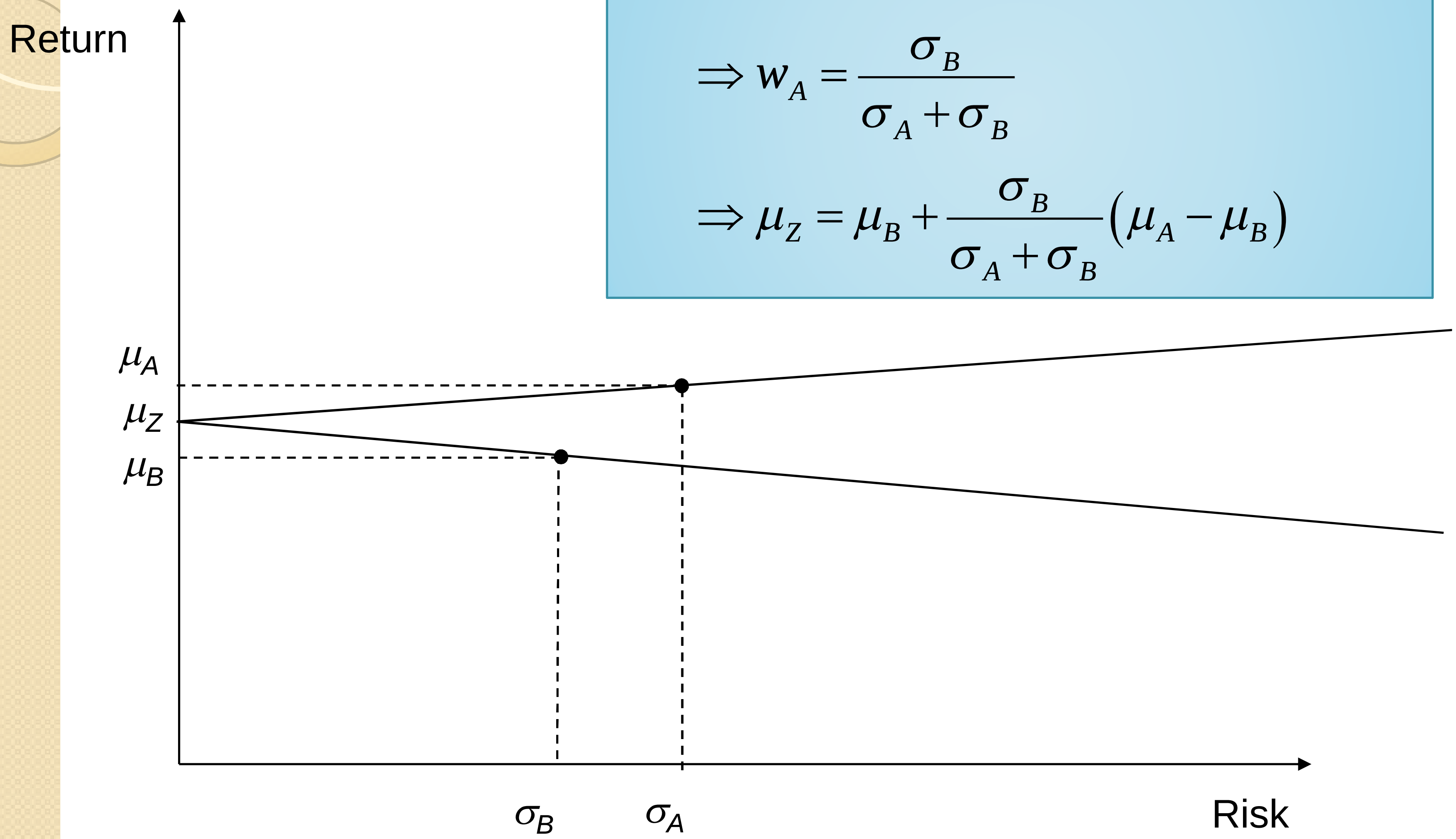


The zero-variance portfolio (assuming $\sigma_A > \sigma_B$)

$$\sigma_Z = 0$$

$$\Rightarrow w_A = \frac{\sigma_B}{\sigma_A + \sigma_B}$$

$$\Rightarrow \mu_Z = \mu_B + \frac{\sigma_B}{\sigma_A + \sigma_B}(\mu_A - \mu_B)$$



Case 3: ρ_{AB} = 0

Portfolio characteristics:

$$\mu_{\Pi} = w_A \mu_A + w_B \mu_B = \mu_B + w_A(\mu_A - \mu_B)$$

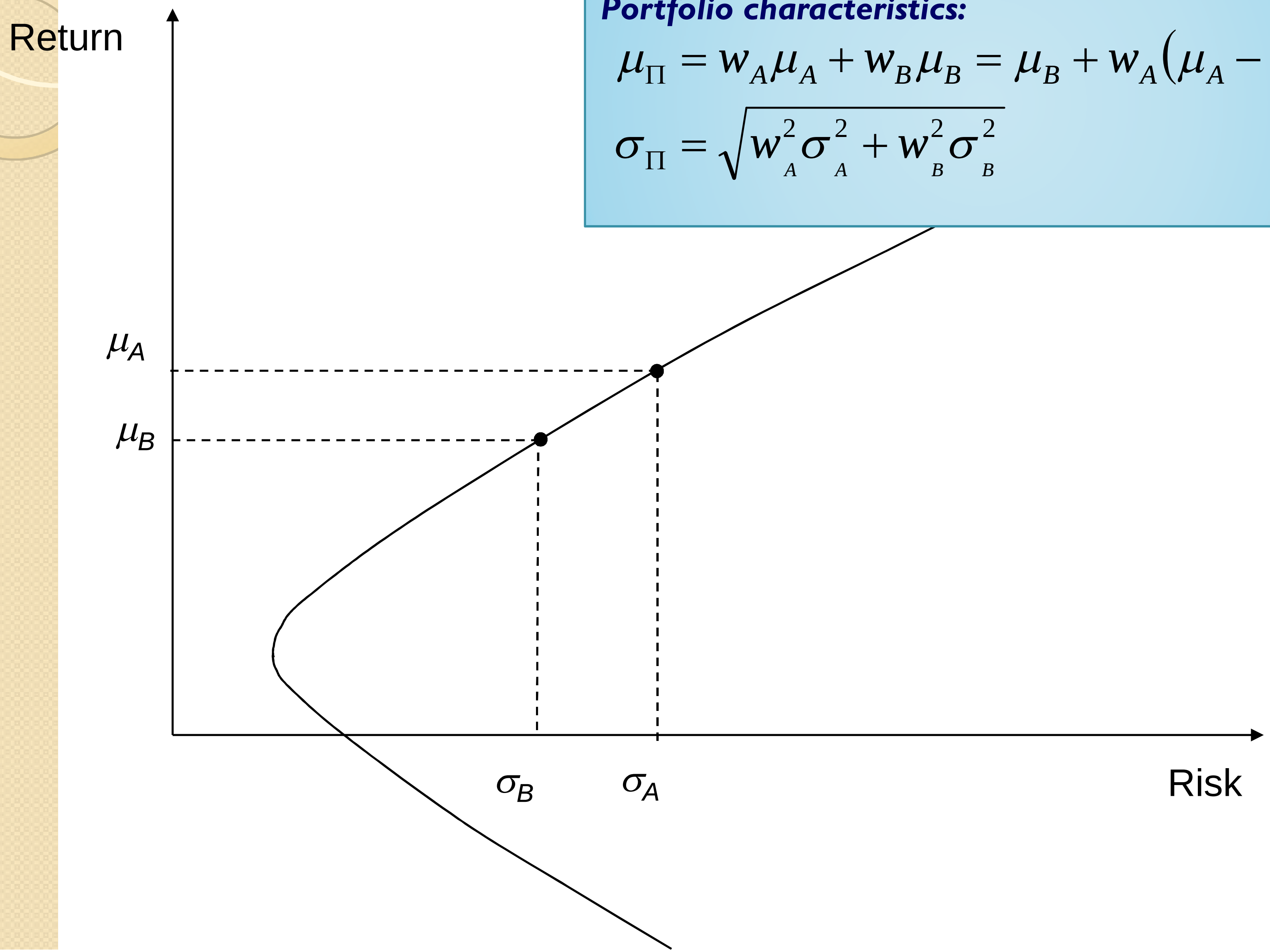
$$\sigma_{\Pi} = \sqrt{w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2}$$

Case 3: $\rho_{AB} = 0$

Portfolio characteristics:

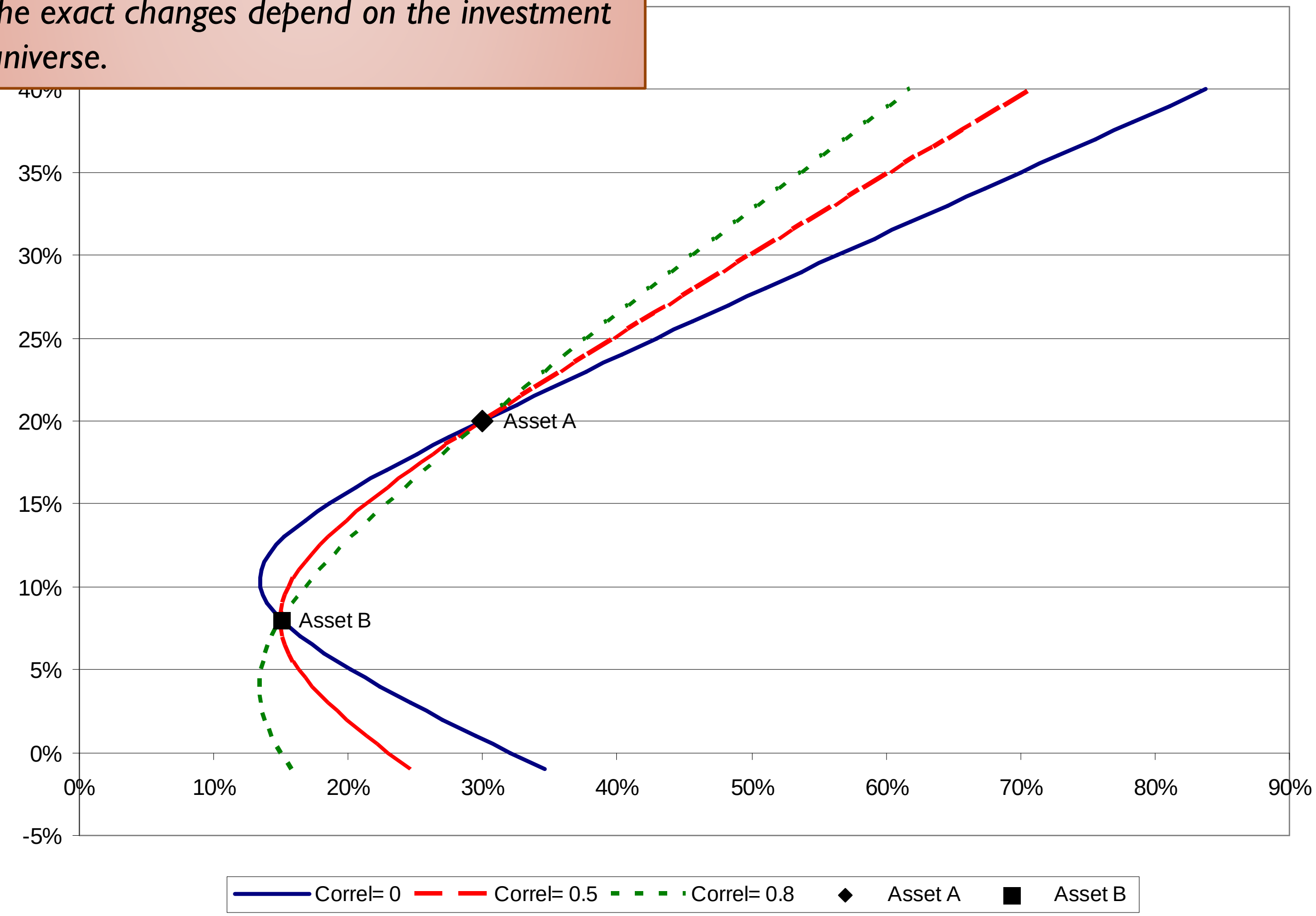
$$\mu_\Pi = w_A\mu_A + w_B\mu_B = \mu_B + w_A(\mu_A - \mu_B)$$

$$\sigma_\Pi = \sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2}$$



Conclusion: so what happens as the correlation changes?

As the correlation changes, the opportunity set and efficient frontiers change, although the exact changes depend on the investment universe.

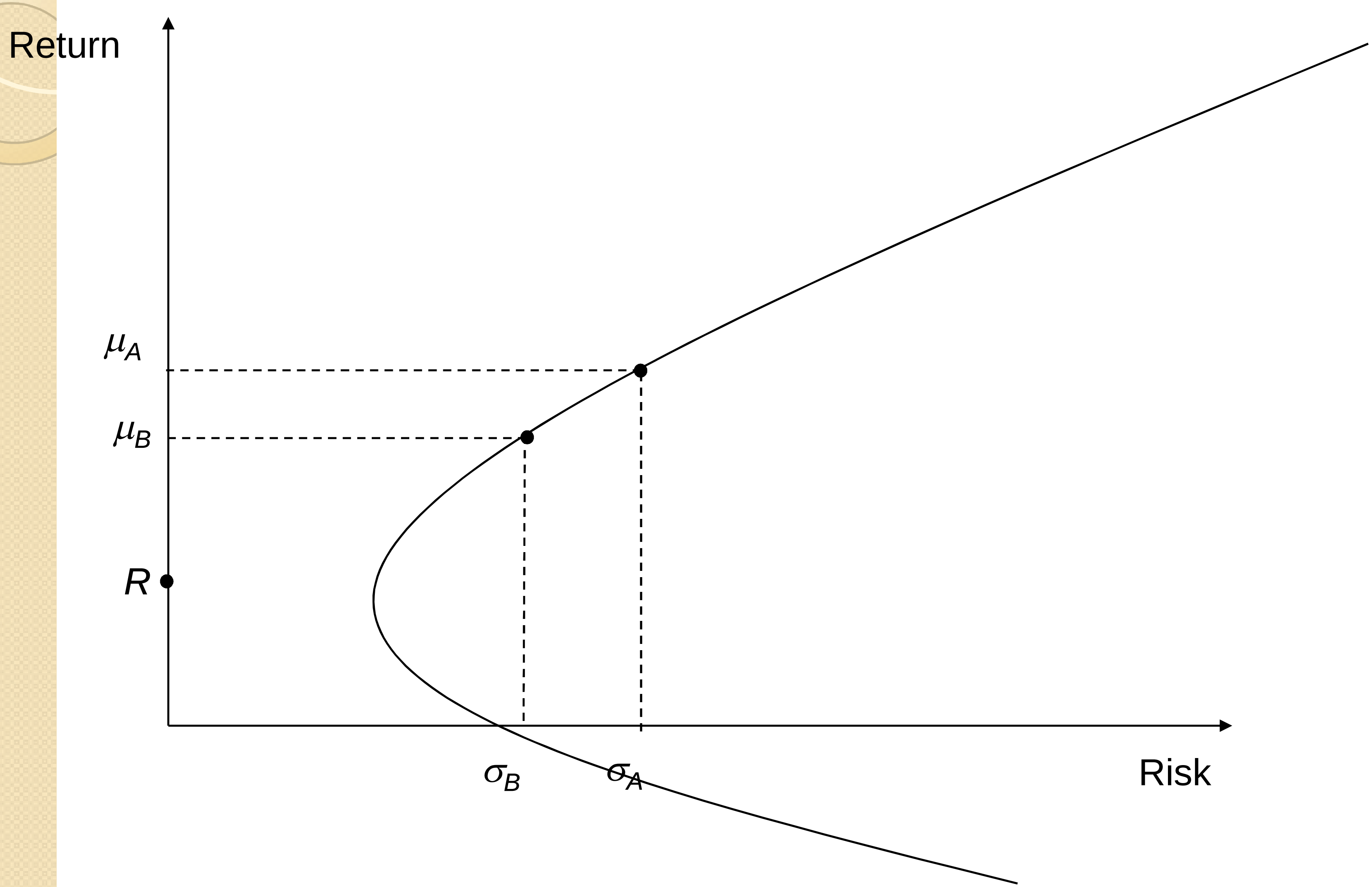


(Re)introducing the risk-free asset

- What happens if we now consider an allocation between the risk-free asset and the two risky securities?

- Surely, this new problem should be the same thing as:
  - Selecting a risky portfolio P made of positions in securities A and B, and then;
  - Allocating funds between the risk-free asset and the portfolio P.

Building the new efficient frontier



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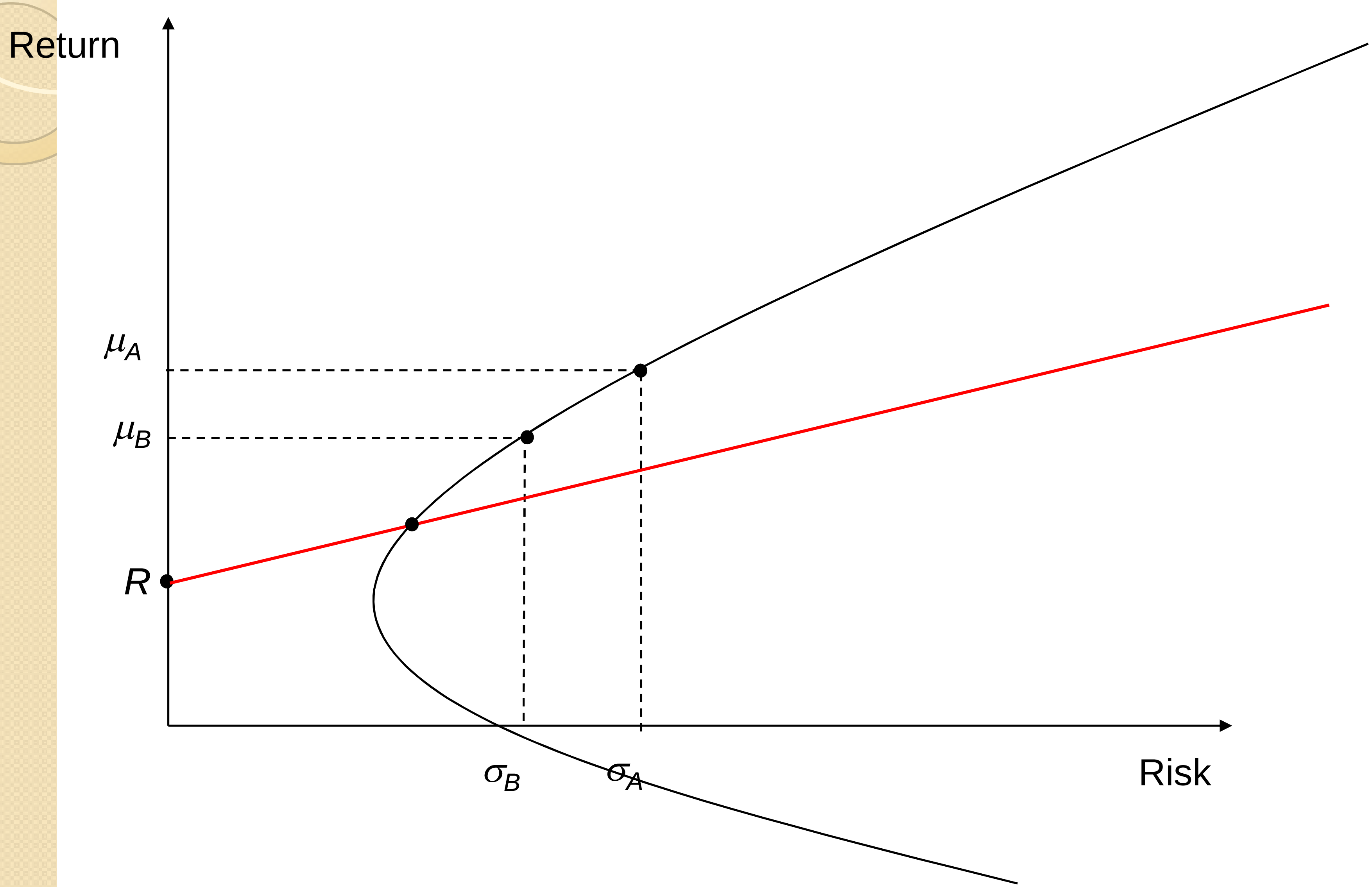

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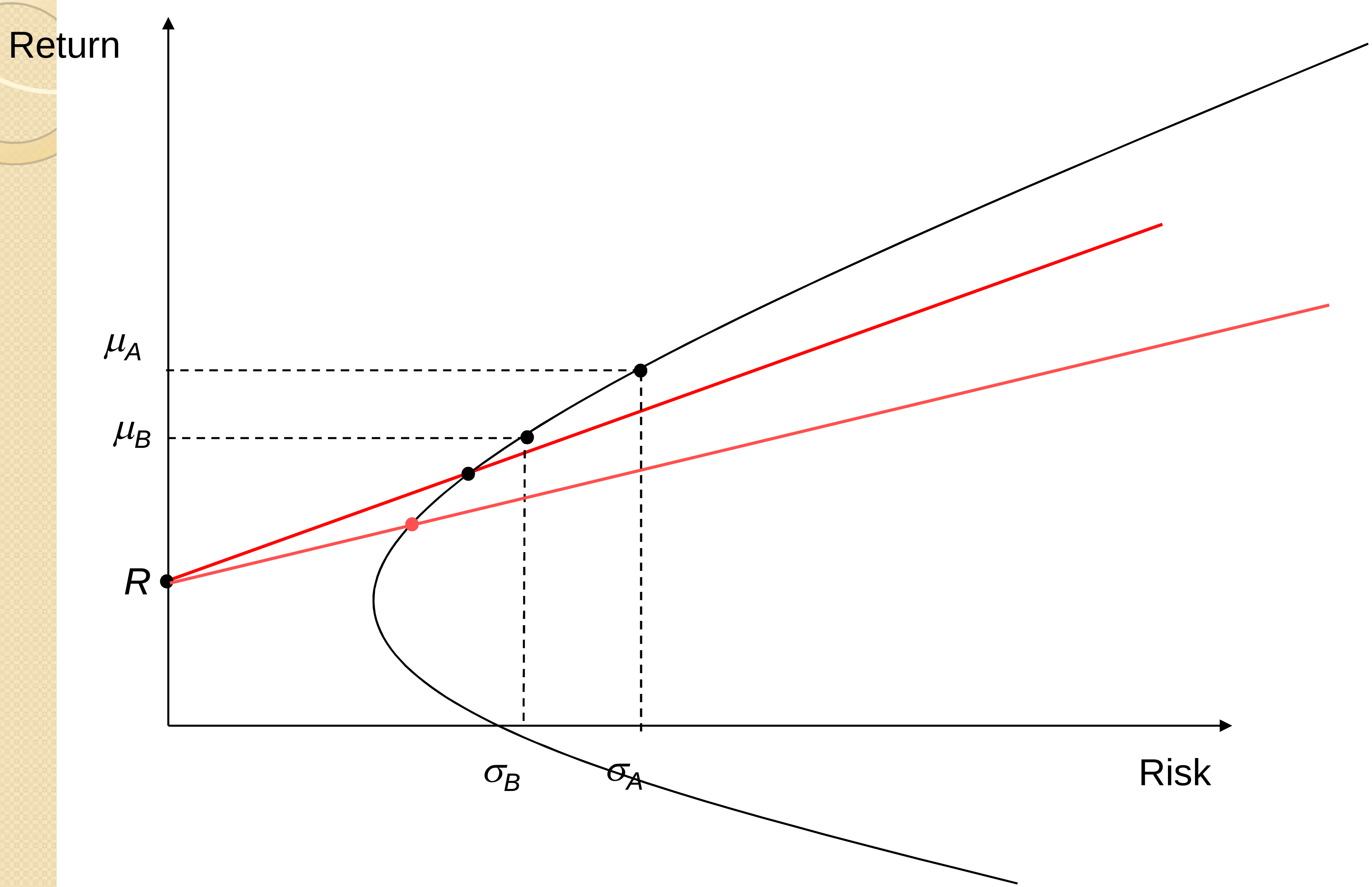

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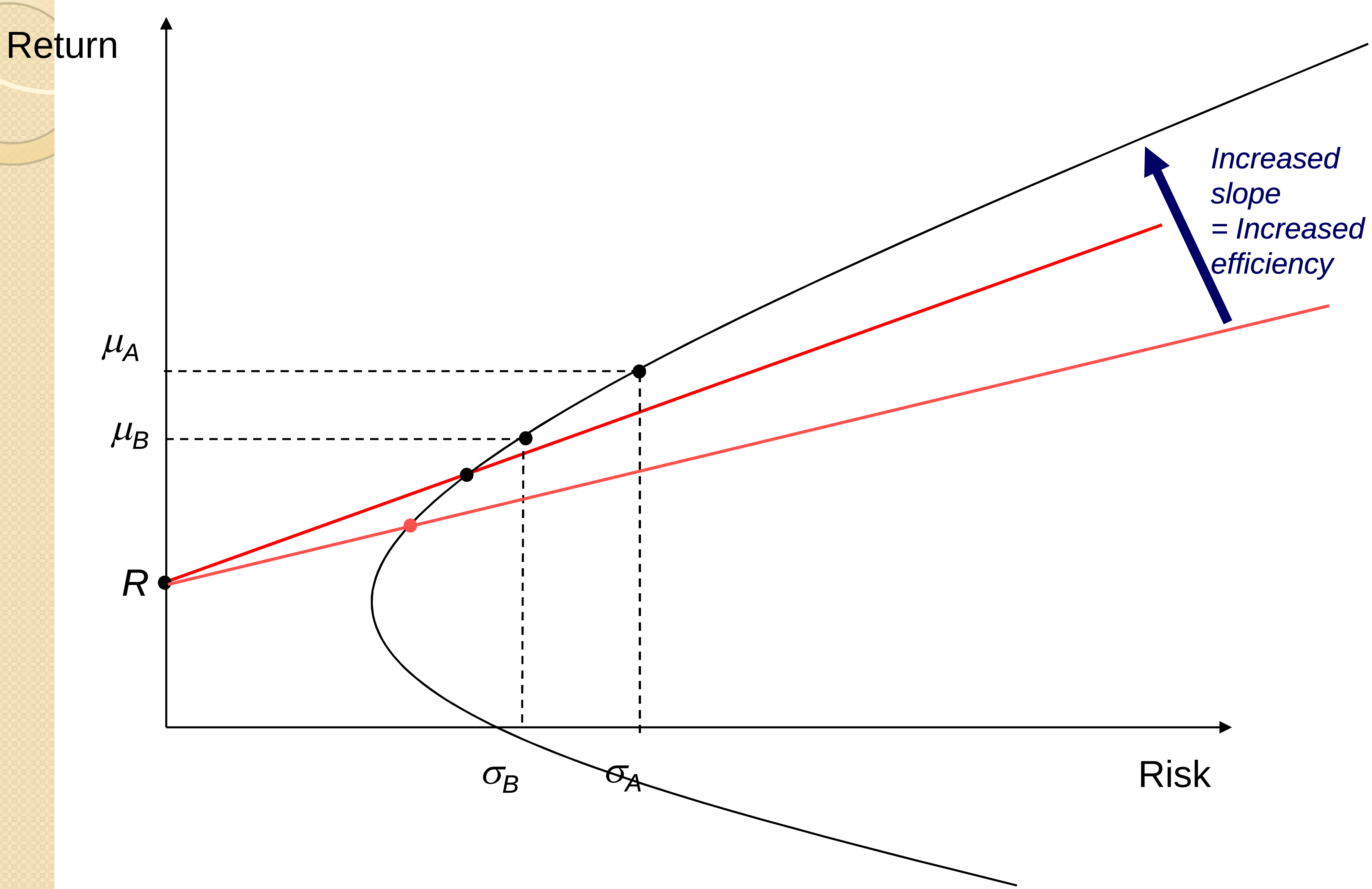
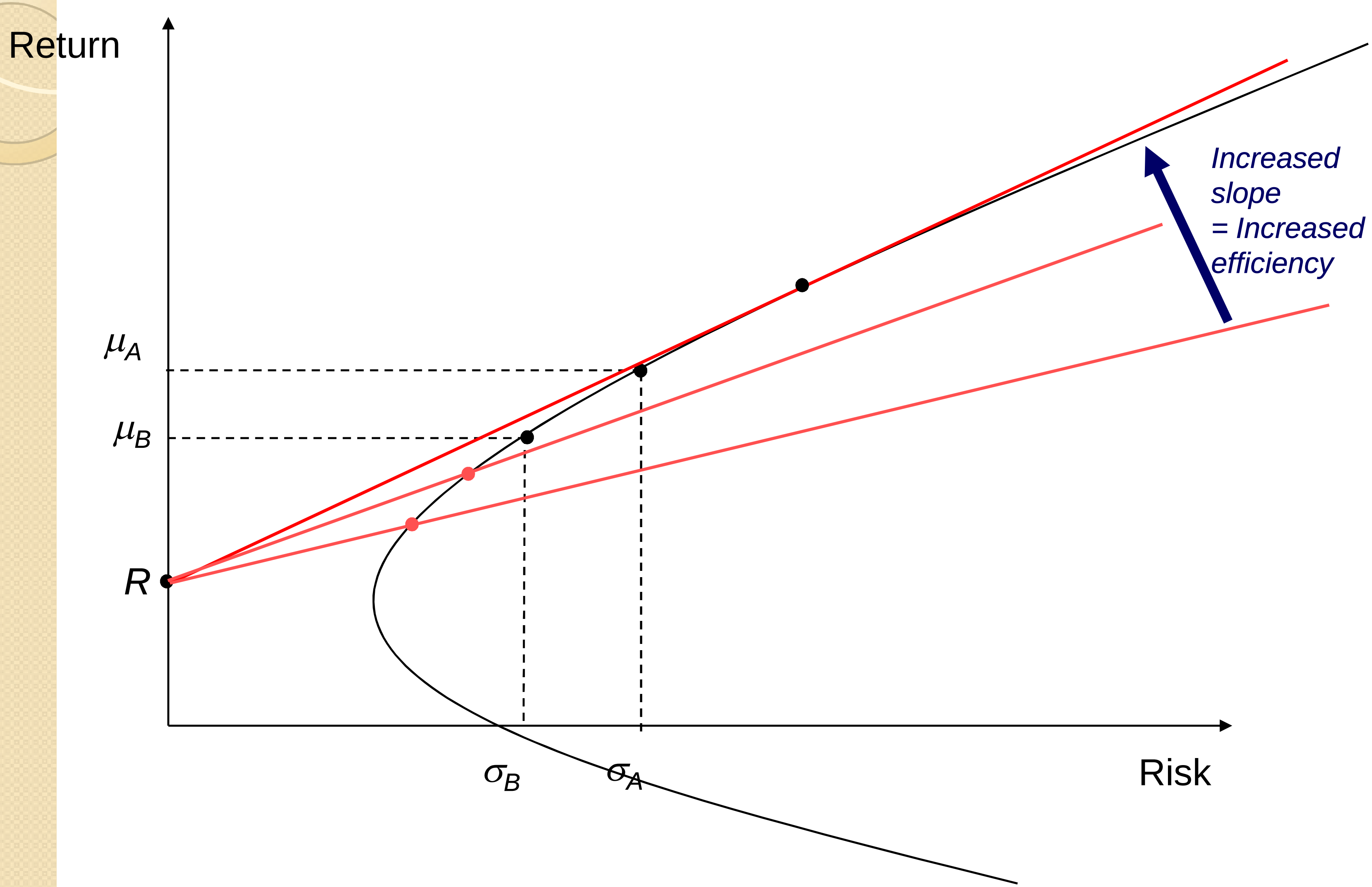
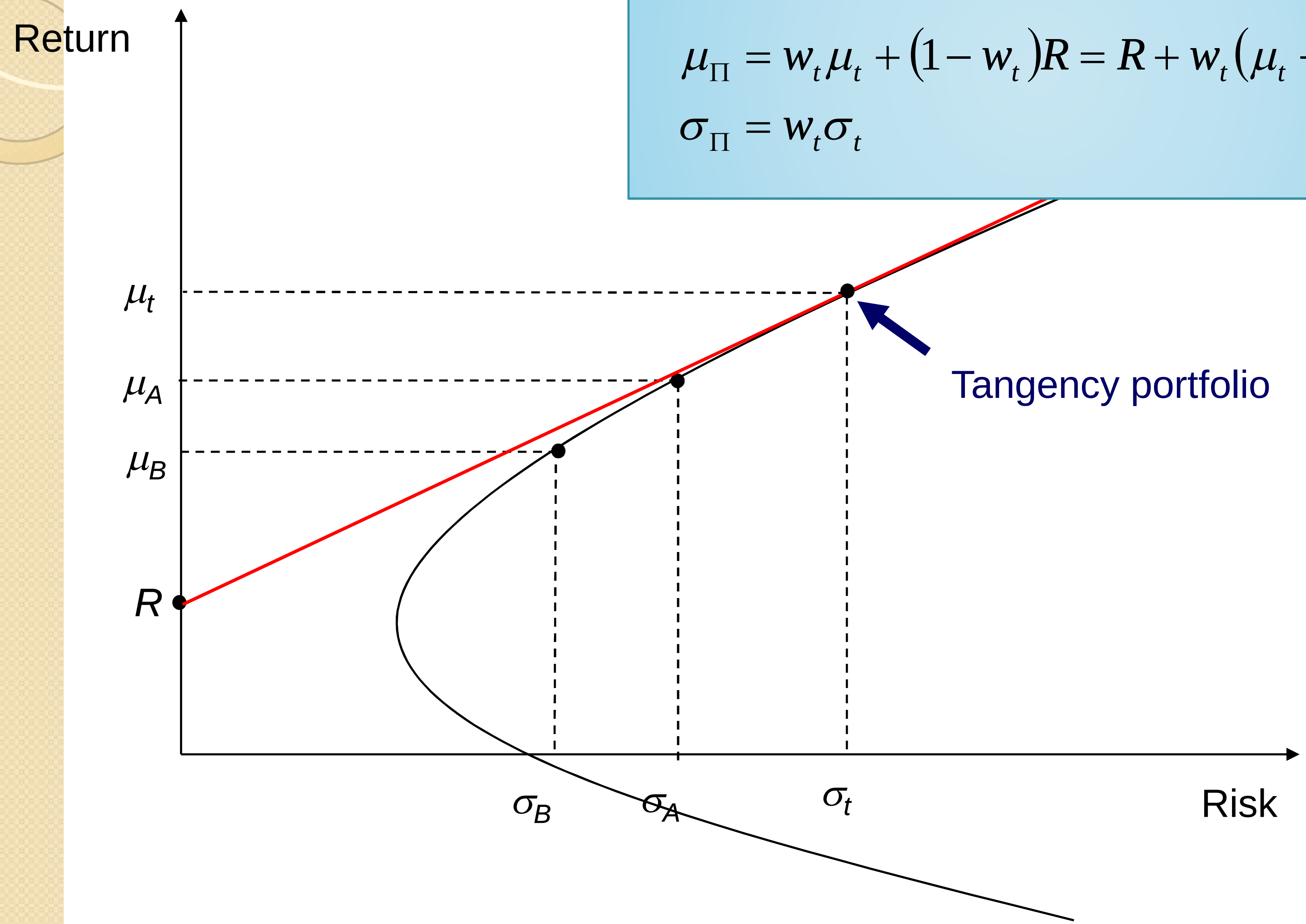


Building the new efficient frontier



The tangency portfolio



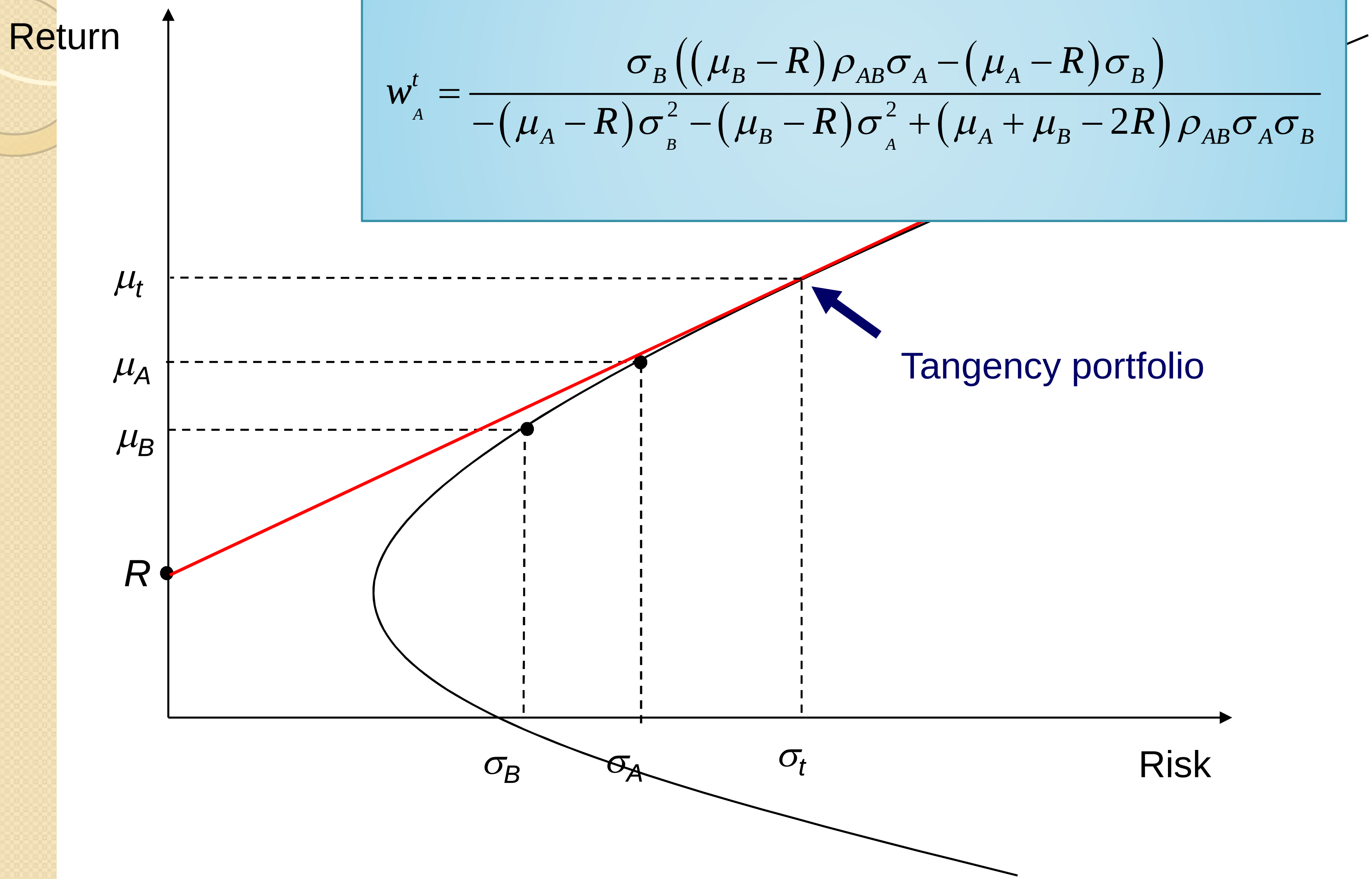
Line parametrized by w_t:

$$\mu_\Pi = w_t \mu_t + (1 - w_t) R = R + w_t(\mu_t - R)$$

$$\sigma_\Pi = w_t \sigma_t$$

The tangency portfolio's allocation

$$w_A^t = \frac{\sigma_B\left(\left(\mu_B - R\right)\rho_{AB}\sigma_A - \left(\mu_A - R\right)\sigma_B\right)}{-\left(\mu_A - R\right)\sigma_B^2 - \left(\mu_B - R\right)\sigma_A^2 + \left(\mu_A + \mu_B - 2R\right)\rho_{AB}\sigma_A\sigma_B}$$



Slope of the efficient frontier and tangency portfolio

- We can now express the risk-return relationship more directly.
- By the “risk equation”, of the previous slide

$$\mu_\Pi = R + \sigma_\Pi \frac{\mu_t - R}{\sigma_t} = R + S_t \sigma_\Pi$$

- Substituting in the return “equation” of the previous slide

$$w_t = \frac{\sigma_\Pi}{\sigma_t}$$

where

$$S_t = \frac{\mu_t - R}{\sigma_t}$$

- This confirms our insights: the tangency portfolio is the risky portfolio for which the slope S_t is maximized.

The Sharpe Ratio

- For any investment C, one could consider the line of all portfolios made up of C and the RFA.

$$\mu_{\Pi} = R + \sigma_{\Pi} \frac{\mu_C - R}{\sigma_C} = R + S_C \sigma_{\Pi}$$

where

$$S_C = \frac{\mu_C - R}{\sigma_C}$$

- The slope S_c is called the **Sharpe ratio** of investment C:
  - It is a key measure of risk-adjusted return representing the excess return (over the risk free rate) per unit of **total** risk taken;
  - The higher the Sharpe ratio of a portfolio, the more efficient the portfolio is.
  - The risky portfolio with highest Sharpe ratio is the tangency portfolio.

Case ρ_{AB} = -1 Revisited

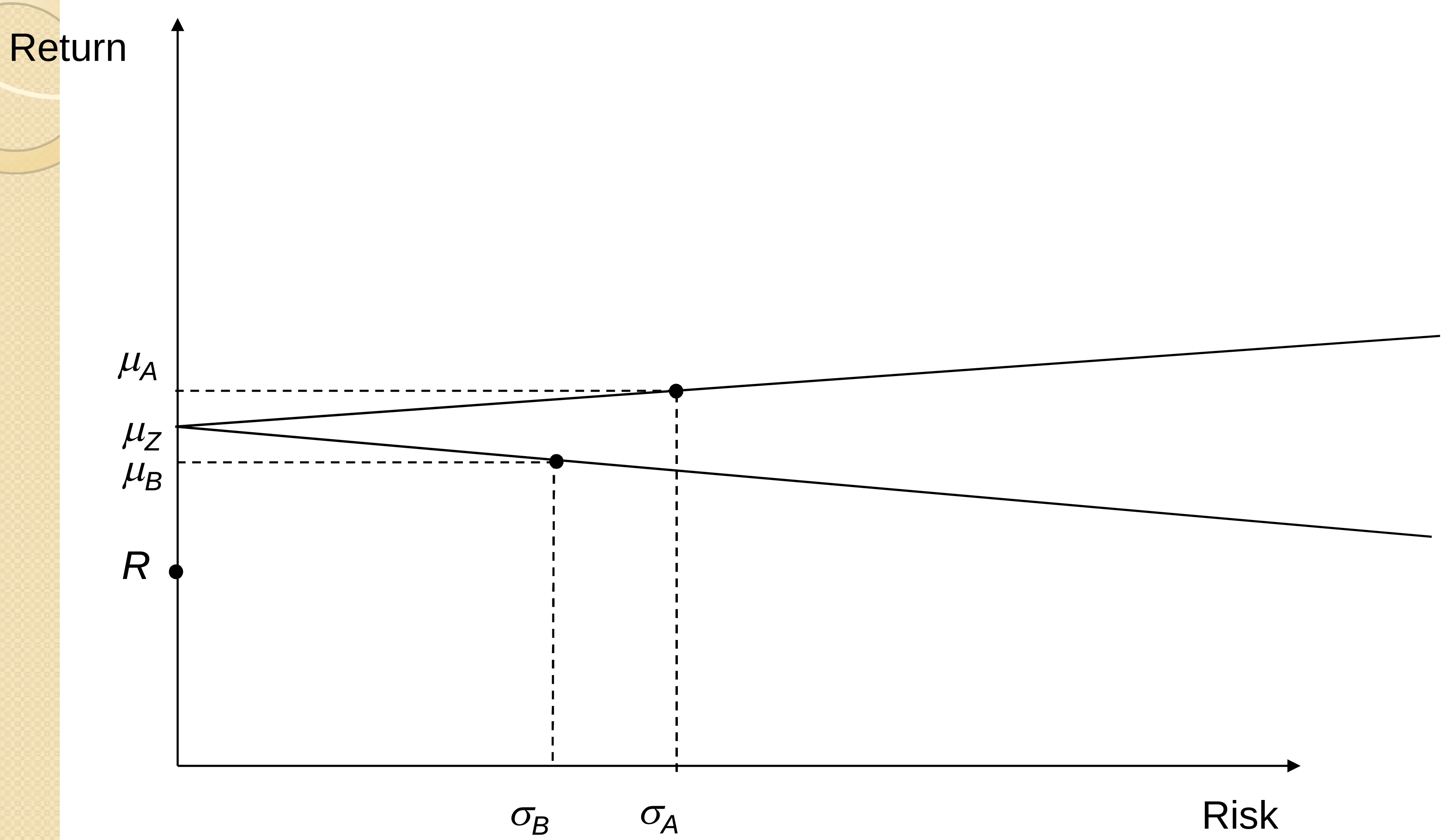
- We have already seen the perfectly negatively correlated case earlier.

- What does the introduction of a risk-free rate change?

- The introduction of a risk-free rate brings a new concept which is essential to both hedging and derivatives pricing: **arbitrage**.

- Let's illustrate…
  - We denote by Z the zero-risk portfolio generated by investing in an optima amount of assets A and B.
  - We will consider three cases:
    - $\mu_Z > R$;
    - $\mu_Z < R$;
    - $\mu_Z = R$.

What if $\mu_Z > R$?

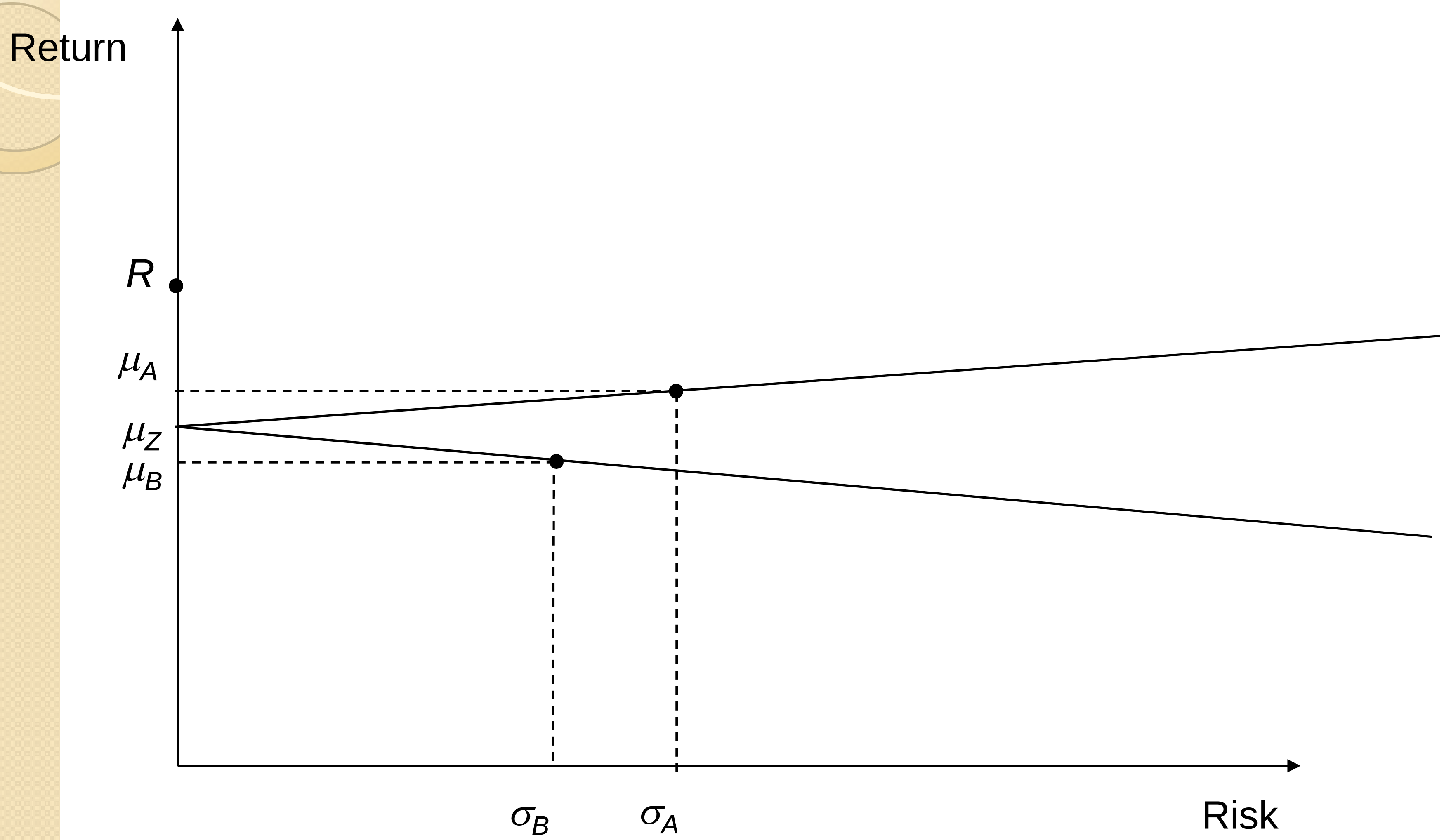


What if $\mu_Z > R$?

- If $\mu_Z > R$, there is a clear opportunity to make a riskless profit by:
  - Buying as much of the Zero-risk portfolio as we can,
  - And financing this purchase by borrowing at the risk-free rate.

- This brings an immediate (and certain) return of $\mu_Z - R > 0$ per unit of the Zero-risk portfolio purchased.

What if μ_Z < R?



What if $\mu_Z < R$?

- If $\mu_Z < R$, there is a clear opportunity to make a riskless profit by:
  - Buying as much of the risk-free asset as possible
  - And financing this purchase by shorting as many units of the Zero-risk portfolio as we can.
- This brings an immediate (and certain) return of $R - \mu_Z > 0$ per unit of the Zero-risk portfolio purchased.

We should have μ_Z = R

- The previous two situations generate a riskless profit. Financial economists call this outcome **arbitrage**.

- Now, in theory arbitrage cannot last long. As more traders become aware of it, they will
  - Buy more and more of the high-return assets, therefore increasing its price and lowering its return potential;
  - Short more and more of the low-return assets, therefore pushing its price down and increasing its return potential;

- This constant trading will bring us to **equilibrium**, which corresponds to the situation where all the assets are correctly priced and in particular μ_Z = R.

So what?

- The practical implication of all this is:

  As soon as you can construct a zero-risk portfolio by appropriately trading two or more assets, then in equilibrium this portfolio will generate a return equal to the risk-free rate.

- This insight is the key to options and derivatives pricing.

Back to the General Problem: *N* risky assets and the risk-free asset

Back to the general problem: N risky assets

- We now return to the general case in which the market has $N \geq 2$ risky assets and one risk-free asset.

- All the concepts derived in the special case $N = 2$
  - Opportunity set,
  - Efficient frontier,
  - Tangency portfolio,
  - Sharpe ratio

  are still valid in the general setting.

- We will consider the following two cases:
  - Portfolios of risky securities only;
  - Portfolios of risk-free and risky securities;

Case 1: Risky securities portfolio (Part 1)

- First, consider a portfolio fully invested in risky assets. Recall that the weight w_i invested in asset *i*, *i* =1,…,*N* is defined as

$$w_i = \frac{\text{Market Value of Asset } i}{\text{Total Market Value of the Portfolio}}$$

- Since all of the wealth must be invested in the assets, the proportion of wealth invested or "weights" invested in the various assets must equal 100% of wealth. This leads to the budget equation

$$\sum_{i=1}^{N} w_i = 1$$

- In matrix notation, the budget equation can be expressed as

$$\mathbf{w}^T \mathbf{1}_N = 1$$

where

  - $\mathbf{w}$ is the n-element column vector of weights;
  - $\mathbf{v}^{\mathrm{T}}$ denotes the transpose of vector $\mathbf{v}$;
  - $\mathbf{1}_{\mathrm{N}}$ is the N-element column unit vector, i.e. the vector with all entries set to 1.

Case 1: Risky securities portfolio (Part 2)

- The expected return of Portfolio Π is

$$\mu_\Pi := E[r_\Pi] = \sum_{i=1}^{N} w_i \mu_i$$

- The standard deviation of portfolio returns is

$$\sigma_\Pi = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \operatorname{Cov}(R_i, R_j)} = \sqrt{\underbrace{\sum_{i=1}^{N} w_i^2 \sigma_i^2}_{\substack{\text{weighted sum of variances}\\ \text{of asset returns}}} + \underbrace{2\sum_{\substack{i=1\\ j>i}}^{N} w_i w_j \rho_{ij} \sigma_i \sigma_j}_{\substack{\text{weighted sum of covariances}\\ \text{of asset returns}}}}$$

- In matrix notation, we have respectively

$$\mu_\Pi = \mathbf{w}^T \boldsymbol{\mu}$$

$$\sigma_\Pi = \sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}$$

where

  - $\boldsymbol{\mu}$ is the n-element column vector of expected returns;
  - $\boldsymbol{\Sigma}$ is the covariance matrix.

Mean-Variance Analysis

- While working on his thesis, Markowitz figured out that you can decide how to invest if you know either your return objective or your risk constraint… it is just a matter of solving a fairly simple optimization problem:
  - Return objective:

$$\underset{w_1, w_2, \ldots, w_n}{\text{minimize}}\, \sigma_P^2\left(w_1, w_2, \ldots, w_n\right)$$

Subject to

$$E\left[R_P\right] = m$$

$$\sum_{i=1}^{N} w_i = 1$$

Mean-Variance Analysis

- Risk constraint:

$$\underset{w_1, w_2, \ldots, w_n}{\text{maximize}}\, E\left[R_P\left(w_1, w_2, \ldots, w_n\right)\right]$$

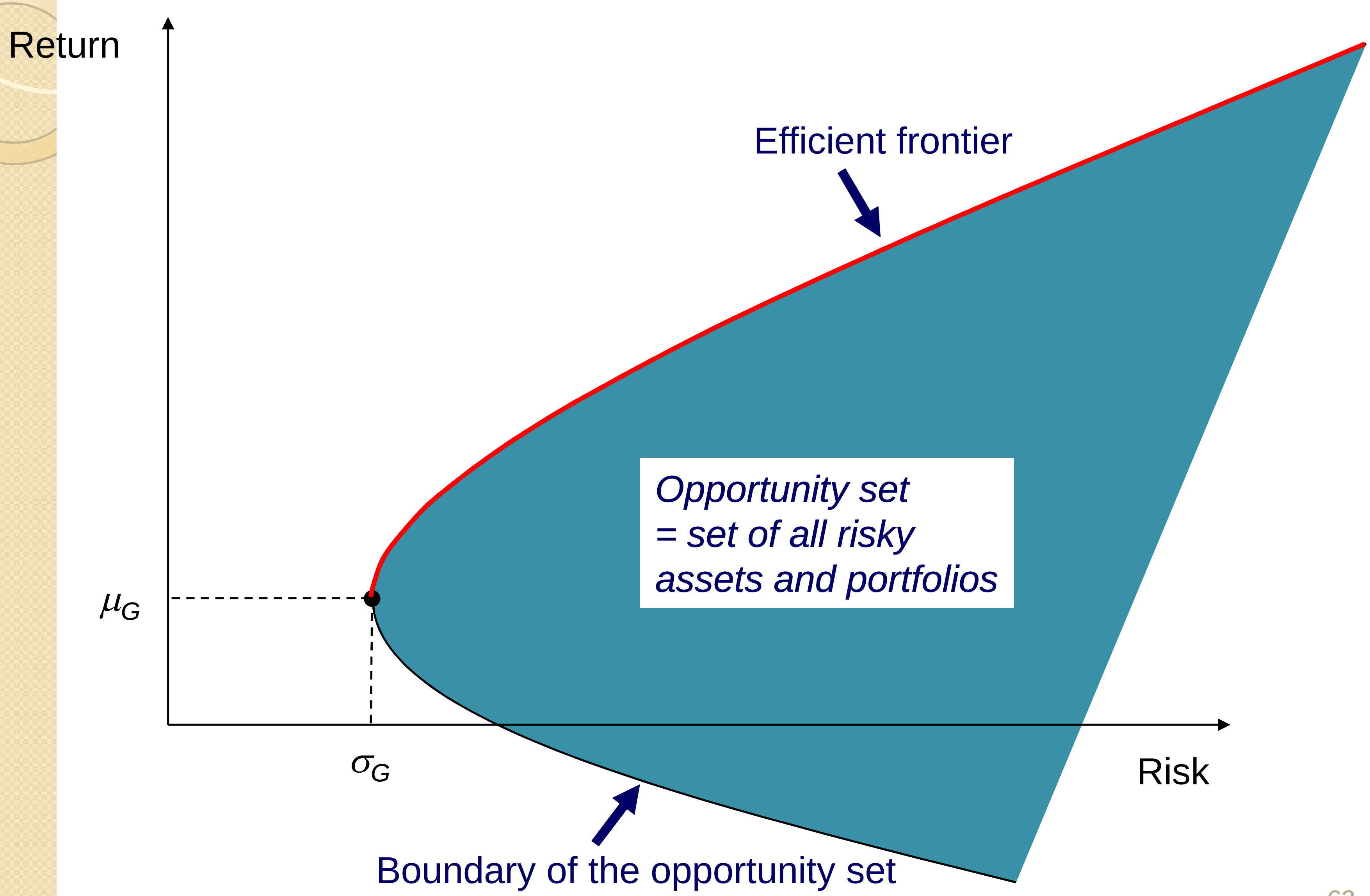
Subject to

$$\sigma_P^2\left(w_1, w_2, \ldots, w_n\right) = v^2$$

$$\sum_{i=1}^{N} w_i = 1$$

- Of the two formulations, the first one (return objective) is the most intuitive and easiest to workout ➔ it is by far the dominant formulation.

Case 1: Risky securities portfolio – efficient frontier



Case 1: Quantifying diversification (Part 1)

- We will now illustrate how diversification works.

- For convenience, assume that the market is homogeneous
  - All the securities have the same expected return $\mu_i = \mu$, i=1,…,N;
  - All the securities have the same standard deviation of return $\sigma_i = \sigma$, i=1,…,N;
  - The securities returns have the same correlation $\rho_{ij} = \rho$, i,j=1,…,N.

- And we decide to invest equally in all *N* risky securities so that $w_i = 1/N$.

- What happens to the portfolio return μ_i and the portfolio risk σ_i ?

Case 1: Quantifying diversification (Part 2)

- The expected return of the portfolio is

$$\mu_\Pi = \sum_{i=1}^{N} w_i \mu_i = N \times \frac{1}{N} \times \mu = \mu$$

The portfolio return stays the same irrespective of the value of *N*: we say that it is **invariant** in *N.*

Case 1: Quantifying diversification (Part 3)

- The variance of portfolio returns is

$$\sigma_\Pi^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2\sum_{\substack{i=1 \\ j>1}}^{N} w_i w_j \rho_{ij} \sigma_i \sigma_j$$

$$= \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{\substack{i=1 \\ j\neq 1}}^{N} w_i w_j \rho_{ij} \sigma_i \sigma_j$$

$$= N \times \frac{1}{N^2} \times \sigma^2 + N \times \left(N-1\right) \frac{1}{N^2} \times \rho \times \sigma^2$$

$$= \frac{N + \rho N\left(N-1\right)}{N^2} \sigma^2$$

$$= \left(\rho + \frac{1-\rho}{N}\right) \sigma^2$$

which shrinks to $\rho\sigma^2$ as N gets large.

Case 1: Quantifying diversification (Part 4)

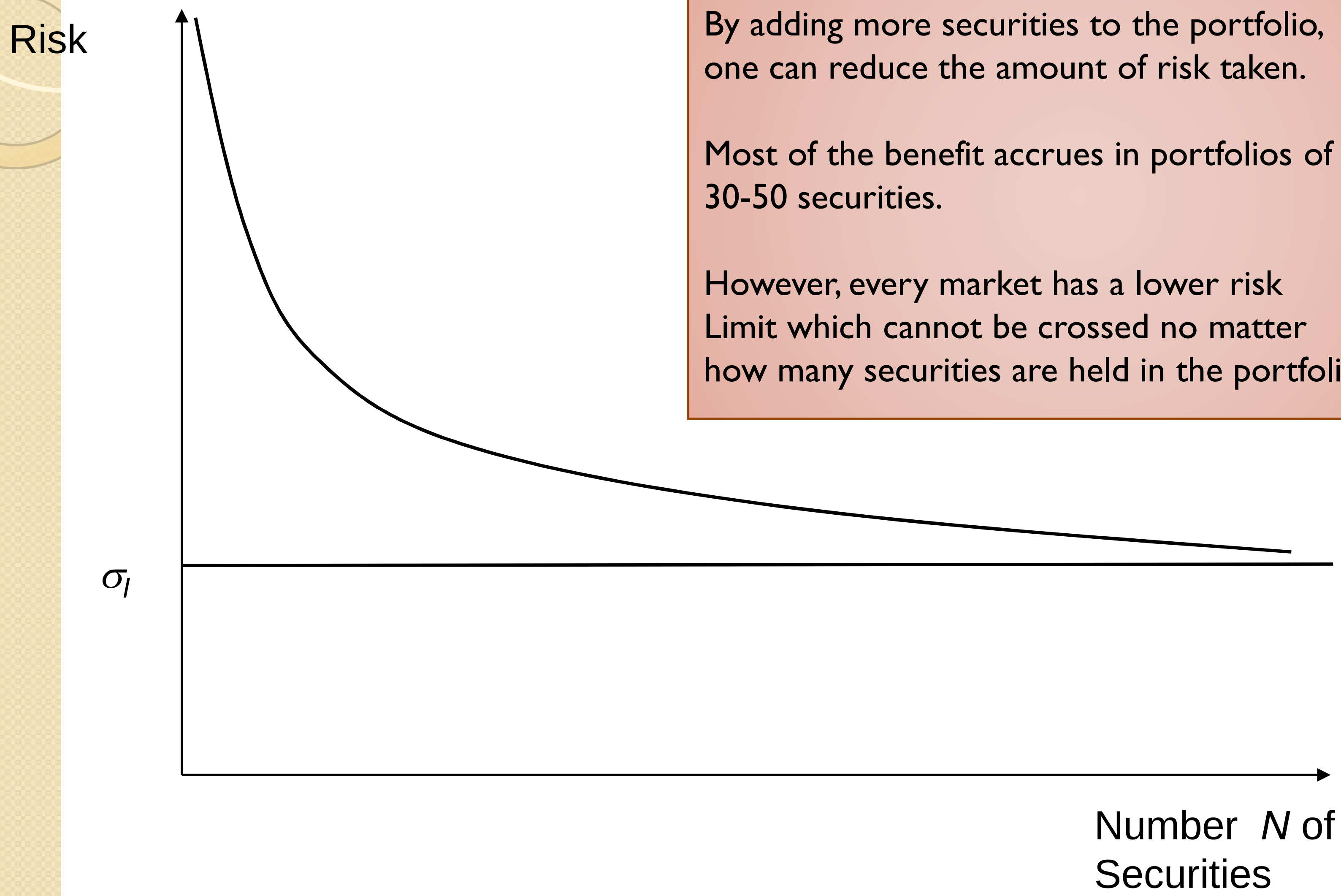
- If $\rho = 0$, then the variance of portfolio returns is

$$\sigma_{\Pi}^{2} = \frac{1}{N}\sigma^{2}$$

which is $O(N^{-1})$.

- When returns are uncorrelated, the standard deviation of portfolio returns actually shrinks like $N^{-1/2}$ as N gets larger.

Case 1: How far do diversification benefits extend?



By adding more securities to the portfolio, one can reduce the amount of risk taken.

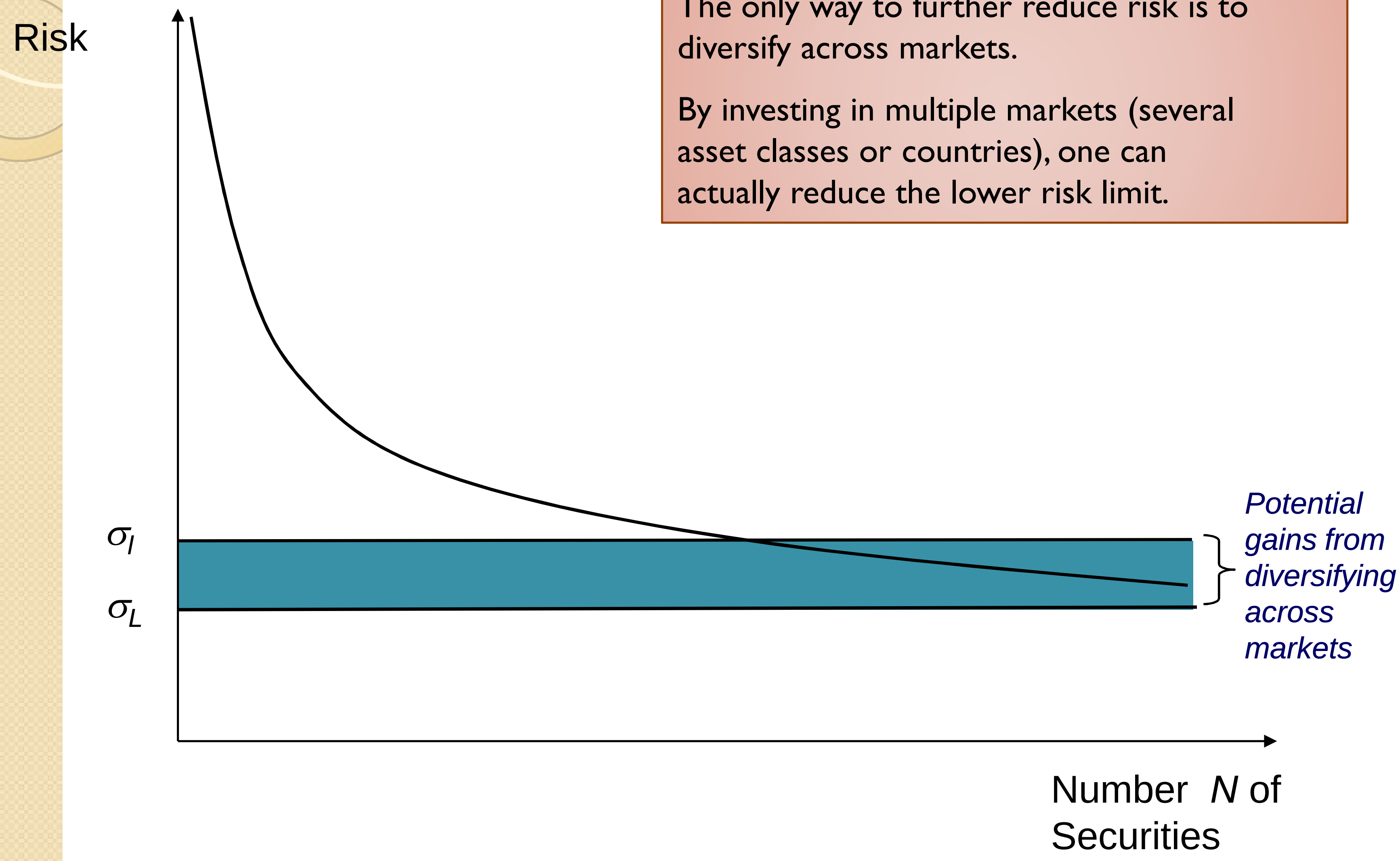
Most of the benefit accrues in portfolios of 30-50 securities.

However, every market has a lower risk Limit which cannot be crossed no matter how many securities are held in the portfolio

Case 1: Diversifying across markets

The only way to further reduce risk is to diversify across markets.

By investing in multiple markets (several asset classes or countries), one can actually reduce the lower risk limit.



Case 2: Risk-free and risky portfolio (Part 1)

- Denote by w_0 the weight of the risk-free asset in the portfolio. The budget equation in this case is

$$w_0 = 1 - \sum_{i=1}^{N} w_i$$

and we consider the allocation to the risk-free asset as a residual of the allocation of wealth to the risky assets

- In matrix notation, the budget equation can be expressed as

$$w_0 = 1 - \mathbf{w}^T \mathbf{1}_N$$

Case 2: Risk-free and risky portfolio (Part 2)

- The expected return of the portfolio is

$$E[r_\Pi] := \mu_\Pi = w_0 R + \sum_{i=1}^{N} w_i \mu_i = R + \sum_{i=1}^{N} w_i \underbrace{(\mu_i - R)}_{\substack{\text{Excess return} \\ \text{of security } i}}$$

- The standard deviation of portfolio returns is still

$$\sigma_\Pi = \sqrt{\sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{\substack{i=1 \\ j>1}}^{N} w_i w_j \rho_{ij} \sigma_i \sigma_j}$$

- In matrix notation,

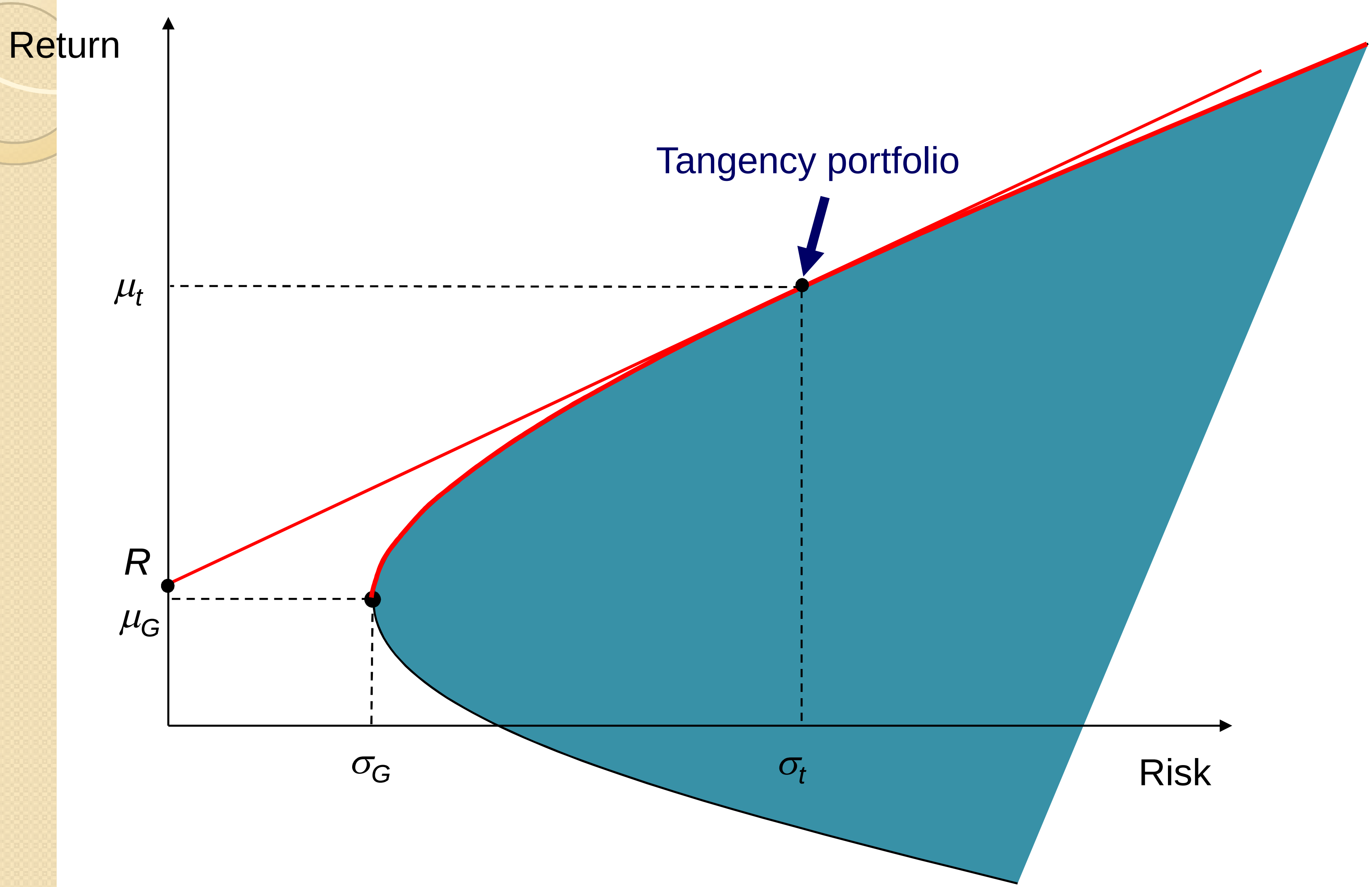
$$\mu_\Pi = R + \mathbf{w}^T (\boldsymbol{\mu} - \mathbf{1}_N R)$$

$$\sigma_\Pi = \sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}$$

where

  - $\boldsymbol{\mu}$ is the n-element column vector of expected returns;
  - $\boldsymbol{\Sigma}$ is the covariance matrix.

Case 2: Risk-free and risky portfolio – efficient frontier



Further MPT

Further MPT

- In the last part of this presentation, we introduce some further ideas about the MPT:
  - The market portfolio and the market price of risk;
  - Computational efficiency of the mean-variance analysis;
  - The factor model;
  - The CAPM;

Homogeneous beliefs and the market portfolio

- Let's now assume that all the investors on the market share:
  - The same investment universe of N risky securities and one risk-free asset returning R;
  - The same time horizon T;
  - The same estimations for the market parameters (expected return, standard deviation and correlation);

- Then, all the investors will identify (and buy) the same tangency portfolio.
- The relative market values of all the securities on the market will adjust to reflect their allocation within the tangency portfolio.
- Consequently, the tangency portfolio becomes a perfect representation of the underlying asset market.
- In these equilibrium conditions, the tangency portfolio is called the **market portfolio**.

Sharpe ratio and market price of risk

- The market portfolio is the risky portfolio which maximizes the Sharpe ratio (i.e. the slope of the efficient frontier).

- Since everyone now invests in the market portfolio and the RFA, the Sharpe ratio is interpreted as the ***market price of risk***.

- Indeed, the Sharpe ratio now measures the number of units of extra return generated (above the risk free- rate) per unit of ***market risk*** taken.

- *Note:* You will see more about market price of risk later in the CQF when you learn about interest rate modelling and bond pricing.

The market in practice

- At the beginning of this presentation, we saw that the investment universe, what we now call “the market”, is comprised of all traded assets.

- However, there currently does not exist any financial index or economic time series capable of tracking the price of all tradable assets.

- The solution adopted in practice is to use a “proxy”: a financial index (such as the S&P 500, or the MSCI World Index) which represents a sizeable share of the assets traded on financial markets.

- The solution is not perfect since it only reflects a small portion of tradable assets, but it is deemed good enough in practice since few portfolio managers venture beyond a few asset classes.

Computational efficiency of mean-variance analysis

- One of the main problems with what we have done so far is the dimensionality of the problem.

- If we had N risky securities (as opposed to two so far), we would need to estimate:
  - N expected returns;
  - N standard deviations;
  - N(N-1)/2 correlations.

- Hence, as N gets larger the number of parameters grows at a quadratic rate, i.e. $O(N^2)$.

- This pace is too fast to enable efficient computations.

The linear factor model: definition

- To reduce the number of parameters, Sharpe postulated a simpler linear model linking portfolio returns to market returns, such as

$$r_i = \alpha_i + \beta_i r_M + \varepsilon_i$$

where

  - r_i is the actual return of asset i in the period of reference.
  - β_i represents the exposure of the asset i to the market return and measures the exposure to **systematic risk**;
  - α_i represents the base return generated by asset i;
  - ε_i represents the **idiosyncratic risk** of asset i, a type of residual risk proper to asset i only and unrelated to any other asset or to the market. The assumption is that $\varepsilon_i \sim N(0,e_i^2)$ and $Cov[\varepsilon_i, \varepsilon_j] = 0$ for $i \neq j$.

- Once the market portfolio (or a proxy) has been identified, the parameters can be estimated using a linear regression.

The factor model: computational efficiency

- With such model, one would only need
  - N values of α ;
  - N values of β ;
  - N values of ε ;

  to parametrize a market with *N* risky assets.

- Hence, as *N* gets larger the number of parameters grows at a linear rate, i.e. $O(N)$, which enables efficient computations.

The factor model: some relations

- Consider investment (i.e portfolio or asset) *C*, then

$$r_C = \alpha_C + \beta_C r_M + \varepsilon_C$$

and

$$E[r_C] := \mu_C = \alpha_C + \beta_C \mu_M$$

- The total risk of *C*, σ_C, is equal to:

$$\sigma_C = \sqrt{\beta_C^2 \sigma_M^2 + e_C^2}$$

by the properties of the variance.

- Considering in addition an investment *D*, then the covariance of returns between *C* and *D* is

$$Cov(C, D) := \sigma_{CD} = \beta_C \beta_D \sigma_M^2$$

by applying the properties of the covariance.

The factor model: some more relations

- In particular, if investment C is a portfolio of all the risky assets with respective weights w_i in asset i, i=1,…,N, we can apply these relations to deduce that

$$r_C = \sum_{i=1}^{N} w_i \alpha_i + \sum_{i=1}^{N} w_i \beta_i \, r_M + \sum_{i=1}^{N} w_i \varepsilon_i$$

$$E[r_C] := \mu_C = \sum_{i=1}^{N} w_i \, \alpha_i + \sum_{i=1}^{N} w_i \, \beta_i \, \mu_M$$

and

$$\sigma_C = \sqrt{\left(\sum_{i=1}^{N} w_i \, \beta_i \right)^2 \sigma_M^2 + \sum_{i=1}^{N} w_i^2 e_i^2}$$

since by independence of the random variables ε_i , we have

$$e_C^2 = \sum_{i=1}^{N} w_i^2 e_i^2$$

Quantifying the diversification benefits (Part 1)

- The factor model sheds a different light on the diversification question.

- For convenience, we will assume that:

  - all of the idiosyncratic risks are not only independent, but IID, i.e. for all i, $\varepsilon_i \sim N(0, e^2)$ for some constant e and $Cov[\varepsilon_i, \varepsilon_j] = 0$ for $i \neq j$;

  - when we invest in a portfolio, we invest an equal proportion in each security, so that $w_i = w = 1/N$, and;

  - all the securities have the same systematic risk[1], so that $\beta_i = \beta$.

[1] *This last assumption is not necessary, but it makes the argument clearer.*

Quantifying the diversification benefits (Part 2)

- The formula $\sigma_\Pi = \sqrt{\left(\sum_{i=1}^{N} w_i \beta_i\right)^2 \sigma_M^2 + \sum_{i=1}^{N} w_i e_i^2}$

  shows that the variance of an investment in a security *C* is comprised of both systematic and idiosyncratic risk.

- In the case of an investment in a portfolio of N securities, we have

$$\sigma_C = \sqrt{\beta^2 \sigma_M^2 + e^2}$$

- Taking the limit as $N \to \infty$, and by the Central Limit Theorem, this last equation becomes

$$\sigma_\Pi = \beta \sigma_M$$

- Idiosyncratic risk has vanished!

The factor model: an adhoc model

- The linear factor model is an "adhoc" model, i.e.
  - it is practically convenient,…
  - … but it is not theoretically justified.

- Because it is not theoretically justified, adhoc models do not have any predictive power and should not be used for forecasting purpose[1].

- However, Sharpe also developed a very similar economic model: the Capital Asset Pricing Model.

[1] *Although they are very much used in practice!*

The CAPM

- The Capital Asset Pricing Model (CAPM)
  - Is a linear factor model, in which the factor is the market return;
  - Is derived directly from the mean-variance analysis (see "Fundamentals of Optimization and Application to Portfolio Selection" for more details);
  - Is an equilibrium model: it can be used to predict asset prices;
  - Can be applied to any security or portfolio;
  - Is expressed in terms of **expectations**.

- For an investment *I*, the CAPM takes the form

$$E\left[r_I - R\right] = \beta_I E\left[r_M - R\right]$$

or, alternatively

$$E\left[r_I\right] = R + \beta_I E\left[r_M - R\right]$$

The CAPM

- The CAPM states that the risk premium on any investment is:
  - Proportional to the risk premium of the market;
  - And the proportionality constant is the **degree of systematic risk** of the investment.
- In short, “on average the market is compensating us for taking on systematic risk”.
- Because of the Expectation operator, the CAPM can be used as a predictive model.

The CAPM

- One variable does not appear in the CAPM: idiosyncratic risk. Where did it go?
- Because we take the expectation, idiosyncratic risk vanishes.
- Read differently, the CAPM implies that only systematic risk should be rewarded, not idiosyncratic risk.
- This is quite logical: since we can diversify away all of our idiosyncratic risk, the market should not compensate us for taking this type of risk.
- This idea is central to financial economics. You will see it again later in the CQF when you learn about the implication of using jump-diffusion processes to price options.

The APT

- We can easily extend the idea of the linear factor model to build a multifactor model.

- In 1976, Stephen Ross introduced an equilibrium multifactor model, the Arbitrage Pricing Theory (APT), which takes the form:

$$k_e = R_F + \sum_{i=1}^{m} \lambda_i \beta_i$$

where
  - λ_i is risk premium associated with factor *i*;
  - β_i is the sensitivity of the stock to the i^{th} factor;
  - R_F is the risk-free rate.

[1] *Ross, Stephen. 1976. "The arbitrage theory of capital asset pricing". Journal of Economic Theory 13 (3): 341–360.*

The APT

- Comparison with the CAPM:
  - The assumptions of the APT are much weaker than that of the CAPM.
  - In particular, the APT does not require all investors to identify and hold the same market portfolio;
  - However, the APT does not specify what the factors should be, making it difficult to apply in practice.

- Comparison with other factor models:
  - The APT is an equilibrium model while other multifactor models are ad hoc;
  - In the APT, the intercept term is the risk-free rate (just like in the CAPM);
  - In the APT, the factors are risk premia (i.e. neither surprises nor raw financial data).

Testing the CAPM and the APT

- Testing empirically the validity of the APT and of the CAPM has proved difficult.

- Indeed, a joint test of
  - The model's equation
  - The model parameters (equity risk premium, number of factors, etc…)

  is required.

- In case of rejection, should we blame:
  - The model?
  - The parameters? or
  - Both?

Measuring Risk-Adjusted Performance

Efficiency Ratios

- It is useful to express risk-adjusted performance as an **efficiency ratio**:

$$\text{Efficiency Ratio} = \frac{\text{Return}}{\text{Risk}}$$

- Efficiency ratios measure how much return is produced per unit of risk taken.

- A number of efficiency ratios are commonly used.
  - Principle is the same…
  - Difference is the definition of “risk” and “return”.

Sharpe Ratio vs. Treynor Ratio

- The Sharpe ratio and Treynor ratio are two of the oldest and best known ratios.
  - Both are based on expected excess return:

$$\text{Excess return} = r_I - R$$

  - The Sharpe Ratio measures the excess return achieved per unit of <u>total</u> risk:

$$\text{Sharpe Ratio} = \frac{E[r_I] - R}{\sigma_I}$$

  - The Treynor ratio measures the excess return achieved per unit of <u>systematic</u> risk:

$$\text{Treynor Ratio} = \frac{E[r_I] - R}{\beta_I}$$

Jensen's Alpha

- **Jensen's alpha** is a measure of <u>risk-adjusted excess return</u>.
- It is defined as the difference between
  - the actual return realized by an investment *I*, and;
  - The return predicted by the CAPM.

$$\begin{aligned}\text{Jensen's alpha }(\alpha) &= r_I^{Actual} - r_I^{CAPM} \\ &= r_I^{Actual} - R - \beta_I E\left[r_M - R\right]\end{aligned}$$

- Today, Jensen's alpha is used to measure the **active return** of a manager, that is the return generated by taking active positions that deviates from a benchmark or from the market.
  - This "alpha" is what hedge funds promise to deliver.

Alpha Hunters and Beta Grazers

- Rearranging the definition of Jensen's alpha, we obtain

$$
\begin{aligned}
r_I^{Actual} &= r_I^{CAPM} + \alpha \\
&= \underbrace{R + \beta_I E\left[r_M - R\right]}_{\text{Passive return}} + \underbrace{\alpha}_{\text{Active return}}
\end{aligned}
$$

- Jensen's alpha is therefore useful to separate
  - **"passive" return**: generated by the degree of exposure to systematic risk;
  - **"active" return**: generated by the manager's ability to buy/short the right security.

- This idea has led to a view that fund managers should focus on either of two strategies:
  - **Beta grazer**: passively managed funds (α = 0) targeting a level of systematic risk exposure (typically β = 1);
  - **Alpha hunter**: funds with no directional bet (ideally β = 0) dedicated to generating positive alpha.

Information Ratio (Grinold & Kahn)

- Alpha captures active return. We measure **active risk** as the standard deviation of the alpha, that is, as the standard deviation of the difference between the realized return and the return predicted by the CAPM
  - Grinold and Kahn call this measure omega (ω):

$$\omega_I = \sigma_{\alpha_I} = SD\left[\alpha_I\right]$$

- Grinold and Kahn's **information ratio** is an efficiency ratio measuring the active return achieved per unit of active risk:

$$IR = \frac{\alpha_I}{\omega_I}$$

- This measure is central to the evaluation of active managers and hedge funds.

MPT in Practice

How is MPT used in practice?

- The impact of MPT on our understanding of financial risks and of the mechanics of portfolio construction cannot be understated.

- In the industry, MPT is routinely used has a frame of reference to
  - Understand portfolio construction;
  - Evaluate financial risks;
  - Compute the cost of equity in corporate finance.

- In a survey of trends in quantitative equity management, Fabozzi, Foccardi and Jonas[1] found 30 out of the 36 firms polled (i.e. 83%) actively used Mean-variance optimization.

- However, historically, MPT has suffered from two main drawbacks:
  - Dimensionality;
  - Parameter estimation.

[1] *Fabozzi, F, S. Focardi and C. Jonas. Trends in quantitative equity management: survey Results. Quantitative Finance. 7(2): 115-122. April 2007*

Dimensionality

- **Dimensionality** was and is still an important concern due to the vast size of financial markets. Although factor models can be used to reduce the dimensionality of the problem, they still do not hold all the answers:
  - What to do with non-linear assets (bonds, securities with embedded options…)?
  - What index/indexes should be used?
  - Are the parameters stable over time?

Parameter estimation

- Because optimizers are particularly efficient at taking advantage of the smallest discrepancy in data to reach their objective, **parameter estimation** is critical to get workable investment policies. This phenomenon is often called the "garbage in, garbage out" syndrome.

- The good news is that the variance and covariance of returns tend to be quite stable over long periods of time…

- …but the bad news is that it would take hundreds of years of financial data to get a reasonably accurate estimates of expected returns.
  - few assets have been traded long enough;
  - in any case, market conditions change over time which cause "breaks" in the time series of returns.

What are the solutions?

- Two school of thoughts developed practical ways of improving the MPT:
  - The first one, advocates staying in a 1-period framework and improving the optimization process through either
    - improved parameter estimation techniques (i.e. Bayesian techniques), or;
    - the use of more robust optimization techniques.
  - The second school promotes the design a multi-period multi-scenario **stochastic programming models**. This method has the important advantage of acknowledging
    - That financial markets are dynamic in nature, and;
    - That it is generally more important to avoid financial disaster in difficult times than generating considerable returns in good times. Thus, scenarios are chosen to model more accurately the left tail of the return distribution.

Conclusion

In this lecture, we have seen…

- The key concepts of MPT:
  - Risky and risk-free assets;
  - Mean-variance analysis;
  - Optimal portfolio;
  - Diversification;
  - Opportunity set and efficient frontier;
  - Tangency and market portfolio;
  - Sharpe ratio and market price of risk;
  - The linear model and the CAPM.
  - The APT
  - Measuring risk-adjusted performance

- The drawbacks of MPT: dimensionality and parameter estimation.

Further Readings

- Financial concepts:
  - Chapters 4 to 12 in E.J. Gruber and M.J. Gruber. *Modern Portfolio Theory*. 1995. John Wiley & Sons.
- Mathematics of portfolio selection:
  - Chapters 6 and 7 in David Luenberger. *Investment Science*. 1997. Oxford University Press
  - Atillio Meucci. *Risk and Asset Allocation*. 2009. Springer.
  - Bernd Scherer. *Portfolio Construction and Risk Budgeting*. 2010. Risk Books. 4th ed.
- Financial economics:
  - Chapter 4 in Jonathan E. Ingersoll. *Theory of Financial Decision Making*. 1987. Rowman & Littlefield.
- Introduction to stochastic programming:
  - William T. Ziemba. *The Stochastic Programming Approach to Asset, Liability and Wealth Management*. 2003. Research Foundation of the CFA Institute.