Real Analysis

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Dedicated to Lili Yu and Ricky Liu.

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Preface

This lecture note is a collection of basic definitions and theorems in real analysis. This note is taken upon two excellent books: principle of mathematical analysis by Walter Rubin and introduction to real analysis by Lee Larson.

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- A special word of thanks goes to my family.

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1

Set Theory

"This is a quote and I don't know who said this."
– Pablo Picasso, *Source of this quote*

1.1 Sets

In naive set theory, any collection of numbers is a set. But this definition may lead to a serious problem. For example, suppose Ω is a set of all sets that are not members of themselves. Is Ω a member of itself? No matter what the answer is (yes or no), it contradicts with the definition of Ω. Thus, Ω is not a well defined set. This is so-called Russell's paradox. To avoid this paradox, Russell proposed to construct axiomatic set theory. How do we define a set? There isn't a simple answer to this question. Interested readers refer to the ZFC (Zermelo–Fraenkel set theory with the axiom of choice).

Definition 1.1. A set Ω is a well-defined collection of distinct objects.

- $x \in \Omega$ indicates that x is an element of Ω.
- $x \notin \Omega$ indicates that x is not an element of Ω.
- $A \subset \Omega$ indicates that A is a subset of Ω.
- The empty set $\emptyset$ has no element.
- The power set $\wp(S)$ of a set S is the set of all subsets of S.
- Two sets A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$

1.2 Basic operations on sets

- The **union** of two sets A and B is the set containing all the elements in either A or B, written as $A \cup B = \{x : x \in A \vee x \in B\}$.

- The **intersection** of A and B is the set containing the elements contained in both A and B, written as $A \cap B = \{x : x \in A \wedge x \in B\}$.
- The **difference** of A and B is the set of elements in A and not in B, written as $A \backslash B = \{x : x \in A \wedge x \notin B\}$.
- The **complement** of a set $A \subset S$ is $A^c = S \backslash A$.
- The **symmetric difference** of A and B is the set of elements in one of the sets, but not the other, written as $A \triangle B = (A \cup B) \backslash (A \cap B)$.
- For sets A and B, the **Cartesian product** $A \times B$ is the set of all ordered pairs $A \times B = \{(a, b) : a \in A \wedge b \in B\}$.

Theorem 1.2. (DeMorgan's Laws) For a collection of sets $\{A_i, i = 1, 2, ...\}$, $(\cup A_i)^c = \cap A_i^c$ and $(\cap A_i)^c = \cup A_i^c$.

1.3 Relations and Functions

Definition 1.3. For the sets A and B, any subset $R \subset A \times B$ is a relation from A to B. If $(a, b) \in R$, we write aRb.

Definition 1.4. The domain of a relation $R \subset A \times B$ is $dom(R) = \{a : (a, b) \in R\}$, and the range of R is $ran(R) = \{b : (a, b) \in R\}$.

- R is symmetric, if $aRb \Leftrightarrow bRa$ or equivalently $(a, b) \in R \Leftrightarrow (b, a) \in R$.
- R is reflexive if aRa or $(a, a) \in R$.
- R is transitive, if $aRb \wedge bRc \Rightarrow aRc$.

Definition 1.5. R is an equivalence relation on A, i.e., $R \subset A \times A$, if it is symmetric, reflexive and transitive.

Definition 1.6. A relation $<$ on X is called a partial order if it is transitive $a < b$ and $b < c$ imply $a < c$, but not reflexive and symmetric for all $x \in X$. The relation is called a total order if for all $x, y \in X$, we have either $x < y$ or $x = y$ or $y > x$.

For the real numbers, the order $x < y$ is defined by $x - y < 0$.

Definition 1.7. A relation $R \subset A \times B$ is a function if $aRb_1 \wedge aRb_2 \Rightarrow b1 = b2$.

- For a function $f \subset A \times B$, we often write $f : A \to B$ to indicate the relation $f(a) = b$, where $a \in A$ and $b \in B$. The set A is the domain of f, but $ran(f) \subset B$.
- If $f : A \to B$ and $g : B \to C$, then the composition $g \circ f : A \to C$ is given by $g \circ f(a) = g(f(a))$.

- f is surjective, if $r(f) = B$.
- f is injective, if $f(a) = f(b)$ implies $a = b$.
- f is bijective, if it is both surjective and injective.
- If $f : A \to B$ is bijective, the inverse of f is the function $f^{-1} : B \to A$.

Theorem 1.8. A function $f : X \to Y$ is bijective iff there is a function $g : Y \to X$ such that $f(g(y)) = y$ for all $y \in Y$ and $g(f(x)) = x$ for all $x \in X$.

Corollary 1.8.1. If f and g are bijective, then $g \circ f$ is bijective.

Theorem 1.9. (Schroder-Bernstein) If there are injective functions $f : A \to B$ and $g : B \to A$, then there is a bijective function $h : A \to B$.

Corollary 1.9.1. There is a bijective function $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

1.4 Image and preimage

Definition 1.10. Let $f : X \to Y$ and $A \subset X$, $B \subset Y$. The image of A under f is the set $f(A) = \{f(x) : x \in A\}$ and the preimage of B under f is the set $f^{-1}(B) = \{x \in X : f(x) \in B\}$.

Note that f does not have to be invertible for the preimage to be defined, and $f^{-1}(B)$ is a set while f^{-1} is a relation.

Theorem 1.11. Let $A, B \subset X$ and $E, F \subset Y$ and a function $f : X \to Y$. Then, $f(A \cup B) = f(A) \cup f(B)$, $f(A \cap B) \subset f(A) \cap f(B)$, $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$ $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$.

1.5 Cardinality

Definition 1.12. The cardinality of a set E is $n \in N$, i.e., $card(E) = n$, if there is a bijection $f : E \to \{1, ..., n\}$.

Definition 1.13. A set E is finite if there is an $n \in N$ and a surjection $f : \{1, ..., n\} \to E$

Definition 1.14. We say $card(A) \leq card(B)$, if there is an injective function $f : A \to B$.

- If $card(A) \leq card(B)$ and $card(B) \leq card(A)$, then $card(A) = card(B)$, because there is a bijective function $f : A \to B$.
- $card(\mathbb{N}) = card(\mathbb{N} \times \mathbb{N}) = card(\mathbb{N} \times \mathbb{N} \times \mathbb{N})$.

- If S is a set, $card(S) < card(\wp(S))$.

Definition 1.15. A set S is countable, if $card(S) \leq card(\mathbb{N})$

- If two sets A and B are countable, then $A \times B$ and $A \cup B$ are countable.
- Every infinite subset of a countable set A is countable.
- The countable union of countable sets is countable.
- The set of all rational numbers is countable.

Theorem 1.16. Let A be the set of all sequences whose elements are the digits 0 and 1. This set is uncountable.

2

The Real Number System

"Sometimes the questions are complicated and the answers are simple."

– Dr. Seuss, *Source of this quote*

2.1 Ordered sets

Definition 2.1. An order on a set S is a relation, denoted by $<$, with the following two properties: (1) If $x, y \in S$, then one and only one of the statements $x < y, x = y, x > y$ is true, (2) if $x, y, z \in S$ and if $x < y$ and $y < z$, then $x < z$.

Definition 2.2. An ordered set is a set in which an order is defined.

Definition 2.3. S is an ordered set and $E \subset S$. If there exists a $\mu \in S$ such that $x \leq \mu$ for every $x \in E$, then μ is an upper bound of E.

Definition 2.4. An upper bound μ is the least upper bound, if $\alpha < \mu$ indicates that α is not an upper bound. We write $\mu = \sup E$.

Definition 2.5. S is an ordered set and $E \subset S$. If there exists a $\mu \in S$ such that $x \geq \mu$ for every $x \in E$, then μ is a lower bound of E.

Definition 2.6. A lower bound μ is the greatest lower bound, if $\alpha < \mu$ indicates that α is not a lower bound. We write $\mu = \inf E$.

Definition 2.7. A set S has the least-upper-bound property if and only if every non-empty subset $E \subset S$ with an upper bound has a least upper bound in S.

Theorem 2.8. An ordered set S has the least-upper-bound property if and only if it has the greatest-lower-bound property.

2.2 Fields

Definition 2.9. A field is a set F with two operations, addition and multiplication, which satisfies the following axioms.

- Associative laws
- Commutative laws
- Distributive laws
- Existence of Identity
- Existence of additive inverse
- Existence of multiplicative inverse

Definition 2.10. A field F is an ordered field if F is an ordered set such that (1) $x + y < x + z$ if $x, y, z \in F$ and $y < z$, (2) $xy > 0$ if $x, y \in F$ and $x, y > 0$.

2.3 The Real Field

Theorem 2.11. There exists an ordered field with the least-upper-bound property, which contains rational numbers as a subfield.

Theorem 2.12. (Archimedean Principle) If $x, y \in R$ and $x > 0$, there is a positive integer n such that $nx > y$.

Theorem 2.13. If $x, y \in R$ and $x < y$, there is a rational number r such that $x < r < y$.

Definition 2.14. An ordered field F is complete if for all $S \in F$ whenever S has an upper bound it also has a least upper .

The real numbers R are the complete ordered field. That is, if F is a complete ordered field then there is an isomorphism $\phi : F \to R$ (an isomorphism preserves all the properties of the field including the arithmetic and order properties.

Theorem 2.15. For every real $x > 0$ and every integer $n > 0$, there is one and only one positive real y such that $y^n = x$.

Definition 2.16. The extended real number system consists of the real field R and two symbols, $-\infty$ and $+\infty$, such that $-\infty < x < +\infty$ for every $x \in R$.

3

Topology

3.1 Euclidean spaces

For each positive integer k, R^k is the set of all ordered k-tuples $\mathbf{x} = (x_1, ..., x_k)$, where $x_1, ..., x_k$ are real numbers. The elements of R^k are called vectors. For two vectors $\mathbf{x}$ and $\mathbf{y}$, we define

- addition: $\mathbf{x}+\mathbf{y} = (x_1 + y_1, ..., x_k + y_k)$.
- scalar multiplication: $a\mathbf{x} = (ax_1, ..., ax_k)$.
- inner product: $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{k} x_i y_i$.
- norm: $|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$.

Definition 3.1. The vector space R^k equipped with the inner product and norm is called Euclidean k-space.

Theorem 3.2. Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^k$ and a is real. Then

- $|\mathbf{x}| \geq 0$
- $|\mathbf{x}| = 0$ if and only if $\mathbf{x} = 0$
- $|a\mathbf{x}| = |a||\mathbf{x}|$
- $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$
- $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$
- $|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$

3.2 Metric spaces

Definition 3.3. A set Ω is said to be a metric space if with any two elements $x, y \in \Omega$ there is associated a real number $d(x, y)$, called the distance from x to y, such that (1) $d(x, y) > 0$ for $x \neq y$ and $d(x, x) = 0$, (2) $d(x, y) = d(y, x)$, (3) $d(x, z) \leq d(x, y) + d(y, z)$ for any $z \in \Omega$.

Definition 3.4. A set $E \subset R^k$ is convex if $\lambda x + (1 - \lambda)y \in E$, whenever $x, y \in E$ and $0 < \lambda < 1$.

- A neighborhood of a point p is a set $N_r(p)$ consisting of all points q such that $d(p, q) < r$ for some $r > 0$. The number r is called the radius of $N_r(p)$.
- A point p is a limit point of the set E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.
- If $p \in E$ is not a limit point of E, then p is called an isolated point of E.
- A set E is closed if every limit point of E is a point of E.
- A point p is an interior point of E, if there is a neighborhood N of p such that $N \subset E$.
- E is open if every point of E is an interior point of E.
- A set E is perfect if E is closed and every point of E is a limit point of E.
- E is bounded if there is a real number M and a point p such that $d(p, q) < M$ for all $q \in E$.
- E is dense in X if every point of X is a limit point of E or a point of E.

Theorem 3.5. Every neighborhood is an open set.

Theorem 3.6. Every neighborhood of a limit point of E contains infinitely many points of E.

Theorem 3.7. A set E is open if and only if its complement is closed.

- The union of any collection of open sets is open.
- The intersection of any finite collection of open sets is open.
- The intersection of any collection of closed sets is closed.
- The union of any finite collection of closed sets is closed.

Theorem 3.8. Suppose $Y \subset X$. A subset E is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.

3.3 Compact sets

Definition 3.9. An open cover of a set E in a metric space Ω is a collection of open subsets $\{G_\alpha\}$ of Ω such that $E \subset \cup_\alpha G_\alpha$

Definition 3.10. A subset X of a metric space Ω is compact if every open cover of X contains a finite subcover.

- Suppose $E \subset Y \subset Z$. Then E is compact relative to Y if and only if E is compact relative to Z.
- Compact subsets of metric spaces are closed.
- Closed subsets of compact sets are compact.

Theorem 3.11. If a set E in R^k has one of the following three properties, then it has the other two, (1) E is closed and bounded, (2) E is compact, (3) Every infinite subset of E has a limit point in E.

Theorem 3.12. (Weierstrass) Every bounded infinite subset of R^k has a limit point in R^k.

Theorem 3.13. If E is a non-empty perfect set in R^k, then it is uncountable.

The cantor set is a non-empty perfect set. Thus, it is uncountable, but its measure is 0.

3.4 Connected sets

Definition 3.14. Two subsets A and B of a metric space X are said to be separated if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty.

Definition 3.15. A set E is said to be connected if E is not a union of two nonempty separated sets.

Theorem 3.16. A subset E of the real line is connected if and only if it has the following property: if $x, y \in E$ and $x < z < y$, then $z \in E$.

4

Sequences and Series

"Good friends, good books, and a sleepy conscience: this is the ideal life."

– Mark Twain, *Source of this quote*

4.1 Convergent sequences

Definition 4.1. A sequence p_n in a metric space X is said to converge if there is a point $p \in X$ with the following property: for every $\epsilon > 0$, there is an integer N such that $n \geq N$ implies $d(p_n, p) < \epsilon$. We write $\lim_{n\to\infty} p_n = p$.

- $\{p_n\}$ converges to $p \in X$ if and only if every neighborhood of p contains p_n for all but finitely many n.
- If $p \in X$, $p' \in X$, and if $\{p_n\}$ converges to $p \in X$ and to $p' \in X$, then $p = p'$.
- If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.
- If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n\to\infty} p_n$.

Theorem 4.2. A sequence of vectors $X_n = (x_{1,n}, ..., x_{k,n})$ in R^k converges to a vector $X = (x_1, ..., x_k)$ in R^k if and only if $\lim_{n\to\infty} x_{j,n} = x_j$ for $1 \leq j \leq k$.

Theorem 4.3. X_n and Y_n are two sequences of vectors in R^k and a_n is a sequence of real numbers. Suppose $X_n \to X$, $Y_n \to Y$, and $a_n \to a$, then $X_n + Y_n \to X + Y$, $X_n \cdot Y_n \to X \cdot Y$, $a_n X_n \to aX$.

4.2 Subsequences

Definition 4.4. Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers such that $n_1 < n_2 < n_3....$ Then the sequence $\{p_{n_i}\}$ is called a subsequence of $\{p_n\}$. If $\{p_{n_i}\}$ converges, its limit is called a subsequential limit of $\{p_n\}$.

- $\{p_n\}$ converges p if and only if every subsequence of $\{p_n\}$ converges to p.
- If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X.
- Every bounded sequence in R^k contains a convergent subsequence.

Theorem 4.5. The subsequential limits of $\{p_n\}$ in a metric space X form a closed subset of X.

4.3 Cauchy sequences

Definition 4.6. A sequence $\{p_n\}$ in a metric space X is said to be a Cauchy sequence if for every $\epsilon > 0$ there is an integer N such that $d(p_n, p_m) < \epsilon$ if $n \geq N$ and $m \geq N$.

Definition 4.7. E is a subset of a metric space X. Let S be the set of $d(p, q)$ with $p, q \in E$. The $\sup S$ is called diameter of E.

Theorem 4.8. If k_n is a sequence of compact sets in a metric space X such that $k_{n+1} \subset k_n$ and if $lim_{n\to\infty} d(k_n) = 0$, where $d(k_n)$ is the diameter of set k_n, then $\cap_1^\infty k_n$ consists of exactly one point.

- In any metric space X, every convergent sequence is a Cauchy sequence.
- If X is a compact metric space, then every Cauchy sequence is a convergent sequence.
- In R^k, every Cauchy sequence converges.

Definition 4.9. A metric space in which every Cauchy sequence converges is said to be complete.

Theorem 4.10. Suppose p_n is monotonic. Then p_n converges if and only if it is bounded.

4.4 Upper and lower limits

Definition 4.11. The limit inferior and superior of a sequence x_n is defined by $\liminf_{x\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{m>n} x_m\right)$ and $\limsup_{x\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{m>n} x_m\right)$.

Definition 4.12. A real number η is a subsequential limit of a sequence (x_n) if there exists a strictly increasing sequence of natural numbers (n_k) such that $\eta = \lim_{k\to\infty} x_{n_k}$.

If $E \subset \overline{\mathbb{R}}$ is the set of all subsequential limits of (x_n), then $\limsup_{n\to\infty} x_n = \sup E$ and $\liminf_{n\to\infty} x_n = \inf E$.

- If the sequence (x_n) are real numbers, then $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ always exist, because real numbers are complete.
- The limit superior of x_n is the smallest real number b such that, for any positive real number ε , there exists a natural number N such that $x_n < b+\varepsilon$ for all $n > N$. In other words, any number larger than the limit superior is an eventual upper bound for the sequence. Only a finite number of elements of the sequence are greater than $b+\varepsilon$.
- The limit inferior of x_n is the largest real number b such that, for any positive real number ε , there exists a natural number N such that $x_n > b-\varepsilon$ for all $n > N$. In other words, any number below the limit inferior is an eventual lower bound for the sequence. Only a finite number of elements of the sequence are less than $b-\varepsilon$.

4.5 Series

Definition 4.13. Given a sequence $\{a_n\}$, we associate a sequence of partial sums $s_n = \sum_{k=0}^{n} a_k$. Then the symbol $\sum_{k=0}^{\infty} a_k$ is called a series.

Theorem 4.14. (Cauchy) $\sum_{k=0}^{\infty} a_k$ converges if and only if for every $\epsilon > 0$ there is an integer N such that $|\sum_{k=n}^{m} a_k| \le \epsilon$ if $m \ge n \ge N$.

- If $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{n\to\infty} a_n = 0$.
- A series of non negative terms converges if and only if the series is bounded.
- If $|a_n| \le c_n$ for $n \ge N_0$ where N_0 is some fixed integer, and if $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.
- root test: Given $\sum_{n=0}^{\infty} a_n$, let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then, $\sum_{n=0}^{\infty} a_n$ converges if $\alpha < 1$.

- ratio test: $\sum_{n=0}^{\infty} a_n$ converges if $\limsup_{n\to\infty} |a_{n+1}/a_n| < 1$.
- If $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$, then $\sum_{n=0}^{\infty}(a_n + b_n) = A + B$ and $\sum_{n=0}^{\infty}(ca_n) = cA$.

Definition 4.15. The series $\sum_{n=0}^{\infty} c_n z^n$, where $\{c_n\}$ and $\{z_n\}$ are complex numbers, is called a power series.

- The series $\sum_{n=0}^{\infty} c_n z^n$ converges if $|z| < \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}$.
- The partial sums A_n of $\sum_{n=0}^{\infty} a_n$ is a bounded sequence and $b_0 \geq b_1 \geq ...$ with $b_n \to 0$. Then $\sum b_n a_n$ converges.

Definition 4.16. A series $\sum_{n=0}^{\infty} a_n$ is said to converge absolutely if the series $\sum_{n=0}^{\infty} |a_n|$ converges.

Theorem 4.17. The series $\sum_{n=0}^{\infty} a_n$ converges absolutely to A and the series $\sum_{n=0}^{\infty} b_n$ converges to B. Then the series $\sum_{n=0}^{\infty} c_n$ converges to AB, in which $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Theorem 4.18. If $\sum_{n=0}^{\infty} a_n$ is a series of complex numbers which converges absolutely, then every rearrangement of $\sum_{n=0}^{\infty} a_n$ converges to the same sum.

5

Continuity

"There are only two ways to live your life. One is as though nothing is a miracle. The other is as though everything is a miracle."
– Albert Einstein, *Source of this quote*

5.1 Limits of Functions

Definition 5.1. Let X and Y be metric spaces. Suppose $E \subset X$, f maps E into Y and p is a limit point of E, we write $\lim_{x \to p} f(x) = q$ if there is a point q such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ for all points $x \in E$ for which $0 < d_X(x, p) < \delta$.

- $\lim_{x \to p} f(x) = q$ if and only if $f(x_n) \to q$ for every sequence $\{x_n\}$ which converges to p.
- If f has a limit, then the limit is unique.
- If $\lim_{x \to p} f(x) = A$ and $\lim_{x \to p} g(x) = B$, then $\lim_{n \to p}(f(x) + g(x)) = A + B$, $\lim_{n \to p}(f(x)g(x)) = AB$, $\lim_{n \to p}(f(x)/g(x)) = A/B$ if $B \neq 0$.

5.2 Continuous functions

Definition 5.2. Let X and Y be metric spaces. Suppose $E \subset X$, f maps E into Y and p is a point of E. Then f is continuous at p if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ for all points $x \in E$ for which $0 < d_X(x, p) < \delta$.

- If f is continuous at every point of E, then f is said to be continuous on E.
- If f and g are continuous functions, then $g \circ f$ is continuous.

- A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(v)$ is open for every open set v in Y.

Theorem 5.3. Let $f_1(x), ..., f_k(x)$ be real functions on a metric space X. $f = (f_1(x), ..., f_k(x))$ is a mapping of X into R^k. Then f is continuous if and only if each of the functions $f_1(x), ..., f_k(x)$ is continuous. If f and g are continuous mapping of X into R^k, then $f + g$ and $f \cdot g$ are continuous on X.

5.3 Continuity and Compactness

Definition 5.4. A mapping f of a set E into R^k is said to be bounded if there is a real number M such that $|f(x)| \leq M$ for all $x \in E$.

- A continuous mapping of a compact set is compact.
- If f is a continuous mapping of a compact metric space X, then $f(X)$ is closed and bounded.
- f is a continuous real function on a compact metric space X. Let $M = \sup_{p \in X} f(p)$ and $m = \inf_{p \in X} f(p)$. Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.
- f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. Then the inverse mapping $f^{-1}(f(x)) = x$ is a continuous mapping.

Definition 5.5. f is a continuous mapping of a metric space X into a metric space Y. f is said to be uniformly continuous on X if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d_Y(f(p) - f(q)) < \epsilon$ for all $p, q \in X$ for which $d_X(p, q) < \delta$.

- If f is a continuous mapping of a compact metric space X into a metric space Y, then f is uniformly continuous on X.

5.4 Continuity and Connectness

Theorem 5.6. f is a continuous mapping of a metric space X into a metric space Y. If E is a connected subset of X, then $f(E)$ is connected.

Theorem 5.7. f is a continuous real function on an interval $[a, b]$. If $f(a) < f(b)$ and if $f(a) < c < f(b)$ is a connected subset of X, then there exists a point $x \in (a, b)$ such that $f(x) = c$.

5.5 Monotonic functions

Definition 5.8. f is a real function on (a, b). f is said to be monotonically increasing if $a < x < y < b$ implies $f(x) \leq f(y)$.

Theorem 5.9. If f is monotonically increasing on (a, b), then $f(x+)$ and $f(x-)$ exist at every point of $x \in (a, b)$, and $\sup\limits_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf\limits_{x<t<b} f(t)$

Theorem 5.10. If f is monotonic on (a, b), then the set of points of (a, b) at which f is discontinuous is at most countable.

6

Differentiation

"You only live once, but if you do it right, once is enough."
– Mae West, *Source of this quote*

6.1 Derivative

Definition 6.1. A real-valued function f is defined on $[a, b]$. Define

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

for any $x \in [a, b]$ and $a < t < b$ and $t \neq x$. We say that f is differentiable at x if the limit exists.

- If f is differentiable at x, then f is continuous at x.
- If f and g are differentiable, then $f + g$, fg, and f/g are differentiable.
- If f is differentiable at x and g is differentiable at $f(x)$, then $g \circ f$ is differentiable at x.

Theorem 6.2. (mean value theorem) If f is a continuous real function on $[a, b]$ and f is differentiable in (a, b), then there is a point $x \in (a, b)$ such that $f(b) - f(a) = f'(x)(b - a)$

Theorem 6.3. (generalized mean value theorem) If f and g are continuous real functions on $[a, b]$ and are differentiable in (a, b), then there is a point $x \in (a, b)$ such that $[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$

Theorem 6.4. (generalized mean value theorem) If f is a real differentiable function on $[a, b]$ and $f'(a) < c < f'(b)$, then there exists a $x \in (a, b)$ such that $f'(x) = c$.

Theorem 6.5. (L'Hospital's rule) f and g are differentiable in (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$. If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ or $g(x) \to \infty$ as $x \to a$, then $\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f'(x)/g'(x)$.

Theorem 6.6. (Taylor theorem) f is a real function on $[a, b]$, n is a positive integer, f^{n-1} is continuous on $[a, b]$, f^n exists on $[a, b]$. Then, there exists a point β between x and α such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^k(a)}{k!}(x-a)^k + \frac{f^n(\beta)}{n!}(x-a)^n$$

Definition 6.7. (vector-valued functions) $\mathbf{f} = (f_1, ..., f_k)$ is a mapping of $[a, b]$ into R^k. $\mathbf{f}'(x)$ is the derivative at x if there exists a point $\mathbf{f}'(x)$ in R^k such that

$$\lim_{t \to x} \left| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x) \right| = 0$$

Theorem 6.8. $\mathbf{f}$ is a continuous mapping of $[a, b]$ into R^k and $\mathbf{f}$ is differentiable in (a, b). Then there exists $x \in (a, b)$ such that

$$|\mathbf{f}(b) - \mathbf{f}(a)| = |\mathbf{f}'(x)|(b - a)$$

7

Riemann Integral

"It does not do to dwell on dreams and forget to live."
– J.K. Rowling, *Harry Potter and the Sorcerer's Stone*

Definition 7.1. $x_1, ..., x_n$ are points in the interval $[a, b]$ such that $a = x_0 \leq x_1 \leq x_2 ... \leq x_n = b$. Let $\Delta x_i = x_i - x_{i-1}$. Suppose f is a bounded real function on $[a, b]$. We define $M_i = \sup\limits_{x_{i-1} \leq x \leq x_i} f(x)$, $m_i = \inf\limits_{x_{i-1} \leq x \leq x_i} f(x)$, $U = \sum_{i=1}^{n} \Big(M_i \Delta x_i\Big)$, $L = \sum_{i=1}^{n} \Big(m_i \Delta x_i\Big)$, and finally

$$\overline{\int_a^b} f dx = \inf U(P, f)$$

and

$$\underline{\int_a^b} f dx = \sup L(P, f)$$

The inf and sup are taken over all partitions. If the upper and lower integrals are equal, we say that f is Riemann integrable on $[a, b]$ and write $\int_a^b f dx$.

Definition 7.2. (Riemann-Stieltjes integral) Let α be a monotonically increasing function on $[a, b]$. Corresponding each partition P of $[a, b]$, we write $\Delta \alpha_i = \alpha_i - \alpha_{i-1}$. For any function f which is bounded on $[a, b]$, define $U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta \alpha_i$ and $L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta \alpha_i$. Finally, we define

$$\overline{\int_a^b} f d\alpha = \inf U(P, f, \alpha)$$

and

$$\underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha)$$

The inf and sup are taken over all partitions. If inf $=$ sup, we say that f is integrable with respect to α and write $f(x) \in R(\alpha)$.

Since f is bounded, $L(P, f, \alpha)$ and $U(P, f, \alpha)$ are bounded for every partition P. Thus, the lower integral $\sup L(P, f, \alpha)$ and the upper integral $\inf U(P, f, \alpha)$ are defined for every bounded function f. The key question is their equality. Observe that the refinement of partitions P will decrease $U(P, f, \alpha)$, increase $L(P, f, \alpha)$, decrease $M_i - m_i$. Thus, we have $L(P_1, f, \alpha) \leq supL(P, f, \alpha) \leq \inf U(P, f, \alpha) \leq U(P_2, f, \alpha)$.

- $f(x) \in R(\alpha)$ on $[a, b]$ if and only if for every $\epsilon > 0$ there exists a partition such that $U(p, f, \alpha) - L(p, f, \alpha) < \epsilon$.
- If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.
- If f is monotonic on $[a, b]$ and if α is continuous on $[a, b]$, then f is integrable.
- If f has only finitely many points of discontinuity on $[a, b]$ and α is continuous at every point at which f is discontinuous, then f is integrable.
- Suppose f is integrable on $[a, b]$ and $m \leq f \leq M$. If g is a continuous function on $[m, M]$, then $g \circ f$ is integrable on $[a, b]$.
- If f and g are integrable on $[a, b]$, then fg integrable on $[a, b]$.
- If f is integrable $[a, b]$, then $|f|$ is integrable $[a, b]$.

Theorem 7.3. (integration and differentiation) f is integrable on $[a, b]$. For $a \leq x \leq b$, define $F(x) = \int_a^x f(t)dt$. Then, $F(x)$ is continuous on $[a, b]$. If f is continuous at a point $x_0 \in [a, b]$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Theorem 7.4. (the fundamental theorem of calculus) If f is integrable on $[a, b]$ and there is a differentiable function F on $[a, b]$ such that $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Integration by parts: $\int_a^b f(x)G(x)dx + \int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a)$

Definition 7.5. Integration of vector-valued functions: Let $f_1, ..., f_k$ be real functions on $[a, b]$. $\mathbf{f} = (f_1, ..., f_k)$ is the mapping of $[a, b]$ into R^k. If α increases monotonically on $[a, b]$. $\mathbf{f}$ is integrable if and only if $f_1, ..., f_k$ are integrable on $[a, b]$. If so, then we define

$$\int_a^b \mathbf{f} d\alpha = \Big(\int_a^b f_1 d\alpha, ..., \int_a^b f_k d\alpha \Big).$$

- $\int_a^b \mathbf{f} d\alpha = \mathbf{F}(b) - \mathbf{F}(a)$.
- If $\mathbf{f}$ is integrable, then $|\mathbf{f}|$ is integrable, and $|\int_a^b \mathbf{f} d\alpha| \leq \int_a^b |\mathbf{f}| d\alpha$.

7.1 Rectifiable curves

Definition 7.6. A continuous mapping γ of $[a, b]$ into R^k is called a curve in R^k. If γ is 1-1, it is called an arc. If $\gamma(a) = \gamma(b)$, it is called a closed curve. Let P be a partition of $[a, b]$. We define

$$\Phi(P, \gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$$

$\Phi(P, \gamma)$ is the length of a polygonal path defined by the partition P. We define the length of γ by

$$\Phi(\gamma) = \sup \Phi(P, \gamma)$$

We say that γ is rectifiable if $\Phi(\gamma) < \infty$.

Theorem 7.7. If the derivative γ' of the curve γ is continuous, then γ is rectifiable, and $\Phi(\gamma) = \int_a^b |\gamma'(t)|dt$.

8

Sequences of functions

"Never doubt that a small group of thoughtful, committed, citizens can change the world. Indeed, it is the only thing that ever has."
– Margaret Mead, *Source of this quote*

8.1 Convergence

Definition 8.1. (pointwise convergence) $\{f_n\}$ is a sequence of functions defined on a set E. If $\{f_n(x)\}$ converges for every $x \in E$, then we define a function f by $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in E$.

Definition 8.2. (uniform convergence) $\{f_n\}$ is a sequence of functions defined on a set E. $\{f_n(x)\}$ is said to converge uniformly on E to a function f if for every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies $|f_n(x) - f(x)| \leq \epsilon$ for all $x \in E$.

- Uniform convergence implies pointwise convergence, but pointwise convergence does not imply uniform convergence.
- $\{f_n(x)\}$ is uniformly convergent if and only if for every $\epsilon > 0$ there is an integer N such that $m, n \geq N$ and $x \in E$ implies $|f_n(x) - f_m(x)| \leq \epsilon$ for all $x \in E$.
- Suppose $|f_n| \leq M_n$. If $\sum M_n$ converges, then $\sum f_n$ uniformly converges.
- If f_n uniformly converges to f, then $\lim_{n\to\infty} \lim_{x\to a} f_n(x) = \lim_{x\to a} \lim_{n\to\infty} f_n(x)$
- f_n is a sequence of continuous functions. Suppose f_n converges uniformly to f, then f is continuous.

Theorem 8.3. $\{f_n\}$ is a sequence of continuous functions on a compact set E, which converges pointwise to a continuous function f. If $f_n(x) \geq f_{n+1}(x)$ for all $x \in E$, $n = 1, 2, 3....$ Then $f_n \to f$ uniformly.

8.2 Uniform convergence and Integration

Theorem 8.4. $\{f_n\}$ is a sequence of integrable functions on $[a, b]$. If f_n converges uniformly to f on $[a, b]$, then $\lim\limits_{n\to\infty} \int_a^b f_n d\alpha = \int_a^b f d\alpha$.

8.3 Uniform convergence and Differentiation

Theorem 8.5. $\{f_n\}$ is a sequence of functions differentiable on $[a, b]$. If $\{f'_n\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly to a function f and $f'(x) = \lim\limits_{n\to\infty} f'_n(x)$.

Theorem 8.6. (STONE-WEIERSTRASS theorem) If f is a continuous complex function on $[a, b]$, there is a sequence of polynomials P_n such that $\lim\limits_{n\to\infty} P_n(x) = f(x)$ uniformly on $[a, b]$. If f is real, then P_n may be taken real.

9

Special functions

"Life is a book and there are a thousand pages I have not yet read."
– Cassandra Clare, *Clockwork Princess*

9.1 Power series

Definition 9.1. f is said to be a power series function if it has the form $f(x) = \sum_{n=0}^{\infty} c_n x^n$

Theorem 9.2. Suppose the series $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$, and define $f(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $\sum_{n=0}^{\infty} c_n x^n$ converges uniformly on $[-R+\epsilon, R-\epsilon]$. The function f is continuous and differentiable on $(-R, R)$ and $f'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1}$.

Theorem 9.3. Suppose the series $\sum c_n$ converges and define $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $-1 < x < 1$. Then, $\lim\limits_{x \to 1} \sum_{n=0}^{\infty} c_n x^n = \sum c_n$

Theorem 9.4. Suppose $\sum_j |a_{ij}| = b_i$ and $\sum b_i$ converges. Then, $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$.

9.2 Fourier series

Definition 9.5. A function of the form $p(x) = \sum_{k=0}^{n} \alpha_k \cos kx + \beta_k \sin kx$ is called a trigonometric polynomial.

- $p(x)$ is 2π periodic and closed under addition and multiplication by real numbers.
- $\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0$, $\int_{-\pi}^{\pi} \sin nx \sin mx dx = 0$ or π, $\int_{-\pi}^{\pi} \cos nx \cos mx dx = 0$ or π, or 2π.
- $\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} p(x) \cos nx dx$ and $\beta_n = \frac{1}{\pi} \int_{-\pi}^{\pi} p(x) \sin nx dx$

Definition 9.6. Let f be a 2π-periodic function which is integrable on $[-\pi, \pi]$. The Fourier series of f is

$$\alpha_0/2 + \sum_{n=1}^{\infty} \alpha_n \cos nx + \beta_n \sin nx$$

where the Fourier coefficients are given by $\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$ and $\beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$.

Theorem 9.7. Under certain conditions, the Fourier series of f converges to f.

Definition 9.8. The Fourier transform of an integrable function $\hat{f} : \mathbb{R} \to \mathbb{C}$ is defined as

$$\hat{f}(\eta) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\eta x}dx$$

for any real number η, where x often represents time and η is frequency.

Under certain conditions, $f(x)$ is determined by $\hat{f}(\eta)$ via the inverse transform:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\eta)e^{2\pi i\eta x}d\eta$$

10

Functions of several variables

Definition 10.1. A set $X \subset R^n$ is a vector space if $x + y \in X$ and $cx \in X$ for all $x, y \in X$ and for all scalars c. If $x_1, ..., x_k \in X$ and $c_1, ..., c_k$ are scalars, the vector $c_1x_1 + ... + c_kx_k$ is called a linear combination of $x_1, ..., x_k \in X$. If $S \subset R^n$ and E is the set of all linear combinations of elements of S, we say that S spans E or that E is the span of S. Every span is a vector space.

Definition 10.2. The vectors $\{x_1, ..., x_k\}$ are said to be independent if $c_1x_1 + c_2x_2 + ... + c_kx_k = 0$ implies $c_1 = c_2 ... = c_k = 0$.

Definition 10.3. An independent subset of a vector space X which spans X is called a basis of X.

Definition 10.4. A mapping A of a vector space X into a vector space Y is said to be linear transformation if $A(x_1 + x_2) = A(x_1) + A(x_2)$ and $A(cx) = cA(x)$ for all $x, x_1, x_2 \in X$ and all scalars c.

Definition 10.5. Linear transformation of X into X is called linear operators on X. If A is a linear operator on X which is 1-1 and onto X, we say that A is invertible.

Definition 10.6. The norm $||A||$ of a linear transformation A is defined as the sup of all numbers $|Ax|$, where x ranges over all vectors in R^n with $|x| \leq 1$.

- $|Ax| \leq ||A|||x|$
- $||A|| < \infty$ and A is a uniformly continuous mapping of R^n into R^m.
- If A and B are linear transformations and c is a scalar, then $||A+B|| \leq ||A|| + ||B||$, ||cA||=|c|||A||. With the distance between A and ||B|| defined as $||A - B||$, $L(R^n, R^m)$ is a metric space.
- $||AB|| \leq ||A||||B||$.

11

The Lebesgue Theory

11.1 Set functions

Definition 11.1. A family Ω of sets is called a ring (or field) if $A \in \Omega$ and $B \in \Omega$ implies $A \cup B \in \Omega$ and $A - B \in \Omega$. In addition, a ring Ω is called a σ-ring if $\cup_{n=1}^{\infty} A_n \in \Omega$ whenever $A_n \in \Omega$ $(n = 1, 2, ...)$.

Definition 11.2. We say that ψ is a set function defined on Ω if ψ assigns to each $A \in \Omega$ a number of the extended real system. The set function ψ is additive if $\psi(A + B) = \psi(A) + \psi(B)$ for $A \cap B = 0$ and ψ is countably additive if $\psi(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \psi(A_n)$ for $A_j \cap A_i = 0$ $(i \neq j)$.

11.2 Construction of the Lebesgue measure

Definition 11.3. An interval in R^p is defined as the set of points $x = (x_1, ..., x_p)$ such that $a_i \leq x_i \leq b_i$. If A is the union of a finite number of intervals, A is said to be an elementary set. If an interval is denoted by I, then $A = \cup_{i=1}^{n} I_i$. We define a set function $m(I) = \prod_{i=1}^{n}(b_i - a_i)$. If I_i are disjoint intervals, then $m(A) = \sum_{i=1}^{n} m(I_i)$.

- The collection Ω of all elementary subsets of R^p is a ring. The set function m is additive on Ω.
- If $A \in \Omega$, A is the union of a finite number of disjoint intervals.

Definition 11.4. A non-negative additive set function ψ on Ω is said to be regular if for every $\epsilon > 0$ and every $A \in \Omega$ there exist sets $F, G \in \Omega$ such that F is closed, G is open, $F \subset A \subset G$, and $\psi(F) + \epsilon \geq \psi(A) \geq \psi(G) - \epsilon$.

- m is regular

Every regular set function can be extended to a countably additive set function on a σ-ring which contains Ω.

Definition 11.5. Let μ be a non-negative, additive, finite, regular on Ω and A_n is a countable open cover of a set $E \in R^p$ such that $E \subset \cup_{n=1}^{\infty} A_n$. Define $\mu^*(E) = \inf \sum_{n=1}^{\infty} \mu(A_n)$. $\mu^*(E)$ is called the outer measure of E.

Theorem 11.6. For every $A \in \Omega$, $\mu^*(A) = \mu(A)$. If $E = \cup_{n=1}^{\infty} E_n$, then $\mu^*(E) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$.

- If a sequence A_n of elementary sets such that $A_n \to A$, we say A is finitely μ-measurable and write $A \in R_F(\mu)$
- If E is the union of a countable collection of finitely μ-measurable sets, we say that E is μ-measurable and write $E \in R(\mu)$.
- $R(\mu)$ is σ-ring and μ^* is countable additive on $R(\mu)$.

Thus, μ, originally an additive set function defined on a ring Ω, is extended to a countably additive set function on the σ-ring $R(\mu)$. If $\mu = m$, it is called Lebesgue measure.

11.3 Measurable spaces

Definition 11.7. A set X is said to be a measure space if there exists a σ-ring R of subsets of X and a non-negative countably additive set function μ on R.

Definition 11.8. A function f defined on a measurable space X is said to be a measurable function if the set $\{x : f(x) > a\}$ is measurable for every real a.

- If f is measurable, then $|f|$ is measurable.
- If f_n is a sequence of measurable functions, then $g = \sup f_n(x)$ and $h = \lim_{n\to\infty} \sup f_n(x)$ are measurable functions.
- If f and g are measurable, then $f + g$, fg, $max(f, g)$ and $min(f, g)$ are measurable
- If f is measurable, then $f^+ = max(f, 0)$ and $f^- = -min(f, 0)$ are measurable.
- The limit of a convergent sequence of measurable functions is measurable.

Definition 11.9. A simple function s is a finite linear combination of indicator functions $s(x) = \sum_{i=1}^{n} C_i I_{(x \in E_i)}$.

Theorem 11.10. f is a real function on X. There exists a sequence of simple functions such that $s_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$. If f is measurable, $\{s_n(x)\}$ can be chosen to be a sequence of measurable functions. If $f \geq 0$, $\{s_n(x)\}$ can be chosen to be a sequence of monotonically increasing measurable functions.

11.4 Integration

Definition 11.11. If f is a measurable and non-negative, we define $\int_E f d\mu = \sup I_E(s)$, where sup is taken over all simple measurable functions s such that $0 \leq s \leq f$.

Definition 11.12. If f is measurable, then $\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$. If both integrals on the right side of the equation are finite, we say that f is integrable on E in the Lebesgue sense with respect to μ.

Theorem 11.13. (Lebesgue monotone convergence theorme) $\{f_n\}$ is a sequence of monotonically increasing non-negative measurable functions $0 \leq f_1 \leq f_2 \leq ...$ and $f_n \to f$, then $\int_E f_n d\mu \to \int_E f d\mu$.

Theorem 11.14. (Fatou's theorem) If f_n is a sequence of non-negative measurable functions. If $f = \liminf_{n \to \infty} f_n$, then $\int_E f d\mu \leq \liminf_{n \to \infty} \int_E f_n d\mu$.

Theorem 11.15. (Lebesgue dominated convergence theorem) f_n is a sequence of measurable functions such that $f_n \to f$. If there exists an integrable function g such that $|f_n| \leq g$, then $\int_E f_n d\mu \to \int_E f d\mu$.