Math 215-800 **PS #9**
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Writing Exercises. Type up your solution to #1 using LaTeX. Start by going to the course webpage and downloading the .tex file for this assignment; pop that .tex file into your favorite latex editor (probably Overleaf.com) and type your responses just after the corresponding problem in the .tex file.

1. Prove using induction, for all $n \geq 1$, that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

   Proof. Let $s(n)$ be the statement:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

   for all $n \geq 1$.

   When $n = 1$,

$$LHS : \frac{1}{1(1+1)} = \frac{1}{1(2)} = \frac{1}{2}$$

$$RHS : \frac{1}{1+1} = \frac{1}{2}$$

   The left hand side and right hand side are equal, thus $s(1)$ is true.

   Now, suppose $s(k)$ is true for some k, to show that $s(k+1)$ is true, let's look at the left hand side:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

   by the hypothesis that $s(k)$ is true.

   Finally by simplifying the left hand side to check for equality with the right hand side:

$$LHS = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{(k+2)k+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)(k+1)}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2} = RHS$$

   Thus, if $s(k)$ is true, then $s(k+1)$ is true. By PMI, $s(n)$ is true for all $n \geq 1$. ∎

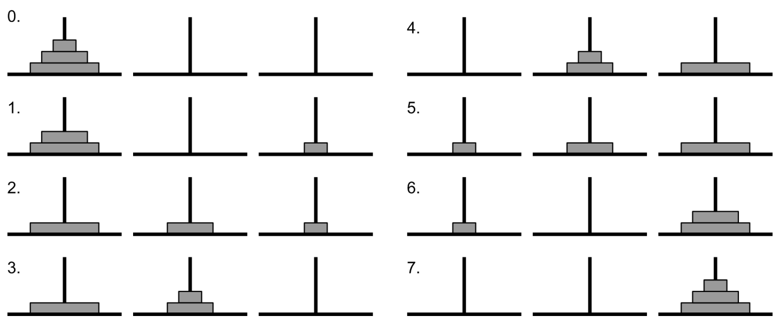
Additional Exercises. Complete the rest of these problems. You need not typeset your answers, unless you want to. Staple your answers to your write-up for the Writing Exercise and turn in one homework with your name on the front, at the top. **For #4 and #5, show some work to explain your answer. A number on its own will earn you no credit, even if the number is correct!**

2. Let $f^{(n)}(x)$ denote the n-th derivative of the function $f(x)$ and fix $f(x) = e^{ax}$ for a non-zero real constant a. Prove using induction, for all $n \geq 1$, that $f^{(n)}(x) = a^n e^{ax}$.

3. Prove, for all integers $n \geq 1$, that the Tower of Hanoi game with n disks can be solved in $2^n - 1$ moves.

   If you haven't heard of the Tower of Hanoi game, you might like to! It's a logic puzzle where there are three pegs and n disks stacked by size on one peg on the left. The largest disk is on the bottom, the smallest on the top. The goal is to move the disks all over to the right-most peg, one at a time. As you move the disks, you can never put a larger disk on top of a smaller disk. There's a picture of the Tower of Hanoi game with $n = 3$, from starting position to finish, in Figure 1.

Figure 1: Here's the algorithm for solving the Tower of Hanoi game with 3 disks. It takes 7 moves, and it's no coincidence that $2^3 - 1 = 7$.



4. (a) How many 5-letter strings can be created using the English alphabet?
   (b) How many 5-letter strings, using letters from the English alphabet, can be created if no letter appears more than once?
   (c) How many ways are there to rearrange a 5-letter string?

5. We are heading to dinner at the Red Sun Cafe in Oswego. This month, the Red Sun is serving stretched bread, lettuce wraps, and wings as appetizers. They also offer a delicious gnocci meal, a Thai curry with tofu dish, a spicy saucy rigatoni meal, and margherita pizza. Dessert! Yum! This month the dessert offerings include chocolate mousse, blueberry cream pie, carrot cake, and also a house-made butter pecan ice cream (To die for!).
   (a) How many different combinations of a main meal and a dessert are possible?
   (b) How many different combinations of an appetizer, a main meal and a dessert are possible?
   (c) How many different combinations of an appetizer, a main meal, and a dessert that include wings are possible?
   (d) I am a vegetarian. I'm not sure who put wings on this description, but dude, that's not something I'll ever order. How many appetizer, main meal, and dessert combinations are really possible for me?

6. The notation $\binom{n}{k}$ is defined to equal the following:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where n, k are non-negative integers with $k \leq n$. Remember that $0! = 1$, so this formula is defined even when $n = k = 0$. For each choice of n and k given below, evaluate $\binom{n}{k}$ by expanding the factorials *symbolically* and cancelling common factors. Show the work to do this expansion and cancellation!

(a) $n = 52$, $k = 52$

(b) $n = 52$, $k = 0$

(c) $n = 108$, $k = 104$

(d) $n = 256$, $k = 5$

(e) $n = 10$, $k = 3$