1. Let $S(n)$ be the statement
$n^3 < 3^n$ for all $n \geq 4$.

n	n^3	3^n
0	0	1
1	1	3
2	8	9
3	27	27
4	64	81
5	125	243

Base case: when $n=4$

LHS: $4^3 = 64$
RHS: $3^4 = 81$
LHS < RHS

Thus $S(4)$ is true.

Hints: 1) If $K \geq 4$ then $K^3 > 3K^2$.
2) If $K \geq 3$ then $K^3 > 4K > 3K+1$

Now suppose $S(K)$ is true for some K. To show that $S(K+1)$ is true, let's look at the LHS

$S(K)$: $K^3 < 3^K$

LHS: $(K+1)^3 = (K^2+2K+1)(K+1)$
$= K^3 + 3K^2 + 3K + 1$
$< K^3 + K^3 + 3K + 1$ if $K \geq 4$
$< K^3 + K^3 + 4K$ if $K \geq 3$
$< K^3 + K^3 + K^3$ if $K \geq 3$
$< 3^K + 3^K + 3^K$
$= 3^1 \cdot 3^K$
$= 3^{K+1}$ (RHS)

Thus if $S(K)$ is true then $S(K+1)$ is true.
By PMI, $S(n)$ is true for all $n \geq 4$.

	n	1
n	n^2	n
1	n	1

$(n^2+2n+1)(n+1)$

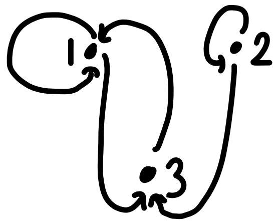
	n^2	$2n$	1
n	n^3	$2n^2$	n
1	n^2	$2n$	1

$n^3 + 3n^2 + 3n + 1$

2: V is a set

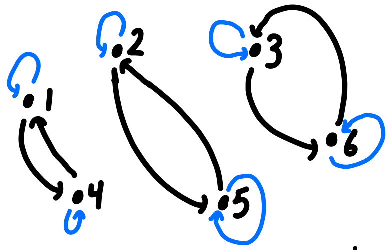
$\varphi \subset V \times V$

(a) $V = \{1,2,3\}$ and $\varphi = \{(1,1),(1,3),(3,1),(2,3),(2,2)\}$



(b) $V = \{1,2,3,4,5,6\}$ and $\varphi = \{(v_1,v_2) \in V \times V \mid v_1$ and v_2 have the same remainder after division by $3\}$

$\varphi = \{(1,4),(2,5),(3,6),$
$(4,1),(5,2),(6,3),(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$



(c) $V = \{0,1,2,3,4,5\}$ and $\varphi = \{(v_1,v_2) \in V \times V \mid v_2 = v_1 + 1$ or $v_1, v_2 \in \{5,0\}$ or $v_1 = v_2 + 1\}$

$\left.\begin{array}{l}(5,0)\\(0,5)\\(0,0)\end{array}\right]$ From case 2

case 1: $v_2 = v_1 + 1$

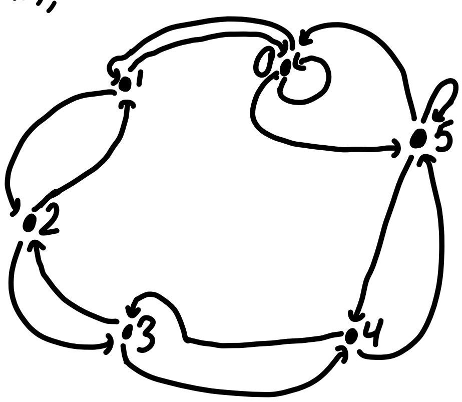
$\varphi = (0,1),$
$(1,2),$
$(2,3),$
$(3,4),$
$(4,5),$

case 2: $v_1, v_2 \in \{5,0\}$

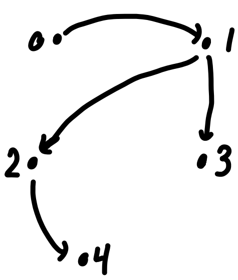
$(0,0),$
$(0,5),$
$(5,0),$
$(5,5),$

case 3: $v_1 = v_2 + 1$

$(1,0),$
$(2,1),$
$(3,2),$
$(4,3),$
$(5,4)$



(d) $V = \{0,1,2,3,4\}$ and $\varphi = \{(0,1),(1,2),(2,4),(1,3)\}$



3 (b)

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

3(c)

$$\begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

3(d)

$$\begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$