Math 215-800 **PS #11 Corrected**
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Writing Exercises. Type up your solution to #1 using LaTeX. Start by going to the course webpage and downloading the .tex file for this assignment; pop that .tex file into your favorite latex editor (probably Overleaf.com) and type your responses just after the corresponding problem in the .tex file.

1. Prove that $3^n > n^3$ using induction, for all $n \geq 4$. You can use the following two hints to help you, without proving them:
   - If $k \geq 4$ then $k^3 \geq 3k^2$.
   - If $k \geq 3$ then $k^3 > 4k > 3k + 1$.

Proof. Let $s(n)$ be the statement:

$$3^n > n^3$$

for all $n \geq 4$.

Base Case: When $n = 4$:

$$LHS : 3^4 = 81,$$

$$RHS : 4^3 = 64,$$

$$LHS > RHS.$$

Thus $s(4)$ is true. Now suppose $s(k)$ is true for some k. To show that $s(k+1)$ is true, let's look at the RHS:

$$RHS : (k+1)^3 = k^3 + 3k^2 + 3k + 1.$$

By Hint 1, if $k \geq 4$, then $k^3 \geq 3k^2$:

$$RHS \leq k^3 + k^3 + 3k + 1.$$

By Hint 2, If $k \geq 3$ then $k^3 > 4k > 3k + 1$:

$$RHS < k^3 + k^3 + 4k < k^3 + k^3 + k^3,$$

$$RHS < 3^k + 3^k + 3^k,$$

by the hypothesis that $s(k)$ is true.

$$3^k + 3^k + 3^k = 3 \times 3^k = 3^{k+1} = LHS.$$

Then, putting these together, we have

$$LHS > RHS.$$

Thus if $s(k)$ is true, then $s(k+1)$ is true. By PMI, $s(n)$ is true for all $n \geq 4$. ∎

Additional Exercises. Complete the rest of these problems. You need not typeset your answers, unless you want to. Staple your answers to your write-up for the Writing Exercise and turn in one homework with your name on the front, at the top.

2. Let V be a set and let $\varphi \subset V \times V$. One way to visualize how φ pairs up the elements of V is to draw "dots", or *vertices* or *nodes*, to represent the elements of V and to draw an arrow from the vertex $v_1 \in V$ to the vertex $v_2 \in V$ if and only if $(v_1, v_2) \in \varphi$. For each set V and set φ, draw the corresponding picture.

   (a) $V = \{1, 2, 3\}$ and $\varphi = \{(1, 1), (1, 3), (3, 1), (2, 3), (2, 2)\}$

   (b) $V = \{1, 2, 3, 4, 5, 6\}$ and
   $$\varphi = \{(v_1, v_2) \in V \times V \mid v_1 \text{ and } v_2 \text{ have the same remainder after division by } 3\}$$

   (c) $V = \{0, 1, 2, 3, 4, 5\}$ and
   $$\varphi = \{(v_1, v_2) \in V \times V \mid v_2 = v_1 + 1 \text{ or } v_1, v_2 \in \{5, 0\} \text{ or } v_1 = v_2 + 1\}$$

   (d) $V = \{0, 1, 2, 3, 4\}$ and $\varphi = \{(0, 1), (1, 2), (2, 4), (1, 3)\}$

   Another way to visualize subsets of $V \times V$ is to create a matrix, with one row and one column for each element of V and then if $(v_1, v_2) \in \varphi$, put a 1 in the entry that marks the intersection of row v_1 and column v_2. For example, the set φ from #2(a) is associated with the matrix in Figure 1.

   Figure 1: Capturing φ from #2(a) in matrix form.

   $$\left(\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 3 & 1 & 0 & 0 \end{array}\right)$$

3. Make the matrix associated with each of the relations in #2(b), #2(c), and #2(d).