Math 215-800 **PS #8**
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Writing Exercises. Type up your solutions to #1 using LaTeX. Start by going to the course webpage and downloading the .tex file for this assignment; pop that .tex file into your favorite latex editor (probably Overleaf.com) and type your responses just after the corresponding problem in the .tex file.

1. Using the terminology and language of MAT 215 properly, give a clear and concise argument why $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ defined by $f(x) = \tan(x)$ is a bijection. You can use a graph of this function to aid in your argument and avoid the technicalities of computing with the tangent function, but your intended reader is a senior math major who you are trying to impress with your abstract proof-writing competency.

   Proof. Let $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be defined by $f(x) = \tan(\mathrm{x})$. We'll show f is a bijection. To show f is injective, let $f(a) = f(b)$. Since the graph of this function passes the horizontal line test, this means that every possible element in the domain will produce a different element in the codomain. This information indicates that the only possible case in which $f(a) = f(b)$ is true is in the case where $a = b$. Therefore $a = b$, so f is injective. To show that f is surjective, let $b \in \mathbb{R}$, in which according to the definition of f, $f(a) = \tan(a) = \mathrm{b}$. Since $b \in \mathbb{R}$ and the graph of this function indicates every element of the codomain is the output of some input from the domain, $a \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Thus f is surjective. Since it is now known that f is both injective and surjective, f is thus a bijection.

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Additional Exercises. Complete the rest of these problems. You need not typeset your answers, unless you want to. Staple your answers to your write-up for the Writing Exercise and turn in one homework with your name on the front, at the top. **You definitely should read the next few paragraphs so that you can do #2-4.**

Let A be an arbitrary set with elements indexed by $\mathbb{N}$. This means that $A = \{a_1, a_2, a_3, a_4, \dots\}$; the set A could be finite or infinite – it is *countable*, though, because we can create a bijection from either $\mathbb{N}$ or a finite subset $\{1, 2, \dots, n\}$ into A using the indices on the elements of A.

Let $\mathcal{F} = \{f \mid f : A \to \{0, 1\}\}$; then $\mathcal{F}$ is the set of functions with A as the domain and $\{0, 1\}$ as the codomain. Since a function $f \in \mathcal{F}$ must *do something* to each $a \in A$, one way to write down f is to write a sequences of 0s and 1s corresponding to the elements of A in index order.

For instance, if $A = \{\alpha, \beta, 52, 7.92, \square\}$ then one function $f \in \mathcal{F}$ would be

$$f = 01001; \text{ or alternatively: } f(\alpha) = 0, f(\beta) = 1, f(52) = 0, f(7.92) = 0, \text{ and } f(\square) = 1.$$

From this function, you can sort the elements of A into two subsets: those elements that output a 1, and those elements that output a 0. *The subset of A associated with f* is the set of elements for which f outputs a 1, so in our example $Y = \{\beta, \square\}$. This association goes both ways, though — if I give you a subset Y of A then you can create a function associated with Y just by defining f to output 1 for those elements in Y and 0 for those elements that are not in Y.

Now that you have a better understanding of the functions $f \in \mathcal{F}$ and their associated subsets for a specific set A, we return to the general setting to formalize this idea. Define $\varphi : \mathcal{P}(A) \to \mathcal{F}$ by $\varphi(Y) = f_Y$ where

$$\varphi(Y)(x) = f_Y(x) = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}.$$

3. You need to figure out how φ works. To help with that, we'll fix a concrete set A and an ordering of its elements. Let $A = \{\pi, \beta, -8.76, \heartsuit, \square, a, q\}$.

   (a) Let $Y = \{\beta, -8.76, \square\}$. Write out f_Y as a sequence of 0s and 1s.

   (b) Let $Y = \emptyset$. Write out f_Y as a sequence of 0s and 1s.

   (c) Suppose that $k \in \mathcal{F}$ such that $k = 1111111$. What set Y will φ assign to k? (What $Y \subseteq A$ is such that $k = f_Y$?)

   (d) Suppose $g \in \mathcal{F}$ such that $g = 1011010$. What set Y will φ assign to g?

4. Let $A = \{a_1, a_2, a_3, \cdots\}$ be an arbitrary countable set. Write out f_Y as a sequence of 0s and 1s for each Y.

   (a) $Y = \{a_1, a_3\}$

   (b) $Y = \{a_i \mid i \text{ is odd}\}$

   (c) $Y = \{a_i \mid \exists k \in \mathbb{Z} \text{ with } k \geq 0 \text{ and } i = 3k + 2\}$

5. Let $A = \{a_1, a_2, a_3, \cdots\}$ be an arbitrary countable set. What set Y would have $f_Y = 101001000100001\cdots$?

Further Reading. The function φ is a bijection. For each $f \in \mathcal{F}$, there is a $Y \subseteq A$ such that $\varphi(Y) = f$ and since $Y \in \mathcal{P}(A)$, this means that φ is surjective. On the other hand, the elements of a set uniquely determine the set; therefore if $\varphi(Y) = \varphi(Z)$ then f_Y and f_Z always produce the same outputs as one another ... when f_Y produces 1 then f_Z produces 1, and therefore $Y \subseteq Z$. Similarly, $Z \subseteq Y$ and thus $Y = Z$. This means that φ is injective. Together, we now have that φ is a bijection and therefore $\mathcal{F}$ and $\mathcal{P}(A)$ are equinumerous.

This is why some people never think about power sets — instead, they think about strings of 0s and 1s of length $|A|$. For many reasons, thinking about strings is a little easier than thinking about power sets. And, you can define set operations on the strings to make computations more conceptually palatable.