1: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Step 1:

Let $S(n)$ be the statement

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1} \text{ for all } n \geq 1$$

Step 2:

Evaluate base case:

When $n=1$

LHS: $\frac{1}{1(1+1)} = \frac{1}{1(2)} = \frac{1}{2}$

RHS: $\frac{1}{1+1} = \frac{1}{2}$, thus $S(1)$ is true.

Step 3:

Goal: Show $S(K+1)$ is true

Suppose $S(K)$ is true for some K

To show that $S(K+1)$ is true, let's look at the LHS

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots \frac{1}{(K+1)(K+2)} = \frac{K}{K+1} + \frac{1}{(K+1)(K+2)}$$ by the hypothesis that $S(K)$ is true.

$$\frac{(3)}{(3)}\frac{1}{2} + \frac{1}{6} = \frac{3+1}{6} = \frac{4}{6} = \frac{2}{3} \leftarrow K=2$$

$$\frac{(K+2)}{(K+2)}\frac{K}{K+1} + \frac{1}{(K+1)(K+2)} \cancel{= \frac{K+1}{K+2}}$$

$$= \frac{(K+2)K + 1}{(K+1)(K+2)} \cancel{= \frac{K+1}{K+2}}$$

$$= \frac{K^2 + 2K + 1}{(K+1)(K+2)} \cancel{= \frac{K+1}{K+2}}$$

$$\begin{array}{c|c|c} & K & 1 \\ \hline K & K^2 & K \\ \hline 1 & K & 1 \end{array}$$

K^2

$K \cdot K$

$$= \frac{\cancel{(K+1)}(K+1)}{\cancel{(K+1)}(K+2)} \cancel{= \frac{K+1}{K+2}}$$

$$= \boxed{\frac{K+1}{K+2} = RHS}$$

Thus, if $S(K)$ is true, then $S(K+1)$ is true. By PMI $S(n)$ is true for all $n \geq 1$.

2: Proof.

Let $S(n)$ be the statement

$f^{(n)}(x) = a^n e^{ax}$ for all $n \geq 1$ and let $f^{(n)}(x)$ denote the n^{th} derivative of the function $f(x)$ and fix $f(x) = e^{ax}$ for a nonzero real constant a.

When $n = 1$

LHS: $f^{(1)}(x) = ae^{ax}$

RHS: $a^1 e^{ax} = ae^{ax}$

, thus $S(1)$ is true.

Now, suppose $S(K)$ is true for some K, to show that $S(K+1)$ is true, let's look at the left hand side:

$f^{(1)}(x) = ae^{ax}$

$f^{(2)}(x) = a^2 e^{ax}$

$f^{(3)}(x) = a^3 e^{ax}$

$\vdots$

$f^{(K+1)}(x) = a^{K+1} e^{ax}$ by the hypothesis that $S(K)$ is true.

Finally, by checking the left hand side and right hand side for equality:

$a^{K+1} e^{ax} = a^{K+1} e^{ax}$

Thus, if $S(K)$ is true, then $S(K+1)$ is true. By PMI, $S(n)$ is true for all $n \geq 1$. $\square$

3: Proof.

Let $S(n)$ be the statement

The Tower of Hanoi game with n disks can be solved in $2^n - 1$ moves for all integers $n \geq 1$.

When $n = 1$

The game is solved by just moving the 1 and only disk to the right-most peg, which takes 1 move.

$1 = 2^1 - 1$

$1 = 1$, thus $S(1)$ is true.

Now suppose $S(K)$ is true for some K, to show that $S(K+1)$ is true we need to first understand how to solve the game with $K+1$ disks.

Firstly, we can move the top K disks to the middle peg, which takes $2^K - 1$ moves and then we can move the largest block to the right-most peg followed by the disks on the middle peg, which will take 2^K moves.

Adding these values indicates that it takes $2^K - 1 + 2^K = 2^1 \cdot 2^K - 1 = 2^{K+1} - 1$ moves to solve The Tower of Hanoi game with $K+1$ disks.

Finally, by checking $S(K+1)$ and $2^{K+1} - 1$ for equality:

$2^{K+1} - 1 = 2^{K+1} - 1$

Thus, if $S(K)$ is true, then $S(K+1)$ is true. By PMI, $S(n)$ is true for all integers $n \geq 1$. $\square$

4: a: 5 letter strings using the English alphabet

$$\underline{26}\cdot\underline{26}\cdot\underline{26}\cdot\underline{26}\cdot\underline{26} = 26^5 = \boxed{11{,}881{,}376 \text{ possible strings}}$$

← This value is based on having 26 choices for all 5 letters, thus the total number of possible strings is 26^5.

b: No letter appears more than once (No repetition)

$$\underline{26}\cdot\underline{25}\cdot\underline{24}\cdot\underline{23}\cdot\underline{22} = \boxed{7{,}893{,}600 \text{ possible strings}}$$

← This value is based on having 26 choices for the first letter, followed by 25 for the second, 24 for the third, 23 for the fourth, and 22 for the fifth, thus the total number of possible strings is $26\cdot25\cdot24\cdot23\cdot22$.

c: ways to rearrange a 5-letter string

$$5\cdot4\cdot3\cdot2\cdot1 = \boxed{120 \text{ possible strings}}$$

← This value is based on the fact that the number of rearrangements of a string is equal to the factorial of the string length. This is also assuming that swapping 2 of the same letter counts as a rearrangement, thus the total number of rearrangements is $5!$.

5: a: $\underbrace{4}_{\text{\# of main meals}}\cdot\underbrace{4}_{\text{\# of desserts}} = 16$ different combinations of a main meal and a dessert.

b: $\underbrace{3}_{\text{\# of appetizers}}\cdot\underbrace{4}_{\text{\# of main meals}}\cdot\underbrace{4}_{\text{\# of desserts}} = 48$ different combinations of an appetizer, a main meal, and a dessert.

c: $\underbrace{1}_{\text{wings as our only appetizer}}\cdot\underbrace{4}_{\text{\# of main meals}}\cdot\underbrace{4}_{\text{\# of desserts}} = 16$ different combinations of an appetizer, a main meal, and a dessert that include wings

d: $\underbrace{2}_{\text{\# of appetizers excluding wings}}\cdot\underbrace{4}_{\text{\# of main meals}}\cdot\underbrace{4}_{\text{\# of desserts}} = 32$ different combinations of an appetizer, a main meal, and a dessert that does not include wings.

6: $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ where n,k are non-negative integers with $k \le n$.

a: $\binom{52}{52} = \dfrac{\cancel{52!}}{\cancel{52!}(52-52)!} = \dfrac{1}{0!} = \dfrac{1}{1} = \boxed{1}$

b: $\binom{52}{0} = \dfrac{52!}{0!(52-0)!} = \dfrac{52!}{1(52)!} = \dfrac{52!}{52!} = \boxed{1}$

c: $\binom{108}{104} = \dfrac{108!}{\cancel{104!}(108-104)!} = \dfrac{108\cdot107\cdot106\cdot105}{4!} = \dfrac{108\cdot107\cdot106\cdot105}{4\cdot3\cdot2\cdot1} = \boxed{5359095}$

$108\cdot107\cdot106\cdot105\cdot\cancel{104}\cdot\cancel{103}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}$

$\cancel{104}\cdot\cancel{103}\cdot\cancel{102}\cdot\cancel{101}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}$

d: $\binom{256}{5} = \dfrac{256!}{\cancel{5!}(256-5)!} = \dfrac{256\cdot255\cdot254\ldots\cdot6}{251!} = \dfrac{256\cdot255\cdot254\cdot253\cdot252}{5\cdot4\cdot3\cdot2\cdot1} = \dfrac{256\cdot255\cdot254\cdot253\cdot252}{5!} = \boxed{8809549056}$

$256\cdot255\cdot254\ldots\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}$

$\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}$

$256\cdot255\cdot254\cdot253\cdot252\cdot\cancel{251}\cdot\cancel{250}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}$

$\cancel{251}\cdot\cancel{250}\cdot\cancel{249}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}\cdot5\cdot4\cdot3\cdot2\cdot1$

e: $\binom{10}{3} = \dfrac{10!}{\cancel{3!}(10-3)!} = \dfrac{10\cdot9\cdot8\cdot\cancel{7}\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}}{7!} = \dfrac{10\cdot9\cdot8}{3\cdot2\cdot1} = \dfrac{10\cdot9\cdot8}{3!} = \boxed{120}$

$\cancel{3}\cdot\cancel{2}\cdot\cancel{1}$

$10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}$

$\cancel{7}\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot3\cdot2\cdot1$