Name:

Directions: These are some problems to help you prepare for Exam #1. Some are ridiculously hard, some are very easy. Don't attempt to do every single one – focus instead on the problems that will help you practice concepts and techniques that you know need strengthening. Solutions may or may not ever become available, as time permits!

1. Complete a truth table for each of the following statements and decide which, if any, are tautologies. Explain your conclusion by referring to the truth table.
   (a) $A \Rightarrow (B \Rightarrow C)$ (b) $(A \Rightarrow B) \Rightarrow C$ (c) $A \Leftrightarrow (B \wedge C)$ (d) $((\sim A \Rightarrow B) \wedge (\sim A \Rightarrow \sim B)) \Rightarrow A$

2. Determine whether each pair of statements are logically equivalent or not by constructing a truth table for each statement. Explain your conclusion by referring to the truth table.
   (a) $(A \wedge B) \vee B)$ and $(A \vee B) \wedge B$
   (b) $A \wedge (B \vee \sim A)$ and $A \wedge B$
   (c) $(A \wedge \neg B) \vee (\neg A \wedge \neg B)$ and $(A \vee \neg B) \wedge (\neg A \vee \neg B)$

3. Explain why the statement, "If pigs can fly then I am the king of Spain." is true.

4. Write the converse, negation, and contrapositive of each statement.
   (a) If you can't do the time, don't do the crime.
   (b) You'll get a good job, if you do well in school.
   (c) If it is raining and you are outside then there must be clouds.
   (d) If for all $n \in \mathbb{N}$, $a_n \leq b_n$ and $\sum_{n=0}^{\infty} b_n$ is a convergent series then $\sum_{n=0}^{\infty} a_n$ is a convergent series.

5. Negate each quantified statement.
   (a) All dogs are animals.
   (b) Some kids do not like video games.
   (c) Every element of S is odd.
   (d) For every $x \in \mathbb{Q}$ and for all $y \in \mathbb{Q}$, the product xy is a rational number.
   (e) The cube of a real number is a real number.
   (f) The set of integers is closed under addition.

6. Let $P(y) =$ "y is brown" and $Q(y) =$ "y is four years old", and let $R(y) =$ "y has white spots." Translate each of the following statements into symbols and then provide a negation.
   (a) There is a brown cow.
   (b) All cows are four years old.
   (c) There is a brown cow with white spots.
   (d) All four year old cows have white spots.
   (e) There exists a cow such that if it is four years old then it has no white spots.
   (f) All cows are brown if and only if they are not four years old.
   (g) There are no brown cows.

7. Negate each statement. Do not write the word "not" applied to any of the objects being quantified. For example, do not write "Not all boys are good" for the first part.
   (a) All boys are good.
   (b) There are bats that weight 50 lbs. or more.
   (c) The equation $x^2 - 2x > 0$ holds for all real numbers x.

(d) Every flying saucer is aiming to conquer some galaxy.

(e) Every parent has to change diapers.

(f) There is a house in Kansas such that every one who enters the house goes blind.

(g) Every house has a door that is white.

8. Translate each statement into English and determine whether or not the statement is true. If true, explain why. If false, provide a counterexample.

   (a) $\forall x \in \mathbb{R}\ x^2 > 0$

   (b) $\forall x \in \mathbb{R}\ \exists n \in \mathbb{N}\ x^n \geq 0$

   (c) $\exists a \in \mathbb{R}\ \forall x \in \mathbb{R},\ ax = x$

   (d) $\forall z \in \mathbb{Z}$ if z is even then z is not divisible by 5

   (e) $\exists m \in \mathbb{Z}\ \forall n \in \mathbb{Z},\ m = n + 5$

9. Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 8, 9\}$. Calculate each of the following sets.

   (a) $A \cup B$ (b) $A \cap B$ (c) $A - B$ (d) $B - A$ (e) $A - \{1\}$ (f) $B - \{7, 89\}$ (g) $B - \emptyset$
   (h) $B \cup \emptyset$ (i) $A \cap \emptyset$ (j) $A \cup \mathbb{Z}$ (k) $A \cap \mathbb{Z}$ (l) $A - \mathbb{Z}$ (m) $\mathbb{Z} - A$ (n) $\mathbb{R} - A$ (o) $A \times \emptyset$
   (p) $A \times B$ (q) $B \times A$ (r) $A \times A$ (s) $\mathcal{P}(A)$ (t) $\mathcal{P}(B - \emptyset)$

10. Write a rough proof for each of the claims below, where $A = \{1, 2, 3\}$ and $B = \{2, 3, 8, 9\}$.

    (a) $A \neq B$ (b) $A \not\subseteq B$ (c) $B \not\subseteq A$

11. If possible, provide an an example of sets A and/or B meeting each description. If it is not possible to provide sets A and/or B that meet a condition, then explain why.

    (a) $A \cap B$ has exactly three elements

    (b) $A \cup B$ is infinite

    (c) $A \cap B \neq \emptyset$ but $A \not\subseteq B$ and $B \not\subseteq A$

    (d) $A \times B$ has exactly 4 elements

    (e) $A \times B$ has exactly 5 elements

    (f) $\mathcal{P}(A)$ has exactly 4 elements

    (g) $\mathcal{P}(A)$ has exactly 5 elements

    (h) $\mathcal{P}(A)$ has exactly 1 element

    (i) $\mathcal{P}(A)$ has exactly 0 elements

    (j) $A \in B$ and $A \subseteq B$

12. Prove each set equality using an "element-chasing" proof, for arbitrary sets A, B, and C.

    (a) $A \cap B = B \cap A$

    (b) $A \subseteq B$ implies that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

    (c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

    (d) $A - (B \cup C) = (A - B) \cap (A - C)$ (You are proving DeMorgan's Law, so you can't use this law in your proof.)

    (e) If $A \subseteq B$ then $B - (B - A) = A$.

    (f) $A \cap \emptyset = \emptyset$

    (g) $A - B = \emptyset$ if and only if $A \subseteq B$

13. For each statement, decide if the statement is true or false. If you think the statement is true, try to prove it. If you think the statement is false, find a counterexample.

(a) For all sets A and B, we have $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.
(b) For all sets A and B, we have $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.
(c) For all sets A and B, we have $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$.
(d) For all sets X, Y, A, and B, we have $(X - A) \times (Y - B) = (X \times Y) - (A \times B)$.
(e) For all sets A and B, we have $(A \times B) \times A = A \times (B \times A)$.

14. Give two different examples of partitions of $\mathbb{N}$. Make one example have infinitely many parts and make one example have parts that with infinitely many elements.

15. Give two different examples of partitions of $\mathbb{R}$. Make one example have infinitely many parts and make one example have parts that with infinitely many elements.