MAT 373 Homework 12

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Problem 1: Show that $10^n + 2$ is not the sum of two squares for all positive integers n.

Proof. For the sake of contradiction, let's suppose that $10^n + 2$ is the sum of two squares for all positive integers n. This means that $10^n + 2 = a^2 + b^2$ for some non-negative integers a, b.

Let's first consider the base case of $n = 1$, which gives us $10^1 + 2 = 12$, which we know is not expressible as the sum of two squares. Specifically, if we look at the canonical prime factorization of 12, we get $12 = 3^1 \cdot 2^2$, and we know that any positive integer n is expressible as a sum of two squares if and only if in the canonical prime factorization of n that any prime congruent to 3 (mod 4) has an even exponent. Because of the odd exponent with the 3 in the prime factorization of 12, 12 is not a sum of two squares.

For all other cases of n, we notice that the sequence for $10^n + 2$ is $\{102, 1002, 10002, ...\}$. Looking carefully at all of these results (including the base case result) helps introduce the claim that $10^n + 2$ is divisible by 6 for all n. The idea behind this claim initially started by noting that 10^n is divisible by 2 for all n, so $10^n + 2$ is also divisible by 2 for all n. We now want to test that $10^n + 2$ is divisible by 3 by checking if the congruence $10^n + 2 \equiv 0 \pmod{3}$ is true for all n. This is actually the case because $10^n \equiv 1 \pmod{3}$ for all n and adding two to both sides of the equivalence gives us $10^n + 2 \equiv 1 + 2 \equiv 3 \equiv 0 \pmod{3}$. So now we have $10^n + 2 \equiv 0 \pmod{3}$ and thus $10^n + 2$ is also divisible by 3 for all n. Since we now know $10^n + 2$ is divisible by 2 and 3 for all n, it must also be divisible by $lcm(2, 3) = 6$, thus proving the claim.

Finally, we know that in the canonical prime factorization of 6, we will get $2^1 \cdot 3^1$. Since we know $10^n + 2$ is divisible by 6 for all n, the resulting prime factorization will be in the form of $2^1 \cdot 3^1 \cdot p_1^{m_1} \cdot p_2^{m_2} \cdot ... \cdot p_k^{m_k}$, where $p_1, p_2, ..., p_k$ are unique primes and $m_1, m_2, ..., m_k$ are positive integers.

Once again, because of 3^1, $10^n + 2$ is not a sum of two squares for all n, which contradicts our original assumption. $\square$

Problem 2: Write 10001 as a sum of two squares in 2 different ways (Hint: $10001 = 137 \cdot 73$)

$10001 = 137 \cdot 73$
$137 \cdot 73 = (11^2 + 4^2)(8^2 + 3^2)$
Way 1: $((11)(3) + (4)(8))^2 + ((11)(8) - (4)(3))^2 = 65^2 + 76^2$
Way 2: $((11)(3) - (4)(8))^2 + ((11)(8) + (4)(3))^2 = 1^2 + 100^2$