MAT 373 Homework 10

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Problem 1: Find all PPT that have an in circle of radius:

a) 12

$12 = t(s-t)$

We need to find combinations of t and $s-t$ such that $s-t$ is an odd, t is even or odd, and $\gcd(t,\, s-t) = 1$. The only possible combinations are $t = 12$, $s-t = 1$ and $t = 4$, $s-t = 3$.

For $t = 12$ and $s-t = 1$:
$s = 13$
$x = 2(13)(12) = 312$
$y = (13)^2 - (12)^2 = 25$
$z = (13)^2 + (12)^2 = 313$

For $t = 4$ and $s-t = 3$:
$s = 7$
$x = 2(7)(4) = 56$
$y = (7)^2 - (4)^2 = 33$
$z = (7)^2 + (4)^2 = 65$

Thus, $(312, 25, 313)$ and $(56, 33, 65)$ are all of the PPT that have an in circle of radius 12.

b) 2^k for any positive integer k

$2^k = t(s-t)$

We need to find combinations of t and $s-t$ such that $s-t$ is an odd, t is even or odd, and $\gcd(t,\, s-t) = 1$. The only possible combination is $t = 2^k$, $s-t = 1$ because 2^k for all possible k will only be divisible by $2^0, 2^1, ..., 2^{k-1}, 2^k$, and $2^0 = 1$, the only odd from this set.

$s = 2^k + 1$
$x = 2(2^k + 1)(2^k) = 2^{k+1}(2^k + 1) = 2^{2k+1} + 2^{k+1}$

$y = (2^k + 1)^2 - (2^k)^2 = 2^{2k} + 2^{k+1} + 1 - 2^{2k} = 2^{k+1} + 1$
$z = (2^k + 1)^2 + (2^k)^2 = 2^{2k} + 2^{k+1} + 1 + 2^{2k} = 2^{2k+1} + 2^{k+1} + 1$

Thus, all PPT that have an incircle of radius 2^k for any positive integer k are of the form $(2^{2k+1} + 2^{k+1}, 2^{k+1} + 1, 2^{2k+1} + 2^{k+1} + 1)$

Problem 2: Let (x, y, z) be a PPT. Show $3|x$ or $3|y$.

Proof. Suppose (x, y, z) is a PPT. We want to be able to say that $3|x$ or $3|y$. Since (x, y, z) is a PPT, we know that $x^2 + y^2 = z^2$ and x, y, z are pairwise relatively prime. With this information, let's consider the possible residuals for any square number (mod 3):

$$a^2 \equiv 0 \pmod 3 \rightarrow (0)^2 = 0$$

$$a^2 \equiv 1 \pmod 3 \rightarrow (1)^2 = 1$$

$$a^2 \equiv 2 \pmod 3 \rightarrow (2)^2 - 3(1) = 1$$

We can see that for any square number (mod 3), the residual will either be 0 or 1. For the sake of contradiction, let's suppose that $3 \nmid x$ and $3 \nmid y$. From this situation, we know that $3 \nmid x^2$ and $3 \nmid y^2$, which means x^2 and y^2 will only have remainders of 1. However, $x^2 + y^2 = z^2$, so z^2 will have a remainder of $1 + 1 = 2$, which is impossible. Therefore, we can conclude that $3|x$ or $3|y$. □

Problem 3: Find all PPT (x, y, z) whose perimeter equals their area.

Let P represent the perimeter of the PPT of (x, y, z):
$P = x + y + z$
Let A represent the area of the PPT of (x, y, z):
$A = \frac{1}{2}xy$

We need to solve for all (x, y, z) such that:
$x + y + z = \frac{1}{2}xy$

Since (x, y, z) is a PPT, we can utilize $x = 2st, y = s^2 - t^2, z = s^2 + t^2$ for some integers s, t where one of them is even and the other is odd, which then gives us:

$2st + s^2 - t^2 + s^2 + t^2 = \frac{1}{2}(2st)(s^2 - t^2)$

Let's simplify:
$2st + s^2 + s^2 = \frac{1}{2}(2st)(s^2 - t^2)$
$2st + 2s^2 = \frac{1}{2}(2st)(s^2 - t^2)$
$2st + 2s^2 = st(s^2 - t^2)$
$2st + 2s^2 = st(s - t)(s + t)$
$2s(t + s) = st(s - t)(s + t)$
$2 = \frac{st(s-t)(s+t)}{s(t+s)}$

$2 = t(s - t)$

Recall that with a PPT (x, y, z), $s - t$ is odd, t is even or odd, and $\gcd(t,\ s - t) = 1$. So, with this information, we can only get $2 = t(s - t)$ with $t = 2$ and $s - t = 1$, which means $s = 3$. So we can find x, y, z by:
$x = 2(3)(2) = 12$
$y = (3)^2 - (2)^2 = 5$
$z = (3)^2 + (2)^2 = 13$

Therefore, now we know that $(12, 5, 13)$ represents all PPTs whose perimeter is equal to its area.