MAT 373 Homework 4 (Revised)

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Problem 1: Use Mathematical Induction to prove that if $x > -1$ and $x \neq 0$, then $(1+x)^n > 1 + nx$ for every positive integer $n > 1$.

Proof. Let $s(n)$ be the statement:

$$(1+x)^n > 1 + nx$$

for every positive integer $n > 1$ if $x > -1$ and $x \neq 0$.

Base Case: When $n = 2$:

$$LHS = (1+x)^2 = 1 + 2x + x^2$$

$$RHS = 1 + 2x$$

Since we know $x > -1$ and $x \neq 0$, we can state that $LHS > RHS$, thus proving that $s(2)$. Now let's suppose $s(k)$ is true for some k. To show that $s(k+1)$ is true, let's look at its LHS:

$$LHS = (1+x)^{k+1} = (1+x)^k(1+x)$$

By the hypothesis that $s(k)$ is true, we can say:

$$LHS > (1+kx)(1+x) = kx^2 + kx + x + 1 = kx^2 + (k+1)x + 1$$

Finally, looking at the RHS:

$$RHS = 1 + (k+1)x = (k+1)x + 1$$

Since we determined that $LHS > kx^2 + (k+1)x + 1$ and we already know that $x > -1$, $x \neq 0$, and k is a positive integer such that $k > 1$, we can also mention that $LHS > (k+1)x + 1$, and $LHS > RHS$, proving that if $s(k)$ is true, then $s(k+1)$ is true. By PMI, $s(n)$ is true for every positive integer $n > 1$ if $x > -1$ and $x \neq 0$. $\square$

Problem 2: Assuming that $gcd(a,b) = 1$, prove that $gcd(a+b, a-b) = 1$ or 2.
Hint: Let $d = gcd(a+b, a-b)$ and show that $d|2a, d|2b$ and thus that $d \leq gcd(2a, 2b) = 2gcd(a,b)$

Proof. Suppose $gcd(a, b) = 1$ and $d = gcd(a + b, a - b)$. Therefore, $d|a + b$ and $d|a - b$. Additionally, $d|2a$ and $d|2b$ because these statements represent a linear combination of $a + b$ and $a - b$. As a result, we can then say $d|gcd(2a, 2b)$, and $d|2gcd(a, b) = 2$. Thus, $d|2$, meaning the only positive integers of d in which this statement is true are 1 or 2, proving that if $gcd(a, b) = 1$, then $gcd(a + b, a - b) = 1$ or 2.

□

Problem 3a: Determine all solutions in the integers of the Diophantine equation, $56x + 72y = 40$.

Determine gcd(56, 72):

(72) = (56)1 + (16)
(56) = (16)3 + (8)
(16) = (8)2 + 0
gcd(56, 72) = 8 and 8|40. Thus this Diophantine equation has integer solutions for x, y.

(8) = (56) - (16)3
(8) = (56) - 3((72) - (56))
(8) = 56(4) - 72(3)
5(8) = 5(56(4) - 72(3))
40 = 56(20) + 72(-15) $\implies x_0 = 20, y_0 = -15$

The general solution for this Diophantine equation is:
$(x, y) = (20 + \frac{72}{8}t, -15 - \frac{56}{8}t) = (20 + 9t, -15 - 7t)$, where $t \in \mathbb{Z}$.

Problem 3b: Determine all solutions in the positive integers of the Diophantine equation, $54x + 21y = 906$.

Determine gcd(54, 21):

(54) = (21)2 + (12)
(21) = (12)1 + (9)
(12) = (9)1 + (3)
(9) = (3)3 + 0
gcd(54, 21) = 3 and 3|906. Thus this Diophantine equation has integer solutions for x, y.

(3) = (12) - (9)
(3) = ((54) - 2(21)) - ((21) - (12))
(3) = ((54) - 2(21)) - ((21) - ((54) - 2(21)))
(3) = 54(2) + 21(-5)
(3)(302) = 54(2)(302) + 21(-5)(302)
906 = 54(604) + 21(-1510) $\implies x_0 = 604, y_0 = -1510$

The general solution for this Diophantine equation is:
$(x, y) = (604 + \frac{21}{3}t, -1510 - \frac{54}{3}t) = (604 + 7t, -1510 - 18t)$, where $t \in \mathbb{Z}$.

Problem 3c: A man has \$4.55 in change composed entirely of dimes and quarters. What are the maximum and minimum number of coins that he can have? Is it possible for the number of dimes to equal the number of quarters? (Show all work!)

We can write this equation in terms of cents. Since \$4.55 is equal to 455 cents, a dime is equal to 10 cents, and a quarter is equal to 25 cents, we can express this problem as the Diophantine equation:

$$10x + 25y = 455$$

Where x represents the number of dimes and y represents the number of quarters.

Determine gcd(10, 25):

(25) = (10)2 + (5)
(10) = (5)2 + 0
gcd(10, 25) = 5 and 5|455. Thus this Diophantine equation has integer solutions for x, y.

(5) = (25) - (10)2
(5)(91) = (25)(91) - (10)(2)(91)
455 = 10(-182) + 25(91) $\implies x_0 = -182, y_0 = 91$

The general solution for this Diophantine equation is:
$(x, y) = (-182 + \frac{25}{5}t, 91 - \frac{10}{5}t) = (-182 + 5t, 91 - 2t)$, where $t \in \mathbb{Z}$.

However, we cannot have a negative amount of coins, so we can only consider cases of t where both of these statements are true: $-182 + 5t \geq 0$ and $91 - 2t \geq 0$

First Inequality: $-182 + 5t \geq 0 \implies t \geq \frac{182}{5} = 36.4$
Second Inequality: $91 - 2t \geq 0 \implies t \leq \frac{91}{2} = 45.5$
Combined Interval for t: $36.4 \leq t \leq 45.5$

Since we know t must be an integer, $t = 37, 38, 39, 40, 41, 42, 43, 44, 45$.

Finally, let's analyze each of these values of t:
$t = 37 \implies (-182 + 5(37), 91 - 2(37)) = (3, 17) \implies$ 20 coins
$t = 38 \implies (-182 + 5(38), 91 - 2(38)) = (8, 15) \implies$ 23 coins
$t = 39 \implies (-182 + 5(39), 91 - 2(39)) = (13, 13) \implies$ 26 coins
$t = 40 \implies (-182 + 5(40), 91 - 2(40)) = (18, 11) \implies$ 29 coins
$t = 41 \implies (-182 + 5(41), 91 - 2(41)) = (23, 9) \implies$ 32 coins
$t = 42 \implies (-182 + 5(42), 91 - 2(42)) = (28, 7) \implies$ 35 coins
$t = 43 \implies (-182 + 5(43), 91 - 2(43)) = (33, 5) \implies$ 38 coins
$t = 44 \implies (-182 + 5(44), 91 - 2(44)) = (38, 3) \implies$ 41 coins
$t = 45 \implies (-182 + 5(45), 91 - 2(45)) = (43, 1) \implies$ 44 coins

Therefore, the maximum number of coins the man can have is 44 (based on $t = 45$), the minimum is 20 (based on $t = 37$), and it is possible for the number of dimes to equal the number of quarters (based on $t = 39$).