MAT 373 Homework 7

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Problem 1: For $k \geq 2$, show that $n = 2^{k-1}$ satisfies the equation $\sigma(n) = 2n - 1$

Proof. Suppose we have integers k, n where $k \geq 2$ and $n = 2^{k-1}$. Since we know that 2 is a prime and $\sigma(p^m) = \frac{p^{m+1}-1}{p-1}$, where p is a prime and m is a positive integer, we can then apply this formula to n to give us:

$$\sigma(n) = \sigma(2^{k-1}) = \frac{2^{k-1+1}}{2-1} = 2^k - 1 \tag{1}$$

Recall $n = 2^{k-1}$. If we log both sides of this equation with base 2, we get:

$$\log_2(n) = \log_2(2^{k-1})$$

$$\log_2(n) + 1 = k \tag{2}$$

After plugging the left side of equation (2) for k into equation (1), we now get this equation:

$$\sigma(n) = 2^{\log_2(n)+1} - 1 = 2^{\log_2(n)} \cdot 2^1 - 1 = 2n - 1 \tag{3}$$

Thus, equation (3) helps us conclude that $n = 2^{k-1}$ satisfies the equation $\sigma(n) = 2n - 1$ for all integers $k \geq 2$. □

Problem 2: For $k \geq 2$, show that if $2^k - 3$ is prime, then $n = 2^{k-1}(2^k - 3)$ satisfies the equation $\sigma(n) = 2n + 2$. It is not known if there are any positive integers n for which $\sigma(n) = 2n + 1$

Proof. Suppose we have a positive integer k such that $k \geq 2$ and $2^k - 3$ is prime. Now suppose that positive integer $n = 2^{k-1}(2^k - 3)$. We want to be able to eventually say that n satisfies $\sigma(n) = 2n + 2$. Since $k \geq 2$, we know that 2^{k-1} and $2^k - 3$ will both be positive integers. This allows us to state that:

$$\sigma(n) = \sigma(2^{k-1})\sigma(2^k - 3) \tag{4}$$

Since we know $\sigma(2^{k-1})$ is of the form $\sigma(p^k)$, where p is a prime and k is a positive integer since 2 is prime and $k - 1$ will always be a positive integer for all k, we can say that $\sigma(2^{k-1})$ is equal to:

$$\sigma(2^{k-1}) = \frac{2^{k-1+1} - 1}{2-1} = 2^k - 1 \tag{5}$$

For $\sigma(2^k - 3)$, recall that we have a hypothesis of $2^k - 3$ being a prime, so this is of the form $\sigma(p^1)$, where prime $p = 2^k - 3$, and thus can be calculated as:

$$\sigma(2^k - 3) = \frac{(2^k - 3)^2 - 1}{(2^k - 3) - 1} = \frac{(2^k - 2)(2^k - 4)}{(2^k - 4)} = 2^k - 2 \tag{6}$$

Now we can plug in the results of equations (5) and (6) into equation (4):

$$\sigma(n) = (2^k - 1)(2^k - 2) = 2^{2k} - 2^k - 2^{k+1} + 2 \tag{7}$$

Now we shall perform two rounds of factoring on the right side of equation (7):

$$\sigma(n) = 2(2^{2k-1} - 2^{k-1} - 2^k) + 2$$

$$\sigma(n) = 2(2^{k-1}(2^k - 1 - 2^1)) = 2(2^{k-1}(2^k - 3)) + 2 \tag{8}$$

Since we know that $n = 2^{k-1}(2^k - 3)$, we can replace the right side of equation (8) with $2n + 2$, which then finally tells us that $\sigma(n) = 2n + 2$.

□

Problem 3: Prove that the Goldbach conjecture implies that for each even integer $2n$, there exist integers n_1 and n_2 with $\sigma(n_1) + \sigma(n_2) = 2n$

Proof. Let's suppose that the Goldbach conjecture is true for all even integers larger than 2. This indicates that every even integer > 2 can be represented as a sum of two primes. With this, we can say $2n = p_1 + p_2$, where n is a positive integer such that $n > 1$ and p_1, p_2 are primes. We want to show that there exists integers n_1 and n_2 that satisfy $\sigma(n_1) + \sigma(n_2) = 2n$ for all possible n.

Looking at the base case $n = 2$, we can simply use an example like $n_1 = 1, n_2 = 2$, which gives us $\sigma(1) + \sigma(2) = 1 + 1 + 2 = 4 = 2n$, which allows us to say that there exists integers n_1 and n_2 that satisfy $\sigma(n_1) + \sigma(n_2) = 2n$ for $n = 2$.

Now let's consider the rest of the cases. Subtracting 2 from the left-hand-side of $2n = p_1 + p_2$ is possible since $2n - 2$ is still an even integer for all n, but now we have to consider $n > 2$ in order to ensure $2n - 2 > 2$. With this idea, we are able to say:

$$2n - 2 = p_1 + p_2$$

$$2n = p_1 + p_2 + 2 \tag{9}$$

Now, let's find $\sigma(p_1) + \sigma(p_2)$. Since p_1, p_2 are primes, we have:

$$\sigma(p_1) = 1 + p_1 \tag{10}$$

$$\sigma(p_2) = 1 + p_2 \tag{11}$$

Adding equation (10) and (11) together gives us:

$$\sigma(p_1) + \sigma(p_2) = p_1 + p_2 + 2 \tag{12}$$

Finally, with the use of equation (9), equation (12) is equivalent to:

$$\sigma(p_1) + \sigma(p_2) = 2n$$

Since we know p_1, p_2 are primes, they are also integers, which tells us that there exists integers n_1 and n_2 that satisfy $\sigma(n_1) + \sigma(n_2) = 2n$ for $n > 2$. Now that we confirmed all of the cases of n, we know this is also true for all possible values of n. $\square$