MAT 373 Homework 2 (Optional) (Revised)

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Problem 1: Show that if n is any integer such that $3 \nmid n$, then $3|(n^2 + 2)$. Hint: Division Algorithm

Proof. Suppose that n is any integer such that $3 \nmid n$. This means $n \neq 0$ and n is not a multiple of 3. Now, let's apply the Division Algorithm which states that if we have integers a, b with $a > 0$, then there are unique integers q, r such that:

$$b = aq + r \tag{1}$$

where $0 \leq r < a$. Since we know that 3 and n are both integers, we can apply the Division Algorithm. Additionally, since we know $3 \nmid n$, we can plug in $a = 3$ and $b = n$ into equation (1) and state that:

$$n = 3q + r \tag{2}$$

where $0 < r < 3$. Since we also know that r is an integer, it is only possible that either $r = 1$ or $r = 2$. We thus consider those two cases individually.

For our first case, let's plug in $r = 1$ and square both sides of equation (2), which results in:

$$n^2 = (3q + 1)^2 \tag{3}$$

By expanding the right side of equation (3), we then get:

$$n^2 = 9q^2 + 6q + 1 \tag{4}$$

Finally, let's add 2 to both sides:

$$n^2 + 2 = 9q^2 + 6q + 3 = 3(3q^2 + 2q + 1) \tag{5}$$

Notice that equation (5) can be factored to be in the form of $3k$, where $k = 3q^2 + 2q + 1$, thus making k an integer since we already know q is an integer. Therefore, this allows us to indicate that for when $r = 1$, $3|(n^2 + 2)$.

Applying the same plan for our second case for when $r = 2$ gives us:

$$n^2 = (3q + 2)^2 \tag{6}$$

$$n^2 = 9q^2 + 12q + 4 \tag{7}$$

$$n^2 + 2 = 9q^2 + 12q + 6 = 3(3q^2 + 4q + 2) \tag{8}$$

Equation (8) can also be factored to be in the form of $3k$, where $k = 3q^2 + 4q + 2$. Once again, k is an integer since q is an integer. Thus, for all possible cases of r, $3|(n^2 + 2)$. We can then state that if $3 \nmid n$, then $3|(n^2 + 2)$.

$\square$