MAT 373 Homework 6

Justyce Countryman

February 26, 2024

Problem 1: Let p, q be distinct odd primes. Prove that $n = pq$ is not perfect.

Proof. Suppose p, q are distinct odd primes. So we know that p, q are odd integers such that $p, q > 1$ and that p, q have exactly two positive divisors, 1 and themselves. Calculating $\sigma(pq)$ would then give us:

$$\sigma(pq) = \sigma(p)\sigma(q) = (1+p)(1+q) = 1 + p + q + pq \tag{1}$$

We can only say that pq is perfect provided that $\sigma(pq) = 2pq$. Combining this information with equation (1), we are looking to see if these statements are true:

$$1 + p + q + pq = 2pq$$

$$1 + p + q = pq$$

Recall that p, q are distinct odd primes (meaning the smallest values of p, q are $p = q = 3$), so we can say that $(p-1)(q-1) > 2$, which allows us to say:

$$pq - p - q + 1 > 2$$

$$pq > 1 + p + q \tag{2}$$

Equation (2) shows that the two prior statements we were looking at are false. This finally allows us to indicate that pq is not perfect. Therefore, if we have p, q as distinct odd primes, then $n = pq$ is not perfect.

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Problem 2: Let n be a positive integer such that $\tau(n)$ is odd. Prove that n is a perfect square (That is, prove $n = a^2$ for some integer a.)

Proof. Suppose n is a positive integer such that $\tau(n)$ is odd. This means that the number of positive divisors of n is odd. Let's suppose the prime factorization of n is:

$$n = p_1^{a_1} p_2^{a_2} ... p_k^{a_k} \tag{3}$$

, where $p_1, p_2, ...p_k$ represent unique primes and $a_1, a_2, ...a_k$ are positive integers. We can also get the general form of $\tau(n)$ as well:

$$\tau(n) = (a_1 + 1)(a_2 + 1)...(a_k + 1) \tag{4}$$

Since we know that $\tau(n)$ is odd, every $(a_i + 1)$ term in equation (4), where $i = 1, 2, ..., k$ must also be odd, which also indicates that every a_i term must be even. Let's then consider this only case of a_i, meaning $a_i = 2b_i$ for some positive integer b_i. Plugging in $2b_i$ for every a_i in equation (3) gives us:

$$n = p_1^{2b_1} p_2^{2b_2} ... p_k^{2b_k} = (p_1^{b_1} p_2^{b_2} ... p_k^{b_k})^2 \tag{5}$$

It is clear from equation (5) that $p_1^{b_1} p_2^{b_2} ... p_k^{b_k}$ is an integer since we are multiplying prime factors that are raised to the power of unique positive integers. Thus, since we can represent $n = a^2$ for some integer a, we now know that n is a perfect square.

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Problem 3: Suppose that n is a positive integer such that $2^n - 1$ is a prime. Prove that n itself must be a prime.

Proof. Suppose that there is a positive integer n such that $2^n - 1$ is a prime. We want to be able to eventually say that n is prime. For the purpose of contradiction, let's suppose this is false, meaning n is not a prime. With this statement, we can say that $n = ab$ for some integers a, b where $a, b > 1$. We are able to eliminate $n = 1$ from the possible choices of n because $2^1 - 1 = 1$ is not a prime, thus making our initial hypothesis false. Plugging in ab for n into $2^n - 1$, gives us $2^{ab} - 1$. This statement is equal to:

$$2^{ab} - 1 = (2^a)^b - 1 \tag{6}$$

Since we know a is an integer and exponentiation of two integers gives another integer, 2^a must also an integer. As a result, the right hand side of equation (6) is of the form $x^y - 1$ for some integers x, y. In this case $x = 2^a$ and $y = b$. We are then able to factor equation (6) into:

$$2^{ab} - 1 = (2^a - 1)((2^a)^{b-1} + (2^a)^{b-2} + ... + 2^a + 1) \tag{7}$$

, where $(2^a - 1)$ is our factor. Since $a > 1$, we can say $(2^a - 1) > 1$, meaning our factor is not 1. We also know $b > 1$, which tells us that $(2^a - 1) \neq 2^{ab} - 1$, so our factor is also not $2^{ab} - 1$. Since we have found a factor of $2^{ab} - 1$ that is not 1 or itself, equation (7) reveals that we run into a contradiction because $2^{ab} - 1$ is not prime, which means that $2^n - 1$ is also not prime when n is not prime. Therefore, we are then able to state that if n is a positive integer such that $2^n - 1$ is a prime, then n itself must be a prime.

□

Problem 4: Given an integer N, let M be the integer formed by reversing the order of the digits of N (For example, if $N = 6923$, the $M = 3296$). Verify that $N - M$ is divisible by 9.

Proof. Suppose we have an integer N. We also have an integer M where M is formed by reversing the order of the digits of N. Let's suppose the value of N is represented by:

$$N = (a_n \cdot 10^n) + (a_{n-1} \cdot 10^{n-1}) + ... + (a_1 \cdot 10^1) + (a_0 \cdot 10^0) \tag{8}$$

, where a_i represents the integer digit of N at index i (starting from the ones place), and $0 \leq a_i \leq 9$. M would then be of the form:

$$M = (a_0 \cdot 10^n) + (a_1 \cdot 10^{n-1}) + ... + (a_{n-1} \cdot 10^1) + (a_n \cdot 10^0) \tag{9}$$

Now, we can represent $N - M$ by:

$$N - M = (a_n - a_0) \cdot 10^n + (a_{n-1} - a_1) \cdot 10^{n-1} + ... + (a_1 - a_{n-1}) \cdot 10 + (a_0 - a_n) \tag{10}$$

Since we know that $10 \equiv 1 \pmod 9$ and $10^x \equiv 1 \pmod 9$ for when $x \geq 0$, we can then establish the congruence statement:

$$(a_n - a_0) + (a_{n-1} - a_1) + ... + (a_1 - a_{n-1}) + (a_0 - a_n) \equiv 0 \pmod 9 \tag{11}$$

Equation (11) is valid because all a_i terms will cancel out, thus leaving us with 0 on the left hand side of equation (10), which clearly is congruent to 0 (mod 9). This indicates that $9 | N - M$.

However, this case only satisfies positive integers. We also have to verify this is true if N, M are negative integers. Using the same logic, the sign of each term from the right hand side of equations (8) and (9) would simply be flipped, which will still result in all a_i terms cancelling out when considering $N - M$ and the congruence statements $10 \equiv 1 \pmod 9$ and $10^x \equiv 1 \pmod 9$, once again resulting in $0 \equiv 0 \pmod 9$.

Therefore, we can then verify that $N - M$ is divisible by 9.

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