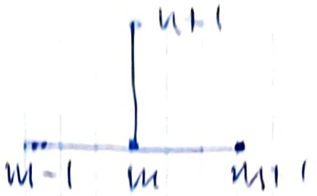
Первое задание

[illegible]

$\boxed{XII.7.2}$

$u_t + cu_x = 0$



$\alpha u^n_{m-1} + \beta u^n_m + \gamma u^n_{m+1} + \delta u^{n+1}_m = 0$

$u^{n+1}_m = u^n_m + \dot u \tau + \ddot u \frac{\tau^2}{2} + O(\tau^3)$

$u^n_{m+1} = u^n_m + u'^n_m h + u''^n_m \frac{h^2}{2} + O(h^3)$

$u^n_{m-1} = u^n_m - u'^n_m h + u''^n_m \frac{h^2}{2} + O(h^3)$

$u^n_m = u^n_m$

$\alpha u^n_m - \alpha u'^n_m h + \alpha u''^n_m \frac{h^2}{2} + \beta u^n_m + \gamma u^n_m + \gamma u'^n_m h + \gamma u''^n_m \frac{h^2}{2} +$
$+ \delta u^n_m + \delta \dot u^n_m \tau + \delta \ddot u^n_m \frac{\tau^2}{2} + O(h^3) + O(\tau^3) = 0 = u_t + cu_x$

$$\left|\begin{array}{l} \alpha + \beta + \gamma + \delta = 0 \\ \delta\tau = 1 \\ -\alpha h + \gamma h = c \\ (\alpha+\gamma)\frac{h^2}{2} = 0 \end{array}\right. \quad \left|\begin{array}{l} \alpha + \beta + \gamma + \delta = 0 \\ \delta\tau = 1 \\ -\alpha + \gamma = \frac{c}{h} \\ \alpha + \gamma = 0 \end{array}\right. \Rightarrow \left|\begin{array}{l} \beta = -\frac{1}{\tau}, \\ \delta = \frac{1}{\tau} \\ \gamma = \frac{c}{2h} \\ \alpha = -\gamma = -\frac{c}{2h} \end{array}\right.$$

$$\boxed{\frac{u^{n+1}_m - u^n_m}{\tau} + \frac{c}{2h}(u^n_{m+1} - u^n_{m-1}) = 0}$$ 2 **Часть ответа**

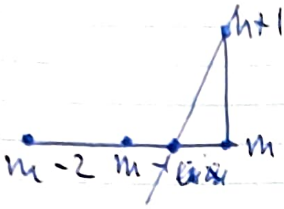
$$\frac{\dot u \tau + \ddot u \frac{\tau^2}{2} + O(\tau^3)}{\tau} + \frac{c}{2h}(\cancel{u} + u'h + \cancel{u''}\frac{h^2}{2} + - \cancel{u} + u'h - \cancel{u''}\frac{h^2}{2} + O(h^3)) = 0$$

$\Rightarrow r = \dot u + O(\tau^2) + \frac{1}{2h} 2h\, cu' + O(h^2).$

$\dot u + u'c = 0 \Rightarrow \boxed{\|r\| \le C\tau + Ch^2}$

$\boxed{XII\ 7.4}$

$u_t + cu_x = 0$



$\frac{dx}{dt} = c \Rightarrow x - ct = const$ – характеристика.

$x^* = x_m - c\tau$

~~$u^* = f(x^*_m - ct_n)$~~

~~$u^{n+1}_m = f(x_m - ct_n)$~~

$u^{n+1}_m = u^*$

x_{m-2} u_{m-2}

$\frac{u_{m-2} - u_{m-1}}{-h}$

x_{m-1} u_{m-1} $\frac{u_{m-2} - 2u_{m-1} + u_m}{2h^2}$

$\frac{u_{m-1} - u_m}{-h}$

x_m u_m

~~$\Rightarrow P(x) = u_{m-2} - \frac{u_{m-2} - u_{m-1}}{-h}(x - x_{m-2})$~~

~~$P(x) = u_{m-1} + \frac{u_m - u_{m-1}}{h}(x - x_{m-1}) + \frac{u_{m-2} - 2u_{m-1} + u_m}{2h^2} \cdot (x - x_{m-1})(x - x_{m-2})$~~

~~$u^* = u_m + \frac{u_m - u_{m-1}}{h}(-c\tau) + \frac{u_{m-2} - 2u_{m-1} + u_m}{2h^3} \cdot (-c\tau) \cdot (-c\tau - h)$~~

~~$= u_m + \frac{u_m - u_{m-1}}{h}(-c\tau) + \frac{u_{m-2} - 2u_{m-1} + u_m}{2h^2} \cdot ($~~

$\Rightarrow u^* = u^n_m + \frac{(u^n_m - u^n_{m-1})}{h}(x^* - x_m) + \frac{u^n_m - 2u^n_{m-1} + u^n_{m-2}}{2h^2}(x^* - x_m)(x^* - x_{m-1})$

$= u^n_m + \frac{(u^n_m - u^n_{m-1})}{h}(-c\tau) + \frac{u^n_m - 2u^n_{m-1} + u^n_{m-2}}{2h^2}(-c\tau)\cdot(-c\tau + h)$

$= u^n_m - c\tau \frac{3u^n_m - 4u^n_{m-1} + u^n_{m-2}}{2h} + \frac{c^2\tau^2}{2h^2}(u^n_m - 2u^n_{m-1} + u^n_{m-2})$

$= u^{n+1}_m$ / 2 Чаев Ответ

$\boxed{№П.7.2}$ - повышение порядка.

$$\begin{cases} \ddot{u} + c\dot{u}' = 0 \\ \dot{u}' + cu'' = 0 \end{cases} \Rightarrow \begin{cases} \ddot{u} - c^2u'' = 0 \\ \dot{u}' = -cu'' \end{cases}$$

$$\Rightarrow \ddot{u} = c^2u''$$

~~$\alpha + \beta + \gamma + \delta = 0$~~

~~$\alpha + \beta + \gamma + \delta = 0$~~

~~$\delta\tau = 1$~~

~~$-\alpha h + \gamma h = c$~~

~~$(\alpha + \gamma)\frac{h^2}{2} + \delta\frac{\tau^2}{2} \cdot c^2 = 0$~~.

~~$\delta = \frac{1}{\tau}$~~

~~$\alpha = \gamma$~~

~~$2\alpha\frac{h^2}{2} + \delta\frac{\tau^2}{2} \cdot c^2 = 0$~~

~~$2\alpha h^2 + \delta\tau^2c^2 = 0$~~

~~$\Rightarrow \alpha = -\frac{\delta\tau^2c^2}{2h^2} = -\frac{c^2\tau}{2h^2} = \gamma$~~

~~$\delta = \frac{1}{\tau}$~~

~~$(\gamma - \alpha) = \frac{c}{h}$~~

~~$(\alpha + \gamma)$~~

Вычтем из схемы $\frac{\tau^2}{2}\ddot{u} = \frac{\tau^2}{2}c^2u'' \Rightarrow$

Итоговая схема: $\frac{u_m^{n+1} - u_m^n}{\tau} + \frac{c}{2h}(u_{m+1}^n - u_{m-1}^n) - \frac{c^2\tau}{2h^2}(u_{m+1}^n - 2u_m^n + u_{m-1}^n) = 0$

$\boxed{XII.7.4}$ (сравнение с МНК)

МНК: $a u_m^{n+1} + b u_m^n + e u_{m-1}^n + d u_{m-2}^n = 0 = u_t + c u_x$

$u_m^{n+1} \to$

$u_m^{n+1} = u_m^n + \dot{u}_m^n \tau + \ddot{u}_m^n \frac{\tau^2}{2} + O(\tau^3)$

$u_m^n = u_m^n$

$u_{m-1}^n = u_m^n - u'^n_m h + u''^n_m \frac{h^2}{2} + O(h^3)$

$u_{m-2}^n = u_m^n - u'^n_m 2h + u''^n_m 2h^2 + O(h^3)$

$a u_m^n + a \dot{u}_m^n \tau + a \ddot{u}_m^n \frac{\tau^2}{2} +$

$+ b u_m^n +$

$+ e u_m^n - e u'^n_m h + e u''^n_m \frac{h^2}{2} +$

$+ d u_m^n - d u'^n_m 2h + d u''^n_m 2h^2 = u_t + c u_x = 0$

$$\begin{cases} a+b+e+d=0 \\ a\tau = 1 \\ -eh - 2dh = c \\ a c^2 \frac{\tau^2}{2} + e\frac{h^2}{2} + d\,2h^2 = 0 \end{cases}$$

$$\begin{cases} a+b+e+d=0 \\ a = \frac{1}{\tau} \\ -e-2d=0 \Rightarrow e+2d=0 \Rightarrow e=-2d \\ c^2\frac{\tau}{2} + \left(\frac{e}{2}+2a\right)h^2 = 0 \to c^2\frac{\tau}{2} + dh^2 = 0 \end{cases}$$

$d = -\frac{c^2\tau}{2h^2}$

$e = +\frac{c^2\tau}{h^2}$

$\frac{1}{\tau} + b + \frac{c^2\tau}{h^2} - \frac{c^2\tau}{2h^2} = 0$

$b = -\frac{1}{2}\frac{c^2\tau}{h^2} - \frac{1}{\tau}$

$$\frac{1}{\tau}u_m^{n+1} - \left(\frac{1}{2}\frac{c^2\tau}{h^2} + \frac{1}{\tau}\right)u_m^n + \frac{c^2\tau}{h^2}u_{m-1}^n - \frac{c^2\tau}{h^2}u_{m-2}^n = 0$$

Получается что-то другое.

← нужен только пункт в)

$\boxed{\text{XII. 7. 15}}$

$$\frac{y_m^{n+1} - y_m^{n-1}}{2\tau} = \gamma\frac{y_{m+1}^n - y_m^{n+1} - y_m^{n-1} + y_{m-1}^n}{h^2}$$

Исследуем на устойчивость:

$$y_m^n = \lambda^n e^{ikmh}$$

$$\frac{\lambda^{n+1}e^{ikmh} - \lambda^{n-1}e^{ikmh}}{2\tau} = \gamma\frac{\lambda^n e^{ik(m+1)h} - \lambda^{n+1}e^{ikmh} - \lambda^{n-1}e^{ikmh} + \lambda^n e^{ik(m-1)h}}{h^2}$$

~~$e^{ik(m+1)h} = e^{ikmh}e^{ikh}$~~

$$e^{ik(m-1)h} = e^{ikmh}/e^{ikh}$$

Разделим обе части на $\lambda^{n-1}\cdot e^{ik(m-1)h}$.

$$\cancel{\lambda^2 e^{ikh}} \frac{\lambda e^{ikh}(\lambda^2 - 1)}{2\tau} = \gamma\frac{\lambda e^{2ikh} - \lambda^2 e^{ikh} - e^{ikh} + \lambda}{h^2}.$$

$$\lambda e^{2ikh} - \lambda^2 e^{ikh} - e^{ikh} + \lambda = e^{ikh}\lambda(e^{ikh} - \lambda) - (e^{ikh} - \lambda) =$$
$$= (e^{ikh}\lambda - 1)(e^{ikh} - \lambda)$$

$$\frac{\lambda e^{ikh}(\lambda - 1)}{2\tau} = \gamma\frac{(e^{ikh}\lambda - 1)(e^{ikh} - \lambda)}{h^2}$$

~~Решаем в лоб формулой:~~

~~$\lambda_{1,2} = 2\left(-\frac{e^{ikh}}{h^2} - \frac{e^{ikh}}{2\tau} * -\frac{\gamma}{h^2}\right)^2 - 4e^{ikh}\gamma\left(\frac{e^{ikh}}{h^2} + \frac{e^{ikh}}{2\tau}\right)$~~

Давайте поделим на $\lambda^n e^{ikmh}$

$$\frac{\lambda - \frac{1}{\lambda}}{2\tau} = \gamma\frac{e^{ikh} - \lambda - \frac{1}{\lambda} + \frac{1}{e^{ikh}}}{h^2}$$

$$\frac{\lambda-\frac{1}{\lambda}}{2\tau} = \gamma\,\frac{e^{ikh}-\lambda-\frac{1}{\lambda}+e^{-ikh}}{\cancel{2}h^2}$$

$$\frac{(\lambda^2-1)}{2\lambda\tau} = \gamma\,\frac{e^{ikh}-e^{-ikh}}{h^2} - \gamma\,\frac{(\lambda^2+1)}{\lambda h^2}$$

$$\frac{\lambda^2-1}{2\lambda\tau}^{\backslash h^2} + \gamma\frac{\lambda^2+1}{\lambda h^2}^{\backslash 2\tau} = \gamma\,\frac{e^{ikh}-e^{-ikh}}{h^2}$$

$$\frac{\lambda^2h^2-h^2+2\gamma\tau\lambda^2+2\gamma\tau}{2\lambda\tau h^2} = \gamma\,\frac{e^{ikh}\cancel{-e}-e^{-ikh}}{h^2}$$

$$\frac{\lambda^2(h^2+2\gamma\tau)+(2\gamma\tau-h^2)}{2\lambda\tau h^2} = \gamma\,\frac{e^{ikh}-e^{-ikh}}{h^2}$$

$$\cancel{4}\lambda^2(h^2+2\gamma\tau)h^2+(2\gamma\tau-h^2)h^2 = 2\lambda\tau h^2\gamma(e^{ikh}-e^{-ikh})$$

$$\lambda^2(h^4+2\gamma\tau h^2)+(2\gamma\tau h^2-h^4)\cancel{+} -$$
$$-2\lambda\tau h^2\gamma(\underbrace{\underbrace{\underbrace{e^{ikh}-e^{-ikh}}_{2\,\mathrm{sh}(ikh)}}_{-2i\sin(\underbrace{ikh\cdot i}_{-kh})}}_{2i\sin kh})=0$$

$$\Rightarrow \lambda^2(h^4+2\gamma\tau h^2)+(2\gamma\tau h^2-h^4)-\cancel{4}i\lambda\tau h^2\gamma\sin kh=0$$

$$\lambda = \frac{2i\tau h^2\gamma\sin kh \pm \sqrt{-4\tau^2h^4\gamma^2\sin^2kh-\cancel{4}(4\gamma^2\tau^2h^4-h^8)}}{h^4+2\gamma\tau h^2}$$

$$|\lambda|^2 = \frac{\cancel{4}\,4\tau^2h^4\gamma^2\sin^2kh}{\cancel{h^4+2\gamma\tau h^2}\;(h^4+2\gamma\tau h^2)^2} + \frac{-4\tau^2h^4\gamma^2\sin^2kh-(4\gamma^2\tau^2h^4-h^8)}{(h^4+2\gamma\tau h^2)^2} =$$

$$= \frac{1}{(h^4+2\gamma\tau h^2)^2}\cdot\left[\cancel{4\tau^2 h^4\gamma^2\sin^2 kh - 4\tau^2 h^4\gamma^2 \sin^2 kh} + 4\gamma^2\tau^2 h^4 - h^8\right] =$$

$$= \frac{4\gamma^2\tau^2 h^4 - h^8}{(h^4+2\gamma\tau h^2)^2} = \frac{(2\gamma\tau h^2 - h^4)\cancel{(2\gamma\tau h^2 + h^4)}}{\cancel{(h^4+2\gamma\tau h^2)^2}} =$$

$$= \frac{2\gamma\tau h^2 - h^4}{2\gamma\tau h^2 + h^4} \le 1$$ (h=0 не рассматриваем)

$\Rightarrow$ ~~устойчиво всегда~~ устойчиво безусловно

$\boxed{XII.7.19}$

1) 2)

$$\frac{3y_m^{n+1} - 4y_m^n + y_m^{n-1}}{2\tau} = \gamma\,\frac{y_{m+1}^{n+1} - 2y_m^{n+1} + y_{m-1}^{n+1}}{h^2}$$

$$y_m^n = y$$

$$y_m^{n+1} = y_m^n + \dot{y}_m^n\tau + \ddot{y}_m^n\frac{\tau^2}{2} + O(\tau^3)$$

$$y_m^{n-1} = y_m^n - \dot{y}_m^n\tau + \ddot{y}_m^n\frac{\tau^2}{2} + O(\tau^3)$$

$$y_{m+1}^{n+1} = y_m^{n+1} + y'^{\,n+1}_m h + y''^{\,n+1}_m\frac{h^2}{2} + O(h^3)$$

$$y_{m-1}^{n+1} = y_m^{n+1} - y'^{\,n+1}_m h + y''^{\,n+1}_m\frac{h^2}{2} + O(h^3)$$

1) $$\frac{3\cancel{y_m^n} + 3\dot{y}_m^n\tau + \frac{3}{2}\ddot{y}_m^n\tau^2 - 4\cancel{y_m^n} + \cancel{y_m^n} - \dot{y}_m^n\tau + \ddot{y}_m^n\frac{\tau^2}{2} + O(\tau^3)}{2\tau} =$$

$$= \frac{2\dot{y}_m^n\tau + 2\ddot{y}_m^n\tau^2 + O(\tau^3)}{2\tau} = \dot{y}_m^n + \ddot{y}_m^n\tau + O(\tau^2)$$

2) $$\gamma\,\frac{\cancel{y_m^{n+1}} + \cancel{y'^{\,n+1}_m h} + y''^{\,n+1}_m\frac{h^2}{2} - 2\cancel{y_m^{n+1}} + \cancel{y_m^{n+1}} - \cancel{y'^{\,n+1}_m h} + y''^{\,n+1}_m\frac{h^2}{2} + O(h^3)}{h^2} =$$

$$= \gamma \frac{y''^{\,n+1}_m h^2 + O(h^3)}{h^2} = \gamma y''^{\,n+1}_m + O(h)$$

$$\Rightarrow r = \dot y''^{\,n}_m + \ddot y''^{\,n}_m \tau + O(\tau^2) - \gamma y''^{\,n+1}_m + O(h)$$

$$y''^{\,n+1}_m = y''^{\,n}_m + \underbrace{\dot y''^{\,n}_m \tau + \ddot y''^{\,n}_m \frac{\tau^2}{2} + O(\tau^3)}_{\text{отбросим}}$$

$\Rightarrow$ нужное уравнение это

$$\frac{\partial y}{\partial t} - \gamma \frac{\partial^2 y}{\partial x^2} = 0.$$

$$\Rightarrow \dot y''^{\,n}_m - \gamma y''^{\,n+1}_m = 0.$$

$$\Rightarrow r = \ddot y''^{\,n}_m \tau + O(\tau^2) + O(h)$$

По хорошему не до конца разложили

$\Rightarrow$ Возьмем до y^{IV}.

$$\Rightarrow r = \ddot y''^{\,n}_m \tau + O(\tau^2) - \gamma\left(y''^{\,n+1}_m + \frac{h^2}{24} y^{IV\,n+1}_m + O(h^3)\right)$$

$$\Rightarrow \|r\| \le C\tau + Ch^2.$$

№ $\boxed{XIV.8.5}$

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad a \ne \text{const}$$

Ищем дисперсионное соотношение в виде $y^n_m = e^{\lambda(\omega) t_n} e^{i\omega x_m} = e^{\lambda n\tau + i\omega m h}$

Для левого уголка первого порядка:

$$\frac{1}{\tau}(e^{\lambda\tau} - 1) + \frac{a}{h}(1 - e^{-i\omega h}) = 0 \Rightarrow$$

$$\lambda(\tau, h, \omega) = \frac{1}{\tau}\ln\left(1 - \frac{a\tau}{h} + \frac{a\tau}{h} e^{-i\omega h}\right)$$

[illegible]

Сравним дисперсионные соотношения для дифф. и соотв. разн. ур-ий.

$\exists\ \frac{a\tau}{h} < 1$, $kh \ll 1$, h - мало =>

$$\lambda(\tau,h,k) \approx -iak - \frac{ahk^2}{2}\left(1 - \frac{a\tau}{h}\right) = \lambda(k) - \frac{k^2}{2}ah(1-\sigma)$$ // 1 часть ответа

Частное решение: $\exp(ik(mh - ank\tau)) \cdot \exp\left(-\frac{1}{2}ak^2(h - a\tau)n\tau\right)$

в ~~[illegible] не возрастает~~.

Л-В: $(y_m^{n+1} - y_m^n) + \frac{\sigma}{2}(y_{m+1}^n - y_{m-1}^n) - \frac{\sigma^2}{2}(y_{m-1}^n - 2y_m^n + y_{m+1}^n) = 0$

=> нужное соотн. есть $\lambda = \frac{1}{\tau}\ln\left(1 - i\sigma\sin kh - 2\sigma^2\sin^2\frac{kh}{2}\right)$

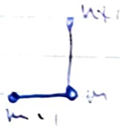
$\exists kh \ll 1$ => $\lambda(\tau,h,k) \approx -ika + ika\frac{k^2h^2}{6}(1 - 3\sigma^2)$

2

Часть Ответа

$\boxed{\text{XIV. 9.2}}$ $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$ $u(x,0) = \varphi(x)$ $x \in (-\infty;\infty)$
$t \in [0;T]$

а)



$$\left|\begin{array}{l} \frac{u_m^{n+1} - u_m^n}{\tau} + \frac{u_m^n - u_{m-1}^n}{h} = 0 \\ u_m^0 = \varphi_m \end{array}\right.$$

Устойчивость: ~~ξ_m^n~~ $u_m^n = \lambda^n e^{ikmh}$ =>

$$\frac{\lambda^{n+1}e^{ikmh} - \lambda^n e^{ikmh}}{\tau} + \frac{\lambda^n e^{ikmh} - \lambda^n e^{ik(m-1)h}}{h} = 0$$

Разделим на $\lambda^n e^{ik(m-1)h}$ =>

$$\frac{\lambda e^{ikh} - e^{ikh}}{\tau} + \frac{e^{ikh}-1}{h} = 0$$

$$\Rightarrow \frac{e^{ikh}(\lambda-1)}{\tau} = \frac{1-e^{ikh}}{h}$$

$$\lambda - 1 = \frac{\tau}{h}\frac{(1-e^{ikh})}{e^{ikh}}$$

$$\Rightarrow \lambda = 1 + \frac{\tau}{h}\frac{(1-e^{ikh})}{e^{ikh}} = 1 + \frac{\tau}{h}\underbrace{e^{-ikh}}_{\text{по модулю} \leq 1} - \frac{\tau}{h}$$

$\Rightarrow$ при $\frac{\tau}{h} \leq 1$ устойчива.

Аппроксимация:

$u^n_m = u^n_m$

$u^{n+1}_m = u^n_m + \dot{u}^n_m \tau + \ddot{u}^n_m \frac{\tau^2}{2} + O(\tau^3)$

$u^n_{m-1} = u^n_m - u'^n_m h + u''^n_m \frac{h^2}{2} + O(h^3)$

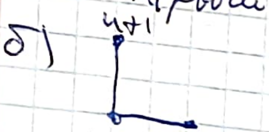
$$\Rightarrow \frac{\cancel{u^n_m} + \dot{u}^n_m \tau + \ddot{u}^n_m \frac{\tau^2}{2} + O(\tau^3) - \cancel{u^n_m}}{\tau} + \frac{\cancel{u^n_m} - (\cancel{u^n_m} - u'^n_m h + u''^n_m \frac{h^2}{2} + O(h^3))}{h} =$$

$= \dot{u}^n_m + \ddot{u}^n_m \frac{\tau^2}{2} + O(\tau^2) + u'^n_m - u''^n_m \frac{h}{2} - O(h^2)$

$\dot{u}^n_m + u'^n_m = 0$ ~~по услов~~

$\Rightarrow$ первый порядок

б)



~~по КФЛ~~

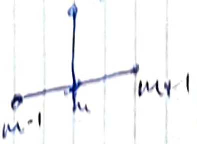
$$\left| \begin{array}{l} \frac{u^{n+1}_m - u^n_m}{\tau} + \frac{u^n_{m+1} - u^n_m}{h} = 0 \\ u^0_m = \varphi_m \end{array} \right.$$

Устойчивость: по КФЛ нет устойчивости

Аппроксимация $u^n_{m+1} = u^n_m + u'^n_m h + u''^n_m \frac{h^2}{2} + O(h^3)$

При подстановке получим тоже самое, что и в а), но у $u''^n_m \frac{h}{2}$ будет другой знак $\Rightarrow$ первый порядок.

б)



$$\begin{cases} \frac{u_m^{n+1}-u_m^n}{\tau} + \frac{u_{m+1}^n - u_{m-1}^n}{2h} = 0, \\ u_m^0 = \varphi_m \end{cases}$$

Устойчивость: $u_m^n = \lambda^n e^{ikmh}$

$$\Rightarrow \frac{\lambda^{n+1}e^{ikmh} - \lambda^n e^{ikmh}}{\tau} + \frac{\lambda^n e^{ik(m+1)h} - \lambda^n e^{ik(m-1)h}}{2h} = 0$$

Разделим на $\lambda^n e^{ikmh}$ $\Rightarrow$

$$\frac{\lambda - 1}{\tau} + \frac{e^{ikh} - e^{-ikh}}{2h} = 0$$

$$\frac{\lambda - 1}{\tau} + \frac{i}{h}\sin kh = 0,$$

$$\lambda - 1 = -\frac{i\tau}{h}\sin kh$$

$$\Rightarrow \lambda = 1 - \frac{i\tau}{h}\sin kh$$

$$|\lambda|^2 = 1 + \frac{\tau^2}{h^2}\sin^2 kh \Rightarrow |\lambda| = \sqrt{1+\frac{\tau^2}{h^2}\sin^2 kh} \geq 1 \Rightarrow$$ отсутствует устойчивость

Аппроксимация:

~~$u^n_m = u^n_m$~~

$$\frac{\cancel{u^n_m} + \dot{u}^n_m\tau + \ddot{u}^n_m\frac{\tau^2}{2} + O(\tau^3) - \cancel{u^n_m}}{\tau} + \frac{\cancel{u^n_m} + u'^n_m h + \cancel{u''^n_m\frac{h^2}{2}} - \cancel{u^n_m} + u'^n_m h - \cancel{u''^n_m\frac{h^2}{2}} + O(h^3)}{2h} = 0$$

$$r = \dot{u}^n_m + \ddot{u}^n_m\frac{\tau}{2} + O(\tau^2) + u'^n_m + O(h^2)$$

$$\dot{u}^n_m + u'^n_m = 0 \Rightarrow$$

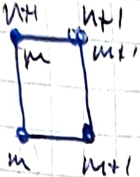
$r = \ddot{u}^n_m\frac{\tau}{2} + O(\tau^2) + O(h^2) \Rightarrow$ не до конца разложили

Возьмем $u^n_{m+1} = u^n_m + u'^n_m h + u''^n_m\frac{h^2}{2} + u'''^n_m\frac{h^3}{6} + O(h^4)$

$u^n_{m-1} = u^n_m - u'^n_m h + u''^n_m\frac{h^2}{2} - u'''^n_m\frac{h^3}{6} + O(h^4)$

$$\Rightarrow r = \ddot{u}^n_m\frac{\tau}{2} + u'''^n_m\frac{h^2}{6} + O(\tau^2 + h^3)$$

$\Rightarrow$ первый по τ и второй по h.

2) 

$$\begin{cases}\dfrac{0.5(u_m^{n+1}-u_m^n)+0.5(u_{m+1}^{n+1}-u_{m+1}^n)}{\tau}+\dfrac{0.5(u_{m+1}^{n+1}-u_m^{n+1})+0.5(u_{m+1}^n-u_m^n)}{h}=0\\ \overset{0}{u}_m=\varphi_m\end{cases}$$

Устойчивость: $u_m^n=\lambda^n e^{imh}$

~~⇒ Разд [illegible] сокр~~

~~$\frac{\lambda-1+(\lambda-1)e^{ih}}{2\tau}+\frac{\lambda(e^{ih}-1)+(e^{ih}-1)}{2h}=0$~~

~~$\frac{(\lambda-1)(1+e^{ih})}{2\tau}+\frac{(\lambda+1)(e^{ih}-1)}{2h}=0$~~

~~$\lambda\frac{1+e^{ih}}{2\tau}$~~

$$\frac{\lambda^{n+1}e^{imh}-\lambda^n e^{imh}+\lambda^{n+1}e^{i(m+1)h}-\lambda^n e^{i(m+1)h}}{2\tau}+\frac{\lambda^{n+1}e^{i(m+1)h}-\lambda^{n+1}e^{imh}+\lambda^n e^{i(m+1)h}-\lambda^n e^{imh}}{2h}=0$$

Разделим на $\lambda^n e^{imh}$:

$$\frac{\lambda-1+\lambda e^{ih}-e^{ih}}{2\tau}+\frac{\lambda e^{ih}-\lambda+e^{ih}-1}{2h}=0$$

$$\frac{(\lambda-1)+e^{ih}(\lambda-1)}{2\tau}+\frac{\lambda(e^{ih}-1)+(e^{ih}-1)}{2h}=0$$

$$\frac{(\lambda-1)(1+e^{ih})}{2\tau}+\frac{(\lambda+1)(e^{ih}-1)}{2h}=0$$

Разделим на $e^{\frac{ikh}{2}}$

$$\Rightarrow \frac{(\lambda-1)\left(e^{-\frac{ikh}{2}}+e^{\frac{ikh}{2}}\right)}{2\tau}+\frac{(\lambda+1)\left(e^{\frac{ikh}{2}}-e^{-\frac{ikh}{2}}\right)}{2h}=0$$

$$\frac{(\lambda-1)}{\tau}\cos\frac{kh}{2}+i\frac{(\lambda+1)}{h}\sin\frac{kh}{2}=0$$

$$\frac{\lambda}{\tau}\cos\frac{kh}{2}+i\frac{\lambda}{h}\sin\frac{kh}{2}-\frac{\cos\frac{kh}{2}}{\tau}+\frac{i}{h}\sin\frac{kh}{2}=0.$$

$$\lambda\left(\frac{\cos\frac{kh}{2}}{\tau}+\frac{i}{h}\sin\frac{kh}{2}\right)=\frac{\cos\frac{kh}{2}}{\tau}-\frac{i}{h}\sin\frac{kh}{2}$$

$$\backslash\frac{\cos\frac{kh}{2}}{\tau}-\frac{i}{h}\sin\frac{kh}{2}$$

$$\Rightarrow\lambda=\frac{\frac{\cos\frac{kh}{2}}{\tau}-\frac{i}{h}\sin\frac{kh}{2}}{\frac{\cos\frac{kh}{2}}{\tau}+\frac{i}{h}\sin\frac{kh}{2}}$$

$$\cancel{\lambda=\frac{\left(\frac{\cos\frac{kh}{2}}{\tau}-\frac{i}{h}\sin\frac{kh}{2}\right)^2}{\frac{\cos^2\frac{kh}{2}}{\tau^2}+\frac{\sin^2\frac{kh}{2}}{h^2}}=\frac{\frac{\cos^2\frac{kh}{2}}{\tau^2}-\frac{2i}{h\tau}\cos\frac{kh}{2}\sin\frac{kh}{2}-\frac{\sin^2\frac{kh}{2}}{h^2}}{\frac{\cos^2\frac{kh}{2}}{\tau^2}+\frac{\sin^2\frac{kh}{2}}{h^2}}}$$

$|\lambda|=1$. Судя по Демченко - устойчива абс.

аппроксимация:

$$0.5\,\frac{\cancel{u^n_m}+\dot u^n_m\tau+\ddot u^n_m\frac{\tau^2}{2}-\cancel{u^n_m}+\cancel{u^n_{m+1}}+\dot u^n_{m+1}\tau+\ddot u^n_{m+1}\frac{\tau^2}{2}+o(\tau^3)-\cancel{u^n_{m+1}}}{2\tau}+$$

$$+\frac{\cancel{u^{n+1}_m}+u'^{n+1}_m h+u''^{n+1}_m\frac{h^2}{2}+\cancel{u^n_m}-\cancel{u^{n+1}_m}\cancel{+u^n_{m+1}+u^{n+1}_{m+1}}+\cancel{u^n_m}+u'^n_m h+u''^n_m\frac{h^2}{2}+o(h^3)-\cancel{u^n_m}}{2h}=0$$

$\dot u^n_m=\dot u^n_{m+1}$

$\ddot u^n_m=\ddot u^n_{m+1}$

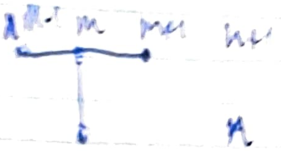
$u'^{n+1}_m=u'^n_m$

$u''^{n+1}_m=u''^n_m$

$\Rightarrow \dot u^n_m+\ddot u^n_m\frac{\tau}{2}+o(\tau^2)+u'^n_m+u''^n_m\frac{h}{2}+o(h^2)$

$\dot u^n_m+u'^n_m=0\Rightarrow$ первый порядок.

7)



$$\begin{cases} \frac{u_m^{n+1}-u_m^n}{\tau} + \frac{u_{m+1}^{n+1}-u_{m-1}^{n+1}}{2h} = 0 \\ u_m^0 = \varphi_m \end{cases}$$

Д. устойчивость. $u_m^n = \lambda^n e^{iumh}$

$$\frac{\lambda^{n+1} e^{iumh} - \lambda^n e^{iumh}}{\tau} + \frac{\lambda^{n+1} e^{iu(m+1)h} - \lambda^{n+1} e^{iu(m-1)h}}{2h} = 0$$

Разделим на $\lambda^n e^{iumh}$:

$$\frac{\lambda-1}{\tau} + \frac{\lambda e^{iuh} - \lambda e^{-ikh}}{2h} = 0$$

$$\frac{\lambda-1}{\tau} + \frac{\lambda}{h} \frac{e^{iuh}-e^{-iuh}}{2} = 0$$

$$\frac{\lambda-1}{\tau} + \frac{\lambda}{h} i \sin uh = 0 \quad | \cdot \tau$$

$$\lambda - 1 + \lambda \frac{i\tau}{h} \sin uh = 0$$

$$\lambda \left(1 + \frac{i\tau}{h} \sin uh\right) = 1$$

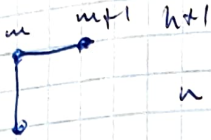
$$\lambda = \frac{1}{1+\frac{i\tau}{h}\sin uh} \quad \backslash 1 - \frac{i\tau}{h}\sin uh$$

$$\lambda = \frac{1-\frac{i\tau}{h}\sin uh}{1-\frac{i^2\tau^2}{h^2}\sin^2 uh} = \frac{1-\frac{i\tau}{h}\sin uh}{1+\frac{\tau^2}{h^2}\sin^2 uh}$$

$$|\lambda| = \sqrt{\frac{1+\frac{\tau^2}{h^2}\sin^2 uh}{\left(1+\frac{\tau^2}{h^2}\sin^2 uh\right)^2}} = \sqrt{\frac{1}{1+\frac{\tau^2}{h^2}\sin^2 uh}} \le 1 \Rightarrow$$ устойчиво абс.

аппроксимация: порядок аппроксимации будет аналогичен пункту в), так как в силу

$u'^{n+1}_m = u'^{n}_m$ и
$u''^{n+1}_m = u''^{n}_m$ => второе слагаемое раскладывается одинаково

е) 

$$\begin{cases} \dfrac{u^{n+1}_m - u^n_m}{\tau} + \dfrac{u^{n+1}_{m+1} - u^{n+1}_m}{h} = 0. \\ u^0_m = \varphi_m \end{cases}$$

~~$u^n_m =$~~

устойчивость $u^n_m = \lambda^n e^{ikmh}$

$$\frac{\lambda^{n+1} e^{ikmh} - \lambda^n e^{ikmh}}{\tau} + \frac{\lambda^{n+1} e^{ik(m+1)h} - \lambda^{n+1} e^{ikmh}}{h} = 0.$$

Разделим на $\lambda^n e^{ikmh}$ =>

$$\frac{\lambda - 1}{\tau} + \frac{\lambda e^{ikh} - \lambda}{h} = 0$$

$$\frac{\lambda - 1}{\tau} + \frac{\lambda(1 - e^{ikh})}{h} = 0 \quad | \cdot \tau$$

$$\lambda - 1 + \lambda \frac{\tau}{h}(1 - e^{ikh}) = 0.$$

$$\lambda\left(1 + \frac{\tau}{h}(1 - e^{ikh})\right) = 1 \Rightarrow \lambda = \frac{1}{1 + \frac{\tau}{h}(1 - e^{ikh})}$$

~~$|\lambda| = \dfrac{1}{\sqrt{\frac{\tau^2}{h^2}\sin^2 kh + (\frac{\tau}{h}(1-\cos kh) + \frac{\tau}{h} + 1)^2}}$~~

$$|\lambda|^2 = \frac{1}{1 + 2\frac{\tau}{h}(\frac{\tau}{h} - 1)(1 - \cos kh)} \Rightarrow$$ устойчивость при ~~$\frac{\tau}{h}$~~ $\frac{\tau}{h}(\frac{\tau}{h} - 1) \geq 0$

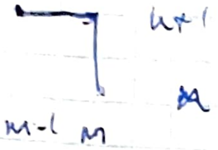
$$\underset{\geq 0}{(1 - \cos kh)}$$

не
~~физично~~
[illegible]

=> при $\frac{\tau}{h} \geq 1$

Аппроксимация: аналогично пред. пункту делаем вывод о сложности порядка с а) и б) => первый по τ и h.

м)



$$\begin{cases} \frac{u_m^{n+1}-u_m^n}{\tau} + \frac{u_m^{n+1}-u_{m-1}^{n+1}}{h} = 0 \\ u_m^0 = 0 \end{cases}$$

Аппроксимация: Аналогично пред. пункту первый порядок по τ и h,

Устойчивость: $u_m^n = \lambda^n e^{i\alpha mh}$

$$\frac{\lambda^{n+1}e^{i\alpha mh}-\lambda^n e^{i\alpha mh}}{\tau} + \frac{\lambda^{n+1}e^{i\alpha mh}-\lambda^{n+1}e^{i\alpha(m-1)h}}{h} = 0$$

Разделим на $\lambda e^{i\alpha mh}$ =>

$$\frac{\lambda-1}{\tau} + \frac{\lambda-\lambda e^{-i\alpha h}}{h} = 0 \quad |\cdot\tau$$

$$\lambda-1+\lambda\frac{\tau}{h}(1-e^{-i\alpha h}) \Rightarrow$$

$$\lambda = \frac{1}{1+\frac{\tau}{h}(1-e^{-i\alpha h})}$$

$$|\lambda| = \frac{1}{\sqrt{1+((1+\cos\alpha h)\frac{\tau}{h})^2+(\frac{\tau}{h}\sin\alpha h)^2}} \le 1 \Rightarrow$$

абс. устойчиво.

XIV.9.6

$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$, $a = const > 0$

Схема: $\frac{y^{n+1}_m - y^n_m}{\tau} + \underset{const}{a}\,\frac{y^n_{m+1} - y^n_{m-1}}{2h} = \xi\tau\,\frac{y^n_{m+1} - 2y^n_m + y^n_{m-1}}{h^2}$

Лакс-Вендрофф при $\xi = a^2$

Лакс при $\xi = \frac{h^2}{2\tau^2}$, тогда тогда=

$$\frac{y^{n+1}_m - y^n_m}{\tau} + a\frac{y^n_{m+1} - y^n_{m-1}}{2h} = \frac{y^n_{m+1} - 2y^n_m + y^n_{m-1}}{2\tau}$$

$$\frac{y^{n+1}_m - \frac{1}{2}(y^n_{m+1} + y^n_{m-1})}{\tau} + a\frac{y^n_{m+1} - y^n_{m-1}}{2h} = 0$$

← Лакс

~~КИР при $\xi = \frac{ha}{2\tau}$~~

~~КИР: $\frac{y^{n+1}_m - y^n_m}{\tau} + a\, y^n_{m+1} - y^n_m$~~

КИР при $a > 0$ есть $\frac{u^{n+1}_m - u^n_m}{\tau} + a\frac{u^n_m - u^n_{m-1}}{h} = 0$

$\Rightarrow \xi = \frac{ha}{2\tau}$

Исследуем на устойчивость:

$y^n_m = \lambda^n e^{inmh} \Rightarrow$

$$\frac{\lambda^{n+1} e^{inmh} - \lambda^n e^{inmh}}{\tau} + a\frac{\lambda^n e^{in(m+1)h} - \lambda^n e^{in(m-1)h}}{2h} = \xi\tau\,\frac{\lambda^n e^{in(m+1)h} - 2\lambda^n e^{inmh} + \lambda^n e^{\dots}}{h^2}$$

Разделим на $\lambda^n e^{inmh} \Rightarrow$

$$\frac{\lambda - 1}{\tau} + a\frac{e^{inh} - e^{-inh}}{2h} = \xi\tau\,\frac{e^{inh} - 2 + e^{-inh}}{h^2}$$

$$\frac{\lambda-1}{\tau} + a\frac{a}{h} i \sin kh = \frac{\xi\tau}{h^2} 2\cos kh - \frac{2\xi\tau}{h^2} \quad |\cdot\tau$$

$$\lambda = 1 - \cancel{a} i\frac{a\tau}{h}\sin kh + \frac{2\xi\tau^2}{h^2}\cos kh - \frac{2\xi\tau^2}{h^2}$$

$$\lambda = \left(1 + \frac{2\xi\tau^2}{h^2}(\cos kh - 1)\right) - i\frac{a\tau}{h}\sin kh$$

$$|\lambda|^2 = \left(1 + \frac{2\xi\tau^2}{h^2}(\cos kh - 1)\right)^2 + \frac{a^2\tau^2}{h^2}\sin^2 kh$$

~~$\cos kh - 1 \le 0$~~

(3) $$|\lambda|^2 = 1 + \frac{4\xi\tau^2}{h^2}(\cos kh - 1) + \overset{\geq 0}{\boxed{\frac{4\xi^2\tau^4}{h^4}(\cos kh - 1)^2}} + \overset{\geq 0}{\boxed{\frac{a^2\tau^2}{h^2}\sin^2 kh}}$$

Рассмотрим $f(\tau, h, \xi) = \frac{4\xi\tau^2}{h^2}(\cos kh - 1) + \frac{4\xi^2\tau^4}{h^2}(\cos kh - 1)^2 + \frac{a^2\tau^2}{h^2}\sin^2 kh$

~~$\exists\ f(\tau$~~

Для устойчивости $f(\tau, h, \xi) \le 0$

Разделим сразу на $\frac{\tau^2}{h^2} > 0 \Rightarrow$

$$4\xi(\cos kh - 1) + 4\xi^2\frac{\tau^2}{h^2}(\cos kh - 1)^2 + a^2\sin^2 kh \le 0$$

~~Пусть $\frac{\cancel{4\xi}(\cos kh - 1)}{4\xi^2\frac{\tau^2}{h^2}(\cos kh - 1)^2} = \varphi = \frac{1}{\xi\frac{\tau^2}{h^2}(\cos kh - 1)}$~~ ~~$\begin{cases} > 0 & \text{при } \xi < 0 \\ < 0 & \text{при } \xi > 0 \end{cases}$~~

~~$\varphi + \frac{a^2\sin^2 kh}{4\xi^2\frac{\tau^2}{h^2}(\cos kh - 1)^2} \le 0$~~

~~$\frac{a^2}{4\xi^2\frac{\tau^2}{h^2}} > 0 = C$~~

~~$\xi + 1\frac{c\sin^2 kh}{4a(\cos kh - 1)^2} \le 0$~~

~~$\xi \le 1 - \frac{c\sin^2 kh}{(\cos kh - 1)^2}$~~

~~При $kh = \pi + 2\pi n$ второе слагаемое есть~~ 0

$4\xi(\cos kh - 1) + 4\xi^2\frac{\tau^2}{h^2}(\cos kh - 1)^2 + a^2(1 - \cos^2 kh) \le 0.$

$4\xi(\cos kh - 1) + 4\xi^2\frac{\tau^2}{h^2}(\cos^2 kh - 2\cos kh + 1) + a^2(1 - \cos^2 kh) \le 0.$

~~$4\xi(\cos kh - 1) + 4\xi^2\frac{\tau^2}{h^2}\cos^2 kh - a^2\cos^2 kh$~~

$4\xi\cos kh - 4\xi + 4\xi^2\frac{\tau^2}{h^2}\cos^2 kh - 8\xi^2\frac{\tau^2}{h^2}\cos kh + 4\xi^2\frac{\tau^2}{h^2} + a^2 - a^2\cos^2 kh \le 0.$

$\cos^2 kh\left(4\xi^2\frac{\tau^2}{h^2} - a^2\right) + \cos kh\left(4\xi - 8\xi^2\frac{\tau^2}{h^2}\right) + (a^2 - 4\xi) \le 0$

Давайте посмотрим на начальную $\lambda^2 - (1)$.

$|\lambda|^2 \le 1$ ~~только~~ может быть только при $\xi \le 0$ очевидно.

$\Rightarrow a^2 - 4\xi \ge 0$

$\cos kh \in [-1;1]$; ~~$\cos^2 kh \ge 0.$~~ $\cos^2 kh \in [0;1]$

$4\xi - 8\xi^2\frac{\tau^2}{h^2} \le 0.$

~~При $4\xi^2\frac{\tau^2}{h^2} \ge a^2$~~

~~Найдем критическую [illegible]~~

$-4\xi(\cos kh - 1) \ge 4\xi^2\frac{\tau^2}{h^2}(\cos kh - 1)^2 + a^2\sin^2 kh$

$4\xi(1 - \cos kh) \ge 4\xi^2\frac{\tau^2}{h^2}\underset{}{(\cos kh - 1)}^{\overset{0}{\shortparallel}\,2} + a^2\sin^{\overset{0}{\shortparallel}\,2} kh$

~~$\xi \le 0$ – очевидное решение~~ ~~и~~ Неравенство нарушается при $\xi \le 0$

$$\underbrace{4\xi(1-\cos kh)}_{\substack{\vee\\ 0}}\left(1-\frac{\tau^2}{h^2}\xi(1-\cos kh)\right)\geq \overbrace{a^2\sin^2 kh}^{\wedge\, 0}$$

~~Условие нарушается если~~

~~$1-\frac{\tau^2}{h^2}\xi(1-\cos kh)\leq 0$~~

~~т.е $\frac{\tau^2}{h^2}\xi(1-\cos kh)\geq 1$~~

$$\underset{\substack{\vee\\ 0}}{4\xi(1-\cos kh)}\ \frac{\overset{\substack{1\\ \vee}}{1-\frac{\tau^2}{h^2}\xi(1-\cos kh)}}{\underset{\substack{\wedge\\ a^2}}{a^2\sin^2 kh}}\geq 1$$

$\xi \geq a$

$\Rightarrow \quad \xi > 0$

$1-\frac{\tau^2}{h^2}\xi(1-\cos kh)\geq 0$

$\frac{\tau^2}{h^2}(1-\cos kh)\xi\cancel{4}\leq 1$

$(1-\cos kh)\xi\leq \cancel{\frac{1}{4}}\frac{h^2}{\tau^2}$

$\cancel{\frac{h^2}{\xi^2}}$

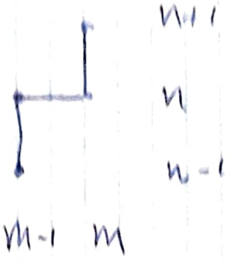
Далее не особо понятно, как исследовать неравенство, потому условие можно оставить недоказанным

Устойчивость: $\left|\begin{array}{l}\xi>0\\ 1-\cos kh\leq\frac{h^2}{\tau^2}\\ 4\xi(1-\cos kh)\ \frac{1-\frac{\tau^2}{h^2}(1-\cos kh)}{a^2\sin^2 kh}\geq 1\end{array}\right.$

Д.З.

$\boxed{\text{XIV, 9.8}}$

$$\frac{\partial u}{\partial t} + \underset{\underset{\text{const}>0}{\parallel}}{a}\frac{\partial u}{\partial x} = 0$$



$\Downarrow$

Аналогично получению схемы □ нетрудно получить

$$\frac{\frac{1}{2}(u_m^{n+1}-u_m^n)+\frac{1}{2}(u_{m-1}^n-u_{m-1}^{n-1})}{\tau} + a\frac{u_m^n-u_{m-1}^n}{h} = 0.$$

2
Часть Ответа

~~Исследуем на устойчивость:~~ $u_m^n = \lambda^n e^{ikmh}$

$$\frac{\lambda^{n+1}e^{ikmh} - \lambda^n e^{ikmh} + \lambda^n e^{ik(m-1)h} - \lambda^{n-1}e^{ik(m-1)h}}{2\tau} + a\frac{\lambda^n e^{ikmh} - \lambda^n e^{ik(m-1)h}}{h} = 0$$

~~Разделим на~~ λe^{ikmh}

$$\frac{\lambda - 1 + e^{-ikh} - \frac{1}{\lambda}e^{-ikh}}{2\tau} + a\frac{1-e^{-ikh}}{h} = 0.$$

$$\frac{\lambda-1 + e^{-ikh}(1-\frac{1}{\lambda})}{2\tau} + a\frac{1-e^{-ikh}}{h} = 0.$$

$$\frac{\lambda-1}{2\tau} + e^{-ikh}\,\lambda-1$$

$$\frac{(\lambda-1) + \frac{\lambda-1}{\lambda}e^{-ikh}}{2\tau} = -\frac{a(1-e^{-ikh})}{h}$$

$$h(\lambda-1) + h\frac{\lambda-1}{\lambda}e^{-ikh} = +2\tau(e^{-ikh}-1)$$

$\boxed{XII.7.2}$ (устойчивость)

$$\frac{u_m^{n+1}-u_m^n}{\tau}+\frac{c}{2h}\left(u_{m+1}^n-u_{m-1}^n\right)-\frac{c^2\tau}{2h^2}\left(u_{m+1}^n-2u_m^n+u_{m-1}^n\right)=0$$

$$u_m^n=\lambda^n e^{i\alpha mh}$$

$$\frac{\lambda^{n+1}e^{i\alpha mh}-\lambda^n e^{i\alpha mh}}{\tau}+\frac{c}{2h}\left(\lambda^n e^{i\alpha(m+1)h}-\lambda^n e^{i\alpha(m-1)h}\right)-$$

$$-\frac{c^2\tau}{2h^2}\left(\lambda^n e^{i\alpha(m+1)h}-2\lambda^n e^{i\alpha mh}+\lambda^n e^{i\alpha(m-1)h}\right)=0.$$

Разделим на $\lambda^n e^{i\alpha mh}$ =>

$$\frac{\lambda-1}{\tau}+\frac{c}{2h}\,\frac{e^{i\alpha h}-e^{-i\alpha h}}{1}-\frac{c^2\tau}{2h^2}\,\frac{e^{i\alpha h}-2+e^{-i\alpha h}}{1}=0$$

$$\frac{\lambda-1}{\tau}+\frac{c}{h}\,i\sin\alpha h-\frac{c^2\tau}{\cancel{2}h^2}\cos\alpha h+\frac{c^2\tau}{h^2}=0.$$

$$\lambda=1+\frac{c^2\tau^2}{h^2}\cos\alpha h-\frac{c^2\tau^2}{h^2}-\frac{c\tau}{h}\,i\sin\alpha h.$$

$$\lambda=\left(1+\frac{c^2\tau^2}{h^2}(\cos\alpha h-1)\right)-i\,\frac{c\tau}{h}\sin\alpha h$$

$$|\lambda|^2=\left(1+\frac{c^2\tau^2}{h^2}(\cos\alpha h-1)\right)^2+\frac{c^2\tau^2}{h^2}\sin^2\alpha h\le 1$$

$$\frac{2c^2\tau^2}{h^2}(\cos\alpha h-1)+\frac{c^4\tau^4}{h^4}(\cos\alpha h-1)^2+\frac{c^2\tau^2}{h^2}\sin^2\alpha h\le 0.$$

$$2(\cos\alpha h-1)+\frac{c^2\tau^2}{h^2}(\cos\alpha h-1)^2+\sin^2\alpha h\le 0.$$

↘ отсюда определяется условие устойчивости.

VIN:48) (недописано)

Аппроксимация:

~~$\frac{u^{n}_{m} - u^{n}_{m}}{\tau} \dots$~~ [illegible]

~~$a\, \frac{u^{n}_{m} - u^{n}_{m} + u^{n}_{m} h - u^{n}_{m} \frac{h^2}{2} + O(h^3)}{h} = 0$~~

~~$r = \dot{u}^{n}_{m} + \ddot{u}^{n}_{m} \frac{\tau}{2} + O(\tau^2) + a u'^{n}_{m} - a u''^{n}_{m} \frac{h}{2} + O(h^2)$~~

~~[illegible]~~

Схема симметрична относительно точки

~~$((m-\frac{1}{2})h, n\tau)$~~

$((m-\frac{1}{2})h, n\tau)$ => раскладая все относительно этой точки получим аппроксимацию с $O(\tau^2+h^2)$

Устойчивость: $\exists\, u^{n}_{m} = e^{i(\omega\tau n - kh(m-1))}$

$q = e^{i\omega\tau}$

=> $q_k^2 - q_k(1-e^{ikh})(1-2r) - e^{ikh} = 0$

=> $q_{k,1} = -\frac{b}{2} + \sqrt{D}$ $\quad b = -(1-e^{ikh})(1-2r)$

$q_{k,2} = -\frac{b}{2} - \sqrt{D}$ $\quad c = -e^{ikh}$

$D = \frac{b^2}{4} - c$

$$D = (2r-1)^2(1-e^{i\varphi})^2+4e^{i\varphi} = (2r-1)^2\left[2\cos\varphi(\cos\varphi-1)+\right.$$

$$\left.+2i\sin\varphi(\cos\varphi-1)\right]+4\cos\varphi+4i\sin\varphi$$

$$\Rightarrow \operatorname{Re} q_{k,1} = (1-2r)\sin^2\frac{\varphi}{2}+\sqrt{|1-(1-2r)^2\sin^2\frac{\varphi}{2}|}\cos\frac{\varphi}{2}$$

$$\operatorname{Im} q_{k,1} = -(1-2r)\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}+\sqrt{|1-(1-2r)^2\sin^2\frac{\varphi}{2}|}\sin\frac{\varphi}{2}$$

$$\operatorname{Re} q_{k,2} = (1-2r)\sin^2\frac{\varphi}{2}-\sqrt{|1-(1-2r)^2\sin^2\frac{\varphi}{2}|}\cos\frac{\varphi}{2}$$

$$\operatorname{Im} q_{k,2} = -(1-2r)\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}-\sqrt{|1-(1-2r)^2\sin^2\frac{\varphi}{2}|}\sin\frac{\varphi}{2}$$

$$\Rightarrow |q_{k,1}| = \begin{cases} 1, & (1-2r)^2\sin^2\frac{\varphi}{2}\le 1 \\ \overset{\overset{1}{\wedge}}{\boxed{2(1-2r)^2\sin^2\frac{\varphi}{2}+1}}, & (1-2r)^2\sin^2\frac{\varphi}{2}>1 \end{cases}$$

$$|q_{k,2}| = \begin{cases} 1 & (1-2r)^2\sin^2\frac{\varphi}{2}\le 1 \\ \underset{\underset{1}{\vee}}{\boxed{2(1-2r)^2\sin^2\frac{\varphi}{2}+1}} & (1-2r)^2\sin^2\frac{\varphi}{2}>1 \end{cases}$$

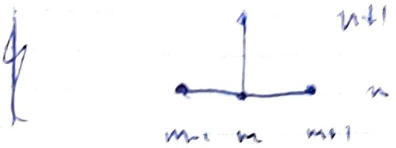
$\Rightarrow$ условие устойчивости есть

$$(1-2r)^2\sin^2\frac{\varphi}{2}\le 1$$

$\Rightarrow$ по основной теореме выч. устойчивость при $(1-2r)^2\sin^2\frac{\varphi}{2}\le 1$ и аппроксимация с $O(\tau^2+h^2)$ $\Rightarrow$ сходимость такого же порядка при выполнении устойчивости.

$\boxed{XIV.9.11(a)}$

$$\begin{cases} \dfrac{u_m^{n+1}-u_m^n}{\tau} + \dfrac{-3u_{m+1}^n + 4u_m^n - u_{m-1}^n}{2h} + \dfrac{-3v_{m+1}^n + 3v_{m-1}^n}{2h} = f_m^n \\ \dfrac{v_m^{n+1}-v_m^n}{\tau} + \dfrac{-u_{m+1}^n + u_{m-1}^n}{2h} + \dfrac{-v_{m+1}^n + 4v_m^n - 3v_{m-1}^n}{2h} = g_m^n \end{cases}$$



Исследуем на устойчивость по спектр. признаку

$] \; u_m^n = \cancel{\lambda} \, a\lambda^n e^{i\varphi m}$

$v_m^n = b\lambda^n e^{i\varphi m}$

$$\Rightarrow \begin{cases} a\dfrac{\lambda^{n+1}e^{i\varphi m} - \lambda^n e^{i\varphi m}}{\tau} + a\dfrac{-3\lambda^n e^{i\varphi(m+1)} + 4\lambda^n e^{i\varphi m} - \lambda^n e^{i\varphi(m-1)}}{2h} + b\dfrac{-3\lambda^n e^{i\varphi(m+1)} + 3\lambda^n e^{i\varphi}}{2h} \\ b\dfrac{\lambda^{n+1}e^{i\varphi m} - \lambda^n e^{i\varphi m}}{\tau} + a\dfrac{-\lambda^n e^{i\varphi(m+1)} + \lambda^n e^{i\varphi(m-1)}}{2h} + b\dfrac{-\lambda^n e^{i\varphi(m+1)} + 4\lambda^n e^{i\varphi m} - 3\lambda^n e^{i\varphi m}}{2h} \end{cases}$$

Разделим уравнения на $\lambda^n e^{i\varphi m} \Rightarrow$

$$\begin{cases} a\dfrac{\lambda-1}{\tau} + a\dfrac{-3e^{i\varphi} + 4 - e^{-i\varphi}}{2h} \cancel{+b} + 3b\dfrac{-e^{i\varphi} + e^{-i\varphi}}{2h} = 0 \\ b\dfrac{\lambda-1}{\tau} + b\dfrac{-e^{+i\varphi} + 4 - 3e^{-i\varphi}}{2h} + a\dfrac{-e^{i\varphi} + e^{-i\varphi}}{2h} = 0 \end{cases}$$

$$\Rightarrow \cancel{ab}\cdot\left(\frac{\lambda-1}{\tau}\right)^2 + \cancel{ab}\underbrace{\frac{(-3e^{i\varphi} + 4 - e^{-i\varphi})(-e^{i\varphi} + 4 - 3e^{-i\varphi})}{4h^2}} + 3\cancel{ba}\frac{(-e^{i\varphi} + e^{-i\varphi})^2}{4h^2} = 0$$

$$\left(\frac{\lambda-1}{\tau}\right)^2 + (3e^{2i\varphi} - 4e^{i\varphi}$$

$$3e^{2i\varphi} - 4e^{i\varphi} + 1 - 12e^{i\varphi} + 16 - 4e^{-i\varphi} + 9 - 12e^{-i\varphi} + 3e^{-2i\varphi} =$$

$$= 3e^{2i\varphi} + 3e^{-2i\varphi} - 16e^{i\varphi} - 16e^{-i\varphi} + 26.$$

$$(-e^{i\varphi}+e^{-i\varphi})^2 = e^{2i\varphi} - 2e^{i\varphi}\cdot e^{-i\varphi} + e^{-2i\varphi} =$$

$$= e^{2i\varphi} + e^{-2i\varphi} - 2$$

$$\Rightarrow \left(\frac{\lambda-1}{\tau}\right)^2 + \frac{7(e^{2i\varphi}+e^{-2i\varphi}) - 16(e^{i\varphi}+e^{-i\varphi}) + 26 + 3(e^{2i\varphi}+e^{-2i\varphi}) - 6}{4h^2} = 0$$

$$\left(\frac{\lambda-1}{\tau}\right)^2 + \frac{6(e^{2i\varphi}+e^{-2i\varphi}) - 16(e^{i\varphi}+e^{-i\varphi}) + 20}{4h^2} = 0$$

$$\left(\frac{\lambda-1}{\tau}\right)^2 + \frac{\cancel{6}\,3\cos 2\varphi - \cancel{16}\,8\cos\varphi + 20}{4h^2} = 0 \quad /\cdot\tau^2$$

$$(\lambda-1)^2 + \frac{\tau^2}{4h^2}(3\cos 2\varphi - 8\cos\varphi + 20) = 0$$

$$\cancel{\lambda^2 - 2\lambda + 1 + \tau^2}$$

$$(\lambda-1)^2 = -\frac{\tau^2}{4h^2}(3\cos 2\varphi - 8\cos\varphi + 20)$$

$$\Rightarrow \lambda - 1 = \pm i\frac{\tau}{2h}\sqrt{\cancel{2}\,20 + 8\cos\varphi - 3\cos 2\varphi}$$

$$\lambda = 1 \pm i\frac{\tau}{2h}\sqrt{20 + 8\cos\varphi - 3\cos 2\varphi}$$

$\Rightarrow$ неустойчивость, так как $\forall\varphi\ \exists\lambda : |\lambda| > 1 + \frac{\tau}{2h}$

$\boxed{\text{XIV, 9.14(б)}}$

$$\begin{cases} \dot u + 2u' - 2v' + 3w' = f \\ \dot v + u' + v' + w' = g \\ \dot w + u' + 3v' \cancel{x} - w' = h \end{cases} \Rightarrow A = \begin{Vmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{Vmatrix}$$

1) Найдем с.числа A: $\begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0 \Leftrightarrow \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = -2 \\ \lambda_3 = 3 \end{matrix}$

Заметим, что все λ действительные $\Rightarrow$ система гиперболическая

Найдем левые собственные вектора

$$\lambda_1 = 1 \Rightarrow \quad h^1 \cdot \begin{Vmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{Vmatrix} = 0$$

$$\begin{Vmatrix} h_1 & h_2 & h_3 \end{Vmatrix} \cdot \begin{Vmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{Vmatrix} = 0$$

$$\begin{cases} h_1 + h_2 + h_3 = 0 \\ -2h_1 + 3h_3 = 0 \Rightarrow h_1 = \frac{3}{2} h_3 \\ 3h_1 + h_2 - 2h_3 = 0 \end{cases}$$

$$\begin{cases} \frac{3}{2} h_3 + h_2 + h_3 = 0 \\ \frac{9}{2} h_3 + h_2 - \frac{4}{2} h_3 = 0 \end{cases} \qquad \begin{cases} \frac{5}{2} h_3 + h_2 = 0 \\ \frac{5}{2} h_3 + h_2 = 0 \end{cases}$$

$$3 \quad h_3 = -2 \Rightarrow h_1 = -3, \quad h_2 = 5$$

$$\Rightarrow h^1 = \begin{Vmatrix} -3 & 5 & -2 \end{Vmatrix}$$

$$\lambda_2 = -2 \Rightarrow \quad h^2 \cdot \begin{Vmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{Vmatrix} = 0$$

$$\begin{cases} ~~4h_1 + h_2 + h_3 = 0~~ \\ -2h_1 + 3h_2 + 3h_3 = 0 \\ 3h_1 + h_2 + h_3 = 0 \end{cases}$$

$$h^2 = \begin{Vmatrix} 0 \\ 1 \\ -1 \end{Vmatrix}^T$$

$$\lambda_3 = 3 \quad h^3 \cdot \begin{Vmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{Vmatrix}$$

$$h^3 = \|$$

$$\begin{cases} -h_1 + h_2 + h_3 = 0 \\ -2h_1 - 2h_2 + 3h_3 = 0 \\ 3h_1 + h_2 - 4h_3 = 0 \end{cases}$$

$h_1 = h_2 + h_3$

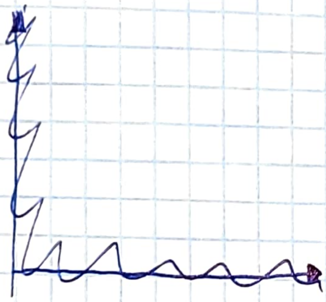
$\swarrow \; -2h_2 - 2h_3 - 2h_2 + 3h_3 = 0 \Rightarrow -4h_2 + h_3 = 0$

$h_3 = 4h_2$

$\Rightarrow \begin{matrix} h_1 = 5h_2 \\ h_3 = 4h_2 \end{matrix} \rightarrow$ ~~$15h_1 + h_2 - 16h_2 = 0$~~

$\exists\, h_2 = 1 \Rightarrow h^3 = \| 5 \;\; 1 \;\; 4 \|$

$$\Rightarrow \Omega = \begin{Vmatrix} -3 & 5 & -2 \\ 0 & 1 & -1 \\ 5 & 1 & 4 \end{Vmatrix} \Rightarrow \underset{\begin{Vmatrix} r_1 \\ r_2 \\ r_3 \end{Vmatrix}}{\underset{\parallel}{\exists r}} = \Omega \cdot \begin{Vmatrix} u \\ v \\ w \end{Vmatrix} = \begin{Vmatrix} -3u + 5v - 2w \\ v - w \\ 5u + v - 4w \end{Vmatrix}$$ – инв Римана.



$\lambda_1, \lambda_3 \rightarrow$ на правую границу, $\lambda_2 \rightarrow$ на левую $\Rightarrow$ 1-ая некорректна
2-ая корректна
3-я и 4-я некорректны

2) Нет шаблона!!! Или шаблон сверху, то Лакс-Вендрофф.

3)
$$\begin{cases} -3u_m^{n+1} + 9v_m^{n+1} - 2w_m^{n+1} = r_{1\,m}^{\;n+1} \\ v_m^{n+1} - w_m^{n+1} = r_{2\,m}^{\;n+1} \\ 5u_m^{n+1} + v_m^{n+1} - 4w_m^{n+1} = r_{3\,m}^{\;n+1} \end{cases}$$

$$\Rightarrow \begin{cases} u_m^{n+1} = \frac{1}{2} r_{1\,m}^{\;n+1} - 3 r_{2\,m}^{\;n+1} + \frac{1}{2} r_{3\,m}^{\;n+1} \\ v_m^{n+1} = \frac{5}{6} r_{1\,m}^{\;n+1} - \frac{11}{3} r_{2\,m}^{\;n+1} + \frac{1}{2} r_{3\,m}^{\;n+1} \\ w_m^{n+1} = \frac{5}{6} r_{1\,m}^{\;n+1} - \frac{14}{3} r_{2\,m}^{\;n+1} + \frac{1}{2} r_{3\,m}^{\;n+1} \end{cases}$$

Лакс-Вендрофф не даст решение на границах => используем Бима-Уорминга

Но если схема устойчива, то решить нельзя, потому что она не представлена в задачнике.